2.3 Linear Approximation, Tangent Planes, and the Differential

In single-variable Calculus, you should have encountered linear approximation: if \( f = f(x) \) is differentiable at \( a \), then \( f(x) \) is approximately equal to \( f(a) + f'(a)(x - a) \), provided that \( x \) is “close to \( a \”).

Linear approximation is also referred to as the tangent line approximation, since the graph of \( y = f(a) + f'(a)(x - a) \) is the tangent line to the graph of \( y = f(x) \) at the point \((a, f(a))\). Graphically, linear/tangent line approximation means simply that the line which best approximates the graph of \( f \) at the point \((a, f(a))\) is the line given by \( y = f(a) + f'(a)(x - a) \).

We can look at linear approximation from the point of view of the changes in \( x \) and \( f \). The linear approximation, above, can be rewritten as

\[
\Delta f = f(x) - f(a) \approx f'(a)(x - a) = f'(a) \Delta x,
\]

when \( \Delta x \) is close to 0.

Looking at this last form, it is common to formally define new “variables” \( df \) and \( dx \), which are related by \( df = f'(a)dx \), and then to say that, near \( a \), if \( dx \) is a small \( \Delta x \) (small, meaning close to 0), then the differential \( df \) is approximately the change in \( f \), i.e., if \( |dx| \) is small, then \( \Delta f \approx df \). In this guise, the approximation is normally referred to as differential approximation.

We wish to look at all of this again — linear approximation, tangent sets, and differential approximations — but, now, for multivariable functions.

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Basics:

Throughout this section, we assume that \( f \) is a real-valued function, whose domain is a neighborhood of a point \( p \) in \( \mathbb{R}^n \), and that \( f \) is differentiable at \( p \).
The definition of the total derivative in Definition 2.2.1 tells us that
\[
\lim_{|h| \to 0} \frac{|f(p + h) - f(p) - d_pf(h)|}{|h|} = 0. \quad (2.5)
\]

The limit in Formula (2.5) implies that, when $|h|$ is close to 0, the quantity
\[
|f(p + h) - f(p) - d_pf(h)|
\]
must also be close to 0 or, equivalently,
\[
f(p + h) - f(p) - d_pf(h)
\]
must be close to the zero vector, 0. Writing $x$ for $p + h$, so that $h = x - p$, we immediately conclude

**Proposition 2.3.1. (Linear Approximation)** If $x$ is close to $p$, i.e., if $|x - p|$ is close to 0, then
\[
f(x) \approx f(p) + d_pf(x - p) = f(p) + \nabla f(p) \cdot (x - p).
\]

*In particular, suppose $f = f(x, y)$, and $p = (a, b)$. In this case, if $(x, y)$ is close to $(a, b)$, then
\[
f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).
\]*

The point of multi-variable linear approximation is the same as it was in the single variable case; we want to approximate a complicated function by a very simple function, for values of the independent variable(s) near some given point.
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**Example 2.3.2.** Consider the function

\[ f(x, y) = 4 + x - x^2 - y^3. \]

Let’s use linear approximation at (1, 1) to approximate \( f(0.9, 1.2) \) and \( f(1.01, 1.05) \).

We apply the formula

\[ f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b), \]

where \((a, b) = (1, 1)\), and first using \( (x, y) = (0.9, 1.2) \), and then using \( (x, y) = (1.01, 1.05) \).

We find \( f(1, 1) = 3 \). Now we calculate

\[ f_x(x, y) = 1 - 2x, \quad \text{and} \quad f_y(x, y) = -3y^2, \]

and so

\[ f_x(1, 1) = -1, \quad \text{and} \quad f_y(1, 1) = -3. \]

Therefore, for both of our \((x, y)\) pairs, which are both near \((1, 1)\), we will use that

\[ f(x, y) \approx 3 - 1(x - 1) - 3(y - 1). \]

Now, we easily find

\[ f(0.9, 1.2) \approx 3 - 1(0.9 - 1) - 3(1.2 - 1) = 2.5 \]

and

\[ f(1.01, 1.05) \approx 3 - 1(1.01 - 1) - 3(1.05 - 1) = 2.84. \]

The actual values of \( f(0.9, 1.2) \) and \( f(1.01, 1.05) \) are 2.362 and 2.832275, respectively. Not surprisingly, the second approximation is significantly better; after all, \((1.01, 1.05)\) is significantly closer to \((1, 1)\) than \((0.9, 1.2)\) is.
We give a name to the affine linear function on the right in the formulas of Proposition 2.3.1.

**Definition 2.3.3.** The affine linear function on \( \mathbb{R}^n \) given by

\[
L_f(x; p) = f(p) + \nabla f(p) \cdot (x - p)
\]

is called the **linearization of** \( f \) **at** \( p \).

If it is clear what \( f \) and \( p \) are, we sometimes write simply \( L(x) \) for the linearization.

Using our new notation and terminology, we can rephrase Proposition 2.3.1

**Proposition 2.3.4.** (Linear Approximation) If \( x \) is close to \( p \), i.e., if \( |x - p| \) is close to 0, then

\[
f(x) \approx L_f(x; p).
\]

Let’s look again at Example 2.3.2 but now using our new terminology.

**Example 2.3.5.** Consider the function

\[
f(x, y) = 4 + x - x^2 - y^3.
\]

Find the linearization of \( f \) at \((1, 1)\), and use it to approximate the values of \( f(0.9, 1.2) \) and \( f(1.01, 1.05) \).

**Solution:**

We did all of the work for this in Example 2.3.2. We found that \( f(1, 1) = 3 \) and \( \nabla f(1, 1) = (f_x(1, 1), f_y(1, 1)) = (-1, -3) \).

Thus,

\[
L(x, y) = f(1, 1) + \nabla f(1, 1) \cdot ((x, y) - (1, 1)) = 3 + (-1, -3) \cdot (x - 1, y - 1),
\]
and so

$$L(x, y) = 3 - (x - 1) - 3(y - 1).$$

Of course, we end up with the same approximations as we did in Example 2.3.2:

$$f(0.9, 1.2) \approx L_f((0.9, 1.2); (1, 1)) = 3 - 1(0.9 - 1) - 3(1.2 - 1) = 2.5$$

and

$$f(1.01, 1.05) \approx L_f((1.01, 1.05); (1, 1)) = 3 - 1(1.01 - 1) - 3(1.05 - 1) = 2.84.$$  

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**Example 2.3.6.** Suppose that $g : \mathbb{R}^4 \to \mathbb{R}$ is given by

$$g(x_1, x_2, x_3, x_4) = x_2e^{x_1} + x_3\cos(2\pi x_4).$$

Find the linearization of $g$ at $(0, 1, -2, 1)$, and use it to approximate $g(0.01, 1.02, -2.005, 0.99)$.

**Solution:** Writing $x$ for $(x_1, x_2, x_3, x_4)$, the linearization of $g$ at $(0, 1, -2, 1)$ is

$$L_g(x; (0, 1, -2, 1)) = g(0, 1, -2, 1) + \nabla g(0, 1, -2, 1) \cdot (x - (0, 1, -2, 1)).$$

We first find $g(0, 1, -2, 1) = 1 \cdot e^0 - 2\cos(2\pi \cdot 1) = -1$. Now, we calculate

$$\frac{\partial g}{\partial x_1} = x_2e^{x_1}, \quad \frac{\partial g}{\partial x_2} = e^{x_1}, \quad \frac{\partial g}{\partial x_3} = \cos(2\pi x_4),$$

and

$$\frac{\partial g}{\partial x_4} = -2\pi x_3\sin(2\pi x_4).$$
Therefore,

\[
\nabla g(0, 1, -2, 1) = (1 \cdot e^0, e^0, \cos(2\pi \cdot 1), -2\pi(-2)\sin(2\pi 1)) = (1, 1, 1, 0).
\]

Thus,

\[
L_g(x; (0, 1, -2, 1)) = -1 + (1, 1, 1, 0) \cdot (x - (0, 1, -2, 1)) = -1 + 1(x_1 - 0) + 1(x_2 - 1) + 1(x_3 - (-2)) + 0(x_4 - 1),
\]

that is

\[
L(x) = L_g(x; (0, 1, -2, 1)) = -1 + x_1 + (x_2 - 1) + (x_3 + 2),
\]

where we have left parentheses around quantities that will be close to 0.

Finally, we look at the linear approximation

\[
g(0.01, 1.02, -2.005, 0.99) \approx L(0.01, 1.02, -2.005, 0.99) =
\]

\[
-1 + 0.01 + (1.02 - 1) + (-2.005 + 2) = -1 + 0.01 + 0.02 - 0.005 = -0.975.
\]

You may check that the actual answer, to 8 decimal places, is \(-0.97079242\), so our estimate is reasonably close.

Recall that, if \(f\) is a function of one variable, then the graph of the linearization of \(f\), at \(p\), is the tangent line to the graph of \(f\) at \((p, f(p))\); that is, the tangent line is the graph of 
\[y = f(p) + f'(p)(x - p)\] . Why should this be true? Because the linearization is the affine linear function that best approximates \(f\) at points near \(p\); so the line you get from graphing the linearization should be the line which best approximates the graph of \(f\) near points with \(x\)-coordinate close to \(p\), and that line is the tangent line to the graph.

In the same way, if we have \(z = f(x, y)\), then the graph of the linearization \(z = L_f((x, y); (a, b))\) is the tangent plane to the graph of \(z = f(x, y)\) at the point \((a, b, f(a, b))\).
More generally, in any number of dimensions, we can consider the tangent set to the graph of $f$:

**Definition 2.3.7.** Suppose that $f$ is a real-valued function on a subset of $\mathbb{R}^n$, and that $f$ is differentiable at $p$. Then, the graph of the linearization of $f$ at $p$ is called the tangent set to the graph of $f$ at the point $(p, f(p))$. 

In other words, the tangent set to the graph of $f$ at the point $(p, f(p))$ is the set of points $(x, z)$ in $\mathbb{R}^{n+1}$ such that

$$z = f(p) + \nabla f(p) \cdot (x - p).$$

In particular, suppose that $f = f(x, y)$ is a real-valued function on a subset of $\mathbb{R}^2$, and that $f$ is differentiable at $(a, b)$. Then, the tangent plane to the graph of $f$ at the point $(a, b, f(a, b))$ is the set of points $(x, y, z)$ in $\mathbb{R}^3$ such that

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

**Example 2.3.8.** Consider again the function $z = g(x, y) = 4 + x - x^2 - y^3$ from Example 2.3.2 and Example 2.3.5, but now let’s look at the tangent plane to its graph at the point where $(x, y) = (1, 1)$.

We already produced the linearization of $g$ at $(1, 1)$ in Example 2.3.5; the linearization of $g$ at $(1, 1)$ is

$$L(x, y) = 3 - (x - 1) - 3(y - 1),$$

where we remind you that you do **not** want to simplify this; you want to have the quantities $x - 1$ and $y - 1$ in parentheses.

Therefore, the tangent plane to the graph of $g$ at the point where $(x, y) = (1, 1)$ is the graph of

$$z = 3 - (x - 1) - 3(y - 1).$$

We give you two views of the graph of $g$ together with the graph of $L$ in Figure 2.3.1 and Figure 2.3.2. The bold black dot is at the point $(1, 1, 3)$.

Figure 2.3.1: The tangent plane to $z = 4 + x - x^2 - y^3$ at $(1, 1, 3)$.

Figure 2.3.2: Another view of the same tangent plane.
Remark 2.3.9. It is important that the tangent plane to the graph of function $f$, at a point where $(x, y) = (a, b)$, contains the tangent lines to the corresponding cross sections of the graph of $f$.

How do you see this? Let’s consider the $x = a$ cross section. This cross section of the graph of $f$ is just the graph of the function obtained from $f$ by setting $x$ equal to $a$; let’s call this cross section function $h(y)$. In other words, we let $h(y) = f(a, y)$. We would like to see that the tangent line (inside the copy of the $yz$-plane given by $x = a$) to the graph of $h$, at the point where $y = b$, is contained in the tangent plane to the graph of $f$ at the point where $(x, y) = (a, b)$.

In fact, the tangent line to the graph of $h$ isn’t just contained in the tangent plane, it’s the $x = a$ cross section of the tangent plane. Why? Because the tangent line to the graph of $h$, where $y = b$, is given by

$$z = h(b) + h'(b)(y - b),$$

inside the plane $x = a$.

In terms of $f$, the two equations above for the tangent line become

$$x = a \quad \text{and} \quad z = f(a, b) + f_y(a, b)(y - b).$$

On the other hand, the tangent plane to the graph of $f$ is the graph of

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

The $x = a$ cross section of the tangent plane is thus given by $x = a$ and

$$z = f(a, b) + f_x(a, b)(a - a) + f_y(a, b)(y - b) = f(a, b) + f_y(a, b)(y - b);$$

the same equations that we had before, which shows that the tangent line to the $x = a$ cross section of the graph of $f$ is the $x = a$ cross section of the tangent plane. The same
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reasoning applies to the cross section where \( y = b \).

Consider, for instance, the function \( z = g(x, y) = 4 + x - x^2 - y^3 \) from the previous example, Example 2.3.8. We found that the tangent plane was the graph of the linearization, i.e., the graph of \( z = 3 - (x - 1) - 3(y - 1) \).

If we look inside the cross section where \( x = 1 \), then we find that the graph of \( g \) becomes the graph of \( z = 4 - y^3 \) and the tangent line, at \( y = 1 \), is given by \( z = 3 - 3(y - 1) \), inside the plane where \( x = 1 \). This cross section and corresponding tangent line can easily be seen in Figure 2.3.3.

Example 2.3.10. Let us return to Example 2.2.14 in which we considered

\[
z = f(x, y) = \sqrt{x^2 + y^2}.
\]

We mentioned at the time that \( f \) is not differentiable at the origin, and that the graph has a sharp point there. Thus, the tangent plane to the graph at the point \((0,0,0)\) is undefined.

Thinking about Remark 2.3.9 it is instructive to look at the \( x = 0 \) and \( y = 0 \) cross sections. The \( x = 0 \) cross section is the graph of \( z = \sqrt{0^2 + y^2} = |y| \). Similarly, the \( y = 0 \) cross section is the graph of \( z = |x| \). As you should recall from single-variable Calculus, there is no tangent line to the graph of the absolute value function at the origin.

Linear approximation tells us that, if \( \mathbf{x} \) is close to \( \mathbf{p} \), then

\[
f(\mathbf{x}) \approx f(\mathbf{p}) + d_\mathbf{p}f (\mathbf{x} - \mathbf{p}),
\]

or, equivalently,

\[
f(\mathbf{x}) - f(\mathbf{p}) \approx d_\mathbf{p}f (\mathbf{x} - \mathbf{p}).
\]

Writing \( \Delta \mathbf{x} \) in place of \( \mathbf{x} - \mathbf{p} \), and writing \( \Delta f \) in place of \( f(\mathbf{x}) - f(\mathbf{p}) \), we arrive at:
Proposition 2.3.11. (Differential Approximation) If \( x \) is close to \( p \), so that \( \Delta x \) is close to 0, then

\[
\Delta f \approx d_pf(\Delta x) = \nabla f(p) \cdot \Delta x.
\]

Of course, this is just a different version of linear approximation, but one that is used when what you’re worried about is the change in various quantities, not the actual values.

Example 2.3.12. Suppose that \( h(x, y, z) = 2z \ln(e + xy) \). Then, \( h(0, -2, 5) = 10 \). Use differential approximation to estimate the change in \( h \), if \( \Delta(x, y, z) = (0.01, 0.05, -0.2) \).

Solution:

We use that

\[
\Delta h \approx \nabla h(0, -2, 5) \cdot (0.01, 0.05, -0.2),
\]

and so we need to calculate the gradient vector.

We find

\[
\frac{\partial h}{\partial x} = \frac{2zy}{e + xy}, \quad \frac{\partial h}{\partial y} = \frac{2zx}{e + xy}, \quad \text{and} \quad \frac{\partial h}{\partial z} = 2 \ln(e + xy),
\]

and so

\[
\nabla h(0, -2, 5) = \left( -\frac{20}{e}, 0, 2 \right).
\]

Therefore,

\[
\Delta h \approx \left( -\frac{20}{e}, 0, 2 \right) \cdot (0.01, 0.05, -0.2) = -\frac{0.2}{e} + 0 - 0.4 \approx -0.473575888.
\]
Example 2.3.13. Suppose that we have a right circular cylinder of radius 0.5 feet and height 1 foot. If we increase the radius by 0.1 feet and decrease the height by 0.1 feet, then differential approximation tells us that the change in the volume is approximately what?

Solution:

The volume of the cylinder is given by \( V = \pi r^2 h \). We will use that

\[
\Delta V \approx \nabla V(p) \cdot \Delta (r, h) = \left( \frac{\partial V}{\partial r}, \frac{\partial V}{\partial h} \right) \bigg|_p \cdot (\Delta r, \Delta h),
\]

where \( p = (\text{original radius, original height}) = (0.5, 1) \), \( \Delta r = 0.1 \), and \( \Delta h = -0.1 \).

We find

\[
\Delta V \approx \left( 2\pi rh, \pi r^2 \right) \bigg|_{(r, h) = (0.5, 1)} \cdot (0.1, -0.1) = (\pi, 0.25\pi) \cdot (0.1, -0.1) = 0.075\pi \text{ ft}^3.
\]

More Depth:

Before we give more examples, we first want to make more precise the way in which \( f(p + h) \) gets “close to” \( f(p) + d_pf(h) \) as \( h \) gets “close to” \( 0 \). We will show that the difference between \( f(p + h) \) and \( f(p) + d_pf(h) \) is the product of \( |h| \) with a function that approaches \( 0 \) as \( h \) approaches \( 0 \); hence, the difference approaches \( 0 \) faster than just \( |h| \) does.

Theorem 2.3.14. Suppose that \( E \) is a subset of \( \mathbb{R}^n \), that \( p \) is a point in \( E \), and that \( f : E \to \mathbb{R} \) is differentiable at \( p \).

Then, there exists a real-valued function \( \mathcal{E} \) on a neighborhood \( W \) of \( 0 \) in \( \mathbb{R}^n \), which is continuous at \( 0 \), such that \( \mathcal{E}(0) = 0 \), and such that, for all \( h \) in \( W \), \( p + h \) is in the domain of \( f \), and

\[
f(p + h) = f(p) + d_pf(h) + |h|\mathcal{E}(h).
\]
Proof. Actually, this is quite easy. Define

\[ E(h) = \begin{cases} \frac{f(p + h) - f(p) - d_p f(h)}{|h|}, & \text{if } h \neq 0; \\ 0, & \text{if } h = 0. \end{cases} \]

Then the continuity of \( E \) is immediate from the fact that \( f \) is differentiable at \( p \). The theorem follows immediately. \( \square \)

Now let’s consider a slightly complicated tangent plane question.

**Example 2.3.15.** Determine all of the points on the graph of

\[ z = f(x, y) = \frac{5x}{x^2 + y^2 + 1} \]

at which the tangent planes are horizontal.

**Solution:**

A horizontal plane has an equation of the form \( z = c \), where \( c \) is a constant. In order for \( z = f(p) + \overrightarrow{\nabla}f(p) \cdot (x - p) \) to be of this form, \( \overrightarrow{\nabla}f(p) \) would have to be \( 0 \). Thus, we need to find those points where both partial derivatives of \( f \) are 0, i.e., we need to simultaneously solve

\[ \frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0. \]

We calculate

\[ \frac{\partial f}{\partial x} = 5 \cdot \frac{(x^2 + y^2 + 1)(1) - x(2x)}{(x^2 + y^2 + 1)^2} = \frac{5(y^2 - x^2 + 1)}{(x^2 + y^2 + 1)^2} \]

and

\[ \frac{\partial f}{\partial y} = 5x \cdot \frac{\partial}{\partial y} \left[ (x^2 + y^2 + 1)^{-1} \right] = \frac{-10xy}{(x^2 + y^2 + 1)^2}. \]
Thus, both partial derivatives are 0 if and only if

\[ y^2 - x^2 + 1 = 0 \quad \text{and} \quad xy = 0. \]

Hence, we must have \( x = 0 \) or \( y = 0 \). But if \( x = 0 \), then \( y^2 + 1 = 0 \), which is impossible (in the real numbers, where we’re working). Therefore, we must have \( y = 0 \), and then \( -x^2 + 1 = 0 \), so that \( x = \pm 1 \).

Finally, we conclude that the tangent plane to the graph is horizontal precisely when \((x, y) = (1, 0)\) and \((x, y) = (-1, 0)\). The points in space are thus \((1, 0, f(1, 0)) = (1, 0, 5/2)\) and \((-1, 0, f(-1, 0)) = (-1, 0, -5/2)\).

---

**Example 2.3.16.** Let’s look at the tangent set in a higher-dimensional example. Consider the function from Example 2.3.6:

\[ g(x_1, x_2, x_3, x_4) = x_2 e^{x_1} + x_3 \cos(2\pi x_4). \]

The graph of \( z = g(x_1, x_2, x_3, x_4) \) is the set of points \((x_1, x_2, x_3, x_4, g(x_1, x_2, x_3, x_4))\) in \(\mathbb{R}^5\). So, obviously, we’re not going to try to picture this!

We found the linearization of \( g \) at \((0, 1, -2, 1)\) to be

\[ L(x) = -1 + x_1 + (x_2 - 1) + (x_3 + 2). \]

So the tangent set to the graph of \( g \) at \((0, 1, -2, 1, -1)\) is the set of points in \(\mathbb{R}^5\) of the form

\((x_1, x_2, x_3, x_4, -1 + x_1 + (x_2 - 1) + (x_3 + 2)).\)
Recall Example 2.2.9 in which we saw that, if \( x_i \) (which we’ll write instead of \( X_i \)) is the \( i \)-th coordinate function on \( \mathbb{R}^n \), then \( d_p x_i(v_1, v_2, \ldots, v_n) = v_i \). Since this result does not depend on the point \( p \), it is standard to write simply \( dx_i \) in place of \( d_p x_i \) (though we shall include the \( p \) subscript initially below).

Now, if \( f \) is differentiable at \( p \), then

\[
d_p f(v_1, v_2, \ldots, v_n) = \nabla f(p) \cdot v = \left. \frac{\partial f}{\partial x_1} \right|_p v_1 + \left. \frac{\partial f}{\partial x_2} \right|_p v_2 + \cdots + \left. \frac{\partial f}{\partial x_n} \right|_p v_n.
\]

Combining this with the \( d_p x_i \)'s, and writing \( v \) for \((v_1, v_2, \ldots, v_n)\), we obtain

\[
d_p f(v) = \frac{\partial f}{\partial x_1} \bigg|_p d_p x_1(v) + \frac{\partial f}{\partial x_2} \bigg|_p d_p x_2(v) + \cdots + \frac{\partial f}{\partial x_n} \bigg|_p d_p x_n(v).
\]

Since we now have functions applied to \( v \) on both sides, we can suppress the \( v \), we just write an equality of functions, from \( \mathbb{R}^n \) to \( \mathbb{R} \):

\[
d_p f = \frac{\partial f}{\partial x_1} \bigg|_p d_p x_1 + \frac{\partial f}{\partial x_2} \bigg|_p d_p x_2 + \cdots + \frac{\partial f}{\partial x_n} \bigg|_p d_p x_n.
\]

Finally, since everything is now calculated at an arbitrary point \( p \), the explicit reference to \( p \) is usually omitted; we arrive at

**Definition 2.3.17.** The differential of \( f \) is

\[
df = \frac{\partial f}{\partial x_1} \, dx_1 + \frac{\partial f}{\partial x_2} \, dx_2 + \cdots + \frac{\partial f}{\partial x_n} \, dx_n = \nabla f \cdot dx.
\]

Thus, \( df \) gives you \( d_p f \) for each point \( p \) at which \( f \) is differentiable.

When you use the differential in approximation problems, what you do is “replace” each \( dx_i \) by the actual small change \( \Delta x_i \), and then the differential yields the approximate change in \( \Delta f \). Of course, what we think of as “replacing” is actually evaluating both sides at the vector \( \Delta x \), and then using differential approximation at the point \( p \) in...
question, i.e., using $\Delta f \approx d_p f(\Delta x) = \nabla f(p) \cdot \Delta x$.

**Example 2.3.18.** Suppose that

$$\omega(x, y, z) = e^{x^2 y} \sin \left( \frac{\pi z}{2} \right).$$

Calculate the general differential $d\omega$, then calculate, $d_{(3,2,1)}\omega$ and $d_{(3,2,1)}\omega(-2,5,7)$. Finally, if $(x, y, z)$ is initially $(3, 2, 1)$ and changes by $\Delta(x, y, z) = (-0.02, 0.05, 0.07)$, then approximate the corresponding change in $\omega$.

**Solution:**

We first calculate

$$d\omega = \frac{\partial \omega}{\partial x} dx + \frac{\partial \omega}{\partial y} dy + \frac{\partial \omega}{\partial z} dz =$$

$$2xye^{x^2 y} \sin \left( \frac{\pi z}{2} \right) dx + xe^{x^2 y} \sin \left( \frac{\pi z}{2} \right) dy + \frac{\pi xe^{x^2 y}}{2} \cos \left( \frac{\pi z}{2} \right) dz.$$  

This the general differential.

To find $d_{(3,2,1)}\omega$, we now plug $(x, y, z) = (3, 2, 1)$ into the coefficients of $dx$, $dy$, and $dz$. We obtain:

$$d_{(3,2,1)}\omega = 12e^{18} dx + 9e^{18} dy + 0 dz = 3e^{18}(4dx + 3dy).$$

Now we quickly obtain

$$d_{(3,2,1)}\omega(-2,5,7) = 3e^{18}(4 \cdot (-2) + 3 \cdot 5) = 21e^{18}.$$
The approximate change in $f$ is given by differential approximation

$$
\Delta f \approx d_{(3,2,1)}\omega(-0.02, 0.05, 0.07) = d_{(3,2,1)}\omega(0.01(-2, 5, 7)) =
$$

$$
0.01 d_{(3,2,1)}\omega(-2, 5, 7) = 0.01 \cdot 21e^{18} = 0.21e^{18}.
$$

Of course, this result is precisely what you get from taking

$$
d_{(3,2,1)}\omega = 12e^{18} dx + 9e^{18} dy + 0 dz,
$$

and replacing $(dx, dy, dz)$ by the actual changes $(\Delta x, \Delta y, \Delta z) = (-0.02, 0.05, 0.07)$.

Note that, in the previous example, our approximate change in $\omega$ was $0.21e^{18}$; this number is huge. Consequently, it would be reasonable to expect that the error using linear approximation would be large, in an absolute sense, but might be small relative to the size of $\omega$ itself. The point is that, for instance, a change in length of 1 foot for something that’s 0.5 feet long seems like a big change, while a change of 1 foot for something that’s 10,000 feet long doesn’t seem like much of a change.

It is frequently the case that what you care about isn’t just the change in a function $f$, but its change, relative to its initial size or value; that is, rather than calculating just $\Delta f$, you want to calculate $\frac{\Delta f}{f}$. Of course, this is approximated by $\frac{df}{f}$, the relative differential.

**Example 2.3.19.** The power loss across a resistor is given by

$$
P = I^2 R,
$$

where $P$ is the power loss in watts, $I$ is the current, in amps, and $R$ is the resistance, in ohms.
Express the relative differential of $P$ in terms of the relative differentials of $I$ and $R$, and use this to approximate the relative change in $P$, if $I$ goes up by 10%, but $R$ goes down by 5%.

**Solution:**

We find

$$dP = \frac{\partial P}{\partial I} dI + \frac{\partial P}{\partial R} dR = 2IR dI + I^2 dR.$$ 

Thus,

$$\frac{dP}{P} = \frac{2IR dI + I^2 dR}{I^2 R} = 2 \frac{dI}{I} + \frac{dR}{R},$$

which expresses the relative differential of $P$ in terms of the relative differentials of $I$ and $R$.

We now replace $dI$ with the actual $\Delta I$ and $dR$ with the actual $\Delta R$ or, here, $dI/I$ and $dR/R$ with $\Delta I/I$ and $\Delta R/R$. We are given that

$$\frac{\Delta I}{I} = 0.1 \quad \text{and} \quad \frac{\Delta R}{R} = -0.05.$$ 

Therefore, we find that

$$\frac{\Delta P}{P} \approx \frac{dP}{P} = 2(0.1) + (-0.05) = 0.15,$$

i.e., the relative change in $P$ is approximately 15%.

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### 2.3.1 Exercises

1.