0.1 Preface

Welcome to the Worldwide Integral Calculus textbook; the second textbook from the Worldwide Center of Mathematics.

Our goal with this textbook is, of course, to help you learn Integral Calculus (and power series methods) – the Calculus of integration. But why publish a new textbook for this purpose when so many already exist? There are several reasons why we believe that our textbook is a vast improvement over those already in existence.

• Even if this textbook is used as a classic printed text, we believe that the exposition, explanations, examples, and layout are superior to every other Calculus textbook. We have tried to write the text as we would speak the material in class; though, of course, the book contains far more details than we would normally present in class. In the book, we emphasize intuitive ideas in conjunction with rigorous statements of theorems, and provide a large number of illustrative examples. Where we think it will be helpful to you, we include proofs, or sketches of proofs, in the midst of the sections, but the extremely technical proofs are contained in the Technical Matters appendices to chapters, or are contained in referenced external sources. This greatly improves the overall readability of our textbook, while still allowing us to give mathematically precise definitions and statements of theorems.

• Our textbook is an Adobe pdf file, with linked/embedded/accompanying video content, annotations, and hyperlinks. With the videos contained in the supplementary files, you effectively possess not only a textbook, but also an online/electronic version of a course in Integral Calculus. Depending on the version of the files that you are using, clicking on the video frame to the right of each section title will either open an online, or an embedded, or a locally installed video lecture on that section. The annotations replace classic footnotes, without affecting the readability or formatting of the other text. The hyperlinks enable you to quickly jump to a reference elsewhere in the text, and then jump back to where you were.

• The pdf format of our textbook makes it incredibly portable. You can carry it on a laptop computer, on many handheld devices, e.g., an iPad, or can print any desired pages.

• Rather than force you to buy new editions of textbooks to obtain corrections and minor revisions, updates of this textbook are distributed free of cost.

• Because we have no print or dvd costs for the electronic version of this book and/or videos, we can make them available for download at an extremely low price. In addition, the printed, bound copies of this text and/or disks with the electronic files are priced as low as possible, to help reduce the burden of excessive textbook prices.

In this book, we assume you are already familiar with Differential Calculus. Specifically, we assume that you know the definition of the derivative of a function, that it represents the instantaneous rate of change, and that you know the "rules" for calculating derivatives. We will also need l'Hôpital's Rule and parameterized curves. Referring to the Worldwide Differential Calculus textbook [2], this means that you should know the contents of Chapters 1 and 2, Section 3.5, and Appendix A.

Our discussion of definite integrals, and their applications, is fairly traditional. However, our approach to infinite series is somewhat unusual. Our approach is motivated by two factors. First, we believe that the primary use that students will have for infinite series, outside of a Calculus class, is that many important functions have convergent power series representations, and these power series representations allow the student to mathematically manipulate and estimate the functions involved, in ways that would be difficult/impossible without power series. Second, statistical data that we collected over several years has made it clear that, in general, students do not grasp the basic idea that, when x is close to zero, smaller powers of x are more significant than larger powers of x in a power series or, even, in a polynomial function.

Consequently, we place emphasis on polynomial approximations and power series representations for functions, and, in a sense, view the classic convergence tests for sequences and series of constants as the "technical details" required to understand power series. We still include a chapter, Chapter 5, on sequences and series of constants, but that chapter comes **after** Chapter 4, which is on power series and approximating functions with polynomials. We firmly believe that this ordering of topics is better for the student and for applications, even though it may seem a bit awkward not to have the rigorous mathematical foundations of sequences and series come before their use in discussing power series.

This book is organized as follows:

Other than the Technical Matters sections, each section is accompanied by a video file, which is either a separate file, or an embedded video. Each video contains a classroom lecture of the essential contents of that section; if the student would prefer not to read the section, he or she can receive the same basic content from the video. Each non-technical section ends with exercises. The answers to all of the odd-numbered exercises are contained in Appendix C, at the end of the book.

Important definitions are boxed in green, important theorems are boxed in blue. Remarks, especially warnings of common misconceptions or mistakes, are shaded in red. Important conventions or fundamental principles, that will be used throughout the book, are boxed in black.

Very technical definitions and proofs from each section are contained in the Technical Matters appendices at the ends of some chapters, or in external sources. Our favorite external technical source is the excellent textbook by William F. Trench, *Introduction to Real Analysis*, [4], which is available, courtesy of the author, as a free pdf. For producing answers to various exercises or for help with examples or visualization, you may find the free web site wolframalpha.com very useful.

Internal references through the text are hyperlinked; simply click on the boxed-in link to go to the appropriate place in the textbook. If you have activated the “forward” and ”back” buttons in your pdf-viewer software, clicking on the “back” button will return you to where you started, before you clicked on the hyperlink.

Some terms or names are annotated; these are clearly marked in the margins by little blue “balloons”. Comments will pop up when you click on such annotated items.

Occasionally, when looking at approximations, we write an equals sign in quotes, as in “=”. We use this to denote “equal as far as a calculator is concerned”, i.e., equal to the precision of many/most/all calculators.

We sincerely hope that you find using our modern, multimedia textbook to be as enjoyable as using a mathematics textbook can be.

David B. Massey
August 2009

Chapter 1

Anti-differentiation: the Indefinite Integral

In this chapter, we discuss anti-differentiation, which is also called *indefinite integration*. This is the process for "undoing" differentiation. In the first section, we start with the basic techniques/results, and then in the remaining sections, we include some more-complicated methods.

The indefinite integral should not be confused with the definite integral, which is the topic of the next chapter. The definite integral is the mathematically precise notion of what it means to "take a continuous sum of infinitesimal contributions." The reason that both indefinite and definite integration are referred to as "integration" is because calculating continuous sums and finding anti-derivatives are related by the *Fundamental Theorem of Calculus*, Theorem 2.4.10.

1.1 Basic Anti-Differentiation

This section is about the process and formulas involved in un-doing differentiation, that is, in *anti-differentiating*. This means that you are given a function $f(x)$ and are asked to produce some/all functions $F(x)$ which have $f(x)$ as their derivative. This comes up often in applications, such as when you're given the acceleration $a(t)$ of an object and want the velocity $v(t)$, or when calculating *definite integrals* via the Fundamental Theorem of Calculus (see Section 2.3 and Theorem 2.4.10).

Since you know differential Calculus, you know what it means to have a function $F(x)$ and then be asked to calculate its derivative $F'(x)$. For instance, if $F(x) = x^3$, then $F'(x) = 3x^2$.

But what about the "reverse" question? What if you are given the function $f(x) = 3x^2$ and asked to produce an *anti-derivative* of $f(x)$, that is, if you are asked to find a function $F(x)$ whose derivative equals the given $f(x)$?

Certainly, $F(x) = x^3$ is **one** anti-derivative of $3x^2$. Are there any others? According to a corollary to the Mean Value Theorem, the only other anti-derivatives of $3x^2$ are functions that differ by a constant from the one anti-derivative that we produced, i.e., every other anti-derivative $F(x)$ of $f(x) = 3x^2$ is of the form $F(x) = x^3 + C$, for some constant C.

Definition 1.1.1. *Given a function $f(x)$, defined on an open interval I, a function $F(x)$, on I, such that $F'(x) = f(x)$ is called an* **anti-derivative of** *$f(x)$, with respect to x.*

Thus, an anti-derivative $y = F(x)$ of $f(x)$ is a solution to the differential equation $dy/dx = f(x)$.

If $F(x)$ is an anti-derivative of $f(x)$, on an open interval, then every anti-derivative of $f(x)$, on that interval, is given by $y = F(x) + C$, where C is a constant. The collection $y = F(x) + C$ is called the **(general) anti-derivative of** *$f(x)$, with respect to x; it is the general solution y to the differential equation $dy/dx = f(x)$.*

The notation for the general anti-derivative of $f(x)$, with respect to x, is

$$\int f(x)\,dx.$$

This is also called the **(indefinite) integral of** *$f(x)$, with respect to x.*

Remark 1.1.2. We have several important comments to make.

• First, it is important that $\int f(x)\,dx$ is not one particular function, but it **almost** is; $\int f(x)\,dx$ is actually a collection, or set, of functions, any two of which differ by a constant.

We write

$$\int 3x^2\,dx \;=\; x^3 \;+\; C,$$

where including the $+C$ is extremely important, for changing the value of C changes which element of the set of all anti-derivatives of $3x^2$ you are talking about. Technically, we ought to write

$$\int 3x^2\,dx \;=\; \{x^3 + C \mid C \in \mathbb{R}\},$$

but this is very cumbersome, and no one (well...no one that we know of) ever writes this.

• Second, you should notice that it follows from the definition that the units of $\int f(x)\,dx$ are the units of $f(x)$ times the units of x.

For instance, if $f(x)$ is in kilograms per cubic meter, and x is in cubic meters, then $\int f(x)\,dx$ is in kilograms.

• Third, you should think of the anti-differentiation, with respect to x, operator, $\int (\quad)\,dx$, as essentially being the inverse operator of $\frac{d}{dx}(\quad)$, differentiation with respect to x. That is, the anti-differentiation operator is a compound symbol; it starts with a $\int$, and ends with a differential, like dx, which, together, tell you to anti-differentiate whatever is in-between with respect to the variable which appears in the differential.

We wrote "essentially" above because, if you first differentiate and then anti-differentiate, you get what you started with, except that there is an additional $+C$; that is, you end up with a collection of functions that all differ by constants, instead of simply the one function that you started with.

• We should also comment on the term "indefinite integral." There is another notion, called the *definite integral* of a function over a closed interval; see Section 2.3. The definite integral is defined in such a way that it agrees with one's intuitive idea of what a "continuous sum of infinitesimal contributions" should mean. This would seem to be unrelated to anti-differentiating. However, there is a theorem, the **Fundamental Theorem of Calculus**, which tells us: i) every continuous function possesses an anti-derivative (Theorem 2.4.7), and ii) the primary step used to obtain a nice formula for a definite integral is to produce an anti-derivative of the given function (Theorem 2.4.10).

Hence, anti-differentation is frequently referred to simply as "integration", and definite integration is also simply referred to as "integration"; the context should always make it clear whether the meaning is anti-differentiation or definite integration. In addition, the symbols for anti-differentiating $\int (\ )\, dx$ are essentially the same as the symbols used for definite integration.

All of the differentiation formulas which you have learned yield corresponding anti-differentiation formulas; it's just a matter of reading things "in reverse", for, if $F'(x) = f(x)$, then the corresponding integration rule is $\int f(x)\, dx = F(x) + C$, where C denotes an arbitrary constant. In this context, $f(x)$ is frequently referred to as the *integrand*.

For instance, we have a **Power Rule for Integration**:

Theorem 1.1.3. *For all x in an open interval for which the functions involved are defined,*

1. $\int 0\, dx = C$;
2. $\int 1\, dx = x + C$;
3. *if* $p \neq -1$, $\displaystyle\int x^p\, dx = \frac{x^{p+1}}{p+1} + C$; *and*
4. $\displaystyle\int x^{-1}\, dx = \int \frac{1}{x}\, dx = \ln|x| + C$.

Remark 1.1.4. In many books, only the third formula above is referred to as the Power Rule for Integration.

As is frequently the case, you should try to remember this rule not in symbols, but in words; it says that, as long as the exponent is not -1, you obtain the anti-derivative of x raised to a constant exponent by adding one to the exponent, and dividing by the new exponent (and then adding a C).

Remark 1.1.5. There are two fairly common, horrific mistakes associated with the integration rule $\displaystyle\int x^{-1}\, dx = \int \frac{1}{x}\, dx = \ln|x| + C$.

The first big mistake is to treat the $p = -1$ case in the same manner as the cases where $p \neq -1$. If you were to do this, you would obtain

$$\int x^{-1}\, dx \;=\; \frac{x^{-1+1}}{-1+1} + C \;=\; \frac{x^0}{0} + C.$$

The undefined division by 0 should immediately tell you that you've done something wrong, and remind you that you must treat the $p = -1$ case differently.

The second big mistake may come later, when we have more integration rules. It will then be tempting to look at the formula $\int \frac{1}{x}\, dx = \ln|x| + C$ and think that it implies that

$$\int \frac{1}{\text{anything}}\, dx \;=\; \ln|\text{anything}| + C.$$

This is **completely wrong** (in general); it is not true that the derivative of the expression on the right would be the integrand. The problem is that, if you differentiate the expression on the right, you do, in fact, get a $1/\text{anything}$ factor, but then the Chain Rule tells you that that is multiplied by $d(\text{anything})/dx$.

Example 1.1.6. Find the function $P(r)$, with domain $r > 0$, such that

$$\frac{dP}{dr} \;=\; \sqrt{r} \qquad \text{and} \qquad P(9) = -7.$$

Solution:

We find that

$$P \;=\; \int r^{1/2}\, dr = \frac{r^{3/2}}{3/2} + C \;=\; \frac{2}{3}\, r^{3/2} + C.$$

We need to determine C. We have

$$-7 \;=\; P(9) \;=\; \frac{2}{3}\,(9)^{3/2} + C \;=\; 18 + C.$$

Therefore, $C = -25$, and so

$$P = \frac{2}{3} r^{3/2} - 25.$$

The linearity of the derivative gives us the linearity of the anti-derivative.

Theorem 1.1.7. (**Linearity of Anti-differentiation**) *If a and b are constants, not both zero, then*

$$\int af(x) + bg(x)\,dx = a \cdot \int f(x)\,dx + b \cdot \int g(x)\,dx.$$

Remark 1.1.8. The prohibition against $a = b = 0$ in Theorem 1.1.7 is there for just one reason: we do not want both of the arbitrary constants on the right to be eliminated by multiplying by zero. We will explain this more fully.

Since each indefinite integral actually yields a set, or collection, of functions, there may be some question in your mind about what it means to multiply a set of functions by a constant, like a or b, and what it means to add two such sets. In other words, you may wonder exactly what the right-hand side of the equality in Theorem 1.1.7 means.

For instance, what does it mean to write that

$$\int (5x^3 - 7\sqrt{x})\,dx = 5 \cdot \int x^3\,dx - 7 \cdot \int \sqrt{x}\,dx?$$

We know, from the Power Rule for Integration, that $\int x^3\,dx = x^4/4 + C_1$ and

$$\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{x^{3/2}}{3/2} + C_2 = 2x^{3/2}/3 + C_2,$$

where we have used C_1 and C_2, in place of using simply C twice, since we don't want to assume that we have to pick the two arbitrary constants to be the same thing.

So, what does $5 \cdot \int x^3\,dx$ mean? It means the collection of functions obtained by taking 5 times any function from the collection of functions $x^4/4 + C_1$; that is, the collection of functions

About the Author:

David B. Massey was born in Jacksonville, Florida in 1959. He attended Duke University as an undergraduate mathematics major from 1977 to 1981, graduating *summa cum laude*. He remained at Duke as a graduate student from 1981 to 1986. He received his Ph.D. in mathematics in 1986 for his results in the area of complex analytic singularities.

Professor Massey taught for two years at Duke as a graduate student, and then for two years, 1986-1988, as a Visiting Assistant Professor at the University of Notre Dame. In 1988, he was awarded a National Science Foundation Postdoctoral Research Fellowship, and went to conduct research on singularities at Northeastern University. In 1991, he assumed a regular faculty position in the Mathematics Department at Northeastern. He has remained at Northeastern University ever since, where he is now a Full Professor.

Professor Massey has won awards for his teaching, both as a graduate student and as a faculty member at Northeastern. He has published 32 research papers, and two research-level books. In addition, he was a chapter author of the national award-winning book on teaching: "Dear Jonas: What can I say?, Chalk Talk: E-advice from Jonas Chalk, Legendary College Teacher", edited by D. Qualters and M. Diamond, New Forums Press, (2004).

Professor Massey founded the Worldwide Center of Mathematics, LLC, in the fall of 2008, in order to give back to the mathematical community, by providing free or very low-cost materials and resources for students and researchers.