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THE SHEAF $\alpha^\bullet_X$

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Abstract. We introduce, in a reduced complex space, a "new coherent sub-sheaf" of the sheaf $\omega^\bullet_X$ which has the "universal pull-back property" for any holomorphic map, and which is, in general, bigger than the usual sheaf of holomorphic differential forms $\Omega^\bullet_X/\text{torsion}$. We show that the meromorphic differential forms which are sections of this sheaf satisfy integral dependence equations over the symmetric algebra of the sheaf $\Omega^\bullet_X/\text{torsion}$. This sheaf $\alpha^\bullet_X$ is closely related to the normalized Nash transform.

We also show that these $q-$meromorphic differential forms are locally square-integrable on **any** $q-$dimensional cycle in X and that the corresponding functions obtained by integration on an analytic family of $q-$cycles are locally bounded and continuous on the complement of a closed analytic subset.

Introduction

In this article, we discuss the following question: given a reduced complex space X, the normalization of X consists in building a proper modification $\nu : \tilde{X} \to X$ such that meromorphic locally bounded functions on X becomes holomorphic after pull-back to $\tilde{X}$. Moreover this process gives a desingularization process for curves, that is to say for X of pure dimension 1.

It seems then natural to define an analogous process for meromorphic locally bounded differential forms. The main trouble is to define what means "locally bounded" for a meromorphic differential form of positive degree on a reduced complex space. To define this notion is the purpose of this paper. Of course, this does not lead to a simple proof of a desingularization process for a reduced complex space, but we will show that the natural process associated to "normalization of meromorphic differential forms" is simply the classical **normalized Nash transform**, and it is an old (an probably very difficult) conjecture that this process leads to a desingularization. We hope that the introduction of this "new sheaf" $\alpha^\bullet_X$ will be useful in that direction.

But in fact, the main reason to introduce this sheaf is the look for the **"universal pull-back property"** which means to define a coherent sheaf of meromorphic differential forms which admits a natural pull-back for **any** holomorphic map between reduced complex spaces and which is "maximal" with this property. Note that if we only consider complex manifolds the sheaf $\Omega^\bullet_X$ has this property, but we will show that this is no longer maximal when X admits singularities.

Our main result is the theorem 4.1.1 (and its precise formulation 4.1.2) giving the "universal pull-back property" for these sheaves. We obtain also two other results which may be useful:

- The fact that for any section α of the sheaf α^q_X the form $\alpha \wedge \bar{\alpha}$ is locally integrable on **any** holomorphic cycle of dimension q and also the local boundness and the "generic" continuity of such an integral when the $q-$cycle moves in an analytic family (see theorem 5.1.7);

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- The existence of a local integral dependence equation for a section of α_X^q over the symmetric algebra of the sheaf Ω_X^q/torsion (see proposition 5.2.1).

We conclude this article by computing some simple examples showing that the sheaf $\alpha_X^\bullet$ may be different from other classical sheaves of meromorphic differential forms which are used on singular complex spaces.

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1. Universal pull-back for $\Omega_X^\bullet$/torsion

It is well known that the sheaves $\Omega_X^\bullet$ of holomorphic differential forms on complex spaces have a functorial pull-back. To begin we shall prove that the sheaves $\Omega_X^\bullet$/torsion still have this "universal pull-back" property on reduced complex spaces.

Proposition 1.0.1. *Let X be a reduced complex space and consider a torsion holomorphic $p-$form α on X (meaning that α vanishes at smooth points in X). Let Z be an analytic subset in X. Then the $p-$holomorphic form induced by α on Z is again a torsion form on Z.*

Proof. Without any lost of generality we can assume that Z is irreducible. Let S be the singular set of X. If Z is not contained in S the result is obvious. Also if the dimension of Z is less that p the conclusion is again obvious. So let dim $Z = p + q$ with $q \geq 0$ and let Z' be the dense open set of smooth points x in Z for which the multiplicity of x in X is minimal. It is enough to show that the restriction of α to Z' vanishes. As the problem is local on Z', we can assume that we have an open neighbourhood X' of x_0 in X and a local parametrisation $\pi : X' \to U$ on a polydisc U of $\mathbb{C}^n$ with the following properties:

i) $\pi(x_0) = 0$.
ii) $U = V \times W$ where V and W are polydiscs with center 0 respectively in $\mathbb{C}^{p+q}$ and $\mathbb{C}^{n-p-q}$.
iii) $Z'' := Z' \cap X' = \pi^{-1}(V \times \{0\})$ set theoretically and $\pi : Z'' \to V \times \{0\}$ is an isomorphism.

Define the analytic family of $(p + q)-$cycles $(Z_w)_{w\in W}$ in X' parametrized by W by letting $Z_w := \pi^*(V \times \{w\})$, where the pull-back by π is taken in the sense of cycles[1]. Then, if k is the degree of π (which is the multiplicity in X of each point in Z'') we have $Z_0 = k.Z''$ as a cycle in X'. Remark that for w generic in W the intersection of the cycle Z_w with the ramification set of π has no interior point in Z_w which is a reduced cycle. So the restriction of the holomorphic form α to Z_w for w generic is a torsion form.

Now choose a non-negative continuous function with compact support ρ on X', a holomorphic $q-$form β on X' and define the function on W

$$\varphi : W \to \mathbb{R}^+, \quad w \mapsto \varphi(w) := \int_{Z_w} \rho.(\alpha \wedge \beta) \wedge \overline{(\alpha \wedge \beta)}.$$

It is a continuous function (see [B-M 1] ch.IV) and it vanishes for w generic in W as α generically vanishes on Z_w for such a w. Then it vanishes for $w = 0$ and this shows that the restriction of α to an open dense subset of Z' vanishes. □

Corollary 1.0.2. *Consider a holomorphic map $f : X \to Y$ where X and Y are reduced complex spaces. Then, if α is a $p-$holomorphic form on Y which is a torsion form, the $p-$holomorphic form $f^*(\alpha)$ is a torsion form on X.*

[1] This means that if $f : U \to \mathrm{Sym}^k(X')$ is the holomorphic map classifying the fibers of π, the cycle Z_w is the cycle-graph of the analytic family of $k-$tuples in X' defined by the restriction of f to $V \times \{w\}$; see [B-M 1] ch.IV.

Proof. It is enough to consider the case where X is a connected complex manifold. Let X' be the open dense subset of X where f has maximal rank. On X' the map f is locally a submersion on a locally closed complex sub-manifold of Y and the previous proposition applies to show that the pull-back of α on this locally closed sub-manifold vanishes. So the holomorphic form $f^*(\alpha)$ vanishes on X'. Then it is a torsion form on X. $\square$

Definition 1.0.3. *Let $f : X \to Y$ a holomorphic map between two reduced complex spaces. We have a natural graded pull-back $\mathcal{O}_X-$morphism*

$$(*) \qquad f^* : f^*(\Omega^\bullet_Y/\text{torsion}) \to \Omega^\bullet_X/\text{torsion}$$

We shall denote $f^{**}(\Omega^\bullet_Y)$ the image of this graded sheaf morphism.

We shall also denote $f^{**}(\mathcal{G})$ for any sub-sheaf $\mathcal{G}$ of $\Omega^\bullet_Y/\text{torsion}$ its image by the morphism f^* above (or also when $\mathcal{G}$ is a subs-sheaf of $\Omega^\bullet_Y$).

So, by definition, $f^{**}(\Omega^\bullet_Y)$ (and more generally $f^{**}(\mathcal{G})$) is a sub-sheaf of the sheaf $\Omega^\bullet_X/\text{torsion}$, so it has no $\mathcal{O}_X-$torsion.

Lemma 1.0.4. *Let $f : X \to Y$ and $g : Y \to Z$ two holomorphic maps between reduced complex spaces. Then we have equality of the sub-sheaves $f^{**}(g^{**}(\mathcal{H}))$ and $(g \circ f)^{**}(\mathcal{H})$ for any sub-sheaf $\mathcal{H}$ of the sheaf $\Omega^\bullet_Z/\text{torsion}$.*

Proof. The pull-back by g gives a morphism

$$g^*(\Omega^\bullet_Z/\text{torsion}) \to \Omega^\bullet_Y/\text{torsion}$$

with image $g^{**}(\Omega^\bullet_Z/\text{torsion})$ and the pull-back by f gives a morphism

$$f^*(g^*(\Omega^\bullet_Z/\text{torsion})) \to f^*(\Omega^\bullet_Y/\text{torsion})$$

which, by right-exactness of the tensor product, is surjective on $f^*(g^{**}(\Omega^\bullet_Z/\text{torsion}))$. Then we have the following commutative diagram

$$\begin{array}{ccccc} f^*(g^*(\Omega^\bullet_Z/\text{torsion})) & \xrightarrow{\alpha} & f^*(g^{**}(\Omega^\bullet_Z/\text{torsion}) & \xrightarrow{u} & f^*(\Omega^\bullet_Y/\text{torsion})\ . \\ \simeq\downarrow & & & \searrow^{\beta} & \downarrow v \\ (g\circ f)^*(\Omega^\bullet/\text{torsion}) & & \xrightarrow{\gamma} & & \Omega^\bullet_X/\text{torsion} \end{array}$$

Here α is surjective and the image of β is the sub-sheaf $f^{**}(g^{**}(\Omega_Z))$ by definition. Also the image of γ is in $(g \circ f)^{**}(\Omega^\bullet_Z)$ by definition. Now the commutativity of the diagram allows to conclude. $\square$

Conclusion.

- The usual pull-back for holomorphic differential forms induced a natural pull-back for the sheaf $\Omega^\bullet_X/\text{torsion}$ by any holomorphic map between reduced complex spaces. The previous lemma shows that this pull-back is functorial.

2. Normalization of a coherent sheaf

2.1. **Definition.** Let $\mathcal{F}$ be a coherent sheaf on a reduced complex space X and let $pr : F \to X$ be the associated linear bundle over X. Recall that, if on the open set U in X we have a presentation

$$\mathcal{O}^m_X \xrightarrow{M} \mathcal{O}^n_X \to \mathcal{F} \to 0$$

where M is a matrix with holomorphic entries, then $F_{|U}$ is given as the kernel

$$Ker[{}^tM : U \times \mathbb{C}^n \to U \times \mathbb{C}^m].$$

Then a section of $\mathcal{F}$ over an open set U in X is a holomorphic map over U, $F_{|U} \to U\times\mathbb{C}$, which is linear on the fibres of $pr_{|U}$; and conversely if $f : F_{|U} \to U \times \mathbb{C}$ is a holomorphic map which is linear on the fibres of $pr_{|U}$, let $g : V \times W \to \mathbb{C}$ be a holomorphic function on a neighbourghood of $(x_0, 0) \in U \times \mathbb{C}^n$ inducing f on $F \cap (V \times W)$. Write $g = \sum_{\nu=0}^{+\infty} \gamma_\nu$ be the Taylor expansion of g in homogeneous polynomials in the $\mathbb{C}^n$ −variables. Then γ_1, the homogeneous part of degree 1 in the $\mathbb{C}^n$ −variables, induces f on $X \cap (V \times \mathbb{C}^n)$. And γ_1 is a holomorphic function which is linear on fibres.

For the notion of linear bundle see [F.76], [A-M.86] or [B-M 2].

The symetric algebra of a linear bundle. We define the symetric algebra

$$S_\bullet(\mathcal{F}) := \oplus_{h=0}^{+\infty} S_h(\mathcal{F}),$$

where $S_h(\mathcal{F})$ is the sheaf of holomorphic functions on F which are homogeneous of degree h along the fibres of F. If $\sigma_1, \dots, \sigma_N$ is a local generator of $\mathcal{F}$ near a point $x_0 \in X$ then, for $\alpha \in \mathbb{N}^N$ such that $|\alpha| = \sum_{j=1}^N \alpha_j = h$, the $\sigma^\alpha := \sigma_1^{\alpha_1}\dots\sigma_N^{\alpha_N}$ for all such α generate locally $S_h(\mathcal{F})$ near x_0.

Note that if F is, on an open set $U \subset X$, the kernel of ${}^tM : U \times \mathbb{C}^n \to U \times \mathbb{C}^m$, the linear bundle $S_h(F)$ associated to the coherent sheaf $S_h(\mathcal{F})$ is defined as the kernel of the holomorphic map, linear on the fibers:

$$S_h({}^tM) : U \times S_h(\mathbb{C}^n) \to U \times S_h(\mathbb{C}^m).$$

As the complex space F is not reduced in general, the vanishing of a holomorphic function homogeneous of degree h on the fibres of F is not given, in general, by generic vanishing on F of such a function. But, when we assume that X is reduced and $F_{X\setminus S}$ is a vector bundle, the vanishing of a section of $S_h(\mathcal{F})$ on an open set $U \setminus S$ is just pointwise vanishing.

Recall that the exterior algebra of a coherent sheaf may be defined in the same way using the kernel of the map $\Lambda^q({}^tM) : U \times \Lambda^q(\mathbb{C}^n) \to U \times \Lambda^q(\mathbb{C}^m)$ on the "linear bundle side", or directly as a quotient of the tensor product $\mathcal{F}^{\otimes q}$.

Proposition 2.1.1. *Let X be a reduced complex space and let $\mathcal{F}$ be a coherent sheaf on X. Let $S \subset X$ be a closed analytic subset with no interior point in X such that on $X \setminus S$ the sheaf $\mathcal{F}$ is locally free. Then there exists a modification $\tau : \tilde{X} \to X$ with the following properties :*

i) The center of τ is contained in S.
ii) The sheaf $\tau^*(\mathcal{F})/\mathrm{torsion}$ is locally free on $\tilde{X}$.
iii) The reduced complex space $\tilde{X}$ is normal.
iv) For any holomorphic map $f : Y \to X$ from a normal complex space Y such that $f^{-1}(S)$ has no interior point in Y and such that the coherent sheaf $f^*(\mathcal{F})/\mathrm{torsion}$ is locally free, there exists an unique holomorphic lifting $\tilde{f} : Y \to \tilde{X}$ such that $\tau \circ \tilde{f} = f$. And in this situation we have

$$\tilde{f}^*(\tau^*(\mathcal{F})/\mathrm{torsion}) = f^*(\mathcal{F})/\mathrm{torsion}.$$

Proof. Note that, without any lost of generality, we may assume that $\mathcal{F}$ has no torsion. Consider first an open set U in X such that on U the coherent sheaf $\mathcal{F}$ has a presentation

$$\mathcal{O}_X^m \xrightarrow{M} \mathcal{O}_X^n \to \mathcal{F} \to 0.$$

Let $n-p$ be the generic rank of the holomorphic matrix M. Then the linear bundle L associated to $\mathcal{F}_{|U}$ is the kernel of the holomorphic map, linear on the fibres

$$\mathrm{id}_X \times {}^tM : U \times \mathbb{C}^n \to U \times \mathbb{C}^m.$$

Then we have a holomorphic map $g : U \setminus S \to Gr(p,n)$ which sends the $p-$dimensional vector sub-space $Ker^t M_x$ to the corresponding point in $Gr(p,n)$, the grassmannian of $p-$vector sub-spaces in $\mathbb{C}^n$. Consider then the closed analytic subset

$$Z := \{(x,P) \in U \times Gr(p,n) \ / \ P \subset Ker^t M(x)\}.$$

Over $U \setminus S$ the set Z coincides with the graph of the holomorphic map g. Then define $\tilde{X}_U$ as the normalization of the union of the irreducible components of Z which dominate an irreducible component of U. The projection map $\tau : \tilde{X}_U \to U$ is clearly a (proper) modification of U with center contained in S.

Let $V \to Gr(p,n)$ be the universal bundle of $Gr(,p,n)$ and let $\mathcal{U}$ the associated coherent sheaf. Let $p_2 : \tilde{X}_U \to Gr(p,n)$ the composition of the normalization with the projection on $Gr(p,n)$. Then let us show that there is a natural isomorphism $\tau^*(\mathcal{F})\big/\text{torsion} \to p_2^*(\mathcal{U})$. For that purpose it is equivalent to prove that there is a natural holomorphic map, linear on the fibres

$$p_2^*(V) \to \tau^*(F)$$

of linear bundles from the pull-back on $\tilde{X}_U$ of the tautological rank $p-$vector bundle V on $Gr(p,n)$ to the linear bundle $\tau^*(F)$ associated to $\tau^*(\mathcal{F})$. But this map is obvious as the fiber of $p_2^*(V)$ at $\tilde{x} \in \tilde{X}_U$ is, by definition, a $p-$vector subspace of the fibre of $\tau^*(F)$ at $\tilde{x}$. Moreover, this map is an isomorphism on $U \setminus S$ by construction, so it is injective. The corresponding morphism of coherent sheaves $\tau^*(\mathcal{F}) \to p_2^*(\mathcal{U})$ is then surjective and its kernel is supported by $\tau^{-1}(S)$. This implies that it induces an isomorphism $\tau^*(\mathcal{F})\big/\text{torsion} \simeq p_2^*(\mathcal{U})$.

To complete the proof of the assertions $i)$ to $iv)$, it is enough now to prove that the property $iv)$ holds for the modification $\tau : \tilde{X}_U \to U$ because this will imply the globalisation of this construction, thanks to the patching of these local pieces via the "universal property".

So let $f : Y \to U$ be a holomorphic map from a normal complex space Y such that $f^{-1}(S)$ has no interior point in Y and such that $f^*(\mathcal{F})\big/\text{torsion}$ is locally free on Y. Then by right exactness of the tensor product we have on Y the exact sequence

$$\mathcal{O}_Y^m \overset{f^*(M)}{\longrightarrow} \mathcal{O}_Y^n \to f^*(\mathcal{F}) \to 0.$$

This implies that the rank p vector bundle G associated to the locally free sheaf $f^*(\mathcal{F})\big/\text{torsion}$ is a sub-vector bundle of the linear bundle $f^*(F)$ which is the kernel of the holomorphic map, linear in the fibres

$$\mathrm{id}_Y \times f^*({}^tM) : Y \times \mathbb{C}^n \to Y \times \mathbb{C}^m\,.$$

This induces a holomorphic map $\tilde{g} : Y \to Gr(p,n)$ which sends $y \in Y$ to the fibre at y of $G \subset Y \times \mathbb{C}^n$. As G and $f^*(F)$ are isomorphic over $Y \setminus f^{-1}(S)$ which is a dense open set by assumption, this proves the uniqueness of $\tilde{g}$ and then of the map $\tilde{f} := (f, \tilde{g}) : Y \to \tilde{X}_U$. $\square$

Definition 2.1.2. *In the situation of the previous proposition we shall call the modification* $\tau : \tilde{X} \to X$ *the* **normalization of the coherent sheaf $\mathcal{F}$ on X**.

We shall say that a holomorphic map $f : Y \to X$ is **normalizing for the coherent sheaf** *$\mathcal{F}$ on X which is locally free outside the closed analytic subset S with no interior point in X, when it satisfies the following conditions:*

i) The complex space Y is normal.
ii) The closed analytic subset $f^{-1}(S)$ has no interior point in Y.
iii) The sheaf $f^*(\mathcal{F})\big/$torsion is locally free on Y.

Thanks to the universal property of the normalization $\tau : \tilde{X} \to X$ of $\mathcal{F}$, the holomorphic map f is normalizing for $\mathcal{F}$ if and only if the map f factorizes through the modification τ.

Remark. The normalization of a coherent sheaf $\mathcal{F}$ on a reduced complex space X is always a locally projective modification, as , by construction, it is locally contained in a product of an open set in X by a grassmannian.

Note that the proposition 2.1.1 is consequence of rather elementary results and do not use the desingularization theorem of H. Hironaka. But thanks to Hironaka, for any X and any coherent sheaf $\mathcal{F}$ on X there always exists a proper modification $\tau : \hat{X} \to X$ which is smooth and normalizes the sheaf $\mathcal{F}$: it is enough to apply the desingularization theorem to the normalization of $\mathcal{F}$ constructed above. Moreover, we may always assume that such a "normalizing" desingularization is a projective modification of X. This remark will be used in the next section.

Note that, in general, a desingularization of X is not necessarily normalizing for the sheaf Ω^1_X, see for instance the case of S_3 in example 6.2.

For a pure dimensional reduced complex space X the **Nash transform** (resp. the normalized Nash transform) is simply the previous results applied to the coherent sheaf Ω^1_X. Note that the corresponding linear bundle on X is the Zariski tangent linear bundle on X. See section 5.

Lemma 2.1.3. *In the situation of the proposition 2.1.1, consider an integer $q \geq 1$ and the coherent sheaf $\Lambda^q(\mathcal{F})$ and its normalization $\tau_q : \tilde{X}_q \to X$. Then we have a natural holomorphic map*

$$\varphi_q : \tilde{X} \to \tilde{X}_q$$

satisfying the following properties

1. *φ_q is a modification with center contained in S and $\tau_q \circ \varphi_q = \tau$.*
2. *We have a natural isomorphism of locally free sheaves on $\tilde{X}$*

$$e_q : \Lambda^q(\tau^*(\mathcal{F})\big/\text{torsion})) \to \varphi_q^*(\tau_q^*(\Lambda^q(\mathcal{F})\big/\text{torsion}).$$

Proof. Note that we may assume without any lost of generality that $\mathcal{F}$ has no torsion. As the sheaf $\tau^*(\mathcal{F})\big/$torsion is locally free on $\tilde{X}$ the sheaf $\Lambda^q(\tau^*(\mathcal{F})\big/\text{torsion})$ is also locally free on $\tilde{X}$.

The natural surjective morphism

$$\Lambda^q(\tau^*(\mathcal{G})) \to \tau^*(\Lambda^q(\mathcal{G})), \quad \tau^*(g_1) \wedge \cdots \wedge \tau^*(g_q) \mapsto \tau^*(g_1 \wedge \cdots \wedge g_q)$$

for any coherent sheaf $\mathcal{G}$ induces an isomorphism

$$(@) \qquad \Lambda^q(\tau^*(\mathcal{F})\big/\text{torsion}) \to \tau^*(\Lambda^q(\mathcal{F})\big/\text{torsion})$$

because the kernel must be a torsion sub-sheaf of $\Lambda^q(\tau^*(\mathcal{F})\big/\text{torsion})$ which is locally free. Then the universal property of the normalization of the sheaf $\Lambda^q(\mathcal{F})$ gives the holomorphic map $\varphi_q : \tilde{X} \to \tilde{X}_q$ such that $\tau_q \circ \varphi_q = \tau$, and the isomorphism (@) allows to obtain the isomorphism e_q. $\square$

Consequence. If the holomorphic map $f : Y \to X$ is normalizing for the coherent sheaf $\mathcal{F}$ it is normalizing for the sheaf $\Lambda^q(\mathcal{F})$ for any integer $q \geq 0$.

This will be useful for instance for $\mathcal{F} = \Omega^1_X$, because a normalizing map for Ω^1_X is then normalizing for each $\Omega^q_X \quad \forall q \geq 1$.

2.2. **Locally bounded sections.** Let X be a reduced complex space, $\mathcal{F}$ a coherent sheaf on X which is locally free outside the closed analytic subset $S \subset X$ with no interior point in X. Consider the linear bundle on X, $pr : F \to X$, associated to $\mathcal{F}$. For any open set U in X a section $\sigma \in \Gamma(U, \mathcal{F})$ corresponds to a holomorphic function $f : F_U \to \mathbb{C}$ which is linear on the fibres of F.

Definition 2.2.1. *We shall say that* $\sigma \in \Gamma(U \setminus S, \mathcal{F})$ *is a* **locally bounded section** *of* $\mathcal{F}$ *near the point* $s_0 \in U$ *when there exist an open neighbourhood* U_0 *of* s_0 *in* U*, sections* $\sigma_1, \dots, \sigma_N$ *sections of* $\mathcal{F}$ *on* U_0 *and continuous bounded functions* $\rho_1, \dots, \rho_N$ *on* $U_0 \setminus S$ *such that the function* f *on* $F_{U_0 \setminus S}$ *corresponding to* σ *is given by*

$$f = \sum_{j=1}^{N} \rho_j(x).f_j(x, v) \quad \forall (x, v) \in F_{U_0 \setminus S}$$

where, for each $j \in [1, N]$, $f_j : F_{U_0} \to \mathbb{C}$ *is the holomorphic function linear on the fibres of* F *which corresponds to* $\sigma_j \in \Gamma(U_0, \mathcal{F})$

Remark that, by definition of S, $F_{U_0 \setminus S}$ is a reduced complex space: it is a holomorphic vector bundle on a reduced complex space. So the equality above is a "pointwise equality".

Of course, if σ is the restriction to $U_0 \setminus S$ of a section $\sigma \in \Gamma(U_0, \mathcal{F})$, it is locally bounded near each point in U_0: take $\sigma_1 = \sigma$ and $\rho_1 \equiv 1$!

Note that the function $f : F_{U_0 \setminus S} \to \mathbb{C}$ corresponding to a locally bounded section

$$\sigma \in \Gamma(U_0 \setminus S, \mathcal{F})$$

is locally bounded near each point of $F_{U_0 \cap S}$ which belongs to the irreducible components of F_{U_0} which surject onto an irreducible component of U_0. So, in general, such a f is not a locally bounded function on F_{U_0} but only on the closure in F_{U_0} of $F_{U_0 \setminus S}$.

Lemma 2.2.2. *Let* $S \subset X$ *be a closed analytic subset with no interior point in* X *containing the singular set in* X *and assume that the coherent sheaf* $\mathcal{F}$ *is locally free on* $X \setminus S$*. Let*

$$\sigma \in \Gamma(U_0 \setminus S, \mathcal{F})$$

and $f : F_{U_0 \setminus S} \to \mathbb{C}$ *the corresponding holomorphic function linear on the fibers of* F*.Then the fonction* f *is bounded in a neighbourhood of the point* $\{s_0\} \times \{0\}$ *in the closure of* $F_{U_0 \setminus S}$ *in* F[2] *if and only if the section* σ *is locally bounded near* s_0 *as a section of* $\mathcal{F}$ *on* $U_0 \setminus S$ *in the sense of the definition 2.2.1.*

Proof. Let first consider a section σ of $\mathcal{F}$ which is locally bounded near s_0 in the sense of the definition 2.2.1. Then we can find holomorphic sections $\sigma_1, \dots, \sigma_N$ on an open neighbourhood U_0 of s_0 in U and continuous bounded functions $\rho_1, \dots, \rho_N$ on $U_0 \setminus S$, such that $\sigma = \sum_{n=1}^{N} \rho_n.\sigma_n$ on $U_0 \setminus S$. Then, if $f_1, \dots, f_N$ are the holomorphic functions (linear on the fibres) on $F_{|U_0}$ corresponding to $\sigma_1, \dots, \sigma_N$, we have $f = \sum_{j=1}^{N} \rho_j.f_j$ on $F_{|U_0 \setminus S}$. This implies that the function f is locally bounded near points in the intersection of $pr^{-1}(s_0)$ with the closure of $F_{U_0 \setminus S}$. In particular near $\{s_0\} \times \{0\}$.

Conversely, if f is locally bounded on the intersection with $F_{|U_0 \setminus S}$ of a neighbourghood of $\{s_0\} \times \{0\}$ in the closure of $F_{U_0 \setminus S}$, remark that, as an obvious consequence of its homogeneity on the fibres of $\overline{F_{U_0 \setminus S}}$, it is locally bounded in a neighbourhood of each point of $pr^{-1}(s_0) \cap \overline{F_{U_0 \setminus S}}$.

Consider now a modification $\tau : \tilde{X} \to X$ with center contained in S such that $\tilde{X}$ is normal and such that the strict transform $\tilde{\tau} : \tilde{F} \to F$ of F is a holomorphic vector bundle. Then the function $f \circ \tilde{\tau}$ is locally bounded near $\tau^{-1}((s_0) \times \{0\})$ in $\tilde{F}$. As $\tilde{F}$ is a holomorphic vector bundle over the normal complex space $\tilde{X}$, it is a normal complex space and then $f \circ \tilde{\tau}$ extends to $\tilde{F}_{|\tau^{-1}(U_0)}$ to a holomorphic function $\tilde{f}$ which is linear on the fibres. If $\sigma_1, \dots, \sigma_N$ are sections of $\mathcal{F}$ on an open

[2]Note that if G is an irreducible component of F_V which is contained in $pr^{-1}(V \cap S)$ then G does not meet the open set where f is defined. So we obtain the same condition on σ if we replace $\mathcal{F}$ by $\mathcal{F}/$torsion.

neighbourhood U_0 of s_0 in U which generate $\mathcal{F}$ at each point of U_0, their pull-back by τ generate the coherent sheaf on $\tilde{X}$ associated to $\tilde{F}$ at each point of $\tau^{-1}(U_0)$. Near each such point we can write $\tilde{f} = \sum_{j=1}^N \ c_j \otimes \tau^{-1}(\sigma_j)$ where $c_1, \dots, c_N$ are local holomorphic functions on $\tilde{X}$. Using a continuous partition of unity along the compact fibre $\tau^{-1}(s_0)$ we obtain that f can be written as $\sum_{j=1}^N \ \rho_j.\sigma_j$ on $U_0 \setminus S$ where $\rho_1, \dots, \rho_N$ are bounded continuous functions on $U_0 \setminus S$. □

Corollary 2.2.3. *Let X be a reduced complex space and let $\mathcal{F}$ be a coherent sheaf on X. Let $S \subset X$ be a closed analytic subset with no interior point in X such that $\mathcal{F}$ is locally free on $X \setminus S$. Note $j : X \setminus S \to X$ the inclusion. Let Y be a normal complex space and consider a (proper) modification $\tau : Y \to X$ normalizing the sheaf $\mathcal{F}$. Then the sheaf $\tau_*(\tau^*(\mathcal{F})\big/\mathrm{torsion})$ is the sub-sheaf of the sheaf $j_*j^*(\mathcal{F})$ of sections which are locally bounded along S.*
So this sheaf is independent of the choice of such a τ.

Proof. First consider a section of $\theta \in \tau_*(\tau^*(\mathcal{F})\big/\mathrm{torsion})$. It can be written locally on Y as a sum $\sum_{j=1}^N \ g_j.\tau^*(\sigma_j)$ where $\sigma_1, \dots, \sigma_N$ generate locally $\mathcal{F}$ and where $g_1, \dots, g_N$ are local holomorphic functions on Y. Then using a continuous partition of unity along the fibres of τ we see that θ satisfies the definition 2.2.1.

Conversely, if η is a section of the sheaf $j_*j^*(\mathcal{F})$ which is locally bounded along S, its lifting gives a holomorphic function on $\tau^*(F)$ on the complement of $\tau^{-1}(S)$, which is linear on the fibres and locally bounded near the points of $\tau^*(F)$ which are in the closure of the restriction of $\tau^*(F)$ to $Y \setminus \tau^{-1}(S)$. But this closure is a vector bundle, by our assumption on τ. As a vector bundle on a normal complex space is a normal complex space, the Riemann extension theorem holds, and this holomorphic function extends holomorphically to this vector bundle. Then it is a section of the sheaf $\tau_*(\tau^*(\mathcal{F})\big/\mathrm{torsion})$ concluding the proof. □

Proposition 2.2.4. *Let $S \subset X$ be a closed analytic subset with no interior point in X containing the singular set in X and assume that the coherent sheaf $\mathcal{F}$ is locally free on $X \setminus S$. Consider a holomorphic function f on $F_{|U\setminus S}$ which is linear on the fibres of F and which is locally bounded along $pr^{-1}(S) \cap \overline{pr^{-1}(U \setminus S)}$ corresponding to a locally bounded section σ of $\mathcal{F}$ on $U \setminus S$. Then for each point s_0 in S there exists an open neighbourhood U_0 of s_0 in X, an integer $h \geq 1$ and sections $s_1, \dots, s_h$ on U_0 respectively of the sheaves $S_1(\mathcal{F}), \dots, S_h(\mathcal{F})$ such that the equality of sections of $S_h(\mathcal{F})$:*

$$\sigma^h + \sum_{a=1}^h s_a.\sigma^{h-a} = 0$$

is satisfied on the open set $U_0 \setminus S$.

proof. We keep the notations of the proof of the previous lemma 2.2.2. As the function f is locally bounded on F_1, the conic bundle over X which is the union of the irreducible components of F near the point $\{s_0\} \times \{0\}$ which dominate an irreducible component of X at s_0, there exist an open neighbourhood U_0 of s_0 in X, an integer $h \geq 1$ and holomorphic functions $\tilde{s}_1, \dots, \tilde{s}_h$ on an open neighbourhood W of $F_1 \cap pr^{-1}(U_0 \times \{0\})$ such that $\sigma^h + \sum_{a=1}^h \tilde{s}_a.\sigma^{h-a} = 0$ on $W \cap pr^{-1}(U_0 \setminus S)$. Taking the homogeneous degree h parts of the expansions of this equality in the fibres of $pr : F \to X$ leads to sections $s_1, \dots, s_h$ of the sheaves $S_a(\mathcal{F})$, where s_a is the homogeneous degree a part of $\tilde{s}_a$[3] concluding the proof. □

[3]which is in fact well-defined only modulo torsion in $S_a(\mathcal{F})(U_0)$, but this torsion is concentrated on S, so is irrelevant for the desired equality on $U_0 \setminus S$.

3. Definition of the sheaf $\alpha_X^\bullet$

It will be convenient to use the following definition in the sequel.

Definition 3.0.1. *Let X be a reduced complex space. We say that a modification $\tau : \tilde{X} \to X$ is a* **special desingularization of** X *when the following properties are satisfied:*

i) $\tilde{X}$ *is a complex manifold.*
ii) The modification τ is projective.
iii) The sheaf $\tau^*(\Omega_X^1)\big/\text{torsion}$ is locally free on $\tilde{X}$.

We have already remark that, thanks to Hironaka and to the fact that the normalization of the sheaf Ω_X^1 is a projective modification of X, for any modification $\theta : Y \to X$ there exists a special desingularization $\tau : \tilde{X} \to X$ which factors through θ.

The following result is the key of the definition of the sheaf $\alpha_X^\bullet$ on a reduced complex space X.

Theorem 3.0.2. *Let X be a reduced complex space and let S be a closed analytic subset with no interior point in X containing the singular set of X. Let α be a section on X of the sheaf ω_X^p. The following properties are equivalent for α:*

- *There exists locally on X a normalizing modification for the sheaf Ω_X^1*[4] *$\tau : \tilde{X} \to X$ such that α extends to a section on X of the sub-sheaf $\tau_*\tau^{**}(\Omega_X^p)$ of ω_X^p.* (A)
- *There exists,* **locally on** X*, a finite collection $(\rho_j)_{j\in J}$ of continuous functions on $X \setminus S$ which are bounded near S and holomorphic $p-$forms $(\omega_j)_{j\in J}$ in $\Omega_X^p\big/\text{torsion}$ such that $\alpha = \sum_{j\in J} \rho_j.\omega_j$ as a $(p,0)$ currents on X.* (B)

Note that under the second property stated in the theorem, the $(p,0)-$current on X associated to the form $\sum_{j\in J} \rho_j.\omega_j$ on $X \setminus S$ is defined by

$$\mathscr{C}_c^\infty(X)^{n-p,n} \ni \varphi \mapsto \int_X \varphi \wedge (\sum_{j\in J} \rho_j.\omega_j)$$

and this integral is absolutely convergent as the functions ρ_j are locally bounded near each point in S. It defines a $(p,0)-$current on X with order 0. The assumption that α is a section of the sheaf ω_X^p implies that this current is $\bar{\partial}-$closed on X.

Proof. Let us begin by the implication $(A) \Rightarrow (B)$. By definition, a section $\alpha \in \omega_X^p$ is in the sub-sheaf $\tau_*\tau^{**}(\Omega_X^\bullet)$ if, locally on $\tilde{X}$, it can be written as a linear combination of pull-back of holomorphic forms on X with holomorphic coefficients in $\mathcal{O}_{\tilde{X}}$. Using the properness of the modification τ and a $\mathscr{C}^\infty$ partition of the unity on $\tilde{X}$ we obtain the first part of (B) because τ induces an isomorphism $\tilde{X} \setminus \tau^{-1}(S) \to X \setminus S$ by hypothesis. The last property in (B), that is to say the fact that the current defined on X by the right hand-side coincides with α, is consequence of the fact that both are sections of the sheaf ω_X^p and are equal on $X \setminus S$.

To prove the implication $(B) \Rightarrow (A)$ consider the pull-back to $\tilde{X} \setminus \tau^{-1}(S)$ of the form $\sum_{j\in J} \rho_j.\omega_j$. We obtain a holomorphic section on $\tilde{X} \setminus \tau^{-1}(S)$ of the locally free sheaf

$$\tau^*(\Omega_X^p)\big/\text{torsion}$$

[4] In fact normalizing for the sheaf Ω_X^p would be enough; see lemma 2.1.3.

which has locally bounded coefficients along $\tau^{-1}(S)$ when we compute it in a local trivialisation near a point of $\tau^{-1}(S)$. So, by normality of $\tilde{X}$, it extends to a holomorphic section $\tilde{\alpha}$ of $\tau^*(\Omega_X^p)\big/\text{torsion}$ and then defines a section of $\tau_*(\tilde{\alpha})$ of the sheaf $\tau_*(\tau^*(\Omega_X^p)\big/\text{torsion})$. Note that the pull-back of holomorphic forms gives an injective morphism $\tau^*(\Omega_X^p)\big/\text{torsion} \to \Omega_{\tilde{X}}^p\big/\text{torsion}$ with image $\tau^{**}(\Omega_X^p)$. So $\tau^*(\alpha)$ defines a holomorphic form on $\tilde{X}$ and the direct image of this form and α coincide on $X \setminus S$, and then on X as sections of the sheaf ω_X^p because this sheaf has no non-zero section supported in S. $\square$

REMARKS.

1. The condition (B) does not depend on the choice of the modification τ normalizing the sheaf Ω_X^1.
2. Let $L_X^\bullet$ be the direct image of the sheaf $\Omega_Y^\bullet$ where $\tau : Y \to X$ is a desingularization of X. Using a special desingularization of X in the proof above we obtain that the form α is in the coherent sub-sheaf $L_X^p \subset \omega_X^p$, so it gives the inclusion $\alpha_X^\bullet \subset L_X^\bullet$.

Corollary 3.0.3. *Let $\tau : \tilde{X} \to X$ be any proper modification of X which is normalizing the sheaf Ω_X^1. The graduate sub-sheaf $\alpha_X^\bullet := \tau_*(\tau^{**}(\Omega_X^\bullet))$ of the sheaf $L_X^\bullet$ is independent of the choice of the modification of X normalizing Ω_X^1.* $\square$

Corollary 3.0.4. *Let X be a pure dimensional reduced complex space and let*

$$X := \cup_{i\in I} X_i$$

be its decomposition in irreducible components. Then the sheaf $\alpha_X^\bullet$ has a natural injection in the locally finite direct sum of the direct images in X of the sheaves $\alpha_{X_i}^\bullet$ for $i \in I$.

PROOF. This is an easy consequence of the fact that a section of the sheaf $L_X^\bullet$ is a section of $\alpha_X^\bullet$ if and only if it satisfies the condition (B) in the previous theorem, because we have an isomorphism $L_X^\bullet \simeq \oplus_{i\in I}\,(j_i)_*(L_{X_i}^\bullet)$, where $j_i : X_i \to X$ is the inclusion. $\square$

Note that when X is not irreducible the injective map $\alpha_X^\bullet \hookrightarrow \oplus_{i\in I}\,(j_i)_*(\alpha_{X_i}^\bullet)$ is not an isomorphism, in general, because the injective map

$$\Omega_X^\bullet/\text{torsion} \hookrightarrow \oplus_{i\in I}\,(j_i)_*(\Omega_{X_i}^\bullet/\text{torsion})$$

is not an isomorphism, in general.

But, for each $i \in I$, and any point $x \in X_i$, the "restriction" map

$$\alpha_{X,x}^\bullet \to \alpha_{X_i,x}^\bullet$$

is surjective because each restriction map $\Omega_{X,x}^\bullet/\text{torsion} \to \Omega_{X_i,x}^\bullet/\text{torsion}$ is surjective.

4. UNIVERSAL PULL-BACK FOR $\alpha_X^\bullet$

4.1. **Statement of the theorem.** The main result of this paragraph is the following theorem.

Theorem 4.1.1. *For any holomorphic map $f : X \to Y$ between reduced complex spaces, there exists a functorial*[5] *graduate $\mathcal{O}_X-$morphism*

$$\hat{f}^* : f^*\alpha_Y^\bullet \to \alpha_X^\bullet$$

which is compatible with the usual pull-back of the sheaf $\Omega_Y^\bullet/\text{torsion}$.
For any holomorphic maps $f : X \to Y$ and $g : Y \to Z$ between reduced complex spaces we have

$$\hat{f}^*(\hat{g}^*(\alpha)) = \widehat{g\circ f}^*(\alpha) \qquad \forall \alpha \in \alpha_Z^\bullet. \tag{1}$$

[5]We shall make this precise in the theorem 4.1.2 below.

Let now give a precise formulation of this result. For that purpose let $\mathcal{C}$ be the category of reduced complex spaces with morphisms all holomorphic maps. We may enrich this category, using the universal pull-back property for the graded sheaf $\Omega_X^\bullet/\text{torsion}$:

Let $\mathcal{C}_{diff}$ be the category whose objects are pairs $(X, \Omega_X^\bullet/\text{torsion})$ where X is an object in $\mathcal{C}$ and where the morphisms are given by pairs (f, f^*) where $f : X \to Y$ is a morphism in $\mathcal{C}$ and $f^* : f^*(\Omega_Y^\bullet/\text{torsion}) \to \Omega_X^\bullet/\text{torsion}$ is the graded pull-back by f of holomorphic forms modulo torsion (see section 1). Of course the forget-full functor $G_0 : \mathcal{C}_{diff} \to \mathcal{C}$ obtained by $(X, \Omega_X^\bullet/\text{torsion}) \mapsto X, (f, f^*) \mapsto f$ is an equivalence of category.

Then the precise content of the theorem above is the following result.

Theorem 4.1.2. [Precise formulation] *There exists a category $\mathcal{C}_{b-diff}$ whose objects are pairs $(X, \alpha_X^\bullet)$ where X is in $\mathcal{C}$ and where the graded coherent sheaf $\alpha_X^\bullet$ has been defined in section 3 for any object X in $\mathcal{C}$. The morphisms are given by pairs $(f, \hat{f}^*)$ for each $f : X \to Y$ a morphism in $\mathcal{C}$ where $\hat{f}^* : f^*(\alpha_Y^\bullet) \to \alpha_X^\bullet$ is the graded $\mathcal{O}_X-$linear sheaf map defined by f. Moreover, the following properties holds:*

(1) For each $X \in \mathcal{C}$ we have a graded $\mathcal{O}_X-$linear injection

$$\eta_X : \Omega_X^\bullet/\text{torsion} \to \alpha_X^\bullet.$$

(2) For any morphism $f : X \to Y$ in $\mathcal{C}$ we have a commutative diagram of graded $\mathcal{O}_X-$linear maps of sheaves

$$\begin{array}{ccc} f^*(\Omega_Y^\bullet/\text{torsion}) & \xrightarrow{f^*} & \Omega_X^\bullet/\text{torsion} \\ \downarrow{\scriptstyle f^*(\eta_Y)} & & \downarrow{\scriptstyle \eta_X} \\ f^*(\alpha_Y^\bullet) & \xrightarrow{\hat{f}^*} & \alpha_X^\bullet \end{array} \tag{2}$$

where $\hat{f}^*$ is the graded $\mathcal{O}_X-$linear map of coherent sheaves on X associated to the holomorphic map f.

Of course the interest of this result comes from the fact that the sheaf $\alpha_X^\bullet$ is, in general, strictly bigger that the sheaf $\Omega_X^\bullet/\text{torsion}$; see section 6.

For any holomorphic map $f : X \to Y$ between reduced complex spaces a pull-back morphism $f^\sharp : f^*(L_Y^\bullet) \to L_X^\bullet$ is defined in [K. 00]. But this pull-back is not functorial on these sheaves: let $\tau : \tilde{X} \to X$ be a desingularization of $X \in \mathcal{C}$ and let $x \in X$ be a point such that $\tau^{-1}(x)$ has dimension ≥ 1. Let ω be a holomorphic form near $\tau^{-1}(x)$ in $\tilde{X}$ which does not induce a torsion form on an irreducible component Γ of $\tau^{-1}(x)$. Then, because the map $\tau_{|\Gamma} : \Gamma \to X$ factorizes by the constant map to $\{x\}$ the functoriality of the pull-back of ω on Γ would imply that the pull-back has to be zero. But this map factorizes also by the inclusion of Γ in $\tilde{X}$ and τ. As the pull-back by τ is injective (by definition of $L_X^\bullet$), this gives a contradiction. Such an example is given in section 6.3.

4.2. **The proof.**

Preliminaries. Consider the following situation: let Z be a connected complex manifold and consider a proper holomorphic map $\pi : Z \to X$ which is surjective on a reduced (irreducible) complex space X. Let $q := \dim Z - \dim X$ and let k be the number of connected components of the generic fibre of π. Assume that we have a kähler form ω on Z.

Claim. After a suitable normalization of ω, the smooth $(q,q)-$form $w := \frac{1}{k}.\omega^{\wedge q}$ is $d-$closed and satisfies the condition $\pi_*(w) = 1$ as a $d-$closed $(0,0)-$current on X.

Proof. Consider the Stein factorization $\pi_0 : Z \to Y, \theta : Y \to X$ of π; the reduced complex space Y is irreducible. We have a meromorphic fibre-map $Y - -- > \mathcal{C}_q(Z)$ for π_0 (see [B-M 1] ch.IV Th. 9.1.1) and this implies, thanks to the irreducibility of Y, that the generic fibres of π_0 are in the same connected component of the space of $q-$cycles in Z. So the volume computed by $\omega^{\wedge q}$ of the connected components of the generic fibres of π is constant, and we may normalized ω in order that this volume is equal to 1. Then the $d-$closed $(0,0)-$current $\pi_*(w)$ on X is equal to 1 on a dense Zariski open set in X. This implies our claim.

Assume now that the complex manifold Z has finitely many connected components $Z_1, \dots, Z_r$ such that the restriction of π is surjective on each Z_j and such that each Z_j has a kähler form ω_j. We can normalize each ω_j in order that the integral of the form $w_j := \frac{1}{k_j}.\omega^{\wedge q_j}$ is equal to $1/r.k_j$ on each connected component of the generic fibres of $\pi_j := \pi_{|Z_j}$ and then the smooth form $w := \sum_{j=1}^r \ w_j$ satisfies again the condition $\pi_*(w) = 1$ as a $(0,0)-$current on X.

In this situation we shall say that the smooth form w on Z satisfies the condition (@).

The proof of the theorem 4.1.2 will use the following proposition.

Proposition 4.2.1. *Let $X = \cup_{i\in I} \ X_i$ be the decomposition of a reduced complex space X as the union of its irreducible components. Let $Z := \cup_{j\in J} \ Z_j$ be a disjoint union of connected complex kähler manifolds. Assume that we have a map $\theta : J \to I$ which is surjective and has finite fibres. Let $\pi : Z \to X$ be a proper holomorphic map normalizing the sheaf Ω^1_X, such that for each $j \in J$ it induces a surjective map*

$$\pi_j : Z_j \to X_{\theta(j)}$$

and let $q_j := \dim Z_j - \dim X_{\theta(j)}$. For each $j \in J$ let w_j be a smooth $(q_j, q_j)-$form on Z_j which is $d-$closed and satisfies the condition (@) relative to the restriction of π to Z_j (see preliminaries above). Let $w := \sum_{j\in J} w_j$.
Let β be a section on Z of the sheaf $\pi^{**}(\Omega^p_X)$. Then we have:

1. *The $\bar\partial-$closed $(p,0)-$current $\pi_*(\beta \wedge w)$ on X is independent of the choices of the forms w_j, assuming that they are $d-$closed and satisfy the condition (@).*
2. *The section $\pi_*(\beta \wedge w)$ on X of the sheaf ω^p_X is a section of the sub-sheaf α^p_X.*
3. *If there exists a section α of the sheaf $\Omega^p_X/\text{torsion}$ such that $\beta = \pi^{**}(\alpha)$ on Z, then $\alpha = \pi_*(\beta \wedge w)$ as a section on X of the sheaf ω^p_X.*

Remarks.

(1) It is enough to prove assertion 1) and 3) of the proposition above for each map $\pi_j, j \in J$ because the sheaf ω^p_X is a sub-sheaf of the direct sum of the sheaves $\omega^p_{X_i}, i \in I$ and the restriction of β to Z_j is a section of the sheaf $\pi^{**}_j(\Omega^p_{X_{\theta(j)}})$ for each $j \in J$.
This is not the case for the assertion 2) of the proposition: the sheaf α^p_X is a sub-sheaf of the direct sum of the sheaves $\alpha^p_{X_i}, i \in I$ but, in general, strictly smaller than this direct sum. Note also that the condition on β to be a section of the sheaf $\pi^{**}(\Omega^p_X)$ is stronger than the condition on each $\beta_j := \beta_{|Z_j}, j \in J$ to be a section of the sheaf $\pi^{**}_j(\Omega^p_{X_{\theta(j)}})$.

(2) In general, a section $\beta \in \Gamma(Z, \pi^{**}(\Omega^p_X))$ is not equal to some $\pi^{**}(\alpha)$ where α is in $\Gamma(X, \Omega^p_X)$ even in the case where $\pi : Z \to X$ is a special desingularization of X. □

Proof. Thanks to the previous remark, we may assume that X is irreducible to prove assertions 1) and 3) of the proposition.

In the case $q_j = 0$ the map π_j is generically finite and w_j is a locally constant function on Z with a prescribed value on each Z_j. So there is no choice for w_j and the first assertion of the proposition is trivial. As the second assertion is also clear in this case (the sheaf ω_X^p has no torsion on X by definition), we shall assume $q_j \geq 1$ in the sequel.

The fact that the current $\pi_*(\beta \wedge w)$ is $\bar{\partial}-$closed on X is consequence of the fact that on each Z_j the smooth $(p+q_j, q_j)$ form $\beta \wedge w_j$ is $\bar{\partial}-$closed and of the holomorphy of π. Let w' be a smooth form on Z which is $d-$closed and satisfies the condition (@). We want to show that $\pi_*(\beta \wedge (w - w'))$ vanishes as a section of the sheaf ω_X^p. Let X' be the open and dense subset of smooth points in X for which the Stein factorization of each $\pi_j : Z_j \to X$ is a covering of degree k_j. Remember that, as we assume that X is irreducible here, the set I is reduced to one point and so J is a finite set. On this open set X' it is enough to prove that for each $j \in J$ the current $(\pi_j)_*(\beta \wedge (w_j - w'_j))$ vanishes. So we can fix j and replace locally X' by one sheet of the corresponding finite covering and make the proof in this case. That is to say that we may assume that Z is smooth and connected and that $\pi : Z \to X$ has connected fibers on X.

In this case the generic fibres of π are irreducible and of dimension q. For any $x \in X'$ there exists an open neighbourhood $V(x)$ of $\pi^{-1}(x)$ which is a deformation retract of $\pi^{-1}(x)$. Then we have an isomorphism $H^{2q}(V(x), \mathbb{C}) \to \mathbb{C}$ which is given by integration on $\pi^{-1}(x)$. But w and w' have the same integral on $\pi^{-1}(x)$ by the property (@). So there exists a $(2q-1)$ smooth form θ on $V(x)$ such that $d\theta = w - w'$ by de Rham's theorem.

Consider now a small open neighbourhood U of x in X' such that $\pi^{-1}(U) \subset V(x)$. Let $x_1, \dots, x_n$ be a local coordinate system on U. Then the sheaf $\pi^*(\Omega_X^p)$ is a free sheaf of $\mathcal{O}_Z-$modules on $\pi^{-1}(U)$ with basis $\pi^*(dx^L)$ where L runs in all ordered sub-sets of cardinal p in $[1,n]$. If we write $\beta = \sum_{|L|=p} g_L.\pi^*(dx^L)$ on U the holomorphic functions g_L on $\pi^{-1}(U)$ are constant along the fibres of π and so there exists holomorphic functions $f_L, |L| = p$, with $g_L = \pi^*(f_L)$ (recall that U is a smooth open set in X). This means that there exists a holomorphic $p-$form α on U such that $\beta = \pi^*(\alpha)$ on $\pi^{-1}(U)$.

Let $\psi \in \mathscr{C}_c^\infty(U)^{(n-p,n)}$. By definition of the direct image we have

$$\langle \pi_*(\beta \wedge d\theta), \psi \rangle = \int_{\pi^{-1}(U)} \beta \wedge d\theta \wedge \pi^*(\psi).$$

But it follows from the equality $\beta = \pi^*(\alpha)$ on $\pi^{-1}(U)$ that the form

$$\beta \wedge \pi^*(\psi) = \pi^*(\alpha \wedge \psi)$$

is $d-$closed as $\alpha \wedge \psi$ is $d-$closed on U (its degree is $2n$). So by Stokes formula the integral

$$\int_{\pi^{-1}(U)} \beta \wedge d\theta \wedge \pi^*(\psi) = \pm \int_{\pi^{-1}(U)} d\big(\beta \wedge \theta \wedge \pi^*(\psi)\big)$$

vanishes. This implies that the section $\pi_*(\beta \wedge (w - w'))$ of the sheaf ω_X^p vanishes on the open dense subset X', so everywhere on X as the sheaf ω_X^p has no torsion.

Assertion 3) of the proposition is clear, because the equality is obvious at the generic points in X.

Let us prove assertion 2). We no longer assume that I has a unique point.

Let $\tau : \tilde{X} \to X$ be a special desingularization of X, so $\tilde{X}$ is the disjoint union of special desingularizations $\tau_i : \tilde{X}_i \to X_i$ for each $i \in I$, and consider the commutative diagram

$$\begin{array}{ccc} \tilde{X} \times_{X,str} Z & \xrightarrow{\tilde{\tau}} & Z \\ \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \pi} \\ \tilde{X} & \xrightarrow{\tau} & X \end{array}$$

where $\tilde{X} \times_{X,str} Z$ is the strict transform, so the union of irreducible components of $\tilde{X} \times_X Z$ which dominate some $\tilde{X}_i$.

Note that the map $\tau \circ \tilde{\pi}$ is normalizing for the sheaf Ω_X^1 because it is the case for τ (and also for π). Then the $p-$form $\tilde{\tau}^{**}(\beta)$ gives, for each such component, a section of the sheaf $(\pi \circ \tilde{\tau})^{**}(\Omega_X^p)$ and as the $d-$closed form $\tilde{\tau}^*(w)$ satisfies the condition (@) for the map $\tilde{\pi}$, the $\bar{\partial}-$closed current $\tilde{\pi}_*(\tilde{\tau}^{**}(\beta) \wedge \tilde{\tau}^*(w))$ is in fact a $p-$holomorphic form on $\tilde{X}$ thanks to Dolbeault-Grothendieck's lemma. This already proved that $\alpha := \pi_*(\beta \wedge w)$ is a section of the sheaf L_X^p, because $\tau^{**}(\pi_*(\beta \wedge w)) = \tilde{\pi}_*(\tilde{\tau}^{**}(\beta) \wedge \tilde{\tau}^*(w)))$ at the generic points of $\tilde{X}$, so everywhere on $\tilde{X}$.

Now the map $\eta : \Omega^p_{\tilde{X}} \to \Omega^p_{\tilde{X}}$ given by $\gamma \mapsto \tilde{\pi}_*(\tilde{\pi}^{**}(\gamma) \wedge \tilde{\tau}^*(w))$ is the identity map, thanks to the assertion 3). So, if $\tilde{\pi}^{**}(\gamma)$ gives a section of the image of the sub-sheaf $\tilde{\pi}^{**}\big(\tau^{**}(\Omega_X^p)\big)$ of the sheaf $\Omega^p_{\tilde{X} \times_{X,str} Z}\big/$torsion, γ will be a section of the image of the sub-sheaf $\tau^{**}(\Omega_X^p)$ because the map $\tilde{\pi}^* : \tilde{\pi}^*(\Omega^p_{\tilde{X}}) \to \Omega^p_{(\tilde{X} \times_{X,str} Z)}$ is injective.

Apply this to $\gamma := \tau^{**}(\alpha) = \tilde{\pi}_*\big(\tilde{\tau}^{**}(\beta) \wedge \tilde{\tau}^*(w)\big)$ which is a section of $\Omega^p_{\tilde{X}}$ as we already proved that α is a section in L_X^p; we obtain that $\tau^{**}(\alpha)$ is a section of the sheaf $\tau^{**}(\Omega_X^p)$ because, as the diagram above commutes, $\tilde{\tau}^{**}(\beta)$ is a section of the sheaf $\tilde{\tau}^{**}\big(\pi^{**}(\Omega_X^p)\big) = \tilde{\pi}^{**}\big(\tau^{**}(\Omega_X^p)\big)$ thanks to the lemma 1.0.4. □

Remark. If Z is not assumed to be smooth in the previous proposition, replacing Z by a projective desingularization $\sigma : \tilde{Z} \to Z$ (as before, this means that $\tilde{Z}$ is the disjoint union of projective desingularizations $\sigma_j : \tilde{Z}_j \to Z_j$ for $j \in J$), the proposition applies to the proper map $\pi \circ \sigma$ and to $\tilde{\beta} := \sigma^*(\beta)$ which is a section of the sheaf $(\pi \circ \sigma)^{**}(\Omega_X^p)$. Then the result is still true. □

Proof of theorem 4.1.1. The first step in proving the theorem will be the construction of $\hat{f}^*(\alpha) \in \alpha_X^\bullet$ when α is a section of the sheaf $\alpha_Y^\bullet$. So let α be a section on Y of the sheaf α_Y^p. Let $\tau : \tilde{Y} \to Y$ be a special desingularization of Y. Consider the following commutative diagram

$$\begin{array}{ccccc} Z & \xrightarrow{\theta} & \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ & \searrow^{\pi_1} & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tau} \\ & & X & \xrightarrow{f} & Y \end{array}$$

where $\tilde{X} \subset X \times_Y \tilde{Y}$ is the strict transform of X, that is to say the union of irreducible components of $X \times_Y \tilde{Y}$ which dominate an irreducible component of X, and where π and $\tilde{f}$ are induced by the natural projections of $X \times_Y \tilde{Y}$. Then let Z be a special desingularization of X such that π_1 factorizes by π (see the remark following the definition 3.0.1).

Now the problem is local on X and Y and we may assume that $X, \tilde{X}, Y, \tilde{Y}$ and Z are kähler. So we may assume that we have on Z a smooth $d-$closed form w which satisfies the condition (@) for the proper map π_1 (we use a special desingularization to reach the precise situation of the proposition 4.2.1; see the remark above and the remark following the definition 2.1.2).

Let β be the section of $\tau^{**}(\Omega^p_Y)$ defined by α; then the form $(\tilde{f}\circ\theta)^{**}(\beta)$ is a section of $\pi_1^{**}(\Omega^p_X)$ because if we write locally on $\tilde{Y}$

$$\beta := \sum_l \ g_l.\tau^{**}(\omega_l)$$

where ω_l are local sections of Ω^p_Y and g_l are holomorphic functions on $\tilde{Y}$, we obtain

$$(\tilde{f}\circ\theta)^{**}(\beta) = \sum_l \ (\tilde{f}\circ\theta)^*(g_l).(\tilde{f}\circ\theta)^{**}(\tau^{**}(\omega_l))$$

and the equality $(\tilde{f}\circ\theta)^{**}(\tau^{**}(\omega_l)) = \pi_1^{**}(f^{**}(\omega_l))$ due to the commutativity of the diagram and the lemma 1.0.4 shows that $(\tilde{f}\circ\theta)^{**}(\beta)$ is a section of the sheaf $\pi_1^{**}(\Omega^p_X)$. So we can apply the proposition 4.2.1 and obtain that $(\pi_1)_*((\tilde{f}\circ\theta)^{**}(\beta)\wedge w)$ is a section of the sheaf α^p_X. This will give the definition of $\hat{f}^*(\alpha)$ when we shall have proved that it is independent of the choice of the special desingularization $\tau : \tilde{Y} \to Y$.

Note that the proposition 4.2.1 already gives the independence of the choice of w (assumed $d-$closed and satisfying (@)) in this construction.

The proposition 4.2.1 gives also that for α a section of $\Omega^p_Y/\text{torsion}$ $\hat{f}^*(\alpha)$ is a section of $\Omega^p_X/\text{torsion}$ and coincides with the usual pull-back $f^*(\alpha)$ (see section 1).

Remark now that, as the sheaf α^p_X has no torsion on X, to prove the independence of $\hat{f}^*(\alpha)$ on the choice of the special desingularization τ, it is enough to prove it at the generic points of X. Moreover, this problem is local on X and so we may assume that X is smooth and connected.

In our construction, we sum the various direct images $(\pi_j)_*(\tilde{f}^*(\beta)\wedge w_j)$ when j describes the various connected components of the desingularization of $\tilde{X}$. Each such component is sent by $\tilde{f}$ in a connected component of $\tilde{Y}$ and then it is enough to show the invariance of the current $(\pi_j)_*(\tilde{f}^*(\beta)\wedge w_j)$ if we change only one connected component of $\tilde{Y}$ in the given special desingularization, and also if we consider only the corresponding connected components of the special desingularization of $\tilde{X}$. So, in fact, it is enough to prove the following special case of our problem:

Assume that X is smooth and connected and that Y is irreducible. Let $\tau : \tilde{Y} \to Y$ be a special desingularization of Y and let $\theta : \tilde{\tilde{Y}} \to \tilde{Y}$ be a proper smooth modification of $\tilde{Y}$. So our new special desingularization of Y will be $\tau\circ\theta : \tilde{\tilde{Y}} \to Y$.

Now we shall consider the following diagram, where $\tilde{X}$ is a special desingularization of an irreducible component of the strict transform $X\times_Y \tilde{Y}$ and $\tilde{\tilde{X}}$ is a special desingularization of the strict transform of $\tilde{X}\times_{\tilde{Y}} \tilde{\tilde{Y}}$:

$$\begin{array}{ccc} \tilde{\tilde{X}} & \xrightarrow{\tilde{\tilde{f}}} & \tilde{\tilde{Y}} \\ \downarrow{\scriptstyle\tilde{\theta}} & & \downarrow{\scriptstyle\theta} \\ \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ \downarrow{\scriptstyle\tilde{\tau}} & & \downarrow{\scriptstyle\tau} \\ X & \xrightarrow{f} & Y \end{array}$$

Let q the dimension of the generic fibres of $\tilde{\tau}$ and k the number of connected components of its generic fibres. Let ω be a kähler form of $\tilde{Y}$ normalized in order that the form $\tilde{f}^*(\omega^{\wedge q})$ satisfies the condition (@) for the map $\tilde{\tau}$. Let $\tilde{q}$ be the dimension of the generic fibre of $\tilde{\theta}$ and let $\tilde{\omega}$ a kähler form on $\tilde{\tilde{Y}}$ normalized in order that the form $\tilde{\tilde{f}}^*(\tilde{\omega}^{\wedge\tilde{q}})$ satisfies the condition (@)

for the map $\tilde{\theta}$. Now consider the $(q+\tilde{q}, q+\tilde{q})-$smooth form $w := \tilde{\tilde{f}}^*(\theta^*(\omega^{\wedge q}) \wedge \tilde{\omega}^{\tilde{q}})$ on $\tilde{\tilde{X}}$ which is $d-$closed. It satisfies the condition (@) for the map $\tilde{\tau} \circ \tilde{\theta}$.

So the definition of $\hat{f}^*(\alpha)$ using the special desingularization $\tau \circ \theta$ is given by

$$(\tilde{\tau} \circ \tilde{\theta})_*\big((\theta \circ \tilde{\tilde{f}})^*(\beta) \wedge w\big),$$

But, as $\tilde{f}^{**}(\beta)$ is a section of the sheaf $\Omega^p_{\tilde{X}}/\text{torsion}$, we have the equality

$$\tilde{\theta}_*(\tilde{\theta}^{**}(\tilde{f}^{**}(\beta)) \wedge \tilde{\tilde{f}}^*(\tilde{\omega}^{\tilde{q}})) = \tilde{f}^{**}(\beta)$$

and the conclusion follows from the fact that

$$(\tilde{\tau} \circ \tilde{\theta})_*\big((\theta \circ \tilde{\tilde{f}})^{**}(\beta) \wedge w\big) = \tilde{\tau}_*[\tilde{\theta}_*\big(\tilde{\theta}^{**}(\tilde{f}^{**}(\beta)) \wedge \tilde{\tilde{f}}^*(\tilde{\omega}^{\tilde{q}})\big) \wedge \tilde{f}^*(\omega^q))].$$

The compatibility of this construction with the pull-back of holomorphic forms modulo torsion which is given by the last assertion of the proposition 4.2.1 obviously gives that the injective $\mathcal{O}_X-$linear morphism

$$\eta_X : \Omega^\bullet_X/\text{torsion} \to \alpha^\bullet_X$$

for each $X \in \mathcal{C}$ gives the commutative diagram (2) of the precise formulation 4.1.2 of the theorem for each morphism $f : X \to Y$ in $\mathcal{C}$.

Now we have to prove the functoriality of $\hat{f}^*$. Then consider a holomorphic maps $f : X \to Y$ and $g : Y \to Z$. We want to prove the formula (1) of the theorem.
Consider the commutative diagram

$$\begin{array}{ccccc}
\tilde{\tilde{X}} & \xrightarrow{\tilde{\tilde{f}}} & \tilde{\tilde{Y}} & & \\
\downarrow{\scriptstyle\tilde{\theta}} & & \downarrow{\scriptstyle\theta} & & \\
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} & \xrightarrow{\tilde{g}} & \tilde{Z} \\
\downarrow{\scriptstyle\tau_2} & & \downarrow{\scriptstyle\tau_1} & & \downarrow{\scriptstyle\tau} \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z
\end{array}$$

where $\tau : \tilde{Z} \to Z$ is a special desingularization, where $\tilde{g} : \tilde{Y} \to \tilde{Z}$ is the strict transform of g by τ, where $\tilde{f} : \tilde{X} \to \tilde{Y}$ is the strict transform of f by τ_1, where
$\tilde{\tilde{f}} : \tilde{\tilde{X}} \to \tilde{\tilde{Y}}$ is the strict transform of $\tilde{f}$ by $\theta : \tilde{\tilde{Y}} \to \tilde{Y}$ which is a special desingularization of $\tilde{Y}$. Let α be a section of α^p_Z, note $\beta := \tau^{**}(\alpha) \in \tau^{**}(\Omega^p_Z)$[6] and let w_1 and w_2 be smooth $d-$closed forms satisfying the condition (@) of the proposition 4.2.1 for the maps τ_1 and $\tilde{\theta}$ respectively. We have

$$\hat{g}^*(\alpha) = (\tau_1)_*(\tilde{g}^{**}(\beta) \wedge w_1)$$

but we have also, because $\tilde{g}^{**}(\beta)$ is a section of $\tau_1^{**}(\Omega^p_Y)$

$$\hat{g}^*(\alpha) = (\tau_1 \circ \theta)_*(\theta^{**}(\tilde{g}^{**}(\beta) \wedge \theta^*(w_1))).$$

Then we obtain

$$\hat{f}^*(\hat{g}^*(\alpha)) = (\tau_2 \circ \tilde{\theta})_*\big(\tilde{\tilde{f}}^{**}(\theta^{**}(\tilde{g}^{**}(\beta))) \wedge \tilde{\tilde{f}}^*(w_1) \wedge w_2\big).$$

[6]See the simple lemma 4.2.3 below.

As the square

$$\begin{array}{ccc} \tilde X & \xrightarrow{\tilde g\circ\tilde f} & \tilde Z \\ {\scriptstyle\tau_2}\downarrow & & \downarrow{\scriptstyle\tau} \\ X & \xrightarrow{g\circ f} & Z \end{array}$$

is also the strict transform of $g\circ f$ by τ we have

$$\widehat{g\circ f}^*(\alpha) = (\tau_2)_*\big((\tilde g\circ\tilde f)^{**}(\beta)\wedge \tilde f^*(w_1)\big).$$

Then the conclusion follows from the equality

$$\tilde\theta_*\big(\tilde{\tilde f}^{**}(\theta^{**}(\tilde g^{**}(\beta)))\wedge \tilde{\tilde f}^{**}(\theta^*(w_1))\wedge w_2\big) = \tilde f^{**}(\tilde g^{**}(\beta))\wedge \tilde f^*(w_1)$$

obtained by the comparaison of both hand-sides at the generic points of $\tilde X$. $\square$

Our next result shows that the sheaf $\alpha_X^\bullet$ is "maximal" in order to construct the pull-back via the method of the proposition 4.2.1.

Proposition 4.2.2. *Let $\pi : Z \to X$ be a proper surjective holomorphic map between irreducible complex spaces. Put $q := \dim Z - \dim X$. Let $\beta \in \alpha_Z^p$ be equal to $\hat\pi^*(\alpha)$ for a section α of the sheaf α_X^p. Let also w be a smooth $(q,q)-$form on Z which is $d-$closed and satisfies the condition (@) of the proposition 4.2.1 for the map π. Then the $(p,0)-$current $\pi_*(\beta\wedge w)$ on X (which is $\bar\partial-$closed and independent of the choice of w satisfying $dw = 0$ and (@); see proposition 4.2.1) is equal to the image in ω_X^p of the section α of the sheaf α_X^p.*

Using the "pull-back" theorem 4.1.1 the proof of the result above will follow from this simple lemma.

Lemma 4.2.3. *Let X be a reduced complex space and $\tau : \tilde X \to X$ a desingularization of X. Then the image of the pull-back $\hat\tau^* : \tau^*(\alpha_X^\bullet) \to \alpha_{\tilde X}^\bullet = \Omega_{\tilde X}^\bullet$ is the subsheaf $\tau^{**}(\Omega_X^\bullet)$ of $\Omega_{\tilde X}^\bullet$.*

Proof. By definition, a section of this image is locally on $\tilde X$ a $\mathcal{O}_{\tilde X}-$linear combination of holomorphic forms on $\tilde X$ which are locally $\mathcal{O}_{\tilde X}-$linear combinations of pull-back by τ of holomorphic forms on X. So the conclusion is clear. $\square$

Remark. As a consequence of the previous lemma, if τ is a special desingularization of X we have $\tau_*\hat\tau^*$ is the identity on the sheaf $\alpha_X^\bullet$.

Proof of the proposition 4.2.2. Let $\tilde\pi : \tilde Z \to \tilde X$ be the strict transform of π by τ, and denote by $\tilde\tau : \tilde Z \to Z$ the corresponding projection on Z which is a modification. So we have the following commutative diagram

$$\begin{array}{ccc} \tilde Z & \xrightarrow{\tilde\pi} & \tilde X \\ {\scriptstyle\tilde\tau}\downarrow & & \downarrow{\scriptstyle\tau} \\ Z & \xrightarrow{\pi} & X \end{array}$$

The $(q,q)-$form $\tilde\tau^*(w)$ is smooth and $d-$closed in $\tilde Z$ and satisfies the condition (@) of the proposition 4.2.1 for the proper surjective holomorphic map $\tau\circ\tilde\pi$. As we can write $\beta = \hat\pi^*(\alpha)$ where α is a section of α_X^p, we have, by functoriality of the pull-back for the sheaf $\alpha_Y^\bullet$ and the equality $\tau\circ\tilde\pi = \pi\circ\tilde\tau$

$$\hat{\tilde\tau}^*(\beta) = \hat{\tilde\pi}^*(\hat\tau^*(\alpha)).$$

But, thanks to the previous lemma, we have $\hat{\tau}^*(\alpha)$ which is a section of $\tau^{**}(\Omega_X^p)$ and using the smoothness of $\tilde{X}$ we have $\widehat{\tilde{\pi}}^* = \tilde{\pi}^{**}$. Then we obtain, using the lemma 1.0.4, the fact that $\hat{\tilde{\tau}}^*(\beta)$ is a section of the sheaf $(\tau \circ \tilde{\pi})^{**}(\Omega_X^p)$. Then the proposition 4.2.1 applies to the map $\tau \circ \tilde{\pi} : \tilde{Z} \to X$ with the form $\tilde{\tau}^*(w)$ and the section $\hat{\tilde{\tau}}^*(\beta)$ of the sheaf $(\tau \circ \tilde{\pi})^{**}(\Omega_X^p)$ and gives that the $(p, 0)-$current on X given by $\sigma := (\tau \circ \tilde{\pi})_*(\hat{\tilde{\tau}}^*(\beta) \wedge \tilde{\tau}^*(w))$ is $\bar{\partial}-$closed on X and is a section of the sheaf α_X^p.

But the $(p+q, q)-$current $\tilde{\tau}_*(\hat{\tilde{\tau}}^*(\beta) \wedge \tilde{\tau}^*(w))$ is equal to $\beta \wedge w$ at least over the generic points in X, the $(p, 0)-$current $\pi_*(\beta \wedge w)$ is $\bar{\partial}-$closed in X and generically equal to σ and α. So α and σ are equal as sections of the sheaf α_X^p. □

5. Integration on cycles

5.1. **Integrals.**

Notations. Let V be a complex manifold and h be a continuous hermitian form on V. So h is a real continuous positive definite $(1, 1)-$ differential form on V. If ω is a continuous $(p, p)-$form on V, we shall consider ω as a continuous sesqui-linear form on $\Lambda^p(T_V)$ and we shall write

$$\|\omega\|_K \leq C.h^{\wedge p}$$

where K is a subset in V and $C > 0$ a constant, if for any point $x \in K$ and any $v_1, \dots, v_p \in T_{V,x}$ the inequality

$$|\omega(x)[v_1 \wedge \dots \wedge v_p]| \leq C.h^{\wedge p}(x)[v_1 \wedge \dots \wedge v_p]$$

holds. For instance, if $\alpha, \beta \in \Omega_V^p$ we shall write $\|\alpha \wedge \bar{\beta}\|_K \leq C_K.h^{\wedge p}$ when for any $x \in K$ and any $v_1, \dots, v_p \in T_{V,x}$ we have

$$(1) \qquad |\alpha(x)[v_1 \wedge \dots \wedge v_p]|.|\beta(x)[v_1 \wedge \dots \wedge v_p]| \leq C_K.h^{\wedge p}(x)[v_1 \wedge \dots \wedge v_p].$$

Remark. If $f : W \to V$ is a holomorphic map and if (1) holds then we shall have

$$(2) \qquad \|f^*(\alpha) \wedge \overline{f^*(\beta)}\|_{f^{-1}(K)} \leq C_K.f^*(h)^{\wedge p}$$

but, in general, $f^*(h)$ is still positive but no longer definite on W.

Conversely if (2) holds on a set L in W then (1) is satisfied on $f(L)$.

Proposition 5.1.1. *Let X be a reduced complex space, let S be the singular set in X and let h be a continuous hermitian metric on X. Let U be a relatively compact open set in X. For all $\alpha, \beta \in \alpha_X^p$ there exists a constant $C_U > 0$ such that the following inequality holds at each point in $\bar{U} \setminus S$*

$$\|\alpha \wedge \bar{\beta}\|_{\bar{U} \setminus S} \leq C_U.h_{\bar{U} \setminus S}^{\wedge p}.$$

Proof. Remark that the problem is local on the compact set $\bar{U} \cap S$ because near smooth points in X the assertion obviously holds. Let $\tau : \tilde{X} \to X$ be a special desingularization of X. Then we shall show that for each point $y \in \tau^{-1}(\bar{U} \cap S)$ there exists an open neighbourghood W of y in $\tilde{X}$ and a positive constant C_W such that the inequality

$$\|\tau^{**}(\alpha) \wedge \overline{\tau^{**}(\beta)}\|_W \leq C_W.\tau^{**}(h)^{\wedge p}$$

holds: if y is a point in $\tilde{X}$ we can write in an open neighbourghood W of y

$$\alpha = \sum_{|I|=p} g_I.\tau^{**}(dx^I) \quad \text{and} \quad \beta = \sum_{|I|=p} h_I.\tau^{**}(dx^I)$$

where $x_1, \dots, x_N$ are local coordinates in a closed embedding of an open set $U \subset\subset X$ in $\mathbb{C}^N$ near $\tau(y)$. Our estimates is consequence of the facts that the holomorphic functions g_I and h_I are locally bounded and that for any (I, J) there is a constant $c_U^{I,J} > 0$ with

$$\|dx^I \wedge \overline{dx^J})\|_U \leq c_U^{I,J}.h^{\wedge p}$$

because we can assume that h is induced by a continuous hermitain form on $\mathbb{C}^N$.

Now the properness of τ allows to find a a constant C_U such that the inequality

$$\|\tau^{**}(\alpha) \wedge \overline{\tau^{**}(\beta)}\|_K \leq C_U.\tau^{**}(h)^{\wedge p}$$

holds on the compact set $K := \tau^{-1}(\bar{U})$. This allows to conclude thanks to the remark above. □

Corollary 5.1.2. *Let X be a complex space of pure dimension n, and let α, β be sections on X of the sheaf L_X^n. Then, if ρ is a continuous compactly supported function on X the integral*

$$\int_{X \setminus S} \rho.\alpha \wedge \bar{\beta}$$

is absolutely convergent for any closed analytic subset S containing the singular set in X and its value does not depends on the choice of S.
Now fix a continuous hermitian metric h on X and a compact set K in X. If α and β are sections of the sheaf α_X^n, there is constant $C > 0$ depending on α, β, h and K such that for any $\rho \in \mathscr{C}_K^0(X)$ we have

$$\Big|\int_{X \setminus S} \rho.\alpha \wedge \bar{\beta}\Big| \;\leq\; C.\int_X |\rho|.h^{\wedge n} \leq C.||\rho||.\int_{\mathrm{Supp}\,\rho} h^{\wedge n}. \tag{3}$$

Proof. The first part is consequence of the fact that $\tau^{**}(\alpha)$ and $\tau^{**}(\beta)$ are holomorphic $n-$forms on $\tilde{X}$. The estimates when α, β are sections of α_X^n is a direct consequence of the previous proposition. □

Remarks.

(1) Of course, in the second part of the corollary we may replace ρ by the characteristic function of an open subset $V \subset K$ in order to obtain, with the same constant C independent on the choice of V, the estimate

$$|\int_{V \setminus S} \alpha \wedge \bar{\beta}| \;\leq\; C.\int_V h^{\wedge n}. \tag{3 bis}$$

(2) Note that the estimations (3) or ($3bis$) do not hold in general when α and β are sections in the sheaf L_X^n. For instance let

$$X := \{(x, y, z) \in \mathbb{C}^3 \;/\; x.y = z^2\} \quad \text{and} \quad \alpha = \beta = \frac{dx \wedge dy}{z}.$$

They are sections of the sheaf L_X^2 but not sections of the sheaf α_X^2 (see the example with $k = 2$ in the paragraph 6.2); let $K := \{|x| \leq 1\} \cap \{|y| \leq 1\}$ in X and let h be the metric induced on X by the standard kähler form on $\mathbb{C}^3$. Then we have

$$\int_{V(r)} \alpha \wedge \bar{\alpha} = \gamma.r^2$$

where $V(r) := \{|x| \leq r\} \cap \{|y| \leq r\} \cap X$ and $\int_{V(r)} h^{\wedge 2} = \delta.r^4$ for any $r \in]0, 1[$, showing that the estimate ($3bis$) cannot hold.

Definition 5.1.3. *For α, β sections of the sheaf L^n_X the common values of the absolutely convergent integrals $\int_{X\setminus S} \ \rho.\alpha\wedge\bar\beta$ will be denoted simply by $\int_X \ \rho.\alpha\wedge\bar\beta$.*

Lemma 5.1.4. *Let $f : Y \to X$ a proper generically finite and surjective holomorphic map between two complex spaces of pure dimension n; let k be the generic degree of π. Let α, β be sections on X of the sheaf L^n_X and $\rho \in \mathscr{C}^0_c(X)$. Then the holomorphic $n-$forms $f^{**}(\alpha)$ and $f^{**}(\beta)$ are well defined on a dense Zariski open set in Y and extend as sections on Y of the sheaf L^n_Y. We have the equality*

$$\int_X \ \rho.\alpha\wedge\bar\beta = k.\int_Y \ f^*(\rho).f^{**}(\alpha)\wedge\overline{f^{**}(\beta)}.$$

Proof. Remark that it is enough to prove the lemma for $\alpha = \beta$. Let $\tau : \tilde X \to X$ be a desingularization of X. As $\tau^{**}(\alpha)$ is an holomorphic $n-$form on $\tilde X$ the form α is locally L^2 on X. Let H_ε be an open $\varepsilon-$neighbourhood of H a closed analytic subset in X such that the map $f : Y \setminus f^{-1}(H) \to X \setminus H$ is a finite covering between two complex manifolds. Then the usual change of variable gives, if ρ is in $\mathscr{C}^0_c(X)$

$$\int_{X\setminus H_\varepsilon} \ \rho.\alpha\wedge\overline{\alpha} = k.\int_{Y\setminus f^{-1}(H_\varepsilon)} \ f^*(\rho).f^{**}(\alpha)\wedge\overline{f^{**}(\alpha)}.$$

Letting ε goes to 0 shows that $f^{**}(\alpha)$ is locally L^2 on any desingularization of Y and so $f^{**}(\alpha)$ is a section of the sheaf L^n_Y. The conclusion follows easily. □

Definition 5.1.5. *Let X be a complex space and let $Y \subset X$ be an irreducible $p-$dimensional analytic subset in X. We shall denote $j : Y \to X$ the the inclusion. Let α, β be sections of the sheaf α^p_X on X and ρ be a continuous function with compact support in X. We define the number $\int_Y \ \rho.\alpha\wedge\overline{\beta}$ as the integral*

$$\int_Y \ j^*(\rho).\hat j^*(\alpha)\wedge\overline{\hat j^*(\beta)}.$$

Note that this definition makes sense because the pull-back $\hat j^* : j^*(\alpha^p_X) \to \alpha^p_Y$ is well defined and because the inclusion $\alpha^p_Y \subset L^p_Y$ allows to use the definition 5.1.3. Remark that this definition only depends on the irreducible analytic subset Y of X. So we may extend by additivity the definition of the integral

$$\int_Y \ \rho.\alpha\wedge\overline{\beta}$$

to any $p-$dimensional cycle Y in X.

The next lemma shows that the change of variable holds for such a integral.

Lemma 5.1.6. *Let $f : X \to Y$ be a holomorphic map and let α, β be sections on Y of the sheaf α^p_Y. Let ρ be a continuous compactly supported function on Y. Let Z be a $p-$cycle in X and assume that the cycle $f_*(Z)$ is defined in Y[7]. Then the restriction to $|Z|$ of the continuous function $f^*(\rho)$ has compact support and the integral $\int_Z \ f^*(\rho).\hat f^*(\alpha)\wedge\overline{\hat f^*(\beta)}$ is well defined and we have*

$$\int_Z \ f^*(\rho).\hat f^*(\alpha)\wedge\overline{\hat f^*(\beta)} = \int_{f_*(Z)} \ \rho.\alpha\wedge\bar\beta.$$

[7]This means that the restriction of f to $|Z|$ is proper; see [B-M 1] chapter IV.

PROOF. First remark that any irreducible component Γ of Z which has an image of dimension at most equal to $p-1$ does not contribute to the right hand-side and also to the left hand-side because the forms $\hat{f}^*(\alpha)$ and $\hat{f}^*(\beta)$ vanish on such a irreducible component:

Let $g : \Gamma \to f(\Gamma)$ be the map induced by f; by functoriality of the pull-back $\hat{g}^*$ factorizes through $\alpha^p_{f(\Gamma)}$ which is zero.

Then the result is in fact a local statement near each point of the support of the cycle $f_*(Z)$. And because of our previous remark and the fact that closed analytic subsets with no interior point can be neglected in the integrals, it is enough to prove the result when Z is smooth and when f induces an isomorphism of Z on $f(Z)$. In this case, which is not trivial because Z and $f(Z)$ can be contained in the singular sets of X and Y, the functorial property of the pull-back and the fact that for a complex manifold V we have $\alpha^p_V = \Omega^p_V$ allow to conclude. □

Theorem 5.1.7. *Let X be a reduced complex space and let $(Y_t)_{t\in T}$ be an analytic family of $p-$cycles in X parametrized by a reduced complex space T. Fix a compact set K in X and let α, β be sections of the sheaf α^p_X on X. Let ρ be a continuous function with a compact support in K and define the function $\varphi : T \to \mathbb{C}$ by*

$$\varphi(t) := \int_{Y_t} \rho.\alpha \wedge \overline{\beta}.$$

Then φ is locally bounded and for any given hermitian metric h on X and any compact set L in T there exists a constant C depending only on K, α, β, h and L (but not on the choice of ρ) such that the following estimate holds for each $t \in L$:

$$\text{(E)} \qquad |\varphi(t)| \ \leq \ C.\int_{Y_t} |\rho|.h^{\wedge p} \ \leq \ C.||\rho||.\int_{Y_t\cap \mathrm{Supp}\,\rho} h^{\wedge p}.$$

Moreover for each point $t_0 \in T$ there exists an open neighbourhood T_0 of t_0 in T and a closed analytic subset $\Theta_0 \subset T_0$ with no interior point in T_0 such that φ is continuous on $T_0 \setminus \Theta_0$.

PROOF. We shall cut this proof in several steps.

STEP 1. Let $\nu : \tilde{T} \to T$ the normalization of T. The family $(Y_{\nu(\tilde{t})})_{\tilde{t}\in\tilde{T}}$ is an analytic family of $p-$cycles in X parametrized by $\tilde{T}$, and if the theorem is proved for this family it implies the result for the initial family, because the function is constant on the fibres of the normalization map.

So we shall assume that T is normal in the sequel.

STEP 2. If the generic cycle Y_t is not reduced and irreducible, the normality of T allows to write the family $(Y_t)_{t\in T}$ as a finite sum of analytic families of $p-$cycles in X parametrized by T such that the sum of these families is our initial family and such that the generic cycle in each family is reduced and irreducible (see ch. IV theorem 3.4.1 of [B-M 1]). So it is enough to prove the theorem for such a family.

So we shall assume that for t generic in T the cycle Y_t is reduced and irreducible.

STEP 3. Let $G \subset T \times X$ the cycle-graph of our analytic family. It is a reduced and irreducible cycle and the projection $\pi : G \to T$ is (by definition) a geometrically flat map, that is to say that there exists an analytic family of cycles $(Z_t)_{t\in T}$ in G such that for each $t \in T$ we have $|Z_t| = \pi^{-1}(t)$ and such that the generic cycle Z_t is reduced and irreducible. Of course, here we have $Z_t := \{t\} \times Y_t$ for each $t \in T$.

Note $pr : G \to X$ the projection and define on G the sections of the sheaf α^p_G by letting $\alpha_1 := \hat{pr}^*(\alpha)$ and $\beta_1 := \hat{pr}^*(\beta)$. Then, it is enough to prove the theorem for the function $t \mapsto \int_{Z_t} \tilde{\rho}.\alpha_1 \wedge \bar{\beta}_1$ where $\tilde{\rho} := pr^*(\rho)$ thanks to the change variable theorem proved in lemma

5.1.6. Remark that pr induces an isomorphism of $|Z_t|$ onto $|Y_t|$ for each $t \in T$ and also that the continuous function $\tilde{\rho}$ on G has a $\pi-$proper support.

STEP 4. Let $\tau : \tilde{G} \to G$ be a special desingularization of G. Define the subset

$$\Theta := \{t \in T \ / \ \exists y \in K \ \dim_y (pr \circ \tau)^{-1}(t) \geq p+1\}.$$

This is a locally closed analytic subset[8] in T with no interior point. For a given $t_0 \in T$, fix an open neighbourhood T_0 of t_0, small enough in order that $\Theta_0 := \Theta \cap T_0$ is a closed analytic subset. The map

$$q : \tilde{G} \cap (pr \circ \tau)^{-1}(T_0) \setminus (pr \circ \tau)^{-1}(\Theta) \to T_0 \setminus \Theta_0$$

is $p-$equidimensional on a normal basis, so it is geometrically flat and we have an analytic family $(\tilde{Z}_t)_{t \in T_0 \setminus \Theta_0}$ of fibres of q which are $p-$cycles in $\tilde{G}$, and for t generic in $T_0 \setminus \Theta_0$ the cycle $\tilde{Z}_t$ is irreducible.

Note that the pull-back of α_1 and β_1 on $\tilde{G} \cap (pr \circ \tau)^{-1}(T_0)$ are holomorphic $p-$forms. So, by the usual result of the continuity of integration of a continuous form on a continuous family of cycles (see [B-M 1] ch. IV prop. 2.3.1), we conclude using the lemma 5.1.6 that the function φ is continuous on $T_0 \setminus \Theta_0$.

STEP 5. The local boundness on T of the function φ is given by the corollary 5.1.2 which gives the estimate (E) by integration. $\square$

REMARKS.

(1) In the case of a proper family of compact cycles in X, it is easy, using results of [B-M 1] chapter IV, to prove that the function φ becomes continuous after a suitable modification of the complex space T.

(2) Already in the case of the normalization map, if α is a locally bounded meromorphic function on X, the function $x \mapsto |\alpha(x)|^2$ is not continuous on X in general.

5.2. **Normalized Nash transform.** Let us begin by two examples.

TWO EXAMPLES.

(1) We shall show in section 6.2 that for $k \geq 2$ and $k-1 \geq q \geq k/2$ the form

$$\omega_q := z^q.(dx/x - dy/y)$$

is a section of the sheaf $\alpha^1_{S_k}$ where

$$S_k := \{(x,y,z) \in \mathbb{C}^3 \ / \ x.y = z^k\}$$

which are not sections of the sheaf $\Omega^1_{S_k}$/torsion.
But as we have $dx/x + dy/y = k.dz/z$ on S_k we obtain the equality

$$\omega_q^2 = k^2.z^{2q-2}.(dz)^2 - 4z^{2q-k}.dx.dy;$$

so ω_q^2 is equal, for $q \geq k/2$, modulo torsion to a section of $S^2(\Omega^1_{S_k})$, the piece of degree 2 in the symmetric algebra of the sheaf $\Omega^1_{S_k}$.

(2) We shall show in section 6.4 that on $X := \{(x,y,u,v) \in \mathbb{C}^4 \ / \ x.y = u.v\}$ the form $a := u.dv \wedge dx/x$ is a section of the sheaf α^2_X which is not in Ω^2_X/torsion. But using the following identities on X:

$$u.dv \wedge dx/x + u.dv \wedge dy/y = dv \wedge du$$
$$u.dv \wedge dy/y + v.du \wedge dy/y = dx \wedge dy$$

[8] See, for instance, the lemma 2.1.8 in [B.15].

we obtain that

$$a^2 + a.(du \wedge dv + dx \wedge dy) - (dv \wedge dx).(du \wedge dy) = 0$$

which is a homogeneous integral dependence equation for a on the symmetric algebra of the sheaf $\Omega^2_X/\text{torsion}$.

The next proposition will show that these examples are special cases of a general phenomenon.

Proposition 5.2.1. *Let X be a normal complex space. Then for each integer q the sheaf α^q_X is the sub-sheaf of meromorphic sections of the sheaf $\Omega^q_X/\text{torsion}$ which satisfy a homogeneous integral dependence equation over the sheaf $S^\bullet(\Omega^q_X)$, the symmetric algebra of the sheaf $\Omega^q_X/\text{torsion}$.*

Proof. This is a special case of the proposition 2.2.4. □

Notation. For integers $n < N$ we shall denote $Gr(n, N)$ the grassmannian manifold of sub-vector spaces in $\mathbb{C}^N$ of dimension n.

Let X be a reduced complex space pure of dimension n and let S its singular locus. Assuming that X is embedded in an open set U in $\mathbb{C}^N$ we have a holomorphic map

$$\theta : X \setminus S \to Gr(n, N)$$

sending each point $x \in X \setminus S$ to the $n-$dimensional vector sub-space of $\mathbb{C}^N$ which directs the tangent space at x to X. This map is holomorphic on $X \setminus S$ and meromorphic along S: assuming that X is locally defined by $\{f = 0\}$ in an open set in $\mathbb{C}^N$ the analytic subset $G \subset \tilde{G} := \{(x, P) \in X \times Gr(n, N) \ / \ P \subset Ker[df_x]\}$, which is the union of the irreducible components of $\tilde{G}$ which contain an irreducible component of the graph of the map θ, is a proper modification of X which is the closure of the graph of the map θ.

We shall note $\mathcal{N} : \hat{X} \to X$ the projection on X of the normalization of G. We shall call the (local) **normalized Nash transform** of X this modification.

Let $\pi : \mathcal{U} \to Gr(n, N)$ the universal $n-$vector bundle of $Gr(n, N)$ and let $\mathcal{L}^q$ be the sheaf of section of the dual vector bundle to $\Lambda^q(\mathcal{U})$. Let $pr : \hat{X} \to Gr(n, N)$ be the projection.

Proposition 5.2.2. *For each integer q there is a canonical isomorphism*

$$c^q : \mathcal{N}^*(\alpha^q_X)/\text{torsion} \to pr^*(\mathcal{L}^q).$$

Proof. This proposition is an easy consequence of Corollary 2.2.3 and Lemma 2.1.3. □

As a consequence of this proposition we obtain that for a normal complex space we have $\alpha^q_X \simeq \mathcal{N}_*(\mathcal{L}^q)$ for any integer $q \geq 0$.

Lemma 5.2.3. *Let X be a reduced complex space and let $\tau : \tilde{X} \to X$ be any (proper) modification. Then we have a natural inclusion $\alpha^\bullet_X \hookrightarrow \tau_*(\alpha^\bullet_{\tilde{X}})$.*

Proof. Consider a special desingularization $\theta : \tilde{\tilde{X}} \to \tilde{X}$ and remark that $\pi := \tau \circ \theta$ is a desingularization of X. Then we have

$$\alpha^\bullet_X = \pi_*(\pi^{**}(\Omega^\bullet_X) = \tau_*(\theta_*\big(\theta^{**}(\tau^{**}(\Omega^\bullet_X))\big).$$

Now the equality $\alpha^\bullet_{\tilde{X}} = \theta_*(\theta^{**}(\Omega^\bullet_{\tilde{X}}))$ and the inclusion $\tau^{**}(\Omega^\bullet_X) \subset \Omega^\bullet_{\tilde{X}}/\text{torsion}$ give

$$\theta^{**}(\tau^{**}(\Omega^\bullet_X) \subset \theta^{**}(\Omega^\bullet_{\tilde{X}})$$

$$\theta_*\big(\theta^{**}(\tau^{**}(\Omega^\bullet_X)\big) \subset \alpha^\bullet_{\tilde{X}} \quad \text{and then}$$

$$\alpha^\bullet_X \subset \tau_*(\alpha^\bullet_{\tilde{X}})$$

concluding the proof. □

REMARK. This shows that when we consider a sequence of successive modifications over a reduced complex space X, the sequence of coherent sub-sheaves $(\tau_\nu)_*(\alpha^\bullet_{X_\nu})$ is locally stationary on X. For instance, this is the case for iterated normalized Nash transforms over a given X.

6. Some examples

6.1. **Computation of $\omega_X^\bullet$ for hypersurfaces.** We shall need the following elementary lemma.

Lemma 6.1.1. *Let U be an open polydisc in $\mathbb{C}^n$ and D an open disc in $\mathbb{C}$. Let $X \subset U \times D$ be a reduced multiform graph of degree k in $U \times D$ with canonical equation $P \in \mathcal{O}(U)[z]$, which is a monic degree k polynomial in z. Then we have the inclusion*

$$\Gamma(X, \omega_X^q) \subset \sum_{j=0}^{k-1} \frac{z^j}{P'(z)}.\Gamma(U, \Omega_U^q)$$

with equality for $q = n$.

PROOF. First will shall prove the following formula, where $(j,h) \in [0, k-1]^2$:

$$\det_{j,h}\big[\mathrm{Trace}_{X/U}(\frac{z^{j+h}}{P'(z)})\big] = (-1)^{k.(k-1)/2}.$$

Assume, without loss of generality, that D is centered at the origin with radius R. Then for $r > R$ we have, thanks to Cauchy's formula

$$\mathrm{Trace}_{X/U}(\frac{z^m}{P'(z)}) = \frac{1}{2i\pi}.\int_{|z|=r} \frac{z^m.dz}{P(z)}.$$

Then for $m \leq k-2$ put $z = r.e^{i.\theta}$ we obtain

$$\mathrm{Trace}_{X/U}(\frac{z^m}{P'(z)}) = \frac{1}{2\pi}.\int_0^{2\pi} \frac{r^{m+1-k}.e^{i.(m+1-k)}.d\theta}{1 + O(1/r)}$$

and letting $r \to +\infty$ gives 0. For $m = k-1$ the same computation gives

$$\mathrm{Trace}_{X/U}(\frac{z^{k-1}}{P'(z)}) = \frac{1}{2\pi}.\int_0^{2\pi} \frac{d\theta}{1 + Q((1/r).e^{-i.\theta})}$$

Where Q is a polynomial without constant term.

So we obtain that $\mathrm{Trace}_{X/U}(\frac{z^{k-1}}{P'(z)}) = 1$. This is enough to get the formula above.

To prove the inclusion

$$\Gamma(X, \omega_X^q) \subset \sum_{j=0}^{k-1} \frac{z^j}{P'(z)}.\Gamma(U, \Omega_U^q),$$

take $\alpha \in \omega_X^q$ and write

$$\alpha = \sum_{|H|=q} g_H.dt^H$$

where g_H are degree $\leq k-1$ polynomials in z with meromorphic functions on U as coefficients. As we have $P'(z).dz = -\sum_{h=1}^n \frac{\partial P}{\partial t_h}.dt_h$ on X, this is possible. Now for any $f \in \mathcal{O}(X)$ we have

$\mathrm{Trace}_{X/U}[f.\alpha] \in \Omega^q(U)$ and this implies that for any $H \subset [1, n], \mathrm{Trace}_{X/U}[f.g_H]$ is in $\mathcal{O}(U)$. Let g be a meromorphic function on X and assume that we write

$$g = \sum_{j=0}^{k-1} a_j.\frac{z^j}{P'(z)}$$

where $a_j, j \in [0, k-1]$ is a meromorphic function on U. This is always possible for the g_H as we can see in what follows. Let $m_p := \mathrm{Trace}_{X/U}[z^p.g]$ for $p \in [0, k-1]$. Then we have the linear system in the $(a_j), j \in [0, k-1]$:

$$\sum_{j=0}^{k-1} a_j.\mathrm{Trace}_{X/U}[\frac{z^{p+j}}{P'(z)}] = m_p \quad \forall p \in [0, k-1].$$

But the determinant of this linear system is $(-1)^{k.(k-1)/2}$, so this implies, if we assume that the functions m_p are holomorphic on U, that the functions a_j for $j \in [0, k-1]$, are holomorphic in U and so that g is in $\frac{1}{P'(z)}.\mathcal{O}(X)$. Then our inclusion is proved, as $\mathcal{O}(X) = \sum_{j=0}^{k-1} \mathcal{O}(U).z^j$. Note that in the situation above, the condition in order that $\alpha = \sum_{j=0}^{k-1} \frac{z^j}{P'(z)}.\Omega^q(U)$ will be in $\omega^q(X)$ is that for any $j \in [0, k-1]$ the $(q+1)-$forms

$$\mathrm{Trace}_{X/U}[z^j.dz \wedge \alpha]$$

are holomorphic in U for all $j \in [0, k-1]$. This is consequence of the fact that for any $\beta \in \Omega^p(X)$ the $(p+q)-$form $\mathrm{Trace}_{X/U}[\alpha \wedge \beta]$ must be holomorphic (see [B. 78] for this characterization of the sheaf $\omega_X^\bullet$). For $q = n$ this extra condition is empty, so the equality occurs. □

Remark. For a general reduced multiform graph $X \subset U \times B$ where B is now a polydisc in $\mathbb{C}^p$, for any linear form l in $\mathbb{C}^p$ which separates generically the fibres of the projection $\pi : X \to U$, the map $id_U \times l : U \times B \to U \times \mathbb{C}$ is proper and generically injective on X. If we define $Y_l := (id_U \times l)(X)$, we are in the situation of the lemma above, and, as the direct image by π induces an injective sheaf map $\pi_* : \omega_X^\bullet \to \pi^*(\omega_{Y_l}^\bullet)$, we obtain the inclusion

$$\pi_*\omega_X^\bullet \subset \sum_{j=0}^{k-1} \frac{l(x)^j}{P_l'}.\Omega_U^\bullet$$

for any such l, where P_l is the canonical equation for Y_l (see [B-M 1] chapter II). Note that the canonical equation P_l is obtained from the canonical equation of the reduced multiform graph X by the evaluation at l (with $z = l(x)$); see *loc. cit.* □

Note that, if X is a reduced complex space of pure dimension n, a section $\alpha \in \omega_X^n$ is in L_X^n iff $\alpha \wedge \bar{\alpha}$ is locally integrable on X. The analogous characterization, for $p < n$, involves local integrability of $\alpha \wedge \bar{\alpha}$ on all $p-$dimensional irreducible analytic subset $Y \subset X$ not contained in the singular set of X; so it may be useful as a necessary condition but very difficult to check as a sufficient condition.

Preliminary remark. Let $\tau : \tilde{X} \to X$ be a desingularization of a reduced complex space X. Note S the singular set in X and assume that the center of τ is contained in S.

- Let $\alpha \in \omega_X^p$. To check if α is in L_X^p is equivalent to check if $\tau^*(\alpha)$, as a section of $\Omega_{\tilde{X}}^p$ on $\tau^{-1}(X \setminus S)$, extends to a section of $\Omega_{\tilde{X}}^p$ on $\tilde{X}$.
- For $\alpha \in L_X^p$ to check if α is a section of α_X^p is equivalent to check if $\tau^{**}(\alpha)$ extends to a section of $\tau^{**}(\Omega_X^p)$ **when τ is a special desingularization of** X. But this not true, in general, for an arbitrary desingularization of τ.

- But for any desingularization, it is a necessary condition in order that $\alpha \in L_X^p$ belongs to α_X^p that $\tau^{**}(\alpha)$ is a section of $\tau^{**}(\Omega_X^p)$ on $\tilde{X}$.

So, in order to have a complete description of the sheaf $\alpha_X^\bullet$, we shall use a special desingularization of X.

6.2. **The case** $X := \{(x, y, z) \in \mathbb{C}^3 \ / \ x.y = z^k\}, k \geq 2$**.**
NOTATION. After blow-up (x, y, z) in $\mathbb{C}^3$ the homogeneous coordinates in $\mathbb{P}_2$ will be (α, β, γ). The symetry between x and y allows to consider only the chart $\{\alpha \neq 0\}$ on which we put $b := \beta/\alpha, c := \gamma/\alpha$ and the chart $\{\gamma \neq 0\}$ on which we put $a := \alpha/\gamma, b := \beta/\gamma$.

Our first example will be the normal complex spaces, where $k \in \mathbb{N}, k \geq 2$

$$X := S_k := \{(x, y, z) \in \mathbb{C}^3 \ / \ x.y = z^k\}.$$

Note that S_0 and S_1 are smooth complex surfaces.

Lemma 6.2.1. *For any $k \geq 2$ the normal complex space S_k is nearly smooth*[9]*. So we have $L_{S_k}^\bullet = \omega_{S_k}^\bullet$ for any k.*

PROOF. Let ζ be a $k-$th primitive root of 1. Then S_k is isomorphic to the quotient of $\mathbb{C}^2$ by the action of the automorphism $\theta(u, v) = (\zeta.u, \zeta^{-1}.v)$. The quotient map is given by $q(u, v) = (u^k, v^k, u.v) \in \mathbb{C}^3$. $\square$

Now compute the sheaf ω_X^h for $h \in [0, 2]$. We have $\omega_X^0 = \mathcal{O}_X$ as X is normal, and $\omega_X^2 = \mathcal{O}_X.\frac{dx\wedge dy}{z^{k-1}}$. A rather easy computation shows that the quotient ω_X^1/Ω_X^1 is generated on $\mathcal{O}_X$ by the image of $x.dy/z^{k-1} = -y.dx/z^{k-1} + k.dz$ which is annihilated in this quotient by x, y and z^{k-1}.

Lemma 6.2.2. *For any $k \geq 2$, the sheaf $\alpha_{S_k}^2$ coincides with $\Omega_{S_k}^2/$torsion.*

PROOF. Remark that for $k = 0, 1$ the lemma is obvious as S_k is smooth. We shall prove the lemma by induction on $k \geq 2$.
We have to consider the case $k = 2$ first because it appears that the computation is special in this case (see the denominator $k - 2$ in the computation for $k \geq 3$).

For $k = 2$ after blowing-up the origin we have a smooth manifold:
CLAIM. This a special desingularization of S_2.
PROOF. In the chart $\{\alpha \neq 0\}$ we have $y = x.b, \quad z = x.c, \quad b = c^2 \quad$ so (x, c) is a coordinate system in this chart and the sheaf $\tau^*(\Omega_{S_2}^1)/$torsion is generated by dx and $x.dc$, so it is free.
In the chart $\{\gamma \neq 0\}$ we have $x = z.a, \quad y = z.b, \quad a.b = 1$ and so (z, a) is a coordinate system with $a \neq 0$. Then the sheaf $\tau^*(\Omega_{S_2}^1)/$torsion is generated by dz and $z.da$ which is also free. By symetry in x and y, the proof of the claim is complete.

Let us come back to the computation of $\alpha_{S_2}^2$.

In the chart $\{\alpha \neq 0\}$, we have

$$\frac{dx \wedge dy}{z} = 2.dx \wedge dc$$

[9]See [B-M. 17]

which is holomorphic but not in $\tau^*(\Omega^2_{S_2}) \simeq \mathcal{O}_{\tilde{X}}.x.dx\wedge dc$.

In the chart $\{\gamma \neq 0\}$, we have $x = z.a, \quad y = z.b, \quad a.b = 1$ and

$$\frac{dx\wedge dy}{z} = -2dz\wedge da/a$$

which is holomorphic but not in $\tau^{**}(\Omega^2_{S_2}) \simeq \mathcal{O}_{\tilde{X}}.z.dz\wedge da$.

The assertion is proved for $k = 2$.

Consider now the case $k = 3$. Then the blowing-up the origin gives a smooth manifold:

Claim. This desingularization of S_3 is not special.

Proof. In the chart $\{\alpha \neq 0\}$ we have $y = x.b, \quad z = x.c, \quad b = x.c^3 \quad$ so (x, c) is a coordinate system in this chart and the sheaf $\tau^*(\Omega^1_{S_3})\big/$torsion is generated by dx and $x.dc$ so it is free. But in the chart In the chart $\{\gamma \neq 0\}$ we have $x = z.a, y = z.b$ and $a.b = z$ and (a, b) is a coordinate system. As $x = a^2.b$ and $y = a.b^2$, the sheaf $\tau^*(\Omega^1_{S_2})\big/$torsion is generated by $d(a^2.b), d(a.b^2), d(a.b)$ and it is not locally free near the point $a = b = 0$.

But blowing-up the point $a = b = 0$ in the second chart make the pull-back of the sheaf $\tilde{\tau}^*(\Omega^1_{S_3})\big/$torsion locally free, where $\tilde{\tau}$ is the composition of τ and the blow-up of the point $a = b = 0$ in the second chart:

In the chart $a = \theta.b$ of this second blow-up the coordinate system is given by (b, θ) so $x = \theta^2.b^3, y = \theta.b^3, z = \theta.b^2$. Then the sheaf $\tilde{\tau}^*(\Omega^1_{S_3})\big/$torsion is generated by dx, dy, dz. An easy computation shows that $dx = -\theta.dy + 3\theta.b.dz$ so sheaf $\tilde{\tau}^*(\Omega^1_{S_3})\big/$torsion is free in this chart. The other chart is obtained by exchanging a and b.

Consider now the section $\frac{dx\wedge dy}{z}$ of $\omega^3_{S_3}$. Its pull-back by $\tilde{\tau}$ is given by

$$\frac{d(\theta^2.b^3)\wedge d(\theta.b^3)}{\theta.b^2} = -3\theta.b^3.db\wedge d\theta$$

and the generator of the sheaf $\tilde{\tau}^*(\Omega^2_{S_3})\big/$torsion is given by

$$\tilde{\tau}^*(dy\wedge dz) = \theta.b^4.db\wedge d\theta.$$

So we conclude that neither $\frac{dx\wedge dy}{z}$ nor $\frac{dx\wedge dy}{z^2}$ are in $\alpha^2_{S_3}$.

As the assertion is proved for $k = 2, 3$ we may assume that, for $k \geq 4$ the equality is proved for S_{k-2}. Then let $\tilde{X} \to X := S_k$ be the blow-up of S_k at the singular point $x = y = z = 0$. In the chart $\{\gamma \neq 0\}$ of $\tilde{X}$ we have the relations

$$x = a.z, \quad y = b.z \quad a.b = z^{k-2}$$

and we find a copy of S_{k-2}. For $k \geq 4$ we have

$$dx\wedge dy = \frac{k}{k-2}.z^2.da\wedge db = k.\frac{a.b}{k-2}.\frac{da\wedge db}{z^{k-4}},$$

$$dx\wedge dz = \frac{a}{k-2}.\frac{da\wedge db}{z^{k-4}}, \quad dy\wedge dz = \frac{b}{k-2}.\frac{da\wedge db}{z^{k-4}}.$$

So in this chart

$$\tau^{**}(\Omega^2_{S_k}/\text{torsion}) = \mathcal{O}_{S_{k-2}}.\big(a, b\big).\frac{da\wedge db}{z^{k-4}}$$

and, as a consequence of the fact that z^{k-q-2} is not in the ideal $\big(a, b\big).\mathcal{O}_{S_{k-2}}$ for $q \geq 1$, for each $q \geq 1$ the $2-$form $dx\wedge dy/z^q$ is not a section of the sheaf $\tau^{**}(\Omega^2_{S_k}/\text{torsion})$ near the origin

$a = b = z = 0$ in this chart. So the sheaf $\alpha^2_{S_k}$ is equal to $\Omega^2_{S_k}\big/$torsion. □

Lemma 6.2.3. *For all $k \geq 0$ the vector space $L^1_{S_k}\big/\alpha^1_{S_k}$ has dimension $p = [(k-1)/2]$ the integral part of $(k-1)/2$. A basis is given by the $1-$forms $x.dy/z^q$ for q in $[[k/2]+1, k-1]$, for $k \geq 2$.*

Proof. We shall begin by a simple remark.

Assume that $k \geq 2$ and let $p := [(k-1)/2]$. Then for any $q \in [1,p]$ the form $x.dy/z^q$ satisfies an integral dependence equation on $\Omega^1_{S_k}$. We have

$$x.dy/z^q + y.dx/z^q = d(z^k)/z^q = k.z^{k-q-1}.dz$$

and also

$$(x.dy/z^q).(y.dx/z^q) = z^{k-2q}.(dx).(dy).$$

This implies that $x.dy/z^q$ is solution of the integral dependence equation

$$X^2 - (k.z^{k-q-1}.dz).X + z^{k-2q}.(dx).(dy) = 0$$

in $S_2(\Omega^1_{S_k})\big/$torsion. So these sections of the sheaf ω^1_X are in fact sections of the sheaf α^1_X.

Now remark also that with the weights $x \to k, y \to k, z \to 2$ the form $x.dy/z^q$ has weight $2(k-q)$. Then they have different quasi-homogeneities, so they are linearly independent over $\mathbb{C}$. Let now prove that for $k-1 > q > p$ the form $x.dy/z^q$ is not in $\alpha^1_{S_k}$ by induction on $k \geq 0$. As the assertion is empty for $k = 0, 1$ assume $k \geq 2$ and the assertion proved for $k-2$.

We have seen that after blowing-up the singular point in S_k for any $k \geq 2$ we find only one singular point of the type S_{k-2} in the chart $\{\gamma \neq 0\}$ and that the form $x.dy/z^q$ is given by the following computation in this chart $\{\gamma \neq 0\}$:

$$x = z.a, \quad y = z.b, \quad a.b = z^{k-2} \quad x.dy/z^q = a.db/z^{q-2} + z^{k-q-1}.dz.$$

But on S_{k-2} we know, by the induction hypothesis, that the form $a.db/z^{q-2}$ is not a section of $\alpha^1_{S_{k-2}}$ for $q-2 > [\frac{k-3}{2}] = p-1$. So only the case $q = p+1$ is left.

Assume first that $k = 2p+1$. In the last chart $\{\gamma \neq 0\}$ in the desingularization process of S_{2p+1} by blowing up the unique singular point at each step, we reach the following relations:

$$x = u^p.v^{p+1}, y = u^{p+1}.v^p, z = u.v \quad x.dy/z^{p+1} = (p+1).u^{p-1}.v^p.du + p.u^p.v^{p-1}.dv$$

where $(u,v) \in \mathbb{C}^2$ is a local coordinate system.

But, as we have seen for $k = 3$ this desingularization is not special. So we have to blow up the origin one more time and chek that we obtain now a special desingularization of S_{2k+1}. In the chart $u = \theta.v$ we obtain $x = \theta^p.v^{2p+1}, y = \theta^{p+1}.v^{2p+1}, z = \theta.v^2$ which gives

$$\begin{aligned} dx &= \theta^{p-1}.v^{2p}.(p.v.d\theta + (2p+1).\theta.dv) := \theta^{p-1}.v^{2p}.A \\ dy &= \theta^p.v^{2p}.((p+1).v.d\theta + (2p+1).\theta.dv) := \theta^p.v^{2p}.B \\ dz &= v.(v.d\theta + 2\theta.dv) := v.C \end{aligned}$$

Now remark that $B = -A + (2p+1).C$ which implies that

$$dy = -\theta.dx + (2p+1).\theta^p.v^{2p-1}.dz$$

and so the pull-back of $\Omega^1_{S_{2p+1}}$ is locally free after this last blow-up.
Now the pull-back of the form $x.dy/z^{p+1}$ is given by

$$\theta^{p-1}.v^{2p-1}.B = \theta^{p-1}.v^{2p-1}.(-A + (2p+1).C)$$

and it is now easy to see that this does not belong to the sub-sheaf generated by dx and dz.

Now assume that $k = 2p$ with $p \geq 2$ then in the last chart $\{\gamma \neq 0\}$ we shall have, with coordinates (z, u) with $u \neq 0$

$$x = z^p.u, \quad y = z^p/u.$$

We again have to check that this is a special desingularization of S_{2p}. But as $u \neq 0$ in this chart, (dx, dz) generate the pull-back of $\Omega^1_{S_{2p}}$.

Now, as $x.dy/z^{p+1} = u.dy/z$ to see if this form belongs to sub-sheaf generated by (dx, dz) is equivalent to see if $z^{p-1}.du$ is a section of this sub-sheaf. This is clearly not the case as $z^{p-1}.du = dx/z - p.z^{p-2}.u.dz$. □

6.3. **The case $X := \{(x, y, z) \in \mathbb{C}^3 \ / \ x^3 + y^3 + z^3 = 0\}$.** Now consider

$$X := \{(x, y, z) \in \mathbb{C}^3 \ / \ x^3 + y^3 + z^3 = 0\}.$$

The lemma 6.1.1 gives the inclusion

$$\omega^1_X \subset \frac{1}{z^2}.\Omega^1_{\mathbb{C}^2} + \frac{1}{z}.\Omega^1_{\mathbb{C}^2} + \Omega^1_{\mathbb{C}^2}$$

where x, y are the coordinates on $\mathbb{C}^2$. An easy computation shows that the forms

$$\alpha := (x.dy - y.dx)/z^2$$

and $z.\alpha$ generate ω^1_X/Ω^1_X.

Let $\tau : \tilde{X} \to X$ the blowing-up at the origin of X.

Claim. This is a special desingularization:

In the chart $\{\alpha \neq 0\}$ we have

$$y = u.x, \quad z = v.x, \quad u^3 + v^3 + 1 = 0.$$

Then we can choose (x, u) or (x, v) as local coordinates when $v \neq 0$ or $u \neq 0$. The sheaf $\tau^*(\Omega^1_X)$ is generated by dx and $x.du$ when $v \neq 0$ and so is free on this open set. So the sheaf $\tau^*(\Omega^1_X)/\text{torsion}$ is locally free on this blow-up, proving the claim.

In the chart $\{\gamma \neq 0\}$ let $a := \alpha/\gamma$ and $b := \beta/\gamma$; then we have the relations

$$x = z.a, \quad y = z.b, \quad a^3 + b^3 + 1 = 0$$

and then we can choose (z, a) or (z, b) as local coordinates. Then we have

$$\alpha = a.db - b.da = db/a^2 = -da/b^2.$$

In the chart $\{\alpha \neq 0\}$ we have

$$y = u.x, z = v.x, u^3 + v^3 + 1 = 0.$$

Then we can choose (x, u) or (x, v) as local coordinates and $\alpha = du/v^2 = -dv/u^2$. This shows that $\omega^1_X = L^1_X$. But α does not vanish on the exceptional divisor, so α is not a section of α^1_X.

But, in the first chart,

$$z.\alpha = z.a.db - z.b.da = a.dy - a.b.dz - b.dx + a.b.dz = a.dy - b.dx \in \tau^{**}(\Omega^1_X)$$

and in the second chart

$$x.\alpha = x.du/v^2 = dy/v^2 - u.dx/v^2 = -dz/u^2 + v.dx/u^2$$

also belong to $\tau^{**}(\Omega^1_X)$.

Then $x.\alpha, y.\alpha$ and $z.\alpha$ are sections of α^1_X and the quotient L^1_X/α^1_X is a vector space of dimension 1 with basis α. □

Note that $x.y.z.\alpha$ is not a section of $\Omega_X^1/\text{torsion}$ because if we assume that $x.y.z.\alpha$ is a section of $\Omega_X^1/\text{torsion}$, we can write

$$x.y.(x.dy - y.dx) = z.\big[\lambda.dx + \mu.dy + \nu.dz + \rho.df + \sigma.f\big]$$

in $\mathbb{C}^3$, where λ, μ, ν where homogeneous of degree 2, ρ is a complex number and where

$$\sigma := u.dx + v.dy + w.dz$$

with u, v, w complex numbers. This gives, for instance $-x.y^2 = z.\lambda + 3z.\rho.x^2 + u.f$ which is impossible.

So the vector space α_X^1/Ω_X^1 has dimension at least 2. The complete determination of the quotient α_X^1/Ω_X^1 is a non-trivial exercise left to the reader.

Lemma 6.3.1. *For $X := \{(x, y, z) \in \mathbb{C}^3 \ / \ x^3 + y^3 + z^3 = 0\}$ we have*

$$\dim_{\mathbb{C}} \alpha_X^2/\Omega_X^2 = 2, \quad \dim_{\mathbb{C}} L_X^2/\alpha_X^2 = 3 \quad \dim_{\mathbb{C}} \omega_X^2/L_X^2 = 1.$$

Proof. After blowing-up (x, y, z) in $\mathbb{C}^3$ we consider the chart $\{\gamma \neq 0\}$ as above. We have

$$\omega := \frac{dx \wedge dy}{z^2} = -\frac{dz}{z} \wedge \frac{db}{a^2} = \frac{dz}{z} \wedge \frac{da}{b^2}.$$

Then $x.\omega, y.\omega, z\omega$ are holomorphic in this chart, as we have $x = z.a$ and $y = z.b$ and this chart is enough as $dx \wedge dy/z^2 = dy \wedge dz\big/x^2 = dz \wedge dx\big/y^2$ so $x.\omega, y.\omega, z.\omega$ belongs to L_X^2.

But this is not the case for ω. So $\dim \omega^2/L_X^2 = 1$.

The sheaf $\tau^{**}(\Omega_X^2/\text{torsion})$ in this chart is generated by

$$z.(da/a^2) \wedge dz = -z.(db/b^2) \wedge dz.$$

Then it is equal to $z.\Omega_{\tilde{X}}^2$ in this chart. So a section in L_X^2 is in α_X^2 if and only if it belongs to $(x.L_X^2) \cap (y.L_X^2) \cap (z.L_X^2)$. This intersection is generated by $x.y.\omega, y.z.\omega, z.x.\omega$ as a $\mathcal{O}_X-$module. The vector space L_X^2/α_X^2 is generated by $x.\omega, y.\omega, z.\omega$ because $x^2.\omega, y^2.\omega, z^2.\omega$ are in $\Omega_X^2 \subset \alpha_X^2$. We let to the reader the proof that they give a basis of L_X^2/α_X^2.

Let us prove that $x.y.z.\omega$ is not in $\Omega_X^2/\text{torsion}$.

Assume that $x.y.z.\omega \in \Omega_X^2/\text{torsion}$. Then we can write on $\mathbb{C}^3$:

$$x.y.dx \wedge dy - z\big[\lambda.dx \wedge dy + \mu.dy \wedge dz + \nu.dz \wedge dx + (a.dx + b.dy + c.dz) \wedge df\big] = 0$$

where we can assume that λ, μ, ν are linear forms on $\mathbb{C}^3$ and a, b, c are complex number, using the homogeneity of the situation. The coefficient of $dx \wedge dy$ in this identity is equal to

$$x.y - z.\lambda - a.y^2 + b.x^2,$$

which cannot be identically zero. Contradiction.

As it is easy to see that $x.y.\omega = y.z.\omega = z.x.\omega$ and $x.y.z.\omega$ are linearly independent over $\mathbb{C}$ (different homogeneities) we conclude that $\dim \alpha_X^2/\Omega_X^2 = 2$. □

Remark. We have on X

$$\omega := \frac{dx \wedge dy}{z^2} = \frac{dy \wedge dz}{x^2} = \frac{dz \wedge dx}{y^2}$$

so

$$(x.y.\omega)^2 = \frac{x^2.y^2.(dx \wedge dy)^2}{z^4} = \frac{x^2.dx \wedge dy}{z^2}.\frac{y^2.(dx \wedge dy)}{z^2} = (dz \wedge dy).(dx \wedge dz),$$

because on X we have $x^2.dx \wedge dy = -z^2.dz \wedge dy$ and $y^2.dx \wedge dy = -z^2.dx \wedge dz$. This gives an integral dependence relation for $x.y.\omega$ in the symetric algebra of $\Omega_X^2/\text{torsion}$.

6.4. **The case $X := \{(x,y,u,v) \in \mathbb{C}^4 \ / \ x.y = u.v\}$.** Let us begin by the verification that blowing-up the origin gives a special desingularization for X.

Write X as $\{(x_1,x_2,x_3,x_4) \in \mathbb{C}^4 \ / \ x_1^2+x_2^2+x_3^2+x_4^2 = 0\}$ and look at the chart $\alpha \neq 0$. So we have $x_2 = b.x_1, x_3 = c.x_1, x_4 = e.x_1$ with the relation $1+b^2+c^2+e^2 = 0$ and coordinates (x_1,b,c) on the subset $e \neq 0$. The the sheaf $\tau^*(\Omega^1_X)$ is generated by $dx_1, x_1.db, x_1.dc$ because for $e \neq 0$ we have $x_1.de = -e^{-1}[c.x_1.dc + b.x_1.db]$. So modulo its torsion, the sheaf $\tau^*(\Omega^1_X)$ is locally free.

Lemma 6.4.1. *The sheaf L^3_X is equal to ω^3_X and is given by $\mathcal{O}_X.\omega$ where we define*

$$\omega := \frac{dy\wedge du\wedge dv}{y}.$$

Moreover, ω does not belong to α^3_X.

PROOF. On X we have $x.dy + y.dx = u.dv + v.du$

$$\omega = -\frac{dx\wedge du\wedge dv}{x} = \frac{du\wedge dx\wedge dy}{u} = \frac{dv\wedge dx\wedge dy}{v}.$$

To see that $\omega^3_X = \mathcal{O}_X.\omega$ it is enough (X is a hypersurface !) to see that

$$\omega\wedge df/f = dx\wedge dy\wedge du\wedge dv/f$$

where $f := x.y - u.v$. This is clear.

Using the symetries between the coordinates, it is enough to see that $\tau^*(\omega)$ is holomorphic in the first chart of the strict transform $\tilde{X}$ of X by the blow-up at the origin in $\mathbb{C}^4$ to show that ω is a section of L^3_X. Let $y = \lambda.x, u = \mu.x, v = \nu.x$. Then

$$\tau^*(\omega) = -\frac{dx}{x}\wedge x.d\mu\wedge x.d\nu = -x.dx\wedge d\mu\wedge d\nu$$

where x,μ,ν are the coordinates for $\tilde{X}$ in this chart (and $\lambda = \mu.\nu$). So $\omega \in L^3_X$.

To see that ω is not in α^3_X it is enough to see that ω does not belongs to $\tau^{**}(\Omega^3_X)$ in the first chart above. An easy computation show that $\tau^{**}(\Omega^3_X)$ is generated by

$$\pi^{**}(dx\wedge du\wedge dv) = x^2.dx\wedge d\mu\wedge d\nu$$

and so $\omega = -x.dx\wedge d\mu\wedge d\nu$ does not belong to $\tau^{**}(\Omega^3_X)$. □

Lemma 6.4.2. *The meromorphic form $w := u.dv\wedge dx/x$ is a section of α^2_X but it is not a section of $\Omega^2_X/\mathrm{torsion}$ and its differential is not a section of α^3_X.*

PROOF. As

$$u.dv\wedge dx/x + v.du\wedge dx/x = -dx\wedge dy$$

is holomorphic on X, u and v play the same role for this form modulo holomorphic forms. Also $u.dv\wedge(dx/x+dy/y) = dv\wedge du$ so x and y play also the same role modulo holomorphic forms on X. So it is enough to see that in the first chart of the strict transform $\tilde{X}$ of X by the blow-up at the origin in $\mathbb{C}^4$ the form $\tau^{**}(w)$ is a section of $\tau^{**}(\Omega^2_X)$ to prove that w is a section of α^2_X. Using the same coordinates as above we obtain

$$\tau^{**}(w) = \mu.x.d(\nu.x)\wedge dx/x = \mu.x.d\nu\wedge dx = \mu.dv\wedge dx$$

which is a section of $\tau^{**}(\Omega^2_X)$.

To see that w is not a section of $\Omega_X^2/\text{torsion}$ assume the contrary. Then, by symmetry[10] $w' := v.du \wedge dx/x$ is also a section of $\Omega_X^2/\text{torsion}$ and the differential of $w - w'$ must be a section of $\Omega_X^3/\text{torsion}$. But we have already seen that $2.\omega = -d(w - w')$ is not a section of α_X^3. Contradiction. $\square$

Note that an integral dependence relation on the symmetric algebra of the sheaf $\Omega_X^2/\text{torsion}$ for w is given in the second example of the begining of the section 5.1.

Lemma 6.4.3. *We have* $\Omega_X^1/\text{torsion} = \alpha_X^1 = L_X^1 = \omega_X^1$.

Proof. Write $X := \{x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0\} \subset \mathbb{C}^4$. Then thanks to the lemma 6.1.1 we have:

$$\omega_X^1 \subset \Omega_{\mathbb{C}^3}^1 + \frac{1}{x_4}.\Omega_{\mathbb{C}^3}^1.$$

To prove that $\Omega_X^1/\text{torsion} = \omega_X^1$ it is enough to consider a section in ω_X^1, let

$$w := (a.dx_1 + b.dx_2 + c.dx_3)$$

and put $v := w/x_4$ and to show that v is a section of $\Omega_X^1/\text{torsion}$. But then

$$\text{Trace}_\pi(v \wedge dx_4) = w \wedge \text{Trace}_\pi(dx_4/x_4)$$

must be a holomorphic form on $\mathbb{C}^3$, where $\pi : X \to \mathbb{C}^3$ is the projection which makes X a branched covering of degree 2. This condition implies $df \wedge w$ is in $f.\Omega_{\mathbb{C}^3}^2$ where $f := x_1^2 + x_2^2 + x_3^2$. As the sheaf $\Omega_{S_2}^1$ has no torsion[11], this implies that $w = u.df + f.\xi$ where $u \in \mathcal{O}_{\mathbb{C}^3}$ and $\xi \in \Omega_{\mathbb{C}^3}^1$. But $f = -x_4$ on X, so this gives $v = -\xi - u.dx_4$ on X and v is in Ω_X^1. $\square$

Lemma 6.4.4. *We have* $\omega_X^2 = \Omega_X^2/\text{torsion} \oplus \mathbb{C}.\eta$ *where*

$$\eta := \frac{x_1.dx_2 \wedge dx_3 + x_2.dx_3 \wedge dx_1 + x_3.dx_1 \wedge dx_2}{x_4}.$$

Proof. Write $\omega := (a.dx_1 \wedge dx_2 + b.dx_2 \wedge dx_3 + c.dx_3 \wedge dx_1)/x_4$ where a, b, c are holomorphic on $\mathbb{C}^3$. Then ω is in ω_X^2 if and only if $\text{Trace}_\pi(dx_4 \wedge \omega)$ is a section of $\Omega_{\mathbb{C}^3}^3$. This is satisfyed if and only if $a.x_3 + b.x_1 + c.x_2$ is a multiple of $\xi := x_1^2 + x_2^2 + x_3^2$ in $\mathcal{O}_{\mathbb{C}^3}$. This gives the relation $(a - g.x_3).x_3 + (b - g.x_1).x_1 + (c - g.x_2).x_2 = 0$. And, as x_1, x_2, x_3 is a regular sequence, this implies

$$a = g.x_3 + \lambda.x_1 + \mu.x_2, \quad b = g.x_1 + \lambda'.x_2 - \lambda.x_3, \quad c = g.x_2 - \lambda'.x_1 - \mu.x_3$$

where λ, λ', μ are in $\mathcal{O}_{\mathbb{C}^3}$. This shows that ω_X^2 is generated as a $\mathcal{O}_X-$module by Ω_X^2 and η. Note that we already know that η is not a section of $\Omega_X^2/\text{torsion}$ as we have shown that ω_X^2 is not equal to $\Omega_X^2/\text{torsion}$

Claim. For $i = 1, 2, 3, 4 \quad x_i.\eta$ is in $\Omega_X^2/\text{torsion}$:

for instance:

$$\begin{aligned}
\frac{x_1.\eta}{x_4} &= \frac{x_1}{x_4}.(x_1.dx_2 \wedge dx_3 + x_2.dx_3 \wedge dx_1 + x_3.dx_1 \wedge dx_2) \\
&= \frac{1}{x_4}.(-(x_2^2 + x_3^2 + x_4^2).dx_2 \wedge dx_3 + (x_2.dx_3 - x_3.dx_2) \wedge x_1.dx_1) \\
&= \frac{1}{x_4}.(-(x_2^2 + x_3^2 + x_4^2).dx_2 \wedge dx_3 + (x_2.dx_3 - x_3.dx_2) \wedge (-x_2.dx_2 - x_3.dx_3 - x_4.dx_4)) \\
&= -x_4.dx_2 \wedge dx_3 - x_2.dx_3 \wedge dx_4 + x_3.dx_2 \wedge dx_4 \in \Omega_X^2
\end{aligned}$$

[10] or using $(u.dv + v.du) \wedge dx/x = dy \wedge dx$.

[11] This is easy to see using the fact that $S_2 = \{f = 0\}$ is the quotient of $\mathbb{C}^2$ by ± 1.

proving our claim. □

6.5. **The case $X := \{(x,y,z,t) \in \mathbb{C}^4 \ / \ x.y.z = t^3\}$.** Remark first that the form $\omega_1 := y.z.dx/t^2$ is in ω^1_X because we have, with the notation $f := x.y.z - t^3$:

$$\omega_1 \wedge df = \frac{z.t^3.dx \wedge dy + y.t^3.dx \wedge dz + 3t^2.y.z.dx \wedge dt}{t^2} \in \Omega^2_{\mathbb{C}^4} \quad \text{modulo}(f/t^2).\Omega^2_{\mathbb{C}^4}$$

which allows to conclude as t is not a zero divisor in X (see [B.78]).

Consider now the following sections of ω^1_X:

$$u := t.\omega_1 \quad v := t.\omega_2 \quad w := t.\omega_3$$

where ω_2 and ω_3 are deduced from ω_1 respectively by

$$x \to y, y \to z, z \to x \qquad \text{and} \qquad x \to z, y \to x, z \to y.$$

Then we have in the symmetric algebra of Ω^1_X:

$$u + v + w = 3t.dt \quad u.v + v.w + w.u = t.(z.dx.dy + x.dy.dz + z.dx.dy) \quad u.v.w = t^3.dx.dy.dz.$$

This shows that u, v, w satisfy the following integral dependence relation over the symmetric algebra of Ω^1_X:

$$\Theta^3 - 3t.dt.\Theta^2 + t.(z.dx.dy + x.dy.dz + z.dx.dy).\Theta - t^3.dx.dy.dz = 0.$$

Note that, because the coefficient of Θ does not belong to (t^2), we do not obtain an integral dependence relation over the symmetric algebra of Ω^1_X for Θ/t so for the forms $\omega_i, i = 1,2,3$! In fact they are not sections of the sheaf α^1_X (for instance the restriction of ω_1 to the surface $S_3 \simeq \{z = 1\} \cap X$ is not in $\alpha^1_{S_3}$ (see sub-section 6.2).

Let us now verify that $t.u$ is not a section of Ω^1_X/torsion. Assume that we can write

$$y.z.dx = t.\big(\lambda.dx + \mu.dy + \nu.dz + \theta.dt\big) \quad \text{modulo } f.\Omega^1_X + \mathcal{O}_X.df$$

then, by homogeneity, we may assume that λ, μ, ν are homogeneous of degree 2 and

$$y.z.dx = t.\big(\lambda.dx + \mu.dy + \nu.dz + \theta.dt\big) + \sigma.df$$

where σ is a constant. This implies

$$y.z.(1-\sigma) - t.\lambda = 0, \quad t.\mu + \sigma.x.z = 0$$

which is already enough to obtain a contradiction, as these equations imply $\sigma = 1$ and $\sigma = 0$ respectively. □

Remark. Using the map $((x,y,z) \mapsto (x + y, x + j.y, x + j^2.y, -z)$ which sends the previous $Y := \{x^3 + y^3 + z^3 = 0\}$ to $X = \{x.y.z = t^3\}$ allows to find an integral equation over the symmetric algebra of Ω^1_Y of the section

$$\frac{(x^2 + y^2 - x.y).d(x+y)}{z}$$

of α^1_Y.

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