Representation theory of symplectic singularities

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March 22, 2013

Quantization of symplectic singularities

In my abstract, I said that I wanted to view some objects from algebraic/symplectic geometry from the perspective of a representation theorist. That initially might sound like a strange thing to do.

It makes perfect sense to me; what I want to argue is that lots of interesting noncommutative algebras, especially $U(\mathfrak{g})$, are most naturally seen in the context of *quantization of symplectic singularities.*

So, my job is this talk to try to explain

- what those words mean,
- in what sense that's true and
- a bit about what it buys us: both what geometry can tell representation theory, and maybe what representation theory can give back.

Quantization

Quantization can mean a lot of things. For me, it's the idea that you take something commutative and make it non-commutative.

	classical		quantum
position	x	$\longrightarrow$	x
momentum	p	$\longrightarrow$	$\frac{d}{dx}$

The basic relation between these is $[\frac{d}{dx}, x] = 1$. Thus, if we give x and $\frac{d}{dx}$ "degree 1," these commute "modulo lower order terms." Taking "classical limit" means throwing away those terms, so the variables commute again.

I express this as **associated graded**. The algebra $A = \mathbb{C}\langle x, \frac{d}{dx}\rangle$ has a filtration with $A_0 = \mathbb{C}, A_1 = \mathrm{span}(1, x, \frac{d}{dx})$ and $A_n = A_1^n$. We have an isomorphism

$$\mathbb{C}[x,p] \cong \mathrm{gr}(A) := \bigoplus_n A_n/A_{n-1}.$$

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Poisson geometry

This doesn't look like something you can do very systematically. But it is if you remember the next order term of the multiplication.

You pass between formulas in quantum and classical mechanics by matching commutator with **Poisson bracket**:

$$\{f,g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial x} \qquad \{p,x\} = 1$$

Hamiltonian's equation (for an observable): $\dfrac{\partial f}{\partial t} = \{H,f\}$

Schrödinger's equation (for an operator): $i\hbar\dfrac{\partial \hat{f}}{\partial t} = [\hat{H},\hat{f}]$

Commutator is intrinsic to the multiplication of operators, but Poisson bracket is a new "semi-classical" structure to remember. The Poisson bracket on $\mathbb{C}[x,p]$ can be obtained as "the first order piece of $[-,-]$ on A."

Poisson geometry

The algebra A is the only filtered non-commutative algebra $\mathrm{gr}(A) = \mathbb{C}[x, p]$ with this grading and Poisson bracket.

There's a notion of a **Poisson bracket** on any commutative algebra $\mathbb{C}[Y]$: it's a bilinear map

$$\{-,-\}: \mathbb{C}[Y] \times \mathbb{C}[Y] \to \mathbb{C}[Y] \qquad \text{which satisfies}$$

- anti-symmetric + Jacobi
- Leibnitz rule $\{ab, c\} = a\{b, c\} + b\{a, c\}$.

So any algebra which is positively graded and carries a homogeneous Poisson structure looks like it might be the classical limit of some interesting non-commutative algebra. We call these **conical Poisson** algebras/varieties.

The word "cone" here comes from the fact you can think of a grading on $\mathbb{C}[Y]$ as a $\mathbb{C}^*$-action on Y, where the action on f of degree i as $t \cdot f = t^i f$. Thus, you can it embed into $\mathbb{A}^m$ in a way that's invariant under (weighted) scaling.

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Quantizations

Definition

If Y is an affine conical Poisson variety, then we call a filtered algebra A a **quantization** *of Y if we have an isomorphism of conical Poisson algebras* $\mathrm{gr}(A) \cong \mathbb{C}[Y]$.

So, A is the unique quantization of $\mathbb{C}[x,p]$. What happens when we consider other Poisson varieties?

In general, finding all quantizations is not easy; Kontsevich got a Fields Medal in large part for doing so for a *real* Poisson structure on $\mathbb{R}^n$.

But the thing that made his job hard is that he took arbitrary Poisson structures; quantization is usually easy for *symplectic* things, or things that are somehow morally symplectic.

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The KKS bracket and UAE's

One of the most important examples of a non-symplectic Poisson structure is the **Kostant-Kirillov-Souriau bracket** on $\mathbb{C}[\mathfrak{g}^*]$, $\mathfrak{g}$ a Lie algebra; it induces the slightly more famous symplectic structure on coadjoint orbits.

If $X, Y \in \mathfrak{g}$, then we have functions f_X, f_Y on $\mathfrak{g}^*$ given by pairing. Under KKS:

$$\{f_X, f_Y\} = f_{[X,Y]}.$$

This is quantized by the universal enveloping algebra

$$U = U(\mathfrak{g}) = T(\mathfrak{g})/\langle XY - YX = [X,Y]\rangle$$

The PBW theorem says that $U(\mathfrak{g})$ is a quantization of $\mathfrak{g}^*$:

$$\operatorname{gr}(U) \cong \bigoplus_n U^n/U^{n-1} \cong \mathbb{C}[\mathfrak{g}^*] \qquad U^n = \mathfrak{g}^n + \mathfrak{g}^{n-1} + \cdots + \mathfrak{g} + \mathbb{C}.$$

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The KKS bracket and UAE's

Exercise to the bored

Assume $\mathfrak{g}$ is semi-simple. Prove that $U(\mathfrak{g})$ is the unique quantization of $\mathfrak{g}^*$.

Hmmm, well, that wasn't hard, and $\mathfrak{g}^*$ is definitely not symplectic. I would claim, however, that it's *morally* symplectic.

Imagine $\mathfrak{g} = \mathfrak{gl}_n$, that is, $n \times n$ matrices. This can be identified with its dual by coordinatewise inner product. You can split up into the subvarieties where you fix the eigenvalues. These are symplectic if they are smooth (distinct eigenvalues) and always have a symplectic structure on their smooth locus.

What's "symplectic" about $\mathfrak{g}^*$ is the way it's built out these symplectic pieces: it's universal in a certain sense. One hint is that the different pieces give (almost) every symplectic structure on G/T.

UAE's and quantization

There's only one of these pieces which is a cone: the nilpotent cone $\mathcal{N}$ (again, you can just think of this as nilpotent matrices); this is very singular, but its smooth locus is symplectic.

You can actually reconstruct all of $\mathfrak{g}^*$ from $\mathcal{N}$ using some tricks from deformation theory. The Poisson $\mathfrak{g}^*$ is the *universal* way of deforming $\mathcal{N}$ keeping the Poisson structure (you might call this a **semi-classical deformation**).

So, I've added a new step to my treasure map:

1. start with $\mathcal{N}$ and its grading and Poisson structure.
2. deform it in every way you can while still having a Poisson structure; this will give you $\mathfrak{g}^*$.
3. now quantize the only way you can.

Congrats! You've found $U(\mathfrak{g})$.

Symplectic varieties

So, is $\mathcal{N}$ completely unique? What can we stick in the same slot?

Symplectic varieties

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Definition

Let Y be a conical Poisson variety. We call Y is a **conical symplectic variety** *if the Poisson bracket induces a symplectic structure on the smooth locus (+silly technical condition).*

- If $\mathbb{C}^{2n}$ has the usual symplectic structure, and Γ is a finite group preserving ω, then $Y \cong \mathbb{C}^{2n}/\Gamma$ is an example.
- There are interesting examples from Lie theory like nilpotent orbits and and Slodowy slices.
- There are examples related to toric geometry called hypertoric varieties. These have interesting relations to hyperplane arrangements.
- Other examples come from geometric representation theory: slices between Gr_λ and Gr_μ in the affine Grassmannian and Nakajima quiver varieties; both these are a geometric avatars of the representation theory of a Lie algebra $\mathfrak{g}$ in different ways.

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Deformations

Theorem (Namikawa)

Every conic symplectic variety Y has a universal deformation $\mathcal{Y}$ that smoothes it out as much as possible while staying symplectic. The base of this deformation is a vector space H.

For example, consider

$$\mathbb{C}^2/(\mathbb{Z}/\ell\mathbb{Z}) \cong \{(x,y,z) | xy = z^\ell\} \qquad \deg x = \deg y = \ell, \deg z = 2$$

The universal deformation is given by changing the equation to $xy = z^\ell + a_1 z^{\ell-1} + \cdots + a_\ell$, where a_i is a new variable of degree $2i$. Thus, a_i are the coordinates on H.

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Quantizations

So, how do we get an analogue of the universal enveloping algebra?

Theorem (Bezrukavnikov-Kaledin, Braden-Proudfoot-W.)

Every conical symplectic variety Y has a canonical quantization A of its universal Poisson deformation $\mathcal{Y}$; this is a non-commutative filtered algebra such that $\operatorname{gr} A \cong \mathbb{C}[\mathcal{Y}]$.

The center $Z(A)$ is the polynomial ring $\mathbb{C}[H]$; the quotient A_λ for a maximal ideal $\lambda \in H$ has an isomorphism $\operatorname{gr} A_\lambda \cong \mathbb{C}[Y]$. These are all the different quantizations of Y.

That is, the parameter space of quantizations and of semi-classical deformations coincide!

Quantizations

Examples:

- If Y is a nilcone, then A is the universal enveloping algebra; other examples from Lie theory give things like finite W-algebras (Gan-Ginzburg).
- If $Y \cong \mathbb{C}^{2n}/\Gamma$, then A is a symplectic reflection algebra (Etingof-Ginzburg).
- If Y is a slice between Gr_ν and Gr_μ in the affine Grassmannian, then A is (probably) a quotient of a shifted Yangian (Kamnitzer-W-Weekes-Yacobi).
- If Y is a hypertoric variety, A is what's called the "hypertoric enveloping algebra" (Braden-Licata-Proudfoot-W).

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How generalizable is Lie theory?

Symplectic singularities are the Lie algebras of the 21st century. - Okounkov

So, how far can we carry this analogy? What theorems that we know about universal enveloping algebras carry over?

Possibilities:

1. Are finite dimensional representations semi-simple? No.
2. Character formulas (Weyl, Kazhdan-Lusztig)? At least sometimes.
3. Categorical actions of braid groups? Yes, if you change the group.
4. Localization theorem of Beilinson-Bernstein? Mostly.

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How generalizable is Lie theory?

One of the things that I'm most interested in carrying over is a particular category of representations called **category** $\mathscr{O}$.

Theorem (BLPW)

You can define a version of category $\mathscr{O}$ *for any conical symplectic variety* Y *and Hamiltonian* $\mathbb{C}^*$*-action on* Y *(this plays the role that a Borel did before; it allows you to define "highest-weight vectors").*

These category $\mathscr{O}$'s match categories already defined in the case of symplectic reflection algebras.

Explicit descriptions

In many cases, they can also be related to combinatorially defined algebras; this gives a new supply of interesting finite dimensional algebras which are pretty interesting on their own terms.

- For $U(\mathfrak{g})$, this follows from results of Elias, Khovanov and Williamson on categories of Soergel bimodules.
- For hypertoric varieties, hypertoric category $\mathcal{O}$ is equivalent to an explicit algebra constructed by BLPW.
- For quiver varieties, this is given by a **weighted Khovanov-Lauda-Rouquier algebra**; in particular, this is a totally new description of category $\mathcal{O}$ for a symplectic reflection algebra.

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Explicit descriptions

The category $\mathscr{O}$'s for different $\mathbb{C}^*$-actions are related by derived equivalences called **shuffling functors**.

Theorem (W.)

For certain special category $\mathscr{O}$'s for quiver varieties, these functors form an action of the braid group. This can be used to construct homological knot invariants which categorify the Reshetikhin-Turaev invariants for any representation of $\mathfrak{g}$.

This construction also has a purely algebraic description.

Categories $\mathcal{O}$

Theorem

Category $\mathcal{O}$ is **highest weight (quasihereditary)**, **Koszul**, *and the classes of simples in the Grothendieck group are a* **"canonical basis"** *(which is an implicit character formula) if*

- *$Y = \mathcal{N}$ is a nilcone (CPS, BGS, BK, KL, etc.)*
- *Y is a hypertoric variety (BLPW)*
- *Y is a finite or affine type A quiver variety (W)*

Furthermore, the Koszul dual is again a category $\mathcal{O}$ of the same type of variety with the combinatorial data given by Langlands/Gale/rank-level duality.

This equivalence switches shuffling functors with another family of functors called **twisting functors** *that arise from tensor product with natural bimodules.*

Symplectic duality

This (and other pieces of evidence) led us to suggest that:

Conjecture (BLPW)

Conic symplectic singularities have a duality operation $X \leftrightarrow X^!$ generalizing Langlands duality of nilcones. We call this **symplectic duality**.

Unfortunately, at the moment, I can't tell you a definition of $X^!$ based on X; "the variety with a Koszul dual category $\mathscr{O}$" is about the best we've got.

On the other hand, in the cases we know a **lot** of interesting combinatorics match up; something is going on, though it may be years before we really know what it is.

However, this is slightly less insane than what I've said thus far might suggest. In particular, physicists seem to know about the same duality operation; for them, it's a reflection of some bizarre duality of 3-dimensional field theories.

Thanks for your attention.