ON REAL ANTI-BICANONICAL CURVES WITH ONE DOUBLE POINT ON THE 4-TH REAL HIRZEBRUCH SURFACE

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ABSTRACT. We list up all the candidates for the real isotopy types of real anti-bicanonical curves with one real nondegenerate double point on the 4-th real Hirzebruch surface $\mathbb{R}\mathbb{F}_4$ by enumerating the connected components of the moduli space of real 2-elementary K3 surfaces of type $(S, \theta) \cong ((3,1,1), -\mathrm{id})$. We also list up all the candidates for the non-increasing simplest degenerations of real nonsingular anti-bicanonical curves on $\mathbb{R}\mathbb{F}_4$. We find an interesting correspondence between the real isotopy types of real anti-bicanonical curves with one real nondegenerate double point on $\mathbb{R}\mathbb{F}_4$ and the non-increasing simplest degenerations of real nonsingular anti-bicanonical curves on $\mathbb{R}\mathbb{F}_4$. This correspondence is very similar to the one provided by the rigid isotopic classification of real sextic curves on $\mathbb{R}\mathbb{P}^2$ with one real nondegenerate double point by I. Itenberg.

1. INTRODUCTION

This paper is a continuation of [10] and [11]. In [10] the moduli spaces of "($\mathcal{D}\mathbb{R}$)-nondegenerate" real K3 surfaces with non-symplectic holomorphic involutions (namely, real 2-elementary K3 surfaces) are formulated and it is shown that the connected components of such a moduli space are in **one to one correspondence** with the isometry classes of integral involutions of the K3 lattice of certain type (see Theorem 2.11 below and [10] for more precise statements). As its applications, we obtain the real isotopic classifications (more precisely, deformation classifications) of real nonsingular sextic curves on the real projective plane $\mathbb{R}\mathbb{P}^2$, real nonsingular curves of bidegree $(4, 4)$ on the real $\mathbb{P}^1 \times \mathbb{P}^1$ (hyperboloid, ellipsoid), and real nonsingular anti-bicanonical curves on the real Hirzebruch surfaces $\mathbb{R}\mathbb{F}_m$ ($m = 1, 4$). Here the m-th Hirzebruch surface $\mathbb{F}_m$ ($m \geq 0$) means the ruled surface over $\mathbb{P}^1$ having an exceptional section s with $s^2 = -m$. $\mathbb{F}_4$ has 2 real structures (anti-holomorphic involutions). The real part of $\mathbb{F}_4$ is homeomorphic to the 2-dimensional torus or the empty set. On the other hand, $\mathbb{F}_3$ has a unique real structure and its real part is homeomorphic to the Klein's bottle (see [2]).

We say a complex curve A on a real Hirzebruch surface is **real** if its anti-holomorphic involution can be restricted to the curve A. We say two real curves $\mathbb{R}A$, $\mathbb{R}A'$ on a real nonsingular surface $\mathbb{R}B$ are **real isotopic** if there exists a continuous map $\Phi : \mathbb{R}B \times [0, 1] \to \mathbb{R}B$ (a "real isotopy from $\mathbb{R}A$ to $\mathbb{R}A'$") such that $\Phi_t := \Phi(\ , t) : \mathbb{R}B \to \mathbb{R}B$ is a homeomorphism for any $t \in [0, 1]$, $\Phi_0 = \mathrm{id}_{\mathbb{R}B}$, and $\Phi_1(\mathbb{R}A) = \mathbb{R}A'$. Moreover, two real curves $\mathbb{R}A$, $\mathbb{R}A'$ in a fixed class on a real nonsingular surface $\mathbb{R}B$ are **rigidly isotopic** if there exists a real isotopy $\Phi : \mathbb{R}B \times [0, 1] \to \mathbb{R}B$ from $\mathbb{R}A$ to $\mathbb{R}A'$ such that $\Phi_t(\mathbb{R}A)$ is contained in the same class for any $t \in [0, 1]$.

Using the same method as above, we obtain the real isotopic classifications of real nonsingular anti-bicanonical curves on the real Hirzebruch surfaces $\mathbb{R}\mathbb{F}_2$ and $\mathbb{R}\mathbb{F}_3$ (see also Theorem 2.13 and Remark 2.26 of this paper) in [11]. Especially, all the connected components of the moduli space of real 2-elementary K3 surfaces of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$, which are defined below,

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are enumerated in [11]. Here we should remark that any real 2-elementary K3 surfaces of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$ are $(\mathcal{D}\mathbb{R})$-**nondegenerate** in the sense of [10].

Any 2-elementary K3 surface (X, τ) of type $S \cong (3, 1, 1)$ has an elliptic fibration and a unique reducible fiber $E + F$, where the curves E and F are defined in [11] (see also Subsection 2.2 of this paper). There are two types of F, which are reducible and irreducible. Let A be the fixed point set (curve) of τ on X, and $\pi : X \to Y := X/\tau$ be the quotient map. A intersects $\pi(E)$ at a single point. If we contract the curve $\pi(E)$ on Y to a point, then we get the 3-th Hirzebruch surface $\mathbb{F}_3$, and the image of the curve A is a real **nonsingular** anti-bicanonical curve on $\mathbb{F}_3$. This enables us to enumerate up (see [11]) all the real isotopy types of real nonsingular anti-bicanonical curves on $\mathbb{R}\mathbb{F}_3$.

However, on the other hand, if we contract the curve $\pi(F)$ on Y to a point, then we get the 4-th Hirzebruch surface $\mathbb{F}_4$, and the image of the curve A is real anti-bicanonical and has **one real double point**. Even though a real 2-elementary K3 surface of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$ is always $(\mathcal{D}\mathbb{R})$-**nondegenerate**, the double point is possibly **degenerate**, namely, a real cusp point. Thus, the main difficulty of the real isotopic classification of real anti-bicanonical curves on $\mathbb{F}_4$ with one real double point is that the connected components (equivalently, the isometry classes of integral involutions of the K3 lattice) of the moduli space ([10]) of real 2-elementary K3 surfaces of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$ cannot distinguish degenerate and nondegenerate double points, equivalently, reducible F and irreducible F. Hence, the connected components cannot distinguish the topological types (node, cusp, or isolated point) of the real double points (Remark 2.25). This problem was left to the readers in [11] (see Remark 8).

Thus, the aim of this paper is: *Classify the real isotopy types of real anti-bicanonical curves with one real nondegenerate double point on $\mathbb{R}\mathbb{F}_4$.*

First we list up all the candidates for the real isotopy types of real anti-bicanonical curves with one real nondegenerate double point on $\mathbb{R}\mathbb{F}_4$ (Theorem 2.24) using some well-known topological interpretations ([10], [11]) of some arithmetic invariants of integral involutions of the K3 lattice.

Unfortunately, the realizability of each real isotopy type listed in Theorem 2.24 has not been resolved in this paper. We only know that at least one of real isotopy types with nondegenerate double points can be realized for **each** isometry class (see Remark 2.25).

In order to distinguish the real isotopy types, we should remove real 2-elementary K3 surfaces which yield anti-bicanonical curves with degenerate double points on $\mathbb{R}\mathbb{F}_4$ from the moduli space (period domain) in the sense of [10]. We follow Itenberg's argument ([4],[5],[6]) for the rigid isotopic classification of real sextic curves on $\mathbb{R}\mathbb{P}^2$ with one nondegenerate double point. Lemma 4.6 provides a sufficient condition for the double point to be non-degenerate.

Moreover, according to his papers [4] and [5], real curves of degree 6 on $\mathbb{R}\mathbb{P}^2$ with one nondegenerate double point are obtained by "non-increasing simplest degenerations" of real nonsingular curves of degree 6 on $\mathbb{R}\mathbb{P}^2$. Hence, we next list up the candidates for the non-increasing simplest degenerations of real nonsingular anti-bicanonical curves on $\mathbb{R}\mathbb{F}_4$ (Theorem 3.4).

Then, we find an obvious interesting correspondence between the real isotopy types of curves with one real nondegenerate double point on $\mathbb{R}\mathbb{F}_4$ (Theorem 2.24) and the non-increasing simplest degenerations of nonsingular curves on $\mathbb{R}\mathbb{F}_4$ (Theorem 3.4). See Remark 3.5. We also get some similar properties (Lemma 3.7) to the degenerations of real nonsingular curves of degree 6 on $\mathbb{R}\mathbb{P}^2$.

In the final section 4, we review and confirm the periods of marked real 2-elementary K3 surfaces $((X, \tau, \varphi),\ \alpha)$ of type (S, θ) satisfying $\alpha \circ \varphi_* \circ \alpha^{-1} = \psi$ for a fixed integral involution ψ, and give some further problems which are inspired by the argument in [4] and [5].

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2. Real 2-elementary K3 surfaces of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$

2.1. **Real 2-elementary K3 surfaces.**

Definition 2.1 (Real 2-elementary K3 surface). A triple (X, τ, φ) is called a **real K3 surface with non-symplectic (holomorphic) involution** (or **real 2-elementary K3 surface**) if
(1) (X, τ) is a K3 surface X with a non-symplectic holomorphic involution τ, i.e., "2-elementary K3 surface" ([8]).
(2) φ is an anti-holomorphic involution on X.
(3) $\varphi \circ \tau = \tau \circ \varphi$

Note that any K3 surface with a non-symplectic holomorphic involution is algebraic.

For a real K3 surface with non-symplectic involution (X, τ, φ), we call $\widetilde{\varphi} := \tau \circ \varphi = \varphi \circ \tau$ the **related (anti-holomorphic) involution** of φ ([10]). The triple $(X, \tau, \tau \circ \varphi)$ is also a real K3 surface with non-symplectic involution.

For a 2-elementary K3 surface (X, τ), we denote by $H_{2+}(X, \mathbb{Z})$ the fixed part of

$$\tau_* : H_2(X, \mathbb{Z}) \to H_2(X, \mathbb{Z}).$$

Note that $H_{2+}(X, \mathbb{Z}) \subset N(X)$, where $N(X)$ is the Picard lattice of X.

We fix an even unimodular lattice $\mathbb{L}_{K3}$ of signature $(3, 19)$. The isometry class of such lattices is unique (the K3 lattice).

Let $\alpha : H_2(X, \mathbb{Z}) \to \mathbb{L}_{K3}$ be an isometry (marking). If we temporary set $S := \alpha(H_{2+}(X, \mathbb{Z}))$, then S is a primitive (,i.e., $\mathbb{L}_{K3}/S$ is free,) hyperbolic (i.e., S is of signature $(1, \mathrm{rank}\, S - 1)$) 2-elementary (i.e., $S^*/S \cong (\mathbb{Z}/2\mathbb{Z})^a$ for some nonnegative integer a) sublattice of $\mathbb{L}_{K3}$.

Now let $S\ \ (\subset \mathbb{L}_{K3})$ be a primitive hyperbolic 2-elementary sublattice of the K3 lattice $\mathbb{L}_{K3}$.

Definition 2.2. We set $r(S) := \mathrm{rank}\, S$. The non-negative integer $a(S)$ is defined by the equality

$$S^*/S \cong (\mathbb{Z}/2\mathbb{Z})^{a(S)}.$$

We define

$$\delta(S) := \begin{cases} 0 & \text{if } z \cdot \sigma(z) \equiv 0 \bmod 2 \quad (\forall z \in \mathbb{L}_{K3}) \\ 1 & \text{otherwise,} \end{cases}$$

where we define $\sigma : \mathbb{L}_{K3} \to \mathbb{L}_{K3}$ to be the unique integral involution whose fixed part is S.

It is known that the triplet

$$(r(S), a(S), \delta(S))$$

determines the isometry class of the lattice S ([8]).

If S and S' are primitive hyperbolic 2-elementary sublattices of the K3 lattice $\mathbb{L}_{K3}$, and S is isometric to S'; then there exists an automorphism f of $\mathbb{L}_{K3}$ such that $f(S') = S$ ([1], [7]). Hence, if (X, τ) and (X', τ') are two 2-elementary K3 surfaces, $\alpha : H_2(X, \mathbb{Z}) \to \mathbb{L}_{K3}$ is an isometry, and $H_{2+}(X, \mathbb{Z})$ is isometric to $H_{2+}(X', \mathbb{Z})$; then there exists an isometry (marking) $\alpha' : H_2(X', \mathbb{Z}) \to \mathbb{L}_{K3}$ such that $\alpha'(H_{2+}(X', \mathbb{Z})) = \alpha(H_{2+}(X, \mathbb{Z}))$. Thus, only the isometry class of $H_{2+}(X, \mathbb{Z})$ is essential for 2-elementary K3 surfaces (X, τ). Henceforth, we often fix an isometry class, equivalently, the invariants $(r(S), a(S), \delta(S))$ instead of fixing a particular sublattice.

We quote the following formulation from [10]. We additionally fix a half-cone $V^+(S)$ of the cone $V(S) := \{x \in S \otimes \mathbb{R} \mid x^2 > 0\}$. We also fix a fundamental chamber $\mathcal{M} \subset V^+(S)$ for the group $W^{(-2)}(S)$ generated by reflections in all elements with square (-2) in S. This is equivalent to fixing a fundamental subdivision

$$\Delta(S) = \Delta(S)_+ \cup -\Delta(S)_+$$

of all elements with square -2 in S. $\mathcal{M}$ and $\Delta(S)_+$ define each other by the condition

$$\mathcal{M} \cdot \Delta(S)_+ \geq 0.$$

Let (X, τ) be a 2-elementary K3 surface, and $\alpha : H_2(X, \mathbb{Z}) \to \mathbb{L}_{K3}$ be a marking such that $\alpha(H_{2+}(X, \mathbb{Z})) = S$. We always assume that $\alpha_{\mathbb{R}}^{-1}(V^+(S))$ contains a hyperplane section of X and the set $\alpha^{-1}(\Delta(S)_+)$ contains only classes of effective curves of X.

Now let θ be an integral involution of S.

Definition 2.3 (the action of φ on $H_{2+}(X, \mathbb{Z})$). A real 2-elementary K3 surface (X, τ, φ) is called that of **type** (S, θ) if there exists an isometry (marking)

$$\alpha : H_2(X, \mathbb{Z}) \cong \mathbb{L}_{K3}$$

such that $\alpha(H_{2+}(X, \mathbb{Z})) = S$ and the following diagram commutes:

$$
\begin{array}{ccc}
H_{2+}(X, \mathbb{Z}) & \xrightarrow{\ \alpha\ } & S \\
\varphi_* \downarrow & & \downarrow \theta \\
H_{2+}(X, \mathbb{Z}) & \xrightarrow{\ \alpha\ } & S.
\end{array}
$$

Definition 2.4 (marked real 2-elementary K3 surfaces). A pair $((X, \tau, \varphi),\ \alpha)$ of a real 2-elementary K3 surface (X, τ, φ) of type (S, θ) and an isometry (marking) $\alpha : H_2(X, \mathbb{Z}) \cong \mathbb{L}_{K3}$ such that

- $\alpha(H_{2+}(X, \mathbb{Z})) = S$,

- $\alpha \circ \varphi_* = \theta \circ \alpha$ on $H_{2+}(X, \mathbb{Z})$,

- $\alpha_{\mathbb{R}}^{-1}(V^+(S))$ contains a hyperplane section of X and

- the set $\alpha^{-1}(\Delta(S)_+)$ contains only classes of effective curves of X

is called a **marked real 2-elementary K3 surface of type** (S, θ).

Note that we have $\theta(V^+(S)) = -V^+(S)$ and $\theta(\Delta(S)_+) = -\Delta(S)_+$ for a marked real 2-elementary K3 surface of type (S, θ).

Definition 2.5 (Integral involution ψ of $\mathbb{L}_{K3}$ of type (S, θ)). Let S be a hyperbolic 2-elementary sublattice of $\mathbb{L}_{K3}$, $\theta : S \to S$ be an integral involution of the lattice S, and $\psi : \mathbb{L}_{K3} \to \mathbb{L}_{K3}$ be an integral involution of the lattice $\mathbb{L}_{K3}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
S & \subset & \mathbb{L}_{K3} \\
\theta \downarrow & & \downarrow \psi \\
S & \subset & \mathbb{L}_{K3}.
\end{array}
$$

We say such a pair $(\mathbb{L}_{K3}, \psi)$ (or ψ) an **integral involution of $\mathbb{L}_{K3}$ of type** (S, θ).

Let $((X, \tau, \varphi),\ \alpha)$ be a marked real 2-elementary K3 surface of type (S, θ). If we set

$$\psi := \alpha \circ \varphi_* \circ \alpha^{-1} : \mathbb{L}_{K3} \to \mathbb{L}_{K3},$$

then we have $\psi(S) = S$, and $\psi(x) = \theta(x)$ for every $x \in S$ because $\alpha^{-1}(x) \in H_{2+}(X, \mathbb{Z})$. Hence, $(\mathbb{L}_{K3}, \psi)$ is an integral involution of $\mathbb{L}_{K3}$ of type (S, θ).

Definition 2.6 (the associated integral involution)**.** We call the integral involution ψ of $\mathbb{L}_{K3}$ of type (S, θ) **the associated integral involution** of $\mathbb{L}_{K3}$ with a marked real 2-elementary K3 surface $((X, \tau, \varphi),\ \alpha)$ of type (S, θ) if the following diagram commutes:

$$
\begin{array}{ccc}
H_2(X, \mathbb{Z}) & \xrightarrow{\ \alpha\ } & \mathbb{L}_{K3} \\
\varphi_* \downarrow & & \downarrow \psi \\
H_2(X, \mathbb{Z}) & \xrightarrow{\ \alpha\ } & \mathbb{L}_{K3}.
\end{array}
$$

Note that the fixed part $\mathbb{L}_{K3}^{\psi}$ of ψ is hyperbolic for any associated integral involution of $\mathbb{L}_{K3}$.

Definition 2.7 (($\mathcal{D}\mathbb{R}$)-nondegenerate, [10])**.** $x \in N(X) \otimes \mathbb{R}$ is **nef** if $x \neq 0$ and $x \cdot C \geq 0$ for any effective curve on X. We say that a 2-elementary K3 surface (X, τ) of type S is $(\mathcal{D})$**-degenerate** if there exists $h \in \mathcal{M}$ such that h is not nef. This is equivalent to the existence of an exceptional curve (i.e., irreducible and having negative self-intersection) with square -2 on the quotient surface $Y := X/\tau$. This is also equivalent to have an element $\delta \in N(X)$ with $\delta^2 = -2$ such that $\delta = (\delta_1 + \delta_2)/2$ where $\delta_1 \in S$, $\delta_2 \in S_{N(X)}^{\perp}$ and $\delta_1^2 = \delta_2^2 = -4$.

A real 2-elementary K3 surface $(X, \tau,\ \varphi)$ of type (S, θ) is $(\mathcal{D}\mathbb{R})$**-degenerate** if there exists a real element $h \in S_- \cap \mathcal{M}$ which is not nef, where we set $S_{\pm} := \{x \in S \mid \theta(x) = \pm x\}$. This is equivalent to have an element $\delta \in N(X)$ with $\delta^2 = -2$ such that $\delta = (\delta_1 + \delta_2)/2$ where $\delta_1 \in S$, $\delta_2 \in S_{N(X)}^{\perp}$ and $\delta_1^2 = \delta_2^2 = -4$, and δ_1 must be orthogonal to an element $h \in S_- \cap \text{int}(\mathcal{M})$ with $h^2 > 0$. Here $\text{int}(\mathcal{M})$ denote the interior part of $\mathcal{M}$, i. e., the polyhedron $\mathcal{M}$ without its faces.

Let $\Delta(S, L)^{(-4)}$ be the set of all elements δ_1 in S such that $\delta_1^2 = -4$ and there exists $\delta_2 \in S_L^{\perp}$ such that $(\delta_2)^2 = -4$ and $\delta = (\delta_1 + \delta_2)/2 \in L$. Let $W^{(-4)}(S, L) \subset O(S)$ be the group generated by reflections in all roots from $\Delta(S, L)^{(-4)}$, and $W^{(-4)}(S, L)_{\mathcal{M}}$ be the stabilizer subgroup of $\mathcal{M}$ in $W^{(-4)}(S, L)$. Let G be the subgroup generated by reflections s_{δ_1} in all elements $\delta_1 \in \Delta(S, L)^{(-4)}$ which are contained either in S_+ or in S_- and satisfy $s_{\delta_1}(\mathcal{M}) = \mathcal{M}$. (Then G is a subgroup of $W^{(-4)}(S, L)_{\mathcal{M}}$.)

Definition 2.8 (Isometries with respect to the group G)**.** Let $(\mathbb{L}_{K3}, \psi_1)$ and $(\mathbb{L}_{K3}, \psi_2)$ be two integral involutions of $\mathbb{L}_{K3}$ of type (S, θ). An **isometry with respect to the group** G from $(\mathbb{L}_{K3}, \psi_1)$ to $(\mathbb{L}_{K3}, \psi_2)$ means an isometry $f : \mathbb{L}_{K3} \to \mathbb{L}_{K3}$ such that $f(S) = S$, $f|_S \in G$, and the following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{L}_{K3} & \xrightarrow{\ f\ } & \mathbb{L}_{K3} \\
\psi_1 \downarrow & & \psi_2 \downarrow \\
\mathbb{L}_{K3} & \xrightarrow{\ f\ } & \mathbb{L}_{K3}.
\end{array}
$$

In the above definition, remark that $\theta \circ f|_S = f|_S \circ \theta$ on S, hence, $f|_S$ is at least an automorphism of (S, θ). However, we require the condition "$f|_S \in G$".

Definition 2.9. We say two integral involutions $(\mathbb{L}_{K3}, \psi_1)$ and $(\mathbb{L}_{K3}, \psi_2)$ of type (S, θ) are **isometric with respect to the group** G if there exists an isometry with respect to the group G from $(\mathbb{L}_{K3}, \psi_1)$ to $(\mathbb{L}_{K3}, \psi_2)$.

By an **automorphism** of an integral involution $(\mathbb{L}_{K3}, \psi)$ of type (S, θ) **with respect to the group** G we mean an isometry with respect to the group G from $(\mathbb{L}_{K3}, \psi)$ to itself. Namely, an isometry $f : \mathbb{L}_{K3} \to \mathbb{L}_{K3}$ which satisfies that

$$\psi \circ f = f \circ \psi, \ f(S) = S \text{ and } f|_S \in G.$$

Definition 2.10 (analytically isomorphic with respect to G). Two marked real 2-elementary K3 surfaces $((X, \tau, \varphi), \ \alpha)$ and $((X', \tau', \varphi'), \alpha')$ of type (S, θ) are **analytically isomorphic with respect to the group** G if there exists an analytic isomorphism $f : X \to X'$ such that $f \circ \tau = \tau' \circ f$, $f \circ \varphi = \varphi' \circ f$ and $\alpha' \circ f_* \circ \alpha^{-1} | S \in G$.

By considering their associated integral involutions, we get a natural map from the moduli space of marked real 2-elementary K3 surfaces of type (S, θ) to the set of isometry classes with respect to G of integral involutions of $\mathbb{L}_{K3}$ of type (S, θ) such that the fixed part $\mathbb{L}_{K3}^\psi$ of ψ is hyperbolic.

Theorem 2.11 ([10]). *The natural map above gives a bijective correspondence between the connected components of the moduli space of ($\mathcal{DR}$)-nondegenerate marked real 2-elementary K3 surfaces of type (S, θ) and the set of isometry classes with respect to G of integral involutions of $\mathbb{L}_{K3}$ of type (S, θ) such that the fixed part $\mathbb{L}_{K3}^\psi$ of ψ is hyperbolic.*

2.2. **Elliptic fibrations of** 2-**elementary K3 surfaces of type** $S \cong (3, 1, 1)$. We quote some basic facts from [1]. Let (X, τ) be a 2-elementary K3 surface of type S with invariants

$$(r(S), a(S), \delta(S)) = (3, 1, 1),$$

and $A := X^\tau$ be the fixed point set (curve) of τ. Then we have

$$A = A_0 \cup A_1 \ \text{(disjoint union)},$$

where A_0 is a nonsingular rational curve ($\cong \mathbb{P}^1$) with $A_0^2 = -2$ and A_1 is a nonsingular curve of genus 9. (X, τ) has a structure of an elliptic pencil $|E + F|$ with its section A_0 and the unique reducible fiber $E + F$ having the following properties below:

(i): E is an irreducible nonsingular rational curve with $E^2 = -2$ and $E \cdot A_0 = 1$.

(ii): $E \cdot F = 2$, $F^2 = -2$, $F \cdot A_0 = 0$, and F is either an irreducible nonsingular rational curve (type **IIa**), or the union of two irreducible nonsingular rational curves F' and F'' which are conjugate by τ and $F' \cdot F'' = 1$ (type **IIb**) (that is, the reducible fiber $E + F$ corresponds to the extended root system $\widetilde{\mathbb{A}_1}$ or $\widetilde{\mathbb{A}_2}$). We have $(F')^2 = (F'')^2 = -2$. See also §2.4 of [1].

(iii): The classes $[A_0]$, $[E]$ and $[F]$ generate the lattice $H_{2+}(X, \mathbb{Z})$ ($\cong S$). Moreover, $A_1 \cdot E = 1$, $A_1 \cdot F = 2$. The Gram matrix of the lattice $H_{2+}(X, \mathbb{Z})$ with respect to the basis $[E]$, $[F]$ and $[A_0]$ is as follows:

$$
\begin{array}{c|ccc}
 & [E] & [F] & [A_0] \\
\hline
[E] & -2 & & \\
[F] & 2 & -2 & \\
[A_0] & 1 & 0 & -2 \\
\end{array}
$$

We next consider the quotient complex surface $Y := X/\tau$ and let $\pi : X \to Y$ be the quotient map. Then, A, considered in Y, is contained in $|-2K_Y|$ (an anti-bicanonical curve on Y).

We define the curves:

$$e := \pi(E) \quad \text{and} \quad f := \pi(F).$$

If F is the union of two nonsingular rational curves F' and F'' which are conjugate by τ and $F' \cdot F'' = 1$, then

$$f = \pi(F) = \pi(F' \cup F'') = \pi(F') = \pi(F'').$$

We use the same symbols A_0 and A_1 for their images in Y by π. Then, the Picard group $\mathrm{Pic}(Y)$ of Y is generated by the classes $[e]$, $[f]$ and $[A_0]$. The Gram matrix of $\mathrm{Pic}(Y)$ with respect to the basis $[e]$, $[f]$ and $[A_0]$ is as follows:

$$
\begin{array}{cccc}
 & [e] & [f] & [A_0] \\
[e] & -1 & & \\
[f] & 1 & -1 & \\
[A_0] & 1 & 0 & -4
\end{array}
$$

For A_1, we have

$$A_1 \cdot e = 1 \quad \text{and} \quad A_1 \cdot f = 2.$$

We have:

(1) F is a nonsingular rational curve if and only if A_1 intersects f in **two distinct points**.

(2) F is a union of two nonsingular rational curves if and only if A_1 **touches** f.

Remark 2.12 ([11])**.** For any **real** 2-elementary K3 surface (X, τ, φ) of type (S, θ) with $S \cong (3, 1, 1)$, we have $\theta = -\mathrm{id}$ and $G = \{\mathrm{id}\}$, where id stands for the identity map on S.

2.3. **Enumeration of the connected components of the moduli space.** For any 2-elementary K3 surface (X, τ) of type $S \cong (3, 1, 1)$, all exceptional curves on Y are exactly the curves e, f and A_0. Hence, (X, τ) is $(\mathcal{D})$-nondegenerate and any real 2-elementary K3 surface (X, τ, φ) of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$ is $(\mathcal{DR})$-nondegenerate (see Definition 2.7). By Theorem 2.11, we have:

Theorem 2.13 ([11]CTheorem 1)**.** *The connected components of the moduli space (in the sense of [10]) of marked real 2-elementary K3 surfaces $((X, \tau, \varphi), \alpha)$ of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$ are in bijective correspondence with the isometry classes of integral involutions of $\mathbb{L}_{K3}$ of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$ with respect to $G = \{\mathrm{id}\}$.*

The complete isometry invariants of integral involutions ψ of $\mathbb{L}_{K3}$ of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$ are the data

$$(2.1) \qquad\qquad\qquad (r(\psi), \ a(\psi), \ \delta_{\psi S}, \ H(\psi)).$$

See [10], Subsection 2.3 (after the equation (2.20)) for the definition of the invariant $\delta_{\psi S}$ and $H(\psi)$. Remark that $H(\psi)$ is a subgroup of the discriminant group $S_-/2S_- = S/2S = \mathbb{Z}/2\mathbb{Z}(\alpha([F]))$. Hence, $H(\psi) = 0$ or isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

All realizable data (2.1) are enumerated in [11]. We have 12 data of "Type 0" $(\Rightarrow H(\psi) = 0)$, 12 data of "Type Ia" , 39 data of "Type Ib" with $H(\psi) = 0$, and 39 data of "Type Ib" with $H(\psi) \cong \mathbb{Z}/2\mathbb{Z}$. See also TABLE 1 for $H(\psi) = 0$ case and TABLE 2 for $H(\psi) \cong \mathbb{Z}/2\mathbb{Z}$ case, where 0 or 1 in each cell stands for the value of $\delta_{\psi S}$.

Thus we have exactly **102** connected components of the moduli of real 2-elementary K3 surfaces of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$. By [10], we find how invariants of an involution ψ and those of its "related integral involution" $\sigma \circ \psi$ are calculated from each other. Here $\sigma : \mathbb{L}_{K3} \to \mathbb{L}_{K3}$ is defined to be the integral involution of $\mathbb{L}_{K3}$ whose fixed part is S. Identifying related pairs of anti-holomorphic involutions on each K3 surface, we have exactly **51** connected components.

a(ψ) / r(ψ)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
9									1									
8								1		0,1								
7							1		1		1							
6						1		1		0,1		1						
5					1		1		1		1		1					
4				1		0,1		1		0,1		1		0,1				
3			1		1		1		1		1		1		1			
2		0,1		1		0		1		0,1		1		0		1		
1	1		1						1		1						1	
0		0								0								0

TABLE 1. All the data $(r(\psi),\ a(\psi),\ \delta_{\psi S})$ for $H(\psi) = 0$.

a(ψ) / r(ψ)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
10										1								
9									0,1		1							
8								1		1		1						
7							1		0,1		1		1					
6						1		1		1		1		1				
5					0,1		1		0,1		1		0,1		1			
4				1		1		1		1		1		1		1		
3			1		0		1		0,1		1		0		1		0,1	
2		1						1								1		1
1	0								0								0	

TABLE 2. All the data $(r(\psi),\ a(\psi),\ \delta_{\psi S})$ for $H(\psi) \cong \mathbb{Z}/2\mathbb{Z}$.

2.4. **The regions A_+ and A_- in the real part $Y(\mathbb{R})$ of Y.** Let (X, τ, φ) be a real 2-elementary K3 surface of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$. We use the notation in Subsection 2.2. Let $X_\varphi(\mathbb{R})$ denote the real part of (X, φ), i.e., the fixed point set of φ, and let $Y(\mathbb{R})$ be the real part of the quotient surface Y with the real structure $\varphi_{\mathrm{mod}\ \tau}$.

The real part

$$\mathbb{R}A = \mathbb{R}A_0 \cup \mathbb{R}A_1$$

of the branch curve A divides $Y(\mathbb{R})$ into two regions [1] A_+ and A_-.

Either A_+ or A_- is doubly covered by the real part $X_\varphi(\mathbb{R})$, and the other by the real part $X_{\widetilde{\varphi}}(\mathbb{R})$, where we set $\widetilde{\varphi} := \tau \circ \varphi$. Since $X_\varphi(\mathbb{R})$ is always non-empty ([11], p.27), $Y(\mathbb{R})$, A_+, and A_- are also non-empty. Regions $A_\pm$ could have several connected components.

We distinguish two regions A_+ and A_- as follows.

Definition 2.14 (The regions A_+ and A_-). For a real 2-elementary K3 surface (X, τ, φ) of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$, we define the regions A_+ and A_- ($\subset Y(\mathbb{R})$) as follows:

$$\left\{ \begin{array}{ll} A_+ & := \pi(X_\varphi(\mathbb{R})) \quad \text{if } H(\psi) \cong \mathbb{Z}/2\mathbb{Z}, \\ A_- & := \pi(X_\varphi(\mathbb{R})) \quad \text{if } H(\psi) = 0. \end{array} \right.$$

---

[1]These two regions are called "positive curves" in [10].

Note that if $H(\psi) \cong \mathbb{Z}/2\mathbb{Z}$, then $H(\sigma \circ \psi) = 0$.

An important topological characterization of the invariant $H(\psi)$ is as follows.

Lemma 2.15 ([10]). *Suppose that the curve F is irreducible. Then we have $H(\psi) \cong \mathbb{Z}/2\mathbb{Z}$ if and only if $[F_\varphi(\mathbb{R})] = 0$ in $H_1(X_\varphi(\mathbb{R}); \mathbb{Z}/2\mathbb{Z})$, where we set $F_\varphi(\mathbb{R}) := X_\varphi(\mathbb{R}) \cap F$.*

2.5. **Real anti-bicanonical curves with one real double point on $\mathbb{R}\mathbb{F}_4$.** We now contract the exceptional curve $f = \pi(F)$ to a point. We get a blow up $\mathrm{bl} : Y \to \mathbb{F}_4$, where $\mathbb{F}_4$ is the 4-th Hirzebruch surface.

We set
$$s := \mathrm{bl}(A_0), \quad A_1' := \mathrm{bl}(A_1), \quad c := \mathrm{bl}(e),$$
and we have
$$\mathrm{bl}(A) = \mathrm{bl}(A_0) + \mathrm{bl}(A_1) = s + A_1' \quad \in |-2K_{\mathbb{F}_4}|,$$
i.e., $\mathrm{bl}(A)$ is an anti-bicanonical curve, where s is the exceptional section of $\mathbb{F}_4$ with $s^2 = -4$ and c is a fiber of the fibration $\mathbb{F}_4 \to s$ with $c^2 = 0$. Since $s \cdot A_1' = 0$, A_1' **does not intersect the section** s.

We have $-2K_{\mathbb{F}_4} \sim 12c + 4s$. It follows that $A_1' \in |12c + 3s|$.

Let
$$\mathbb{R}\mathrm{bl}(A) = \mathbb{R}s \cup \mathbb{R}A_1'$$
be the real part of the curve $\mathrm{bl}(A)$, where $\mathbb{R}s$ and $\mathbb{R}A_1'$ are the real parts of s and A_1' respectively.

We set
$$P_0 := \mathrm{bl}(f) \quad (\in \mathbb{R}c).$$

Since $A_1 \cdot f = 2$ in Y, A_1' has **one real double point** P_0. If A_1 intersects with f at two distinct points in Y, then they are real points or non-real conjugate points. In the former case P_0 is a **real node**, and in the latter case it is a **real isolated point**. Anyway it is a **nondegenerate double point**. If A_1 touches to f in Y, then P_0 is a **real cusp**. (a **degenerate** double point)

Thus, there are three topological types of the curve $\mathbb{R}A_1'$ near the double point P_0.

Node case: P_0 is a real node of $\mathbb{R}A_1'$.

Cusp case: P_0 is a real cusp of $\mathbb{R}A_1'$. (degenerate double point)

**Isolated point case:** P_0 is a real isolated point of $\mathbb{R}A_1'$.

See Figure 1 below.

We have $A_1' \cdot s = 0$ and $A_1' \cdot c = 3$ (A_1' is a trigonal curve). Since $f \cdot A_0 = 0$ in Y, the section s does not meet the double point P_0. We may assume that e does not pass through any intersection point of A_1 and f in Y. (See Figure 1.) Since $A_1 \cdot e = 1$ in Y, the intersection point of A_1 with e is **real** and does not meet f. Via the map bl, this point goes to a real intersection point of A_1' with c with multiplicity 1. Thus we set
$$P_1 := \mathrm{bl}(A_1 \cap e). \quad \text{(the intersection point)}$$

Since $A_1' \cdot c = 3$, A_1' intersects with c at P_0 with multiplicity 2 and at P_1 with multiplicity 1.

Since $A_1' \cap s = \emptyset$, any **non-contractible (possibly real singular)** components of $\mathbb{R}A_1'$ are "parallel" to $\mathbb{R}s$. See Figure 2, 3 and 4. But two types of non-contractible singular components in Figure 2 are real isotopic, and three types of non-contractible components in Figure 4 are real isotopic.

Definition 2.16. We call a connected component of $\mathbb{R}A_1'$ an **oval** if it has no real singular points and contractible in $\mathbb{R}\mathbb{F}_4$ (a torus), namely, realizes 0 in $H_1(\mathbb{R}\mathbb{F}_4; \mathbb{Z})$.
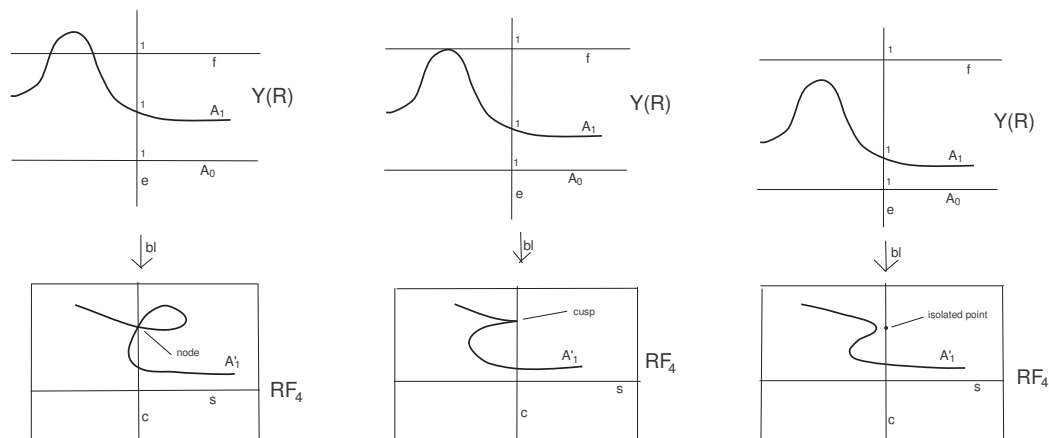
FIGURE 1. The real double point P_0 of the curve A_1'
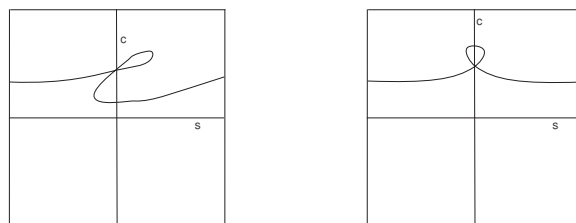
FIGURE 2. A non-contractible component with a node of type **Node (1)**
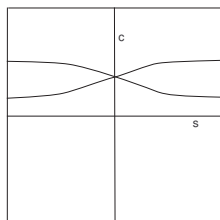
FIGURE 3. A non-contractible component with a node of type **Node (*)**

Remark 2.17. Since $A_1' \cdot c = 3$, A_1' is a **trigonal** curve on $\mathbb{F}_4$. See [3], [12] for related results. We see that the interior of each oval of $\mathbb{R}A_1'$ does not contain any other ovals. Moreover, the ovals are canonically ordered according to their projections to the base $\mathbb{R}s$.

We now get the following possibilities.
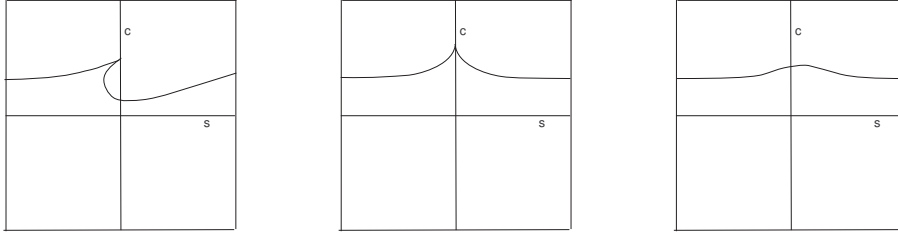
FIGURE 4. A non-contractible component with a **real cusp** or without singular points

Node case:

Node (1) case: We consider the case when **both** the node P_0 and the intersection point P_1 are contained in **the same** connected singular component of $\mathbb{R}A_1'$. Then $\mathbb{R}A_1'$ meets $\mathbb{R}c$ at only P_0 and P_1, and the singular component is **not contractible**. See Figure 2.

Note that $\mathbb{R}A_1'$ might have some ovals. Since $A_1' \cdot c = 3$, the interior of any oval of $\mathbb{R}A_1$ does not contain other ovals. The interior of the node also does not contain ovals.

The non-contractible singular component containing P_0 and P_1 and the section $\mathbb{R}s$ divide $\mathbb{R}\mathbb{F}_4$ (a torus) into three parts, which are **the interior of the node** and **two non-contractible regions**.

Definition 2.18 (The regions R_1 and R_2 in Node (1) case)**.** Let R_1 denote **the non-contractible region which is connected with the interior of the node in the blow up of** $\mathbb{R}\mathbb{F}_4$, and let R_2 denote the other non-contractible region. We define the integers α and β as follows:

$$(2.2) \qquad \begin{aligned} \alpha &:= \#\{\text{ovals contained in } R_1\}, \\ \beta &:= \#\{\text{ovals contained in } R_2\}. \end{aligned}$$

See **the left** figure of Figure 5.
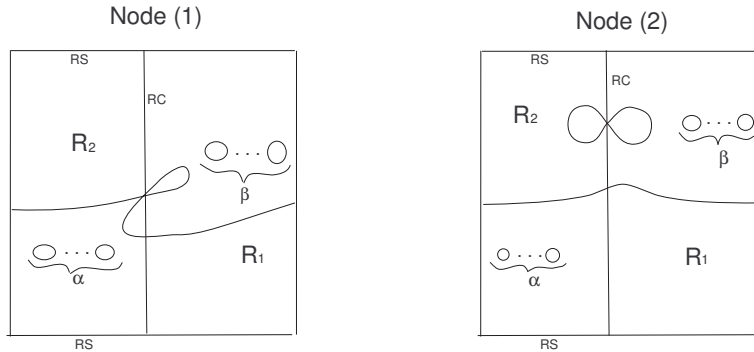
FIGURE 5. The regions R_1 and R_2 in **Node (1)** or **Node (2)** case

Node (2) case and Node (*) case: When the node P_0 and the intersection point P_1 respectively are contained in **different** connected components of $\mathbb{R}A_1'$, the component containing P_1 is nonsingular and not contractible like the rightmost figure of Figure 4 above. The component containing P_0 can be either **contractible** (the left figure of Figure 6) or **non-contractible** (Figure 3).
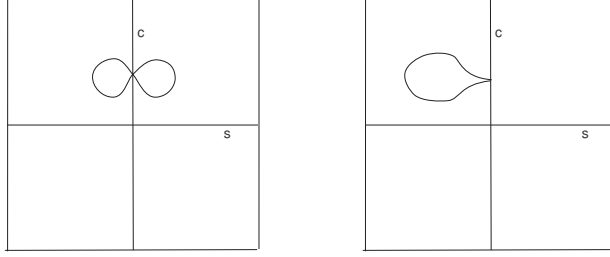
FIGURE 6. Contractible components with a real node or a real cusp

Node (2) case: When the component containing the node P_0 is **contractible** (the left figure of Figure 6), $\mathbb{R}A_1'$ might have some ovals. **The component containing P_1 and the section $\mathbb{R}s$ divide $\mathbb{R}\mathbb{F}_4$ into two regions.**

Definition 2.19 (The regions R_1 and R_2 in Node (2) case)**.** Let R_1 denote **the region which does not contain the contractible component containing the node P_0**, and let R_2 denote the other region. (Since $A_1' \cdot c = 3$, the interior of the contractible component containing P_0 and the interior of any oval of $\mathbb{R}A_1$ cannot contain any other ovals.)

We define the integers α and β by (2.2). See **the right** figure of Figure 5.

Node (*) case: If the component containing the node P_0 is **non-contractible** (see Figure 3), then $\mathbb{R}A_1'$ has **no ovals** (see Figure 7).
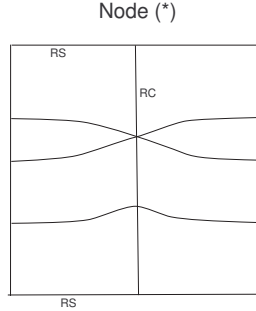
Node (*)

FIGURE 7. **Node (*)**

Cusp case:
   Cusp (1):   When both the cusp P_0 and the point P_1 are contained in **the same** connected component of $\mathbb{R}A_1'$, $\mathbb{R}A_1'$ meets $\mathbb{R}c$ at only P_0 and P_1, and the component containing P_0 and P_1 is not contractible (the leftmost figure of Figure 4). $\mathbb{R}A_1'$ might have some ovals. **The non-contractible component containing P_0 and P_1 and the section $\mathbb{R}s$ divide $\mathbb{R}\mathbb{F}_4$ into two regions** (the left figure of Figure 8). **One of these regions goes to a non-orientable region via the blow up of $\mathbb{R}\mathbb{F}_4$.**

Definition 2.20 (The regions R_1 and R_2 in Cusp (1) case)**.** Let R_2 denote the region which goes to a **non-orientable region** via the blow up of $\mathbb{R}\mathbb{F}_4$. and let R_1 denote the other region. (Since $A_1' \cdot c = 3$, the interior of any oval of $\mathbb{R}A_1$ does not contain any other ovals.)

We define the integers α and β by (2.2). See the left figure of Figure 8.
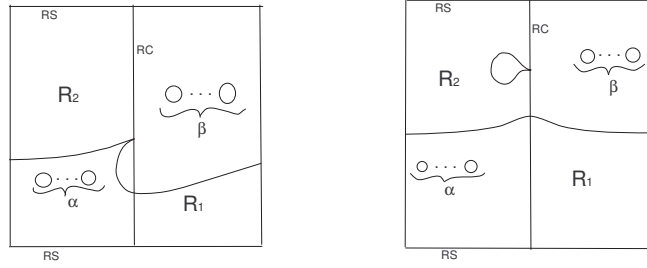
FIGURE 8. The regions R_1 and R_2 in **Cusp (1)** or **Cusp (2)** case

Cusp (2)   When the cusp P_0 and P_1 respectively are contained in **different** connected components of $\mathbb{R}A_1'$, the component containing P_1 is **not contractible** (the rightmost figure of Figure 4). The component containing the cusp P_0 should be **contractible** (the right figure of Figure 6). (If not, then this component would be like the middle figure of Figure 4 and the number of the intersection points of $\mathbb{R}A_1'$ with $\mathbb{R}c$ would be even. This contradicts with $A_1' \cdot c = 3$.) $\mathbb{R}A_1'$ might have some ovals. **The component containing P_1 and the section $\mathbb{R}s$ divide $\mathbb{R}\mathbb{F}_4$ into two regions.**

Definition 2.21 (The regions R_1 and R_2 in Cusp (2) case). Let R_1 denote the region which does not contain the contractible component which contains the cusp P_0, and let R_2 denote the other region. (the right figure of Figure 8) (Since $A_1' \cdot c = 3$, the interior of any oval of $\mathbb{R}A_1$ does not contain any ovals.)

We define the integers α and β by (2.2). See the right figure of Figure 8 above.

Isolated point case:

In this case the connected component containing P_1 is nonsingular and non-contractible like the right figure of Figure 4. $\mathbb{R}A_1'$ might have some ovals. **The component containing P_1 and the section $\mathbb{R}s$ divide $\mathbb{R}\mathbb{F}_4$ into two regions.**

Definition 2.22 (The regions R_1 and R_2 in Isolated point case). Let R_1 denote the region which **does not contain the isolated point**, and let R_2 denote the other region. (Since $A_1' \cdot c = 3$, the interior of any oval of $\mathbb{R}A_1$ does not contain any other ovals.)

We define the integers α and β by (2.2). See Figure 9.
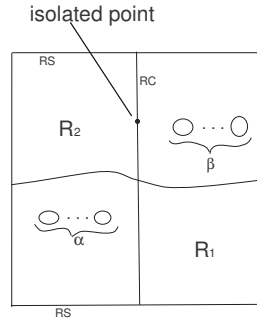
FIGURE 9. The regions R_1 and R_2 in **Isolated point** case

From the above argument, we have:

Proposition 2.23. *The real isotopy type of the singular connected component, which has one double point, of the curve* $\mathbb{R}A_1'$ *on* $\mathbb{R}\mathbb{F}_4$ *is one of the following* 6 *types:*

- Node (1), Node (2), Node (*),
- Cusp (1), Cusp (2), *or*
- Isolated point.

See TABLE 3. $\square$
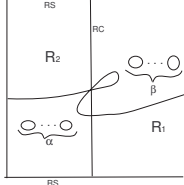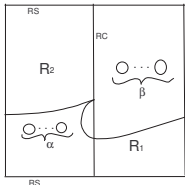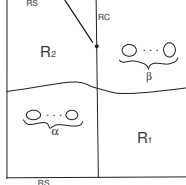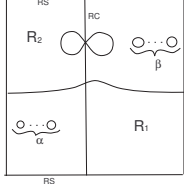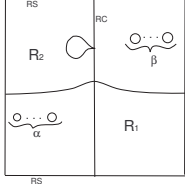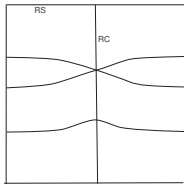
TABLE 3. Real isotopy types of the singular component of the curve $\mathbb{R}A_1'$.

2.6. **Topology of the real parts of K3 surfaces** X **viewed via the blow up** $Y \to \mathbb{F}_4$. We determine the topology of the real parts $X_\varphi(\mathbb{R})$ and $X_{\widetilde{\varphi}}(\mathbb{R})$ of a real 2-elementary K3 surface (X, τ, φ) of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$, and the real part $Y(\mathbb{R})$ of the quotient surface Y with the real structure $\varphi_{\mathrm{mod}\ \tau}$. Recall Proposition 2.23.

I. Node (1), Cusp (1) and **Isolated point** cases.

In these cases, by the definitions of the regions $A_\pm$ and R_1, R_2, we see that:

- A_+ is homeomorphic to the disjoint union of (an annulus with α holes) and (β disks), and
- A_- is homeomorphic to the disjoint union of ((((an annulus $\setminus D^2$)$\cup$ **Möbius band**) with β holes) and (α disks).

For example, see Figure 10 for **Node (1)** case.
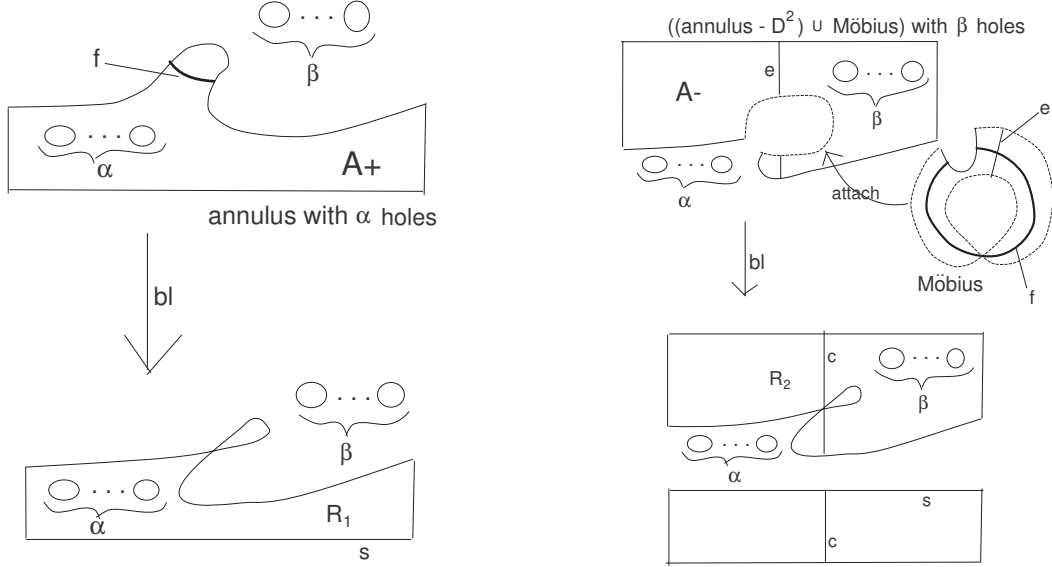
FIGURE 10. The regions A_+ and A_- in **Node (1)** case

Suppose that the invariant $H(\psi) = 0$ for the involution φ.

If F is **irreducible**, then we have $[X_\varphi(\mathbb{R}) \cap F] \neq 0$ and $\pi(X_\varphi(\mathbb{R})) = A_-$. On the other hand, for $\widetilde{\varphi}$, we have $[X_{\widetilde{\varphi}}(\mathbb{R}) \cap F] = 0$ and $\pi(X_{\widetilde{\varphi}}(\mathbb{R})) = A_+$.

We can say that

$$\alpha = \#\{\text{ovals whose \textbf{interiors} are contained in } \mathrm{bl}(A_-)\}, \text{ and}$$
$$\beta = \#\{\text{ovals whose \textbf{interiors} are contained in } \mathrm{bl}(A_+)\}.$$

Thus we have

$$X_\varphi(\mathbb{R}) \sim \Sigma_{2+\beta} \cup \alpha S^2.$$

Moreover, we have $(r(\psi), a(\psi), \delta_{\psi S}) \neq (10, 10, 0), (10, 8, 0)$,

$$r(\psi) = 9 + \alpha - \beta, \quad a(\psi) = 9 - \alpha - \beta.$$

On the other hand, we have

$$X_{\widetilde{\varphi}}(\mathbb{R}) \sim \Sigma_{1+\alpha} \cup \beta S^2.$$

Hence, we have $H(\sigma \circ \psi) \cong \mathbb{Z}/2\mathbb{Z}$ and

$$r(\sigma \circ \psi) = 10 - \alpha + \beta, \quad a(\sigma \circ \psi) = 10 - \alpha - \beta.$$

We omit the cusp and isolated point cases.

II. Node (2) and **Cusp (2)** cases.

In these cases, by the definitions of the regions $A_\pm$ and R_1, R_2, we see that

- A_+ is homeomorphic to the disjoint union of (an annulus with α holes) and $((\beta + 1)$ disks), and
- A_- is homeomorphic to the disjoint union of $(((\text{an annulus} \setminus D^2) \cup \textbf{Möbius band})$ with $(\beta + 1)$ holes) and (α disks).

For example, See Figure 11 for **Node (2)** case.
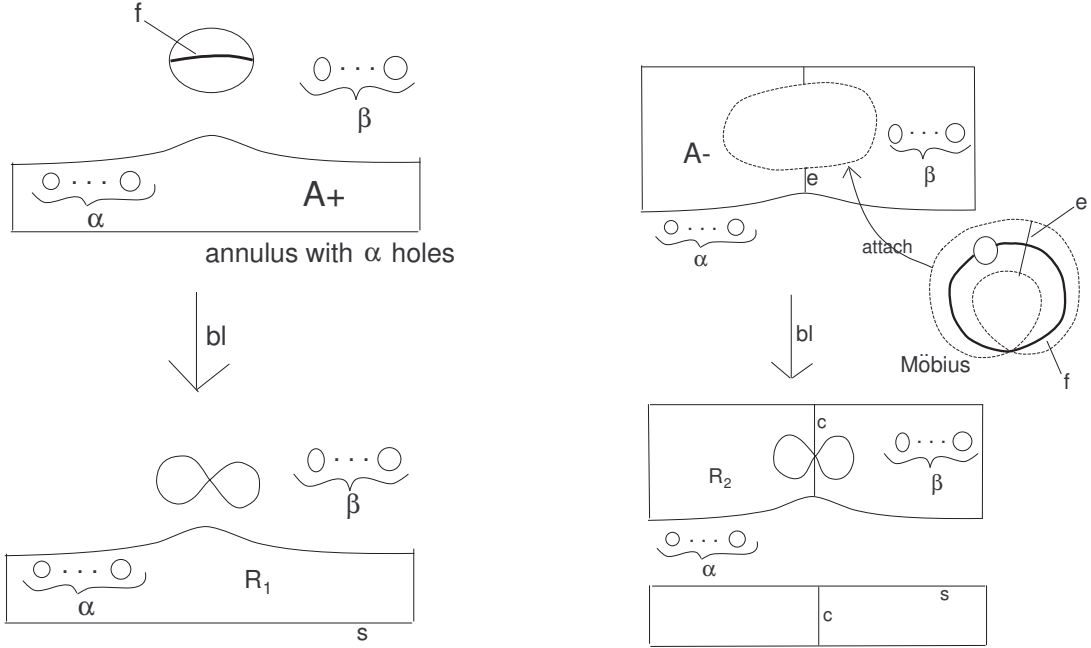
FIGURE 11. The regions A_+ and A_- in **Node (2)** case

Suppose that $A_- = \pi(X_\varphi(\mathbb{R}))$, namely, the invariant $H(\psi) = 0$ for the involution φ. Then we have

$$X_\varphi(\mathbb{R}) \sim \Sigma_{2+(\beta+1)} \cup \alpha S^2.$$

Moreover, we have $(r(\psi), a(\psi), \delta_{\psi S}) \neq (10, 10, 0), \ (10, 8, 0),$

$$r(\psi) = 8 + \alpha - \beta, \quad a(\psi) = 8 - \alpha - \beta.$$

On the other hand, we have $A_+ = \pi(X_{\widetilde{\varphi}}(\mathbb{R}))$ and

$$X_{\widetilde{\varphi}}(\mathbb{R}) \sim \Sigma_{1+\alpha} \cup (\beta + 1)S^2,$$

Moreover, we have $H(\sigma \circ \psi) \cong \mathbb{Z}/2\mathbb{Z}$ and

$$r(\sigma \circ \psi) = 11 - \alpha + \beta, \quad a(\sigma \circ \psi) = 9 - \alpha - \beta.$$

We omit the cusp cases.

III. Node (*) case.

In this case, we see that

- A_+ is homeomorphic to $D^2 \setminus 2D^2$ and
- A_- is the disjoint union of an **Möbius band** and an annulus.

See Figure 12.
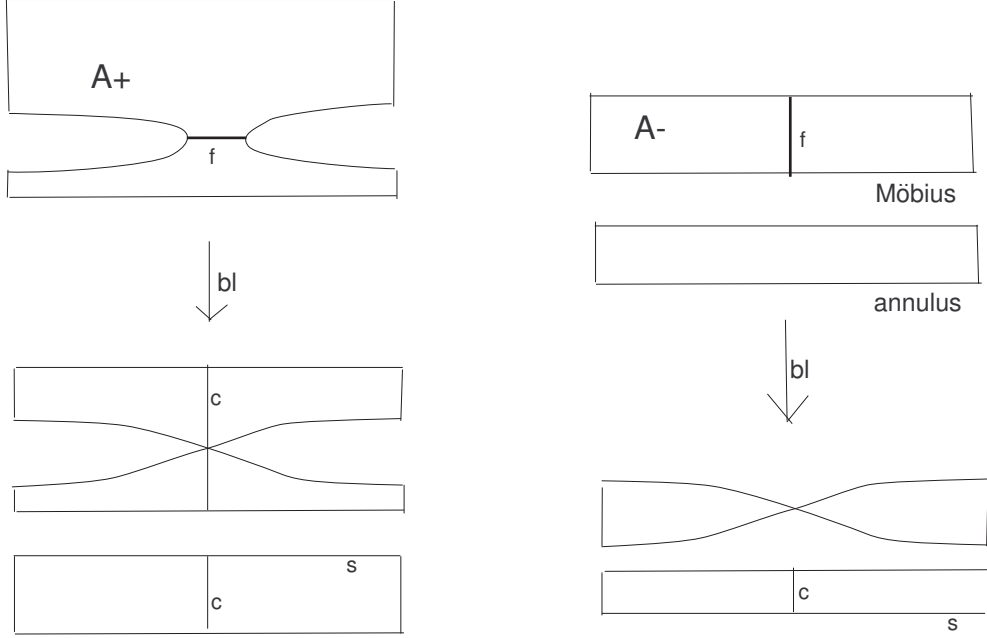
FIGURE 12. The regions A_+ and A_- in **Node (*)** case

Suppose that, for φ, $A_- = \pi(X_\varphi(\mathbb{R}))$. Then we see that

$$X_\varphi(\mathbb{R}) \sim T^2 \cup T^2,$$

and

$$A_+ = \pi(X_{\widetilde{\varphi}}(\mathbb{R})), \quad X_{\widetilde{\varphi}}(\mathbb{R}) \sim \Sigma_2.$$

Moreover, we have

$$H(\psi) = 0 \ \text{ and } \ (r(\psi), a(\psi), \delta_{\psi S}) = (10, 8, 0).$$

On the other hand, we have

$$H(\sigma \circ \psi) \cong \mathbb{Z}/2\mathbb{Z}$$

and

$$(r(\sigma \circ \psi), a(\sigma \circ \psi), \delta_{\sigma \circ \psi S}) = (9, 9, 0).$$

2.7. **Real isotopy types of real anti-bicanonical curves** $\mathbb{R}\mathrm{bl}(A)$ **with one real double point on** $\mathbb{R}A_1'$ **on** $\mathbb{R}\mathbb{F}_4$**.** In Node and Cusp cases, the number of connected components of $\mathbb{R}A_1'$ equals to that of connected components of $\mathbb{R}A_1$. Hence, we have

$$1 \leq \#\{\text{Connected components of } \mathbb{R}A_1'\} \leq 10.$$

In **Node (1)** and **Cusp (1)** cases, we have

$$0 \leq \alpha + \beta \leq 9.$$

If for φ, $A_- = \pi(X_\varphi(\mathbb{R}))$, namely, the invariant $H(\psi) = 0$, then $\alpha + \beta = 9 - a(\psi)$.

In **Node (2)** and **Cusp (2)** cases, we have

$$0 \leq \alpha + \beta \leq 8.$$

If for φ, $A_- = \pi(X_\varphi(\mathbb{R}))$, namely, the invariant $H(\psi) = 0$, then $\alpha + \beta = 8 - a(\psi)$.

In Isolated point case, the number of connected components of $\mathbb{R}A_1'$ equals to that of connected components of $\mathbb{R}A_1$ plus 1.

Hence, we have

$$2 \leq \#\{\text{Connected components of } \mathbb{R}A_1'\} \leq 11.$$

Hence, we have

$$0 \leq \alpha + \beta \leq 9.$$

If for φ, $A_- = \pi(X_\varphi(\mathbb{R}))$, namely, the invariant $H(\psi) = 0$, then $\alpha + \beta = 8 - a(\psi)$.

We already have all the isometry classes. Recall TABLE 1 AND TABLE 2 in Subsection 2.3.

Theorem 2.24. *We have the following.*

- *For* **each** *isometry class with* $H(\psi) = 0$, *the real isotopy type of a real anti-bicanonical curve* $\mathbb{R}\mathrm{bl}(A) = \mathbb{R}s \cup \mathbb{R}A_1'$ *on* $\mathbb{R}\mathbb{F}_4$ *with* **one real nondegenerate double point** *on* $\mathbb{R}A_1'$ *on* $\mathbb{R}\mathbb{F}_4$ *is one of the data listed up in Table 4.*
- *For* **each** *isometry class with* $H(\psi) \cong \mathbb{Z}/2\mathbb{Z}$, *the real isotopy type of a real anti-bicanonical curve* $\mathbb{R}\mathrm{bl}(A) = \mathbb{R}s \cup \mathbb{R}A_1'$ *on* $\mathbb{R}\mathbb{F}_4$ *with* **one real nondegenerate double point** *on* $\mathbb{R}A_1'$ *on* $\mathbb{R}\mathbb{F}_4$ *is one of the data listed up in Table 5.*

Note that the isometry class No.k and the isometry class No.k′ are related integral involutions for each $k = 1, \ldots, 50$. *The isometry class* $(10, 8, 0, H(\psi) = 0)$ *and* $(9, 9, 0, H(\psi) \cong \mathbb{Z}/2\mathbb{Z})$ *are also related integral involutions.* $\square$

Isometry class of type $((3,1,1),-\mathrm{id})$						Node (1)		Isolated point		Node (2)		Node (*)
No.	$r(\psi)$	$a(\psi)$	$\delta_{\varphi S}$	g	k	α	β	α	β	α	β	
1	1	1	1	10	0	0	8	0	8	0	7	
2	2	0	0	10	1	1	8	1	8	1	7	
3	2	2	0	9	0	0	7	0	7	0	6	
4	2	2	1	9	0	0	7	0	7	0	6	
5	3	1	1	9	1	1	7	1	7	1	6	
6	3	3	1	8	0	0	6	0	6	0	5	
7	4	2	1	8	1	1	6	1	6	1	5	
8	4	4	1	7	0	0	5	0	5	0	4	
9	5	3	1	7	1	1	5	1	5	1	4	
10	5	5	1	6	0	0	4	0	4	0	3	
11	6	2	0	7	2	2	5	2	5	2	4	
12	6	4	0	6	1	1	4	1	4	1	3	
13	6	4	1	6	1	1	4	1	4	1	3	
14	6	6	1	5	0	0	3	0	3	0	2	
15	7	3	1	6	2	2	4	2	4	2	3	
16	7	5	1	5	1	1	3	1	3	1	2	
17	7	7	1	4	0	0	2	0	2	0	1	
18	8	2	1	6	3	3	4	3	4	3	3	
19	8	4	1	5	2	2	3	2	3	2	2	
20	8	6	1	4	1	1	2	1	2	1	1	
21	8	8	1	3	0	0	1	0	1	0	0	
22	9	1	1	6	4	4	4	4	4	4	3	
23	9	3	1	5	3	3	3	3	3	3	2	
24	9	5	1	4	2	2	2	2	2	2	1	
25	9	7	1	3	1	1	1	1	1	1	0	
26	9	9	1	2	0	0	0	0	0			
27	10	0	0	6	5	5	4	5	4	5	3	
28	10	2	0	5	4	4	3	4	3	4	2	
29	10	2	1	5	4	4	3	4	3	4	2	
30	10	4	0	4	3	3	2	3	2	3	1	
31	10	4	1	4	3	3	2	3	2	3	1	
32	10	6	0	3	2	2	1	2	1	2	0	
33	10	6	1	3	2	2	1	2	1	2	0	
	10	8	0	2	1							$T^2 \cup T^2$
34	10	8	1	2	1	1	0	1	0			
35	11	1	1	5	5	5	3	5	3	5	2	
36	11	3	1	4	4	4	2	4	2	4	1	
37	11	5	1	3	3	3	1	3	1	3	0	
38	11	7	1	2	2	2	0	2	0			
39	12	2	1	4	5	5	2	5	2	5	1	
40	12	4	1	3	4	4	1	4	1	4	0	
41	12	6	1	2	3	3	0	3	0			
42	13	3	1	3	5	5	1	5	1	5	0	
43	13	5	1	2	4	4	0	4	0			
44	14	2	0	3	6	6	1	6	1	6	0	
45	14	4	0	2	5	5	0	5	0			
46	14	4	1	2	5	5	0	5	0			
47	15	3	1	2	6	6	0	6	0			
48	16	2	1	2	7	7	0	7	0			
49	17	1	1	2	8	8	0	8	0			
50	18	0	0	2	9	9	0	9	0			

TABLE 4. Candidates for real isotopy types of real anti-bicanonical curves $\mathbb{R}\mathrm{bl}(A)$ with one real **nondegenerate** double point on $\mathbb{R}A_1'$ on $\mathbb{R}\mathbb{F}_4$ for each isometry class of type $(S,\theta) \cong ((3,1,1),-\mathrm{id})$ with $H(\psi) = 0$

Isometry class of type $((3,1,1),-\mathrm{id})$						Node (1)		Isolated point		Node (2)		Node (*)
No.	$r(\psi)$	$a(\psi)$	$\delta_{\varphi S}$	g	k	α	β	α	β	α	β	
1'	18	2	1	1	8	0	8	0	8	0	7	
2'	17	1	0	2	8	1	8	1	8	1	7	
3'	17	3	0	1	7	0	7	0	7	0	6	
4'	17	3	1	1	7	0	7	0	7	0	6	
5'	16	2	1	2	7	1	7	1	7	1	6	
6'	16	4	1	1	6	0	6	0	6	0	5	
7'	15	3	1	2	6	1	6	1	6	1	5	
8'	15	5	1	1	5	0	5	0	5	0	4	
9'	14	4	1	2	5	1	5	1	5	1	4	
10'	14	6	1	1	4	0	4	0	4	0	3	
11'	13	3	0	3	5	2	5	2	5	2	4	
12'	13	5	0	2	4	1	4	1	4	1	3	
13'	13	5	1	2	4	1	4	1	4	1	3	
14'	13	7	1	1	3	0	3	0	3	0	2	
15'	12	4	1	3	4	2	4	2	4	2	3	
16'	12	6	1	2	3	1	3	1	3	1	2	
17'	12	8	1	1	2	0	2	0	2	0	1	
18'	11	3	1	4	4	3	4	3	4	3	3	
19'	11	5	1	3	3	2	3	2	3	2	2	
20'	11	7	1	2	2	1	2	1	2	1	1	
21'	11	9	1	1	1	0	1	0	1	0	0	
22'	10	2	1	5	4	4	4	4	4	4	3	
23'	10	4	1	4	3	3	3	3	3	3	2	
24'	10	6	1	3	2	2	2	2	2	2	1	
25'	10	8	1	2	1	1	1	1	1	1	0	
26'	10	10	1	1	0	0	0	0	0			
27'	9	1	0	6	4	5	4	5	4	5	3	
28'	9	3	0	5	3	4	3	4	3	4	2	
29'	9	3	1	5	3	4	3	4	3	4	2	
30'	9	5	0	4	2	3	2	3	2	3	1	
31'	9	5	1	4	2	3	2	3	2	3	1	
32'	9	7	0	3	1	2	1	2	1	2	0	
33'	9	7	1	3	1	2	1	2	1	2	0	
	9	9	0	2	0	1	0	1	0			Σ_2
34'	9	9	1	2	0	1	0	1	0			
35'	8	2	1	6	3	5	3	5	3	5	2	
36'	8	4	1	5	2	4	2	4	2	4	1	
37'	8	6	1	4	1	3	1	3	1	3	0	
38'	8	8	1	3	0	2	0	2	0			
39'	7	3	1	6	2	5	2	5	2	5	1	
40'	7	5	1	5	1	4	1	4	1	4	0	
41'	7	7	1	4	0	3	0	3	0			
42'	6	4	1	6	1	5	1	5	1	5	0	
43'	6	6	1	5	0	4	0	4	0			
44'	5	3	0	7	1	6	1	6	1	6	0	
45'	5	5	0	6	0	5	0	5	0			
46'	5	5	1	6	0	5	0	5	0			
47'	4	4	1	7	0	6	0	6	0			
48'	3	3	1	8	0	7	0	7	0			
49'	2	2	1	9	0	8	0	8	0			
50'	1	1	0	10	0	9	0	9	0			

TABLE 5. Candidates for real isotopy types of real anti-bicanonical curves $\mathbb{R}\mathrm{bl}(A)$ with one real **nondegenerate** double point on $\mathbb{R}A_1'$ on $\mathbb{R}\mathbb{F}_4$ for each isometry class of type $(S,\theta) \cong ((3,1,1),-\mathrm{id})$ with $H(\psi) \cong \mathbb{Z}/2\mathbb{Z}$

Remark 2.25. By Theorem 2.24, we find that each isometry class of integral involutions of the K3 lattice $\mathbb{L}_{K3}$ of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$ may contain several real isotopy types (for example, Node (1), Isolated point, and Node (2)) of real anti-bicanonical curves $\mathbb{R}\mathrm{bl}(A)$ with one real nondegenerate double point on $\mathbb{R}A_1'$ on $\mathbb{R}\mathbb{F}_4$. Hence, the realizability of all the real isotopy types listed in Table 4,5 has not been solved yet. However, we can take a real 2-elementary K3 surface for which F is irreducible in the same connected component of the moduli (see [11], notes after Theorem 2). Hence, for each isometry class, at least one of real isotopy types is realizable. Especially, Node (*) with the isometry class $(10, 8, 0, H(\psi) = 0)$ is realizable. It is conjectured that Node (1) with $(\alpha, \beta) = (1, 0)$ and Isolated point with $(\alpha, \beta) = (1, 0)$ do not exist for the isometry class $(9, 9, 0, H(\psi) \cong \mathbb{Z}/2\mathbb{Z})$. See also Remark 3.5 below.

Remark 2.26 ([11])**.** On the other hand, if we contract the exceptional curve $e = \pi(E)$ in Y, then we get a map onto the 3-th Hirzebruch surface $\mathbb{F}_3$: $\mathrm{bl}_1 : Y \to \mathbb{F}_3$. Then $s := \mathrm{bl}_1(A_0)$ is the exceptional section of $\mathbb{F}_3$ with $s^2 = -3$ and $c := \mathrm{bl}_1(f)$ is a fiber. We have

$$\mathrm{bl}_1(A) = s + \mathrm{bl}_1(A_1) \in |-2K_{\mathbb{F}_3}|.$$

$A_1 := \mathrm{bl}_1(A_1)$ is a real **nonsingular** curve of genus 9. The real isotopic classification of $\mathbb{R}A_1$ on $\mathbb{R}\mathbb{F}_3$ was already done in [11], Theorem 1.

In order to **distinguish** the real isotopy types of real anti-bicanonical curves with one real double point on $\mathbb{R}\mathbb{F}_4$, we expect that Itenberg's argument of the rigid isotopic classification of real curves of degree 6 on $\mathbb{R}\mathbb{P}^2$ with one nondegenerate double point are helpful. See [4],[5] and [6], and also the last section 4 of this paper. This classification corresponds to that of real **nonsingular** curves A in $|-2K_{\mathbb{F}_1}|$ on the first real Hirzebruch surface $\mathbb{R}\mathbb{F}_1$ when we blow up $\mathbb{P}^2$ at the nondegenerate double point to $\mathbb{F}_1$. The double coverings X of $\mathbb{F}_1$ ramified along the nonsingular curves A are real 2-elementary K3 surfaces of type $(S, \theta) \cong (\langle 2 \rangle \oplus \langle -2 \rangle, -\mathrm{id})$. Moreover, the classification of real curves of degree 6 on $\mathbb{R}\mathbb{P}^2$ with one nondegenerate double point is related to that of "non-increasing simplest degenerations" (conjunctions and contractions) of real **nonsingular** curves of degree 6 on $\mathbb{R}\mathbb{P}^2$. See [4] and [5].

Thus we next pay attention to the degenerations of real nonsingular anti-bicanonical curves on $\mathbb{R}\mathbb{F}_4$.

3. Degenerations of nonsingular real anti-bicanonical curves on $\mathbb{R}\mathbb{F}_4$

3.1. **Review of nonsingular real anti-bicanonical curves on $\mathbb{R}\mathbb{F}_4$.** The contents of this subsection are quoted from the last section of [10].

Let $\mathbb{U}$ be the even unimodular lattice of signature $(1, 1)$ (the hyperbolic plane). Consider real 2-elementary K3 surfaces (X, τ, φ) of type $(S, \theta) \cong (\mathbb{U}, -\mathrm{id})$. All these real 2-elementary K3 surfaces are $(\mathcal{D})$-nondegenerate.

Let A be the fixed point set of τ. Then A is a real nonsingular curve. We have $Y := X/\tau = \mathbb{F}_4$. Let $\pi : X \to \mathbb{F}_4$ be the quotient map. We use the same symbol A for its image in $\mathbb{F}_4$ by π. Then $A \in |-2K_{\mathbb{F}_4}|$. Let s be the exceptional section with $s^2 = -4$ of $\mathbb{F}_4$, and c the fiber of the fibration $f : \mathbb{F}_4 \to s$, where $c^2 = 0$. We have $-2K_{\mathbb{F}_4} \sim 12c + 4s$. The nonsingular curve A has **two irreducible components** s and A_1;

$$A = s \cup A_1 \quad \text{(disjoint union)}.$$

Conversely, any nonsingular curve A_1 in $|12c + 3s|$ gives a nonsingular curve $A = s + A_1$ in $|-2K_{\mathbb{F}_4}|$.

We set $C := \pi^*(c)$ and $E := \pi^*(s)/2$ in $H_2(X, \mathbb{Z})$. Then $C^2 = 0$, $E^2 = -2$ and $C \cdot E = 1$. C and E generate the fixed part of $\tau_* : H_2(X, \mathbb{Z}) \to H_2(X, \mathbb{Z})$. Hence, $S \cong \mathbb{Z}C + \mathbb{Z}E \cong \mathbb{U}$.

An isometry class of an integral involution $(\mathbb{L}_{K3}, \psi)$ of type $(S, \theta) \cong (\mathbb{U}, -\mathrm{id})$ is determined (see [10]) by the data

(3.1) $(r(\psi),\ a(\psi),\ \delta_\psi = \delta_{\psi S}).$

The complete list of the data (3.1) is given in Section 7 (Figure 30) of [10].

There are 14 isometry classes with $\delta_\psi = 0$ and 49 isometry classes with $\delta_\psi = 1$. Thus we have 63 classes.

If we identify related integral involutions, there are 10 isometry classes with $\delta_\psi = 0$ and 27 isometry classes with $\delta_\psi = 1$. Thus we have 37 classes.

If $(r(\psi), a(\psi), \delta_\psi) = (10, 10, 0)$, then $\mathbb{R}\mathbb{F}_4 = \emptyset$. Hence, $\mathbb{R}A = \emptyset$.

If $(r(\psi), a(\psi), \delta_\psi) \neq (10, 10, 0)$, then $\mathbb{R}\mathbb{F}_4$ is not empty and homeomorphic to a 2-torus. We have $\mathbb{R}A \supset \mathbb{R}s \neq \emptyset$. The real curve $\mathbb{R}A_1$ is contained in the open cylinder $\mathbb{R}\mathbb{F}_4 \setminus \mathbb{R}s$. The region $A_- = \pi(X_\varphi(\mathbb{R}))$ with the invariants (3.1) has the real isotopy type given in Figure 13.

When $(r(\psi), a(\psi), \delta_\psi) \neq (10, 10, 0)$ and $\neq (10, 8, 0)$, we set

$$g := (22 - r(\psi) - a(\psi))/2 \quad \text{and} \quad k := (r(\psi) - a(\psi))/2.$$
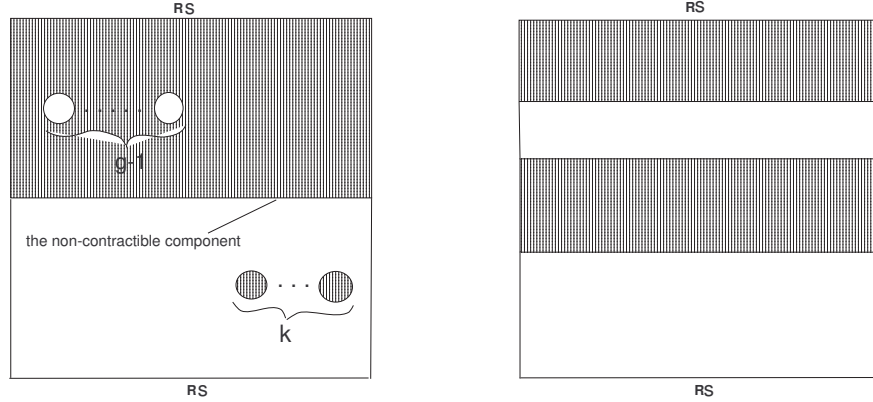
FIGURE 13. A region A_- with $(r(\psi), a(\psi), \delta_\psi) \neq (10, 8, 0), (10, 10, 0)$ and a region A_- with $(r(\psi), a(\psi), \delta_\psi) = (10, 8, 0)$

Since all real 2-elementary K3 surfaces of type $(S, \theta) \cong (\mathbb{U}, -\mathrm{id})$ are $(\mathcal{D})$-nondegenerate, by Theorem 2.11, we have:

Theorem 3.1 ([10], Theorem 27). *A connected component of the moduli of the regions*

$$A_- := \pi(X_\varphi(\mathbb{R})),$$

curves $A \in |-2K_{\mathbb{F}_4}|$, up to the action of the automorphisms group of $\mathbb{F}_4$ over $\mathbb{R}$ is determined by the data (3.1), equivalently, the real isotopy types of A_- and the invariants $\delta_\psi = \delta_{\psi S}$. All the data are given in Section 7 (Figure 30) of [10]. See also Figure 13.

3.2. **Degenerations of nonsingular real anti-bicanonical curves on** $\mathbb{R}\mathbb{F}_4$**.** We next introduce the notions of "non-increasing simplest degenerations" (conjunctions, contractions) of **nonsingular** real anti-bicanonical curves on $\mathbb{R}\mathbb{F}_4$ as analogies of Section 3 (pp.284–285) of [4].

As stated in the previous subsection, a nonsingular curve in $|-2K_{\mathbb{F}_4}|$ has two irreducible components s and A_1, where A_1 is a nonsingular curve in $|12c + 3s|$.

Let C_0 be a real curve in $|12c + 3s|$ on $\mathbb{F}_4$ with one nondegenerate double point, and

$$C_t \quad (-\varepsilon < t < \varepsilon)$$

be a smoothing (, where every C_t ($t \neq 0$) is a real nonsingular curve in $|12c + 3s|$ on $\mathbb{F}_4$,) such that

$$\sharp\{\text{ovals of a nonsingular curve } C_{t_{-1}} \} \;\geq\; \sharp\{\text{ovals of a nonsingular curve } C_{t_1} \}$$

for any $t_{-1} < 0$ and any $t_1 > 0$.

We call such a family C_t ($t_{-1} \leq t \leq 0$) a **non-increasing simplest degeneration** of $C_{t_{-1}}$ to C_0.

We do not know whether non-increasing simplest degenerations of $C_{t_{-1}}$ to C_0 are realizable for **any pair** $C_{t_{-1}}$ and C_0.

We define 8 kinds of non-increasing simplest degenerations.

Definition 3.2 (Conjunctions 1), 2), 1'), 2') and Contractions 3), 3'))**.** First we fix **an isometry class** of integral involutions of type $(S, \theta) \cong (\mathbb{U}, -\text{id})$ with $(r(\psi), a(\psi), \delta_\psi) \neq (10, 8, 0)$ and $\neq (10, 10, 0)$.

Take a corresponding real 2-elementary K3 surfaces (X, τ, φ) of type $(S, \theta) \cong (\mathbb{U}, -\text{id})$. Let $\pi : X \to X/\tau = \mathbb{F}_4$ be the quotient map. Then we get a real nonsingular curve $A = s + A_1$ in $|-2K_{\mathbb{F}_4}|$ where A_1 is a real nonsingular curve in $|12c + 3s|$ on $\mathbb{R}\mathbb{F}_4$. Suppose that *the fixed point set $X_\varphi(\mathbb{R})$ is homeomorphic to $\Sigma_g \cup kS^2$.* Then the region $\pi(X_\varphi(\mathbb{R}))$ $(\subset \mathbb{R}\mathbb{F}_4)$ is the disjoint union of an annulus with $(g - 1)$ holes and k disks. The boundary of the annulus with $(g - 1)$ holes consists of (see Figure 13) *the non-contractible component of $\mathbb{R}A_1$, $\mathbb{R}s$, and the $(g - 1)$ empty ovals.*

- **Conjunction 1)** The conjunction of the non-contractible component and one of the $(g - 1)$ empty ovals. (Recall that the annulus with $(g - 1)$ holes is covered by the fixed point set of the anti-holomorphic involution φ.)
- **Conjunction 1')** The conjunction of the non-contractible component and one of the k empty ovals. (Remark that the region (annulus) surrounded by the non-contractible component, $\mathbb{R}s$ and the k empty ovals is covered by the fixed point set of the related involution $\widetilde{\varphi}$ of the anti-holomorphic involution φ.)
- **Conjunction 2)** The conjunction of two of the $(g - 1)$ empty ovals.
- **Conjunction 2')** The conjunction of two of the k empty ovals.
- **Contraction 3)** The contraction of one of the $(g - 1)$ empty ovals.
- **Contraction 3')** The contraction of one of the k empty ovals.

Definition 3.3 (Conjunctions 4), 4'))**.** Consider **the isometry class** of integral involutions of type $(S, \theta) \cong (\mathbb{U}, -\text{id})$ with $(r(\psi), a(\psi), \delta_\psi) = (9, 9, 1)$ or $(11, 9, 1)$. Remark that these two involutions are **related**.

If $(r(\psi), a(\psi), \delta_\psi) = (9, 9, 1)$, then we have $g = 2$, $k = 0$. Then the region $\pi(X_\varphi(\mathbb{R}))$ $(\subset \mathbb{R}\mathbb{F}_4)$ is an annulus with one hole. The boundary of the annulus consists of *the non-contractible component of $\mathbb{R}A_1$, $\mathbb{R}s$, and the empty oval.*

- **Conjunction 4)** The empty oval conjuncts with itself and becomes the union of two real lines (Node (*)) on $\mathbb{R}\mathbb{F}_4$.

If $(r(\psi), a(\psi), \delta_\psi) = (11, 9, 1)$, then we have $g = 1$, $k = 1$. Then the region $\pi(X_\varphi(\mathbb{R}))$ $(\subset \mathbb{RF}_4)$ is the disjoint union of an annulus and one disk. The boundary of the annulus consists of *the non-contractible component of* $\mathbb{R}A_1$ *and* $\mathbb{R}s$.

- **Conjunction 4')** The empty oval conjuncts with itself and becomes the union of two real lines (Node (*)) on $\mathbb{RF}_4$.
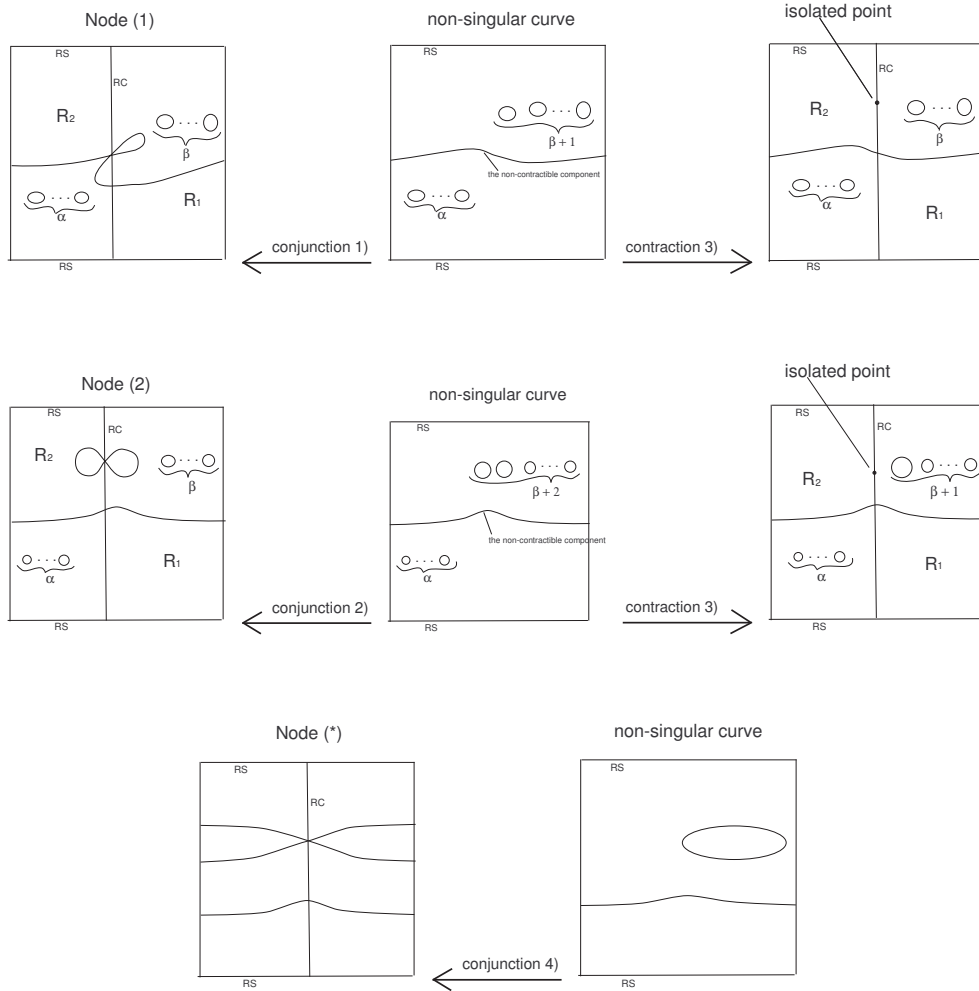
See Figure 14. Compare with Figure 13.

FIGURE 14. Non-increasing simplest degenerations of nonsingular real anti-bicanonical curves on $\mathbb{RF}_4$.

isometry class of type $(\mathbb{U}, -\mathrm{id})$ (nonsingular curve)						conjunction 1) $\rightarrow$ Node (1)	conjunction 2) $\rightarrow$ Node (2)	contraction 3) $\rightarrow$ Isolated point	
$r(\psi)$	$a(\psi)$	δ_ψ	g	k	$g-1$	α, β	α, β	α, β	No.
1	1	1	10	0	9	0, 8	0, 7	0, 8	1
2	0	0	10	1	9	1, 8	1, 7	1, 8	2
2	2	0	9	0	8	0, 7	0, 6	0, 7	3
2	2	1	9	0	8	0, 7	0, 6	0, 7	4
3	1	1	9	1	8	1, 7	1, 6	1, 7	5
3	3	1	8	0	7	0, 6	0, 5	0, 6	6
4	2	1	8	1	7	1, 6	1, 5	1, 6	7
4	4	1	7	0	6	0, 5	0, 4	0, 5	8
5	3	1	7	1	6	1, 5	1, 4	1, 5	9
5	5	1	6	0	5	0, 4	0, 3	0, 4	10
6	2	0	7	2	6	2, 5	2, 4	2, 5	11
6	4	0	6	1	5	1, 4	1, 3	1, 4	12
6	4	1	6	1	5	1, 4	1, 3	1, 4	13
6	6	1	5	0	4	0, 3	0, 2	0, 3	14
7	3	1	6	2	5	2, 4	2, 3	2, 4	15
7	5	1	5	1	4	1, 3	1, 2	1, 3	16
7	7	1	4	0	3	0, 2	0, 1	0, 2	17
8	2	1	6	3	5	3, 4	3, 3	3, 4	18
8	4	1	5	2	4	2, 3	2, 2	2, 3	19
8	6	1	4	1	3	1, 2	1, 1	1, 2	20
8	8	1	3	0	2	0, 1	0, 0	0, 1	21
9	1	1	6	4	5	4, 4	4, 3	4, 4	22
9	3	1	5	3	4	3, 3	3, 2	3, 3	23
9	5	1	4	2	3	2, 2	2, 1	2, 2	24
9	7	1	3	1	2	1, 1	1, 0	1, 1	25
9	9	1	2	0	1	0, 0	impossible	0, 0	26
10	0	0	6	5	5	5, 4	5, 3	5, 4	27
10	2	0	5	4	4	4, 3	4, 2	4, 3	28
10	2	1	5	4	4	4, 3	4, 2	4, 3	29
10	4	0	4	3	3	3, 2	3, 1	3, 2	30
10	4	1	4	3	3	3, 2	3, 1	3, 2	31
10	6	0	3	2	2	2, 1	2, 0	2, 1	32
10	6	1	3	2	2	2, 1	2, 0	2, 1	33
10	8	1	2	1	1	1, 0	impossible	1, 0	34
11	1	1	5	5	4	5, 3	5, 2	5, 3	35
11	3	1	4	4	3	4, 2	4, 1	4, 2	36
11	5	1	3	3	2	3, 1	3, 0	3, 1	37
11	7	1	2	2	1	2, 0	impossible	2, 0	38
12	2	1	4	5	3	5, 2	5, 1	5, 2	39
12	4	1	3	4	2	4, 1	4, 0	4, 1	40
12	6	1	2	3	1	3, 0	impossible	3, 0	41
13	3	1	3	5	2	5, 1	5, 0	5, 1	42
13	5	1	2	4	1	4, 0	impossible	4, 0	43
14	2	0	3	6	2	6, 1	6, 0	6, 1	44
14	4	0	2	5	1	5, 0	impossible	5, 0	45
14	4	1	2	5	1	5, 0	impossible	5, 0	46
15	3	1	2	6	1	6, 0	impossible	6, 0	47
16	2	1	2	7	1	7, 0	impossible	7, 0	48
17	1	1	2	8	1	8, 0	impossible	8, 0	49
18	0	0	2	9	1	9, 0	impossible	9, 0	50

TABLE 6. Conjunction 1), Conjunction 2) and Contraction 3)

We list up candidates for possible non-increasing simplest degenerations for each of 63 isometry classes of integral involutions of type $(S, \theta) \cong (\mathbb{U}, -\mathrm{id})$ with the invariant $(r(\psi), a(\psi), \delta_\psi)$.

Theorem 3.4. *We can enumerate up the following candidates:*

- *For Conjunction 1) Conjunction 2) Contraction 3), see* TABLE 6.
- *For Conjunction 1') Conjunction 2') Contraction 3'), see* TABLE 7.
- *For Conjunction 4) and 4'), see* TABLE 8. $\square$

isometry class of type $(\mathbb{U}, -\mathrm{id})$ (nonsingular curve)						conjunction 1') $\rightarrow$ Node (1)	conjunction 2') $\rightarrow$ Node (2)	contraction 3') $\rightarrow$ Isolated point	
$r(\psi)$	$a(\psi)$	δ_ψ	g	k	$g-1$	α, β	α, β	α, β	No.
19	1	1	1	9	0	0, 8	0, 7	0, 8	1'
18	0	0	2	9	1	1, 8	1, 7	1, 8	2'
18	2	0	1	8	0	0, 7	0, 6	0, 7	3'
18	2	1	1	8	0	0, 7	0, 6	0, 7	4'
17	1	1	2	8	1	1, 7	1, 6	1, 7	5'
17	3	1	1	7	0	0, 6	0, 5	0, 6	6'
16	2	1	2	7	1	1, 6	1, 5	1, 6	7'
16	4	1	1	6	0	0, 5	0, 4	0, 5	8'
15	3	1	2	6	1	1, 5	1, 4	1, 5	9'
15	5	1	1	5	0	0, 4	0, 3	0, 4	10'
14	2	0	3	6	2	2, 5	2, 4	2, 5	11'
14	4	0	2	5	1	1, 4	1, 3	1, 4	12'
14	4	1	2	5	1	1, 4	1, 3	1, 4	13'
14	6	1	1	4	0	0, 3	0, 2	0, 3	14'
13	3	1	3	5	2	2, 4	2, 3	2, 4	15'
13	5	1	2	4	1	1, 3	1, 2	1, 2	16'
13	7	1	1	3	0	0, 2	0, 1	0, 2	17'
12	2	1	4	5	3	3, 4	3, 3	3, 4	18'
12	4	1	3	4	2	2, 3	2, 2	2, 3	19'
12	6	1	2	3	1	1, 2	1, 1	1, 2	20'
12	8	1	1	2	0	0, 1	0, 0	0, 1	21'
11	1	1	5	5	4	4, 4	4, 3	4, 4	22'
11	3	1	4	4	3	3, 3	3, 2	3, 3	23'
11	5	1	3	3	2	2, 2	2, 1	2, 2	24'
11	7	1	2	2	1	1, 1	1, 0	1, 1	25'
11	9	1	1	1	0	0, 0	impossible	0, 0	26'
10	0	0	6	5	5	5, 4	5, 3	5, 4	27'
10	2	0	5	4	4	4, 3	4, 2	4, 3	28'
10	2	1	5	4	4	4, 3	4, 2	4, 3	29'
10	4	0	4	3	3	3, 2	3, 1	3, 2	30'
10	4	1	4	3	3	3, 2	3, 1	3, 2	31'
10	6	0	3	2	2	2, 1	2, 0	2, 1	32'
10	6	1	3	2	2	2, 1	2, 0	2, 1	33'
10	8	1	2	1	1	1, 0	impossible	1, 0	34'
9	1	1	6	4	5	5, 3	5, 2	5, 3	35'
9	3	1	5	3	4	4, 2	4, 1	4, 2	36'
9	5	1	4	2	3	3, 1	3, 0	3, 1	37'
9	7	1	3	1	2	2, 0	impossible	2, 0	38'
8	2	1	6	3	5	5, 2	5, 1	5, 2	39'
8	4	1	5	2	4	4, 1	4, 0	4, 1	40'
8	6	1	4	1	3	3, 0	impossible	3, 0	41'
7	3	1	6	2	5	5, 1	5, 0	5, 1	42'
7	5	1	5	1	4	4, 0	impossible	4, 0	43'
6	2	0	7	2	6	6, 1	6, 0	6, 1	44'
6	4	0	6	1	5	5, 0	impossible	5, 0	45'
6	4	1	6	1	5	5, 0	impossible	5, 0	46'
5	3	1	7	1	6	6, 0	impossible	6, 0	47'
4	2	1	8	1	7	7, 0	impossible	7, 0	48'
3	1	1	9	1	8	8, 0	impossible	8, 0	49'
2	0	0	10	1	9	9, 0	impossible	9, 0	50'

TABLE 7. Conjunction 1'), Conjunction 2') and Contraction 3')

isometry class of type $(\mathbb{U}, -\mathrm{id})$ (nonsingular curve)						conjunction 4)	isometry class of type $(\mathbb{U}, -\mathrm{id})$ (nonsingular curve)						conjunction 4')
$r(\psi)$	$a(\psi)$	δ_ψ	g	k	$g-1$		$r(\psi)$	$a(\psi)$	δ_ψ	g	k	$g-1$	
9	9	1	2	0	1	Node (*)	11	9	1	1	1	0	Node (*)

TABLE 8. Conjunctions 4) and 4')

Remark 3.5. We observe an obvious interesting correspondence between the data of Theorem 2.24 (Tables 4 — 5) and those of Theorem 3.4 (Tables 6 — 8).

If we prove the existence of all the non-increasing simplest degenerations listed in Tables 6, 7, and 8, then we can say that every real isotopy type with an anti-holomorphic involution in Theorem 2.24 except Node (1) with $(\alpha, \beta) = (1, 0)$ and Isolated point with $(\alpha, \beta) = (1, 0)$ in the isometry class $(9, 9, 0, H(\psi) \cong \mathbb{Z}/2\mathbb{Z})$ can be obtain by certain non-increasing simplest degeneration of a real nonsingular curve in $|12c + 3s|$ on $\mathbb{F}_4$. Recall Remark 2.25.

To state the following lemma 3.7, we distinguish the two anti-holomorphic involutions $\varphi_\pm$ on (X, τ) of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$ as follows.

Definition 3.6 (Definition of $\varphi_\pm$)**.** We define the anti-holomorphic involutions $\varphi_\pm$ on a 2-elementary K3 surface (X, τ) of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$ such that

$$\pi(X_{\varphi_\pm}(\mathbb{R})) = A_\pm$$

respectively. Here recall Definition 2.14 of the two regions $A_\pm$.

Thus, if $H(\psi) = 0$ for a real 2-elementary K3 surface (X, τ, φ) of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$, then we set $A_- = \pi(X_\varphi(\mathbb{R}))$, and

$$\varphi_- := \varphi \quad \text{and} \quad \varphi_+ := \widetilde{\varphi} = \tau \circ \varphi.$$

Comparing the data in Theorem 2.24 and Theorem 3.4, we have:

Lemma 3.7. *Fix an* **isometry class** *of integral involutions of* $\mathbb{L}_{K3}$ *of type* $(S, \theta) \cong (\mathbb{U}, -\mathrm{id})$ *with* $(r(\psi), a(\psi), \delta_\psi) \neq (10, 8, 0), (10, 10, 0)$*, and take a corresponding real 2-elementary K3 surface of type* $(S, \theta) \cong (\mathbb{U}, -\mathrm{id})$ *and a real* **nonsingular** *curve* $A = s + A_1$ *in* $|-2K_{\mathbb{F}_4}|$ *where* A_1 *is a real nonsingular curve in* $|12c + 3s|$ *on* $\mathbb{R}\mathbb{F}_4$*.*

Then we have the following:

- *Take real curves* A_1' *with* **one nondegenerate double point** *on* $\mathbb{F}_4$ *from the degenerations of types 1)—3) of the real* **nonsingular** *curve* A_1*.* **Choose** *real 2-elementary K3 surfaces* (X, τ, φ_-) *of type* $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$*. (See Definition 3.6 and also Remark 4.7 below.)*

  Then, for all such marked real 2-elementary K3 surfaces $((X, \tau, \varphi_-), \alpha)$ *obtained from* A_1'*, their associated integral involutions* $\alpha \circ (\varphi_-)_* \circ \alpha^{-1}$ *of* $\mathbb{L}_{K3}$ *are* **isometric** *with respect to* $G = \{\mathrm{id}\}$*.*
- *Take real curves* A_1' *with* **one nondegenerate double point** *on* $\mathbb{F}_4$ *from the degenerations of types 1')—3') of the real* **nonsingular** *curve* A_1*.* **Choose** *real 2-elementary K3 surfaces* (X, τ, φ_-) *of type* $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$*. (See Definition 3.6 and also Remark 4.7 below.)*

  Then, for all such marked real 2-elementary K3 surfaces $((X, \tau, \varphi_-), \alpha)$ *obtained from* A_1'*, their associated integral involutions* $\alpha \circ (\varphi_-)_* \circ \alpha^{-1}$ *of* $\mathbb{L}_{K3}$ *are* **isometric** *with respect to* $G = \{\mathrm{id}\}$*.* $\square$

Lemma 3.7 is an analogy of Proposition 3.3 of [4].

4. Appendix: Period domains and further problems

We use the terminology defined in Subsection 2.1, and review the formulations of period domains of marked real 2-elementary K3 surfaces in [10], [4].

Lemma 4.1. *Let $((X, \tau, \varphi),\ \alpha)$ and $((X', \tau', \varphi'),\ \alpha')$ be two marked real 2-elementary K3 surfaces of type (S, θ). Let $(\mathbb{L}_{K3}, \psi)$ and $(\mathbb{L}_{K3}, \psi')$ be their associated integral involutions respectively. Suppose that f is an* **isometry** *with respect to the group G from $(\mathbb{L}_{K3}, \psi)$ to $(\mathbb{L}_{K3}, \psi')$. Then $((X', \tau', \varphi'),\ f^{-1} \circ \alpha')$ is also a marked real 2-elementary K3 surface of type (S, θ), and its associated integral involution is also $(\mathbb{L}_{K3}, \psi)$.*

Especially, if f is an **automorphism** *of $(\mathbb{L}_{K3}, \psi)$ with respect to the group G, then $((X, \tau, \varphi),\ f \circ \alpha)$ is also a marked real 2-elementary K3 surface of type (S, θ), and its associated integral involution is also $(\mathbb{L}_{K3}, \psi)$. (Recall Definition 2.9.)*

Proof. We have $\psi' \circ f = f \circ \psi$, $f(S) = S$, and $f|_S \in G$. We have $\alpha' \circ {\varphi'}_* \circ {\alpha'}^{-1} \circ f = f \circ \alpha \circ \varphi_* \circ \alpha^{-1}$, and hence, $f^{-1} \circ \alpha' \circ {\varphi'}_* \circ {\alpha'}^{-1} \circ f = \alpha \circ \varphi_* \circ \alpha^{-1}$. We have

$$(f^{-1} \circ \alpha') \circ {\varphi'}_* \circ (f^{-1} \circ \alpha')^{-1} = \alpha \circ \varphi_* \circ \alpha^{-1} = \psi.$$

Here $(f^{-1} \circ \alpha') : H_2(X', \mathbb{Z}) \to \mathbb{L}_{K3}$ is an isometry with

$$(f^{-1} \circ \alpha')(H_{2+}(X', \mathbb{Z})) = f^{-1}(S) = S$$

and $(f^{-1} \circ \alpha') \circ {\varphi'}_* = f^{-1} \circ (\alpha' \circ {\varphi'}_*) = f^{-1} \circ (\psi' \circ \alpha') = \psi \circ f^{-1} \circ \alpha'$. Since $f^{-1} \circ \alpha'(H_{2+}(X', \mathbb{Z})) = S$ and $\psi|_S = \theta$, we have

$$(f^{-1} \circ \alpha') \circ {\varphi'}_* = \theta \circ f^{-1} \circ \alpha' \text{ on } H_{2+}(X', \mathbb{Z}).$$

Hence,

$$(f^{-1} \circ \alpha') : H_2(X', \mathbb{Z}) \to \mathbb{L}_{K3}$$

is another marking of (X', τ', φ'). If we take this new marking of (X', τ', φ'), then its associated integral involution is ψ, which is the same as $((X, \tau, \varphi),\ \alpha)$. Moreover, since $f|_S \in G$, we have $f(\mathcal{M}) = \mathcal{M}$. Hence, if we set $\beta := (f^{-1} \circ \alpha')$, then $\beta_{\mathbb{R}}^{-1}(V^+(S))$ contains a hyperplane section of X' and the set $\beta^{-1}(\Delta(S)_+)$ contains only classes of effective curves of X'. $\qquad\square$

Let us **fix** an integral involution $(\mathbb{L}_{K3}, \psi)$ of type (S, θ) through this subsection.

We set

$$\Omega_\psi := \{v\ (\neq 0) \in \mathbb{L}_{K3} \otimes \mathbb{C} \mid v \cdot v = 0,\ v \cdot \overline{v} > 0,\ v \cdot S = 0,\ \psi_{\mathbb{C}}(v) = \overline{v}\}/\mathbb{R}^{\times}.$$

Let $((X, \tau, \varphi),\ \alpha)$ be a marked real 2-elementary K3 surface of type (S, θ) satisfying

$$\alpha \circ \varphi_* \circ \alpha^{-1} = \psi,$$

namely, ψ is **the associated integral involution with** $((X, \tau, \varphi,)\ \alpha)$. We denote by

$$H\ \ (\subset H_2(X, \mathbb{C}))$$

the Poincare dual of the complex 1-dimensional space $H^{2,0}(X)$. Then we have the complex 1-dimensional subspace $\alpha_{\mathbb{C}}(H)$ in $\mathbb{L}_{K3} \otimes \mathbb{C}$. Then we have

$$\alpha_{\mathbb{C}}(H)\ \in \Omega_\psi.$$

Definition 4.2 (Periods). We say $\alpha_{\mathbb{C}}(H)$ the **period** of a marked real 2-elementary K3 surface $((X, \tau, \varphi),\ \alpha)$ of type (S, θ) satisfying $\alpha \circ \varphi_* \circ \alpha^{-1} = \psi$.

By Lemma 4.1, all marked real 2-elementary K3 surfaces whose associated integral involutions are isometric to $(\mathbb{L}_{K3}, \psi)$ with respect to G can be found in Ω_ψ if we change their markings appropriately.

A point in Ω_ψ is not necessarily the period of some marked real 2-elementary K3 surface of type (S, θ) satisfying $\alpha \circ \varphi_* \circ \alpha^{-1} = \psi$.

Definition 4.3 (Equivalence). We say a point $[v]$ $(\in \Omega_\psi)$ is **equivalent** to a point $[v']$ $(\in \Omega_\psi)$ if $[v'] = f_{\mathbb{C}}([v])$ for an automorphism f of $(\mathbb{L}_{K3}, \psi)$ of type (S, θ) with respect to the group G.

Lemma 4.4. *If a point $[v]$ $(\in \Omega_\psi)$ is equivalent to $[v']$ $(\in \Omega_\psi)$ and $[v]$ is the period of some marked real 2-elementary K3 surface $((X, \tau, \varphi),\ \alpha)$ of type (S, θ) satisfying $\alpha \circ \varphi_* \circ \alpha^{-1} = \psi$, then $[v']$ is also the period of a marked real 2-elementary K3 surface $((X, \tau, \varphi),\ \alpha')$ of type (S, θ) satisfying $(\alpha') \circ \varphi_* \circ (\alpha')^{-1} = \psi$ where α' is* **some another marking** *of (X, τ, φ).*

Proof. Since $[v]$ is equivalent to $[v']$, we have $[v'] = f_{\mathbb{C}}([v])$ for an automorphism f of $(\mathbb{L}_{K3}, \psi)$ of type (S, θ) with respect to the group G. By Lemma 4.1, $((X, \tau, \varphi),\ f \circ \alpha)$ is also a marked real 2-elementary K3 surface of type (S, θ) satisfying $(f \circ \alpha) \circ \varphi_* \circ (f \circ \alpha)^{-1} = \psi$. Moreover, the period of $((X, \tau, \varphi),\ f \circ \alpha)$ is

$$(f \circ \alpha)_{\mathbb{C}}(H) = f_{\mathbb{C}}(\alpha_{\mathbb{C}}(H)) = f_{\mathbb{C}}([v]) = [v']$$

where H $(\subset H_2(X, \mathbb{C}))$ is the Poincare dual of $H^{2,0}(X)$. It is sufficient that we set $\alpha' = f \circ \alpha$. $\square$

Remark 4.5 ([10])**.** By the global Torelli theorem, if two periods are equivalent, then corresponding marked real 2-elementary K3 surfaces are analytic isomorphic (see Definition 2.10). The converse is also true.

The domain Ω_ψ has two connected components which are interchanged by $-\psi$. We see $-\psi$ is an **automorphism** of $(\mathbb{L}_{K3}, \psi)$ with respect to the group G. Hence, by Lemma 4.4 and Remark 4.5, it is sufficient that we investigate the quotient space $\Omega_\psi / -\psi$. We set

$$\mathbb{L}_\pm := \{x \in \mathbb{L}_{K3} \mid \psi(x) = \pm x\}.$$

Note that the lattices $\mathbb{L}_\pm$ depend on the integral involution ψ.

For $[v] \in \Omega_\psi$ $(v \in \mathbb{L}_{K3} \otimes \mathbb{C})$, we have the decomposition $v = v_+ + v_-$, where $v_\pm \in \mathbb{L}_\pm \otimes \mathbb{R}$.

We restrict ourselves the case when $S \subset \mathbb{L}_-$, namely, $\theta = -\mathrm{id}$, and set

$$\mathbb{L}_{-,S} := \mathbb{L}_- \cap S^\perp.$$

$\mathbb{L}_{-,S}$ also depends on the integral involution ψ.

Since $v_- \in \mathbb{L}_{-,S} \otimes \mathbb{R}$ and $v_+^2 = v_-^2 > 0$, we see that $\mathbb{L}_+$, $\mathbb{L}_{-,S}$ is a hyperbolic lattice.

Let $\mathcal{L}_+$, $\mathcal{L}_{-,S}$ be the hyperbolic spaces obtained from $\mathbb{L}_+ \otimes \mathbb{R}$, $\mathbb{L}_{-,S} \otimes \mathbb{R}$ respectively. Then we have

$$\Omega_\psi / -\psi\ =\ \mathcal{L}_+ \times \mathcal{L}_{-,S} \quad \text{(a direct product)}.$$

We now fix a primitive hyperbolic 2-elementary sublattice S with $S \cong (3, 1, 1)$ of the K3 lattice $\mathbb{L}_{K3}$ and set $\theta := -\mathrm{id}$. Remark that $G = \{\mathrm{id}\}$ for this case (Remark 2.12).

We use the terminology and facts stated in Subsection 2.2. For a real 2-elementary K3 surface (X, τ, φ) of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$, the fixed point curve A is a disjoint union of A_0 and A_1. X has an elliptic fibration with the exceptional section A_0. Its unique reducible fiber is $E + F$. There are two kinds of the curve F, which is irreducible or not. The classes $[A_0]$, $[E]$ and $[F]$ generate $H_{2+}(X, \mathbb{Z})$. We have an orthogonal decomposition

$$\mathbb{Z}([A_0],\ [E] + [F]) \oplus \mathbb{Z}([F])$$

of $H_{2+}(X, \mathbb{Z})$. The subgroups $\mathbb{Z}([A_0],\ [E] + [F])$, $\mathbb{Z}([F])$ are isometric to the hyperbolic plane and the lattice $\langle -2 \rangle$ respectively.

We fix an integral involution $(\mathbb{L}_{K3}, \psi)$ of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$. Moreover, we fix a decomposition $S = \mathbb{U} \oplus \langle -2 \rangle$, where $\mathbb{U}$ is a sublattice of S isometric to the hyperbolic plane. Let $\mathcal{F}$ be the generator of $\langle -2 \rangle$ with $\mathcal{M} \cdot \mathcal{F} \geq 0$.

We consider marked real 2-elementary K3 surfaces $((X, \tau, \varphi), \alpha)$ of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$ (see Definition 2.4) such that

$$\alpha(\mathbb{Z}([A_0],\ [E] + [F])) = \mathbb{U}, \quad \alpha([F]) = \mathcal{F},$$

and

$$\alpha \circ \varphi_* \circ \alpha^{-1} = \psi.$$

Any real 2-elementary K3 surface (X, τ, φ) of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$ has such a marking α.

We want to know a criterion for the double point of a real anti-bicanonical curve $\mathbb{R}\mathrm{bl}(A)$ with one real double point on $\mathbb{R}A_1'$ on $\mathbb{R}\mathbb{F}_4$ to be **nondegenerate**. At present we can prove the following lemma.

Lemma 4.6. *For* $[\omega] \in \Omega_\psi / -\psi$, $[\omega]$ *is the period of a marked real* 2-*elementary K3 surface of type* $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$ *(see Definition 2.4) such that* $\alpha(\mathbb{Z}([A_0], [E]+[F])) = \mathbb{U},\ \ \alpha([F]) = \mathcal{F}$, *and* $\alpha \circ \varphi_* \circ \alpha^{-1} = \psi$ *obtained from a real anti-bicanonical curve* $\mathbb{R}\mathrm{bl}(A)$ *with one real* **nondegenerate** *double point on* $\mathbb{R}A_1'$ *on* $\mathbb{R}\mathbb{F}_4$ **if** *there are no* $\mathbf{v}$ $(\neq \pm\mathcal{F})$ *in* $\mathbb{L}_{K3}$ *satisfying that* $\mathbf{v} \cdot \omega = 0$, $\mathbf{v} \cdot \mathbb{U} = 0$, *and* $\mathbf{v}^2 = -2$.

Proof. For a real anti-bicanonical curve with one real degenerate double point on $\mathbb{R}\mathbb{F}_4$, the corresponding marked real 2-elementary K3 surface $(X, \tau, \varphi, \alpha)$ has a unique reducible fiber $E + F$ where $F^2 = -2$, F is the union of two nonsingular rational curves F' and F''. Here F' and F'' are conjugate by τ and $F' \cdot F'' = 1$. Hence, $(F')^2 = (F'')^2 = -2$. (see Subsection 2.2) Both $[F']$ and $[F'']$ are orthogonal to $[A_0]$ and $[E] + [F]$. We set $\mathbf{v} := \alpha[F']$ or $\alpha[F'']$. Since $\mathbb{U} = \alpha(\mathbb{Z}([A_0], [E] + [F]))$, we have $\mathbf{v}$ $(\neq \pm\mathcal{F})$, $\mathbf{v} \cdot \omega = 0$, $\mathbf{v} \cdot \mathbb{U} = 0$, and $\mathbf{v}^2 = -2$ for the period $[\omega]$ of $(X, \tau, \varphi, \alpha)$. $\qquad\square$

Problem 1 (cf. [4], the top of p.281)**.** Is the converse of Lemma 4.6 also true?

If the converse of Lemma 4.6 is true, then we can get the precise image $(\subset \mathcal{L}_+ \times \mathcal{L}_{-,S})$ of the period map on the set of all marked real 2-elementary K3 surfaces of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$ such that $\alpha(\mathbb{Z}([A_0],\ [E] + [F])) = \mathbb{U}$, $\alpha([F]) = \mathcal{F}$, and $\alpha \circ \varphi_* \circ \alpha^{-1} = \psi$ obtained from real anti-bicanonical curves $\mathbb{R}\mathrm{bl}(A)$ with one real **nondegenerate** double point on $\mathbb{R}A_1'$ on $\mathbb{R}\mathbb{F}_4$.

Let $\mathbf{v}$ be an element of $\mathbb{L}_+$ with square -2. The reflection on $\mathcal{L}_+$ with respect to the real hyperplane $\mathbf{v}^\perp$ is well-defined and it sends a point in $\Omega_\psi / -\psi$ to its equivalent point. Also, the reflection on $\mathcal{L}_{-,S}$ with respect to the real hyperplane $\mathbf{v}^\perp$ where $\mathbf{v}$ is an element of $\mathbb{L}_{-,S}$ with square -2 is well-defined and it sends a point in $\Omega_\psi / -\psi$ to its equivalent point. Let

$$\Omega_+ \quad (\text{respectively, } \Omega_{-,S})$$

be the open fundamental domains with respect to the groups generated by the reflections with respect to the real hyperplanes $\mathbf{v}^\perp$ satisfying $\mathbf{v}^2 = -2$ and $\mathbf{v} \in \mathbb{L}_+$ (respectively, $\mathbb{L}_{-,S}$) and we consider the direct product $\Omega_+ \times \Omega_{-,S}$.

If the converse of Lemma 4.6 is true, then the periods of marked real 2-elementary K3 surfaces $(X, \tau, \varphi, \alpha)$ of type $((3, 1, 1), -\mathrm{id})$ obtained from some real anti-bicanonical curves with one real **nondegenerate** double point on $\mathbb{R}\mathbb{F}_4$ and satisfying $\alpha \circ \varphi_* \circ \alpha^{-1} = \psi$ are contained in $\Omega_+ \times \Omega_{-,S}$ up to equivalence. Moreover, we should remove more $\omega = \omega_+ + \omega_-$ orthogonal to some $\mathbf{v} \in \mathbb{L}_{K3}$ such that $\mathbf{v}$ $(\neq \pm\mathcal{F})$, $\mathbf{v} \cdot \mathbb{U} = 0$, $\mathbf{v}^2 = -2$, $\mathbf{v} \notin \mathbb{L}_+$ and $\mathbf{v} \notin \mathbb{L}_{-,S}$. Hence, as the argument before Theorem 2.1 of Itenberg's paper [4], we should remove some (-6)-orthogonal real hyperplanes (extra "walls") from $\Omega_{-,S}$ or Ω_+. This seems to be the reason why the connected components of the moduli space in the sense of [10] (equivalently, the isometry classes of integral involutions

of the K3 lattice) cannot distinguish the topological types (node or isolated point) of the real double points (Remark 2.25).

Problem 2 (cf. [4], Theorem 2.1)**.** Formulate some period domain, say it were $\Omega_*^{\mathbb{R}\mathbb{F}_4}$, whose connected components (up to equivalence) are in **bijective correspondence** with the connected components of the moduli space (up to the action of the automorphism group of $\mathbb{F}_4$ over $\mathbb{R}$) of real anti-bicanonical curves with one real nondegenerate double point on $\mathbb{R}\mathbb{F}_4$ which yield marked real 2-elementary K3 surfaces $((X, \tau, \varphi_-), \alpha)$ satisfying $\alpha \circ (\varphi_-)_* \circ \alpha^{-1} = \psi$.

Remark 4.7. For $(\mathbb{L}_{K3}, \psi)$, either $\mathbb{L}_+$ or $\mathbb{L}_{-,S}$ does not contain any element $\mathbf{v}$ such that

$$\mathbf{v} \equiv \mathcal{F} \pmod{2\mathbb{L}_{K3}}.$$

For the anti-holomorphic involution "φ_-" (recall Definition 3.6), $\mathbb{L}_{-,S}$ contains an element $\mathbf{v}$ such that $\mathbf{v} \equiv \mathcal{F} \pmod{2\mathbb{L}_{K3}}$. This phenomenon is similar to the argument in [4].

We are also interested in the correspondences of the Coxeter graphs (see [4], [5]) obtained from isometry classes of integral involutions of the K3 lattice $\mathbb{L}_{K3}$ of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$ and the non-increasing simplest degenerations of nonsingular curves. Problems concerning this topic are as follows.

We fix an integral involution $(\mathbb{L}_{K3}, \psi)$ of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$. Recall

$$\mathbb{L}_\pm := \{x \in \mathbb{L}_{K3} \mid \psi(x) = \pm x\},$$

$\mathbb{U}$, and $\mathbb{L}_{-,\mathbb{U}} := \mathbb{L}_- \cap \mathbb{U}^\perp$. $\mathbb{L}_{-,\mathbb{U}}$ is also hyperbolic. Let $\mathcal{L}_{-,\mathbb{U}}$ be the hyperbolic space obtained from $\mathbb{L}_{-,\mathbb{U}} \otimes \mathbb{R}$. The group generated by the reflections with respect to real hyperplanes $\mathbf{v}^\perp$ such that $\mathbf{v} \in \mathbb{L}_{-,\mathbb{U}}$ with $\mathbf{v}^2 = -2$ acts on the hyperbolic space $\mathcal{L}_{-,\mathbb{U}}$. Let $\widetilde{\Omega_-}$ be one of its fundamental domains which has a face orthogonal to F. Let C be the Coxeter graph of $\widetilde{\Omega_-}$.

Definition 4.8. • Let C' be the graph which is obtained by removing all thick or dotted edges from C.
- Consider the group of symmetries of C' obtained from some automorphism of $(\mathbb{L}_{K3}, \psi, \mathbb{U})$. Let C'' be the quotient graph of C' by the action of the group.
- Let e be the vertex of C corresponding to F and let e' be the class (in C'') containing e.
- Let K be the connected component of C'' containing e'.

Problem 3 (cf. [4], Proposition 3.1)**.** Does the number (up to equivalence) of connected components of $\Omega_*^{\mathbb{R}\mathbb{F}_4}$ coincide with that of vertices of the graph K ?

Degenerations and the graph P**.** We fix an isometry class of integral involutions of type $(S, \theta) \cong (\mathbb{U}, -\mathrm{id})$ with $(r(\psi), a(\psi), \delta_\psi) \neq (10, 8, 0), (10, 10, 0)$. Then we get a real 2-elementary K3 surface (X, τ, φ) of type $(S, \theta) \cong (\mathbb{U}, -\mathrm{id})$ and a real **nonsingular** curve $A = s + A_1$ in $|-2K_{\mathbb{F}_4}|$ where A_1 is a real nonsingular curve in $|12c + 3s|$ on $\mathbb{R}\mathbb{F}_4$.

We define the graphs P as follows (See Definition 3.2).

Definition 4.9. • The **vertices** of P are all the **rigid isotopy classes** of real curves A_1' in $|12c+3s|$ with one nondegenerate double point on $\mathbb{F}_4$ obtained from the degenerations of types 1)—3) of the real nonsingular curve A_1 with φ.
- Two vertices of P are connected by an **edge** if one rigid isotopy class is obtained from the conjunction of the ovals $\mathcal{E}_1$ and the non-contractible component of A_1 (Conjunction 1)), and the other rigid isotopy class is obtained from the contraction of the oval $\mathcal{E}_1$ (Contraction 3)).

- Two vertices of P are connected by an **edge** if one rigid isotopy class is obtained from the conjunction of the ovals $\mathcal{E}_1$ and $\mathcal{E}_2$ of A_1 (Conjunction 2)), and the other rigid isotopy class is obtained from the contraction of the oval $\mathcal{E}_1$ (Contraction 3)).

Problem 4 (cf. [4], Proposition 3.4)**.** Fix a real **nonsingular** curve $A = s + A_1$ in $|-2K_{\mathbb{F}_4}|$ where A_1 is a real nonsingular curve in $|12c + 3s|$ on $\mathbb{RF}_4$ and φ as above.

Let us get an **arbitrary** real curve A_1' in $|12c + 3s|$ with one nondegenerate double point on $\mathbb{F}_4$ obtained from one of the degenerations of types 1)—3) of A_1 with φ, and construct the graph K (see Lemma 3.7 and Definition 4.8) from A_1' and φ_-. Then, is the graph K isomorphic to the graph P ?

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