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SMOOTH MIXED PROJECTIVE CURVES AND A CONJECTURE

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Dedicated to Professor Egbert Brieskorn

Abstract. Let $f(\mathbf{z},\bar{\mathbf{z}})$ be a strongly mixed homogeneous polynomial of three variables $\mathbf{z}=(z_1,z_2,z_3)$ of polar degree q with an isolated singularity at the origin. It defines a smooth Riemann surface C in the complex projective space $\mathbb{P}^2$. The fundamental group of the complement $\pi_1(\mathbb{P}^2\setminus C)$ is a cyclic group of order q if f is a homogeneous polynomial without $\bar{\mathbf{z}}$. We propose a conjecture that this may be even true for mixed homogeneous polynomials by giving several supporting examples.

1. Introduction

Let $f(\mathbf{z},\bar{\mathbf{z}})=\sum_{\nu,\mu}c_{\nu,\mu}\mathbf{z}^\nu\bar{\mathbf{z}}^\mu$ be a mixed polynomial of n-variables $\mathbf{z}=(z_1,\dots,z_n)\in\mathbb{C}^n$. A mixed polynomial $f(\mathbf{z},\bar{\mathbf{z}})$ is called *mixed weighted homogeneous* if there exist integers $q_1,\dots,q_n$ and $p_1,\dots,p_n$ and non-zero integers m_r, m_p such that

$$\gcd(q_1,\dots,q_n)=1,\quad \gcd(p_1,\dots,p_n)=1,$$
$$\textstyle\sum_{j=1}^n q_j(\nu_j+\mu_j)=m_r,\quad \sum_{j=1}^n p_j(\nu_j-\mu_j)=m_p,\quad \text{if } c_{\nu,\mu}\neq 0$$

We say $f(\mathbf{z},\bar{\mathbf{z}})$ is *mixed weighted homogeneous of radial weight type* $(q_1,\dots,q_n;m_r)$ *and of polar weight type* $(p_1,\dots,p_n;m_p)$.

Using polar coordinates r, η of $\mathbb{C}^*$ where $r>0$ and $\eta\in S^1$ with $S^1=\{\eta\in\mathbb{C}\,|\,|\eta|=1\}$, we define *a polar* $\mathbb{C}^*$*-action* on $\mathbb{C}^n$ by

$$(r,\eta)\circ\mathbf{z}=(r^{q_1}\eta^{p_1}z_1,\dots,r^{q_n}\eta^{p_n}z_n),\quad (r,\eta)\in\mathbb{R}^+\times S^1$$
$$(r,\eta)\circ\bar{\mathbf{z}}=\overline{(r,\eta)\circ\mathbf{z}}=(r^{q_1}\eta^{-p_1}\bar z_1,\dots,r^{q_n}\eta^{-p_n}\bar z_n).$$

Then f satisfies the Euler equality

$$f((r,\eta)\circ(\mathbf{z},\bar{\mathbf{z}}))=r^{m_r}\eta^{m_p}f(\mathbf{z},\bar{\mathbf{z}}). \tag{E1}$$

It is easy to see that such a polynomial defines a global fibration

$$f:\mathbb{C}^n-f^{-1}(0)\to\mathbb{C}^* \tag{GM}$$

without further assumption. A mixed polynomial $f(\mathbf{z},\bar{\mathbf{z}})$ is called a *strongly mixed weighted homogeneous polynomial* (respectively *strongly mixed homogeneous polynomial*) of n-variables $\mathbf{z}=(z_1,\dots,z_n)\in\mathbb{C}^n$ with polar degree q and radial degree d if $p_i=q_i$ for $i=1,\dots,n$ and $\sum_{i=1}p_i(\nu_i\pm\mu_i)=d$ and q (resp. $p_i=q_i=1$, $i=1,\dots,n$ and $|\nu|+|\mu|=d$ and $|\nu|-|\mu|=q$) for any ν,μ with $c_{\nu,\mu}\neq 0$. Here q is assumed to be a positive integer. For such a strongly mixed weighted homogenous polynomial, the associated $\mathbb{C}^*$-action on $\mathbb{C}^n$ is the holomorphic action defined by

$$(\zeta,(z_1,\dots,z_n))\mapsto\zeta\circ\mathbf{z}=(\zeta^{p_1}z_1\dots,\zeta^{p_n}z_n)$$

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and f satisfies the equality

$$f(\lambda\circ\mathbf{z},\overline{\lambda\circ\mathbf{z}})=r^d\exp{(iq\theta)}f(\mathbf{z},\bar{\mathbf{z}}),\quad \text{where } \lambda=r\exp(i\theta)\in\mathbb{R}^+\times S^1.$$

Assume that f is strongly mixed homogeneous. Then the action is reduced to

$$\lambda\circ\mathbf{z}\,=\,(\lambda z_1,\dots,\lambda z_n).$$

By the above equality, it defines canonically a real analytic projective variety V in $\mathbb{P}^{n-1}$:

$$V=\{[\mathbf{z}]\in\mathbb{P}^{n-1}\mid f(\mathbf{z},\bar{\mathbf{z}})=0\}.$$

Let $\widetilde{V}$ be the mixed affine hypersurface

$$\widetilde{V}=f^{-1}(0)=\{\mathbf{z}\in\mathbb{C}^n\mid f(\mathbf{z},\bar{\mathbf{z}})=0\}.$$

Let $f:\mathbb{C}^n\setminus\widetilde{V}\to\mathbb{C}^*$ be the global Milnor fibration defined by f and let F be the Milnor fiber, namely $F=f^{-1}(1)\subset\mathbb{C}^n$. The monodromy map $h:F\to F$ is defined by

$$h(\mathbf{z})=(\omega_q z_1,\dots,\omega_q z_n),\quad \omega_q=\exp(\frac{2\pi\, i}{q}).$$

and the restriction of the Hopf fibration to the Milnor fiber $\pi:\ F\to\mathbb{P}^{n-1}\setminus V$ is nothing but the quotient map by the cyclic action induced by h.

Remark 1. *We may also consider the case $q=0$ in the above strongly mixed homogeneous polynomial and consider the corresponding projective variety $V=\{[\mathbf{z}]\,|\,f(\mathbf{z},\bar{\mathbf{z}})=0\}$ but V need not be a real codimension 2 hypersurface. For example, $n=3$ and take*

$$f(\mathbf{z},\bar{\mathbf{z}}):=z_1\bar z_1-z_2\bar z_2+z_3\bar z_3.$$

Then $\dim_{\mathbb{R}}V=3$. Note that f does not have a Milnor fibration if $q=0$. Another extreme case is $g(\mathbf{z},\bar{\mathbf{z}}):=z_1\bar z_1+z_2\bar z_2+z_3\bar z_3$. Then $\tilde V=\{\mathbf{0}\}$ and V is empty. Such a polynomial is called a fake *strongly mixed homogeneous polynomial.*

A strongly mixed homogeneous polynomial is called a *true* strongly mixed homogeneous polynomial if f does not have any fake strongly mixed homogeneous factor in the polynomial ring $\mathbb{C}[z_1,\bar z_1,\dots,z_n,\bar z_n]$ which defines a non-empty projective variety.

In [10, 14], we have shown that

Theorem 2 (Theorem 11, [14])**.** *Assume that $f(\mathbf{z},\bar{\mathbf{z}})$ is a non-degenerate, strongly mixed homogeneous polynomial of n variables such that V is irreducible and mixed non-singular in an open dense subset. Then the embedding degree of V is equal to the polar degree q. In particular, $H_1(\mathbb{P}^{n-1}\setminus V)=\mathbb{Z}/q\mathbb{Z}$.*

Here "irreducible" means as a real algebraic variety.

Proposition 3. *Assume that $f(\mathbf{z},\bar{\mathbf{z}})$ is a non-degenerate, strongly mixed homogeneous polynomial of n variables such that V is irreducible and mixed non-singular in an open dense subset. Then the Euler characteristics satisfy the following equalities.*

(1) *$\chi(F)=q\,\chi(\mathbb{P}^{n-1}\setminus V)$ and $\chi(\mathbb{P}^{n-1}\setminus V)=n-\chi(V)$. In particular, if $n=3$ and V is smooth curve with the genus g, then $\chi(F)=q(1+2g)$.*

(2) *The following sequence is exact.*

$$1\to\pi_1(F)\xrightarrow{\pi_\sharp}\pi_1(\mathbb{P}^{n-1}\setminus V)\to\mathbb{Z}/q\mathbb{Z}\to 1.$$

In particular, F is simply-connected if and only if $\pi_1(\mathbb{P}^{n-1}\setminus V)\cong\mathbb{Z}/q\mathbb{Z}$.

(3) *If $q=1$, the projection $\pi:F\to\mathbb{P}^{n-1}\setminus V$ is a diffeomorphism.*

Remark 4. *The assumption that V is irreducible as a real algebraic variety is different from the irreduciblity of f in $\mathbb{C}[z_1, \bar{z}_1, \dots, z_n, \bar{z}_n]$.*

Using the periodic monodromy argument in [7], we have

Proposition 5. *Assume that f is a strongly mixed homogeneous polynomial of polar degree $q > 0$. The zeta function of the monodromy $h : F \to F$ is given by*

$$\zeta(t) = (1 - t^q)^{-\chi(F)/q}.$$

In particular, if $q = 1$, $h = \mathrm{id}_{\mathrm{F}}$ and $\zeta(t) = (1 - t)^{-\chi(F)}$.

If f is a holomorphic function with an isolated singularity at the origin, F is $(n-2)$-connected and it is homotopic to a bouquet of μ spheres of dimension $n-1$ ([7]). For mixed polynomials, we do not have any connectivity theorem. But we do not have any examples of mixed weighted homogeneous polynomials which break the connectivity which holds in the holomorphic case. Thus we propose the following conjecture as a first working problem.

simply-connectedness Conjecture 6. *Assume $n \geq 3$ and that $f : \mathbb{C}^n \to \mathbb{C}$ is a non-degenerate strongly mixed homogeneous polynomial of polar degree q which has an isolated singularity at the origin. In other words, V is an irreducible mixed non-singular hypersurface of real codimension 2. Then*
(a) The Milnor fiber F is simply-connected.
By (2) of Proposition 3, this conjecture is equivalent to the following.
(b) The fundamental group of the complement $\pi_1(\mathbb{P}^{n-1} - V)$ is a cyclic group of order q.

The purpose of this paper is to give several non-trivial examples for the case $n = 3$ which support this conjecture.

Remark 7. *The condition "strongly non-degenerate" (with respect to the Newton boundary), introduced in [15], is necessary to have a Milnor fibration for a non-mixed weighted homogeneous polynomial. However for a mixed weighted homogeneous polynomial, the Milnor fibration (GM) always exists.*

For a mixed weighted homogneous polynomial, the notion 'non-degenerate' implies 'strongly non-degenerate'. We explain this assertion for a strongly mixed weighted homogenous polynomial f for simplicity. Take any face function f_Δ of f. f_Δ is also strongly mixed homogeneous and satisfies the Euler equality

$$f_\Delta(\lambda \circ \mathbf{z}, \overline{\lambda \circ \mathbf{z}}) = r^d \exp(iq\theta) f_\Delta(\mathbf{z}, \bar{\mathbf{z}}), \quad \text{where } \lambda = r\exp(i\theta) \in \mathbb{R}^+ \times S^1.$$

Take any point $\mathbf{w} \in \mathbb{C}^{*n} \setminus V(f_\Delta)$. Consider two tangent vectors $\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \in T_{\mathbf{w}}\mathbb{C}^{*n}$. By the above equality, it is easy to see that their images by $df_\Delta : T_{\mathbf{w}}\mathbb{C}^{*n} \to T_{f_\Delta(\mathbf{w})}\mathbb{C}^*$ are linearly independent. That is, non-zero numbers are regular values for f_Δ. (For a mixed weighted homogeneous polynomial, we use the equality (E1) and do the same argument.)

In Theorem 2 and Proposition 3, to make $\mathbb{P}^{n-1} \setminus V$ connected, we have to assume that V has no real codimension 1 component. This does not happen if f is non-degenerate or 'true'. To make the Milnor fiber F connected, we have to assume that $\dim_{\mathbb{R}} V = 2n - 4$ and the existence of a mixed smooth point (see (1) of Theorem 9). Thus in Theorem 2 and Proposition 3, we can replace the assumption on f by the assumption that f is a true strongly mixed homogeneous polynomial such that V is irreducible and mixed non-singular in an open dense subset.

2. Easy mixed polynomials

Unlike the holomorphic case, we do not know in general the connectivity of the Milnor fiber even under the assumption that $\tilde{V}$ has an isolated singularity at the origin. In this section, we study easy examples. Suppose that f is either a simplicial mixed polynomial or a join type or twisted join type polynomial of three variables. Then the connectivity behaves just as the holomorphic case. We will first explain these polynomials below.

2.1. **Simplicial polynomial.** Assume that $n = 3$ and $\mathbf{z} = (z_1, z_2, z_3)$. A mixed polynomial $f(\mathbf{z}, \bar{\mathbf{z}})$ is called simplicial if it is a linear sum of three mixed monomials

$$f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{i=1}^{3} c_i \mathbf{z}^{\nu_i} \bar{\mathbf{z}}^{\mu_i}$$

and if the two matrices

$$(\nu_i \pm \mu_i)_{i=1}^3 = \begin{pmatrix} \nu_{11} \pm \mu_{11} & \nu_{12} \pm \mu_{12} & \nu_{13} \pm \mu_{13} \\ \nu_{21} \pm \mu_{21} & \nu_{22} \pm \mu_{22} & \nu_{23} \pm \mu_{23} \\ \nu_{31} \pm \mu_{31} & \nu_{32} \pm \mu_{32} & \nu_{33} \pm \mu_{33} \end{pmatrix}$$

are non-degenerate where $\nu_i = (\nu_{i1}, \nu_{i2}, \nu_{i3})$, $\mu_i = (\mu_{i1}, \mu_{i2}, \mu_{i3})$. In this case, we may assume that $c_i = 1$ for $i = 1, 2, 3$. Among them, the following polynomials are strongly mixed homogeneous and have an isolated singularity at the origin.

$$\begin{aligned}
f_B \quad &:= z_1^{q+r}\bar{z}_1^r + z_2^{q+r}\bar{z}_2^r + z_3^{q+r}\bar{z}_3^r, \text{ (Brieskorn Type)} \\
f_I \quad &:= z_1^{q+r-1}\bar{z}_1^r z_2 + z_2^{q+r-1}\bar{z}_2^r z_3 + z_3^{q+r}\bar{z}_3^r, \text{ (Tree type a)} \\
f_{II} \quad &:= z_1^{q+r-1}\bar{z}_1^r z_2 + z_2^{q+r-1}\bar{z}_2^r z_3 + z_3^{q+r-1}\bar{z}_3^r z_1, \text{ (Cyclic type a)} \\
f_{III} \quad &:= z_1^{q+r-1}\bar{z}_1^r z_2 + z_2^{q+r-1}\bar{z}_2^r z_1 + z_3^{q+r}\bar{z}_3^r, \text{ (Simplicial+Join a)} \\
f_I' \quad &:= z_1^{q+r}\bar{z}_1^{r-1}\bar{z}_2 + z_2^{q+r}\bar{z}_2^{r-1}\bar{z}_3 + z_3^{q+r}\bar{z}_3^r, \text{ (Tree type b)} \\
f_{II}' \quad &:= z_1^{q+r}\bar{z}_1^{r-1}\bar{z}_2 + z_2^{q+r}\bar{z}_2^{r-1}\bar{z}_3 + z_3^{q+r}\bar{z}_3^{r-1}\bar{z}_1, \text{ (Cyclic type b)} \\
f_{III}' \quad &:= z_1^{q+r}\bar{z}_1^{r-1}\bar{z}_2 + z_2^{q+r}\bar{z}_2^{r-1}\bar{z}_1 + z_3^{q+r}\bar{z}_3^r, \text{ (Simplicial+Join b).}
\end{aligned}$$

Here $q \geq 1$ and $r \geq 1$ are positive integers. All above polynomials have simply-connected Milnor fibers ([9]). For f_B, f_I, f_{II}, f_{III}, their Milnor fiberings and links are in fact isotopic to the holomorphic ones by the contraction $z_i^r \bar{z}_i^r \mapsto 1$ ([11, 5]):

$$\begin{aligned}
f_B \quad &:= z_1^q + z_2^q + z_3^q, \text{ (Brieskorn Type)} \\
f_I \quad &:= z_1^{q-1} z_2 + z_2^{q-1} z_3 + z_3^q, \text{ (Tree type a)} \\
f_{II} \quad &:= z_1^{q-1} z_2 + z_2^{q-1} z_3 + z_3^{q-1} z_1, \text{ (Cyclic type a)} \\
f_{III} \quad &:= z_1^{q-1} z_2 + z_2^{q-1} z_1 + z_3^q, \text{ (Simplicial+Join a).}
\end{aligned}$$

Remark 8. *The above list does not cover all possibilities. For example, we can combine f_I and f_I':*

$$f_I{}'' := z_1^{q+r}\bar{z}_1^{r-1}\bar{z}_2 + z_2^{q+r-1}\bar{z}_2^r z_3 + z_3^{q+r}\bar{z}_3^r.$$

2.2. **Join type mixed polynomials.** Let $f(\mathbf{z}, \bar{\mathbf{z}})$ be a true strongly mixed homogeneous convenient polynomial of n-variables $\mathbf{z} = (z_1, \dots, z_n)$ of polar degree q and radial degree $q + 2r$ with an isolated singularity at the origin. Consider the join polynomial $g := f(\mathbf{z}, \bar{\mathbf{z}}) - w^{q+r}\bar{w}^r$ of $(n+1)$-variables. Let F_f, F_g be the respective Milnor fibers of f and g. Consider the projective Mixed hypersurfaces V_f and V_g defined by $f = 0$ and $g = 0$ respectively in $\mathbb{P}^{n-1}$ or $\mathbb{P}^n$.

Theorem 9. *Assume that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a true strongly mixed homogeneous convenient polynomial of n-variables and the corresponding projective variety V_f has a mixed smooth point in V_f and $n \geq 2$. Then*

(1) *F_f is connected and*

(2) $\pi_1(\mathbb{P}^n \setminus V_g) = \mathbb{Z}/q\mathbb{Z}$ *and* F_g *is simply-connected.*

Proof. In this theorem, we do not assume that f is strongly non-degenerate. Note that

$$\begin{aligned} F_f =& \{\mathbf{z} \in \mathbb{C}^n \mid f(\mathbf{z}, \bar{\mathbf{z}}) - 1 = 0\} \\ V_f =& \{[\mathbf{z}] \in \mathbb{P}^{n-1} \mid f(\mathbf{z}, \bar{\mathbf{z}}) = 0\} \\ V_g =& \{[\mathbf{z} : w] \in \mathbb{P}^n \mid f(\mathbf{z}, \bar{\mathbf{z}}) - w^{q+r}\bar{w}^r = 0\}. \end{aligned}$$

Consider the affine chart $U_w := \{w \neq 0\}$ in $\mathbb{P}^n$. In this coordinate space, using affine coordinates $u_j = z_j/w$, $j = 1, \dots, n$, we see that

$$V_g \cap U_w = \{\mathbf{u} \in \mathbb{C}^n \mid f(\mathbf{u}, \bar{\mathbf{u}}) - 1 = 0\}.$$

This expression says that $F_f \cong V_g \cap U_w$. Note that $V_g \cap \{w = 0\} \cong V_f$ has a smooth point p. Consider the projection $\pi : \mathbb{P}^n \to \mathbb{P}^{n-1}$ which is defined by $[\mathbf{z} : w] \mapsto [\mathbf{z}] \in \mathbb{P}^{n-1}$. Then the restriction $\pi : V_g \to \mathbb{P}^{n-1}$ is a q-fold covering branched over $V_f \subset \mathbb{P}^{n-1}$. Take a non-singular point p of $V_f \subset \mathbb{P}^{n-1}$ and consider a small normal disk D centered at p. For simplicity, we assume that $p \in \{z_1 \neq 0\}$ and we choose the affine coordinate chart $\{z_1 \neq 0\}$ with affine coordinates $v_j = z_j/z_1$, $j = 2, \dots, n$ and $x = w/z_1$. In this chart, V_g is defined by $f(\mathbf{v}, \bar{\mathbf{v}}) - x^{q+r}\bar{x}^r = 0$ with $\mathbf{v} = (1, v_2, \dots, v_n)$. Then the covering $(\mathbf{v}, x) \mapsto \mathbf{v}$ is topologically equivalent to the holomorphic cyclic covering defined by $x^q - f = 0$ in a small disk D with center p. (In D, we can take the function $f : D \to \mathbb{C}$ as a real analytic complex-valued coordinate function and we may assume that the image $f(D)$ is a small unit disk Δ_ρ with radius ρ.) Thus the fiber of a boundary point p' , $f(p') = \rho e^{i\theta_0} \in \partial\Delta$, decomposes by $\{Re^{i(\theta_0+2j\pi)/q} \mid j = 0, \dots, q-1\}$ in x-coordinate with $R = |\rho|^{1/(q+2r)}$ and under the local monodromy along ∂D, those q points are cyclically rotated as $Re^{i(\theta_0+2j\pi)/q} \mapsto Re^{i(\theta_0+2(j+1)\pi)/q}$, $j = 0, \dots, q-1$. Thus $\pi^{-1}(D^*)$ is connected, where $D^* := D \setminus \{p\}$. As $\mathbb{P}^{n-1} \setminus V_f$ is connected, any point $y \in V_g \setminus V_f$ can be connected using the covering structure to one of the points $\pi^{-1}(p')$. Here we identify V_f with $V_g \cap \{w = 0\}$. As $V_g - V_f = V_g \cap U_\omega$, $V_g \cap U_\omega$ is connected.

Now we consider the fundamental group, assuming $n = 2$ for simplicity. V_g is defined by $z_3^{q+r}\bar{z}_3^r - f(\mathbf{z}, \bar{\mathbf{z}})$ where $\mathbf{z} = (z_1, z_2)$. Consider the pencil lines $L_\eta = \{z_2 = \eta z_1\}$ and let $b = (0 : 0 : 1)$ be the base point of the pencil. Let $\widetilde{\mathbb{P}}^2$ be the blow-up space at b. Then $\widetilde{\pi} : \widetilde{\mathbb{P}}^2 \to \mathbb{P}^1$ is well defined and $\pi_1(\widetilde{\mathbb{P}}^2 \setminus \widetilde{V}_g) \equiv \pi_1(\mathbb{P}^2 \setminus V_g)$ with $\widetilde{V}_g = \widetilde{\pi}^{-1}(V_g) \cong V_g$. The zero points $f(\mathbf{z}, \bar{\mathbf{z}}) = 0$ are the locus of singular pencil lines. Take a simple zero $p \in V_f$ and take p' nearby as a base line and put $L = \pi^{-1}(p')$. Take generators $\xi_1, \dots, \xi_q$ of $\pi_1(L \setminus V_g \cap L)$ as in Figure 1. They satisfy the vanishing relation at infinity: $\xi_q \dots \xi_1 = e$. The centers of the small circles are the points of $L \cap V_g$. We always orient the small circles counterclockwise. Then the monodromy relations at p are given by

$$\xi_1 = \xi_2 = \cdots = \xi_q, \quad \xi_q \dots \xi_1 = e$$

See [8]. The argument is exactly the same as for a complex algebraic curve with a maximal flex point in Zariski [18]. Thus we get $\xi_1^q = e$ and $\pi_1(\mathbb{P}^2 \setminus V_g) \cong \mathbb{Z}/q\mathbb{Z}$.

The assertion (2) of Theorem 9 is true for any $n \geq 2$. For $n > 2$, we take a generic hyperplane H of type $a_1z_1 + \cdots + a_nz_n = 0$ which contains $[0 : \cdots : 0 : 1]$ and use the surjectivity $\pi_1(H \setminus V \cap H) \to \pi_1(\mathbb{P}^n \setminus V_g)$. The defining polynomial of $V_g \cap H$ is also of join type and use an induction argument. Here we do not use the Zariski Hyperplane section theorem [4] (we do not know if the same assertion holds for mixed hypersurfaces or not) but we only use the surjectivity for a non-singular mixed hypersurface of join type which is easy to be shown. We leave this assertion to reader. ∎

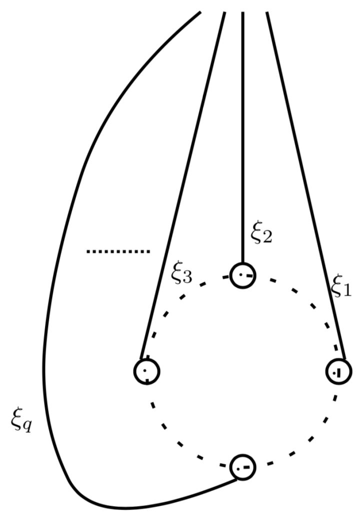

FIGURE 1. Generators of $\pi_1(L - L \cap V)$

Example 10. *Consider Rhie's Lens equation*

$$\varphi_n(z) := \bar{z} - \frac{z^{n-2}}{z^{n-1} - a^{n-1}} - \frac{\varepsilon}{z} = \frac{g(z,\bar{z})}{(z^{n-1} - a^{n-1})z} = 0,\ n \geq 2.$$

We can choose suitable positive numbers a, ε so that $0 < \varepsilon \ll a \ll 1$ and φ_n has $5(n-1)$ simple zeros (see [16] and also [13]). Let $g(z,\bar{z})$ be the numerator of φ_n and take the homogenization of $g(z,\bar{z})$

$$\begin{aligned} G(\mathbf{z}', \bar{\mathbf{z}'}) &:= g(z_1/z_2, \bar{z}_1/\bar{z}_2) z_2^n \bar{z}_2 \\ &= \bar{z}_1(z_1^n - a^{n-1} z_1 z_2^{n-1}) - \bar{z}_2 \left(z_1^{n-1} z_2 + \varepsilon(z_1^{n-1} z_2 - a^{n-1} z_2^{n-1} z_1) \right) \end{aligned}$$

where $\mathbf{z}' = (z_1, z_2)$. Consider the join type polynomial and the associated projective curve C:

$$C:\ f(\mathbf{z}, \bar{\mathbf{z}}) := z_3^n \bar{z}_3 + G(\mathbf{z}', \bar{\mathbf{z}'}) = 0, \quad \mathbf{z} = (z_1, z_2, z_3).$$

Observe that f is strongly mixed homogeneous of polar degree $q = n-1$ and radial degree $n+1$. Consider the affine chart $\{z_2 \neq 0\}$ and consider the affine coordinates $w_3 = z_3/z_2, w_1 = z_1/z_2$. Then the affine equation takes the form $w_3^n \bar{w}_3 - g(w_1, \bar{w}_1) = 0$. Consider the pencil of lines $L_\eta = \{z_1 - \eta z_2 = 0\}$ or in the affine equation, $w_1 = \eta$. There are exactly $5(n-1)$ singular pencil lines corresponding to the zeros of $g(w_1, \bar{w}_1) = 0$. These roots are all simple by the construction. In a small neighborhood of any such zero, the projection $\pi : C \to \mathbb{C}$ is locally equivalent to $w_3^n \bar{w}_3 - w_1 = 0$ or $w_3^n \bar{w}_3 - \bar{w}_1 = 0$ depending on the sign of the zero. Take a point η_0 near some zero of g and take generators $\xi_1, \ldots, \xi_q$ of $\pi_1(L_{\eta_0} \setminus C)$ on the line L_{η_0} as in Figure 1, then we get that $\xi_1 = \cdots = \xi_q$ as the monodromy relation. Thus we get $\pi_1(\mathbb{P}^2 \setminus C) = \mathbb{Z}/q\mathbb{Z}$. Note that $L_\infty \cap C$ consists of q simple points. Thus the Euler number and the genus of C are calculated easily as

$$\begin{aligned} \chi(C) &= (n-1)(2 - (5n-5) - 1) + 5n - 5 + n - 1 = 17n - 5n^2 - 12, \\ g(C) &= \frac{(5n-7)(n-2)}{2}. \end{aligned}$$

In the moduli space of mixed polynomials of polar degree $n-1$ and radial degree $n+1$, the lowest genus is taken by

$$z_1^n\bar{z}_1 + z_2^n\bar{z}_2 + z_3^n\bar{z}_3$$

which is isotopic to the holomorphic curve

$$z_1^{n-1} + z_2^{n-1} + z_3^{n-1} = 0$$

of degree $n-1$ and therefore the genus is $(n-2)(n-3)/2$ by Plücker's formula.

Remark 11. *The genus of a non-singular mixed curve of polar degree q is greater or equal to $\frac{(q-1)(q-2)}{2}$ by the Thom inequality ([6]). In [10], it is shown that for any $g \geq 0$, there exists a mixed non-singular curve of polar degree 1 with genus g.*

2.3. **Twisted join type polynomials.** Let $f(\mathbf{z})$ be a strongly mixed homogeneous polynomial of polar degree q and radial degree $q+2r$ and consider the mixed homogeneous polynomial of $(n+1)$-variables:

$$g(\mathbf{z}, \bar{\mathbf{z}}, w, \bar{w}) \;=\; f(\mathbf{z}, \bar{\mathbf{z}}) + \bar{z}_n w^{q+r}\bar{w}^{r-1}.$$

g is also strongly mixed homogeneous polynomial. Recall that $f(\mathbf{z}, \bar{\mathbf{z}})$ is called to be *1-convenient* if the restriction of f to each coordinate subspace $f_i := f|_{\{z_i=0\}}$ is non-trivial for $i = 1, \dots, n$ ([9])

Theorem 12. *([10]) Assume that $n \geq 2$ and f is 1-convenient with a connected Milnor fiber F_f and let $g(\mathbf{z}, \bar{\mathbf{z}}, w, \bar{w})$ be the twisted join polynomial as above.*

1. *The Milnor fiber $F_g = g^{-1}(1)$ of g is simply-connected.*
2. *The Euler characteristic of F_g is given by the formula:*

$$\chi(F_g) = -(q+r)\chi(F_f) + (q+r+1)\chi(F_{f_n})$$

where $f_n := f|\{z_n = 0\}$ and $F_{f_n} = f_n^{-1}(1)$.

Assume that $n = 2$ and $f(\mathbf{z}, \bar{\mathbf{z}})$ has an isolated singularity at the origin. Then we have

Corollary 13. *$V = \{g = 0\} \subset \mathbb{P}^2$ is a non-singular mixed curve and $\pi_1(\mathbb{P}^2 - V) \cong \mathbb{Z}/q\mathbb{Z}$.*

Example 14.

Consider the mixed curve defined by

$$f_I' = z_1^{q+r}\bar{z}_1^{r-1}\bar{z}_2 + z_2^{q+r}\bar{z}_2^{r-1}\bar{z}_3 + z_3^{q+r}\bar{z}_3^{r}, \text{ (Tree type b)}$$

As f_I' is simplicial and also of twisted join type as

$$f_I' = z_1^{q+r}\bar{z}_1^{r-1}\bar{z}_2 + (z_2^{q+r}\bar{z}_2^{r-1}\bar{z}_3 + z_3^{q+r}\bar{z}_3^{r}),$$

we show that the Milnor fiber is simply-connected and

$$\pi_1(\mathbb{P}^2 \setminus C) \cong \mathbb{Z}/q\mathbb{Z}.$$

Here as $(z_2^{q+r}\bar{z}_2^{r-1}\bar{z}_3 + z_3^{q+r}\bar{z}_3^{r})$ is not 1- convenient, Theorem 12 can not be applied directly. Let us see this assertion directly. We take the coordinate chart $U_2 := \{z_2 \neq 0\}$ and put $w_1 = z_1/z_2, w_3 = z_3/z_2$. Then the affine equation of C in U_2 is

$$f(w_1, w_3) = w_1^{q+r}\bar{w}_1^{r-1} + \bar{w}_3 + w_3^{q+r}\bar{w}_3^{r}.$$

We consider the pencil $L_\eta := \{w_3 - \eta = 0\}$, $\eta \in \mathbb{C}$. It is easy to see that the branching locus is the set of the $q+2$ points given by

$$\Sigma := \{w_3 \mid \bar{w}_3(w_3^{q+r}\bar{w}_3^{r-1} + 1) = 0\}.$$

The base point of the pencil is $b = [1:0:0]$ and note that $b \in C$. $L_\eta \cap C$ has $q+1$ points over $\mathbb{C} \setminus \Sigma$ and 1 point over Σ. Taking a generic pencil L_{η_0} near a branching point $w \in \Sigma$ and taking generators $\xi_1, \dots, \xi_{q+1}$ of $\pi_1(L_{\eta_0} \setminus C)$ similarly as those in Figure 1, we get cyclic monodromy relations at each point of Σ:

$$\xi_1 = \xi_2 = \cdots = \xi_{q+1}.$$

This is enough to conclude that $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian and therefore isomorphic to

$$H_1(\mathbb{P}^2 \setminus C) \cong \mathbb{Z}/q\mathbb{Z}.$$

As for the Euler characteristic, we get $\chi(C) = -(q+1)(q-1) + q + 3 = -q^2 + q + 4$. Thus the genus of C is $(q+1)(q-2)/2$.

3. Non-trivial examples

Let $F(\mathbf{z}, \bar{\mathbf{z}})$ be a true strongly non-degenerate mixed homogeneous polynomial of three variables $\mathbf{z} = (z_1, z_2, z_3)$ of polar degree q and radial degree $q + 2r$ and we consider the projective mixed curve

$$C := \{[\mathbf{z}] \in \mathbb{P}^2 \mid F(\mathbf{z}, \bar{\mathbf{z}}) = 0\}.$$

We study the geometric structure of C and the fundamental group $\pi_1(\mathbb{P}^2 - C)$ using the pencil $L_\eta := \{z_2 = \eta z_3\}$, $\eta \in \mathbb{C}$, or equivalently the projection

$$p : (\mathbb{P}^2, C) \to \mathbb{P}^1, \quad [\mathbf{z}] \mapsto [z_2, z_3].$$

Take the affine chart $U_3 := \{z_3 \neq 0\}$ with coordinate functions (z, w) with $z = z_1/z_3$, $w = z_2/z_3$. Then $C \cap U_3$ is defined by $f(z, w, \bar{z}, \bar{w}) = F(\mathbf{z}, \bar{\mathbf{z}})/z_3^{q+r}\bar{z}_3^r = 0$. Let $\Sigma \subset \mathbb{P}^1$ be the branching locus of p.

3.0.1. *Holomorphic case.* If F is homogeneous polynomial without complex conjugate variables, Σ is described by the discriminant locus of f as a polynomial in z. Put $\sigma(w) := \mathrm{discrim_z f(z,w)}$. Thus Σ is a finite set of points $\Sigma = \{\rho_1, \dots, \rho_\ell\}$ given by $\sigma(w) = 0$. For any $\rho_j \in \Sigma$ and $\rho_{j,k} \in p^{-1}(\rho_j)$, C is locally a cyclic covering of order $s_{j,k}$ at $\rho_{j,k}$ where $s_{j,k}$ is the multiplicity of $\rho_{j,k}$ in $p^{-1}(\rho_j)$ as the root of $f(z, \rho_j) = 0$ which is equal to the intersection multiplicity of L_{ρ_j} and C at $\rho_{j,k}$.

3.0.2. *Mixed polynomial case.* Let F be a mixed homogeneous polynomial. Usually it is not easy to compute Σ. Instead of computing Σ, we proceed as follows. Let $z = x + yi$ and $w = u + vi$ and write f as $f(x, y, u, v) := g(x, y, u, v) + ih(x, y, u, v)$ where g and h are real polynomials which are the real and imaginary part of f respectively. Consider the complex algebraic variety

$$C(\mathbb{C}) := \{(x, y, u, v) \in \mathbb{C}^4 \,|g(x, y, u, v) = h(x, y, u, v) = 0\}$$

which is the complexification of our curve. Note that $C(\mathbb{C}) \cap \mathbb{R}^4 = C$. The branching locus of $p_{\mathbb{C}} : C(\mathbb{C}) \to \mathbb{C}^2$ is obtained by a Groebner basis calculation from the ideal $[g, h, J]$ where $J = \frac{\partial g}{\partial x}\frac{\partial h}{\partial y} - \frac{\partial g}{\partial y}\frac{\partial h}{\partial x}$ and $[g, h, J]$ is the ideal generated by g, h, J. The defining ideal is generated by the polynomials $\mathbb{C}[u, v] \cap [g, h, J]$. It is usually a principal ideal and the generating polynomial $R(u, v)$ of this ideal describes the discriminant locus of the complexified variety. We define the branching locus $\Sigma_{\mathbb{R}}$ by the intersection $\Sigma_{\mathbb{C}} \cap \mathbb{R}^2$. Take a point $w \in \Sigma_{\mathbb{R}}$. It is not always true that a point $w \in \Sigma_{\mathbb{R}}$ is a branching point of $p : C \to \mathbb{R}^2$. It might come from the branching on the complex point of $C(\mathbb{C})$ outside of C. That is $\Sigma \subset \Sigma_{\mathbb{R}}$ but the equality does not hold in general. See Example 2 below. Also it might have some point η_0 such that $L_{\eta_0} \cap C$ contains a 1-dimensional intersection. See Remark 16.

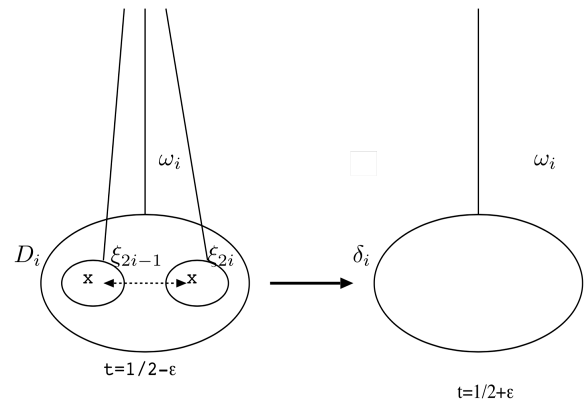

Figure 2. Vanishing loops

There are some cases for which these branching loci are comparatively simple. Suppose that f is a join type polynomial of $z_1^{q+r}\bar{z}_1^r$ and a strongly mixed homogeneous convenient polynomial $K(z_2, z_3, \bar{z}_2, \bar{z}_3)$ of two variables z_2 and z_3. Then the affine equation takes the form

$$f(\mathbf{z}, \bar{\mathbf{z}}) = z^{q+r}\bar{z}^r + k(w, \bar{w}) = 0$$

with respect to the affine coordinates $z = z_1/z_3$ and $w = z_2/z_3$. By the non-degeneracy assumption, the roots of $k(w, \bar{w}) = 0$ are all simple. Then the branching locus Σ is nothing but the set of those roots and over any of these roots, the projection is locally equivalent to the q-cyclic coverings $z^{q+r}\bar{z}^r - w = 0$ or $z^{q+r}\bar{z}^r - \bar{w} = 0$ respectively depending on the sign of the root.

However for a generic mixed polynomial, $\Sigma_{\mathbb{R}}$ and Σ are much more complicated. Usually they have real dimension 1 components and also they can have isolated points. We assume that for each η, $L_\eta \cap C$ is a finite point. We define $\gamma(\eta)$ to be the cardinality of $L_\eta \cap C$. We subdivide $\{\mathbb{C} \setminus \Sigma, \Sigma - S(\Sigma), S(\Sigma)\}$ by γ-values where $S(\Sigma)$ is the singular locus of Σ and let $\mathcal{D}$ be the corresponding subdivision. We call $\mathcal{D}$ *the γ-subdivision* of the parameter space. A 2-dimensional connected component $V \in \mathcal{D}$ (respectively 1-dimensional L, 0-dimensional P) is called a *region* (resp. an *edge*, a *vertex*). A region V is called *regular* if the inclusion map $V \subset \bar{V}$ is a homotopy equivalence. An edge L is called *regular* if there exist exactly two regions, say V_1, V_2 whose boundaries contain L and $\gamma(L) = (\gamma(V_1) + \gamma(V_2))/2$. A vertex P is called *regular* if there exist at most two regions which contain P in their boundary.

For a regular edge $M \in \mathcal{D}$, suppose that two regions S_1, S_2 are touching each other along M and suppose that $\gamma(S_1) > \gamma(S_2)$. Take a point $a \in M$ and a small transversal path

$$\sigma : [0, 1] \to \mathbb{R}^2 = \mathbb{C}$$

so that $\sigma(t) \in S_1$ for $t < 1/2$, $\sigma(1/2) = a$ and $\sigma(t) \in S_2$ for $t > 1/2$. Let $\gamma := (\gamma(S_1) - \gamma(S_2))/2$. Then for a sufficiently small $\varepsilon > 0$ and $1/2 - \varepsilon \le \forall t < 1/2$, $p^{-1}(\sigma(t))$ consists of $\gamma(S_1)$ points, say $\xi_1(t), \dots, \xi_{\gamma(S_1)}(t)$ and among them there exist γ pairs of points $\{\xi_{2i-1}(t), \xi_{2i}(t)\}$, $i = 1, \dots, \gamma$ and we can choose a continuous family of disjoint γ disks $D_i(t), i = 1, \dots, \gamma$ in the pencil line $p^{-1}(\sigma(t)) = \mathbb{R}^2$ which contain only the corresponding pair of roots so that when t goes to $1/2$, two roots approach each other in the disk $D_i(t)$ and collapse to $\delta_i \in L_a \cap M$, a double point and then they disappear for $t > 1/2$. These pairs of roots $\{\xi_{2i-1}(t), \xi_{2i}(t)\}$ as roots of a polynomial equation $f(z, \sigma(t)) = 0$ have opposite signs (one positive and one negative). Take a base point

$b = [1:0:0]$ of the fundamental group at the base point of the pencil. Consider a loop $\omega_i \in \pi_1(L_{\sigma(1/2-\varepsilon)} - C, b)$ represented by the boundary loop of $D_i(1/2-\varepsilon)$, connected to the base point by a path outside of $L_{\sigma(1/2-\varepsilon)} \setminus \cup_{i=1}^k D_i(\sigma(1/2-\varepsilon))$. Then we get the following relation for $t : 1/2 - \epsilon \to 1/2 + \epsilon$

$$\omega_i = e, \quad i = 1, \dots, \gamma.$$

Take elements ξ_{2i-1}, ξ_{2i} as in Figure 2. Then this implies that

$$\omega_i = \xi_{2i-1}\xi_{2i} = e \text{ or equivalently } \xi_{2i-1} = \xi_{2i}^{-1},\ i = 1, \dots, \gamma.$$

We call these relations **vanishing monodromy relations**.

3.1. **Example 1.** Now we present several examples which are neither simplicial nor of join type but the complement has an abelian fundamental group.

3.1.1. *Example 1-1.* Consider the following mixed curve of polar degree 1

$$C_t: \quad F(\mathbf{z}, \bar{\mathbf{z}}) := {z_1}^2 \bar{z}_1 + {z_2}^2 \bar{z}_2 + z_3^2 \bar{z}_3 + t\, {z_1}^2 z_2 \bar{z}_3 = 0$$

with $t \in \mathbb{C}$ and let C_t be the corresponding projective curve. Let $M_t = F^{-1}(1)$ be the corresponding Milnor fiber. Then C_0 is of mixed Brieskorn type and isotopic to the standard line $z_1 + z_2 + z_3 = 0$, namely a sphere S^2 (see [11]) and M_0 is diffeomorphic to the plane $\mathbf{C}$. This is true for any small t. Observe that $\{z_3 = 0\} \cap C_t = \{[1 : -1 : 0]\}$.

We are interested in $C := C(-4)$: $z_1^2\bar{z}_1 + z_2^2\bar{z}_2 + z_3^2\bar{z}_3 - 4\, z_1 z_2 \bar{z}_3$. We use the notation $M_{-4} = M$ for simplicity. Take the affine coordinate $z = z_1/z_3$, $w = z_2/z_3$. Then the affine equation is given as

$$C: \quad z^2\bar{z} + w^2\bar{w} + 1 - 4\, zw = 0.$$

To compute the Euler characteristic $\chi(C)$ and the fundamental group $\pi_1(\mathbf{P}^2 \setminus C)$, we consider the pencil $L_\eta := \{w = \eta\}$, $\eta = u + vi \in \mathbb{C}$. The branching locus $\Sigma_\mathbb{R}$ is given by $R(u, v) = 0$ where

$$\begin{aligned} R(u,v) = 27 &+ 11642\, v^2 u^4 - 2640\, u^7 v^2 + 405\, u^4 v^8 + 162\, u^{10} v^2 + 16438\, v^6 \\ &- 6736\, v^6 u^3 - 350\, u^6 + 405\, v^4 u^8 + 162\, u^2 v^{10} + 27\, v^{12} + 540\, v^6 u^6 \\ &+ 28430\, u^2 v^4 - 148\, u^3 - 2196\, uv^2 - 7032\, u^5 v^4 + 27\, u^{12} - 2196\, uv^8 - 148\, u^9. \end{aligned}$$

See Appendix 1 (§3.3.1) for the practical computation of $R(u, v)$. Its diagram of the zero locus set $R = 0$ is given as Figure 3. Let A be the bounded region of $\mathbb{C} \setminus \Sigma_\mathbb{R}$ and let U be the complement $\mathbb{P}^1 \setminus \bar{A}$. There are four singular points $V_i, i = 1, \dots, 4$ of the boundary of $\bar{A}$. Actually -1 is an isolated point of $\Sigma_\mathbb{R}$ but $L_\eta \cap C$ consists of one simple point and it does not give any branching of the projection $p : C \to \mathbb{C}$. Thus $-1 \notin \Sigma$.

As the polar degree is 1, the number of intersection points of $L_\eta \cap C$ counted with sign is always 1. Observe that $L_\eta \cap C$ consist of 3 simple roots of $f(z, \eta) = 0$ for any $\eta \in A$. Observe further that over any point η of the complement U of $\bar{A}$, $L_\eta \cap C$ has a unique simple root, i.e. $\gamma(U) = 1$ and $\gamma(A) = 3$. For any smooth boundary point η of $\partial\bar{A}$, $L_\eta \cap C$ has two points, one simple and one double point. (Strictly speaking, there does not exist the notion of multiplicity in the mixed roots. See [12]. Here we use the terminology "double root" in the sense that it is a limit of two simple roots). As for four singular points, we have $\gamma(V_i) = 1$, $i = 1, \dots, 4$. Let $\overline{a_1 a_2}$ be the line segment cut by $A \cap \{v = 0\}$ where $a_1 \approx 0.51, a_2 \approx 1.94$. For any $a_1 < \eta < a_2$, $L_\eta \cap C$ has three simple points which are all real. This can be observed by the diagram of $f = 0$ restricted on the real plane section $(w, z) \in \mathbb{R}^2$ (Figure 4). Consider the limit of $L_\eta \cap C$ when η goes to a_1 or a_2 along the real line segment $\overline{a_1 a_2}$. There are two real positive roots and one real negative root and at both ends, two positive roots collapse in the double point, which is clear from Figure 4.

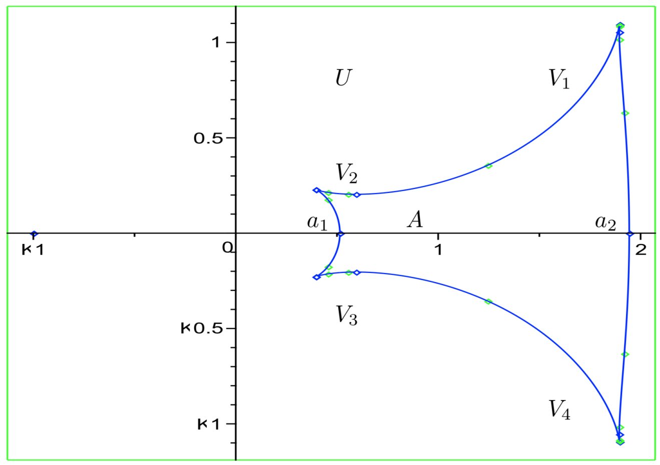

FIGURE 3. Diagram of $R = 0$, Example 1-1

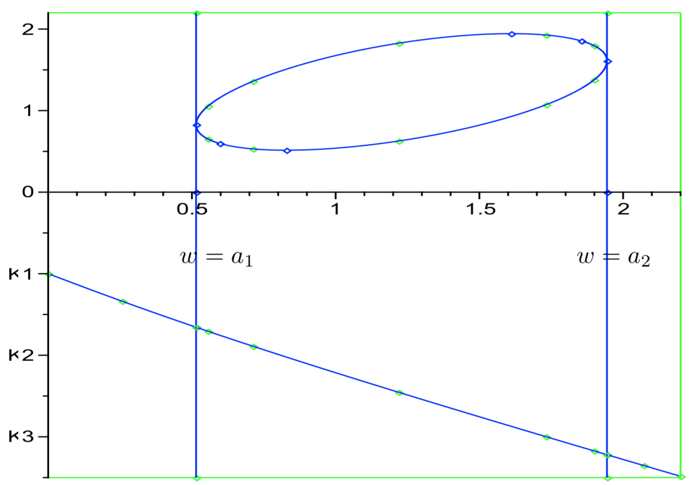

FIGURE 4. Diagram of $f = 0$, Example 1-1

Using these data, we can compute the Euler characteristic as

$$\chi(C) = \chi(p^{-1}(\bar{A})) + \chi(p^{-1}(U)) = -1 + 1 = 0.$$

This implies C is a torus and $\chi(\mathbb{P}^2 - C) = 3 - 0 = 3$. We claim

Proposition 15.

(1) $\pi_1(\mathbb{P}^2 - C) = \{e\}, \quad \pi_1(M) = \{e\}$.
(2) $\chi(M) = 3$, $H_1(M) = 0$, $H_2(M) = 2$.

Proof. We first compute the fundamental group. Put $b_0 = 1$ and we take L_{b_0} as a fixed regular pencil line. Then $L_{b_0} \cap C = \{x_1, x_2, x_3\}$ where

$$x_1 < 0 < x_2 < x_3.$$

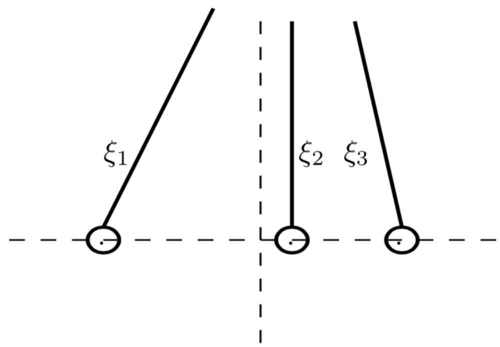

FIGURE 5. Generators of $\pi_1(L_{b_0} - C \cap L_{b_0})$

See Figure 4. It is not hard to see that $\pi_1(L_{b_0} \setminus C \cap L_{b_0}) \to \pi_1(\mathbb{P}^2 \setminus C)$ is surjective. See §3.3 for an explanation in detail. Take generators ξ_1, ξ_2, ξ_3 of $\pi_1(L_{b_0} \setminus C \cap L_{b_0})$ as in Figure 5. They are oriented counterclockwise. First, as a vanishing relation at infinity, they satisfy the relation

$$\xi_1 \xi_2 \xi_3 = e. \tag{1}$$

When η moves on the interval $[a_1, a_2]$ from $\eta = b_0$ to a_1 or a_2, we see that two positive roots collapse in the point for $\eta = a_1$ or $\eta = a_2$ and disappear for $\eta < a_1$ or $\eta > a_2$. Thus as a vanishing relation, we get

$$\xi_2 = \xi_3^{-1}.$$

Now we consider the movement from $\eta = 1$ along the vertical line to $\eta = 1 + v_0 i$ where $(1, v_0) \in \partial A$ and $v_0 \approx 0.26$. The generators are deformed as in Figure 6. Thus as a vanishing monodromy relation, we get $(\xi_2^{-1}\xi_1\xi_2)\xi_3 = e$. Thus combining the above relations, we get

$$\xi_1 = \xi_3,\ \xi_2 = \xi_1, \text{ and } \xi_1 = e.$$

We conclude that $\pi_1(\mathbb{P}^2 - C) = \{e\}$. □

Remark 16. *It can be observed that the set* $\Gamma := \{t = t_1 + t_2 i \in \mathbb{C} \,|\, C_t : \text{ singular}\}$ *is a real one-dimensional semi-algebraic set and the complement* $\mathbb{C} \setminus \Gamma$ *has two connected components in this case. The bounded region contains* 0 *and for any* t *in this region,* C_t *is isotopic to* C_0 *and it is a rational sphere.* Γ *is calculated by Groebner basis calculation. In our case, we found that* Γ *is defined by*

$$t_1^4 - 6t_1^2 + 8t_1 - 3 + 2t_2^2 t_1^2 - 6t_2^2 + t_2^4 = 0.$$

Certainly C_{-4} *is in the outside unbounded region. We may choose another one* $C_{\sqrt[3]{3}}$ *which must be isotopic to* C_{-4} *but the branching locus is very different and defined by* $R = 0$ *and its diagram is given by Figure 7.*

In this example, $\gamma(A) = \gamma(B) = 3$ *but the point* $(u, v) = (-1, 0)$ *is special as* $L_{-1} \cap C$ *has one simple point and one 1-dimensional component which is defined by* $|z| = \sqrt[6]{3}$. *Thus the geometry of the pencil is more complicated and it takes more careful consideration to compute*

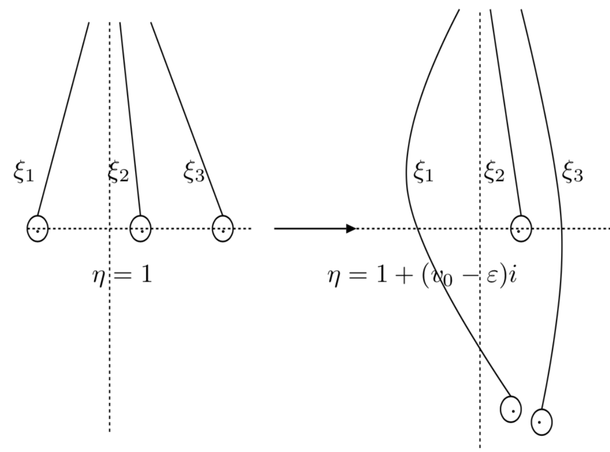

FIGURE 6. movement on $\eta = 1 + si$, Example 1-1

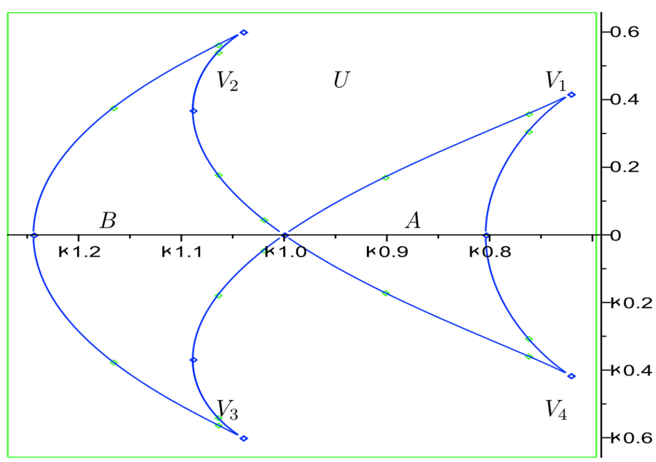

FIGURE 7. Diagram of $R = 0$, Remark16

the fundamental group.

$$R(u,v) := 27+540\,v^6u^6+405\,v^4u^8+162\,u^2v^{10}+120\,u^9+162\,u^{10}v^2+654\,u^4v^2+120\,u^3+216\,uv^2$$
$$+1008\,u^5v^4+216\,uv^8+27\,u^{12}+405\,v^8u^4+768\,v^6u^3+576\,u^7v^2+90\,v^6+558\,u^2v^4+186\,u^6+27\,v^{12}.$$

3.1.2. *Example 1-2.* We consider another example with polar degree 1 and radial degree 3. Let $F(\mathbf{z},\bar{\mathbf{z}}) := z_1^2\bar{z}_1 + z_2^2\bar{z}_2 + z_3^2\bar{z}_3 - 4z_2z_3\bar{z}_3 - 2z_3^2\bar{z}_1$. Taking the affine chart $\{z_3 \neq 0\}$ and coordinates $z = z_1/z_3, w = z_2/z_3$, the affine equation is $f(z,w) = z^2\bar{z} + w^2\bar{w} + 1 - 4w - 2\bar{z}$. Consider the pencil $L_\eta := \{w = \eta\}, \eta \in \mathbb{C}$. Putting $w = u + vi$, the branching locus is described by $R = 0$ where the explicit form is given in Appendix 2(§3.3.2) to show that the equation of R grows exponentially by the number of monomials and degree. However the diagram of $R = 0$ is not so

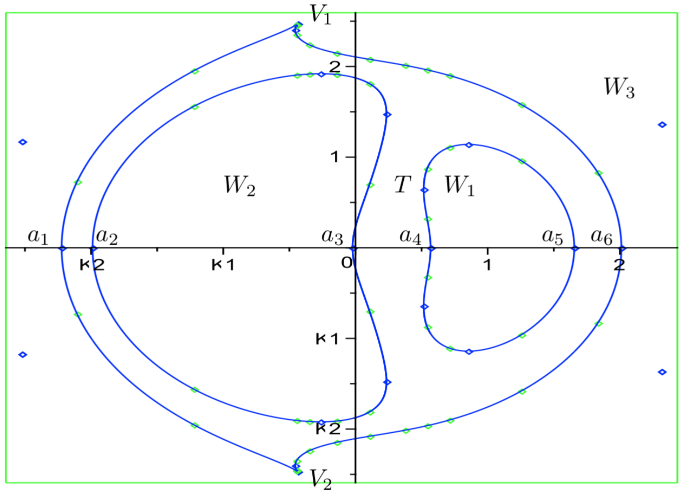

FIGURE 8. Diagram of $R = 0$, Example 1-2

complicated and it is given in Figure 8. We observe that $\gamma(W_i) = 1$, $i = 1, 2, 3$ and $\gamma(T) = 3$ where T is the complement of $\overline{W}_1 \cup \overline{W}_2 \cup \overline{W}_3$. There are two singular points of the boundary of T, V_1, V_2 and $\gamma(V_i) = 1$ and the other boundary points have 2 roots. Let $a_1, \dots, a_6$ be real roots of $R(u, 0) = 0$ and we assume that $a_1 < a_2 < \cdots < a_6$. Note that $a_1 \approx -2.22$ and $a_2 \approx -1.98$. See figure 8. In the Figure, the horizontal line is the w-coordinate axis. Take a base line L_{b_0} with $b_0 = a_2 - \varepsilon$, $0 < \varepsilon \ll 1$. See the diagram of $f = 0$ on $\mathbb{R}^2$ (Figure 9). Two vertical lines are $w = a_1$ and $w = a_2$. We take generators ξ_1, ξ_2, ξ_3 of $\pi_1(L_{b_0} \setminus C)$ as the left side of Figure 5. Considering a movement of $\eta = b_0$ to $\eta = a_1$ and from $\eta = b_0$ to $\eta = a_2$, we get the vanishing monodromy relations

$$\xi_1\xi_2 = e, \quad \xi_2\xi_3 = e. \tag{2}$$

This is also clear from Figure 9. Combining the vanishing relation $\xi_1\xi_2\xi_3 = e$, we get

$$\xi_2 = \xi_1^{-1} = \xi_3^{-1} = e.$$

Thus $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian and we conclude that $\pi_1(\mathbb{P}^2 \setminus C) = H_1(\mathbb{P}^2 - C)$ is trivial. The Euler number is computed as

$$\chi(C) = \chi(W_1) + \chi(W_2) + \chi(W_3) + \chi(T) - \chi(\partial T) = 1 + 1 + 1 - 3 - 2 = -2.$$

Thus the genus of C is 2.

3.2. **Example 2.** Consider the next mixed curve of polar degree 2 and radial degree 4.

$$C_t \quad F(\mathbf{z}, \bar{\mathbf{z}}) := \bar{z}_1 z_1^3 + z_2^3 \bar{z}_2 + z_3^3 \bar{z}_3 + t\, z_1^2 \bar{z}_2 z_3$$

For t small, C_t is isotopic to the conic $z_1^2 + z_2^2 + z_3^2 = 0$ in $\mathbb{P}^2$ and a rational sphere ([11]). We take $t = -4$ and put $C = C_{-4}$ and $M = M_{-4}$ the Milnor fiber. The branching locus is defined by $R = R_1 R_2 = 0$ where

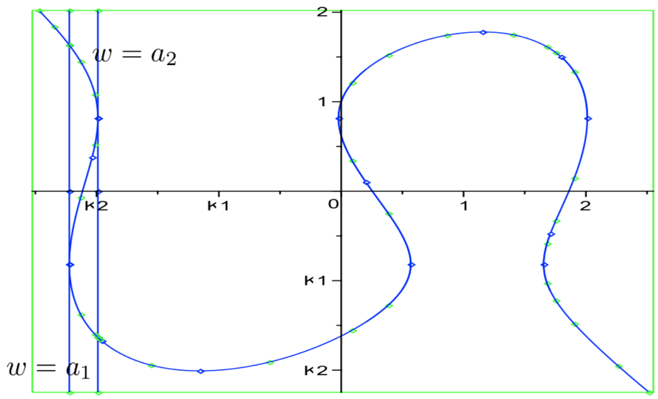

FIGURE 9. Diagram of $f = 0$ with $z, w \in \mathbb{R}$, Example 1-2

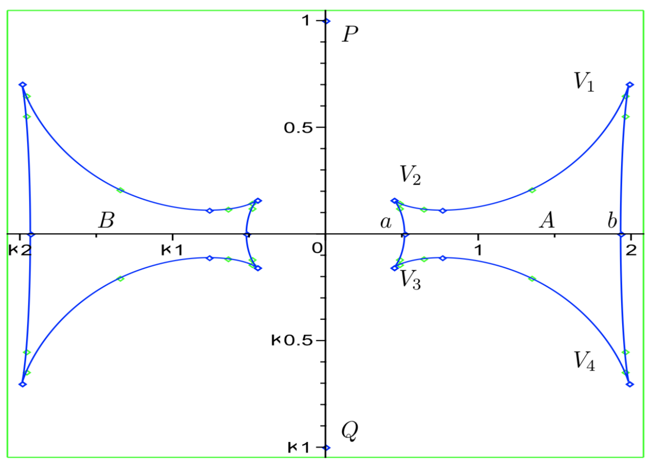

FIGURE 10. Diagram of $R = 0$, Example 2

$$
\begin{aligned}
R_1 &:= 1 + u^8 + 6\,v^4u^4 + 2\,u^4 - 2\,v^4 + 4\,v^6u^2 + 4\,u^6v^2 + v^8,\\
R_2 &:= 1 - 12\,u^4 + 124\,v^4 - 320\,u^2v^2 + 8\,u^{14}v^2 + 12580\,v^4u^4 + 12936\,v^6u^2 + 3464\,u^6v^2 + 70\,v^8u^8\\
&\quad - 1228\,u^8v^4 + 56\,v^{10}u^6 - 1472\,v^6u^6 + 56\,v^6u^{10} - 548\,v^8u^4 - 26\,u^8 + 3846\,v^8 - 12\,u^{12}\\
&\quad + 124\,v^{12} + u^{16} + 28\,v^{12}u^4 + 8\,v^{14}u^2 + 28\,u^{12}v^4 - 368\,u^{10}v^2 + 176\,v^{10}u^2 + v^{16}.
\end{aligned}
$$

We claim that

Proposition 17. (1) *$\pi_1(\mathbb{P}^2 - C) \cong \mathbb{Z}/2\mathbb{Z}$.*
(2) *$\chi(C) = -2$. The genus of C is 2.*

Proof. The locus $R_1 = 0$ gives two isolated points $P = (0, 1)$, $Q = (0, -1)$. Note that the affine equation of C is defined by $f(z, w) = z^3\bar{z} + w^3\bar{w} + 1 - 4z^2\bar{w} = 0$. Recall that the highest degree

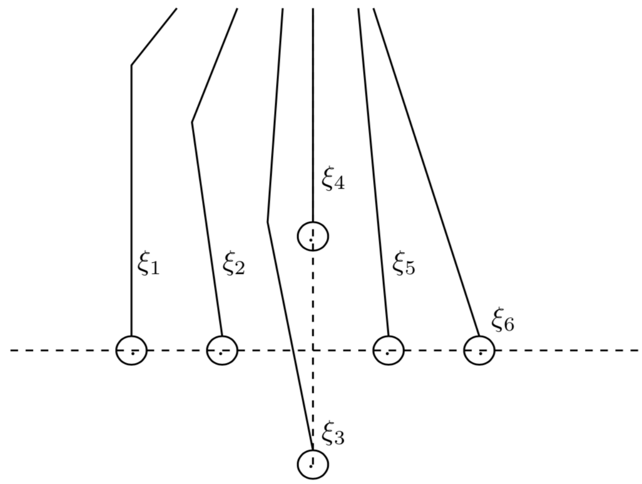

FIGURE 11. Generators of $\pi_1(L_1 - C \cap L_1)\,(\eta = 1)$

part of f as a polynomial of z is $z^3\bar{z}$, and therefore the number of roots counted with sign is two by [12]. We also observe that $f(z,w) = 0 \iff f(-z,w) = 0$. Thus the roots are symmetric with respect to the origin in z-coordinates. R is symmetric with respect to the v-axis but the region B does not give any branching. It comes from the complex part of the curve. Thus $\gamma(B) = 2$. Also we observe that $\gamma(A) = 6$ and $\gamma(\partial A) = 4$ except 4 singular points $V_1, \dots, V_4$ where $L_\eta \cap C$ has 2 multiple points. The complement region $E := \mathbb{P}^1 \setminus (\bar{A} \cup \{P, Q\})$ has 2 simple points for any fiber $L_\eta \cap C$ with $\eta \in E$. We have $\gamma(P) = \gamma(Q) = 1$. Take generators of $\pi_1(L_1 - C)$, $\xi_i, i = 1, \dots, 6$ as in Figure 11. Observe that $f(z,w) = 0$ implies $f(-z,w) = 0$. Thus the roots are always paired by $z, -z$ for a fixed w. Put $\bar{A} \cap \{v = 0\} = \{a, b\}$ with $a \approx 0.51$ and $b \approx 1.93$. First we consider the movement $\eta = 1 \to a$. Consider the diagram of $f_r := z^4 + w^4 + 1 - 4z^2w$ (Figure12) where f_r is the restriction of f to $\mathbb{R}^2$. This says that on $[a, b]$, $L_\eta \cap C$ has exactly four real roots, which are symmetric with respect the origin and at $\eta = a$, they collapse to two double roots. Let f_i be the restriction of $f(iz, w)$ to $(z, w) \in \mathbb{R}^2$ and look at its diagram. See Figure 13. Using the real diagram of $f_i := -z^4 + w^4 + 1 + 4z^2w$, we see also that there are exactly two purely imaginary roots of $f(z, \eta) = 0$ for any $w \in \mathbb{R}$. The above observation says that

$$\xi_1\xi_2 = e, \quad \xi_5\xi_6 = e. \tag{3}$$

(The Figure 12 shows that we get the same degeneration for $\eta \to b$.) Then we consider the movement of the line L_η further to the left until $\eta = 0$. Then we move L_η along the imaginary axis to $\eta = i$ which is a root of multiplicity 2 (P in the diagram). Note that the monodoromy along $|w - i| = \varepsilon$ is topologically the half turn of two roots. Thus we get

$$\xi_3 = \xi_4. \tag{4}$$

Now we will see the vanishing relation along the vertical line for $\eta = 1 \to 1 + c_0 i \dots$ where c_0 is the positive root of $R_2(1, v) = 0$. The root of $f(z, w) = 0$ with $w = 1 + ci$ is given as

$$\pm P_1, \pm P_2, \quad P_1 \approx 1.57 - 1.21i, P_2 \approx 0.76 + 0i$$

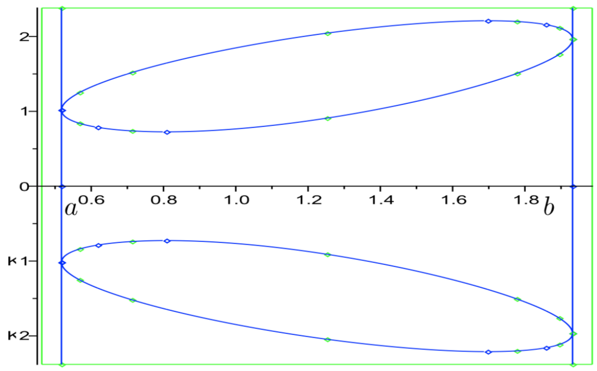

FIGURE 12. Diagram of $f = 0$, Example 2

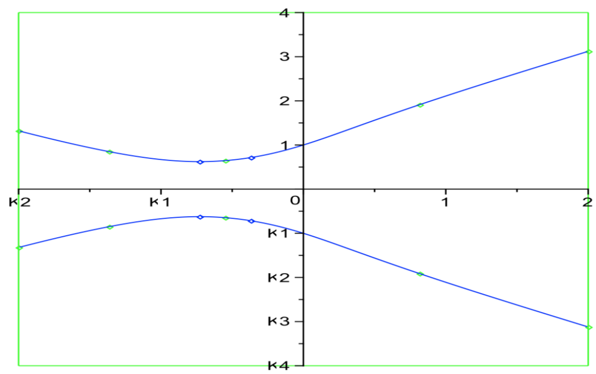

FIGURE 13. Diagram of $f_i(z, w) = 0$, Example 2

where $\pm P_1$ are double roots. Recall that $f(z, 1) = 0$ has roots

$$\pm Q_1, \pm Q_2, \pm Q_3, \quad Q_1 \approx 2.11i,\ Q_2 \approx 0.76, Q_3 \approx 1.84.$$

The movement of generators during the above movement is described in Figure 14. The dotted loops show the situation in $\eta = 1+(c_0-\varepsilon)i$ wit $0 < \varepsilon \ll 1$. They are denoted as $\xi_1', \ldots, \xi_6'$. In this movement, ξ_2, ξ_5 do not move much. Other generators are deformed as indicated with arrows. At $\eta = 1 + c_0 i$, ξ_1', ξ_4' and ξ_3', ξ_6' collapse respectively. Thus $\xi_1 = \xi_1', \xi_4 = \xi_4'$ and $\xi_3 = \xi_3', \xi_6 = \xi_6'$ and we get vanishing relations which are written as

$$\xi_1\xi_4 = e, \quad (\xi_4\xi_5)^{-1}\xi_3(\xi_4\xi_5)\xi_6 = e. \tag{5}$$

Using (3), (4)and (5), we conclude that

$$\xi_2 = \xi_1^{-1},\ \xi_3 = \xi_1,\ \xi_4 = \xi_1^{-1}, \xi_5 = \xi_1,\ \xi_1^2 = e.$$

That is, $\pi_1(\mathbb{P}^2 - C) \cong \mathbb{Z}/2\mathbb{Z} \cong H_1(\mathbb{P}^2 - C)$. □

3.3. **Surjectivity.** Assume that $f(z, \bar{z}, w, \bar{w}) = 0$ is the affine equation of a non-singular mixed curve C of polar degree q and radial degree $q + 2r$. We assume that f is monic in the sense that

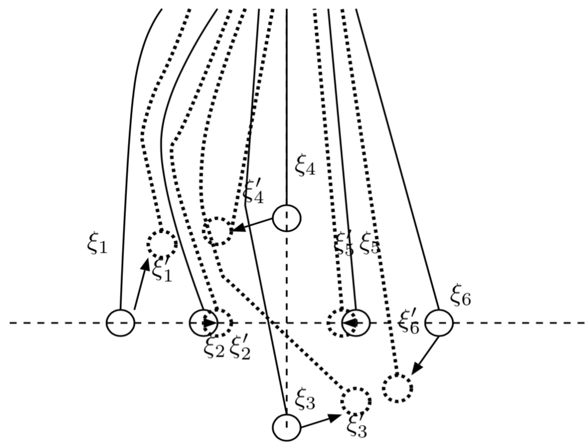

FIGURE 14. Movement of generators, Example 2

it has the monomial $z^{q+r}\bar{z}^r$ with a non-zero coefficient. Consider the pencil line

$$L_\eta = \{w = \eta\}, \eta \in \mathbb{C}$$

and we consider the γ-subdivision $\mathcal{D}$ of $\mathbb{C}$ (the parameter space) by the value of $\gamma(\eta)$ using the diagram of R. We assume that all regions, edges and vertices are regular. We assume also that the base point b of the pencil is not on C. Let $G = \{\gamma(\eta) \,|\, \eta \in \mathbb{C}\} \subset \mathbb{N}$ the possible number of roots of $f(z, \bar{z}, \eta, \bar{\eta}) = 0$ and γ_{max} be the maximum of G. We assume the following two conditions.
(1) The set $U_{max} := \{\eta \,|\, \gamma(\eta) = \gamma_{max}\}$ is connected and it is a region.
(2) Take a region U of $\mathcal{D}$ with $\gamma(U) < \gamma_{max}$. Put $\partial_+ U = \{q \in \partial U \,|\, \gamma(q) \geq \gamma(U)\}$. Then $\partial_+ U$ is connected.

Note that the above condition is satisfied in Example 1-1, Example 1-2, Example 2. Let B be the complement of the union of regions of $\mathcal{D}$, i.e. B is the union of the edges and vertices. We fix a generic line L_{η_0} with $\eta_0 \in U_{max}$ and a base point $b \in L_{\eta_0} \setminus C$. Let $\sigma : (I, \{0, 1\}) \to (\mathbb{P}^2 \setminus C, b)$ be a loop. We may assume that $\{t \,|\, \sigma(t) \in B\}$ is finite. Let $\alpha := \min\{\gamma(\sigma(t)) \,|\, t \in I\}$ and we may assume that α is taken in a region V of $\mathcal{D}$. Put $\mathcal{D}_\beta$ be the union of $\overline{U}$ with $\gamma(U) \geq \beta$. Then we assert:

Assertion 18. *σ is homotopic in $\mathbb{P}^2 \setminus C$ to a loop $\hat{\sigma}$ in the pencil line $L_{\eta_0} \setminus C$.*

Proof. We may assume that $\pi \circ \sigma$ intersects B transversely at smooth points of B if it intersects.

Step 1. Suppose that $\alpha \neq \gamma_{max}$. Then the image of $\pi \circ \sigma$ intersects more than two regions. Take a path segment L of $\pi(\sigma(I)) \cap V$. Let P, Q be the end points of L and assume that $P = \pi(\sigma(t_1))$ and $Q = \pi(\sigma(t_2))$ with $t_1 < t_2$. By the assumption (2), P, Q belongs to the unique boundary component $\partial_+ V$ and there is a path L'' in the boundary $\partial_+ V$ connecting P, Q and $\gamma(\eta) \geq \alpha$ for any $\eta \in L''$. We want replace L by some path $L' \subset V$ which is homotopic to L'' relatively to the end points. See Figure 15. Consider the closed path at Q, $\omega := L^{-1} \cdot L'$. The composition of paths is to be read from the left. Take a lift $\tilde{\omega}$ which is a loop starting at $\sigma(t_2)$, passes through $\sigma(t_1)$ and comes back to $\sigma(t_2)$ which is null homotopic in $\mathbb{P}^2 \setminus C$. We can simply take $\tilde{\omega}$ near the infinity. Then replace σ by $\sigma_{[0,t_2]} \cdot \tilde{\omega} \cdot \sigma_{[t_2,1]}$ which is homotopic to σ. Now σ is clearly homotopic to σ' where $\sigma' := \sigma_{[0,t_2]} \cdot \tilde{\omega} \cdot \sigma_{[t_2,1]}$. Note that the image $\pi(\sigma'(I))$ replaces

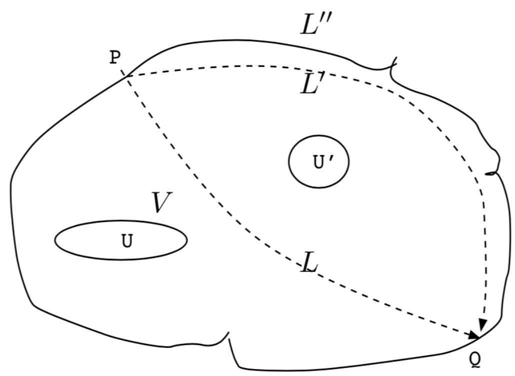

FIGURE 15. Segment L

the segment L by L'. Now we can deform L' to L'' and further to the other side of the region of L'', keeping the homotopy class. Doing this operation for any path segment cut by V, we get a loop σ'' whose image by π is in $\mathcal{D}_\beta$ where $\beta := \min\{G \setminus \{\alpha\}\}$. By induction, we can deform σ keeping the homotopy class to a loop σ_1 in $\pi^{-1}(U_{max})$.

Step 2. Now we assume that σ_1 is a loop in $\pi^{-1}(U_{max})$. We deform σ_1 further to a loop $\hat{\sigma}$ which is a loop in the line L_{η_0}.

If U_{max} is contractible, this is easy to deform using the fibration structure of π over U_{max}. This is the case for Example 1-1 and Example 2. In Example 1-2, $U_{max} = T$ and $\pi_1(U_{max}, \eta_0)$ is a free group of rank 2.

Assume that $\pi_1(U_{max})$ is non-trivial. Put $\tau := \pi \circ \sigma_1$, a loop in U_{max}. Take a lift $\tilde{\tau}$ starting at b which is a contractible closed curve in $\pi^{-1}U_{max} \setminus C$. Consider the loop $\sigma_1 \cdot \tilde{\tau}^{-1}$. This is homotopic to σ_1. The image of this modified loop by π is clearly homotopic to a constant loop at η_0. Using the fibration structure over U_{max}, we can deform this loop to a loop $\hat{\sigma}$ in $L_{\eta_0} \setminus C$. For the detail of lifting argument, see for example Spanier [17]. □

The surjectivity assertion is not true if η_0 does not belong to U_{max}. Also a loop $\tau \in (L_{\eta_0} \setminus C)$ cannot be expressed by a loop in $(L_\eta \setminus C)$ if $\gamma(\eta) < \gamma_{max}$ without using the monodromy relations. An example is given by ξ_{2i-1}, ξ_{2i} in Figure 2 can not deformed on the line $L_{\sigma(1/2+\varepsilon)}$. We close this paper by a question.

Question. Do the conditions (1) and (2) hold for any mixed function?

3.3.1. *Appendix 1.* Let f be a mixed strongly homogeneous polynomial. To compute the defining polynomial of the branching locus R in Example 1-1, Example 1-2 and Example 2, we proceed as follows. Let $z = x + yi$ and $w = u + vi$ and write f as $g + ih$ where g, h are polynomials of x, y, u, v with real coefficients. Let $J = \frac{\partial g}{\partial x}\frac{\partial h}{\partial y} - \frac{\partial g}{\partial y}\frac{\partial h}{\partial x}$ and let $A = [g, h, J]$, the ideal generated by g, h, J. Then we use the MAPLE command: Groebner[Basis](A,plex(x,y,u,v)). For further explanation for Groebner calculation, we refer [3] for example.

Acknowledgement. For the numerical calculation of roots of $f(z, w) = 0$ with fixed various complex numbers w's, we have used the following program on MAPLE which is kindly written by Pho Duc Tai, Hanoi University of Science. I am grateful to him for his help.

Pho's program to compute roots of mixed polynomial on MAPLE:

```
fsol3 := proc (f, z)
local aa, a, b, ff, f1, f2, h, i, j, k, s, temp; print(Factorization_of_Input = factor(f)); ff := factors(f)[2];
temp := {};
for k to nops(ff) do
if 1 < ff[k][2] then RETURN(printf("Input is not squarefree. Please solve each factor.")) end if;
assume(a, real); assume(b, real); h := expand(subs(z = a+I*b, ff[k][1]));
f1 := Re(h); f2 := Im(h); aa := RootFinding[Isolate]([f1, f2], [a, b]);
temp := `union`(temp, seq([[op(aa[i][1])][2], [op(aa[i][2])][2]], i = 1 .. nops(aa))) end do;
RETURN([op(temp)])
end proc
```

3.3.2. *Appendix 2: Equation of R for Example 1.2.*

The equation of the branching locus is the following.

$$\begin{gathered}
R(u,v) = -179685 + 129384576\,u^4v^4 + 2160\,u^{19}v^2 + 27\,u^{24} - 864\,v^{22} + 13590816\,u^7 - 102858240\,v^6u^5 \\
- 47520\,u^{18}v^4 - 7631712\,u + 174564288\,v^6u^4 - 288581376\,v^6u^2 + 193050720\,v^8u^2 + 580608\,v^{14}u^3 \\
+ 2032128\,u^{13}v^4 + 5080320\,u^9v^8 + 4064256\,u^7v^{10} + 72576\,uv^{16} - 142560\,u^6v^{16} - 9504\,u^2v^{20} - 142560\,u^{16}v^6 \\
- 399168\,u^{12}v^{10} - 285120\,u^{14}v^8 - 285120\,u^8v^{14} - 47520\,u^4v^{18} - 399168\,u^{10}v^{12} + 441460992\,v^2u^6 + 27\,v^{24} \\
+ 542688\,v^{16} + 55344648\,v^6u^6 + 216\,u^{21} + 12096\,v^{20} - 6048\,u^{19} + 20877120\,u^2 - 466968\,u^{15} + 4064256\,u^{11}v^6 \\
+ 2032128\,u^5v^{12} + 580608\,u^{15}v^2 + 1550016\,v^2 - 21912872\,u^3 - 103122216\,v^2u + 15611136\,uv^{10} - 2415360\,v^8u^3 \\
- 212093856\,u^3v^4 + 165770304\,u^7v^2 - 93452\,u^{18} + 33480480\,v^8u^5 + 27856256\,v^6u^3 + 138815904\,v^6u \\
- 137971200\,u^7v^4 - 46557076\,v^6 + 134691072\,u^2v^2 + 20322912\,u^4 + 35835552\,v^4 - 15874316\,u^6 \\
+ 588672\,u^5 - 425273772\,v^2u^4 + 404256\,u^{16} + 352376064\,v^2u^3 + 44453280\,v^4u^9 + 57198720\,v^6u^7 \\
+ 6479040\,v^{10}u^3 - 596640\,v^{12}u + 6378146\,v^{12} + 271049040\,u^5v^4 - 55084800\,u^9v^2 - 242721696\,u^8v^2 \\
- 283722816\,u^6v^4 + 85432284\,u^{10}v^2 + 15257280\,u^{11}v^2 - 9504\,u^{20}v^2 - 73189752\,v^8u + 324\,u^{22}v^2 \\
+ 1597920\,u^{13} + 2630208\,v^{14}u^2 - 3738096\,u^{14}v^4 - 12425640\,u^{10}v^8 - 11851032\,u^8v^{10} - 8601296\,u^{12}v^6 \\
- 7583152\,u^6v^{12} - 912372\,u^{16}v^2 - 752448\,u^{14} + 190612542\,u^8v^4 - 52716324\,u^2v^{10} - 107902338\,v^8u^4 \\
+ 1782\,u^{20}v^4 - 50865792\,v^4u - 4232728\,u^9 + 9720\,u^{17}v^4 + 2160\,u^3v^{18} + 9720\,u^5v^{16} + 25920\,u^{15}v^6 \\
+ 54432\,u^{11}v^{10} + 45360\,u^9v^{12} + 45360\,u^{13}v^8 + 25920\,u^7v^{14} + 216\,uv^{20} + 1451520\,u^6v^{14} + 120960\,u^2v^{18} \\
+ 2155584\,v^{12}u^2 - 821676\,u^2v^{16} - 18081480\,u^7v^8 - 11114040\,u^{11}v^4 - 18630120\,u^9v^6 - 10126488\,u^5v^{10} \\
- 3003624\,u^3v^{12} - 3552264\,u^{13}v^2 - 3198096\,u^4v^{14} + 72576\,u^{17} - 337318944\,u^5v^2 - 99220\,v^{18} - 2136768\,v^{14} \\
+ 544320\,u^{16}v^4 + 2540160\,u^{12}v^8 + 3048192\,u^{10}v^{10} + 1451520\,u^{14}v^6 + 2540160\,u^8v^{12} + 120960\,u^{18}v^2 \\
+ 544320\,u^4v^{16} + 12096\,u^{20} + 80914380\,v^4u^2 + 7271904\,u^{10} - 864\,u^{22} + 10619712\,u^4v^{10} + 18579648\,u^{12}v^4 \\
- 22564800\,u^6v^8 + 33143040\,u^8v^8 + 19619136\,u^6v^{10} - 78557760\,u^8v^6 - 71293248\,u^{10}v^4 + 33409728\,u^{10}v^6 \\
+ 7934784\,u^4v^{12} + 4945344\,u^{14}v^2 + 5940\,u^{18}v^6 + 5940\,u^6v^{18} + 324\,u^2v^{22} + 13365\,u^{16}v^8 + 24948\,u^{12}v^{12} \\
+ 21384\,u^{14}v^{10} + 13365\,u^8v^{16} + 1782\,u^4v^{20} + 21384\,u^{10}v^{14} - 217728\,u^{15}v^4 - 762048\,u^{11}v^8 \\
- 762048\,u^9v^{10} - 508032\,u^{13}v^6 - 508032\,u^7v^{12} - 54432\,u^{17}v^2 - 54432\,u^3v^{16} - 217728\,u^5v^{14} - 6048\,uv^{18} \\
- 7068480\,u^8 - 357240\,uv^{14} + 30563520\,v^8 - 15242784\,v^{10} - 1027742\,u^{12} - 1945344\,u^{11} - 22380096\,u^{12}v^2.
\end{gathered}$$

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