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STRATA OF DISCRIMINANTAL ARRANGEMENTS

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In memory of Brieskorn

Abstract. We give an explicit description of the multiplicities of codimension two strata of discriminantal arrangements introduced by Manin and Schechtman. As applications, we discuss the connection of these results with properties of Gale transform and we calculate the fundamental groups of the complements to discriminantal arrangements.

1. Introduction

In 1989, Manin and Schechtman ([13]) introduced a family of arrangements of hyperplanes generalizing classical braid arrangements, which they called the *discriminantal arrangements* (p.209 [13]). Such an arrangement $\mathcal{B}(n,k,\mathcal{A}^0), n,k \in \mathbf{N}$ for $k \geq 2$ depends on a choice $\mathcal{A}^0 = \{H_1^0, ..., H_n^0\}$ of a collection of hyperplanes in the general position in $\mathbb{C}^k$, i.e., such that $\dim \bigcap_{i \in K, \text{Card } K=k} H_i^0 = 0$. It consists of parallel translates of $H_1^{t_1}, ..., H_n^{t_n}, (t_1, ..., t_n) \in \mathbb{C}^n$ which fail to form a general position arrangement in $\mathbb{C}^k$. $\mathcal{B}(n,k,\mathcal{A}^0)$ can be viewed as a generalization of the pure braid group arrangement ([16]) with which $\mathcal{B}(n,1) = \mathcal{B}(n,1,\mathcal{A}^0)$ coincides. These arrangements have several beautiful relations with diverse problems in the areas such as combinatorics (see [13] and also [4], which is an earlier appearance of discriminantal arrangmements), the Zamolodchikov equation with its relation to higher category theory (see Kapranov-Voevodsky [8]), and the vanishing of cohomology of bundles on toric varieties ([17]).

The aim of this note is to study the dependence of $\mathcal{B}(n,k,\mathcal{A}^0)$ on the data $\mathcal{A}^0 = \{H_1^0, ..., H_n^0\}$. Paper [13] concerns the arrangements $\mathcal{B}(n,k,\mathcal{A}^0)$ for which the intersection lattice is constant when $\mathcal{A}^0$ varies within a Zariski open set $\mathcal{Z}$ in the space of general position arrangements. However [13] does not describe the set $\mathcal{Z}$ explicitly. It was shown in [6] that, contrary to what was frequently stated (see for instance [15], sect. 8, [16] or [10]), the combinatorial type of $\mathcal{B}(n,k,\mathcal{A}^0)$ indeed depends on the arrangement $\mathcal{A}^0$. This was done by providing an example of a discriminantal arrangement with a combinatorial type distinct from the one which occurs when $\mathcal{A}^0$ varies within the Zariski open set $\mathcal{Z}$. Few years later, in [1], Athanasiadis provided a full description of combinatorics of $\mathcal{B}(n,k,\mathcal{A}^0)$ when $\mathcal{A}^0$ belongs to $\mathcal{Z}$. In particular, in this case, codimension 2 strata of $\mathcal{B}(n,k,\mathcal{A}^0)$ only have a multiplicity equal to 2 or $k+2$. Following [1], we call arrangements $\mathcal{A}^0$ in $\mathcal{Z}$ *very generic*.

Our main result describes a necessary and sufficient *geometric* condition on arrangement $\mathcal{A}^0$ assuring that $\mathcal{B}(n,k,\mathcal{A}^0)$ admits codimension 2 strata of multiplicity 3.

This condition is given in terms of a notion of *dependency* for the arrangement $\mathcal{A}_\infty$ in $\mathbb{P}^{k-1}$ of hyperplanes $H_{\infty,1}, ...H_{\infty,n}$ which are the intersections of projective closures of $H_1^0, ..., H_n^0 \in \mathcal{A}^0$ with the hyperplane at infinity. Consider three groups of $s \in \mathbb{Z}_{\geq 1}$ hyperplanes in $\mathbb{P}^{2s-2}$ such that together these $3s$ hyperplanes are in general position in $\mathbb{P}^{2s-2}$. If the three subspaces corresponding to this split in groups, each being the intersection of hyperplanes in each group,

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span a hyperplane in $\mathbb{P}^{2s-2}$, we say that the arrangement of $3s$ hyperplanes in $\mathbb{P}^{2s-2}$ is *dependent* (Definition 3.3 in Section 3). This dependence condition defines a proper Zariski closed subset of the space of arrangements of $3s$ hyperplanes in $\mathbb{P}^{2s-2}$ in general position. Our main result (Theorem 3.9) shows that $\mathcal{B}(n,k,\mathcal{A}^0), k > 1$ admits a codimenion two stratum of multiplicity 3 if and only if $\mathcal{A}_\infty$ is an arrangement in $\mathbb{P}^{k-1}$ admitting a restriction[1] which is a dependent arrangement .

Subsequently, in Section 4, we interpret this result in terms of the Gale transform. The relation between discriminantal arrangements and the Gale transform can be seen, at least implicitly, already in paper [6]. From this view point our result asserts an equivalence of certain types of collinearity: the dependency of $\mathcal{A}_\infty$ is equivalent to presence of dependencies in the Gale transform which in turn is equivalent to the presence of strata of multiplicity 3 in an arrangement $\mathcal{B}(n,k,\mathcal{A}^0)$. We shall give a direct verification of such equivalences using the interpretation of Gale transform of six-tuples of point in $\mathbb{P}^2$ in terms of del Pezzo surfaces given in [5]. More precisely, an arrangement $\mathcal{B}(6,3,\mathcal{A}^0)$ depends on arrangement at infinity $\mathcal{A}_\infty$, which in this case is a six-tuple of lines in $\mathbb{P}^2$, or equivalently, a six-tuple $(\mathcal{A}_\infty)^*$ of points in the dual plane. A general position arrangement $\mathcal{A}_\infty$ is dependent if and only if the del Pezzo surface, which is the blow up of $\mathbb{P}^2$ at six-tuple $(\mathcal{A}_\infty)^*$, admits an Eckardt point (cf. subsection 4.2). On the other hand, the interpretation of $\mathcal{B}(6,3,\mathcal{A}^0)$ via Gale transform, described in subsection 4.1, shows that presence in $\mathcal{B}(6,3,\mathcal{A}^0)$ of codimension two strata of multiplicity 3 is equivalent to the following: *the Gale transform of* $(\mathcal{A}_\infty)^*$ is a six-tuple $G(\mathcal{A}_\infty)^*$ such that blow up of $\mathbb{P}^2$ at $G(\mathcal{A}_\infty)^*$ is a del Pezzo surface admitting an Eckardt point. Hence the main result in the Theorem 3.9, in the case of discriminantal arrangments $\mathcal{B}(6,3,\mathcal{A}^0)$, becomes an invariance of existence of Eckardt points in the Gale transform. We show that this can be verified directly (see subsection 4.2).

Finally we supplement R.Lawrence's presentation ([10]) by giving a presentation of the fundamental group in the case of non very generic arrangements (i.e. for which $\mathcal{A}^0 \notin \mathcal{Z}$). In fact, we give calculations yielding the braid monodromy and hence a presentation of the fundamental group of the complement to a discriminantal arrangement in all cases.

Notice that in the case $k = 1$, the complement to the discriminantal arrangement $\mathcal{B}(n,1)$ coincides with the configuration space of ordered n-tuples of points in $\mathbb{C}$. A natural generalization of this configuration space to the case $k \geq 2$ is the space of arrangements of hyperplanes in $\mathbb{C}^k$ in a general position. This is a Zariski open in the product of n copies of spaces of affine hyperplanes in $\mathbb{C}^k$. The fundamental group of this space is another natural candidate for a generalization of the pure braid group. Our result shows the difficulty with a calculation of this fundamental group: natural maps between spaces $\mathcal{B}(n,k,\mathcal{A}^0)$ for various n,k, which in the case $k = 1$ lead to the presentation of the pure braid group fail to be locally trivial fibrations and hence fails to produce an exact sequence of fundamental groups. For example, intersections of projective closures of arrangements in $\mathbb{C}^k$ with the hyperplane at infinity, yields a map from the space of general position arrangements in $\mathbb{C}^k$ to the space of general position arrangements in $\mathbb{P}^{k-1}$. Our result shows that this map is a locally trivial fibration only over the space of very general position arrangements in $\mathbb{P}^{k-1}$. Calculation of the fundamental groups of spaces of general position arrangements of lines will be addressed elsewhere.

The content of the paper is the following. In Section 2, we introduce several notions used later and recall definitions from [13]. Section 3 contains one of the main results of this paper, Theorem 3.9, describing the codimension 2 strata of discriminantal arrangements having multiplicity 3 and showing an absence of codimension 2 strata having a multiplicity different from $2, 3$ and $k+2$. The Section 4 contains the interpretation of the results in Section 3 in terms of the Gale transform.

[1] Here restriction is the standard restriction of arrangements to subspaces as defined in [16] (see also equation (8) in this paper).

The last Section describes the braid mondromy and fundamental groups of the complements to discriminantal arrangements.

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2. Preliminaries

2.1. **Discriminantal arrangements.** Let $H_i^0, i = 1, ..., n$, be a general position arrangement in $\mathbb{C}^k, k < n$, i.e., a collection of hyperplanes such that $\dim \bigcap_{\substack{i\in K,\\ \mathrm{Card}K=k}} H_i^0 = 0$. The space of parallel translates $\mathbb{S}(H_1^0, ..., H_n^0)$ (or simply $\mathbb{S}$ when the dependence on H_i^0 is clear or not essential) is the space of n-tuples $H_1, ..., H_n$ such that either $H_i \cap H_i^0 = \emptyset$ or $H_i = H_i^0$ for any $i = 1, ..., n$. One can identify $\mathbb{S}$ with an n-dimensional affine space $\mathbb{C}^n$ in such a way that $(H_1^0, ..., H_n^0)$ corresponds to the origin.

We will use the compactification of the arrangement $(H_1^0, ..., H_n^0)$ obtained by viewing the ambient space $\mathbb{C}^k$ as $\mathbb{P}^k \setminus H_\infty$ endowed with a collection of hyperplanes $\bar{H}_i^0$ which are projective closures of affine hyperplanes H_i^0. The condition of genericity is equivalent to $\bigcup_i \bar{H}_i^0$ being a normal crossing divisor in $\mathbb{P}^k$. The space $\mathbb{S}$ can be identified with product $\mathcal{L}_1 \times \ldots \times \mathcal{L}_n$ where $\mathcal{L}_i \simeq \mathbb{C}$ is the pencil of hyperplanes spanned by H_∞ and H_i^0 parametrized by $\mathbb{P}^1$ with the deleted point. The latter corresponds to H_∞ and the origin to H_i^0. In particular, an ordering of hyperplanes in $\mathcal{A}$ determines the coordinate system in $\mathbb{S}$.

For a general position arrangement $\mathcal{A}$ in $\mathbb{C}^k$ formed by hyperplanes $H_i, i = 1, ..., n$, *the trace at infinity* (denoted by $\mathcal{A}_\infty$) is the arrangement formed by hyperplanes $H_{\infty,i} = \bar{H}_i^0 \cap H_\infty$.

An arrangement $\mathcal{A}$ (or its trace $\mathcal{A}_\infty$) determines the space of parallel translates $\mathbb{S}(H_1^0, ..., H_n^0)$ (as a subspace in the space of n-tuples of hyperplanes in $\mathbb{P}^k$).

For a general position arrangement $\mathcal{A}_\infty$, we consider the closed subset of $\mathbb{S}(H_1^0, ..., H_n^0)$ formed by those collections which fail to form a general position arrangement. This subset is a union of hyperplanes with each hyperplane corresponding to a subset $K = \{i_1, ..., i_{k+1}\} \subset \{1, ..., n\}$ and consisting of n-tuples of translates of hyperplanes $H_1^0, ..., H_n^0$ in which translates of $H_{i_1}^0, ..., H_{i_{k+1}}^0$ fail to form a general position arrangement (equations are given by (3) below). Such a hyperplane will be denoted D_K. The corresponding arrangement will be denoted $\mathcal{B}(n, k, \mathcal{A})$ and called *the discriminantal arrangement corresponding to* $\mathcal{A}$.

The cardinality of $\mathcal{B}(n, k, \mathcal{A})$ is equal to $\binom{n}{k+1}$. Each hyperplane D_K contains the k-dimensional subspace $\mathbb{T}$ of $\mathbb{S}(H_1^0, ..., H_n^0)$ formed by n-tuples of hyperplanes containing a fixed point in $\mathbb{C}^k$. Clearly, the essential rank, i.e. the dimension of the ambient space minus the dimension of intersection of the hyperlanes of the arrangement (cf. [20]), in the case of $\mathcal{B}(n, k, \mathcal{A})$ is $n - k$ and the arrangement induced by the arrangement of hyperplanes D_K in the quotient of $\mathbb{S}(H_1^0, ..., H_n^0)$ by $\mathbb{T}$ is essential. It is called *the essential part* of the discriminantal arrangement.

2.2. **Hyperplanes in** $\mathcal{B}(n, k, \mathcal{A})$**.** Recall that an arbitrary arrangement $\mathcal{A}$ of hyperplanes $W_1, ..., W_N \subset \mathbb{C}^k$ defines the canonical stratification of $\mathbb{C}^k$ in which strata are defined as follows. Let $L(\mathcal{A})$ be the intersection poset of subspaces in $\mathbb{C}^k$, each being the intersection of a collection of hyperplanes chosen among $W_1, ..., W_N$, and for each $P \in L(\mathcal{A})$, let $\Sigma_P = \{i \in \{1, ..., N\} | P \in W_i\}$ be the set of indices of hyperplanes W_i such that $P = \cap_{i\in\Sigma_P} W_i$. Vice versa, given a subset $\Sigma \subset \{1, ..., N\}$, we denote by w_Σ the subspace $w_\Sigma = \cap_{i\in\Sigma} W_i$. The stratum of P is the

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submanifold of $\mathbb{C}^k$ defined as follows:

(1) $$\mathcal{S}_P = P \setminus \bigcup_{\Sigma_P \subset \Sigma} w_\Sigma.$$

If an arrangement $\mathcal{A} = \{W_1, ..., W_N\}$ in $\mathbb{C}^k$ is *in the general position* then the finite subset in $\mathbb{C}^k$, consisting of 0-dimensional strata, has cardinality $\binom{N}{k}$ and its elements are in one to one correspondence with the subsets of $\{1, ..., N\}$ having cardinality k.

The multiplicity of a point $p \in \mathcal{S}_P$ considered as a point on the subvariety $\bigcup_{i=1,...,N} W_i$ in $\mathbb{C}^k$ is constant along the stratum. We call it *the multiplicity* of the stratum $\mathcal{S}_P$. It is equal to cardinality of the set Σ_P.

As we noted, the hyperplanes of $\mathcal{B}(n, k, \mathcal{A})$ correspond to subsets of cardinality $k+1$ in $\{1, \dots, n\}$. Their equations can be obtained as follows. Let $K, \text{Card } K = k+1$, be a subset in $\{1, ..., n\}$ and let

(2) $$\alpha_1^j y_1 + \ldots + \alpha_k^j y_k = x_j^0, \quad j \in \{1, ..., n\}$$

be the equation of hyperplane H_j^0 of arrangement $\mathcal{A} = \{H_1^0, \dots, H_n^0\} \in \mathbb{C}^n \setminus B(n, k, \mathcal{A})$ in selected coordinates $y_1, ..., y_k$ in $\mathbb{C}^k$. The hyperplanes H_j, $j \in K$, of an arrangement in $\mathbb{S}$ with equations $\alpha_1^j y_1 + \ldots + \alpha_k^j y_k = x_j, \quad j \in K$, will have non-empty intersection iff

(3) $$det \begin{pmatrix} \alpha_1^1 & ... & \alpha_k^1 & x_1 \\ ... & ... & ... & ... \\ \alpha_1^{k+1} & ... & \alpha_k^{k+1} & x_{k+1} \end{pmatrix} = 0.$$

This provides a linear equation in $x_j, j = 1, ..., k+1$, for the hyperplane D_K corresponding to K.

Let J be a subset in $\{1, \dots, n\}$ of cardinality a,

(4) $$D_J = \{(H_1, \dots, H_n) \in \mathbb{S} \text{ such that } \cap_{i \in J} H_i \neq \emptyset\}$$

and

(5) $$\mathcal{P}_{k+1}(J) = \{K \subset J \text{ such that Card } K = k+1\}.$$

Then

(6) $$D_J = \bigcap_{K \in \mathcal{P}_{k+1}(J)} D_K$$

is intersection of $\binom{a}{k+1}$ hyperplanes. In particular D_J, Card $J \geq k+1$, is a linear subspace and the multiplicity of $\bigcup_{\text{Card } K=k+1} D_K$ at its generic point is $\binom{a}{k+1}$. Moreover, codim D_J is $a-k$.

2.3. **Projections of discriminantal arrangements.** Let $\Xi \subset \{1, ..., n\}$ be a subset of the set of indices and let $\mathbb{S}(\Xi) \subset \mathbb{S}$ be the subspace of the space of translates of hyperplanes of a general position arrangement $H_1^0, ..., H_n^0$ consisting of translates of hyperplanes with indices in Ξ. Let us consider the projection $p_\Xi : \mathbb{S} \to \mathbb{S}(\Xi)$ obtained by omitting from a collection of translates from $\mathbb{S}$, the translates of hyperplanes with indices outside of Ξ. The image of a subspace $D_J, J \subset \{1, ..., n\}$ is a proper subspace iff Card $J \cap \Xi \geq k+1$ and in fact $p_\Xi(D_J) = D_{J \cap \Xi}$. In particluar, if D_J is a hyperplane i.e. Card$J = k+1$ then $p_\Xi(D_J)$ is a hyperplane if and only if $J \subset \Xi$.

The maps p_Ξ restricted to the complement to the discriminantal arrangement $\mathbb{S} \setminus \mathcal{B}(n, k, \mathcal{A})$ for $n \geq k+3$ are locally trivial fibrations if and only if $k = 1$. Due to their local triviality they play a prominent role in the study of braid arrangements (cf. [2]). The failure of local triviality for $k \geq 2$ can be seen as follows. Consider, for example, the simplest case $k = 2$. Let $\mathcal{A} = \{l_1^0, .., l_4^0, l_5^0\}$ be a quintuple of lines in $\mathbb{C}^2$ and $\Xi = \{1, 2, 3, 4\} \subset \{1, 2, 3, 4, 5\}$. The fiber of $p_\Xi : \mathbb{C}^5 \to \mathbb{C}^4$ at a generic point $\{l_1, ..., l_4\}$ in the complement $\mathbb{C}^4 \setminus \mathcal{B}(4, 2, \mathcal{A} \setminus \{l_5^0\})$ is given by all general position arrangements $\{l_1, ..., l_4, l_5\}$ such that l_5 does not contain any of the 6 intersection points of $l_i \cap l_j$, $1 \le i < j \le 4$, that is $\mathbb{C}$ with deleted 6 points. On the other hand, one can select a generic point $\{l_1, ..., l_4\}$ in the complement $\mathbb{C}^4 \setminus \mathcal{B}(4, 2, \mathcal{A} \setminus \{l_5^0\})$ such that one of the diagonals of quadrangle formed by lines $l_1, ...l_4$ will be parallel to l_5^0. Hence the fiber of p_Ξ at such a point will be $\mathbb{C}$ with only 5 points deleted. Similar special configurations are inevitable for all $n \ge k+3, k \ge 2$. This failure of local triviality brings serious complication in the study of the topology of the complement $\mathbb{S} \setminus \mathcal{B}(n, k, \mathcal{A})$ (see the last section for a description of the fundamental groups).

Note, that some recent works (see for example [7]) refer to discriminantal arrangements in a more narrow sense than used in this paper i.e. as the restriction arrangements to the fibers of p_Ξ given explicitly as

$$p^{-1}_{\{1,...,l\}}(t_1, ..., t_l) = \{(z_1, ..., z_{n-l}) | z_i = z_j \text{ or } z_i = t_k,\ k = 1, ..., l,\ i, j = 1, ..., n-l\}. \tag{7}$$

3. Codimension two strata having multiplicity 3

In this section we describe necessary and sufficient conditions which should be satisfied by the trace at infinity $\mathcal{A}_\infty$ in order that the corresponding discriminantal arrangement will have codimenion two strata having multiplicity 3. We shall start with the list of notations used throughout this section, some already introduced in the last section.

Notations 3.1. *Let's fix the following notations.*

- *$\mathcal{A}^0$ is a general position arrangement of n hyperplanes in $\mathbb{C}^k$ (we use $\mathcal{A}^0$ for the fixed arrangement to distinguish it from $\mathcal{A}$ which will denote a general translate of $\mathcal{A}^0$),*
- *for each K subset of $\{1, ..., n\}$ of* Card *$K = k+1$, $D_K \subset \mathbb{C}^n$ will denote the hyperplane in $\mathcal{B}(n, k, \mathcal{A}^0)$ corresponding to the subset K.*
- *As in subsection 2.1, hyperplanes in the trace at infinity $\mathcal{A}_\infty$ are denoted by $H_{\infty,i}$.*
- *Let $s \ge 2$. $K_i, i = 1, 2, 3$, denote subsets of $\{1, ..., n\}$ such that*

$$\text{Card } K_i = 2s, \quad \text{Card } K_i \cap K_j = s, \quad i \neq j, \quad \bigcap_{i=1}^{i=3} K_i = \emptyset$$

(in particular Card $\bigcup K_i = 3s$*).*

Lemma 3.2. *Let $s \ge 2$, $n = 3s, k = 2s-1$. Let $\mathcal{A}^0$ be a general position arrangement of n hyperplanes in $\mathbb{C}^k$ and let $K_i, i = 1, 2, 3$ be a triple of subsets of $\{1, ..., n\}$ as described in notations 3.1 above. Consider the triple of codimension s* **subspaces** *of the hyperplane at infinity H_∞ defined as follows: $H_{\infty,i,j} = \cap_{s \in K_i \cap K_j} H_{\infty,s} \cap H_\infty, i \neq j$. If subspaces $H_{\infty,i,j} \subset H_\infty$ span a proper subspace in H_∞ then* codim $\bigcap D_{K_i} = 2$*. Otherwise this codimension is equal to 3.*

This lemma suggests the following:

Definition 3.3. *A general position arrangement in $\mathbb{P}^{2s-2}, s \ge 2$, is called* **dependent** *if it is composed of $3s$ hyperplanes W_i which can be partitioned into 3 groups, each containing s hyperplanes, such that 3 subspaces of dimension $s-2$, each being intersection of hyperplanes in one group, span a proper subspace in $\mathbb{P}^{2s-2}$. We call these three $s-2$-dimensional subspaces* **dependent**.

Remark that, with this terminology, the assumption of Lemma 3.2 is that the trace at infinity of $\mathcal{A}^0$ is a dependent general position arrangement.

If $s = 2$ in Lemma 3.2, then $H_{\infty,i,j}$ are points in the 2-dimensional space $\mathbb{P}^2$. The condition that these points span a proper subspace in H_∞, i.e., are collinear, corresponds to the case of

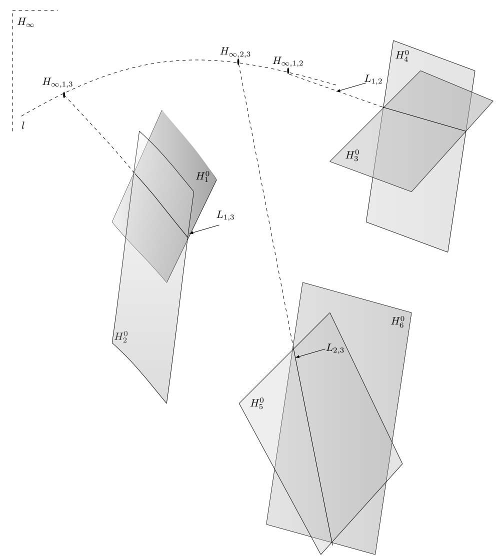

FIGURE 1.

Falk's example of the special discriminantal arrangement in [6]. We shall illustrate the argument in Lemma 3.2 by a discussion of this particular case since the argument for the proof of this lemma is a generalization of the argument used in Example 3.4.

Example 3.4. *Let us consider the case $n = 6$ and $k = 3$, that is a general position arrangement $\mathcal{A}^0 = \{H_1^0, \dots, H_6^0\}$ in $\mathbb{C}^3$. In Lemma 3.2, this corresponds to $s = 2$ and, after possible relabelling, $K_1 = (1, 2, 3, 4), K_2 = (3, 4, 5, 6), K_3 = (1, 2, 5, 6)$. Then subspaces $L_{i,j} = \bigcap_{s \in K_i \cap K_j} H_s^0$ are lines $L_{1,3}^0 = H_1^0 \cap H_2^0, L_{1,2}^0 = H_3^0 \cap H_4^0, L_{2,3}^0 = H_5^0 \cap H_6^0$ with closures $\bar{L}_{i,j}^0$. In this case, (i.e., when $\dim H_\infty = 2$), the assertion of Lemma 3.2 is that the points $H_{\infty,i,j} = \bar{L}_{i,j}^0 \cap H_\infty$ span a line l in H_∞. In other words, the points $H_{\infty,i,j}$ are collinear if and only if codim $_{i=1,2,3} \bigcap D_{K_i} = 2$ (see Figure 1).*

Indeed, an arrangement $\mathcal{A} = \{H_1, \dots, H_6\}$ of translates of planes in $\mathcal{A}^0$ is a point in $D_{K_1} \cap D_{K_2}$ iff pairwise intersections $L_{1,3} \cap L_{1,2}$ and $L_{1,2} \cap L_{2,3}$ in $\mathbb{C}^3$ of lines

$$L_{1,3} = H_1 \cap H_2, \; L_{1,2} = H_3 \cap H_4 \; \text{ and } \; L_{2,3} = H_5 \cap H_6$$

are non-empty. We claim that the collinearity condition implies that two pairs of these three lines $L_{i,j}$ are coplanar if and only if all the three are. Indeed, since $\mathcal{A}$ consists of translates of planes in $\mathcal{A}^0$ the line $L_{i,j}$ has the same point at infinity $H_{\infty,i,j}$ as does the line $L_{i,j}^0$. The condition that $H_{\infty,i,j}$ span a line $l \in H_\infty$ implies that the closure of any plane containing two lines $L_{i,j}$ intersects H_∞ in l. That is two planes containing respectively the pairs of lines $L_{1,3}, L_{1,2}$ and $L_{1,2}, L_{2,3}$ are coincident. This implies that lines $L_{1,3}$ and $L_{2,3}$ have a non-empty intersection i.e. $\bigcap_{i=1,2,5,6} H_i \neq \emptyset$ and hence $\mathcal{A} \in D_{K_3}$.

Vice versa, if the points $H_{\infty,i,j}$ aren't colinear, then it is possible to find configurations in which, for example, $L_{1,3}$ intersects both $L_{1,2}$ and $L_{2,3}$, but $L_{1,2} \cap L_{2,3} = \emptyset$, i.e. $\mathcal{A} \in D_{K_1} \cap D_{K_3}$ and $\mathcal{A} \notin D_{K_2}$.

Proof of Lemma 3.2. Consider first the case when subspaces $H_{\infty,i,j}$ span a proper hyperplane in H_∞ which we shall denote $\mathcal{H}$. Note that dim $H_{\infty,i,j} = s-2$ and, as a consequence of $\mathcal{A}^0$ being in the general position, these subspaces do not intersect. In particular, the subspace which they span has a dimension greater than $2s-4$, i.e., either it is a hyperplane or it is all the space H_∞.

Let $\mathcal{A} = \{H_i\}$ be an arrangement in $\mathbb{C}^k = \mathbb{C}^{2s-1}$ which belongs to D_{K_1} and D_{K_2} (recall that hyperplanes H_i are translates of hyperplanes $H_i^0 \in \mathcal{A}^0$). Hence $\bigcap_{i\in K_1} H_i \neq \emptyset$ and $\bigcap_{i\in K_2} H_i \neq \emptyset$. We claim that $\bigcap_{i\in K_3} H_i \neq \emptyset$, which would imply that codim $\bigcap_{i=1,2,3} D_{K_i} =$ codim $\bigcap_{i=1,2} D_{K_i} = 2$. Let $L_{i,j} = \bigcap_{s\in K_i\cap K_j,} H_s, (i<j)$. Note that the codimension of each linear subspace $L_{i,j}$ of $\mathbb{C}^{2s-1}$ is equal to s and $L_{i,j}\cap H_\infty = H_{\infty,i,j}$.

Since $\mathcal{A} \in D_{K_1}$, the subspaces $L_{1,2}$ and $L_{1,3}$ have a non-empty intersection. Therefore they span in $\mathbb{C}^{2s-1}$ a hyperplane which we denote as $\mathcal{L}_1$. The intersection of $\mathcal{L}_1$ with H_∞ is the hyperplane $\mathcal{H}$ spanned by $H_{\infty,1,2}$ and $H_{\infty,1,3}$. The hyperplane $\mathcal{L}_1$ is spanned by the intersection point $L_{1,2}\cap L_{1,3}$ and the hyperplane $\mathcal{H}$.

Similarly, since $\mathcal{A} \in D_{K_2}$, both $L_{1,2}$ and $L_{2,3}$ have a point in common, they span the hyperplane $\mathcal{L}_2$ spanned by this point and the above hyperplane $\mathcal{H}$ which can be described as the plane spanned by $H_{\infty,1,2}$ and $H_{\infty,1,3}$. Both hyperplanes $\mathcal{L}_1$ and $\mathcal{L}_2$ contain $L_{1,2}$ and $\mathcal{H}$. Hence they coincide. Therefore $L_{1,3}$ and $L_{2,3}$, being both $(s-1)$-dimensional subspaces in $\mathcal{L}_1 = \mathcal{L}_2, \dim\mathcal{L}_1 = \dim\mathcal{L}_2 = 2s-2$, must have a point in common and hence $\mathcal{A} \in \bigcap_{i=1,2,3} D_{K_i}$.

Now assume that the triple $H_{\infty,i,j}$ spans H_∞. Let $\mathcal{A} \in D_{K_1}\cap D_{K_2}$ be sufficiently generic in this space. We show that it does not belong to D_{K_3}. Consider the family of s codimensional subspaces in $\mathbb{C}^{2s-1}$ which compactification intersects the hyperplane at infinity at $H_{\infty,2,3}$. The selection of $\mathcal{A}$ determines subspaces $L_{1,2}, L_{1,3} \subset \mathbb{C}^{2s-1}$ which have a common point and moreover the subspace $L_{2,3}$ which intersects $L_{1,2}$. Since triple $H_{\infty,i,j}$ is *not* in a hyperplane in H_∞, $\mathbb{P}^{2s-1}$, compactifying $\mathbb{C}^{2s-1}$ is spanned by $H_{\infty,2,3}$ and the closures of subspaces $L_{1,2}, L_{1,3}$. Hence the generic subspace L of codimension s containing $H_{\infty,2,3}$ and intersecting $L_{1,2}$ will have an empty intersection with $L_{1,3}$. The corresponding arrangement $\mathcal{A}'$ having L as the subspace $L_{2,3}$ will not belong to D_{K_3} but will be in $D_{K_1}\cap D_{K_2}$. This shows that $D_{K_1}\cap D_{K_2} \notin D_{K_3}$. □

Let's briefly recall here the basic notion of the restriction of an arrangement. For a subset $\mathcal{A}' \subseteq \mathcal{A}$, let us denote by $X_{\mathcal{A}'} = \bigcap_{H\in\mathcal{A}'} H$ the intersection of its hyperplanes. The arrangement

$$\mathcal{A}^{X_{\mathcal{A}'}} = \{H\cap X_{\mathcal{A}'} \mid H \in \mathcal{A}\setminus\mathcal{A}', H\cap X_{\mathcal{A}'} \neq \emptyset\} \tag{8}$$

is called **a restriction** of $\mathcal{A}$ to $X_{\mathcal{A}'}$. Restrictions of $\mathcal{A}$ are in one-to-one correspondence with the splits $\mathcal{A} = \mathcal{A}'\bigcup\mathcal{A}''$ of the set of hyperplanes in $\mathcal{A}$ into a disjoint union. If $\mathcal{A}' = \emptyset$, then the restriction arrangement coincides with $\mathcal{A}$.

Via the restriction of arrangements, the Lemma 3.2 leads to other examples of discriminantal arrangements having codimension two strata with multiplicity 3.

Lemma 3.5. *Let $\mathcal{A}^0$ be a general position arrangement of n hyperplanes in $\mathbb{C}^k$ and $\mathcal{A}'$ be a subarrangement of t hyperplanes in $\mathcal{A}^0$. Assume that the trace at infinity of the restriction $\mathcal{A}^{X_{\mathcal{A}'}}$ of $\mathcal{A}^0$ to $X_{\mathcal{A}'}$ is a dependent arrangement of $3s = n-t$ hyperplanes (in the sense of Def. 3.3). Then $\mathcal{B}(n,k,\mathcal{A})$ admits a codimension two stratum of multiplicity 3.*

Proof. Assume that $\mathcal{A}' = \{H_1^0,...,H_t^0\} \subset \mathcal{A}^0 = \{H_1^0,...,H_n^0\}$ satisfies the conditions of lemma, i.e., the restriction $\mathcal{A}^{X_{\mathcal{A}'}}$ is an arrangement of $3s = n-t$ hyperplanes in $X_{\mathcal{A}'} \simeq \mathbb{C}^{n-t}$ and its trace at infinity $\mathcal{A}_\infty^{X_{\mathcal{A}'}}$ is dependent, i.e., the discriminantal arrangement $\mathcal{B}(n-t,k-t,\mathcal{A}^{X_{\mathcal{A}'}})$ admits a codimension 2 stratum having the multiplicity 3. The dimension of this stratum is $3s-2$ where $n-t = 3s$ and $k-t = 2s-1$. By Lemma 3.2, there are subsets

$$K_i, i=1,2,3, \text{ Card } K_i = 2s = k-t+1 \text{ of } \{t+1,...,n\}$$

such that $D_{K_i} \in \mathcal{B}(n-t,k-t,\mathcal{A}^{X_{\mathcal{A}'}})$ satisfy codim $\bigcap_{i=1,2,3} D_{K_i} = 2$. The above $(3s-2)$-dimensional stratum of the discriminantal arrangement of $n-t$ hyperplanes in $\mathbb{C}^{k-t}$ is the transversal intersection of two submanifolds (each being an open subset of a linear subspace) of $\mathbb{C}^n$. One is the stratum of discriminantal arrangement $\mathcal{B}(n,k,\mathcal{A}^0)$ having the dimension $3s-2+t$ formed by hyperplanes $D_{K_i\cup\{1,...,t\}}, i=1,2,3$, and another is the intersection of t hyperplanes in $\mathbb{S}(H_1^0,...,H_n^0)$ defined by the vanishing of coordinates corresponding to $H_1^0,...,H_t^0$. Hence the multiplicity of this stratum of $\mathcal{B}(n,k,\mathcal{A}^0)$ equals 3. This yields the lemma. □

Corollary 3.6. *If $k\geq 3$ and $n\geq k+3$, then there exists a general position arrangement of n hyperplanes in $\mathbb{C}^k$ such that the corresponding discriminantal arrangement admits a codimension two stratum of multiplicity 3.*

Proof. To apply Lemma 3.5, for a pair (n,k) such that there exist integers $t\geq 0, s\geq 2$ satisfying

$$n=3s+t \qquad k=2s-1+t \tag{9}$$

consider a general position arrangement $\mathcal{A}^0$ of n hyperplanes in $\mathbb{C}^k$ such that the restriction of trace $\mathcal{A}_\infty$ of $\mathcal{A}^0$ on intersection of its t hyperplanes is dependent. By Lemma 3.5, the discriminantal arrangement corresponding to such $\mathcal{A}^0$ will admit the required stratum. Given $(n,k)\in\mathbb{N}^2$, the relation (9) has a unique solution $s=n-k-1, t=3k-2n+3$ which satisfies $s\geq 2, t\geq 0$ iff

$$k+3\leq n\leq \frac{3}{2}(k+1), \qquad k\geq 3. \tag{10}$$

Note that given an arrangement $\mathcal{B}(n,k,\mathcal{A})$ admitting the codimension 2 strata of multiplicity 3, an extension of $\mathcal{A}$ to the arrangement of $N\geq n$ hyperplanes by adding sufficiently generic hyperplanes yields an arrangement $\mathcal{A}'$ such that $\mathcal{B}(n,k,\mathcal{A})$ is the intersection of $\mathcal{B}(N,k,\mathcal{A}')$ and the coordinate subspace. It follows that $\mathcal{B}(N,k,\mathcal{A}')$ admits strata of codimension 2 and the multiplicity 3 as well. On the other hand, for $n=k+2$, $\mathcal{B}(k+2,k,\mathcal{A})$ has only one stratum of multiplicity $k+2$ i.e., the inequality $n\neq k+3$ is sharp. □

The following example illustrates the above two lemmas.

Example 3.7. *Let $\mathcal{A}_\infty$ be a general position arrangement of 8 hyperplanes $H_{\infty,i}$ in $\mathbb{P}^4$ and $\mathcal{A}_\infty^X$ its restriction to the plane $X=H_{\infty,7}\cap H_{\infty,8}$. The restricted arrangement $\mathcal{A}_\infty^X$ is an arrangement in general position since $\mathcal{A}_\infty$ is in general position.*

Assume that the double points $H_{\infty,1}\cap H_{\infty,2}\cap H_{\infty,7}\cap H_{\infty,8}$, $H_{\infty,3}\cap H_{\infty,4}\cap H_{\infty,7}\cap H_{\infty,8}$, $H_{\infty,5}\cap H_{\infty,6}\cap H_{\infty,7}\cap H_{\infty,8}$ are co-linear.

Consider the hyperplanes

$$D_{1,2,3,4,7,8},\ D_{3,4,5,6,7,8},\ D_{1,2,5,6,7,8}$$

in a discriminantal arrangement $\mathcal{B}(8,5,\mathcal{A})$ corresponding to such $\mathcal{A}_\infty$ and the hyperplanes

$$D'_{1,2,3,4},\ D'_{3,4,5,6},\ D'_{1,2,5,6}$$

in the discriminantal arrangement in 3-space $H_7\cap H_8$ for a generic choice of hyperplanes H_7,H_8 intersecting the hyperplane at infinity at $H_{\infty,7},H_{\infty,8}$ respectively. Then the arrangement $\mathcal{A}$ of 8 hyperplanes in $\mathbb{C}^5$ including H_7,H_8 has a common point if and only if the arrangement of 6 planes in 3-space $H_7\cap H_8$ has a common point. Hence

$$\dim D_{1,2,3,4,7,8}\cap D_{3,4,5,6,7,8}\cap D_{1,2,5,6,7,8}=2+\dim D'_{1,2,3,4}\cap D'_{3,4,5,6}\cap D'_{1,2,5,6}=6 \tag{11}$$

(the last equality uses the Example 3.4). Hence the discriminantal arrangement $\mathcal{B}(8,5,\mathcal{A})$ has a codimension two stratum of multiplicity 3.

This case illustrates the case considered in Theorem 3.9 (2) below, corresponding to the dependent restriction arrangement of $\mathcal{A}_\infty$ given by hyperplanes $H_{\infty,i}\cap H_{\infty,7}\cap H_{\infty,8}$, $i=1,\ldots,6$ and $s=2$.

The next Lemma will be useful in the proof below showing the absence of codimension 2 strata having the multiplicity 4.

Lemma 3.8. *For $s \geq 2$, there is no quadruple of subspaces $V_i \subset \mathbb{P}^{3s-2}, i = 1, 2, 3, 4$ having dimension $2s-2$ such that intersections $P_{i,j} = V_i \cap V_j, i \neq j$ satisfy*
a) each $P_{i,j}$ has dimension $s-2$
b) any pair $P_{i,j}, P_{i,k}, i \neq j \neq k \neq i$ spans a hyperplane in V_i, and
c) all three, $P_{i,j}, P_{i,k}, P_{i,l}$, belong to a hyperplane in V_i.

Proof. We shall start with the case $s = 2$. Assume that a configuration as in Lemma 3.8 does exist and consider a quadruple of planes $V_i, i = 1, ..., 4$ in $\mathbb{P}^4$ such that
a) any two intersect at a single point,
b) all 6 points $P_{i,j} = V_i \cap V_j$, $i \neq j$, obtained in this way are distinct, and
c) all three points, $P_{i,j}, P_{i,k}, P_{i,l}$, are colinear, i.e., span a line L_i.

For a fixed k, the triple of points $P_{i,j}$,$i, j \neq k$, outside of V_k, determines the triple of lines $L_i \subset V_i, i \neq k$ spanned by points $P_{i,j}, P_{i,l}$, $i, j, l \neq k$. These lines $L_i, i \neq k$ by their definition are pairwise concurrent ($L_i \cap L_j = P_{i,j}$) and hence belong to a plane H. By assumption c), for each $i \neq k$, the three points $P_{k,i} = V_k \cap V_i$ are points on lines L_i distinct from $P_{i,j}, P_{i,l}$. Hence H and V_k have 3 distinct non-colinear points in common and therefore $H = V_k$, but this contradicts $\dim V_k \cap V_i = 0$.

Now consider the case $s > 2$. Similar to the above, $(s-2)$-dimensional subspaces $P_{i,j} = V_i \cap V_j$ of $\mathbb{P}^{3s-2}$ determine the subspaces $L_i \subset V_i, i \neq k$ (for a fixed k) each being spanned by pairs

$$P_{i,j}, P_{i,l}, i, j, l \neq k$$

which are outside of V_k. Each L_i is a hyperplane in V_i (i.e. $\dim L_i = 2s-3$). Moreover, the dimension of the subspace H of $\mathbb{P}^{3s-2}$ spanned by $L_{i,j}, L_{i,l}, i, j, l \neq k$, is $3s-4$. The subspace H can be described as the subspace of $\mathbb{P}^{3s-2}$ spanned by triple of subspaces $P_{i,j}, i, j \neq k$. Now by our assumption c), V_k contains an $(s-2)$-dimensional subspace of $L_{i,j}, i, j \neq k$, i.e., $P_{i,k}$. The subspace hence is also a subspace of H. This implies that $V_k \subset H$. The dimension of intersection $L_{i,j}$ and V_k, which are both subspaces of H, is equal to $(2s-3) + (2s-2) - (3s-4) = s-1$ and hence $\dim V_i \cap V_k = s-1$. This is a contradiction. $\square$

Now we are ready for the main result of this section. It describes the codimension 2 strata of discriminantal arrangements having the multiplicity 3 and shows an absence of codimension 2 strata having the multiplicity 4 (with obvious exceptions).

Theorem 3.9. *Let $\mathcal{A}_\infty$ be a general position arrangement of hyperplanes in $\mathbb{P}^{k-1}$ which is the trace at infinity of a general position arrangement $\mathcal{A}^0$ in $\mathbb{C}^k$.*
1. The arrangement $\mathcal{B}(n, k, \mathcal{A}^0)$ has $\binom{n}{k+2}$ codimension 2 strata of multiplicity $k+2$.
2. There is a one-to-one correspondence between
a) the dependent restrictions of subarrangements of $\mathcal{A}_\infty$, and
b) triples of hyperplanes in $\mathcal{B}(n, k, \mathcal{A}^0)$ for which the codimension of their intersection is equal to 2.
3. There are no codimension 2 strata having the multiplicity 4 unless $k = 2$. All codimension 2 strata of $\mathcal{B}(n, k, \mathcal{A}^0)$ not mentioned in part 1, have a multiplicity which is either 2 or 3 (the latter corresponding to triples of hyperplanes in b).
4. The codimension 2 strata of $\mathcal{B}(n, 2, \mathcal{A}^0)$ is independent of $\mathcal{A}^0$.

Proof. The statement (1) follows immediately from the discussion after (6) in section 2.2. If $J \subset \{1, ..., n\}$ is a subset of cardinality $k+2$, then D_J is a codimension 2 subspace in $\mathbb{C}^n$ and belongs to $k+2$ hyperplanes $D_K, K \subset J$.

Next we shall determine the conditions on three different sets of $k+1$ indices under which codim $D_{K_i} \cap D_{K_j} \cap D_{K_l} = 2$.

Consider first the case when sets K_i, K_j, K_l, each having the cardinality $k+1$, are such that for one of them, say K_i, one has $K_i \setminus (K_i \cap (K_j \cup K_l)) \neq \emptyset$, i.e., one of the set in this triple is not in the union of other two. If $r \in K_i \setminus (K_i \cap (K_j \cup K_l))$, then the hyperplanes in an arrangement $\mathcal{A} \in D_{K_j} \cap D_{K_l}$ with indices different from the indices in $K_j \cup K_l$ can be chosen as arbitrary parallel translates of hyperplanes in $\mathcal{A}^0$, while $H_r \in \mathcal{A}', \mathcal{A}' \in D_{K_i}$ is fixed by condition $\mathcal{A}'$ being in D_{K_i} and the selection of hyperplanes with indices different from r but in K_i. Hence $D_{K_i} \cap D_{K_j} \cap D_{K_l} \neq D_{K_j} \cap D_{K_l}$, i.e., codim $D_{K_i} \cap D_{K_j} \cap D_{K_l} = 3$.

Now let us consider the alternative to the case considered in the previous paragraph. Hence we have a triple K_i, K_j, K_l such that

$$K_i = (K_i \cap K_j) \bigcup (K_i \cap K_l) \tag{12}$$

for any permutation of (i, j, k). Condition (12) for $k = 2$ implies that either Card$(K_i \cap K_j) = 2$ or Card$(K_i \cap K_l) = 2$ that is either $D_{K_i} \cap D_{K_j} = D_{K_i \cup K_j}$ or respectively $D_{K_i} \cap D_{K_l} = D_{K_i \cup K_l}$. Since this imples that Card$(K_i \cup K_j) = 4$ (resp. Card$(K_i \cup K_l) = 4$), we obtain that $D_{K_i} \cap D_{K_j}$ (resp. $D_{K_i} \cap D_{K_l}$) is a codimension 2 subspace of multiplicity $4 = k + 2$ and part (4) follows.

Let $L_{\alpha,\beta} = (K_\alpha \cap K_\beta) \setminus \bigcap_{s=i,j,k} K_s, t = $ Card $\bigcap_{\alpha=i,j,k} K_\alpha, l_{\alpha,\beta} =$ Card $L_{\alpha,\beta}$. Then (12) implies that $K_\beta \setminus \bigcap_{\alpha=i,j,k} K_\alpha = L_{\alpha,\beta} \bigcup L_{\beta,\gamma}$ and since Card $K_i = k+1$ we have

$$l_{\alpha,\beta} + l_{\beta,\gamma} + t = k + 1, \quad \alpha \neq \beta \neq \gamma. \tag{13}$$

Using these relations for allowable permutations of subscripts, yields:

$$l_{\alpha,\beta} = \frac{k+1-t}{2} \quad \alpha \neq \beta, \alpha, \beta \in \{i, j, k\}. \tag{14}$$

For a triple of subsets K_i, K_j, K_l, Card $K_i \cap K_j \cap K_l = t$ and a fixed arrangement $\mathcal{A}$, consider the map of the spaces of translates:

$$\mathbb{S}(H_1^0, ..., H_n^0) \to \mathbb{C}^t = \mathbb{S}(..., H_r^0, ...) \quad r \in K_i \cap K_j \cap K_l$$

which assigns to a collection of n parallel translates $H_1^{t_1}, ..., H_n^{t_n}$ of $H_1^0, ..., H_n^0$ in $\mathbb{C}^k$, the intersections of the hyperplanes with indices outside of $K_i \cap K_j \cap K_l$ with the linear subspace which is the intersection of t hyperplanes with indices in $K_i \cap K_j \cap K_l$.

This map has as its fiber over the set of translates $H_\beta^{t_\beta}$, the space

$$\mathbb{S}(..., H_\alpha^0 \cap (\bigcap_{\beta \in K_i \cap K_j \cap K_l} H_\beta^{t_\beta}), ...), \qquad \alpha \in [1, ..., n] \setminus K_i \cap K_j \cap K_l$$

of translates in the $(k-t)$-dimenional space $\bigcap_{\beta \in K_i \cap K_j \cap K_l} H_\beta^{t_\beta}$. If s is the dimension of the family of arrangements which is the intersection of hyperplanes $D_{K_\alpha}, \alpha = i, j, l$ then the dimension of the family of restrictions of arrangements to $\mathbb{C}^{k-t}$ is $s - t$. Hence

$$\text{codim } D_{K_i} \cap D_{K_j} \cap D_{K_l} = \text{codim } D_{K_i \setminus \bigcap K_\alpha} \cap D_{K_j \setminus \bigcap K_\alpha} \cap D_{K_l \setminus \bigcap K_\alpha} \qquad \alpha = i, j, l \tag{15}$$

where the intersection on the right is taken in the space of parallel translates in $\bigcap_{j \in \bigcap K_i} H_j^0$.

Clearly $t < k$, and in the case when $t = k - 1$, we have $l_{i,j} = 1$, i.e., Card $\bigcup K_i = k + 2$ and we are in the case (1), i.e., the codimension 2 stratum has the multiplicity $k + 2$. If $t = 0$, then we have the case considered in Lemma 3.2 and we also see from this lemma that the intersection of $D_{K_i}, i = 1, 2, 3$ has a codimension two stratum if and only if the assumptions of the theorem are fulfilled. The rest of the part (2) of the theorem follows from Lemma 3.5 applied to the restriction on $\bigcap_{\alpha \in K_i \cap K_j \cap K_l} H_\alpha^0$ and the relation (15) (with $s = l_{\alpha,\beta}$).

Now consider the existence of a codimension 2 strata of multiplicity 4. Suppose that such stratum exists and $K_i, i = 1, ..., 4$ are the corresponding subsets of $\{1, ..., n\}$. By the quadruples analog of restriction (15), it is enough to consider the case $\bigcap_{i=1,...,4} K_i = \emptyset$. Let

$$l_{i,j,m} = \text{Card } K_i \cap K_j \cap K_m.$$

Then for any i, Card $K_i \cup \bigcap_{j \neq i} K_j = \text{Card } \bigcup K_i$, i.e., $l_{i,j,m} + k + 1$ is independent of (i, j, m). Hence one infers from (13) the relation $l_{i,j,m} = \frac{k+1}{3}$.

Note that codim $\bigcap_{i=1,...,4} D_{K_i} = 2$ if and only if codim $D_{K_{i_1}} \cap D_{K_{i_2}} \cap D_{K_{i_3}} = 2$ for all 4 triple $1 \leq i_j \leq 4$ of distinct integers. Applying part (2) of the theorem to each triple i_1, i_2, i_3, one infers the existence of a quadruple of subspaces as in Lemma 3.8. Hence this lemma implies part (3).

□

Corollary 3.10. *If a discriminantal arrangement $\mathcal{B}(n, k, \mathcal{A})$ satisfies $n > \frac{3}{2}(k+1)$ and admits a codimension 2 stratum of multiplicity 3, then there exists a proper subarrangement $\mathcal{A}' \subset \mathcal{A}$ such that $\mathcal{B}(n, k, \mathcal{A}')$ admits a codimension 2 stratum of multiplicity 3.*

Proof. It follows immediately from the above theorem and inequality (10). □

3.1. **Numerology of singularities of generic plane sections.** Theorem 3.9 contains a complete description of combinatorics of codimension 2 strata of discriminantal arrangements. Indeed, the possible multiplicities of codimension two strata are $\binom{k+2}{k+1}$, 3 and 2. The number of points of multiplicity 3 is the number of triples of strata satisfying condition *2a)*. It is an interesting problem to determine the number of triple points which $\mathcal{B}(n, k, \mathcal{A})$ can have. It is clear from Theorem 3.9 that this number can be arbitrary large when $n \to \infty$, though even the precise asymptotic is not clear.

4. The Gale transform and codimension two strata

4.1. **The Gale transform and associated sets.** In this subsection we shall discuss interpretation of discriminantal arrangements using the Gale transform. Recall the following:

Definition 4.1. *Let V be a vector space over $\mathbb{C}$, $\dim V = k$, $l_i \in V^*, i = 1, ..., n$, be n vectors in the dual of the vector space V and let*

$$0 \to V \xrightarrow{L} \mathbb{C}^n \to W \to 0 \tag{16}$$

be the exact sequence in which $L(v) = (l_1(v), ..., l_n(v))$. The Gale transform of collection l_i is the collection $m_i \in W, i = 1, ..., n$, of images of the vectors e_i of the standard basis in $\mathbb{C}^n$.

The following is suggested by an argument in [6] (see also [17] and [3]).

Proposition 4.2. *Let $\mathcal{A}$ be a central arrangement of* Card $\mathcal{A} = n$ *in a k-dimensional vector space V such that the corresponding arrangement in $\mathbb{P}^{k-1}$ is in the general position. Let l_i be the elements in V^* corresponding to the hyperplanes in $\mathcal{A}$. The essential part of the discriminantal arrangement consists of hyperplanes in W spanned by collections of $n-k-1$ vectors of the Gale transform of vectors $l_i \in V^*$.*

Proof. Let $f_1, ..., f_k$ be a basis in V, $x_j, j = 1, ..., k$, be the coordinates in this basis, and let

$$l_i = \sum a^i_j x_j, j = 1, ..., k, i = 1, ..., n$$

be the equations of the hyperplanes of $\mathcal{A}$. Denote by $A = \{a^i_j\}$ the corresponding matrix. Translates of hyperplanes $l_{i_1} = 0, ..., l_{i_{k+1}} = 0$, by $c_{i_1}, ..., c_{i_{k+1}}$ respectively, have a non-empty intersection if and only if the system of equations $\sum a^{i_s}_j x_j = c_{i_s}, s = i_1, ..., i_{k+1}$, has a solution.

This takes place if and only if the projection $\pi_{i_1,\dots,i_{k+1}}(c)$ of the point $c=(c_1,\dots,c_n)\in\mathbb{C}^n$ on the subspace of $\mathbb{C}^n$ spanned by the vectors $e_{i_1},...,e_{i_{k+1}}$ belongs to the image of projection $\pi_{i_1,\dots,i_{k+1}}(L(V))$ of $L(V)$, which is equivalent to

$$c\in H_{\pi_{i_1,\dots,i_{k+1}}}\simeq Span(V,Ker_{\pi_{i_1,\dots,i_{k+1}}}) \tag{17}$$

(here by an abuse of notation, we identified V with its image $L(V)$ in $\mathbb{C}^n$). The hyperplanes $H_{\pi_{i_1,\dots,i_{k+1}}}$ form the discriminantal arrangement in $\mathbb{C}^n$ and the relation $V\subset H_{\pi_{i_1,\dots,i_{k+1}}}$ shows that the essential part of discriminantal arrangement is its restriction to $W=\mathbb{C}^n/V$. The inclusion (17) is equivalent to $c\in Ker_{\pi_{i_1,\dots,i_{k+1}}} mod\ V$. The image $Ker_{\pi_{i_1,\dots,i_{k+1}}}\in\mathbb{C}^n/V=W$ is spanned by the images of the Gale transform of $l_i, i=1,...,n$, and is the hyperplane in the essential part of discriminantal arrangement. □

Next recall the classical notion of associated sets (cf. [5], Ch.III):

Definition 4.3. *Let V,W be vector spaces such that $\dim V=k,\dim W=n-k$. Let $f_1,...,f_k$ and $g_1,...,g_{n-k}$ be the bases of V,W respectively. The set of vectors $l_1,...,l_n$ in V and $m_1,...,m_n$ in W are called associated if the matrices X and Y of coordinates of $l_i, i=1,...,n$, and $m_j, j=1,...,n$, satisfy:*

$$X\cdot\Lambda\cdot^t Y=0, \tag{18}$$

where Λ is a diagonal matrix.

The sets in V and W are associated if and only if one is the Gale transform of another (see the discussion in [5] p.33 where the association is discussed in projective setting, for example).

4.2. **Discriminantal arrangements of planes in $\mathbb{C}^3$ and the Gale transform.** One can ask for the meaning of a codimension 2 strata with multiplicity three in discriminantal arrangements described in the Theorem 3.9 in terms of the Gale transform. In the case, $n=6,k=3$, one has a geometric interpretation (see [5] for geometric interpretations for some other values n,k) of the Gale transform which allows one to show the following:

Proposition 4.4. *The existence of a partition of 6-tuples of points in $\mathbb{P}^2$ into 3 pairs, each pair defining a line, and such that these lines are concurrent lines, is an invariant of the Gale transform.*

Remark 4.5. *After replacing hyperplanes by the points of a projective dual space, this proposition is equivalent to the case $n=6,k=3$ of the main theorem (cf. also Example 3.4). This equivalence follows from the classical description of the Gale transform recalled in the proof below.*

The more general case, considered in the Lemma 3.2, can be interpreted as the following property of the Gale transform $(\mathbb{P}^{2s-2})^{\times 3s}\to(\mathbb{P}^s)^{\times 3s}$.

The condition that there is a partition of $3s$ points in $\mathbb{P}^s$ into 3 groups of s points, each set spanning a hyperplane in $\mathbb{P}^s$ and that, moreover, such that these hyperplanes belong to a pencil, is equivalent to the condition that the Gale transform of this set of points in $\mathbb{P}^{2s-2}$ admits a partition into 3 groups of cardinality s such that the triple of $(s-1)$-dimensional subspaces, each spanned by an s-tuple in $\mathbb{P}^{2s-2}$, have a non-empty intersection.

This restatement follows immediately from the dualization of hyperplanes of the general position arrangement. Indeed, $3s$ hyperplanes of the general position arrangement in $\mathbb{C}^{2s-1}$ considered in Lemma 3.2 define $3s$ hyperplanes $\mathbb{P}^{2s-2}$ or equivalently $3s$ points in the dual projective space. The assumption of the dependency of $3s$ hyperplanes in $\mathbb{P}^{2s-2}$, after the dualization is equivalent to requiring that 3 $s-1$-dimensional subspaces $\eta_1^{s-1},\eta_2^{s-1},\eta_3^{s-1}\subset\mathbb{P}^{2s-2}$ each spanned by one of 3 subsets of cardinality s (i.e. subsets $K_i\cap K_j$ in notations of definition 3.3) have non-empty intersection. Since $n-k-1=3s-(2s-1)-1=s$, by Proposition 4.2, the hyperplanes *of the essential part of the discriminant arrangment are spanned by s -subsets of the set of $3s$ points in $\mathbb{P}^s$. Lemma 3.2 states that the dependency condition is equivalent to the existence of the triple of hyperplanes in the discriminantal arrangement belonging to a pencil of hyperplanes which gives our claim.*

It would be interesting to have a geometric description of the Gale transform allowing one to show this directly for $s > 2$.

Proof. We shall use the projective setting which, in this case, relates 6-tuples of points in $\mathbb{P}^2$ to another 6-tuples in another copy of $\mathbb{P}^2$. Recall that the smooth cubic surfaces in $\mathbb{P}^3$ (i.e., the del Pezzo surfaces of degree 3) can be viewed as blow ups of a 6-tuples of points in $\mathbb{P}^2$ and the classes of projective equivalence of 6-tuples of points in $\mathbb{P}^2$ correspond to isomorphism classes of cubic surfaces. The 6-tuple of points in $\mathbb{P}^2$ is obtained by contracting 6 of 27 lines having pairwise empty intersections. In terms of the blow up of 6 points in $\mathbb{P}^2$, each of 27 lines is one of the following:

1. 6 exceptional curves of the blow up;
2. proper preimages of 15 lines defined by pairs of points;
3. proper preimages of 6 quadrics determined by a 5 points subset of the blown up 6-tuple.

6-tuples of lines as above on a cubic surface V correspond to the following homology classes in $H^2(V, \mathbb{Z})$:

$$h_i, i = 1, ..., 6, \qquad (h_i, h_j) = -\delta^i_j. \tag{19}$$

Given such 6-tuple h_i, one has a unique additional 6-tuple h'_i characterized by the following: together with h_i the collection h'_i form a double six, i.e., the following relations are satisfied:

$$h_i h_j = h'_i h'_j = -\delta^i_j, \quad h_i h'_j = 1 - \delta^i_j. \tag{20}$$

Using the description of 27 lines above in terms of lines and quadrics on $\mathbb{P}^2$ corresponding to the lines on a del Pezzo surface, the second component h'_i of a double six, in which the first component h_i is formed by the 6-tuple of exceptional curves, can be described as follows. The 6-tuple h'_i consists of the proper preimages of quadrics labeled in the way which assigns to (the class of) exceptional curve h_P contracted to a point $P \in \mathbb{P}^2$ (the class of) the quadric h'_P passing through points of the 6-tuple of points in $\mathbb{P}^2$ distinct from P.

Now the existence of partition of 6-tuples as in Proposition 4.4 is equivalent to existence of Eckardt point (i.e., a point common to a triple of lines on cubic surface) not involving the exceptional curves and to show Proposition 4.4 one needs to show that such Eckardt point exists also for the second component of a double six. But each line containing a pair of points P, P' on the plane $\mathbf{P}$ obtained by contraction of a 6-tuples of disjoint exceptional curves on del Pezzo surface will be passing through a pair of 6 points on the plane $\mathbf{P'}$ obtained by contracting proper preimages of 6 quadrics on $\mathbf{P}$. Indeed, such a line through $P \in \mathbf{P}$ will intersect the proper preimage on the blow up of $\mathbf{P}$ of the quadric not containing P at exactly one point (corresponding to the intersection point with this quadric distinct from the blown up point). Since the blow up of 6 points and contracting proper preimages of 6 quadrics determined by these 6 points is an isomorphism on the complement to quadrics which contains the concurrency point of the triple of lines, the claim follows. □

5. Fundamental groups of the complements to discriminantal arrangements

5.1. **Nilpotent completion of the fundamental group.** In this section we shall describe the nilpotent completion of $\pi_1(\mathcal{B}(n, k, \mathcal{A}))$ in the case when $\mathcal{A}$ is not very generic and the corresponding discriminantal arrangement admits a codimension two strata of multiplicity 3. This is a direct consequence of [9] Prop.2.2 (see also [13]).

Proposition 5.1. *Completion of the group ring $\mathbb{C}[\pi_1(\mathbb{C}^n \setminus \mathcal{B}(n,k,\mathcal{A}))]$ with respect to the powers of the augmentation ideals is the quotient of the algebra of non-commutative power series*

$$\mathbb{C} << X_J >>, \ J \in \mathcal{P}_{k+1}(\{1,\dots,n\})$$

by the two-sided ideal generated by relations

(i) $[X_J, \sum_I X_I]$ for a pair of subsets $J \in \mathcal{P}_{k+1}(K)$, with summation over

$$I \in \mathcal{P}_{k+1}(K), \ K \subset \{1,...,n\}, \text{Card } K = k+2.$$

(ii) $[X_J, X_I + X_J + X_K]$ where I, J, K are subscripts corresponding to triples of hyperplanes in the discriminantal arrangement, such that corresponding hyperplanes in $\mathcal{A}_\infty$ satisfy dependency condition of Theorem 3.9 (2a).

(iii) $[X_J, X_K]$ for any pair of sets with Card $J, K \geq k+3$ *and such that there does not exist subset I such that triple I, J, K satisfies the conditions of Theorem 3.9 (2a).*

5.2. **Braid monodromy of discriminantal arrangements and $\pi_1(\mathbb{S} \setminus \mathcal{B}(n,k,\mathcal{A}))$.** We shall describe the fundamental group of the complement to a discriminantal arrangement. In fact, we shall obtain the braid monodromy of the generic plane section of discriminantal arrangement, which by the classical van Kampen procedure yields the presentation of the fundamental group.

We describe the braid monodromy of the generic section of $\mathcal{B}(n,k,\mathcal{A})$ in terms of a collection of orderings of hyperplanes of $\mathcal{B}(n,k,\mathcal{A})$ constructed in terms of equations (2) of arrangement $\mathcal{A}$ of hyperplanes in the general position $H_j^0, j = 1,...,n$ as follows. The generic plane section of $\mathcal{B}(n,k,\mathcal{A})$ can be described as subset of $\mathbb{C}^2$ with coordinates (s,t) depending on a choice of generic a^n, b^n, c^n (specifying the plane section) consisting of points (s,t) such that the rank of the $(k+1) \times n$ matrix:

$$\begin{pmatrix} \alpha_1^1 & ... & \alpha_k^1 & a^1 t + b^1 s + c^1 \\ ... & ... & ... & ... \\ \alpha_1^n & ... & \alpha_k^n & a^n t + b^n s + c^n \end{pmatrix} \tag{21}$$

is maximal. This plane is given in $\mathbb{S}$ by

$$x_i = a^i t + b^i s + c^i. \tag{22}$$

For fixed $\alpha_j^i, a^i, b^i, c^i, i = 1,...,n, j = 1,...,k$, and generic (t,s), the rank of this matrix is $k+1$. For a generic fixed s, there is a finite collection $t_1(s) < ... < t_{\binom{n}{k+1}}(s)$ of real numbers such that the rank of (21) is k: each $t_i(s)$ corresponds to a $k+1$ subset of $\{1,...,n\}$ labeling a hyperplane in $\mathcal{B}(n,k,\mathcal{A})$. Moreover there will be finite collection of real numbers $s_1 < ... < s_N, N \geq \binom{n}{k+2}$ such that for these s there will be strictly less than $\binom{n}{k+1}$ constants t for which the rank of (21) is less than $k+1$. In fact, these values s correspond to projections on the s-coordinate of multiple points of the arrangement of lines restriction of $\mathcal{B}(n,k,\mathcal{A})$ to the (s,t) plane. In particular, to each s_i corresponds a subset P_i in the sequence $1,...,\binom{n}{k+1}$ corresponding to the set of $(k+1)$-subsets yielding the same value $t(s_i)$. The cardinality of the subset P_i is either $k+2$, 3 or 2 (according to the multiplicity of the singular point corresponding to s_i).

Recall (cf. for example [14] or [11]) that the real line $Im(s) = 0$ in the complex s-line $\mathbb{C}_s \simeq \mathbb{C}$ can be used to define in a canonical way the generators of the fundamental group $\pi_1(\mathbb{C}_s \setminus \bigcup_{i=1}^N s_i)$ of the complement of N points in $\mathbb{C}$. In details, the generator corresponding to the point $s_i, i = 1,....,N$ is the loop from a base point $s_0, Im(s_0) = 0, s_0 << 0$, to the point s_i, circumventing each $s_j, j < i$ as a semi-circle into the halfplane $Im(s) < 0$ and returning back to s_0 after making the full circle around s_i. The braid monodromy for such a path is the product of factors corresponding to each $s_j, j \leq i$, i.e., the half twist β_{P_j} corresponding to P_j for $j < i$ and the full twist $\beta_{P_i}^2$.

Theorem 5.2. *1. The braid monodromy of a generic plane section corresponding to section (22) of $\mathcal{B}(n,k,\mathcal{A})$ is given by*

$$\Pi_{1\le k\le N}\Gamma_i \quad \textit{where} \quad \Gamma_i=\beta_{P_1}^{-1}...\beta_{P_{k-1}}^{-1}\beta_{P_k}^2\beta_{P_{k-1}}...\beta_{P_2}\beta_{P_1}. \tag{23}$$

2. The fundamental group $\pi_1(\mathbb{C}^n\setminus\mathcal{B}(n,k,\mathcal{A}))$ has the following presentation:

$$\Gamma_i(\delta_j)=\delta_j \quad j=1,...,\binom{n}{k+1}, i=1,...,N. \tag{24}$$

These statements are the standard applications of the results from the theory of braid monodromy (cf., among others, [14],[11]). Different presentations can be obtained via Salvetti's presentation or Randell's presentation for complement of hyperplane arrangements (see [19], [18]). For $\mathcal{A}^0$ very generic, this yields a presentation equivalent to the one given in [10].

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