0.1 Preface

Multivariable Calculus refers to Calculus involving functions of more than one variable, i.e., multivariable functions. Not surprisingly, it is important that the reader have a good command of one-variable Calculus, both differential and integral Calculus, before diving into multivariable Calculus.

Several aspects of multivariable Calculus are quite simple. Partial derivatives are just one-variable derivatives, in which you treat all other independent variables as constants. Iterated integrals are the analogous concept for integration; the integrals involved are "partial integrals" (though no one calls them that).

However, the complexity comes in when you consider the different directions in which you can ask for the rates of change of a multivariable function. For a one-variable function, $f(x)$, you are interested in the instantaneous rate of change in f as x moves to the right (i.e., increases) or as x moves to the left (i.e., decreases). For a function of even two variables, $f(x, y)$, there are an infinite number of directions in which (x, y) can move and in which you would want the corresponding rate of change of f.

The fact that you want to look at rates of change in an infinite number of directions means that the derivative, at a given point, of a multivariable function is itself a function of the direction in which the point moves. Once the derivative has to be a function, it is nicest to let the derivative incorporate not only the direction of movement of the point, but also the speed. This leads us to consider the derivative, at a point, as a function that can be applied to arbitrary *vectors*, for vectors are things which have both direction and magnitude.

This point of view of the derivative as a vector function is extremely beautiful, and a large part of its beauty stems from the fact that the derivative is then a *linear transformation*, the fundamental type of function considered in *linear algebra*. For this reason, many statements and results in multivariable Calculus look nicest when given in the language of linear algebra.

Directions and vectors also arise in the most complicated aspects of multivariable integration problems, in which you want, for various reasons, to integrate a *vector field*. The theorems and applications involving integration of vector fields are certainly the most difficult parts of multivariable Calculus.

And so, the question arises of how to best present both the easy and the difficult aspects of multivariable Calculus. To give the rigorous, technical definitions or hypotheses would make even reasonably simple results look difficult, and make the difficult results look nightmarish. The proofs would also complicate the presentation. Finally, there is the dilemma of whether or not to include serious linear algebra in the discussion.

We deal with these issues in a variety of ways.

- First, most sections are divided into subsections, two of which are labeled **Basics** and **More Depth**, so that the "easier" material is separated from the "more difficult" material. Each subsection of **Basics** has an associated video lecture, which can be viewed online by clicking the play button in the margin below the **Basics** box.

• Material in a given section that can be presented nicely in terms of linear transformations and matrices is found in a third subsection, which is labeled + **Linear Algebra**. These subsections can be easily omitted from a course syllabus.

If the + **Linear Algebra** subsections are included in the syllabus, it should be noted that, while vectors are discussed in the body of the textbook, linear transformations and matrices are not. We assume that either the reader already knows linear algebra, or is willing to learn it along the way.

To aid the student in the latter case, rather than include an appendix on this material, we have taken the more modern, but somewhat worrisome, approach of putting in links, in green, to the relevant Wikipedia articles. We say that this is "somewhat worrisome" since Wikipedia articles are open for anyone to edit and, typically, are not a good rigorous reference for deep mathematics. However, we have vetted the linear algebra articles, and they seem to be very good and free from errors. Nonetheless, the Wikipedia articles should be used as an introduction or a refresher, but not as a substitute for a serious linear algebra textbook.

• In those sections which are divided into subsections, the exercises are also divided into **Basics**, **More Depth**, and + **Linear Algebra** for ease in assigning appropriate problems.

• Proofs, other than short ones, which illuminate the material, are **not** contained in this textbook. Instead, the reader is pointed to the appropriate external references. In particular, we refer as often as possible to the excellent, free, pdf textbook of Trench, [8], and provide a link to that pdf. This use of external references, with links, should increase the readability of this textbook, and shorten the book for possible printing, while at the same time providing fully rigorous mathematics. We believe that external links to technical proofs is the future of high school and undergraduate mathematics textbooks.

The background required to read this book is a good understanding of single-variable Calculus, and we assume that you have had courses in differential and integral Calculus. Ideally, you would also be familiar with infinite series, but that material rarely comes up in multivariable Calculus.

Basic references for technical results, and results beyond the scope of this textbook, are Rudin, [7], and Trench, [8].

David B. Massey
February 2011

Chapter 1

Multivariable Spaces and Functions

This chapter is an introduction to Euclidean space, of arbitrarily high dimension, with emphasis on the 2-dimensional case of the xy-plane, $\mathbb{R}^2$, and the 3-dimensional case of xyz-space, $\mathbb{R}^3$.

We define higher-dimensional analogs of concepts that you are familiar with in the real line, $\mathbb{R}$; we generalize such notions as: intervals, open intervals, closed bounded intervals, directions, absolute value/magnitude, functions, continuity, graphs, etc.

We discuss vectors, angles, lines, and planes. This leads us to define two special product operations: the dot product and the cross product.

Finally, this chapter contains a small amount of Calculus. We present the relatively easy, but important, case of derivatives of a function of a single variable, which takes values in a higher-dimensional Euclidean space.

1.1 Euclidean Space

Multivariable Calculus, as the name implies, deals with the Calculus of functions of more than one variable. In this section, we define and discuss basic notions and terminology concerning n-dimensional Euclidean space, where n could be any natural number. We focus on the cases of $\mathbb{R}^2$, the xy-plane, and $\mathbb{R}^3$, xyz-space.

Basics:

The set of real numbers $\mathbb{R}$, or $(-\infty, \infty)$, is frequently referred to, and pictured as, the *real line*.

We assume that you are familiar with the notions of *open intervals*, (a, b), closed intervals, $[a, b]$, and intervals, in general, in the real line.

Also, if a and b are real numbers, you should know that the *distance* between a and b is the absolute value of the difference, i.e.,

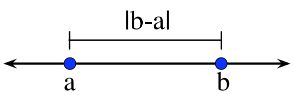

Figure 1.1.1: The distance between points in the real line.

$$\text{dist}(a, b) \ = \ |b-a| \ = \ \sqrt{(b-a)^2},$$

where we have written this last complicated form for the absolute value because it generalizes nicely. The set $\mathbb{R}$, together with its distance function, is known as 1-*dimensional Euclidean space.*

You are also familiar with the xy-plane, in which points are described by pairs of real numbers (x, y). The set of pairs of real numbers is denoted by $\mathbb{R}^2$. Distance in the xy-plane is computed via the Pythagorean Theorem; the distance between points (x_1, y_1) and (x_2, y_2) is given by

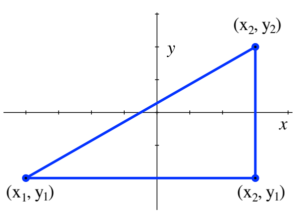

Figure 1.1.2: The distance between points in $\mathbb{R}^2$.

$$\text{dist}\big((x_1, y_1),\, (x_2, y_2)\big) \ = \ \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}.$$

You may already be familiar with $\mathbb{R}^3$, 3-*dimensional Euclidean space*, also known as xyz-space. As a set, this consists of ordered triples of real numbers (x, y, z). This is frequently thought of as representing the space we live in. You could imagine that the floor is the xy-plane, and we have a third axis, the z-axis, which is perpendicular to the floor, i.e., perpendicular to the xy-plane, with the positive z-axis being above the floor, and the negative z-axis being below the floor. Then, z would measure height above the floor, so that a negative z value would indicate that you're below the floor.

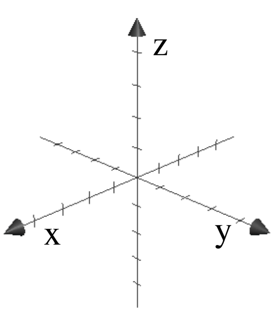

Figure 1.1.3: The x-, y-, and z-axes.

In Figure 1.1.3, we've drawn a typical sketch of the x-, y-, and z-axes (in perspective), with the arrows pointing in the positive directions. Sometimes, to get a better view of some of our later graphs in $\mathbb{R}^3$, it will be convenient to rotate the axes, such as in Figure 1.1.4. Note, however, that, even though it's convenient to rotate the axes, we will always use *right-handed axes*; this means that, if you take your right hand, and point your index finger in the direction of the positive x-axis, while pointing your middle finger in the direction of the positive y-axis, then your thumb will point in the direction of the positive z-axis. See Figure 1.1.5. Of course, the other choice of axes is left-handed, as in Figure 1.1.6; we won't use left-handed axes in this book.

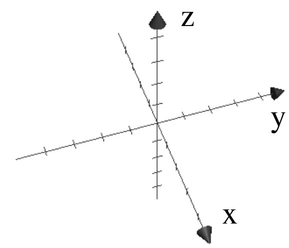

Figure 1.1.4: Rotated x-, y-, and z-axes.

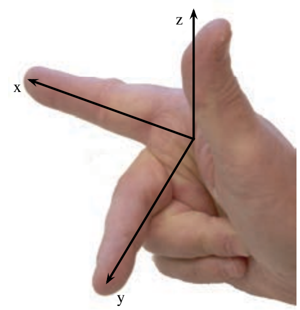

Figure 1.1.5: Right-handed axes.

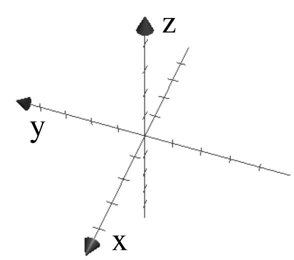

Figure 1.1.6: Left-handed axes.

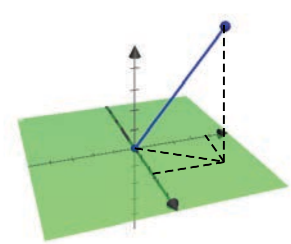

Figure 1.1.7: The Pythagorean Theorem in 3d.

In the xy-plane, you have four quadrants, corresponding to the four choices of positive/negative for the x- and y-coordinates. In $\mathbb{R}^3$, there are eight *octants*, corresponding to the eight choices of positive/negative for the x-, y-, and z-coordinates. However, only one of these octants is given a name; the *1st octant* consists of those points where x, y, and z are ≥ 0.

In three dimensions, by dropping a perpendicular line to the xy-plane, and using the Pythagorean Theorem twice, we can determine the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2); see Figure 1.1.7. We find that

$$\text{dist}\big((x_1, y_1, z_1),\, (x_2, y_2, z_2)\big) \;=\; \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Okay, that's $\mathbb{R}^2$ and $\mathbb{R}^3$, but how do we define $\mathbb{R}^n$, n-dimensional Euclidean space, where n could be any natural number?

It's simple, really. The set $\mathbb{R}^n$, *n-dimensional Euclidean space*, consists of *points* which are n real numbers in order; we call such a point an *ordered n-tuple*. Thus, examples of points in $\mathbb{R}^4$ are the ordered 4-tuples $(2, -1, 0, 3)$ and $(\pi, \sqrt{2}, 7, -e)$. Of course, in place of 2-tuple and 3-tuple, we say pair and triple, respectively. The numbers in the different positions in the n-tuple are called the *components* or *coordinates* of the point. We frequently write a point in $\mathbb{R}^n$ by $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, where the boldface is used to indicate that $\mathbf{x}$ is a point with more than one component.

How do you picture $\mathbb{R}^4$, or $\mathbb{R}^n$ for $n > 4$? You don't. You picture $\mathbb{R}^2$ and $\mathbb{R}^3$, and hope that gives you some intuition for higher-dimensional $\mathbb{R}^n$. In fact, the context of dealing with $\mathbb{R}^n$ might mean that it's completely unreasonable to think of the different coordinates as specifying position. For instance, a company might produce four different types of liquid cleaning products, and keep track of how much of each product they have in stock simply by listing the number of thousands of liters of each, in order, e.g.,

$$(102.7, 34.1, 86.0, 385.4).$$

In this context, it wouldn't be reasonable to think of this point in $\mathbb{R}^4$ as specifying a position; it's just four real numbers in order. Note that we say this point in $\mathbb{R}^4$ has units of thousands of liters only if every coordinate has the same units of thousands of liters.

We referred to $\mathbb{R}^n$ as *Euclidean space*, but, technically, Euclidean space refers to $\mathbb{R}^n$ with a specific notion of *distance*.

Definition 1.1.1. Euclidean n-space, *or* **n-dimensional Euclidean space**, *is the set $\mathbb{R}^n$ of ordered n-tuples of real numbers, together with the notion of the* **distance** *between two points $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ given by*

$$\operatorname{dist}(\mathbf{a}, \mathbf{b}) \;=\; \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots (a_n - b_n)^2}.$$

The **origin in** $\mathbb{R}^n$ *is the ordered n-tuple of all zeroes; we usually write* $\mathbf{0}$ *for the origin (regardless of what dimension we are using).*

Technically, there is a difference between an ordered set with a single real number in one component and the real number itself. We shall not worry about this distinction.

Note that when $n = 1$, 2, or 3, the notion of distance in Euclidean space is exactly what we already discussed.

We need to discuss how to generalize standard notions from the real line to higher dimensions; we need some generalizations/replacements for intervals, open intervals, closed and bounded intervals, etc.

First, we need some even more basic terminology and notation. We sometimes use braces, curly brackets, $\{\cdots\}$, to enclose the elements of a set. We use a vertical line, $|$, as shorthand for the phrase "such that", when describing sets. So, for instance, the set of real numbers greater than 4 is written

$$\{x \in \mathbb{R} \mid x > 4\},$$

which you read as "the set of those x in $\mathbb{R}$ such that $x > 4$". Of course, this is a subset of $\mathbb{R}$, and is the same as the interval $(4, \infty)$.

A *subset* E of $\mathbb{R}^n$ is a collection of some (including, possibly, all or none) of the points in $\mathbb{R}^n$. We write $E \subseteq \mathbb{R}^n$ to indicate that E is a subset of $\mathbb{R}^n$, and we write $\mathbf{p} \in E$ to indicate that $\mathbf{p}$ is a point in, or element of, E. If A and B are both subsets of $\mathbb{R}^n$ (or, are sets, in general), then A is a subset of B, written $A \subseteq B$, if and only if every element of A is also in B.

Now, we can define the most basic generalizations of open and closed intervals in $\mathbb{R}$: open and closed balls in $\mathbb{R}^n$.

Definition 1.1.2. *Suppose that r is a positive real number, and that $\mathbf{p}$ is a point in $\mathbb{R}^n$. Then, the* **n-dimensional open (respectively, closed) ball, $B_r^n(\mathbf{p})$ (respectively, $\overline{B_r^n}(\mathbf{p})$), centered at $\mathbf{p}$, of radius** *r is the set of points in $\mathbb{R}^n$ whose distance from the center is less than (respectively, less than or equal to) r.*

Thus, the open ball is given, in set notation, by

$$B^n_r(\mathbf{p}) \;=\; \{\mathbf{x} \in \mathbb{R}^n \mid \operatorname{dist}(\mathbf{x},\mathbf{p}) < r\},$$

while the closed ball is given by

$$\overline{B^n_r}(\mathbf{p}) \;=\; \{\mathbf{x} \in \mathbb{R}^n \mid \operatorname{dist}(\mathbf{x},\mathbf{p}) \leq r\},$$

The $(n-1)$**-dimensional sphere**, $S^{n-1}_r(\mathbf{p})$*, of radius* $r > 0$*, centered at* $\mathbf{p}$*, is the boundary of the* n*-dimensional ball. Thus,*

$$S^{n-1}_r(\mathbf{p}) \;=\; \{\mathbf{x} \in \mathbb{R}^n \mid \operatorname{dist}(\mathbf{x},\mathbf{p}) = r\}.$$

This means that, if $\mathbf{p} = (p_1, \ldots, p_n)$*, then* $S^{n-1}_r(\mathbf{p})$ *is the set of points* $\mathbf{x} = (x_1, \ldots, x_n)$ *in* $\mathbb{R}^n$ *which satisfy the equation*

$$(x_1 - p_1)^2 + (x_2 - p_2)^2 + \cdots + (x_n - p_n)^2 \;=\; r^2.$$

Note that a 2-dimensional sphere, which lies in $\mathbb{R}^3$*, is what is usually referred to as just "a sphere".*

You may have noticed that this definition means that, what we normally call a circle in $\mathbb{R}^2$, is also referred to as a 1-dimensional sphere. It's also true that n can be 1, which means that a 0-dimensional sphere in the real line consists of two points.

Example 1.1.3. For $r > 0$, the set of points (x, y, z) such that

$$x^2 + y^2 + z^2 = r^2,$$

describes a sphere of radius r, centered at the origin.

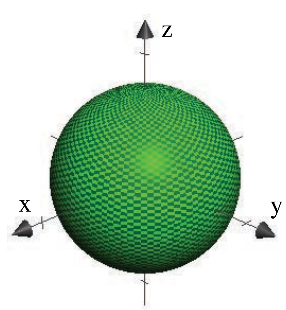

Figure 1.1.8: A sphere of radius $r > 0$, centered at the origin.

About the Author:

David B. Massey was born in Jacksonville, Florida in 1959. He attended Duke University as an undergraduate mathematics major from 1977 to 1981, graduating *summa cum laude*. He remained at Duke as a graduate student from 1981 to 1986. He received his Ph.D. in mathematics in 1986 for his results in the area of complex analytic singularities.

Professor Massey taught for two years at Duke as a graduate student, and then for two years, 1986-1988, as a Visiting Assistant Professor at the University of Notre Dame. In 1988, he was awarded a National Science Foundation Postdoctoral Research Fellowship, and went to conduct research on singularities at Northeastern University. In 1991, he assumed a regular faculty position in the Mathematics Department at Northeastern. He has remained at Northeastern University ever since, where he is now a Full Professor.

Professor Massey has won awards for his teaching, both as a graduate student and as a faculty member at Northeastern. He has published 34 research papers, and two research-level books. In addition, he was a chapter author of the national award-winning book on teaching: "Dear Jonas: What can I say?, Chalk Talk: E-advice from Jonas Chalk, Legendary College Teacher", edited by D. Qualters and M. Diamond, New Forums Press, (2004).

Professor Massey founded the Worldwide Center of Mathematics, LLC, in the fall of 2008, in order to give back to the mathematical community, by providing free or very low-cost materials and resources for students and researchers.