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BRIESKORN AND THE MONODROMY

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To Egbert Brieskorn, in memory.

1. Introduction

Brieskorn's paper "Die Monodromie der isolierten Singularitäten von Hyperflächen," published in 1970 in Manuscripta Mathematica, gave a new insight to the theory of monodromy and Gauß-Manin connections. The paper, written in the framework of isolated hypersurface singularities, has been generalized for isolated complete intersection singularities by G.-M. Greuel in 1975 [10]. In the following times and also more recently, a long list of authors, among them P. Deligne [7], W. Ebeling [8], H. Hamm [12], Lê D. T. [20], B. Malgrange [24], D.Siersma [37] etc. provided generalizations and developments of the monodromy theory. The regularity of the Gauß-Manin connection, proved by Brieskorn in the framework of isolated hypersurface singularities has been proved and developped in various situations by many authors, among them G.-M. Greuel [10], C. Hertling [15], F. Pham [28], K. Saito [29], M. Saito [30], J. Scherk and J.H.M. Steenbrink [31], M. Schulze [32], A. Varchenko [38], etc.

There are many surveys concerning the various aspects of monodromy and including developments of the theory. In particular, Ebeling's survey [8] shows very well the importance of Brieskorn's article as well as developments and generalisations of the Brieskorn's results. Siersma's survey [37] deals with the non-isolated case, and presents new results in this framework.

The present paper, based on ideas of the second author [34, 35, 36], does not pretend any originality. It is not devoted to specialists, but to "beginners". The aim of the paper is to introduce monodromy theory and provide some elementary view about the Brieskorn paper. Our aim is not to replace the reading of this very important Brieskorn article, but hopefully to encourage one to read it.

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2. Connections and monodromy

2.1. **Definitions and notations.** Let $f : (\mathbb{C}^{n+1}, 0) \to \mathbb{C}$ be an analytic function defined in a neighbourhood of the origin 0 in $\mathbb{C}^{n+1}$ and such that $f(0) = 0$. We denote by $(z_0, \dots, z_n)$ the local coordinates of $\mathbb{C}^{n+1}$ at 0. Let us assume that f admits an isolated singularity at 0, that is the partial derivatives $(\partial f/\partial z_i)(z)$ have a common zero at the origin and there is no other singularity in a neighbourhood of 0.

One denotes by $\mathcal{O}$ the local ring of $\mathbb{C}^{n+1}$ at 0 and by I the ideal of $\mathcal{O}$

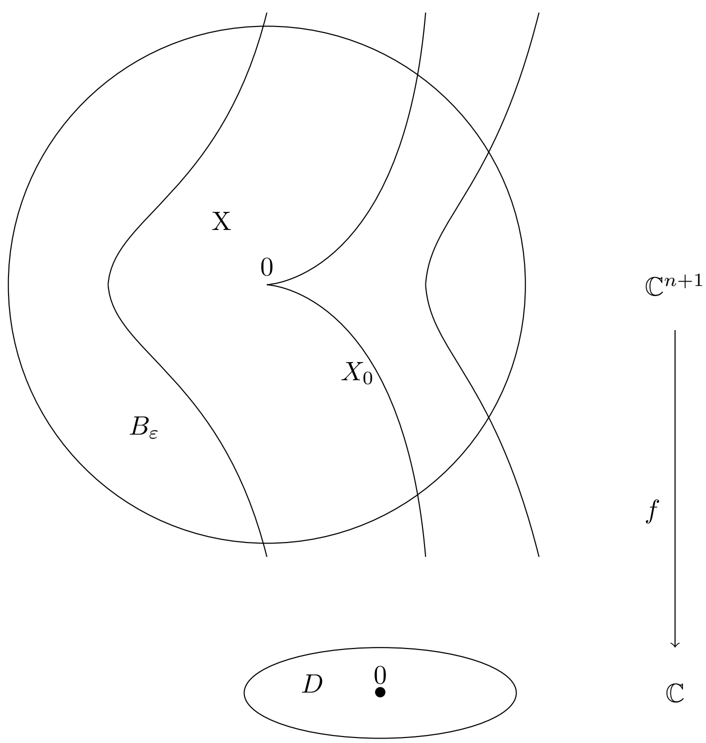
$$I = \left\langle \frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n} \right\rangle.$$

The Milnor number of the singularity is defined by:

$$\mu = \dim_{\mathbb{C}} \mathcal{O}/I,$$

denoted by $b_{f,0}$ in Brieskorn [5].

Let us fix some (classical) notations. Let ε and η be such that $0 < \eta < \varepsilon$ and denote:
$B_\varepsilon \subset \mathbb{C}^{n+1}$ the ball defined by $\|z\| < \varepsilon$, $z \in \mathbb{C}^{n+1}$,
$D \subset \mathbb{C}$ the disk defined by $|t| < \eta$, $t \in \mathbb{C}$, and $D' = D \setminus \{0\}$,
$X = B_\varepsilon \cap f^{-1}(D) = \{z \in \mathbb{C}^{n+1}\ ;\ \|z\| < \varepsilon \text{ and } |f(z)| < \eta\}$,
$X' = X \setminus f^{-1}(0)$ and $X_t = X \cap f^{-1}(t)$ for all $t \in D$.
The following classical picture illustrates the situation.



The fundamental theorem, due to Milnor is the following:

Theorem 2.1 (Milnor)**.** If ε and η are small enough, then:
(i) The map $f : X \setminus f^{-1}(0) \to D \setminus \{0\}$ is a $\mathcal{C}^\infty$ differentiable fibration, locally trivial and whose fibres have the homotopy type of a bouquet of μ spheres with dimension n.
(ii) There exists $\varepsilon_1 < \varepsilon$ such that the intersection of X_t with the sphere $S_r \subset \mathbb{C}^{n+1}$ centered at 0 and with radius r is transverse for all $|t| \leq \eta$ and $\varepsilon_1 \leq r \leq \varepsilon$.

In the following, we intend to make explicit the action of the fundamental group $\pi_1(D')$ on the cohomology of the fibre $H^n(X_t; \mathbb{C})$ for $t \in D'$. We will use some results on connections.

2.2. **Connections.** Let $\pi : E \to B$ a locally trivial fibre bundle, where the fibre F and basis B are locally compact. We assume that F has the homotopy type of a finite complex. One define a (complex) vector fibre bundle $H^n(\pi)$ with basis B in the following way:

The total space is the set of pairs (t, α) where $t \in B$ and $\alpha \in H^n(F_t; \mathbb{C})$, where $F_t = \pi^{-1}(t)$. The projection of $H^n(\pi)$ on B sends (t, α) to t. The vectorial structure of the fibres is clear. The topology of $H^n(\pi)$ is defined by the way of local charts: for every open subset $U \subset B$ which is a trivialization domain of $\pi : E \to B$, one has a homeomorphism $\psi|_U : E|_U \to U \times F$ and then for every $t \in U$, an identification $\psi_t : F_t \xrightarrow{\sim} F$. For all $t \in B$, one has an isomorphism $\psi_t^* : H^*(F;\mathbb{C}) \xrightarrow{\sim} H^*(F_t;\mathbb{C})$. The local chart of $H^n(\pi)$ over U is defined by the bijection

$$\Psi_U : H^n(\pi)|_U \to U \times H^n(F;\mathbb{C})$$

such that $\Psi_U(t,\alpha) = (t, (\psi_t^*)^{-1}(\alpha))$. The vector bundle $H^n(\pi)$ is then well defined.

The charts Ψ_U are given by locally constant maps: if $V = U \cap U'$ is connected, the transition map $V \to \mathrm{Aut}(H^n(F))$ defined by the local charts, is constant. The group $\mathrm{Aut}(H^n(F))$ is a discrete group, that allows to introduce on $H^n(\pi)$ a locally flat connection ∇ (by the way of the parallel transport, see for example [19]).

Definition 2.2. The horizontal sections of the bundle $H^n(\pi)$ are sections for which the covariant derivative for the connection ∇ vanishes, that is sections which, locally, are transformed by each Ψ_U into constant sections of the trivial bundle $U \times H^n(F;\mathbb{C})$.

One notices that, if B is a complex analytic manifold, then $H^n(\pi)$ is a holomorphic vector bundle over B and ∇ is a locally flat holomorphic connection.

2.2.1. *Monodromy.* The parallel transport defines, for all $t_0 \in B$, an action of $\pi_1(B,t_0)$ on $H^n(\pi)_{t_0}$. A practical way to determine this action is the following:

Let $\lambda : [0,1] \to B$ a loop at t_0 and let $\alpha \in H^n(F_{t_0};\mathbb{C})$. One considers a subdivision

$$0 = \tau_0 < \tau_1 < \cdots < \tau_q = 1$$

of $[0,1]$ sufficiently fine so that, for all $i = 1,\ldots,q-1$, there exists a horizontal section v_i of $H^n(\pi)$ defined in an open subset of B containing $\lambda([\tau_i,\tau_{i+1}])$ and such that:

$$v_0(\lambda(0)) = \alpha \qquad v_{i-1}(\lambda(\tau_i)) = v_i(\lambda(\tau_i)), \quad i = 1,\ldots,q-1.$$

The homotopy class of λ in $\pi_1(B,t_0)$ acting on $\alpha \in H^n(F_{t_0};\mathbb{C})$ provides an element

$$v_{q-1}(\lambda(1)) \in H^n(F_{t_0};\mathbb{C}).$$

One has:

$$\begin{array}{ccc} \pi_1(B,t_0) \times H^n(F_{t_0};\mathbb{C}) & \mapsto & H^n(F_{t_0};\mathbb{C}) \\ \lambda \quad , \quad \alpha & \rightsquigarrow & v_{q-1}(\lambda(1)) \end{array}$$

and the result is independent of the performed choices.

This action of the fundamental group on the cohomology of the fibre is called *monodromy* of the fibre bundle $\pi : E \to B$. We can also define it as the holonomy of the bundle $H^n(\pi)$.

2.3. **Application to the Brieskorn-Milnor bundle.** With the notations of section 1, let us denote $\pi = f|_{X'} : X' \to D'$. Then π is the projection of a locally trivial bundle to which the construction of section 2 applies.

One obtains a complex vector bundle $H^n(\pi)$ of rank μ. That is a complex analytic bundle on a non-compact Riemann surface, then, following Grauert [9], an analytically trivial fibre bundle. That implies that $H^n(\pi)$ admits a system of μ holomorphic sections s_j over $D' = D \setminus \{0\}$, linearly independent at each point. In general, they are not horizontal sections. In fact, one can choose them horizontal when the monodromy is identity, and according to A'Campo [1], that implies that the singularity is quadratic and n is odd.

In the case of the Milnor bundle, the connection ∇ defined on $H^n(\pi)$ is called Gauß-Manin connection. We have seen that it defines an action of $\pi_1(D',t_0)$ on $H^n(\pi)_{t_0}$ and that action coincides with the action of $\pi_1(D',t_0) = \mathbb{Z}$ on $H^n(X_{t_0};\mathbb{C}) = \mathbb{C}^\mu$ determined by the Milnor fibration.

In other words, the local solutions of $\nabla(s) = 0$ give a locally constant sheaf of $\mathbb{C}$-vector spaces of dimension μ and the action of $\pi_1(D')$ on a fibre of this sheaf is the monodromy of the singularity.

In order to compute this monodromy, we need to determine the solutions of $\nabla(s) = 0$. That is the reason for which, in section 4.2, we will have to extend ∇ at 0. But, in a first step, we will show that the horizontal sections of $H^n(\pi)$ can be characterized as solutions of a differential equation the monodromy of which coincide with the monodromy of the singularity.

Let U, open subset in D', and $s_1, \dots, s_\mu$ a basis of holomorphic sections of $H^n(\pi)$. Every holomorphic section s of $H^n(\pi)$ over U, can be written as $s = \sum_{j=1}^{\mu} \phi_j s_j$ where the functions $\phi_j : U \to \mathbb{C}$ are holomorphic.

Let us still denote by ∇ the covariant derivative $\nabla_{\frac{\partial}{\partial t}}$ determined by the connection ∇, relatively to the vector field $\frac{\partial}{\partial t}$ of D'. For every $j = 1, \dots, \mu$, then $\nabla(s_j)$ is written

$$\nabla(s_j) = \sum_{k=1}^{\mu} a_{kj} s_k$$

where the a_{kj} are holomorphic functions defined in D'. Then we have

$$\nabla(s) = \sum_j \phi'_j s_j + \sum_j \phi_j \sum_k a_{kj} s_k = \sum_k (\phi'_k + \sum_j a_{kj}\phi_j) s_k.$$

Let us denote $\Phi = (\phi_1, \dots, \phi_\mu)^t$ (column vector) and denote by A the matrix $((a_{kj}))$. One has:

Lemma 2.3. A holomorphic section $s = \sum_{j=1}^{\mu} \phi_j s_j$ of $H^n(\pi)$ over U is a horizontal section if and only if the differential equation

$$\Phi' + A\Phi = 0 \tag{2.4}$$

is satisfied.

The monodromy of the singularity can then be interpreted in the following way:

For initial values given at $t_0 \in D'$, one can define locally solutions of (2.4) which generate the μ-dimensional vector space of solutions of (2.4) over a neighbourhood of t_0. In the same way as before, for every loop $\lambda : [0,1] \to D'$ at t_0, one considers a subdivision $0 = \tau_0 < \tau_1 < \cdots < \tau_q = 1$ of $[0,1]$ sufficiently fine so that, for all $0 \le i \le q-1$, then $\lambda([\tau_i, \tau_{i+1}])$ is contained in an open subset of B, trivialization of $H^n(\pi)$. Then, one can follow, by analytic extension, the μ solutions of (2.4), which are given at t_0, along the loop λ. One obtains in every point of λ a system of μ linearly independent solutions of (2.4). The matrix giving the "new" sections, obtained in that way at the point t_0, in terms of the "old" ones is a monodromy matrix of the singularity.

The monodromy of the solutions of the differential equation (2.4) is then equivalent to the monodromy of the singularity.

Computing the monodromy of the singularity is then equivalent to solving the differential equation (2.4). In order to do that, we need to:

(i) construct a basis of holomorphic sections of $H^n(\pi)$,

(ii) compute the matrix A, given the function f.

That is the aim of the following section.

3. Construction of analytic sections of $H^n(\pi)$

Let us denote by ω a differential form of degree n over X. The restriction of ω to each fibre X_t, for $t \neq 0$, denoted by $\omega|_{X_t}$, has maximum degree and is a closed differential form. We show now that the section $s_\omega : D' \to H^n(\pi)$ defined by

$$s_\omega(t) = [\omega|_{X_t}] \in H^n(X_t; \mathbb{C})$$

is a holomorphic section of $H^n(\pi)$ and we compute $\nabla(s_\omega)$.

The main part of this section comes from [33] and [34].

3.1. **Leray coboundary.** Let X be a complex analytic manifold with (complex) dimension $n+1$ and W a complex analytic submanifold of X with (complex) codimension 1. The long exact sequence in cohomology with compact supports and with coefficients in $\mathbb{C}$ is written:

$$\cdots \longrightarrow H_c^p(X \setminus W) \xrightarrow{i^*} H_c^p(X) \longrightarrow H_c^p(W) \xrightarrow{\delta} H_c^{p+1}(X \setminus W) \longrightarrow \cdots$$

where i^* is induced by the inclusion $X \setminus W \subset X$ and δ is the classical coboundary operator.

By Poincaré duality, applied to $X \setminus W, X$ and W, one obtains the exact sequence:

$$\cdots \longrightarrow H_{q+1}(X \setminus W) \xrightarrow{i_*} H_{q+1}(X) \longrightarrow H_{q-1}(W) \xrightarrow{\partial} H_q(X \setminus W) \longrightarrow \cdots$$

with $p+q+1 = 2n+2$. The map ∂, dual of the coboundary δ, is called *Leray boundary.*

Applying the functor $\mathrm{Hom}(\cdot;\mathbb{C})$, one deduces from the second exact sequence, the following long exact sequence:

$$\cdots \longrightarrow H^q(X \setminus W) \xrightarrow{r} H^{q-1}(W) \longrightarrow H^{q+1}(X) \longrightarrow H^{q+1}(X \setminus W) \longrightarrow \cdots .$$

where $H^{q+1}(X) \longrightarrow H^{q+1}(X \setminus W)$ is induced by the inclusion of $X \setminus W$ into X and where the map r, dual of ∂, is called *Leray coboundary.*

3.2. **Residue - Leray-Norguet Theorem.**

Definition 3.1. Let us consider ω a closed holomorphic form in $X \setminus W$, one says that ω admits a *pole of order less or equal to* 1 on W if, for all $x \in W$ and for all holomorphic function g defined in a neighbourhood U_x of x and vanishing on $U_x \cap W$, then $g\,\omega$ admits a holomorphic extension in U_x.

If ω admits a pole of order less or equal to 1 on W and if U is the domain of a system of local coordinates $z_1, \dots, z_{n+1}$ such that $W \cap U$ is defined by the equation $z_1 = 0$, then the coefficients of ω in this coordinate system are holomorphic functions of $z_2, \dots, z_{n+1}$ and meromorphic with a pole of order ≤ 1 in the coordinate z_1.

As ω is closed, one has: $d(z_1\omega) = dz_1 \wedge \omega$ on $U \setminus W$ and, as $z_1\omega$ is holomorphic on U, then $dz_1 \wedge \omega$ is also holomorphic on U. That implies that ω is of the form

$$\omega = \frac{dz_1}{z_1} \wedge \varphi + \eta \tag{3.2}$$

where φ and η are holomorphic on U.

Lemma 3.3 ([33])**.** The restriction of φ to $U \cap W$, denoted by $\varphi|_{U\cap W}$, depends only on ω.

Then there exists a well determined holomorphic form on W, called *residue* of ω and denoted by $\mathrm{res}_W(\omega)$, characterized by the fact to be locally the restriction of a holomorphic form φ which verifies equation (3.2).

Lemma 3.4 ([33])**.** The form $\mathrm{res}_W(\omega)$ is a closed form on W.

We can now state the Leray-Norguet Theorem:

Theorem 3.5 ([21] and [27])**.** Let ω be a closed holomorphic q-form on $X \setminus W$ with a pole of order ≤ 1 on W, then

$$r\left([\omega]\right) = 2i\pi \left[\mathrm{res}_W(\omega)\right]. \tag{3.6}$$

Corollary 3.7. Under the same hypothesis as in theorem 3.5, one has, for all $(q-1)$-dimensional cycle ξ on W:

$$\int_{\partial(\xi)} \omega = 2i\pi \int_{\xi} \mathrm{res}_W(\omega). \tag{3.8}$$

3.3. **Return to the Brieskorn bundle.** Under the hypothesis of section 1, let us denote by ω a holomorphic form of degree n on X. For all $t \in D'$, the form $\omega|_{X_t}$ is closed. We show now the following theorem:

Theorem 3.9 (Brieskorn)**.** Let s_ω the section of $H^n(\pi)$ defined by $s_\omega(t) = [\omega|_{X_t}]$, one has
(i) s_ω is a holomorphic section of $H^n(\pi)$,
(ii) if $d\omega = df \wedge \varphi$, then $\nabla(s_\omega) = s_\varphi$.

To show the theorem, one proves a preliminary result: Let us consider a holomorphic form α of degree $n+1$ on X. For all $t \in D'$, the form $\alpha/(f-t)$ is a closed holomorphic form on $X \setminus X_t$. According to Lemma 3.3, the form $\mathrm{res}_t\left(\dfrac{\alpha}{f-t}\right) = \mathrm{res}_{X_t}\left(\dfrac{\alpha}{f-t}\right)$ is a closed holomorphic form of degree n on X_t; moreover, the map

$$t \rightsquigarrow \left[\mathrm{res}_t\left(\frac{\alpha}{f-t}\right)\right] \in H^n(X_t : \mathbb{C})$$

defines a section of the bundle $H^n(\pi)$.

Lemma 3.10. a) The map $t \rightsquigarrow \left[\mathrm{res}_t\left(\frac{\alpha}{f-t}\right)\right]$ defines a holomorphic section of $H^n(\pi)$.
b) One has

$$\nabla\left(\left[\mathrm{res}_t\left(\frac{\alpha}{f-t}\right)\right]\right) = \frac{1}{2i\pi}\, r_t \left[\frac{\alpha}{(f-t)^2}\right]$$

where $r_t : H^{n+1}(X \setminus X_t) \longrightarrow H^n(X_t)$ is the Leray coboundary.

Proof. Let t_0 be a point in D' and U a neighbourhood of t_0 in D' which is a trivialization domain of the bundle $\pi : X' \to D'$. For every homology class $\xi_{t_0} \in H_n(X_{t_0})$, there exists a class $\xi \in H_n(\pi^{-1}(U))$ whose restriction to X_{t_0} is ξ_{t_0}. We denote by $\xi_t \in H_n(X_t)$ the restriction of ξ to X_t, for $t \in U$.

In order to prove a) of the Lemma, we show that the map

$$t \rightsquigarrow \langle \mathrm{res}_t\left(\frac{\alpha}{f-t}\right), \xi_t\rangle$$

is holomorphic. According to (3.6), one has:

$$\left\langle \mathrm{res}_t\left(\frac{\alpha}{f-t}\right), \xi_t\right\rangle = \frac{1}{2i\pi}\left\langle r_t\left[\frac{\alpha}{f-t}\right], \xi_t\right\rangle = \frac{1}{2i\pi}\left\langle \frac{\alpha}{f-t}, \partial_t\xi_t\right\rangle$$

because r_t is the dual of the Leray boundary $\partial_t : H_n(X_t) \to H_{n+1}(X \setminus X_t)$.

Let

$$j_t : H_{n+1}(X \setminus \pi^{-1}(U)) \to H_{n+1}(X \setminus X_t)$$

be the morphism induced by the inclusion $X_t \subset \pi^{-1}(U)$; there exists a class

$$z \in H_{n+1}(X \setminus \pi^{-1}(U))$$

such that for all t, one has $j_t(z) = \partial_t(\xi_t)$. In fact, let us assume that U is a closed disk, centered at t_0, then one has a commutative diagram.

$$\begin{tikzcd}
H_c^n(\pi^{-1}(U)) \arrow[rr, "\delta"] \arrow[dd, "i_t"'] \arrow[dr, "P'"', "\sim"] & & H_c^{n+1}(X \setminus \pi^{-1}(U)) \arrow[r, "P", "\sim"'] \arrow[dd] & H_{n+1}(X \setminus \pi^{-1}(U)) \arrow[dd, "j_t"] \\
& H_n(\pi^{-1}(U)) \arrow[dd] & & \\
H_c^n(X_t) \arrow[rr, "\delta_t"] \arrow[dr, "P'_t"', "\sim"] & & H_c^{n+1}(X \setminus X_t) \arrow[r, "P_t", "\sim"'] & H_{n+1}(X \setminus X_t) \\
& H_n(X_t) \arrow[urr, "\partial_t"'] & &
\end{tikzcd}$$

in which the vertical arrows are induced by the inclusion $X_t \subset \pi^{-1}(U)$ and the morphisms P, P', P_t and P'_t are Poincaré duality isomorphisms. Let us denote $\zeta_t = (P'_t)^{-1}(\xi_t) \in H_c^n(X_t)$ and $\zeta = (P')^{-1}(\xi) \in H_c^n(\pi^{-1}(U))$, then one has $i_t(\zeta) = \zeta|_{X_t} = \zeta_t$. The class $z = P\delta(\zeta)$ satisfies, for all $t \in U$, the equality $j_t(z) = \partial_t(\xi_t)$. One has:

$$\left\langle \operatorname{res}_t \left(\frac{\alpha}{f-t} \right), \xi_t \right\rangle = \frac{1}{2i\pi} \int_z \frac{\alpha}{f-t}, \tag{3.11}$$

that is a holomorphic function in t. In fact, the cycle z on which we take integration is fixed (*i.e.* independent of t) and situated in $X \setminus \pi^{-1}(U)$, out of the singularities of $\dfrac{\alpha}{f-t}$. That proves a).

In order to show b), firstly we observe that if s denotes a holomorphic section of $H^n(\pi)$, then one has

$$\langle \nabla(s)(t), \xi_t \rangle = \frac{d}{dt} \langle s(t), \xi_t \rangle. \tag{3.12}$$

In fact, in U, the section s can be written as $s(t) = \sum \varphi_i(t) s_i(t)$ where the sections s_i are a basis of horizontal sections of $H^n(\pi)$. As the classes ξ_t are restriction of the same class ξ in $H_n(\pi{-}1(U))$, then $\langle s_i(t), \xi_t \rangle$ is constant.

That implies:

$$\langle \nabla(s)(t), \xi_t \rangle = \sum \varphi'_i(t) \left\langle s_i(t), \xi_t \right\rangle = \frac{d}{dt} \left\langle s(t), \xi_t \right\rangle.$$

Using the computations performed in the proof of a), one obtains, for the section s of $H^n(\pi)$ defined by $s(t) = \left[\operatorname{res}_t \left(\frac{\alpha}{f-t} \right) \right]$:

$$\left\langle \nabla \left[\operatorname{res}_t \left(\frac{\alpha}{f-t} \right) \right], \xi_t \right\rangle = \frac{d}{dt} \left\langle \operatorname{res}_t \left(\frac{\alpha}{f-t} \right), \xi_t \right\rangle = \frac{1}{2i\pi} \frac{d}{dt} \int_z \frac{\alpha}{f-t} =$$
$$\frac{1}{2i\pi} \int_z \frac{\alpha}{(f-t)^2} = \frac{1}{2i\pi} \left\langle r_t \left[\frac{\alpha}{(f-t)^2} \right], \xi_t \right\rangle.$$

That proves b) of the Lemma. $\square$

Proof of Theorem 3.9. Let ω be a holomorphic form of degree n on X, the lemma 3.10 can be applied to the form $\alpha = df \wedge \omega$. In particular, the section

$$t \rightsquigarrow \left[\operatorname{res}_t \frac{df \wedge \omega}{f-t} \right]$$

of $H^n(\pi)$ is holomorphic. But, by definition of the residue (formula (3.2)), one has:

$$\left[\mathrm{res}_t \frac{df \wedge \omega}{f-t}\right] = \left[\mathrm{res}_t \frac{d(f-t)\wedge\omega}{f-t}\right] = [\omega|_{X_t}] = s_\omega(t). \tag{3.13}$$

That proves (i) of the theorem.

According to (3.13) and (b) of lemma 3.10, one has:

$$\nabla(s_\omega)(t) = \frac{1}{2i\pi}\, r_t \left[\frac{df\wedge\omega}{(f-t)^2}\right] = \frac{1}{2i\pi}\, r_t \left[\frac{d\omega}{f-t} - d\left(\frac{\omega}{f-t}\right)\right]$$

and, as the class of $d\left(\dfrac{\omega}{f-t}\right)$ is zero, one has:

$$\nabla(s_\omega)(t) = \frac{1}{2i\pi}\, r_t \left[\frac{d\omega}{f-t)}\right] = \frac{1}{2i\pi}\, r_t \left[\frac{df\wedge\varphi}{f-t}\right] = [\varphi|_{X_t}] = s_\varphi(t),$$

that is (ii) of the theorem. □

4. Brieskorn's results and the Gauss-Manin connection

In this section, one constructs a complex of sheaves, whose cohomology sheaf, restricted to D', is isomorphic to the sheaf of germs of holomorphic sections of $H^n(\pi)$. That allows us to extend the connection ∇ into a differential operator which is singular at the origin.

4.1. **Relative de Rham complex.** Given a manifold Y, we denote by Ω^*_Y the complex of sheaves of germs of holomorphic forms on Y. We know that, if Y is a Stein manifold, then $H^*(Y;\mathbb{C})$ is the cohomology of Ω^*_Y. That applies in particular for all points t in D to the fibre $X_t = f^{-1}(t)\cap B_\varepsilon$ of $f|_X : X \to D$.

To study the monodromy, that is the action of the parallel transport along a loop in D' on a fibre, we construct a complex of differential forms which, when restricted to a fibre X_t, is $\Omega^*_{X_t}$. That will be the relative de Rham complex of $f|_X : X \to D$, denoted by $\Omega^*_{X/D}$ and defined by:

$$\Omega^p_{X/D} = \Omega^p_X/(df \wedge \Omega^{p-1}_X).$$

We verify that $\Omega^*_{X/D}$ is a complex, because one has:

$$d(df\wedge\omega) = -df\wedge d\omega \in df \wedge \Omega^*_X.$$

We want to study the germs, in D, of differential forms defined along the fibres of the function $f|_X : X \to D$. In other words, we want to consider, for every open subset U in D, the sections of the sheaf $\Omega^p_{X/D}$ over $f^{-1}(U)$. They are, by definition, the sections of the sheaf $f_*\Omega^p_{X/D}$ over U.

Now, it is natural to define the relative de Rham cohomology sheaves of $f|_X : X \to D$ by:

$$\mathcal{H}^p(X/D) = H^p(f_*\Omega^*_{X/D}).$$

Theorem 4.1. [Brieskorn [5, Satz 1.5]] The sheaf $\mathcal{H}^n(X/D)$ is an analytic coherent sheaf on D.

We denote by $\mathcal{H}^n$ the sheaf of germs of holomorphic sections of $H^n(\pi)$ and by $\mathcal{O}_{D'}$ the structural sheaf of D', *i.e.* the sheaf of germs of holomorphic sections on D'. Brieskorn shows the following result:

Theorem 4.2. [Brieskorn [5]] The correspondence $\omega \rightsquigarrow s_\omega$ induces an isomorphism of $\mathcal{O}_{D'}$-modules:

$$\Psi : \mathcal{H}^n(X/D)|_{D'} \to \mathcal{H}^n. \tag{4.3}$$

Here, we will verify only that Ψ is well defined. In fact, an element ω of $\mathcal{H}^n(X/D)|_{D'}$ can be represented by a section of $f_*(\Omega^n_{X/D})$ on an open subset U in D', or, that is equivalent to say, a section of $\Omega^n_{X/D}$ on $f^{-1}(U)$.

For every disk U in D', the inverse image $f^{-1}(U)$ is a Stein manifold. As the sheaves Ω^n_X and $\Omega^n_{X/D}$ are coherent (see [5]), the obtained section can be lifted into a section of Ω^n_X on $f^{-1}(U)$, that is a holomorphic differential form of degree n on $f^{-1}(U)$. We still denote it by ω.

By theorem 3.9, one obtains a holomorphic section s_ω of $H^n(\pi)$ whose germ at the point t is $s_\omega(t)$. That defines Ψ.

4.2. **Gauß-Manin connection.** The previous construction provides an extension of the sheaf $\mathcal{H}^n$ into a sheaf $\mathcal{H}^n(X/D)$ which is defined over all of D. The isomorphism of theorem 4.2 allows us to identify the homomorphism $\nabla : \mathcal{H}^n \to \mathcal{H}^n$, defined at section 2, with a $\mathbb{C}$-linear homomorphism:

$$\nabla : \mathcal{H}^n(X/D)|_{D'} \to \mathcal{H}^n(X/D)|_{D'}.$$

According to section 3, the local solutions of $\nabla(s) = 0$ give a locally constant sheaf of $\mathbb{C}$-vector spaces of dimension μ and the action of $\pi_1(D', t_0)$ on the fibre at t_0 of this sheaf is the monodromy of the singularity.

To compute the monodromy, we will extend ∇ into a singular differential operator ∇_f defined on $\mathcal{H}^n(X/D)_0$ and will prove that its monodromy is equivalent to the one of the singularity.

We will admit the following theorem which provides an interpretation of $\mathcal{H}^n(X/D)_0$:

Theorem 4.4 (Brieskorn [5])**.** Let $\Omega^*_{X/D,0}$ considered as a complex of $\mathcal{O}_{D,0}$-modules. One has a canonical isomorphism:

$$\mathcal{H}^n(X/D)_0 \to H^n(\Omega^*_{X/D,0}) \tag{4.5}$$

induced by the restriction $\Omega^*_{X/D} \to \Omega^*_{X/D,0}$.

More precisely, let U be a neighbourhood of 0 in D and ω be a holomorphic form of degree n on $f^{-1}(U)$. That one represents a cycle of $\Gamma(f^{-1}(U), \Omega^n_{X/D})$ that gives a section of $\mathcal{H}^n(X/D)$ over U. The isomorphism of the theorem sends the value of this section at 0 to the class, in $H^n(\Omega^*_{X/D,0})$, of the cycle represented by ω.

The differential operator ∇_f will be defined on

$$E = H^n(\Omega^*_{X/D,0}) = \frac{\{\omega \in \Omega^n_{X,0} : \exists \eta \in \Omega^n_{X,0},\ d\omega = df \wedge \eta\}}{df \wedge \Omega^{n-1}_{X,0} + d\Omega^{n-1}_{X,0}}. \tag{4.6}$$

As ∇_f is a singular operator, it will take values, not in E, but in a $\mathcal{O}_{D,0}$-module F containing E as sub-$\mathcal{O}_{D,0}$-module. That will be

$$F = \Omega^n_{X/D,0}/d\Omega^{n-1}_{X/D,0} = \Omega^n_{X,0}/df \wedge \Omega^{n-1}_{X,0} + d\Omega^{n-1}_{X,0}. \tag{4.7}$$

We can now define ∇_f:

An element $\overline{\omega}$ in E is represented by a holomorphic form ω of degree n defined in a neighbourhood of 0 in X and such that $d\omega = df \wedge \varphi$ where φ is holomorphic in a neighbourhood of 0. We define $\nabla_f : E \to F$ by:

$$\nabla_f(\overline{\omega}) = \overline{\varphi} \tag{4.8}$$

where $\overline{\varphi}$ is the class of φ in F.

One verifies easily that ∇_f is a *differential operator with polar singularity* in the following sense:

i) ∇_f is $\mathbb{C}$-linear,

ii) $\nabla_f(\overline{g(t)\,\omega}) = g'(t)\,\overline{\omega} + g(t)\,\nabla_f(\overline{\omega})$,

iii) there exists a positive integer k such that $t^k\,\nabla_f(E) \subset E$.

In order to verify (iii), let us recall that, for k large enough, f^k belongs to the ideal generated by $(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n})$ in the local ring of $\mathbb{C}^{n+1}$ at origin. Then for every $(n+1)$-holomorphic form α, there is η such that $f^k\alpha = df \wedge \eta$. For all elements $\overline{\varphi}$ in F, represented by a holomorphic n-form φ, one has:

$$\begin{aligned} d(f^k\varphi) &= f^k d\varphi + kf^{k-1}df \wedge \varphi \\ &= df \wedge \eta' \end{aligned}$$

that shows that $t^k\,\overline{\varphi} \in E$, then (iii).

We observe that this shows more, namely:

Lemma 4.9. *F/E is torsion.*

The result of Sebastiani [36] is the following:

Theorem 4.10. *$\mathcal{H}^n(X/D)_0$ is a free $\mathcal{O}_{D,0}$-module.*

We know (theorem 4.1) that $\mathcal{H}^n(X/D)$ is coherent, that implies that $\mathcal{H}^n(X/D)$ is locally free of rank μ at the point 0. Then we can show that the monodromy of ∇_f is equivalent to the monodromy of the singularity of f at the origin. More precisely:

Theorem 4.11. *Let $\overline{\omega}$ an element in E represented by a holomorphic form ω of degree n on X and such that $\nabla_f(\overline{\omega}) = \overline{\varphi}$, where $\overline{\varphi}$ is the class in F of a holomorphic form φ on X, then $\nabla(s_\omega) = s_\varphi$.*

According to the previous observation, if U denotes an open disk centered at 0, one can find holomorphic forms $\omega_1, \ldots, \omega_\mu$ defined on $f^{-1}(U)$, such that $d\omega_j = df \wedge \varphi_j$ with φ_j holomorphic in $f^{-1}(U)$ and such that the sections $\widetilde{\omega}_1, \ldots, \widetilde{\omega}_\mu$ of $\mathcal{H}^n(X/D)|_U$, induced by $\omega_1, \ldots, \omega_\mu$ generate the sheaf.

Each ω_j represents an element $\overline{\omega}_j$ in E and one has:

$$\nabla_f(\overline{\omega}_j) = \overline{\varphi}_j. \tag{4.12}$$

As F/E is torsion, $\overline{\varphi}_j$ can be written:

$$\overline{\varphi}_j = \sum_{k=1}^{\mu} a_{kj}\,\overline{\omega}_k \tag{4.13}$$

where the a_{kj} are germs of meromorphic functions at the origin in D. If U is small enough, one can assume that the a_{kj} are holomorphic in D'. In the same way as above, let us denote by A the matrix of a_{kj}. The system of differential equations associated to ∇_f in the basis $\overline{\omega}_1, \ldots, \overline{\omega}_\mu$ of E and determined by (4.12) is written:

$$\Phi' + A\Phi = 0. \tag{4.14}$$

Let V be an open subset in D' contained in U. The system (4.14) is the same as the one associated to ∇ in the basis $s_{\omega_1}, \ldots, s_{\omega_\mu}$ of $\mathcal{H}^n|_V$. In fact, according to theorem 3.9, one has:

$$\nabla(s_{\omega_j}) = s_{\varphi_j} \tag{4.15}$$

and, according to (4.13):

$$s_{\varphi_j} = \sum_{k=1}^{\mu} a_{kj}\,s_{\omega_k} \tag{4.16}$$

If $\Phi = (g_1, \dots, g_\mu)$ is a solution of (4.14) on V, let us denote $s = \sum_{j=1}^{\mu} g_j \, s_{\omega_j}$; then one has $\nabla(s) = 0$ and s is a horizontal section of $H^n(\pi)$ over V (see lemma 2.3).

One deduces that the monodromy of ∇_f is the same as the one of ∇ and, according to what we have seen above, the monodromy of solutions of (4.14) coincides with the monodromy of the singularity.

Let us denote by K the field of fractions of $\mathcal{O}_{D,0}$, *i.e.* the field of germs of meromorphic functions on D at 0. As F/E is torsion, then ∇_f can be extended into a connection, still denoted by ∇, on the K-vector space:

$$\mathcal{E} = E \otimes_{\mathcal{O}} K = F \otimes_{\mathcal{O}} K.$$

In the following section, we show that the connection ∇ is regular.

5. Regularity of the Gauss-Manin connection

5.1. **Recall of the theory of differential equations.** Let us denote, as before, K the field of germs of meromorphic functions on D and ∇ a connection on a K-vector space $\mathcal{E}$. Let us denote by $(e_1, \dots, e_\mu)$ a basis for $\mathcal{E}$, one defines the $a_{kj} \in K$ by

$$\nabla(e_j) = \sum_{k=1}^{\mu} a_{kj} \, e_k.$$

A computation (already made, see lemma 2.3), shows that the horizontal sections for the connection ∇ are characterized by a differential system. More precisely, if $\Phi = (g_1, \dots, g_\mu)^t$ are the components of $s \in \mathcal{E}$ in the basis $(e_1, \dots, e_\mu)$ and if $A = ((a_{kj}))$ is the matrix of the a_{kj}, one obtains the differential system:

$$\Phi' + A\Phi = 0 \tag{5.1}$$

whose solutions are the horizontal sections of ∇.

Definition 5.2. One calls *fundamental matrix* $Y(t)$ of (5.1), every matrix $\mu \times \mu$ whose columns are solutions of (5.1) and such that $\det Y(t) \neq 0$.

One knows, by the general theory ([6, p.111], [13, p.70]), that every linear system of differential equations of the type $\Phi' + A(t)\Phi = 0$ where $A(t)$ is a matrix of analytic functions over $0 < |t| < a$, admits fundamental matrices of the form

$$Y(t) = Z(t)t^R \tag{5.3}$$

where $Z(t)$ is a matrix of analytic functions for $0 < |t| < a$ and R a constant matrix.

Then, one can provide the theorem of the classical theory:

Theorem 5.4. The following conditions are equivalent:
(a) By a change of variables of the type $Y = MZ$, where M is an invertible matrix with meromorphic coefficients, equation (5.1) can be transformed into an equation in which the matrix A admits at most a simple pole at the origin.
(b) There exists a fundamental matrix of (5.1) in which $Z(t)$ admits at most a pole at the origin.
(c) In every angular sector $0 \leq \arg t \leq \beta$ of the universal covering of D', the horizontal sections of ∇ have low growing, that means that in one (or all) basis of $\mathcal{E}$, the components g_j verify an estimation of the type $|g_j(t)| \leq C_{\alpha,\beta} \, t^{-N}$.

Definition 5.5. One says that the connection ∇ is *regular* (or *with regular singular points*) if one of the previous conditions is satisfied.

5.2. **Regularity of the Gauß-Manin connection.** Brieskorn proved in [5] the regularity of the Gauß-Manin connection of an isolated hypersurface singularity, using results of Griffiths. The general theorem can be proved by analytic methods (Nilsson [26], Griffiths [11], Malgrange [24]), or arithmetic ones (Katz [17]), or algebraic ones (Deligne [7]). We will adopt the proof by Malgrange [24].

Theorem 5.6. The Gauß-Manin connection is regular.

Let $p : S \to D'$ the universal covering of D'. Let us consider a family of cycles

$$\gamma(u) \in H_n(X_{p(u)}; \mathbb{C})$$

depending continuously on $u \in S$, *i.e.* if u' is near u, then $\gamma(u')$ is image of $\gamma(u)$ by the canonical isomorphism:

$$H_n(X_{p(u')}; \mathbb{C}) \simeq H_n(X_{p(u)}; \mathbb{C}).$$

By abuse of notation, we will denote $\gamma(t)$ instead of $\gamma(u)$, when $p(u) = t$, providing if necessary the argument of t.

Considering, for $\omega \in \Gamma(X; \Omega^n_X)$, the function on S (multiform function on D') defined by $I(t) = \int_{\gamma(t)} \omega$. In a first step, we show that the integrals $I(t)$ verify a regular differential system if and only if (5.1) is regular, then we will show that these integrals verify (c) of the Theorem 5.4.

It results from Theorem 3.9 and from (3.12) that I is holomorphic and one has:

$$\frac{d}{dt}\int_{\gamma(t)} \omega = \int_{\gamma(t)} \nabla(\overline{\omega}).$$

Taking D smaller if necessary, one can find $\omega_1, \ldots, \omega_\mu$ in $\Gamma(X; \Omega^n_X)$ such that $\overline{\omega}_1, \ldots, \overline{\omega}_\mu$ is a basis of $\mathcal{E} = F \otimes_{\mathcal{O}} K$. In this basis, the matrix of the connection is the matrix $A = ((a_{kj}))$ such that:

$$\nabla(\overline{\omega}_j) = \sum_{k=1}^{\mu} a_{kj}\overline{\omega}_k$$

(see 4.12 and 4.13). The equation associated to the Gauß-Manin connection is the equation (4.14): $\Phi' + A\Phi = 0$.

Let us denote

$$I_j(t) = \int_{\gamma(t)} \overline{\omega}_j, \tag{5.7}$$

one has:

$$\frac{dI_j}{dt} = \int_{\gamma(t)} \nabla(\overline{\omega}_j) = \sum_{k=1}^{\mu} a_{kj} \int_{\gamma(t)} \overline{\omega}_k = \sum_{k=1}^{\mu} a_{kj} I_k.$$

In another words, $I = I_1, \ldots, I_\mu$ is solution of the system

$$I' - A^t I = 0 \tag{5.8}$$

dual of (5.1).

Lemma 5.9. The system (5.1) is regular if and only if (5.8) is regular.

Proof. Let Y a fundamental matrix for (5.1); derivating the equality $Y \cdot Y^{-1} = id$ and replacing $Y'(t)$ by $-A(t)Y(t)$, we show that $(Y^{-1})^t$ is a fundamental matrix for (5.8), in other words one has $((Y^{-1})^t)' - A^t(Y^{-1})^t = 0$. That proves the lemma. ☐

Now to prove regularity of the Gauß-Manin connection, it suffices to prove the following result: "When $t \to 0$, with $\alpha \leq \arg t \leq \beta$, the $I_j(t)$ have slow growing."

In fact, Malgrange proves a more precise result based on the following Lemma:

Lemma 5.10. Let $\omega \in \Gamma(X; \Omega^n_X)$, one has:

$$\lim_{t \to 0,\, \arg t = 0} \int_{\gamma(t)} \omega = 0.$$

Proof. Let us choose a strictly positive real number t_0 and denote $T = f^{-1}([0, t_0]) \cap X$. Then T is a semi-analytic set and is contractible (because T can be contracted in a neighbourhood of X_0 and X_0 is contractible). Following Łojaciewicz [22], one can find a semi-analytic triangulation K of T such that X_0 and X_{t_0} are sub-complexes and such that 0 is a vertex.

Let Γ a cycle in X_{t_0} representing $\gamma(t_0)$; as T is contractible, there is a chain Δ in K such that $\partial \Delta = \Gamma$.

Let us recall the result by Herrera [14]: for every chain with integer coefficients $\Lambda = \sum a_j \sigma_j$ where the σ_j are oriented simplices in K, we define:

$$\int_\Lambda \omega = \sum a_j \int_{\sigma_j} \omega \tag{5.11}$$

where $\int_{\sigma_j} \omega = 0$ if $\deg \omega \neq \dim \sigma_j$ and, if $\deg \omega = \dim \sigma_j$, then $\int_{\sigma_j} \omega = \int_{\mathring{\sigma}_j} \omega = \lim_C \int_C \omega$, where C describes the family of compact subsets situated in the interior $\mathring{\sigma}_j$ of σ_j. Following Herrera [14], the integral (5.11) converges and one has

$$\int_{\partial \Lambda} \omega = \int_\Lambda d\omega.$$

Then, the integral $I(t_0)$ is written:

$$I(t_0) = \int_\Gamma \omega = \int_\Delta d\omega.$$

Let us fix $t \in]0, t_0]$ and consider a subdivision $\widetilde{K}$ of K such that X_t and $f^{-1}(]0, t])$ are sub-complexes of $\widetilde{K}$. Denoting by τ_j the oriented simplices of $\widetilde{K}$, one can consider Δ as a chain $\widetilde{\Delta} = \sum n_j \tau_j$ in $\widetilde{K}$. One can write:

$$\widetilde{\Delta} = \Delta'_t + \Delta''_t$$

where $\Delta'_t = \sum m_j \tau_j$ with $m_j = n_j$ if $\tau_j \subset f^{-1}([0, t])$ and $m_j = 0$ otherwise, and where Δ''_t is a chain in $\widetilde{K}$ whose support is contained in $f^{-1}([t, t_0])$. Moreover, one has:

$$\partial \Delta''_t = \partial \widetilde{\Delta} - \partial \Delta'_t.$$

On the one hand, the cycle $\partial \widetilde{\Delta}$ represents $\gamma(t_0)$ in X_{t_0} (in fact we have $\widetilde{\partial \Delta} = \partial \widetilde{\Delta}$). On the other hand the support of $\partial \Delta'_t$ is a cycle of X_t homologous, in $f^{-1}([t, t_0])$, to $\gamma(t_0)$. Then $\partial \Delta'_t$ represents $\gamma(t)$ in X_t. One has:

$$I(t) = \int_{\partial \Delta'_t} \omega = \int_{\Delta'_t} d\omega.$$

The chain Δ is written $\Delta = \sum a_j \sigma_j$ in the triangulation K. We show now the formula:

$$\int_{\Delta'_t} d\omega = \sum a_j \int_{\sigma_j \cap f^{-1}([0,t])} d\omega \tag{5.12}$$

which makes sense, according to Herrera [14], because $\sigma_j \cap f^{-1}([0,t])$ is a semi-analytic set. To prove the lemma, it suffices to show that

$$\lim_{t\to 0}\int_{\sigma_j\cap f^{-1}([0,t])} d\omega = 0. \tag{5.13}$$

If σ_j is in X_0, that is trivial. In fact, as 0 is a vertex in K, then $\mathring{\sigma}_j \subset X_0 \setminus \{0\}$ and $d\omega|_{\mathring{\sigma}_j} = 0$.

If σ_j is not in X_0, then $\mathring{\sigma}_j \cap X_0$ is the empty set. For every compact subset C in $\mathring{\sigma}_j$, one can find a sufficiently small t so that $\mathring{\sigma}_j \cap f^{-1}([0,t]) \subset \mathring{\sigma}_j \setminus C$. One has:

$$\int_{\sigma_j} d\omega = \int_{\mathring{\sigma}_j} d\omega = \int_{\mathring{\sigma}_j\cap f^{-1}([0,t])} d\omega + \int_{\mathring{\sigma}_j\setminus f^{-1}([0,t])} d\omega$$

where the second member of the sum tends to $\int_{\mathring{\sigma}_j} d\omega$ when t tends to 0. That shows (5.13).

Let us prove (5.12). We write all simplices σ_j in K as a sum $\sum \tau_{jk}$ of simplices in $\widetilde{K}$. More precisely, σ_j can be written as

$$\widetilde{\sigma}_j = \sum_{k\in I}\tau_{jk} + \sum_{k\in J}\tau_{jk}$$

where, for $k \in I$, one has $\tau_{jk} \subset \sigma_j \cap f^{-1}([0,t])$ and, for $k \in J$, then τ_{jk} is not contained in $\sigma_j \cap f^{-1}([0,t])$. With the previous notations, $\widetilde{\Delta}$ can be written as:

$$\widetilde{\Delta} = \Delta'_t + \Delta''_t$$

where $\Delta'_t = \sum_j a_j \sum_{k\in I}\tau_{jk}$. One has:

$$\int_{\Delta'_t} d\omega = \sum_j a_j \sum_{k\in I}\int_{\tau_{jk}} d\omega = \sum_j a_j \int_{\sigma_j\cap f^{-1}([0,t])} d\omega.$$

That ends the proof of the lemma. □

Proof of Theorem 5.6. In order to prove the theorem 5.6, it suffices now to prove the following: "With the hypothesis of Lemma 5.10, for $\alpha \le \arg t \le \beta$, then $I(t)$ remains bounded when t tends to 0."

It is sufficient to prove the result for the integrals of type $I_j(t)$ because $I(t)$ is linear combination of $I_j(t)$ with coefficients in $\mathcal{O}_{D,0}$.

From the equation $\frac{dI_j}{dt} = \sum_k a_{kj} I_k$, one deduces that there exists a constant C and an integer $k > 0$ such that:

$$\left|\frac{dI_j}{dt}\right| \le \frac{C}{k+1}\frac{1}{|t|^{k+1}} \sup(|I_1|, \ldots, |I_\mu|).$$

Passing to polar coordinates (in (r,θ)) and integrating in r, one deduces that, when $\arg t$ is bounded one has:

$$|I_j(t)| \le C' \, e^{C|t|^{-k}}.$$

From Lemma 5.10 and from the Phragmen-Lindelöf Theorem [6, p. 162] one obtains the result for $|\beta - \alpha| < \frac{\pi}{k}$. The general case α and β can be deduced immediately. □

5.3. **Development of the integral $I(t)$.** Firstly let us recall the classical results of monodromy theory [18].

Let $t_0 \in D'$, with, for instance $\arg t_0 = 0$. Let us denote by h the endomorphism of $H_n(X_{t_0};\mathbb{C})$ induced by action of the generator of $\pi_1(D', t_0)$ represented by the loop $\lambda \rightsquigarrow e^{2i\pi\lambda}t_0$ with $\lambda \in [0,1]$.

Theorem 5.14. (a) The eigenvalues of h are roots of unity.
(b) If $h = S\cdot U$ with S semi-simple and U unipotent, and $[S,U]=0$, then one has $(U-I)^{n+1} = 0$.

That implies that, in the Jordan decomposition of the matrix of h, the submatrices corresponding to the eigenvalues of h have at most rank $n+1$.

Let us choose $\gamma_1, \dots, \gamma_\mu$ such that the set $\gamma_1(t_0), \dots, \gamma_\mu(t_0)$ is a basis for $H_n(X_{t_0}; \mathbb{C})$ and such that $\int_{\gamma_k(t_0)} \omega_j = \delta_{jk}$. Let us denote

$$I_{jk}(t) = \int_{\gamma_k(t)} \omega_j.$$

The set $I_{1k}, \dots I_{\mu k}$ is a basis of solutions of the equation

$$\frac{dI_j}{dt} = \sum a_{kj} I_k.$$

From theorem 5.6 and from the classical theory of systems of differential equations with regular singular points [13, p. 73], one obtains that the matrix $I = (I_{jk})$ is of the type:

$$I(t) = J(t) t^C = J(t) \cdot e^{C \log(t)}$$

where $J \in GL(\mu, K)$ and $C \in \mathrm{End}(\mathbb{C}^\mu)$.

The action of h on I is translated by the substitution $\log t \rightsquigarrow \log t + 2i\pi$; then, in the basis $\gamma_j(t_0)$, h is expressed by the multiplication by $\exp(2i\pi C)$. Writing C in Jordan form, we obtain the following result:

Proposition 5.15. Let $\omega \in \Gamma(X; \Omega^n_X)$, and let γ defined as above, one has a converging development in D':

$$\int_{\gamma(t)} \omega = \sum_{\alpha, q} C_{\alpha,q}(\omega)\, t^\alpha\, (\log t)^q \tag{5.16}$$

where $\exp(2i\pi\alpha)$ belongs to the set of eigenvalues of h (so that $\alpha \in \mathbb{Q}$) and

$$\alpha > 0 \qquad \text{and} \qquad 0 \le q \le n+1.$$

Moreover, as J is meromorphic, then the set of α has lower bound [13]. One deduces from the lemma 5.10 that one has:

$$C_{\alpha,q}(\omega) \neq 0 \text{ implies } \alpha > 0.$$

On the other hand, let λ be an eigenvalue for h, then, for a certain $p \ge 1$ and for a suitable choice of $\gamma(t_0)$, one has $(h-\lambda)^p \gamma(t_0) = 0$ and $(h-\lambda)^{p-1}\gamma(t_0) \neq 0$.

Lemma 5.17. There are $\eta \in \Omega^n_X$ and $\alpha > 0$ such that $\exp 2i\pi\alpha = \lambda$ and $C_{\alpha,p-1}(\eta) = 0$.

Proof. If that would not be the case , writing $\widetilde{\gamma}(t_0) = (h-\lambda)^{p-1}\gamma(t_0)$, one would have $\int_{\widetilde{\gamma}(t_0)} \eta = 0$, for all $\eta \in \Omega^n_X$. But, as X_{t_0} and X are Stein manifolds, the differential forms $\eta|_{X_{t_0}}$ generate $H^n(X_{t_0}; \mathbb{C})$. That would imply $\widetilde{\gamma}(t_0) = 0$, that is contradictory with hypothesis. □

6. Relation between monodromy and Bernstein polynomials

6.1. **Bernstein polynomials.** Let s be an indeterminate and consider the set of finite summations

$$\sum_{k,\ell} a_{k,\ell}(x)\, s^k\, (f(x))^{s-k}$$

where $a_{k,\ell}$ are germs of analytic functions at the origin in $\mathbb{C}^{n+1}$. With obvious relations $f(x)f(x)^{s-k-1} = f(x)^{s-k}$ and also obvious composition laws, that is a $\mathcal{O}_{X,0}$-algebra.

Let us now consider the differential operators $P(x, s, \frac{\partial}{\partial x})$ with analytic coefficients in x and polynomials in s:

$$P(x, s, \frac{\partial}{\partial x}) = \sum b_{k\alpha}(x)\, s^k \left(\frac{\partial}{\partial x}\right)^\alpha.$$

These operators act on the previous ring, writing

$$\frac{\partial}{\partial x_i} f^{s-k} = (s-k)\,\frac{\partial f}{\partial x_i} f^{s-k-1}.$$

Giving to s integer values, the previous operations are compatible with the classical operations on meromorphic functions. We can now provide the theorem proved by I.N. Bernstein [2] when f is a polynomial and extended by J.E. Björk [3] when f is a germ of an analytic function with any singularity at the origin:

Theorem 6.1. There exists a polynomial $B(s) \neq 0$ and a differential operator $P(x, s, \frac{\partial}{\partial x})$ such that:

$$P\left(x, s, \frac{\partial}{\partial x}\right) f^s = B(s) f^{s-1}. \tag{6.2}$$

It is clear that the set of polynomials $B(s)$ such that one has a relation of type (6.2) is an ideal. We will denote by $b(s)$ and will call *Bernstein polynomial* of f the generator of this ideal whose highest degree term is equal to 1.

One has $P(x, 0, \frac{\partial}{\partial x}) = b(0) f^{-1}$, that implies $b(0) = 0$. We will denote

$$b(s) = s\widetilde{b}(s).$$

The Malgrange's result is the following:

Theorem 6.3 (Malgrange [23])**.** Let λ be an eigenvalue of h whose multiplicity in the minimal polynomial of h equals p, then there are rational numbers $\nu_1, \ldots, \nu_p \in \mathbb{Q}$ with the following properties:
(a) $\exp(2i\pi\nu_j) = \lambda$ for $j = 1, \ldots, p$,
(b) the polynomial $(s+\nu_1)\cdots(s+\nu_p)$ divides $\widetilde{b}$.

We will restrict ourselves to prove the theorem in the case $\lambda \neq 1$. In fact, Malgrange shows that all roots of the Bernstein polynomial can be obtained in the previous way, thus they are rational numbers. In a more precise way, let $\Phi' + A\Phi$ the equivalent form of (5.1) for which tA is holomorphic at 0; then $b(s) = s\widetilde{b}(s)$ where $\widetilde{b}(s)$ is the minimal polynomial of $(tA)(0)$. Many authors extended and generalized these results, let us quote the work of Kashiwara [16] in relation with $\mathcal{D}$-modules.

Example 6.4. Let us consider the polynomial $f = z_1^2 + \cdots z_{n+1}^2$; choosing $P = \sum \frac{\partial^2}{\partial z_i^2}$, one finds $\widetilde{b}(s) = s + \frac{n-1}{2}$. But $H_n(X_1; \mathbb{C})$ has dimension 1 on $\mathbb{C}$ and we have $h = (-1)^{n-1}$.

6.2. **Periods of integrals.** Let α be an $(n+1)$-holomorphic form on X, there is $\omega \in \Omega^n_X$ such that $d\omega = \alpha$. The differential form $\dfrac{\alpha}{f-t}$ is closed and holomorphic in $X - X_t$ and it admits a pole with order 1 along X_t. We denote

$$\frac{\alpha}{df}(t) = \mathrm{res}_t\left(\frac{\alpha}{f-t}\right)$$

and

$$\int_{\gamma(t)} \frac{\alpha}{df} = \int_{\gamma(t)} \frac{\alpha}{df}(t).$$

This integral does not depend on the homology class of $\gamma(t)$ in $H_n(X_t; \mathbb{C})$, moreover one has:

Lemma 6.5.

$$\int_{\gamma(t)} \frac{\alpha}{df} = \frac{d}{dt}\int_{\gamma(t)} \omega. \tag{6.6}$$

Proof. According to Theorem 3.5, one has:

$$\int_{\gamma(t)} \frac{\alpha}{df} = \frac{1}{2i\pi} \int_{\delta\gamma(t)} \frac{\alpha}{f-t}$$

where δ is the Leray boundary. We have:

$$\int_{\gamma(t)} \frac{\alpha}{df} = \frac{1}{2i\pi} \int_{\delta\gamma(t)} \frac{d\omega}{f-t} = \frac{1}{2i\pi} \int_{\zeta} \frac{d\omega}{f-t}$$

where, for t in a small enough open subset U, ζ is a fixed cycle in $H_{n+1}(X \setminus \pi^{-1}(U))$ (see the comments after (3.11)).

From the relation

$$\frac{df \wedge \omega}{(f-t)^2} = -d\left(\frac{\omega}{f-t}\right) + \frac{d\omega}{f-t},$$

one obtains

$$\int_{\zeta} \frac{d\omega}{f-t} = \int_{\zeta} \frac{df \wedge \omega}{(f-t)^2} = \frac{d}{dt} \int_{\zeta} \frac{df \wedge \omega}{f-t}.$$

Using again results of section 3.3, one can write:

$$\frac{1}{2i\pi} \int_{\zeta} \frac{df \wedge \omega}{f-t} = \frac{1}{2i\pi} \int_{\delta\gamma(t)} \frac{df \wedge \omega}{f-t} = \int_{\gamma(t)} \mathrm{res}_t \left(\frac{df \wedge \omega}{f-t}\right) = \int_{\gamma(t)} \omega | X_t = \int_{\gamma(t)} \omega.$$

That proves the Lemma. $\square$

Given the converging development of $\int_{\gamma(t)} \omega$ (see (5.16)), the integral admits a converging development

$$\int_{\gamma(t)} \frac{\alpha}{df} = \sum_{\beta,q} d_{\beta,q}(\alpha)\, t^{\beta} (\log t)^q \tag{6.7}$$

where $\beta \in \mathbb{Q}_{>-1}$, and $\exp(2i\pi(\beta+1)) = \lambda$ is an eigenvalue of h whose multiplicity in the minimal polynomial is p, and $p-1 \geq q \geq 0$. Moreover, there exists an $(n+1)$-holomorphic form α and a rational number β with $d_{\beta,q}(\alpha) \neq 0$. In fact, according to Lemma 5.17, if $\eta \in \Omega^n_X$ and if $\alpha = df \wedge \eta$, one has:

$$\int_{\gamma(t)} \eta = \int_{\gamma(t)} \frac{\alpha}{df} \tag{6.8}$$

which is not zero.

6.3. **Proof of Theorem 6.3.**

A) Proof in the case $\lambda \neq 1$. Let λ be an eigenvalue for h, with multiplicity p in the minimal polynomial of h. For $1 \leq k \leq p$, one defines ν_k as the infimum of β such that there exists $q \leq k-1$ and $\alpha \in \Omega^{n+1}_X$ such that $d_{\beta,q}(\alpha) \neq 0$ and $\exp(2i\pi(\beta+1)) = \lambda$.

In order to show (b) of Theorem 6.3, in the case $\lambda \neq 1$, it is sufficient to show that the polynomial $(s+\nu_1)\cdots(s+\nu_p)$ divides $b(s)$. We will proceed in three steps:

1) Let us consider a fixed point $\tau \in [0,1]$ such that $\tau < \eta$. We consider a $\mathcal{C}^\infty$ singular cycle in X_τ which represents $\gamma(\tau)$, in other words, $\gamma(\tau) = \sum n_i s_i$ where the s_i are applications $s_i : \Delta_n \to X_\tau$, with Δ_n standard simplex in $\mathbb{R}^{n+1}$.

Considering a trivialization of the bundle $\pi : X' \to D'$, restricted to $]0,\tau]$, one defines applications $\widetilde{s}_i$ such that the following diagram commutes:

$$\begin{tikzcd} \Delta_n \times]0,\tau] \arrow[r, "\widetilde{s}_i"] \arrow[dr, "p_2"'] & X \arrow[d, "\pi"] \\ &]0,\tau] \end{tikzcd}$$

and such that $\widetilde{s}_i|_{\{1\}} = s_i$. Here, p_2 is obviously the second projection.

Let us denote

$$\Gamma(t,\tau) = \sum n_i \widetilde{s}_i|_{\Delta_n \times]0,\tau]}\, f^{s-1}\,\alpha.$$

For every $s \in \mathbb{C}$, one has (choosing t_0 such that $\arg t_0 = 0$):

$$\int_{\Gamma(t_0,\tau)} f^{s-1}\,\alpha = \sum n_i \int_{\widetilde{s}_i|_{\Delta_n \times]0,\tau]}} f^{s-1}\,\alpha$$

We can assume that each $\widetilde{s}_i|_{\Delta_n \times]0,\tau]}$ is contained in an open subset U_i in which $\alpha|_{U_i} = df \wedge \eta_i$. In that case,

$$\int_{\widetilde{s}_i|_{\Delta_n \times]0,\tau]}} f^{s-1}\,\alpha = \int_{t_0}^{\tau} t^{s-1} dt \int_{\widetilde{s}_i|_{\Delta_n \times \{t\}}} \eta_i.$$

In fact, one knows that there exists $\omega \in \Omega^n_X$ such that $d\omega = \alpha$. Then

$$\int_{\widetilde{s}_i|_{\Delta_n \times]0,\tau]}} f^{s-1}\,\alpha = \int_{t_0}^{\tau} t^{s-1} (d\omega)^\sharp.$$

where $(d\omega)^\sharp$ is the result of integration of $d\omega$ along the fibres of p_2.

By Stokes, one obtains (for $0 < t \le t_0$):

$$\int_t^{t_0} (d\omega)^\sharp = \int_{\widetilde{s}_i|_{\Delta_n \times [t,t_0]}} d\omega = \int_{\widetilde{s}_i|_{\Delta_n \times \{t_0\}}} \omega - \int_{\widetilde{s}_i|_{\Delta_n \times \{t\}}} \omega.$$

Then, by derivation and using (6.6) and (6.8), one has:

$$(d\omega)^\sharp = \frac{d}{dt} \int_{\widetilde{s}_i|_{\Delta_n \times \{t\}}} \omega = \int_{\widetilde{s}_i|_{\Delta_n \times \{t\}}} \frac{\alpha}{df} = \int_{s_i|_{\Delta_n \times \{t\}}} \eta_i.$$

On the one hand, by construction of the $\widetilde{s}_i$, the cycle $\gamma(t)$ is homologous to $\sum n_i \widetilde{s}_i|_{\Delta_n \times \{t\}}$. On the other hand, $\widetilde{s}_i|_{\Delta_n \times \{t\}}$ is contained in an open subset V_i (contained in U_i) and such that:

$$\eta_i|_{V_i \cap X_t} = \mathrm{res}_t \left(\frac{\alpha}{f-t} \right) \Big|_{V_i}$$

because , in V_i, one has $\dfrac{\alpha}{f-t} = \dfrac{df}{f-t} \wedge \eta_i$.

One obtains that $\sum n_i \displaystyle\int_{s_i|_{\Delta_n \times \{t\}}} = \int_{\gamma(t)} \frac{\alpha}{df}$ and:

$$\int_{\Gamma(t_0,\tau)} f^{s-1}\,\alpha = \int_{t_0}^{\tau} t^{s-1} dt \int_{\gamma(t)} \frac{\alpha}{df}.$$

2) The previous computation allows us to write:

$$b(s) \int_{t_0}^{\tau} t^{s-1} dt \int_{\gamma(t)} \frac{\alpha}{df} = b(s) \int_{\Gamma(t_0,\tau)} f^{s-1}\,\alpha = \int_{\Gamma(t_0,\tau)} [P(x,s,\frac{\partial}{\partial x}) f^s] \alpha.$$

Let us denote by P^* the adjoint operator of P, acting on Ω_X^{n+1}. It is defined, in local coordinates in the following way:

If $P = \sum a_\nu(s,x) D^\nu$ with $D^\nu = \left(\frac{\partial}{\partial z_1}\right)^{\nu_1} \cdots \left(\frac{\partial}{\partial z_{n+1}}\right)^{\nu_{n+1}}$ and if $\alpha = g dz_1 \wedge \cdots \wedge dz_{n+1}$, then $P^*\alpha = \left(\sum (-1)^{|\nu|} D^\nu(a_\nu g)\right) dz_1 \wedge \cdots \wedge dz_{n+1}$. The operator P^* satisfies:

$$[P(x,s,\frac{\partial}{\partial x})f^s]\alpha = f^s(P^*\alpha) + d(f^s \alpha_p)$$

with $P^*\alpha \in \Omega_X^{n+1}[s]$ and $\alpha_p \in \Omega_X^n[s]$.

By Stokes and by construction of the $\widetilde{s}_i$ (see above) one obtains

$$\int_{\Gamma(t_0,\tau)} [P(x,s,\frac{\partial}{\partial x})f^s]\alpha = \int_{\Gamma(t_0,\tau)} f^s(P^*\alpha) + \int_{\gamma(\tau)-\gamma(t_0)} f^s \alpha_p.$$

The same argument as in the first step of the proof shows that

$$\int_{\Gamma(t_0,\tau)} f^s(P^*\alpha) = \int_{t_0}^{\tau} t^s dt \int_{\gamma(t)} \frac{P^*\alpha}{df},$$

and then

$$b(s)\int_{t_0}^{\tau} t^{s-1}dt \int_{\gamma(t)} \frac{\alpha}{df} = \int_{t_0}^{\tau} t^s dt \int_{\gamma(t)} \frac{P^*\alpha}{df} + \int_{\gamma(1)} \alpha_p - t_0^s \int_{\gamma(t_0)} \alpha_p.$$

According to Lemma 5.10, for sufficiently large $\mathcal{R}e(s)$, one can consider the limit for t_0 tending to 0 in the previous equality. One obtains:

$$b(s)\int_0^{\tau} t^{s-1}dt \int_{\gamma(t)} \frac{\alpha}{df} = \int_0^{\tau} t^s dt \int_{\gamma(t)} \frac{P^*\alpha}{df} + \int_{\gamma(1)} \alpha_p. \tag{6.9}$$

3) Let us assume that $\nu_1 = \nu_2 = \ldots = \nu_k < \nu_{k+1}$, and let us choose $\alpha \in \Omega_X^{n+1}$ such that $d_{\nu_1,k-1}(\alpha) \neq 0$ (in (6.7)).

Using the development (6.7)) of $\int_{\gamma(t)} \frac{\alpha}{df}$ and the formula $\int_0^1 t^{\nu+s-1}(\log t)^k dt = \frac{d^k}{ds^k}\left(\frac{1}{s+\nu}\right)$, the integral $\int_0^\tau t^{s-1}dt \int_{\gamma(t)} \frac{\alpha}{df}$ can be extended, for $\mathcal{R}e(s) > 1$, into a meromorphic function of $s \in \mathbb{C}$, with a pole of order k at $-\nu_1$.

In the same way, $\int_0^\tau t^s dt \int_{\gamma(t)} \frac{P^*\alpha}{df}$ admits a development of type (6.7) and can be extended into a meromorphic function of $s \in \mathbb{C}$, without pole at $-\nu_1$.

Finally $\int_{\gamma(\tau)} \alpha_p$ is a polynomial in s.

Equality (6.9) implies that $(-\nu_1)$ is a root of order k of $b(s)$. One works in the same way for $\nu_{k+1}, \ldots, \nu_p$, that implies the result. $\square$

B) The case $\lambda = 1$. In the case $\lambda = 1$, the proof is similar but requires more carefulness. The previous method proves only that $(s+\nu_1)\cdots(s+\nu_p)$ divides $b = s\widetilde{b}$ and there is a risk to "lose" some root of $\widetilde{b}$ (see [23]). $\square$

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