Journal *of* Singularities
Volume 1 (2010), 1-12

received 16 June 2009
in revised form 30 November 2009
DOI: 10.5427/jsing.2010.1a

SPECIAL GENERIC MAPS ON OPEN 4-MANIFOLDS

OSAMU SAEKI

Abstract. We characterize those smooth 1-connected open 4-manifolds with certain finite type properties which admit proper special generic maps into 3-manifolds. As a corollary, we show that a smooth 4-manifold homeomorphic to $\mathbf{R}^4$ admits a proper special generic map into $\mathbf{R}^n$ for some $n = 1, 2$ or 3 if and only if it is diffeomorphic to $\mathbf{R}^4$. We also characterize those smooth 4-manifolds homeomorphic to $L \times \mathbf{R}$ for some closed orientable 3-manifold L which admit proper special generic maps into $\mathbf{R}^3$.

1. Introduction

A *special generic map* $f : M \to N$ between smooth manifolds is a smooth map with at most *definite fold singularities*, which have the normal form

$$(x_1, x_2, \ldots, x_m) \mapsto (x_1, x_2, \ldots, x_{n-1}, x_n^2 + x_{n+1}^2 + \cdots + x_m^2), \tag{1.1}$$

where $m = \dim M \geq \dim N = n$. In particular, submersions are considered special generic maps.

In [24, 25], the author has shown that a smooth connected closed m-dimensional manifold M admits a special generic map into $\mathbf{R}^n$ for every n with $1 \leq n \leq m$ if and only if M is diffeomorphic to the standard m-sphere S^m. Furthermore, certain cobordism groups of special generic maps into $\mathbf{R}$ are naturally isomorphic to the h-cobordism groups of homotopy spheres in higher dimensions [26]. In [27, 28] Sakuma and the author found some pairs of homeomorphic smooth closed 4-manifolds such that one of them admits a special generic map into $\mathbf{R}^3$, while the other does not. These show that special generic maps are sensitive to detecting distinct differentiable structures on a given topological manifold.

On the other hand, it has been known that a smooth m-dimensional manifold is homeomorphic to $\mathbf{R}^m$ if and only if it is diffeomorphic to the standard $\mathbf{R}^m$, provided $m \neq 4$ (see [18, 31]), while for $m = 4$, there exist uncountably many distinct differentiable structures on $\mathbf{R}^4$ (for example, see [4, 8, 10, 32]). In fact, it is known that most open 4-manifolds admit infinitely (and very often, uncountably) many distinct differentiable structures [1, 3, 7, 9].

In this paper, we characterize those smooth 1-connected open 4-manifolds of "finite type" which admit proper special generic maps into 3-manifolds, using the solution to the Poincaré Conjecture in dimension three (see [19, 20, 21] or [17], for example). Here, an open 4-manifold is of finite type if its homology is finitely generated and it has only finitely many ends, whose associated fundamental groups

2000 *Mathematics Subject Classification.* Primary 57N13; Secondary 57R45, 57R55, 58K15.
Key words and phrases. Proper special generic map, differentiable structure, open 4-manifold, end.
The author has been supported in part by Grant-in-Aid for Scientific Research (No. 19340018), Japan Society for the Promotion of Science.

are stable and finitely presentable. As a corollary, we show that a smooth 4-manifold homeomorphic to $\mathbf{R}^4$ is diffeomorphic to the standard $\mathbf{R}^4$ if and only if it admits a proper special generic map into $\mathbf{R}^n$ for some $n = 1, 2$ or 3. We also prove similar results for certain standard 1-connected open 4-manifolds.

Furthermore, in §4 we show that if a smooth 4-manifold M is homeomorphic to $L \times \mathbf{R}$ for some connected closed orientable 3-manifold L and if M admits a proper special generic map into $\mathbf{R}^3$, then M is diffeomorphic to $L \times \mathbf{R}$ and the 3-manifold L admits a special generic map into $\mathbf{R}^2$.

All these results claim that among the (uncountably or infinitely) many distinct differentiable structures on a certain open topological 4-manifold, there is at most one smooth structure that allows the existence of a proper special generic map into a lower dimensional manifold.

Throughout the paper, manifolds and maps between them are differentiable of class C^∞ unless otherwise indicated. The (co)homology groups are with integer coefficients unless otherwise specified. The symbol "$\cong$" denotes a diffeomorphism between smooth manifolds or an appropriate isomorphism between algebraic objects. For a topological space X, the symbol "id_X" denotes the identity map of X.

The author would like to express his sincere gratitude to Kazuhiro Sakuma for stimulating discussions and invaluable comments. He would also like to thank the referee for his/her comments that improved the presentation of the paper.

2. Preliminaries

Let us first recall the following notion of a Stein factorization, which will play an important role in this paper.

Definition 2.1. Let $f : M \to N$ be a smooth map between smooth manifolds. For two points $x, x' \in M$, we define $x \sim_f x'$ if $f(x) = f(x')(= y)$, and the points x and x' belong to the same connected component of $f^{-1}(y)$. We define $W_f = M/\sim_f$ to be the quotient space with respect to this equivalence relation, and denote by $q_f : M \to W_f$ the quotient map. Then, we see easily that there exists a unique continuous map $\bar{f} : W_f \to N$ that makes the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ & {}_{q_f}\searrow \quad \nearrow_{\bar{f}} & \\ & W_f & \end{array}$$

commutative. The above diagram is called the *Stein factorization* of f (see [15]).

The Stein factorization is a very useful tool for studying topological properties of special generic maps. In fact, we can prove the following, which is folklore (for example, see [24]). (In the following, a continuous map is *proper* if the inverse image of a compact set is always compact.)

Proposition 2.2. *Let $f : M \to N$ be a proper special generic map between smooth manifolds with $m = \dim M > \dim N = n$. Then, we have the following.*

1. *The set of singular points $S(f)$ of f is a regular submanifold of M of dimension $n - 1$, which is closed as a subset of M.*
2. *The quotient space W_f has the structure of a smooth n-dimensional manifold possibly with boundary such that $\bar{f} : W_f \to N$ is an immersion.*

(3) *The quotient map* $q_f : M \to W_f$ *restricted to* $S(f)$ *is a diffeomorphism onto* ∂W_f.
(4) *The quotient map* q_f *restricted to* $M \setminus S(f)$ *is a smooth fiber bundle over* $\operatorname{Int} W_f$ *with fiber the standard* $(m-n)$*-sphere* S^{m-n}.

In the following, we recall several notions concerning ends of manifolds. For details, the reader is referred to Siebenmann's thesis [30].

Definition 2.3. Let X be a Hausdorff space. Consider a collection ε of subsets of X with the following properties.

(i) Each $G \in \varepsilon$ is a connected open non-empty set with compact frontier $\overline{G} - G$,
(ii) If $G, G' \in \varepsilon$, then there exists $G'' \in \varepsilon$ with $G'' \subset G \cap G'$,
(iii) $\bigcap_{G \in \varepsilon} \overline{G} = \emptyset$.

Adding to ε every connected open non-empty set $H \subset X$ with compact frontier such that $G \subset H$ for some $G \in \varepsilon$, we produce a collection ε' satisfying (i), (ii) and (iii), which we call the *end* of X determined by ε.

An *end* of a Hausdorff space X is a collection ε of subsets of X which is maximal with respect to the properties (i), (ii) and (iii) above.

A *neighborhood* of an end ε is any set $N \subset X$ that contains some member of ε.

Definition 2.4. Let ε be an end of a topological manifold X. The fundamental group π_1 is *stable* at ε if there exists a sequence of path connected neighborhoods of ε, $X_1 \supset X_2 \supset \cdots$, with $\bigcap \overline{X}_i = \emptyset$ such that (with base points and base paths chosen) the sequence

$$\pi_1(X_1) \xleftarrow{f_1} \pi_1(X_2) \xleftarrow{f_2} \cdots$$

induced by the inclusions induces isomorphisms

$$\operatorname{Im}(f_1) \xleftarrow{\cong} \operatorname{Im}(f_2) \xleftarrow{\cong} \cdots .$$

The following lemma is proved in [30].

Lemma 2.5. *If* π_1 *is stable at* ε *and* $Y_1 \supset Y_2 \supset \cdots$ *is any path connected sequence of neighborhoods of* ε *such that* $\bigcap \overline{Y}_i = \emptyset$, *then for any choice of base points and base paths, the inverse sequence*

$$\mathcal{G} : \qquad \pi_1(Y_1) \xleftarrow{g_1} \pi_1(Y_2) \xleftarrow{g_2} \cdots$$

induced by the inclusions is stable, i.e. there exists a subsequence

$$\pi_1(Y_{i_1}) \xleftarrow{h_1} \pi_1(Y_{i_2}) \xleftarrow{h_2} \cdots$$

inducing isomorphisms

$$\operatorname{Im}(h_1) \xleftarrow{\cong} \operatorname{Im}(h_2) \xleftarrow{\cong} \cdots ,$$

where each h_j *is a suitable composition of* g_i*'s.*

Definition 2.6. When π_1 is stable at an end ε, we define $\pi_1(\varepsilon)$ to be the projective limit $\varprojlim \mathcal{G}$ for some fixed system $\mathcal{G}$ as above. According to [30], $\pi_1(\varepsilon)$ is well defined up to isomorphism.

3. Open 4-manifolds that admit special generic maps

In the following, a manifold is *open* if it has no boundary and each of its component is non-compact, while a manifold is *closed* if it has no boundary and is compact.

Let us begin by the following.

Lemma 3.1. *Let M be a smooth connected open 4-manifold with finitely many ends such that $H_2(M;\mathbf{Z}_2)$ is finitely generated. We assume that for each end ε, π_1 is stable and $\pi_1(\varepsilon)$ is finitely presentable. If $f : M \to N$ is a proper special generic map into a smooth orientable 3-manifold N, then there exists a smooth compact 3-manifold $\widetilde{W}$ possibly with boundary and a smooth embedding $h : W_f \to \widetilde{W}$ such that $h(\operatorname{Int} W_f) = \operatorname{Int} \widetilde{W}$.*

Proof. Suppose that $S(f) \cong \partial W_f$ has infinitely many components. Let S_i, $i = 0, 1, 2, \ldots$, be an infinite family of distinct components of ∂W_f. Since M is connected and q_f is surjective, W_f is connected. Thus, there exists an infinite family of disjointly embedded arcs α_i, $i \geq 1$, connecting S_0 and S_i in the 3-manifold W_f such that each α_i intersects ∂W_f transversely at its end points and $\operatorname{Int} \alpha_i \subset \operatorname{Int} W_f$. Then, $\{q_f^{-1}(\alpha_i)\}_{i\geq 1}$ is an infinite family of disjointly embedded 2-spheres in M. Furthermore, for each $i \geq 1$, $q_f^{-1}(S_i)$ is a submanifold of M which is closed as a subset of M, intersects $q_f^{-1}(\alpha_i)$ transversely at one point, and does not intersect $q_f^{-1}(\alpha_j)$ for $j \neq i$. This implies that the homology classes in $H_2(M;\mathbf{Z}_2)$ represented by $q_f^{-1}(\alpha_i)$, $i \geq 1$, are linearly independent. This contradicts our assumption that $H_2(M;\mathbf{Z}_2)$ is finitely generated. Therefore, ∂W_f has at most finitely many components.

Let the number of ends of M be denoted by r. Let K be an arbitrary compact subset of W_f. Since f is proper, so is q_f, and hence $K' = q_f^{-1}(K)$ is a compact subset of M. Therefore, $M \setminus K'$ has at most r unbounded components[1] (see [30, Lemma 1.8]). Thus, $q_f(M \setminus K') = W_f \setminus K$ has at most r unbounded components, since q_f is proper. Hence, W_f has finitely many ends.

Let ε be an end of W_f and $U_1 \supset U_2 \supset \cdots$ be any path connected sequence of neighborhoods of ε such that $\bigcap \overline{U}_i = \emptyset$. Then, for $V_i = q_f^{-1}(U_i)$, $V_1 \supset V_2 \supset \cdots$ is a path connected sequence of neighborhoods of the corresponding end of M with $\bigcap \overline{V}_i = \emptyset$. By Lemma 2.5 together with our assumption, there exists a subsequence $V_{i_1} \supset V_{i_2} \supset \cdots$ such that the sequence

$$\pi_1(V_{i_1}) \xleftarrow{f_1} \pi_1(V_{i_2}) \xleftarrow{f_2} \cdots$$

induced by the inclusions induces isomorphisms

$$\operatorname{Im}(f_1) \xleftarrow{\cong} \operatorname{Im}(f_2) \xleftarrow{\cong} \cdots .$$

Since U_{i_j} is open in W_f, every V_{i_j} contains an S^1-fiber of q_f. Thus, each f_j induces an isomorphism between the cyclic subgroups generated by the S^1-fibers. Since $(q_f)_* : \pi_1(V_{i_j}) \to \pi_1(U_{i_j})$ is an epimorphism whose kernel coincides with the cyclic subgroup generated by the S^1-fibers, we see that the sequence

$$\pi_1(U_{i_1}) \xleftarrow{g_1} \pi_1(U_{i_2}) \xleftarrow{g_2} \cdots$$

[1] A subset of a topological space is *bounded* if its closure is compact; otherwise, it is *unbounded*.

induced by the inclusions induces isomorphisms

$$\mathrm{Im}(g_1) \xleftarrow{\cong} \mathrm{Im}(g_2) \xleftarrow{\cong} \cdots .$$

Therefore, for each end of W_f, π_1 is stable. Furthermore, by our assumption, π_1 is finitely presentable.

On the other hand, since $\bar{f} : W_f \to N$ is an immersion and N is orientable, W_f is also orientable. Therefore, by [13] (see also [14]), we have the desired conclusion. (In fact, what we need here is [13, Theorem 3] with the condition $\pi_1(\varepsilon_i) \not\cong \mathbf{Z}_2$ for each i being replaced by the orientability of the 3-manifold M. This version of the theorem holds by the same reason as explained in the proof of [13, Corollary 2.1]: when the manifold is orientable, no projective plane appears in the boundary, and the argument works.) □

Remark 3.2. By [5], the compact 3-manifold $\widetilde{W}$ as in Lemma 3.1 is unique up to diffeomorphism.

Using Lemma 3.1, we prove the following.

Theorem 3.3. *Let M be a smooth connected open orientable 4-manifold with finitely many ends such that $H_*(M)$ is finitely generated. We assume that for each end ε, π_1 is stable and $\pi_1(\varepsilon)$ is finitely presentable. If $f : M \to N$ is a proper special generic map into a smooth orientable 3-manifold N, then there exists a smooth connected closed 4-manifold $\widetilde{M}$ and a compact orientable surface F possibly with boundary smoothly embedded in $\widetilde{M}$ such that M is diffeomorphic to $\widetilde{M} \setminus F$.*

Proof. By [24], there exists an orientable linear D^2-bundle $\pi : E_f \to W_f$ such that M is diffeomorphic to ∂E_f, where an ℓ-dimensional disk bundle is *linear* if its structure group can be reduced to a subgroup of the orthogonal group $O(\ell)$. Moreover, if C denotes a small closed collar neighborhood of ∂W_f in W_f, then $N_S = q_f^{-1}(C)$ is a tubular neighborhood of $S(f)$ in M and π restricted to $(\partial E_f) \cap \pi^{-1}(W_f \setminus C)$ can be identified with the smooth S^1-bundle $q_f|_{M \setminus N_S} : M \setminus N_S \to W_f \setminus C$.

Now, let us consider the cohomology exact sequence for the pair $(E_f, M \setminus N_S) \simeq (E_f, M \setminus S(f))$:

$$\widetilde{H}^k(E_f) \to \widetilde{H}^k(M \setminus S(f)) \to \widetilde{H}^{k+1}(E_f, M \setminus S(f)).$$

We have $\widetilde{H}^k(E_f) \cong \widetilde{H}^k(W_f)$, since $E_f \to W_f$ is a D^2-bundle. Furthermore, by the Thom isomorphism theorem (for example, see [16]), we have $\widetilde{H}^{k+1}(E_f, M \setminus S(f)) \cong \widetilde{H}^{k-1}(W_f)$. Therefore, putting $k = 2$, we have the exact sequence

$$H^2(W_f) \to H^2(M \setminus S(f)) \to H^1(W_f).$$

Since $H^*(W_f) \cong H^*(\widetilde{W})$ is finitely generated, so is $H^2(M \setminus S(f))$, where $\widetilde{W}$ is the compact orientable 3-manifold as in Lemma 3.1.

By excision, we have $\widetilde{H}^{k+1}(M, M \setminus S(f)) \cong \widetilde{H}^{k+1}(N_S, \partial N_S)$. Since M and $S(f)$ are orientable, N_S is an orientable D^2-bundle over $S(f)$. Therefore, $\widetilde{H}^{k+1}(N_S, \partial N_S)$ is isomorphic to $\widetilde{H}^{k-1}(S(f))$ by the Thom isomorphism theorem. Thus, we have $\widetilde{H}^{k+1}(M, M \setminus S(f)) \cong \widetilde{H}^{k-1}(S(f))$.

Let us consider the cohomology exact sequence for the pair $(M, M \setminus S(f))$:

$$H^2(M \setminus S(f)) \to H^3(M, M \setminus S(f)) \to H^3(M).$$

Since $H^2(M \setminus S(f))$ and $H^3(M)$ are finitely generated, so is $H^3(M, M \setminus S(f)) \cong H^1(S(f))$. This implies that $H_*(S(f))$ is finitely generated, since $S(f)$ has finitely many components by the proof of Lemma 3.1. Then, we see that $S(f) \cong \partial W_f$ is diffeomorphic to $\partial\widetilde{W} \setminus F_1$, where $F_1 (\subset \partial\widetilde{W})$ is a compact orientable surface possibly with boundary (see [13, Proposition 2]). In fact, we can prove that W_f is diffeomorphic to $\widetilde{W} \setminus F_1$.

Let $\widetilde{\pi} : \widetilde{E} \to \widetilde{W}$ be the linear D^2-bundle which naturally extends $\pi : E_f \to W_f$. Then, by the above arguments, we see that $M \cong \partial E_f$ is diffeomorphic to $\partial\widetilde{E} \setminus \widetilde{\pi}^{-1}(F_1)$. Set $\widetilde{M} = \partial\widetilde{E}$ and let F be the compact surface in $\widetilde{M}$ which corresponds to the zero section of $\widetilde{\pi}$ over F_1. Then the desired conclusion follows. □

Remark 3.4. As the above proof shows, the closed 4-manifold $\widetilde{M}$ in Theorem 3.3 is the boundary of an orientable linear D^2-bundle over the compact orientable 3-manifold $\widetilde{W}$ as in Lemma 3.1. In particular, it admits a special generic map $\widetilde{f} : \widetilde{M} \to \mathbf{R}^3$ whose quotient space can be identified with $\widetilde{W}$ (see [24]). Furthermore, the surface F in Theorem 3.3 is a codimension zero submanifold of $S(\widetilde{f})$ and the quotient map $q_f : M \to W_f$ can be identified with $q_{\widetilde{f}}|_{\widetilde{M} \setminus F}$.

Remark 3.5. Theorem 3.3 holds true even if N is non-orientable, provided that W_f is orientable. If for each end ε, $\pi_1(\varepsilon)$ contains no cyclic subgroup of index two, then even the orientability of W_f is not necessary (but, in this case, the surface F may possibly be non-orientable).

As a corollary, we have the following characterization of smooth 1-connected open 4-manifolds of "finite type" which admit proper special generic maps into 3-manifolds.

Corollary 3.6. *Let M be a smooth 1-connected open 4-manifold with finitely many ends such that $H_*(M)$ is finitely generated. We assume that for each end ε, π_1 is stable and $\pi_1(\varepsilon)$ is finitely presentable. Then there exists a proper special generic map $f : M \to N$ into a smooth 3-manifold N with $S(f) \neq \emptyset$ if and only if M is diffeomorphic to the connected sum of a finite number of copies of the following 4-manifolds:*

(1) $\mathbf{R}^4$,
(2) *the interior of the boundary connected sum of a finite number of copies of $S^2 \times D^2$,*
(3) *the total space of a 2-plane bundle over S^2,*
(4) *the total space of an S^2-bundle over S^2,*

where at least one manifold of the form (1), (2) or (3) should appear in the connected sum. In particular, each end of such an open 4-manifold has a neighborhood diffeomorphic to $L \times \mathbf{R}$, where L is the 3-sphere S^3, a lens space, or a connected sum of a finite number of copies of $S^1 \times S^2$.

Proof. Suppose that there exists a proper special generic map $f : M \to N$ into a 3-manifold N with $S(f) \neq \emptyset$. Since $(q_f)_* : \pi_1(M) \to \pi_1(W_f)$ is an isomorphism (see [24]), W_f is also 1-connected and hence is orientable. Let $\widetilde{W}$ be the compact 3-manifold as in Lemma 3.1 (see also Remark 3.5). Note that $\widetilde{W}$ is 1-connected. Then by the solution to the 3-dimensional Poincaré Conjecture (see [19, 20, 21] or [17], for example), $\widetilde{W}$ is diffeomorphic either to the 3-disk or to the boundary connected sum of a finite number of copies of $S^2 \times I$, where $I = [0, 1]$. By the proof of Theorem 3.3, there exists a compact surface F possibly with boundary in $\partial\widetilde{W}$ such that W_f is diffeomorphic to $\widetilde{W} \setminus F$. Note that $\partial W_f \cong \partial\widetilde{W} \setminus F \neq \emptyset$, since $S(f) \neq \emptyset$.

We can decompose $\widetilde{W}$ as the boundary connected sum of a finite number of compact 3-manifolds W_i such that

(i) each W_i contains at most one component of F, say F_i,
(ii) if W_i contains no component of F, then we put $F_i = \emptyset$ and $W_i \cong S^2 \times I$,
(iii) if $F_i \neq \emptyset$ has no boundary, then $F_i \cong S^2$ is a component of ∂W_i and $W_i \cong S^2 \times I$,
(iv) if F_i has non-empty boundary, then $W_i \cong D^3$.

The 3-manifold W_f can also be decomposed as the boundary connected sum of the manifolds $W'_i = W_i \setminus F_i$. Then, M is decomposed into the connected sum of the 4-manifolds M_i, which is obtained by attaching 4-disks to $q_f^{-1}(W'_i)$ along the boundary 3-spheres (for details, see [24]).

If W_i contains no component of F, then M_i admits a special generic map whose quotient space in the Stein factorization is diffeomorphic to $S^2 \times I$. Therefore, M_i is diffeomorphic to an S^2-bundle over S^2 (see [24]).

If $F_i \neq \emptyset$ has no boundary, then M_i admits a special generic map whose quotient space in the Stein factorization is diffeomorphic to $S^2 \times [0, 1)$. Then, M_i is diffeomorphic to a 2-plane bundle over S^2.

If F_i has non-empty boundary, then by Theorem 3.3 M_i is diffeomorphic to $\partial\widetilde{E}_i \setminus F_i$, where $\widetilde{E}_i$ is a D^2-bundle over $W_i \cong D^3$ and F_i is identified with the zero section over F_i. Therefore, M_i is diffeomorphic to $S^4 \setminus \Sigma$, where Σ is a connected non-empty surface with non-empty boundary embedded in S^4. If Σ is a disk, then M_i is diffeomorphic to $\mathbf{R}^4$. Otherwise, Σ is homotopy equivalent to a bouquet of a finite number of circles. Then, $S^4 \setminus \Sigma$ is diffeomorphic to the interior of the boundary connected sum of a finite number of copies of $S^2 \times D^2$.

Thus, we have proved that M is diffeomorphic to a manifold of a desired form.

Conversely, each 4-manifold in the list admits a proper special generic map into a 3-manifold with non-empty set of singularities. By the connected sum construction with respect to the quotient space (for details, see [24]), we see that their connected sums also admit proper special generic maps into 3-manifolds.

This completes the proof. □

Remark 3.7. The 4-manifold $S^2 \times \mathbf{R}^2$ admits at least two types of proper special generic maps into $\mathbf{R}^3$ as follows. Let $g : S^2 \to \mathbf{R}$ be the standard height function with exactly two critical points, which are non-degenerate. Then, $g \times \mathrm{id}_{\mathbf{R}^2} : S^2 \times \mathbf{R}^2 \to \mathbf{R} \times \mathbf{R}^2$ is a proper special generic map whose quotient space is diffeomorphic to $[-1, 1] \times \mathbf{R}^2$. On the other hand, let $h : \mathbf{R}^2 \to [0, \infty)$ be the proper smooth function defined by $h(x, y) = x^2 + y^2$. Then, $\mathrm{id}_{S^2} \times h : S^2 \times \mathbf{R}^2 \to S^2 \times [0, \infty)$ composed with a proper embedding $S^2 \times [0, \infty) \to \mathbf{R}^3$ is a proper special generic map whose quotient space is diffeomorphic to $S^2 \times [0, \infty)$.

The above observation corresponds to the fact that $S^2 \times \mathbf{R}^2$ appears twice in Corollary 3.6: it is the interior of $S^2 \times D^2$, and at the same time it is the total space of the trivial 2-plane bundle over S^2.

The 4-manifold $(\mathbf{C}P^2 \sharp \overline{\mathbf{C}P^2}) \setminus \{\text{two points}\}$ is another such example. It is the connected sum of a non-trivial S^2-bundle over S^2 and two copies of $\mathbf{R}^4$, and at the same time it is the connected sum of two 2-plane bundles over S^2 with Euler numbers $+1$ and -1.

Remark 3.8. For 4-manifolds as in Corollary 3.6, two are homeomorphic if and only if they are diffeomorphic. Note that every 4-manifold in Corollary 3.6 admits infinitely many distinct differentiable structures by [1]. In fact, most of them admit uncountably many distinct differentiable structures (see [3, 7, 9]).

Remark 3.9. In Corollary 3.6 we assumed that $S(f) \neq \emptyset$. If f is a proper submersion, then W_f is still 1-connected and is diffeomorphic to the interior of the connected sum of a finite number of copies of $S^2 \times [0,1]$. Furthermore, M is diffeomorphic to the total space of an orientable S^1-bundle over W_f. Since M is 1-connected, the Euler class of the S^1-bundle should be primitive.

As a corollary, we have the following.

Corollary 3.10. *Let M be a smooth 4-manifold homeomorphic to $\mathbf{R}^4$. Then there exists a proper special generic map $f : M \to \mathbf{R}^n$ for some $n = 1, 2$ or 3 if and only if M is diffeomorphic to the standard $\mathbf{R}^4$.*

Proof. First note that the standard $\mathbf{R}^4$ admits a special generic map into $\mathbf{R}^n$ for $n = 1, 2$ and 3: just consider the map defined by (1.1) for $m = 4$ globally. Therefore, if $M \cong \mathbf{R}^4$, then M also admits proper special generic maps into $\mathbf{R}^n$ for $n = 1, 2$ and 3.

Suppose that there exists a proper special generic map $f : M \to \mathbf{R}^3$. If f is a submersion, then M must be diffeomorphic to $\mathbf{R}^3 \times S^1$, which is a contradiction. Then by Corollary 3.6, M must be diffeomorphic to $\mathbf{R}^4$.

Suppose now that there exists a proper special generic map $f : M \to \mathbf{R}^2$. Then, the quotient space W_f is a 1-connected non-compact surface with non-empty boundary.

Lemma 3.11. *The boundary ∂W_f is connected and non-compact.*

Proof. Suppose that $S(f) \cong \partial W_f$ is not connected. Let S_1 and S_2 be distinct connected components of ∂W_f. Note that W_f is connected, since so is M. Therefore, there exists an arc α in W_f which intersects S_1 and S_2 at its end points transversely such that $\operatorname{Int} \alpha \subset \operatorname{Int} W_f$. Then, $q_f^{-1}(\alpha)$ is a smooth submanifold of M diffeomorphic to S^3. Furthermore, it intersects the component $q_f^{-1}(S_1)$ of $S(f)$ transversely at one point. Note that $q_f^{-1}(S_1)$ is a 1-dimensional submanifold of M, which is a closed subset of M. This is a contradiction, since M is contractible and $H_3(M) = 0$. Therefore, $S(f)$ must be connected.

Suppose that $S(f)$ is compact. Since M is non-compact and q_f is proper, W_f is non-compact. Therefore, there exists a proper embedding $\gamma : [0, \infty) \to W_f$ which intersects with $\partial W_f (\cong S(f))$ transversely at its end point. Then, $q_f^{-1}(\gamma([0, \infty)))$ is a properly embedded open 3-disk in M which intersects $S(f)$ transversely at one point. This implies that $S(f)$ represents a nontrivial homology class in $H_1(M)$, which is a contradiction, since $H_1(M) = 0$. Therefore, $S(f)$ must be non-compact. □

Therefore, W_f is diffeomorphic to $\mathbf{R} \times [0, \infty)$ (for example, see [13, Proposition 2] or [23]). Then, we see that $M \cong \partial(W_f \times D^3)$ is diffeomorphic to the standard $\mathbf{R}^4$.

Finally, suppose that M admits a proper special generic map into $\mathbf{R}$. Then, W_f is diffeomorphic to $[0,\infty)$ and M is diffeomorphic to the boundary of $[0,\infty)\times D^4$, which is diffeomorphic to the standard $\mathbf{R}^4$. □

Remark 3.12. It has been known that a smooth m-dimensional manifold is homeomorphic to $\mathbf{R}^m$ if and only if it is diffeomorphic to the standard $\mathbf{R}^m$, provided that $m \neq 4$ (see [18, 31]), while for $m = 4$, there exist uncountably many distinct differentiable structures on $\mathbf{R}^4$ (for example, see [4, 8, 10, 32]). This shows that among the uncountably many differentiable structures on $\mathbf{R}^4$, the standard one is the unique structure that allows the existence of a proper special generic map into $\mathbf{R}^n$ for $n \leq 3$.

Remark 3.13. If a smooth 4-manifold M is homeomorphic to $\mathbf{R}^4$, then there always exists a proper special generic map $M \to \mathbf{R}^4$. See [6] and [11, The Folding Theorem (p. 27)] for details.

Remark 3.14. If we omit the properness, then every smooth 4-manifold homeomorphic to $\mathbf{R}^4$ admits a submersion into $\mathbf{R}^n$ for all n with $1 \leq n \leq 4$ (see [22]).

In fact, by virtue of [22], an open 4-manifold admits a submersion into $\mathbf{R}^n$ if and only if it has n everywhere linearly independent vector fields. Therefore, we have the following.

Proposition 3.15. *Let M be a smooth connected open orientable 4-manifold. Then we have the following.*

(1) *There always exists a submersion $M \to \mathbf{R}$.*
(2) *There exists a submersion $M \to \mathbf{R}^2$ if and only if $W_3(M) = \beta w_2(M) = 0$, where W_3 (or w_2) denotes the 3rd Whitney (resp. 2nd Stiefel–Whitney) class.*
(3) *There exists a submersion $M \to \mathbf{R}^3$ if and only if $w_2(M) = 0$.*
(4) *There exists a submersion $M \to \mathbf{R}^4$ if and only if $w_2(M) = 0$.*

Remark 3.16. Let $f : \mathbf{R}^4 \to \mathbf{R}^3$ be a proper special generic map. Then, we can show that the quotient map $q_f : \mathbf{R}^4 \to W_f$ is C^∞ right-left equivalent to the standard map $g : \mathbf{R}^4 \to \mathbf{R}^2 \times [0,\infty)$ defined by (1.1) with $(n,m) = (4,3)$.

Note that the map $\bar{f} : W_f \to \mathbf{R}^3$ is a proper immersion. Since there are plenty of proper immersions $\mathbf{R}^2 \times [0,\infty) \to \mathbf{R}^3$, the C^∞ right-left equivalence class of a proper special generic map $f : \mathbf{R}^4 \to \mathbf{R}^3$ is far from being unique. In fact, we can show that two proper special generic maps $f_i : \mathbf{R}^4 \to \mathbf{R}^3$, $i = 0,1$, are C^∞ right-left equivalent if and only if the proper immersions $\bar{f}_i : W_{f_i} \to \mathbf{R}^3$ are C^∞ right-left equivalent.

Remark 3.17. By [24] together with the solution to the 3-dimensional Poincaré Conjecture, we have the following: a smooth 4-manifold M homeomorphic to S^4 admits a special generic map into $\mathbf{R}^n$ for some $n = 1,2$ or 3 if and only if M is diffeomorphic to the standard S^4. Furthermore, when $n = 3$, the singular set of a special generic map $M \to \mathbf{R}^3$ is always isotopic to the standardly embedded 2-sphere in S^4. (For details, see [29].)

Similarly, we have the following.[2]

[2]Corollaries 3.18 and 3.19, and Theorem 4.1 in §4 were first conjectured by Kazuhiro Sakuma to whom the author would like to express his sincere gratitude.

Corollary 3.18. *Let M be a smooth 4-manifold homeomorphic to $S^3 \times \mathbf{R}$. Then there exists a proper special generic map $f : M \to \mathbf{R}^n$ for some $n = 1, 2$ or 3 if and only if M is diffeomorphic to the standard $S^3 \times \mathbf{R}$.*

Note that $S^3 \times \mathbf{R} \cong \mathbf{R}^4 \sharp \mathbf{R}^4$.

Corollary 3.19. *Let M be a smooth 4-manifold homeomorphic to $S^2 \times \mathbf{R}^2$. Then there exists a proper special generic map $f : M \to \mathbf{R}^n$ for some $n = 2$ or 3 if and only if M is diffeomorphic to the standard $S^2 \times \mathbf{R}^2$.*

4. Manifolds homeomorphic to $L^3 \times \mathbf{R}$

In this section, we prove the following.

Theorem 4.1. *Let L be a smooth connected closed orientable 3-manifold. A smooth 4-manifold M homeomorphic to $L \times \mathbf{R}$ admits a proper special generic map into $\mathbf{R}^3$ if and only if M is diffeomorphic to $L \times \mathbf{R}$ and L is a smooth closed 3-manifold that admits a special generic map into $\mathbf{R}^2$.*

Proof. First suppose that M admits a proper special generic map $f : M \to \mathbf{R}^3$. Note that $S(f) \neq \emptyset$, since otherwise M is diffeomorphic to $S^1 \times \mathbf{R}^3$, which leads to a contradiction.

By the proof of Theorem 3.3, there exist a compact orientable 3-manifold $\widetilde{W}$ and a compact surface F possibly with boundary embedded in $\partial \widetilde{W}$ such that W_f is diffeomorphic to $\widetilde{W} \setminus F$. In particular, for each end of W_f, there exists a neighborhood C_i diffeomorphic to $F_i \times [0, \infty)$ for some compact connected orientable surface F_i possibly with boundary. Then, each $\widetilde{C}_i = q_f^{-1}(C_i)$ is a neighborhood of an end of M. Set $\widetilde{F}_i = q_f^{-1}(F_i \times \{1\})$, which is a connected closed orientable 3-manifold. Since M has exactly two ends and each of them has a neighborhood homeomorphic to $L \times [0, \infty)$, we see that W_f also has exactly two ends and the inclusions $\widetilde{F}_i \to M$ induce homotopy equivalences.

Let us consider the following commutative diagram:

$$\begin{array}{ccc} \pi_1(\widetilde{F}_i) & \xrightarrow{(\widetilde{\iota}_i)_*} & \pi_1(M) \\ {\scriptstyle (q_f)_*}\downarrow & & \downarrow{\scriptstyle (q_f)_*} \\ \pi_1(F_i) & \xrightarrow{(\iota_i)_*} & \pi_1(W_f), \end{array}$$

where $\widetilde{\iota}_i : \widetilde{F}_i \to M$ and $\iota_i : F_i \to W_f$ are the inclusions. Since $(q_f)_* \circ (\widetilde{\iota}_i)_*$ is an isomorphism, $(q_f)_* : \pi_1(\widetilde{F}_i) \to \pi_1(F_i)$ is a monomorphism. Since it is an epimorphism, it must be an isomorphism. Therefore, $(\iota_i)_*$ is also an isomorphism and W_f has a surface fundamental group.

Then by [12, Theorem 10.6] together with the solution to the 3-dimensional Poincaré Conjecture, we see that $\widetilde{W}$ is diffeomorphic to $(F_i \times [0, 1]) \sharp \left(\sharp^k B^3\right)$ for some $k \geq 0$, and hence W_f is diffeomorphic to $(F_i \times \mathbf{R}) \sharp \left(\sharp^k B^3\right)$, where B^3 denotes the 3-dimensional ball. Then, by an argument similar to that in [27], we can show that M is diffeomorphic to the connected sum of $\widetilde{F}_i \times \mathbf{R}$ and S^2-bundles over S^2. Since M is homeomorphic to $L \times \mathbf{R}$ for a closed orientable 3-manifold L, we see that M is diffeomorphic to $\widetilde{F}_i \times \mathbf{R}$.

If F_i has no boundary, then $\widetilde{F}_i$ is an S^1-bundle over F_i. Since $(q_f)_* : \pi_1(\widetilde{F}_i) \to \pi_1(F_i)$ is an isomorphism, we see that $\widetilde{F}_i$ is diffeomorphic to S^3 and the S^1-bundle is the Hopf fibration. If F_i has non-empty boundary, then for any immersion $\eta : F_i \to \mathbf{R}^2$, the composition $\eta \circ q_f : \widetilde{F}_i \to \mathbf{R}^2$ is a special generic map. In either case, $\widetilde{F}_i$ admits a special generic map into $\mathbf{R}^2$.

Note that $\widetilde{F}_i$ has a free fundamental group. Since the inclusion $\widetilde{F}_i \to M$ induces a homotopy equivalence, L also has a free fundamental group. Therefore, L is diffeomorphic to S^3 or the connected sum of some copies of $S^1 \times S^2$ by virtue of [12, Chapter 5] and the solution to the 3-dimensional Poincaré Conjecture. In particular, L admits a special generic map into $\mathbf{R}^2$ (see [2]).

Conversely, if M is diffeomorphic to $L \times \mathbf{R}$ and L admits a special generic map $g : L \to \mathbf{R}^2$, then the map

$$M \cong L \times \mathbf{R} \xrightarrow{g \times \mathrm{id}_{\mathbf{R}}} \mathbf{R}^2 \times \mathbf{R} = \mathbf{R}^3$$

is a proper special generic map. This completes the proof. □

Remark 4.2. As has been seen in the above proof, the 3-manifold L in Theorem 4.1 is diffeomorphic to S^3 or the connected sum of a finite number of copies of $S^1 \times S^2$. For details, see [2].

Remark 4.3. We can also show that if M is homeomorphic to $L \times \mathbf{R}$ for some connected closed orientable 3-manifold L and M admits a proper special generic map into $\mathbf{R}^2$, then L is diffeomorphic to S^3 and M is diffeomorphic to $S^3 \times \mathbf{R}$.

The following conjecture seems plausible.

Conjecture 4.4. For a topological 4-manifold M, there exists at most one differentiable structure on M that allows the existence of a proper special generic map into $\mathbf{R}^3$.

Remark 4.5. In the above conjecture, the properness of the special generic map is essential. Suppose that $f : M \to N$ is a special generic map of a smooth open 4-manifold M into a smooth manifold N with $\dim N < 4$. Let us consider a homeomorphism $h : M' \to M$, where M' is another smooth open 4-manifold. Then, by using h, we can construct a "formal solution" over M' on the jet level for the open differential relation corresponding to special generic maps (see [11]). Then, by virtue of the Gromov h-principle for open manifolds, we see that M' also admits a special generic map into N. Note that even if the original special generic map f is proper, the resulting special generic map on M' may not be proper.

Compare this with the situation in Remark 3.13, where the target has dimension four. In the equidimensional case, the C^0 dense h-principle holds for special generic maps and the properness can be preserved (see [11]).

References

[1] Ž. Bižaca and J. Etnyre, *Smooth structures on collarable ends of 4-manifolds*, Topology **37** (1998), 461–467. DOI: 10.1016/S0040-9383(97)00046-3

[2] O. Burlet and G. de Rham, *Sur certaines applications génériques d'une variété close à 3 dimensions dans le plan*, Enseignement Math. (2) **20** (1974), 275–292.

[3] F. Ding, *Smooth structures on some open 4-manifolds*, Topology **36** (1997), 203–207. DOI: 10.1016/0040-9383(95)00069-0

[4] S.K. Donaldson, *An application of gauge theory to four-dimensional topology*, J. Differential Geom. **18** (1983), 279–315.

[5] C.H. Edwards, Jr., *Concentricity in 3-manifolds*, Trans. Amer. Math. Soc. **113** (1964), 406–423.

[6] J.M. Èliašberg, *On singularities of folding type*, Math. USSR Izv. **4** (1970), 1119–1134. DOI: 10.1070/IM1970v004n05ABEH000946

[7] F. Fang, *Embedding 3-manifolds and smooth structures of 4-manifolds*, Topology Appl. **76** (1997), 249–259. DOI: 10.1016/S0166-8641(96)00118-6

[8] M.H. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geom. **17** (1982), 357–453.

[9] R.E. Gompf, *An exotic menagerie*, J. Differential Geom. **37** (1993), 199–223.

[10] R.E. Gompf and A.I. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics, Vol. 20, American Mathematical Society, Providence, RI, 1999.

[11] M. Gromov, *Partial differential relations*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Vol. 9, Springer-Verlag, Berlin, 1986.

[12] J. Hempel, *3-Manifolds*, Ann. of Math. Studies, No. 86, Princeton Univ. Press, Princeton, N.J.; Univ. of Tokyo Press, Tokyo, 1976.

[13] L.S. Husch and T.M. Price, *Finding a boundary for a 3-manifold*, Ann. of Math. (2) **91** (1970), 223–235. DOI: 10.2307/1970605

[14] L.S. Husch and T.M. Price, *Addendum to: "Finding a boundary for a 3-manifold"*, Ann. of Math. (2) **93** (1971), 486–488. DOI: 10.2307/1970886

[15] H. Levine, *Classifying immersions into* $\mathbf{R}^4$ *over stable maps of 3-manifolds into* $\mathbf{R}^2$, Lecture Notes in Math., Vol. 1157, Springer-Verlag, Berlin, 1985.

[16] J.W. Milnor and J.D. Stasheff, *Characteristic classes*, Ann. of Math. Studies, No. 76, Princeton Univ. Press, Princeton, N. J.; Univ. of Tokyo Press, Tokyo, 1974.

[17] J. Morgan and G. Tian, *Ricci flow and the Poincaré conjecture*, Clay Math. Monographs, Vol. 3, Amer. Math. Soc., Providence, RI; Clay Math. Institute, Cambridge, MA, 2007.

[18] J. Munkres, *Obstructions to the smoothing of piecewise-differentiable homeomorphisms*, Ann. of Math. (2) **72** (1960), 521–554. DOI: 10.2307/1970228

[19] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, preprint, arXiv:math/0211159v1 [math.DG].

[20] G. Perelman, *Ricci flow with surgery on three-manifolds*, preprint, arXiv:math/0303109v1 [math.DG].

[21] G. Perelman, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, preprint, arXiv:math/0307245v1 [math.DG].

[22] A. Phillips, *Submersions of open manifolds*, Topology **6** (1967), 171–206. DOI: 10.1016/0040-9383(67)90034-1

[23] A.O. Prishlyak and K.I. Mischenko, *Classification of noncompact surfaces with boundary*, Methods Funct. Anal. Topology **13** (2007), 62–66.

[24] O. Saeki, *Topology of special generic maps of manifolds into Euclidean spaces*, Topology Appl. **49** (1993), 265–293. DOI: 10.1016/0166-8641(93)90116-U

[25] O. Saeki, *Topology of special generic maps into* $\mathbf{R}^3$, Workshop on Real and Complex Singularities (São Carlos, 1992), Mat. Contemp. **5** (1993), 161–186.

[26] O. Saeki, *Cobordism groups of special generic functions and groups of homotopy spheres*, Japan. J. Math. (N.S.) **28** (2002), 287–297.

[27] O. Saeki and K. Sakuma, *On special generic maps into* $\mathbf{R}^3$, Pacific J. Math. **184** (1998), 175–193. DOI: 10.2140/pjm.1998.184.175

[28] O. Saeki and K. Sakuma, *Special generic maps of 4-manifolds and compact complex analytic surfaces*, Math. Ann. **313** (1999), 617–633. DOI: 10.1007/s002080050275

[29] K. Sakuma, *On special generic maps of simply connected* $2n$*-manifolds into* $\mathbf{R}^3$, Topology Appl. **50** (1993), 249–261. DOI: 10.1016/0166-8641(93)90024-8

[30] L. Siebenmann, *The obstruction to finding a boundary for an open manifold of dimension greater than five*, Doctoral dissertation, Princeton University, 1965.

[31] J. Stallings, *The piecewise-linear structure of Euclidean space*, Proc. Cambridge Philos. Soc. **58** (1962), 481–488. DOI: 10.1017/S0305004100036756

[32] C.H. Taubes, *Gauge theory on asymptotically periodic 4-manifolds*, J. Differential Geom. **25** (1987), 363–430.

Faculty of Mathematics, Kyushu University, Motooka 744, Nishi-ku, Fukuoka 819-0395, Japan

E-mail address: saeki@math.kyushu-u.ac.jp