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The International Conference on Singularity Theory and Applications was held at the University of Science and Technology of China (Hefei, China), during July 25-31, 2011. The conference was focused around recent developments in Singularity Theory, including topics such as: fundamental groups of algebraic varieties, topological and Hodge-theoretic aspects of singularities, characteristic classes of singular varieties, etc. The meeting was attended by about fifty participants from all over the world.

The papers in this volume cover the various subjects discussed during the conference. All manuscripts have been carefully peer-reviewed. I would like to express my gratitude to the authors for their contributions, as well as to the referees for the high quality job. I also thank all the participants – especially the speakers – who made the meeting both successful and fruitful. I am very grateful for the financial support received from the National Research Foundation, as well as from the Department of Mathematics of the University of Science and Technology of China. Last, but not least, I would like to thank Xiuxiong Chen and his staff for a flawless organization of this conference.

May 2012
Laurentiu Maxim, Editor

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MULTIDIMENSIONAL RESIDUE THEORY AND THE LOGARITHMIC DE RHAM COMPLEX

A.G. ALEKSANDROV

Abstract. We study logarithmic differential forms with poles along a reducible hypersurface and the multiple residue map with respect to the complete intersection given by its components. Some applications concerning computation of the kernel and image of the residue map and the description of the weight filtration on the logarithmic de Rham complex for hypersurfaces whose irreducible components are defined by a regular sequence of functions are considered. Among other things we give an easy proof of the de Rham theorem (1954) on residues of closed meromorphic differential forms whose polar divisor has rational quadratic singularities, and correct some inaccuracies in its original formulation and later citations.

Keywords: logarithmic de Rham complex; regular meromorphic forms; multiple residues; complete intersections; weight filtration

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Introduction

The term "residue" (together with its formal definition) appeared for the first time in an article by A. Cauchy (1826), although one can find such a notion as implicit in Cauchy's prior work (1814) about the computation of particular integrals which were related with his research towards hydrodynamics. For the next three-four years, Cauchy developed residue calculus and applied it to the computation of integrals, the expansion of functions as series and infinite products, the analysis of differential equations, and so on.

Though it was already transparent in the pioneer work by N. Abel, a major step towards the elaboration of the residue concept was made by H. Poincaré who introduced in 1887 the notion of differential *residue* 1-*form* attached to any rational differential 2-form in $\mathbf{C}^2$ with simple poles along a smooth complex curve. Such rational form can be considered as the simplest prototype of differential forms called *logarithmic* in the modern terminology. Subsequently É. Picard (1901), G. de Rham (1932/36),

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A. Weil (1947) obtained a series of important results about residues of meromorphic forms of degree 1 and 2 on complex manifolds; such developments were associated with cohomological ideas, leading to the formulation of a cohomological residue formula, and therefore to explicit computations of integrals of rational forms (in the spirit of Cauchy or Abel) along cycles.

Among other things Poincaré has also proved that the residue is, in fact, a *holomorphic* 1-form. A simple generalization of his construction to the case of a complex analytical variety M of dimension $m \geq 2$ leads to the following exact sequence of $\mathcal{O}_M$-modules

$$0 \longrightarrow \Omega^m_M \longrightarrow \Omega^m_M(D) \xrightarrow{\text{rés}} \Omega^{m-1}_D \longrightarrow 0,$$

where $\Omega^m_M(D)$ is a sheaf of meromorphic differential forms of degree m on M with poles of the first order on the *smooth* divisor $D \subset M$, and Ω^m_M and Ω^{m-1}_D are sheaves of regular holomorphic forms on M and D of degrees m and $m-1$, respectively.

In the fifties further cohomological ideas were pursued by G. de Rham (1954) and J. Leray (1959) who defined and studied residues of *d-closed* C^∞-regular differential forms on the complement $M \setminus D$ with poles of the first order along a smooth hypersurface D in complex manifold M. Thus, for any such q-form ω there exists locally the following decomposition

$$\omega = \frac{dh}{h} \wedge \xi + \eta, \tag{1}$$

where h is the germ of a holomorphic function determining the smooth hypersurface D, and ξ, η are germs of regular forms. Moreover, the restriction of ξ to D does not depend on a local equation of the hypersurface; it is globally and uniquely determined and closed on D. The differential form $\xi\big|_D$ is called the *residue-form* denoted by rés(ω). If ω is holomorphic on $M \setminus D$, then the differential form $\xi\big|_D$ is holomorphic on D.

In 1972, J.-B. Poly showed that Leray decomposition (1) as well as the residue form are determined correctly for the so-called *semi-meromorphic* forms (not necessarily closed) if both they and their total differentials have poles of the first order along D (see [16]). Such *meromorphic* forms were called by P. Deligne (1969) differential forms with *logarithmic* poles along D; in fact, he considered the case of divisors with normal crossings. The corresponding coherent sheaves of $\mathcal{O}_M$-modules are denoted by $\Omega^q_M(\log D)$, $q \geq 1$. It is not difficult to see that in these notations there are exact sequences of $\mathcal{O}_M$-modules

$$0 \longrightarrow \Omega^q_M \longrightarrow \Omega^q_M(\log D) \xrightarrow{\text{rés}} \Omega^{q-1}_D \longrightarrow 0,$$

where Ω^{q-1}_D, $q \geq 1$, are sheaves of regular *holomorphic* differential forms on D.

In 1977, making use of decomposition (1) with a multiplier, K.Saito introduced the notion of residue res.(ω) for a meromorphic form ω on M with logarithmic poles along a reduced divisor D with arbitrary singularities (see [19]). Somewhat later the author proved (see [1], [2]) that in this case for all $q \geq 1$ there are exact sequences

$$0 \longrightarrow \Omega^q_M \longrightarrow \Omega^q_M(\log D) \xrightarrow{\text{res.}} \omega^{q-1}_D \longrightarrow 0, \tag{2}$$

where ω^q_D, $q \geq 0$, are sheaves of *regular meromorphic* q-forms on D. Further generalizations of these results are investigated in [3], [4].

For completeness it should be remarked that the original concept of residue is, in fact, a local notion; the classical local residue is given by a variant of Cauchy formula for several complex variables. In the focus of the *global theory* of residue is the *residue formula.* For rational differential 1-forms defined on a compact complex algebraic curve it is one of the fundamental results in the classical analytic and algebraic geometry (see [21]). In the multidimensional case, that is, for meromorphic differential m-forms given on an m-dimensional complex manifold many variants of the residue formula in various situations and different contexts are known (see, for example, [8]). Such a form ω is closed, $d\omega = 0$, by reason of dimension. In this case only meromorphic forms with *polar* singularities, namely *logarithmic* differential m-forms, enter non-trivial contributions in the residue formula.

The paper is organized as follows. In the first two sections some elementary properties of logarithmic differential forms with simple poles along a divisor are considered. Then in the third and fourth sections we discuss properties of *multiple residues* of logarithmic differential forms with poles along reducible hypersurfaces. In particular, it is proved that the residue map determines exact sequences similarly to the above (2) for divisors whose components are defined locally by regular sequences of function germs. The proof is based essentially on the theory of logarithmic and multi-logarithmic differential forms and some properties of the multiple residue studied earlier in [1], [2], [3]. In the next two sections the kernel and image of the multiple residue map are described. Some applications are considered in two final sections, then the obtained results adapt for computing residues of logarithmic differential forms of principal type and for description of the weight filtration on the logarithmic de Rham complex for divisors whose components are defined by a regular sequence of functions. In particular, this allows to compute the mixed Hodge structure on cohomology of the complement of divisors of certain types without using theorems on resolution of singularities or the standard reduction to the case of normal crossings. Among other things in Section 7 we also give an easy proof of the well-known theorem goes back to de Rham (1954) which asserts that the residues of *closed* meromorphic differential forms whose polar divisor has *rational quadratic* singularities are holomorphic on the divisor, and correct also some inaccuracies in its original formulation and later citations.

The author thanks the organizing committee, especially Laurentiu Maxim for a complimentary invitation to participate in the conference as well as all colleagues from Mathematical Department of the University of Science and Technology of China for well provided and excellent organization of this unforgettable meeting.

1. The logarithmic de Rham complex

Let S be a complex analytical variety of dimension $m \geq 1$, and $z = (z_1, \dots, z_m)$ be a local coordinate system in a neighborhood U of the distinguished point $x \in U \subset S$. Further, suppose that a hypersurface $D \subset S$ is defined by a function $h \in \mathcal{O}_U$. We will also assume that h has no multiple factors so that the hypersurface D is *reduced*, that is, the divisor D does not contain multiple components.

Let ω be a meromorphic differential q-form on U with poles along D. Then ω is called *logarithmic* or q-form with *logarithmic poles* along D if $h\omega$ and $hd\omega$ are holomorphic on U.

Let us also denote by $S = (S, x) \cong (\mathbb{C}^m, 0)$ the germ of S at the distinguished point x. For simplification in the record identical notations for the spaces and their germs at this point are often used without additional comments when the sense is clear from the context. Throughout the paper we also use the term divisor for (locally principal) Cartier divisors D in a manifold.

The localization of the concept of logarithmic forms leads to the definition of $\mathcal{O}_{S,x}$-module $\Omega^q_{S,x}(\log D)$ which consists of the germs of meromorphic q-forms on S with poles along D such that $h\omega$ and $hd\omega$ are holomorphic at the point x, that is, $h \cdot \Omega^q_{S,x}(\log D) \subseteq \Omega^q_{S,x}$ and $h \cdot d\Omega^q_{S,x}(\log D) \subseteq \Omega^{q+1}_{S,x}$. Evidently, the second condition is equivalent to the inclusion $dh \wedge \Omega^q_{S,x}(\log D) \subseteq \Omega^{q+1}_{S,x}$. The corresponding coherent analytic sheaves of *logarithmic differential forms* are denoted by $\Omega^q_S(\log D)$, $q \geq 0$. It should be remarked that $\Omega^m_{S,x}(\log D) \cong \mathcal{O}_{S,x}(dz_1 \wedge \ldots \wedge dz_m/h)$. By definition, $\Omega^0_S(\log D) \cong \mathcal{O}_S$, and there are natural inclusions $\Omega^q_S \subseteq \Omega^q_S(\log D)$ for all $q \geq 1$ which are, in fact, isomorphisms $\Omega^q_{S,x} \cong \Omega^q_{S,x}(\log D)$ for all $x \notin D$.

The family $\Omega^q_S(\log D), q \geq 0$, endowed with differential induced by the de Rham differentiation d of $\Omega^\bullet_S$ defines an *increasing* complex called the *logarithmic* de Rham complex. Further, the sheaves of logarithmic differential forms are $\mathcal{O}_S$-modules of finite type, and their direct sum $\oplus^m_{q=0}\Omega^q_S(\log D)$ forms an $\mathcal{O}_S$-exterior algebra closed under the action of d.

Recall that $\mathcal{O}_S$-module of vector fields *logarithmic* along $D \subset S$ consists of germs of holomorphic vector fields $\mathcal{V} \in \mathrm{Der}(\mathcal{O}_S)$ on S such that $\mathcal{V}(h)$ belongs to the principal ideal $(h)\cdot\mathcal{O}_S$. In particular, $\mathcal{V}$ is tangent to D at its non-singular points. This module is denoted by $\mathrm{Der}_S(\log D)$. There is a perfect pairing

$$\mathrm{Der}_S(\log D) \times \Omega^1_S(\log D) \to \mathcal{O}_S$$

induced by the contraction of differential forms along vector fields (see [20]).

Let us also remark that in general $\Omega^q_S(\log D) \ncong \bigwedge^q \Omega^1_S(\log D)$. However, for all $q > 0$ there exist natural inclusions $\bigwedge^q \Omega^1_S(\log D) \to \Omega^q_S(\log D)$. All these inclusions are isomorphisms if $\Omega^1_S(\log D)$ or, equivalently, $\mathrm{Der}_S(\log D)$ is *locally free*. In this case D is called the *free hypersurface* or *Saito free divisor*.

2. Logarithmic forms with poles along reducible hypersurfaces

Let $D = D_1 \cup \ldots \cup D_k$ be any irredundant (not necessarily irreducible) decomposition of a reduced divisor D. It is clear that there are natural inclusions

$$\sum_{i=1}^{k} \Omega^q_S(\log D_i) \hookrightarrow \Omega^q_S(\log D),\ q \geq 0.$$

Analogously, if $\widehat{D_i}$ is the union of all elements of the decomposition excluding D_i, that is, $\widehat{D_i} = D_1 \cup \ldots \cup D_{i-1} \cup D_{i+1} \cup \ldots \cup D_k$, then

$$\sum_{i=1}^{k} \Omega^q_S(\log \widehat{D_i}) \hookrightarrow \Omega^q_S(\log D),$$

and $\Omega^q_{S,x}(\log D_i) \cong \Omega^q_{S,x}(\log D)$ are isomorphisms for all $x \in D_i \setminus (D_i \cap \widehat{D_i})$, and so on.

Claim 1. *Assume that divisors D_i defined by function germs h_i, $i = 1, \ldots, k$, are components of a locally irredundant decomposition of D. Then there is a natural isomorphism*

$$\mathrm{Der}_S(\log D_1) \cap \ldots \cap \mathrm{Der}_S(\log D_k) \cong \mathrm{Der}_S(\log D).$$

Proof. It is clear, that the left side of the relation is contained in the rightist. Conversely, take $\mathcal{V} \in \mathrm{Der}_S(\log D)$. Then $\mathcal{V}(h) = \sum_{i=1}^{k}(h_1 \cdots \widehat{h_i} \cdots h_k)\mathcal{V}(h_i) = fh$, where $f \in \mathcal{O}_S$. After division by h_i the both part of the latter equality one obtains that the function $(h_1 \cdots \widehat{h_i} \cdots h_k)\mathcal{V}(h_i)/h_i$ is holomorphic, that is, h_i divides $\mathcal{V}(h_i)$. Hence, $\mathcal{V}(h_i) \in (h_i)\mathcal{O}_S,\ i = 1, \ldots, k$. QED.

The following assertion one may consider as a dual variant of the above statement.

Claim 2. *Under the same assumptions let us suppose that $\Omega^1_S(\log D)$ is generated by closed forms. Then one has an isomorphism*

$$\Omega^1_S(\log D_1) + \ldots + \Omega^1_S(\log D_k) \cong \Omega^1_S(\log D).$$

Proof. Due to Theorem 2.9 from [20] the conditions of closeness of generators of $\Omega^1_S(\log D)$ is equivalent to the isomorphism $\sum_{i=1}^{k} \mathcal{O}_S \frac{dh_i}{h_i} + \Omega^1_S \cong \Omega^1_S(\log D)$. On the other side, $\frac{dh_i}{h_i} \in \Omega^1_S(\log D_i)$ and there is a natural inclusion $\sum_{i=1}^{k} \Omega^1_S(\log D_i) \hookrightarrow \Omega^1_S(\log D)$. This completes the proof. QED.

Proposition 1. *Under assumptions of Claims above there exist natural inclusions*

$$h_i\Omega^\bullet_S(\log D) \subseteq \Omega^\bullet_S(\log \widehat{D_i}), \quad dh_i \wedge \Omega^\bullet_S(\log D) \subseteq \Omega^{\bullet+1}_S(\log \widehat{D_i}), \quad i = 1, \ldots, k.$$

In other words, the external product by total differentials dh_i as well as multiplication by functions h_i "dissipates" poles of $\omega \in \Omega^\bullet_S(\log D)$ located on D_i.

Proof. Let us first examine the case $k = 2$. Let us set $i = 1$, then take $x \in D_1 \cap D_2$ and show that $h_2\Omega^\bullet_S(\log D) \subseteq \Omega^\bullet_S(\log D_1)$. By assumptions, $h_1(h_2\omega) = h\omega \in \Omega^\bullet_S$. Further,

$$dh \wedge (h_2\omega) = h_2 dh_1 \wedge (h_2\omega) + h_1 dh_2 \wedge (h_2\omega) = h_2 dh_1 \wedge (h_2\omega) + dh_2 \wedge (h\omega).$$

Since the differential form $dh \wedge \omega$ is holomorphic then $dh \wedge (h_2\omega)$ is also a holomorphic form. Analogously, $dh_2 \wedge (h\omega) \in \Omega^\bullet_S$ and, consequently, $h_2 dh_1 \wedge (h_2\omega) = h_2^2 dh_1 \wedge \omega \in \Omega^\bullet_S$. Set $dh_1 \wedge \omega = \vartheta/h_2^2$, where $\vartheta \in \Omega^\bullet_S$. Let us note that $\frac{dh_1}{h_1} \in \Omega^\bullet_S(\log D)$, so that $\frac{dh_1}{h_1} \wedge \omega \in \Omega^\bullet_S(\log D)$ in virtue of $\wedge$-closeness. Therefore, $\frac{dh_1}{h_1} \wedge \omega = \frac{\vartheta}{h_1 h_2^2} \in \Omega^\bullet_S(\log D)$, that is, $\frac{\vartheta}{h_2} \in (h)\Omega^\bullet_S(\log D) \subseteq \Omega^\bullet_S$. Hence, $\vartheta \in (h_2)\Omega^\bullet_S$ and $dh_1 \wedge \omega = \vartheta'/h_2$, where $\vartheta' \in \Omega^\bullet_S$.

Thus, $h_2(dh_1 \wedge \omega) = dh_1 \wedge (h_2\omega) \in \Omega^\bullet_S$, that is, $dh_1 \wedge (h_2\omega)$ is a holomorphic form. It does mean that $h_2\omega \in \Omega^\bullet_S(\log D_1)$. This completes the proof of the first inclusion.

The second inclusion can be proved in the same style. Really, $h_1(dh_2 \wedge \omega)$ is a holomorphic differential form because it is equal to the difference $dh \wedge \omega - h_2 dh_1 \wedge \omega$, where the first form is holomorphic by assumptions, while the holomorphicity of the second form is established similarly to the proof of the first inclusion. Further,

$$dh_1 \wedge (dh_2 \wedge \omega) = d(h_1 dh_2 \wedge \omega) + h_1(dh_2 \wedge d\omega).$$

Since the form $h_1 dh_2 \wedge \omega$ is holomorphic then its total differential is also holomorphic. At last,

$$h_1(dh_2 \wedge d\omega) = dh \wedge d\omega - h_2(dh_1 \wedge d\omega).$$

The differential form $dh \wedge d\omega$ is holomorphic by hypothesis since the external algebra $\Omega_S^\bullet(\log D)$ is closed relative to the de Rham differentiation d, so that $d\omega \in \Omega_S^\bullet(\log D)$. As in the proof of the first inclusion one obtains that $h_2(dh_1 \wedge d\omega)$ is a holomorphic form. This implies that $h_1(dh_2 \wedge d\omega)$ as well as $h_1(dh_2 \wedge \omega)$ are holomorphic forms. Thus, $dh_2 \wedge \omega \in \Omega_S^\bullet(\log D_1)$ as required. The general case $k > 2$ is considered analogously. QED.

Remark 1. By the same reasonings one can see that for all $j = 1, \ldots, k$ there are inclusion

$$(h_1 \cdots \widehat{h_i} \cdots h_k)\Omega_S^\bullet(\log D) \subseteq \Omega_S^\bullet(\log D_i), \quad d(h_1 \cdots \widehat{h_i} \cdots h_k) \wedge \Omega_S^\bullet(\log D) \subseteq \Omega_S^\bullet(\log D_i).$$

Similar relations are also valid for divisors obtained by the exclusion of any collection of components of the decomposition.

Claim 3. *Assume that components D_i, $i = 1, \ldots, k$, of an irredundant decomposition of a reduced divisor D are defined locally by elements of a regular $\mathcal{O}_S$-sequence $(h_1, \ldots, h_k)$. Then*

$$\Omega_S^\bullet(\log \widehat{D_1}) \cap \ldots \cap \Omega_S^\bullet(\log \widehat{D_k}) = \Omega_S^\bullet,$$

and there is an exact sequence of complexes

$$0 \longrightarrow \Omega_S^\bullet \longrightarrow \oplus\, \Omega_S^\bullet(\log \widehat{D_i}) \longrightarrow \sum \Omega_S^\bullet(\log \widehat{D_i}) \longrightarrow 0.$$

Proof. It is sufficient to prove the first relation. Clearly, the right side of the relation is contained in the leftist. Conversely, let us take a differential p-form ω from the left side. Then $(h_1 \cdots \widehat{h_i} \cdots h_k)\omega \in \Omega_S^p$, $i = 1, \ldots, k$. Hence, $\omega \in \bigcap \frac{1}{(h_1 \cdots \widehat{h_i} \cdots h_k)}\Omega_S^p$, or, equivalently, $h\omega \in (h_1)\Omega_S^p \cap \ldots \cap (h_k)\Omega_S^p$. Elementary properties of regular sequences imply that the latter intersection is equal to $(h_1 \cdots h_k)\Omega_S^p$, that is, $\omega \in \Omega_S^p$. QED.

3. A decomposition of meromorphic forms along complete intersections

Let $D = D_1 \cup \ldots \cup D_k$ be a reduced reducible hypersurface. We will denote the $\mathcal{O}_S$-modules of meromorphic differential q-forms, $q \geq 1$, formed by differential q-forms with simple poles and with poles of any order on the divisor $\widehat{D_i} = D_1 \cup \ldots \cup D_{i-1} \cup D_{i+1} \cup \ldots \cup D_k$, by $\Omega_S^q(\widehat{D_i})$ and by $\Omega_S^q(\star\widehat{D_i})$, $i = 1, \ldots, k$, respectively. When $k = 1$ we set $\widehat{D_1} = \emptyset$, so that $\Omega_S^q(\widehat{D_1}) = \Omega_S^q(\star\widehat{D_1}) = \Omega_S^q$.

Let us further assume that the complex analytical space $C = D_1 \cap \ldots \cap D_k$ is a *complete intersection.* This means that the ideal $\mathcal{I}$ defining $C \subset U$ is locally generated by a *regular* $\mathcal{O}_U$-sequence $(h_1, \ldots, h_k)$ and $\dim C = m - k \geq 0$. We also suppose that $C = C_{\mathrm{red}}$ is a *reduced* space when $\dim C > 0$. In other words, the ideal $\mathcal{I} = \sqrt{\mathcal{I}}$ is *radical.* In particular, these conditions imply that the differential k-form $dh_1 \wedge \ldots \wedge dh_k$ is not identically zero on every irreducible component of C. The following statement and its proof are slightly changed versions of considerations from [3], [4].

Theorem 1. *Suppose that in a neighborhood U of $x \in C$ all irreducible components D_i, $i = 1, \ldots, k$, of D are defined by elements of a regular $\mathcal{O}_U$-sequence $(h_1, \ldots, h_k)$. Assume also that a meromorphic differential form $\omega \in \Omega_U^q(D)$ satisfies the following conditions*

$$dh_j \wedge \omega \in \sum_{i=1}^{k} \Omega_U^{q+1}(\widehat{D_i}), \quad j = 1, \ldots, k. \tag{3}$$

Then there is a holomorphic function g, which is not identically zero on every irreducible component of the complete intersection C, a holomorphic differential form $\xi \in \Omega_U^{q-k}$ and a meromorphic q-form $\eta \in \sum_{i=1}^k \Omega_U^q(\widehat{D_i})$ such that there exists the following representation

$$g\omega = \frac{dh_1}{h_1} \wedge \ldots \wedge \frac{dh_k}{h_k} \wedge \xi + \eta. \tag{4}$$

Proof. In a neighborhood of $x \in U$ the differential form ω is represented as follows:

$$\omega = \frac{1}{h_1 \cdots h_k} \sum_{|I|=q} a_I(z) \cdot dz_I,$$

where $I := I^q = (i_1, \ldots, i_q)$, $1 \leq i_1, \ldots, i_q \leq m$, is a multiple index, $dz_I = dz_{i_1} \wedge \ldots \wedge dz_{i_q}$, and $a_I(z) \in \mathcal{O}_U$ is the set of coefficients, skew-symmetric relative to I. It is clear that conditions (3) are equivalent to inclusions

$$dh_j \wedge \sum_I a_I(z) \cdot dz_I \in \sum_{\ell=1}^k h_\ell \, \Omega_U^{q+1}, \quad j = 1, \ldots, k.$$

These inclusions give us the following system of relations between the coefficients a_I and the partial derivatives of h_j :

$$\sum_{\ell=1}^q (-1)^{\ell-1} \frac{\partial h_j}{\partial z_{i_\ell}} a_{I \setminus i_\ell} = b_{jI}^1 h_1 + \ldots + b_{jI}^k h_k, \quad j = 1, \ldots, k, \tag{5}$$

with holomorphic functions $b_{jI}^1, \ldots, b_{jI}^k \in \mathcal{O}_U$.

Let us fix a multi-index $J^p = (j_1, \ldots, j_p)$, $1 \leq j_1, \ldots, j_p \leq m$, $1 \leq p \leq k$, and denote the corresponding minor of Jacobian matrix $\operatorname{Jac}(h_1, \ldots, h_k) = \|\partial h_i / \partial z_j\|$ by

$$\Delta_{J^p} = \delta_{j_1 \ldots j_p} = \det \left\| \frac{\partial h_i}{\partial z_{j_r}} \right\|_{1 \leq i,\, r \leq p}$$

We will prove by induction on index p that the following relations are valid:

$$\Delta_{J^p} a_{I^q} \equiv \sum_{K \subset I^q,\, |K|=p} \operatorname{sgn} \begin{pmatrix} I^q \\ K, I^q \setminus K \end{pmatrix} \Delta_K \, a_{(J^p, I^q \setminus K)} \, \big(\operatorname{mod}(\mathcal{I})\big), \quad p = 1, \ldots, k, \tag{6}$$

where $\mathcal{I} \subseteq \mathcal{O}_U$ is generated by the regular sequence $(h_1, \ldots, h_k)$.

First let us assume that $p = 1$ and set $J^1 = j_1 = j$, $I = (j, I^q) = (j, i_1, \ldots, i_q)$ in formula (5). Then one gets the following relation

$$\frac{\partial h_1}{\partial z_j} a_{I^q} \equiv \sum_{\ell=1}^q (-1)^{\ell-1} \frac{\partial h_1}{\partial z_{i_\ell}} a_{I \setminus i_\ell} \, \big(\operatorname{mod}(\mathcal{I})\big),$$

which coincides with relation (6) for $p = 1$.

Let us suppose that (6) is true for $p-1$ and prove it for p as follows. The cofactor expansion of determinant Δ_{J^p} along the p-th row gives the identity:

$$\Delta_{J^p} a_{I^q} = \sum_{\ell=1}^p (-1)^{p-\ell} \frac{\partial h_p}{\partial z_{j_\ell}} \Delta^{p-1}_{j_1 \ldots \widehat{j_\ell} \ldots j_p} a_{I^q}.$$

By the induction hypothesis there is the congruence

$$\Delta_{j_1 \ldots \hat{j}_\ell \ldots j_p} a_{I^q} \equiv \sum_{\substack{K' \subset I^q \\ |K'|=p-1}} \operatorname{sgn} \begin{pmatrix} I^q \\ K', I^q \setminus K' \end{pmatrix} \Delta_{K'} a_{(j_1 \ldots \widehat{j_\ell} \ldots j_p,\, I^q \setminus K')} \, \big(\operatorname{mod}(\mathcal{I})\big).$$

Let us substitute this expression in the previous identity. Changing then the order of summation, one obtains

$$\Delta_{J^p} a_{I^q} \equiv \sum_{\substack{K' \subset I^q \\ |K'|=p-1}} \operatorname{sgn}\begin{pmatrix} I^q \\ K', I^q \setminus K' \end{pmatrix} \Delta_{K'} \sum_{\ell=1}^{p} (-1)^{p-\ell} \frac{\partial h_p}{\partial z_{j_\ell}} a_{(j_1 \ldots \widehat{j_\ell} \ldots j_p,\, I^q \setminus K')} \ \big(\mathrm{mod}\,(\mathcal{I})\big).$$

The second sum consists of p terms containing in formula (5) with $j = p$, $I = (j_1, \ldots, j_p,\, I^q \setminus K')$.

It is not difficult to rewrite this expression in the form of the sum which contains the remaining $q - p + 1$ terms with opposite signs and an element from the ideal $(h_1, \ldots, h_k)\mathcal{O}_U$. Hence, one obtains the congruence modulo $\mathcal{I}$:

$$\Delta_{J^p} a_{I^q} \equiv \sum_{\substack{K' \subset I^q \\ |K'|=p-1}} \operatorname{sgn}\begin{pmatrix} I^q \\ K', I^q \setminus K' \end{pmatrix} \Delta_{K'} (-1)^{p-1} \sum_{i \in I \setminus K'} (-1)^{\#(i;\, I \setminus K')} \frac{\partial h_p}{\partial z_i} a_{(j_1 \ldots j_p,\, I^q \setminus K' \setminus i)}, \tag{7}$$

where $\#(i;\, I \setminus K')$ is equal to the number of occurrences of the index i in the set $I \setminus K'$. At last, let us put in order all pairs (K', i) in such a way that the multi-index $K' \cup \{i\}$ coincides with the given one $K \subset I$. For any such pair the corresponding coefficient $a_{(j_1 \ldots j_p,\, I \setminus K' \setminus i)}$ is equal to $a_{(J,\, I \setminus K)}$. Then the contribution of the above ordered set to relation (7) is equal to the following:

$$a_{(J^p,\, I^q \setminus K)} (-1)^{p-1} \sum_{i \in K} \operatorname{sgn}\begin{pmatrix} I^q \\ K \setminus i,\, I^q \setminus K,\, i \end{pmatrix} (-1)^{\#(i;\, I \setminus (K \setminus i))} \frac{\partial h_p}{\partial z_i} \Delta_{K \setminus i} =$$

$$= \operatorname{sgn}\begin{pmatrix} I^q \\ K,\, I^q \setminus K \end{pmatrix} a_{J^p,\, I^q \setminus K}\, \Delta_K.$$

This completes the proof of relation (6) for $p \geq 1$.

It remains to show that it is possible to choose the function g in such a way that $g \not\equiv 0$ on each irreducible component of the complete intersection C. For this we examine ideal $\mathcal{G}$ of the ring $\mathcal{O}_U$ generated by all minors $\Delta_{i_1 \ldots i_k}$ of the maximal order of Jacobian matrix $\operatorname{Jac}(h_1, \ldots, h_k)$. Since $dh_1 \wedge \ldots \wedge dh_k$ does not vanish identically on each irreducible component of the complete intersection C, then the image $\widetilde{\mathcal{G}}$ of the ideal $\mathcal{G}$ in the ring $\mathcal{O}_{C,0}$ is not equal to $\operatorname{Ann} \mathcal{O}_{C,0}$. Thus, it is possible to use Theorem 2.4. (1) from [6] which yields that $\mathcal{O}_{C,0}$-depth of the ideal $\widetilde{\mathcal{G}}$ is not less than one. Hence, there is an element $g \in \mathcal{O}_{C,0}$ with the property required by Theorem 1. QED.

Remark 2. It is not difficult to verify that formula (6) implies the following identity

$$\Delta_{i_1 \ldots i_k} \cdot \sum_{|I|=q} a_I dz_I = dh_1 \wedge \ldots \wedge dh_k \wedge \big(\sum_{|I'|=q-k} a_{i_1 \ldots i_k I'}\, dz_{I'} \big) + \nu,$$

where $\nu \in \sum_{j=1}^{k} h_j \Omega_U^{q-k}$. Therefore, by analogy with the case of hypersurface (see [20], Lemma (2.8)) the maximal minors of Jacobian matrix $\operatorname{Jac}(h_1, \ldots, h_k)$ can be considered as *universal denominators* for the complete intersection C.

If $m = k$, that is, $\dim C = 0$ and C is *non-reduced* then the latter formula implies that there exists representation (4) with a function g equal to an element of the one-dimensional socle of the local algebra $\mathcal{O}_{C,0}$ generated over the ground field by the determinant of the Jacobian matrix $\operatorname{Jac}(h)$ (see [23]). In this case the notion of multiple residue of meromorphic differential forms of degree m coincides with the so-called *multidimensional residue;* in the context of Grothendieck local duality theory it can be expressed in terms of projection of elements of a certain finite dimensional vector space to this socle (cf. [8]).

Corollary 1. *Let $\omega \in \Omega_S^q(\log D)$ be a differential form with logarithmic poles along a hypersurface D and let $C = D_1 \cap \ldots \cap D_k$ be a complete intersection. Then there exists representation* (4) *with a differential form $\eta \in \sum_{i=1}^{k} \Omega_S^{\bullet}(\log \widehat{D_i})$.*

Proof. Since for the logarithmic form ω conditions (3) are fulfilled in virtue of Proposition 1 from Section 2, then there is decomposition (4) with $\eta \in \sum_{i=1}^{k} \Omega_U^q(\widehat{D_i})$. For the sake of simplicity, let us examine the case $k=2$. Then $\eta=\eta_1/h_1+\eta_2/h_2$, where $\eta_1,\eta_2\in\Omega^q_U$. Taking the external product by dh of both parts of representation (4), one concludes that the differential form

$$dh\wedge\eta=dh\wedge\Big(\frac{\eta_1}{h_1}+\frac{\eta_2}{h_2}\Big)=dh_2\wedge\eta_1+dh_1\wedge\eta_2+h_2\frac{dh_1}{h_1}\wedge\eta_1+h_1\frac{dh_2}{h_2}\wedge\eta_2$$

is holomorphic. Hence, the sum of the both last terms is also holomorphic. Now let us reduce all the terms of the sum to the common denominator. This gives the inclusion

$$h_2^2(dh_1\wedge\eta_1)+h_1^2(dh_2\wedge\eta_2)\in(h_1h_2)\Omega^\bullet_S,$$

i.e., $h_2^2\alpha+h_1^2\beta=h_1h_2\gamma$, where $\alpha,\beta,\gamma\in\Omega^\bullet_S$. Therefore, $h_2^2\alpha+(h_1\beta-h_2\gamma)h_1=0$. Since (h_1,h_2) is a *regular* sequence, then, comparing the coefficients of the corresponding form for every fixed collection of differentials, one obtains that $\alpha=h_1\alpha'$, $\alpha'\in\Omega^\bullet_S$. Hence, $dh_1\wedge\eta_1\in(h_1)\Omega^\bullet_S$, that is, $\eta_1/h_1\in\Omega^\bullet_S(\log D_1)$. By the same reasonings one can check that $\eta_2/h_2\in\Omega^\bullet_S(\log D_2)$. The general case $k>2$ is investigated analogously. QED.

Corollary 2. *Under conditions of Theorem* 1 *representation* (4) *exists if and only if there are analytical subsets $A_j\subset D_j$, $j=1,\dots,k$, of codimension not less than* 2 *such that the germ ω at any point $x\in\bigcup_{j=1}^k(D_j\setminus A_j)$ belongs to the space*

$$\frac{dh_1}{h_1}\wedge\ldots\wedge\frac{dh_k}{h_k}\wedge\Omega^{q-k}_{U,x}+\sum_{i=1}^k\Omega^q_{U,x}(\widehat{D_i}). \tag{8}$$

Proof. Taking $A_j=D_j\cap\{g=0\}$, $j=1,\dots,k$, one obtains the decomposition of Theorem 1 which implies the desired statement.

The converse is true in view of the following reasonings. If there exists representation (8) for a meromorphic form ω, then $h\omega$ is, in fact, holomorphic outside of subsets $A_i\subset D_i$, $i=1,\dots,k$, of codimension not less than 2. Consequently, according to Riemann extension Theorem, the differential form $h\omega$ is holomorphic everywhere so that $h_j\omega\in\Omega^q_U(\widehat{D_j})$, $j=1,\dots,k$.

Further, $dh_j\wedge\omega$ is represented as the sum of meromorphic forms ω_i, each of which is singular not more than on $k-1$ components of divisor $\widehat{D_i}$ and on the subset $A_i\subset D_i$ of codimension not less than 2. Again, applying Riemann Theorem to $(h_1\cdots\widehat{h_i}\cdots h_k)\omega_i$, one obtains that the differential form ω_i has singularities *only* on $\widehat{D_i}$. As a result $dh_j\wedge\omega\in\sum_{i=1}^k\Omega^q_U(\widehat{D_i})$, $j=1,\dots,k$. QED.

Remark 3. If one takes a decomposition of a reducible divisor D of length $k=1$, so that $C=D$, then representation (4) looks like this

$$g\omega=\frac{dh}{h}\wedge\xi+\eta,\quad \xi,\eta\in\Omega^\bullet_U; \tag{9}$$

it coincides with representation of the basic lemma by K.Saito (see [20], (1.1), *iii*)).

4. The multiple residue map

Let us now discuss the concept of *multiple residues* of meromorphic forms which satisfy conditions of Section 3. In notations of Theorem 1 it is not difficult to see that the function g from representation (4) is a *non-zero divisor* in $\mathcal{O}_{S,0}/(h_1,\dots,h_k)\mathcal{O}_{S,0}\cong\mathcal{O}_{C,0}$. Therefore the restriction of the form ξ/g to the germ of complete intersection $C=D_1\cap\ldots\cap D_k$ is well-defined.

Definition 1. The restriction of differential form ξ/g to the complete intersection C is called the *multiple residue* of the differential form ω; the corresponding map is denoted by Res_C, so that

$$\mathrm{Res}_C(\omega)=\frac{\xi}{g}\Big|_C.$$

Remark 4. The multiple residue of ω is contained in the space $\mathcal{M}_C\otimes_{\mathcal{O}_C}\Omega^{q-k}_C\cong\mathcal{M}_{\widetilde{C}}\otimes_{\mathcal{O}_{\widetilde{C}}}\Omega^{q-k}_{\widetilde{C}}$, $q\geq k$, where $\widetilde{C}$ is the normalization of C.

Proposition 2. *The multiple residue map is well-defined, that is, its values do not depend on representation* (4).

Proof. Let us assume that $q \geq k$ and a differential q-form ω have two local representations

$$g_\ell \omega = \frac{dh_1}{h_1} \wedge \ldots \wedge \frac{dh_k}{h_k} \wedge \xi_\ell + \eta_\ell, \quad \ell = 1, 2.$$

Then

$$dh_1 \wedge \ldots \wedge dh_k \wedge (g_1\xi_2 - g_2\xi_1) = h_1 \cdots h_k (g_1\eta_2 - g_2\eta_1) \in (h_1, \ldots, h_k)\Omega^q_S.$$

Consequently,

$$dh_1 \wedge \ldots \wedge dh_k \wedge (g_1\xi_2 - g_2\xi_1) \equiv 0 \ \big(\mathrm{mod}\, (h_1, \ldots, h_k)\big).$$

Then the first part of the main Theorem from [18] (the generalized de Rham Lemma) with $R = \mathcal{O}_{C,0}$, $M = \Omega^1_{S,0} \otimes \mathcal{O}_{C,0}$, $e_i = z_i$, $i = 1, \ldots, m$, $\omega_j = dh_j$, $j = 1, \ldots, k$, $p = q - k \geq 0$, implies that

$$\mathcal{G}^e(g_1\xi_2 - g_2\xi_1) \subset dh_1 \wedge \Omega^{q-k-1}_{S,0} + \ldots + dh_k \wedge \Omega^{q-k-1}_{S,0} + (h_1, \ldots, h_k)\Omega^{q-k}_{S,0}, \quad e \in \mathbf{Z}_+,$$

where the ideal $\mathcal{G} \subset \mathcal{O}_{S,0}$ is generated by all minors $\Delta_{i_1 \ldots i_k}$ of maximal order of Jacobian matrix $\mathrm{Jac}(h_1, \ldots, h_k)$. As in the end of the proof of Theorem 1 we note that the image $\widetilde{\mathcal{G}}$ of the ideal $\mathcal{G}$ in the ring $\mathcal{O}_{C,0}$ is not equal to $\mathrm{Ann}\, \mathcal{O}_{C,0}$, since the germ C is *reduced*. Therefore $\mathcal{O}_{C,0}$-depth of the ideal $\widetilde{\mathcal{G}}$ is not less than 1. Consequently, there is an element $\Delta \in \widetilde{\mathcal{G}}$, a non-zero divisor in $\mathcal{O}_{C,0}$ such that

$$\Delta^e(g_1\xi_2 - g_2\xi_1) \in dh_1 \wedge \Omega^{q-k-1}_{S,0} + \ldots + dh_k \wedge \Omega^{q-k-1}_{S,0} + (h_1, \ldots, h_k)\Omega^{q-k}_{S,0}.$$

Therefore the class of the element $\Delta^e(g_1\xi_2 - g_2\xi_1)$ in $\Omega^{q-k}_{C,0}$ is equal to zero. It does mean that both elements $\dfrac{1}{g_1}\xi_1$ and $\dfrac{1}{g_2}\xi_2$ determine the same class in $\mathcal{M}_{C,0} \otimes_{\mathcal{O}_{C,0}} \Omega^{q-k}_{C,0}$. QED.

Lemma 1. *The kernel of the multiple residue map coincides with the space* $\sum_{i=1}^k \Omega^\bullet_S(\widehat{D_i})$.

Proof. It is clear that the kernel contains this sum. It remains to prove the converse inclusion. Suppose that $\mathrm{Res}_C(\omega) = 0$ for a certain q-form ω, $q \geq k$. Then there exists a function g in representation (4) of Theorem 1 such that the restriction of meromorphic form ξ/g to C vanishes. Consequently, $\xi = g(\sum h_i\xi_i + \sum dh_i \wedge \xi_i')$, where $\xi_i,\ \xi_i' \in \Omega^\bullet_S$, and

$$h\omega = dh_1 \wedge \ldots \wedge dh_k \wedge (\sum h_i\xi_i) + \frac{1}{g}(\sum h_i\eta_i),\ \ \eta_i \in \Omega^\bullet_S.$$

Since $h\omega$ and the first term in the right side of the identity are holomorphic, then g *divides* $\sum h_i\eta_i$ in $\Omega^\bullet_S$, that is, $g\eta_0 = \sum h_i\eta_i$, $\eta_0 \in \Omega^\bullet_S$. On the other hand, $(h_1, \ldots, h_k)$ is a regular sequence and g is a non-zero divisor in $\mathcal{O}_C = \mathcal{O}_S/(h_1, \ldots, h_k)\mathcal{O}_S$. Therefore, examining coefficients of the differentials dz_I in the coordinate representation of the holomorphic form $\sum h_i\eta_i$, one obtains that $\eta_0 \in (h_1, \ldots, h_k)\Omega^\bullet_S$. This yields $\omega \in \sum_{i=1}^k \Omega^\bullet_S(\widehat{D_i})$. QED.

5. Regular meromorphic differential forms

Let M be a complex variety, $\dim M = m$, and let $X \subset M$ be an analytical subset in a neighborhood of $x \in U \subset M$ defined by a sequence of functions $f_1, \ldots, f_k \in \mathcal{O}_U$. We denote by Ω^q_X, $q \geq 0$, the sheaves of germs of *regular holomorphic* differential q-forms on X; they are defined as restriction to X of the quotient module

$$\Omega^q_X = \Omega^q_U/((f_1, \ldots, f_k)\Omega^q_U + df_1 \wedge \Omega^{q-1}_U + \ldots + df_k \wedge \Omega^{q-1}_U)\Big|_X.$$

Then the usual differential d endows this family of sheaves with structure of a complex; it is called the de Rham complex on X and is denoted by $(\Omega^\bullet_X, d)$.

Throughout this section we assume that X is a *Cohen-Macaulay* space and $\dim X = n$. Then

$$\omega^n_X = \mathrm{Ext}^{m-n}_{\mathcal{O}_M}(\mathcal{O}_X, \Omega^m_M)$$

is called the Grothendieck dualizing module of X.

Definition 2. For any $q \geq 0$ the coherent sheaf of $\mathcal{O}_X$-modules ω_X^q is locally defined as the set of germs of meromorphic differential forms ω of degree q on X such that $\omega \wedge \eta \in \omega_X^n$ for any $\eta \in \Omega_X^{n-q}$. In other words (see [5]),

$$\omega_X^q \cong \mathrm{Hom}_{\mathcal{O}_X}(\Omega_X^{n-q}, \omega_X^n) \cong \mathrm{Ext}_{\mathcal{O}_M}^{m-n}(\Omega_X^{n-q}, \Omega_M^m).$$

Elements of ω_X^q are called *regular meromorphic* differential q-forms on X. There are also other equivalent definitions of such forms in terms of Noether normalization and trace (see [13], [5]), in terms of residual currents (see [3]), and so on. Here are some useful properties of regular meromorphic differential forms.

1) $\omega_X^q = 0$, if $q < 0$ or $q > \dim X$;

2) ω_X^q has no torsion, that is, $\mathrm{Tors}\,\omega_X^q = 0$, $q \geq 0$;

3) de Rham differential d acting on ω_X^q is extended on the family of modules ω_X^q, $0 \leq q \leq n$, and endows this family with structure of complex $(\omega_X^\bullet, d)$;

4) there exists an inclusion $\omega_X^q \subseteq j_* j^* \Omega_X^q$, where $j\colon X \setminus Z \longrightarrow X$ is the canonical inclusion and $Z = \mathrm{Sing}\, X$; moreover, if X is a *normal* space, then $\omega_X^q \cong j_* j^* \Omega_X^q$;

5) if $\pi\colon \widetilde{X} \to X$ is a finite morphism of the *normalization* of X, then the mapping of direct image $\pi_*\colon \omega_{\widetilde{X}}^\bullet \to \omega_X^\bullet$ is injective; if moreover the germ of the normalization is smooth and the codimension of the set of points, in neighborhood of which π is a local isomorphism, is not less than two, then mapping π_* is surjective (see [5]). This means that $\omega_{\widetilde{X}}^\bullet$ and $\omega_X^\bullet$ are isomorphic and, in particular, $\Omega_{\widetilde{X}}^\bullet \cong \omega_X^\bullet$.

6) if X is a *simple rational* singularity of type A_k, D_k, E_6, E_7 or E_8, then the complex $(\omega_X^\bullet, d)$ is acyclic in positive dimensions (see [11]), that is, $\omega_X^\bullet$ is a *resolution* of the constant sheaf $\mathbb{C}_X$.

Let us now assume that $X = C$ is a *complete intersection* given by a regular sequence of functions $f_1, \dots, f_k \in \mathcal{O}_U$ in a neighborhood U of a point $x \in C$. Then $n = m - k$ and

$$\omega_{C,x}^n = \mathrm{Ext}_{\mathcal{O}_{M,x}}^k(\mathcal{O}_{C,x},\ \Omega_{M,x}^m) \cong \mathcal{O}_{C,x}(\omega_0),$$

where ω_0 is the uniquely (modulo $df_1, \dots, df_k$) determined meromorphic differential n-form in $j_* j^* \Omega_{C,x}^n$ for which there is a representation $\omega_0 \wedge df_1 \wedge \ldots \wedge df_k = dz_1 \wedge \ldots \wedge dz_m$ in $j_* j^*(\Omega_{M,x}^m \otimes \mathcal{O}_{C,x})$ with local coordinates $z_1, \dots, z_m$ in U. Thus, the dualizing module ω_C^n is a *locally free* $\mathcal{O}_C$-module of rank one. Furthermore, there are isomorphisms of $\mathcal{O}_M$-modules

$$\omega_C^q \cong \mathrm{Hom}_{\mathcal{O}_C}(\Omega_C^{n-q},\ \mathcal{O}_C) \cong \mathrm{Ext}_{\mathcal{O}_M}^k(\Omega_C^{n-q},\ \mathcal{O}_M), \quad 0 \leq q \leq n.$$

Changing by places the arguments of the extension group Ext^k, one obtains another useful description of regular meromorphic forms [5].

Lemma 2. *Let a subspace $C \subset M$ be a complete intersection. Then there is an exact sequence of $\mathcal{O}_C$-modules*

$$0 \longrightarrow \omega_C^q \xrightarrow{\ \mathcal{C}\ } \mathrm{Ext}_{\mathcal{O}_M}^k\big(\mathcal{O}_C,\ \Omega_M^{q+k}\big) \xrightarrow{\ \mathcal{E}\ } \Big(\mathrm{Ext}_{\mathcal{O}_M}^k\big(\mathcal{O}_C,\ \Omega_M^{q+k+1}\big)\Big)^k, \quad q \geq 0, \tag{10}$$

where $\omega_C^q \subseteq j_* j^* \Omega_C^q$, the morphism $\mathcal{C}$ is the multiplication by the fundamental class C, and the mapping $\mathcal{E}$ is locally defined by $\mathcal{E}(e) = (e \wedge df_1,\ \dots,\ e \wedge df_k)$.

Corollary 3. *Let $C = C_1 \cup \dots \cup C_r$ be an irredundant decomposition of a complete intersection space C. Then there is a canonical inclusion of complexes of regular meromorphic forms*

$$\omega_{C_1}^\bullet \oplus \dots \oplus \omega_{C_r}^\bullet \hookrightarrow \omega_C^\bullet.$$

Proof. It is sufficient to examine the case $r = 2$. Thus, let $C = C' \cup C''$ be the union of two sets which consist of irreducible components of C and have no common elements. One can apply the functor $\mathrm{Ext}_{\mathcal{O}_M}^*$ to the short exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_{C'} \oplus \mathcal{O}_{C''} \to \mathcal{O}_{C' \cap C''} \to 0,$$

then use Lemma 2 and standard properties of functor Ext. QED.

6. Multiple residues of logarithmic forms

As already mentioned before (cf. Corollary 1) for logarithmic differential forms with poles along a divisor satisfying assumptions of Section 3, there exists representation (4), and, consequently, the restriction of multiple residue map Res_C to the subspace of such logarithmic forms is well-defined.

Lemma 3. *Let $\omega \in \Omega^q_S(\log D)$ be a differential form with logarithmic poles along D and let $C = D_1 \cap \ldots \cap D_k$ be a complete intersection. Then the multiple residue map commutes with de Rham differentiation.*

Proof. Let us apply differentiation d to representation (4) for ω :

$$\omega = \frac{dh_1}{h_1} \wedge \ldots \wedge \frac{dh_k}{h_k} \wedge \frac{\xi}{g} + \frac{\eta}{g},$$

Corollary 1 implies that the form η is logarithmic as well as its total differential $d\eta$. Thus, $\mathrm{Res}_C(d\omega) = d(\frac{\xi}{g})\big|_C$. This completes the proof. QED.

The following assertion in the case $k = 1$ has been obtained in [1] (see also [2]); we give a proof in general case $k > 1$ similarly to the proof of Theorem from [3].

Theorem 2. *In notations of Section 3 let $C = D_1 \cap \ldots \cap D_k$ be a complete intersection. Then for $p \geq k$ there is an exact sequence of $\mathcal{O}_S$-modules*

$$0 \longrightarrow \textstyle\sum_{i=1}^k \Omega^p_S(\log \widehat{D_i}) \longrightarrow \Omega^p_S(\log D) \overset{\mathrm{Res}_C}{\longrightarrow} \omega^{p-k}_C \longrightarrow 0.$$

Proof. Let us first compute the kernel of the restriction of the multiple residue morphism Res_C to $\Omega^\bullet_S(\log D)$. In view of Claim 3 from Section 2 and Lemma 1 from Section 4 it is sufficient to verify that for all $j = 1, \ldots, k$ one has

$$\Omega^\bullet_S(\log D) \cap \Omega^\bullet_S(\widehat{D_j}) = \Omega^\bullet_S(\log \widehat{D_j}).$$

Since $\Omega^\bullet_S(\log \widehat{D_j}) \subseteq \Omega^\bullet_S(\log D)$ then the right side is contained in the leftist. To prove the converse inclusion we examine in detail the case $k = 2$. Thus, take $\omega \in \Omega^\bullet_S(\widehat{D_1}) = \Omega^\bullet_S(D_2) \cong \frac{1}{h_2}\Omega^\bullet_S$, that is, $\omega = \xi/h_2$. If $\omega \in \Omega^\bullet_S(\log D)$, then $h\omega = h_1\xi \in \Omega^\bullet_S$ and

$$dh \wedge \omega = dh_1 \wedge \xi + dh_2 \wedge (h_1\omega) \in \Omega^\bullet_S.$$

This implies that $dh_2 \wedge (h_1\omega) \in \Omega^\bullet_S$, that is, $dh_2 \wedge (h_1\xi) \in (h_2)\Omega^\bullet_S$. Therefore, $h_1 dh_2 \wedge \xi = h_2\eta$, where $\eta \in \Omega^\bullet_S$. Since h_1 and h_2 form a regular sequence, then, comparing coefficients of the differential forms $dh_2 \wedge \xi$ and η, one obtains that h_1 divides η, and, therefore, $dh_2 \wedge \xi \in (h_2)\Omega^\bullet_S$, $dh_2 \wedge \omega \in \Omega^\bullet_S$, that is, $\omega \in \Omega^\bullet_S(D_2) = \Omega^\bullet_S(\widehat{D_1})$ as required. The general case $k \geq 2$ is analyzed analogously. QED.

Now we are going to describe the image of morphism Res_C, following the scheme of the proof from [1], § 4. Thus, it suffices to check everything locally. Let us first note that the image of Res_C is an $\mathcal{O}_C$-module, since in view of Proposition 1 of Section 2 there are inclusions $h_j\Omega^\bullet_S(\log D) \subseteq \Omega^\bullet_S(\log \widehat{D_j})$ for all $j = 1, \ldots, k$. In particular, the ideal $\mathcal{I} = (h_1, \ldots, h_k)$ annihilates this image. Further, Remark 2 yields that

$$\Delta_{i_1 \ldots i_k} \cdot \mathrm{Res}_C \Omega^q_S(\log D)\big|_U \subset \Omega^{q-k}_C\big|_{C \cap U},$$

for maximal minors $\Delta_{i_1 \ldots i_k}$, $(i_1, \ldots, i_k) \in [1, \ldots, m]$ of Jacobian matrix $\mathrm{Jac}(h_1, \ldots, h_k)$. Since $\omega^n_{C,0} \cong \mathcal{O}_{C,0}(dz_1 \wedge \ldots \wedge dz_{n+k}/dh_1 \wedge \ldots \wedge dh_k)$, then by definition of regular meromorphic forms one obtains that $\mathrm{Res}_C\big(\Omega^q_{S,0}(\log D)\big) \subseteq \Omega^{q-k}_{C,0}$. Let now $\mathcal{K}_\bullet(\underline{h})$ be the usual Koszul complex associated with the regular sequence $\underline{h} = (h_1, \ldots, h_k)$:

$$0 \to \mathcal{O}_{S,0}\langle e_0 \wedge \ldots \wedge e_{k-1}\rangle \xrightarrow{d_{k-1}} \ldots \xrightarrow{d_1} \sum_{i=0}^{k-1} \mathcal{O}_{S,0}\langle e_i\rangle \xrightarrow{d_0} \mathcal{O}_{S,0} \xrightarrow{d_{-1}} \mathcal{O}_{C,0} \to 0\ ,$$

where $\mathcal{K}_k(\underline{h}) = \mathcal{O}_{S,0}\langle e_0 \wedge \ldots \wedge e_{k-1}\rangle$, $\ldots$, $\mathcal{K}_1(\underline{h}) = \mathcal{O}_{S,0}\langle e_0\rangle + \ldots + \mathcal{O}_{S,0}\langle e_{k-1}\rangle$, $\mathcal{K}_0(\underline{h}) = \mathcal{O}_{S,0}$, $d_0(e_i) = h_{i+1}$, $i = 0, \ldots, k-1$, $d_{-1}(1) = 1$.

The dual exact sequence implies an isomorphism

$$\mathrm{Ext}^k_{\mathcal{O}_{S,0}}(\mathcal{O}_{C,0},\ \Omega^{q+1}_{S,0}) \cong \mathrm{Hom}_{\mathcal{O}_{S,0}}(\mathcal{K}_k(\underline{h}),\ \Omega^{q+1}_{S,0})/d^{k-1}\big(\mathrm{Hom}_{\mathcal{O}_{S,0}}(\mathcal{K}_{k-1}(\underline{h}),\ \Omega^{q+1}_{S,0})\big).$$

Thus, any element from the space $\mathrm{Ext}^k_{\mathcal{O}_{S,0}}(\mathcal{O}_{C,0},\ \Omega^{q+1}_{S,0})$ can be represented as a Čzech $(k-1)$-cochain (more precisely, as a $(k-1)$-cocycle) as follows:

$$\frac{\nu}{h_1\cdots h_k} \in \mathrm{Hom}_{\mathcal{O}_{S,0}}(\mathcal{K}_k(\underline{h}),\ \Omega^{q+1}_{S,0}) \cong C^{k-1}_{(k)}(\Omega^{q+1}_{S,0}),$$

where $\nu \in \Omega^{q+1}_{S,0}$. Let us consider an element $\nu \in \Omega^{q+1}_{S,0}$ such that the meromorphic form

$$\frac{\nu}{h_1\cdots h_k} \wedge dh_j \in \mathrm{Ext}^k_{\mathcal{O}_{S,0}}(\mathcal{O}_{C,0},\ \Omega^{q+2}_{S,0}), \quad j = 1,\ldots,k,$$

corresponds to the *trivial* element.

This means that for any $j = 1,\ldots,k$ the differential form $\nu \wedge dh_j/h_1\cdots h_k$ is determined by a certain element from the space $d^{k-1}\big(\mathrm{Hom}_{\mathcal{O}_{S,0}}(\mathcal{K}_{k-1}(\underline{h}),\ \Omega^{q+2}_{S,0})\big)$. Hence, one gets

$$\nu \wedge dh_j \in \sum_{i=1}^{k} h_i \Omega^{q+2}_S, \quad j = 1,\ldots,k,$$

or, equivalently,

$$\omega \wedge dh_j \in \sum_{i=1}^{k} \Omega^{q+2}_S(\widehat{D_i}), \quad j = 1,\ldots,k, \quad \text{where} \quad \omega = \frac{\nu}{h_1\cdots h_k}.$$

As a result, the differential form ω satisfies conditions (3) of Theorem 1. It remains to use exact sequence (10) of Lemma 2 as follows.

Let $\tilde{\nu} = \mathcal{C}^{-1}(\nu/h_1\cdots h_k)$. Then $\mathcal{C}(\tilde{\nu})$ corresponds to Čzech cocycle $\nu/h_1\cdots h_k$ such that $\nu = \tilde{\nu} \wedge dh_1 \wedge\cdots\wedge dh_k$. Making use of the description for ω^q_C in terms of multiplication by the fundamental class $C \subset S$ in exact sequence (10), one can take $v = \tilde{\nu}$, $w = \nu$, since $\mathcal{C}(v)$ corresponds to Čzech cocycle w/h such that $w = v \wedge dh_1 \wedge\cdots\wedge dh_k$. This implies

$$\omega = \tilde{\nu} \wedge \frac{dh_1}{h_1} \wedge\cdots\wedge \frac{dh_k}{h_k}, \qquad \mathrm{Res}_C(\omega) = \mathrm{Res}_C\left(\frac{\nu}{h_1\cdots h_k}\right) = \tilde{\nu}.$$

Thus, for any element $\tilde{\nu} \in \omega^{q-k}$ there exists a preimage relatively to the residue map Res_C represented by $\omega = \nu/h_1\cdots h_k$ such that the differential form $h\omega$ is holomorphic, and $dh \wedge \omega = 0$. In particular, this means that $\omega \in \Omega^\bullet_S(\log D)$ as required. QED.

Remark 5. In notations of Remark 3 let us take $k = 1$, and $C = D$. Then $\mathrm{Res}_C = \mathrm{Res}_D$; it is, in fact, the residue map res. introduced by K.Saito [20]. In this case there is (see [1]) an exact sequence

$$0 \longrightarrow \Omega^q_S \longrightarrow \Omega^q_S(\log D) \xrightarrow{\text{res.}} \omega^{q-1}_D \longrightarrow 0,\ q \geq 1, \tag{11}$$

supplementing diagram (2.5) of [20] from the right side. Thus, Theorem 2 can be considered as an extension of this sequence for the multiple residue map.

Corollary 4. *Under the same assumptions there is a natural isomorphism*

$$\mathcal{H}^p_{DR}(\Omega^\bullet_S(\log D)) \cong \mathcal{H}^{p-1}_{DR}(\omega^\bullet_D),$$

where $\mathcal{H}^*_{DR}$ is the functor of cohomologies of complexes endowed with de Rham differentiation d. In particular, $\Omega^\bullet_S(\log D)$ is acyclic in dimensions $p > 1$ when D is a simple rational singularity of type A_k, D_k, E_6, E_7 or E_8 of dimension $n \geq 2$.

Proof. The residue map is compatible with differentiation d. Hence, exact sequences (11) for all $q \geq 1$ are composed in the exact sequence of the corresponding complexes. This yields the desired isomorphism. Further, it is known (see [11], Bem. (4.8), (2)) that for rational singularities the complex $(\omega^\bullet_D, d)$ is acyclic in positive dimensions; this implies the second part of the statement. In addition, since the dimension of $\mathcal{H}^0_{DR}(\omega^\bullet_D)$ is equal to the number of irreducible components of D [loc. cite, (4.1)], then $\mathcal{H}^1_{DR}(\Omega^\bullet_S(\log D)) \cong \mathbb{C}$ under our assumptions. For completeness, it should be mentioned that these results can be also obtained by direct computations (see [10]).

7. Closed differential forms and the image of the residue map

As was discussed before the image of Poincaré-Leray residue map consists of holomorphic forms on a smooth hypersurface D; in this case $\Omega_D^\bullet$ and $\omega_D^\bullet$ are naturally isomorphic. Let us prove in the context of the theory of logarithmic differential forms the following statement for singular hypersurfaces due to G. de Rham [17] (see also [14], p.83).

Theorem 3. *Let D be a hypersurface in a manifold S, $\dim S = m \geq 3$. Assume that $\operatorname{Sing} D$ consists of isolated double quadratic points and ω is a holomorphic d-closed p-form on $S \setminus D$ with a pole of the first order on D. Then the residue-form $\operatorname{res}_D(\omega)$ is holomorphic at singular points of D if and only if either $p < m$, or $p = m$ and the functional coefficient of m-form $\omega(z)h(z)$ vanishes on $\operatorname{Sing} D$.*

Proof. At first remark, that $\Omega_S^m(D) \cong \Omega_S^m(\log D)$, and $d\omega = 0$ for all $\omega \in \Omega_S^m(D)$. By assumptions, for any $p < m$ the differential p-form $\omega \in \Omega_S^p(D)$ is closed, $d\omega = 0$. Thus, $dh \wedge \omega = d(h\omega)$ is holomorphic at $x \in S$. That is, ω is *logarithmic,* $\omega \in \Omega_{S,x}^p(\log D)$.

In view of Remark 3 for such differential form ω there exists locally representation (9) with a holomorphic function g, a non-zero divisor of $\mathcal{O}_{S,x}/(h)\mathcal{O}_{S,x}$, where $x \in \operatorname{Sing} D$ and h is equal to the sum of squares of local coordinate functions, $h = z_1^2 + \ldots + z_m^2$, in a suitable neighborhood of x. Moreover, $h\omega$ is a *torsion* element of $\Omega_{D,x}^p$ and there is an exact sequence (see [1], [2])

$$0 \longrightarrow \Omega_{S,x}^p + \frac{dh}{h} \wedge \Omega_{S,x}^{p-1} \longrightarrow \Omega_{S,x}^p(\log D) \xrightarrow{\cdot h} \operatorname{Tors}\Omega_{D,x}^p \longrightarrow 0.$$

Since $m \geq 3$ and $\operatorname{Sing} D$ consists of *isolated* double quadratic points, then D is a *normal* irreducible hypersurface. Hence, $\operatorname{Tors}\Omega_{D,x}^p = 0$ for all $p < \operatorname{codim}(\operatorname{Sing} D, D) = m-1$, and for such p one has $\omega \in \frac{dh}{h} \wedge \Omega_{S,x}^{p-1} + \Omega_{S,x}^p$, that is, the function g in representation (9) is *invertible* at x. Consequently, $\operatorname{res}_D(\omega) = \xi\big|_D$ is holomorphic on D.

When $p = m$, then $\omega = \varphi\, dz_1 \wedge \ldots \wedge dz_m/h$, where φ is a holomorphic function germ. The vanishing of φ at $x \in \operatorname{Sing} D$ yields that $h\omega = dh \wedge \xi$. Hence $\operatorname{res}_D(\omega) = \xi\big|_D$, where ξ is holomorphic at $x \in S$ and vice versa.

It remains to analyze the case $p = m-1$. In this case one has $\operatorname{Tors}\Omega_{D,x}^{m-1} = \Omega_{D,x}^m \neq 0$. To be more precise, if $z_1, \ldots, z_m$ is a local coordinate system at $x \in S$, $x = 0$, then $\operatorname{Tors}\Omega_{D,x}^{m-1}$ is generated over $\mathbb{C}$ by the Euler differential form $\vartheta = \sum (-1)^{\ell-1} z_\ell\, dz_1 \wedge \ldots \wedge \widehat{dz_\ell} \wedge \ldots \wedge dz_m$, the result of contraction of the canonical generator of $\operatorname{Tors}\Omega_{D,x}^m = \Omega_{D,x}^m \cong \mathbb{C}\langle dz_1 \wedge \ldots \wedge dz_m\rangle$ along Euler vector field. The differential form ϑ/h is *not closed,* since $d(\vartheta/h) = (m-2)dz_1 \wedge \ldots \wedge dz_m/h$. Since $\mathcal{O}_{D,x}$ is a domain, then all partial derivatives $\partial h/\partial z_\ell$, $\ell = 1, \ldots, m$, are non-zero divisors. Therefore one can take the multiplier in representation (9) equal to any z_ℓ. Explicit calculations show that for $g = z_\ell$ one has

$$\begin{aligned} \xi &= \tfrac{1}{2} \sum_{j=1,\, j\neq \ell}^{m} (-1)^{\ell+j} \operatorname{sgn}(j-\ell)\, z_j\, dz_1 \wedge \ldots \wedge \widehat{dz_\ell} \wedge \ldots \wedge \widehat{dz_j} \wedge \ldots \wedge dz_m, \\ \eta &= (-1)^{\ell-1} dz_1 \wedge \ldots \wedge \widehat{dz_\ell} \wedge \ldots \wedge dz_m. \end{aligned}$$

It is clear that z_ℓ does not divide ξ; hence, the differential $(m-2)$-form $\operatorname{res}_D(\vartheta/h) = \frac{\xi}{g}\big|_D$ is *not holomorphic* on D. Let us describe conditions under which a logarithmic form $\omega = \eta_1 + \frac{dh}{h} \wedge \xi_1 + \varphi\frac{\vartheta}{h}$ with holomorphic η_1, ξ_1 and φ, has a holomorphic residue on D. Without loss of generality one can suppose that the above differential forms and functions are homogeneous at the distinguished point x. If φ is *invertible* at x, then $\operatorname{res}_D(\vartheta/h) = -\frac{\xi_1}{\varphi}\big|_D$ is holomorphic at x; this contradicts to the above computations. Moreover, in such a case ω is *not closed.* Otherwise, if $d\omega = 0$, then there is an identity

$$d\eta_1 - \frac{dh}{h} \wedge d\xi_1 + (m-2)\varphi \frac{dz_1 \wedge \ldots \wedge dz_m}{h} = 0,$$

or, equivalently,

$$h\eta_1 - dh \wedge d\xi_1 + (m-2)\varphi dz_1 \wedge \ldots \wedge dz_m = 0.$$

However, it is impossible, since h and dh vanish at x, while φ is invertible. Finally, let suppose that φ is *not invertible,* that is, φ is contained in the maximal ideal of $\mathcal{O}_{D,x}$. In this case $\varphi\frac{\vartheta}{h}$ is contained in $\Omega_{S,x}^{m-1} + \frac{dh}{h} \wedge \Omega_{S,x}^{m-2}$ in view of the above calculations. As a result, ω has a holomorphic residue on D. In particular, we also obtain that all closed logarithmic $(m-1)$-forms are contained in $\Omega^{m-1}_{S,x} + \frac{dh}{h} \wedge \Omega^{m-2}_{S,x}$, and, obviously, their residues are holomorphic on D. QED.

Remark 6. It is useful to examine also the case $m = 2$ separately. Thus, $h = z^2 + w^2$, that is, D is a node; it is a divisor with normal crossing in a plane. The module $\Omega^1_S(\log D)$ is generated by differential forms dh/h and ϑ/h, where $\vartheta = -wdz + zdw$. It is not difficult to verify that $d(\vartheta/h) = 0$ in contrast with the case $m \geq 3$ considered in the above Theorem. Furthermore, $\mathrm{res}_D(\vartheta/h) = -\frac{w}{z}\big|_D = \frac{z}{w}\big|_D$ is *not holomorphic* on D. Simple considerations show that this residue is, in fact, a *weakly holomorphic* function on D, that is, it is holomorphic only on the *normalization* $\widetilde{D}$ of D. Really, changing coordinate system, one gets $h = zw$, and $\Omega^1_S(\log D)$ is generated by two *closed* differential forms dz/z and dw/w whose residues are holomorphic on $\widetilde{D}$, but not on D (see [18], (2.11)). In fact, this phenomenon occurs not only for divisors with normal crossings (see [loc.cite, Th. (2.9)]). Curiously that in the original formulation of the theorem as well as in its later citations the restriction $m \geq 3$ is omitted (cf. [14], pp. 84, 103, or [9], § 5).

More generally, in a similar style one can describe the image of the multiple residue map for divisors with normal crossings. In this case this map coincides with the multidimensional Poincaré residue considered in [7], (3.1.5). To be more precise, residues of logarithmic p-forms along the union of any collection $D_{i_1} \cup \ldots \cup D_{i_\ell}$ of irreducible components of D consist of restrictions to the intersection $D_{i_1} \cap \ldots \cap D_{i_\ell}$ of differential $(p-\ell)$-forms holomorphic on the ambient space. Hence they are regular holomorphic on the intersection as well as on its normalization since the map of direct image π_* is an isomorphism in view of property 5) of Section 5.

The next example is a simple modification of the above. By definition, $\mathcal{O}_S$-modules of logarithmic differential p-forms of *principal type* $\Omega^p_S\langle D\rangle$, $p \geq 0$, are defined as follows:

$$\Omega^0_S\langle D\rangle = \mathcal{O}_S, \quad \Omega^1_S\langle D\rangle = \sum_{i=1}^{k} \mathcal{O}_S \frac{dh_i}{h_i} + \Omega^1_S, \quad \Omega^p_S\langle D\rangle = \bigwedge^p \Omega^1_S\langle D\rangle,\ p \geq 2.$$

One can easily verify that the family $\Omega^p_S\langle D\rangle$, $p \geq 0$, forms a subcomplex of the logarithmic de Rham complex $\Omega^\bullet_S(\log D)$ closed under the external differentiation and external product by dh_i/h_i, $1 \leq i \leq k$. Clearly, for divisors with normal crossings the equality $\Omega^p_S\langle D\rangle = \Omega^p_S(\log D)$ holds for all $p \geq 0$. Further, any logarithmic form of principal type has decomposition (4) of Theorem 1 with an *invertible* multiplier g. Similarly to the case of divisors with normal crossings, multiple residues of such forms are *holomorphic* on the corresponding complete intersection.

More generally, if D is a divisor such that there is an isomorphism $\Omega^p_S\langle D\rangle \cong \Omega^p_S(\log D)$ for certain $p \geq k$, then the image of the multiple residue map can be characterized as above. A special class of such divisors considered in cohomology theory of the "twisted" de Rham complex can be described as follows (another examples are also studied in [15]).

Let h_j, $j = 1, \ldots, \ell$, be non-constant homogeneous polynomials on S. Denote the ideal generated by all minors $\Delta_{i_1 \ldots i_r}$ of maximal order of Jacobian matrix $\mathrm{Jac}(h_{i_1}, \ldots, h_{i_r})$ and polynomials $h_{i_1}, \ldots, h_{i_r}$ by $\mathcal{G}_{i_1 \ldots i_r} \subset \mathcal{O}_{S,0}$.

Claim 4. *Assume that for any* $1 \leq r \leq \min\{\ell, m-1\}$, *the algebraic set defined by the ideal* $\mathcal{G}_{i_1 \ldots i_r}$ *is either empty or the origin, and* $h_{i_1}, \ldots, h_{i_s}$ *is a regular sequence for* $1 \leq s \leq \min\{\ell, m\}$. *Then any logarithmic differential p-form,* $0 \leq p \leq m-2$, *has decomposition* (4) *of Theorem* 1 *with an invertible multiplier g, and there are isomorphisms* $\Omega^p_S\langle D\rangle \cong \Omega^p_S(\log D)$.

Proof. It is a slightly modified version of considerations from [12] or [15]. QED.

8. The weight filtration

The concept of weight filtration on the logarithmic de Rham complex for divisors with normal crossings on manifolds was introduced by P.Deligne [7] for computation of the mixed Hodge structure on the cohomologies of the complement. The case of divisors with normal crossings on V-varieties was examined by J.Steenbrink [22]. In this section we construct the weight filtration on the logarithmic de Rham complex for divisors whose components are defined by a regular sequence of functions. In particular, this allows to compute the mixed Hodge structure on cohomology of the complement of divisors of certain types without using theorems on resolution of singularities or the standard reduction to the case of normal crossings.

Let X be an analytical manifold, $D \subset X$ be a reduced divisor with *irreducible* decomposition $D = D_1 \cup \ldots \cup D_k$. It is also assumed that D has no components with self-intersections. For any ordered collection $I = (i_1 \cdots i_n)$, $1 \leq i_1 < \ldots < i_n \leq k$, of length $n = \#(I)$, let us consider the following germs:

$$D_I = D_{(i_1 \cdots i_n)} = D_{i_1} \cup \ldots \cup D_{i_n}, \quad C^I = C^{(i_1 \cdots i_n)} = D_{i_1} \cap \ldots \cap D_{i_n}.$$

We denote by $C^{(n)}$ an analytical subspace of X given by the union of $C^{(i_1 \cdots i_n)}$ for all permissible collections so that $C^{(1)} = D$, $C^{(k)} = C^{(i_1 \cdots i_k)} = C$, and so on. Let us also set $D_0 = C^0 = \emptyset$.

Definition 3. The *weight filtration*, or *filtration of weights* W on the logarithmic de Rham complex $\Omega^p_X(\log D)$ is locally defined as follows:

$$W_n(\Omega^p_{X,x}(\log D)) = \begin{cases} 0, & n < 0; \\ \Omega^p_{X,x}, & n = 0; \\ \sum\limits_{\#(I)=p} \Omega^p_{X,x}(\log D_I), & n \geq p,\ 0 < p < k_x, \\ \sum\limits_{\#(I)=n} \Omega^p_{X,x}(\log D_I), & \text{otherwise}, \end{cases}$$

where k_x is the number of irreducible components of D passing through the point $x \in X$.

First non-trivial elements of the weight filtration in the case $k = 3$.

$$\begin{array}{ccccc}
W_0 & \Omega^1_X & \Omega^2_X & \Omega^3_X & \Omega^4_X \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
W_1 & \sum \Omega^1_X(\log D_i) & \sum \Omega^2_X(\log D_i) & \sum \Omega^3_X(\log D_i) & \sum \Omega^4_X(\log D_i) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
W_2 & \sum \Omega^1_X(\log D_i) & \sum \Omega^2_X(\log(D_i \cup D_j)) & \sum \Omega^3_X(\log(D_i \cup D_j)) & \sum \Omega^4_X(\log(D_i \cup D_j)) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
W_3 & \sum \Omega^1_X(\log D_i) & \sum \Omega^2_X(\log(D_i \cup D_j)) & \sum \Omega^3_X(\log D) & \sum \Omega^4_X(\log D) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

Thus, $W_n(\Omega^p_{X,x}(\log D)) = \Omega^p_{X,x}(\log D)$, if $n \geq p \geq k_x$. Further, W is an increasing filtration and, in view of d- and $\wedge$-closeness of $\Omega^\bullet_X(\log D)$, there exist the following natural inclusions

$$\begin{aligned} d(W_n(\Omega^\bullet_X(\log D))) &\subset W_n(\Omega^\bullet_X(\log D)), \\ W_n(\Omega^p_X(\log D)) \bigwedge W_\ell(\Omega^q_X(\log D)) &\subset W_{n+\ell}(\Omega^{p+q}_X(\log D)) \end{aligned}$$

for all entire numbers p, q, n, ℓ. It should be remarked that for any $n \leq p$ the module $W_n(\Omega^p_{X,x}(\log D))$ contains all differential forms of principal type from $\Omega^n_X\langle D\rangle$ considered in Section 7 :

$$\frac{dh_{i_1}}{h_{i_1}} \wedge \cdots \wedge \frac{dh_{i_\ell}}{h_{i_\ell}} \wedge \Omega^{p-\ell}_{X,x}, \quad 1 \leq i_1 < \ldots < i_\ell \leq k,\ 1 \leq \ell \leq n,$$

where $h_{i_1}, \ldots, h_{i_\ell}$ are local equations of the corresponding components of divisor D passing through the point $x \in X$. In general,

$$\Omega^n_{X,x}\langle D\rangle \wedge \Omega^{p-n}_{X,x} \subseteq W_n(\Omega^p_{X,x}(\log D)) \subseteq \Omega^n_{X,x}(\log D) \wedge \Omega^{p-n}_{X,x},\ n \in \mathbb{Z},\ p \geq n.$$

For divisors with normal crossings two complexes $\Omega^\bullet_X\langle D\rangle$ and $\Omega^\bullet_X(\log D)$ are equal. Therefore, both inclusions in the latter formula are, in fact, equalities and the weight filtration on the complex $\Omega^\bullet_X\langle D\rangle$ is given as follows:

$$W_n(\Omega^p_X\langle D\rangle) = \Omega^n_X\langle D\rangle \wedge \Omega^{p-n}_X,\ n \in \mathbb{Z}.$$

The following assertion can be considered as a generalization of isomorphism (3.1.5.2) from [7] valid for divisors with normal crossings to the case of divisors whose components are given by a regular sequence of functions.

Let $\pi\colon \widetilde{C}^{(n)} \to C^{(n)}$ be a morphism of normalization so that $\widetilde{C}^{(n)}$ coincides with the non-connected sum of normalizations $\widetilde{C}^{(i_1\cdots i_n)}$ for all possible collections of length $n \geq 1$. We denote by ι the projection $\widetilde{C}^{(n)}$ in X, so that $\iota = i \circ \pi$, where $i\colon C^{(n)} \to X$ is a natural inclusion.

Proposition 3. *Let us assume that a divisor D satisfies assumptions of Theorem* 1 *and the morphism of normalization induces an isomorphism of complexes $\pi_*\colon \omega^\bullet_{\widetilde{C}^{(n)}} \cong \omega^\bullet_{C^{(n)}}$. Then the multiple residue map*

$$\mathrm{Res}^\bullet_n\colon W_n(\Omega^\bullet_X(\log D)) \longrightarrow \iota_*\omega^\bullet_{\widetilde{C}^{(n)}}[-n],$$

induces an isomorphism of complexes of $\mathcal{O}_X$-modules

$$\mathrm{Gr}^W_n(\Omega^\bullet_X(\log D)) \cong \iota_*\omega^\bullet_{\widetilde{C}^{(n)}}[-n].$$

Proof. Let firstly note that the morphism of normalization induces the isomorphism of direct image π_* if condition 5) from Section 5 is fulfilled. Furthermore, it suffices to prove our assertion locally, for the germ (X,x) and for all $n \leq p$.

For any ordered collection $I = (i_1 \cdots i_n)$, $1 \leq i_1 < \ldots < i_n \leq k_x$, accordingly Theorem 2 with $D = D_I$ there exists an exact sequence of complexes of $\mathcal{O}_{X,x}$-modules

$$0 \longrightarrow \sum_{\ell=1}^{n} \Omega^\bullet_{X,x}(\log(\widehat{D_I})_{i_\ell}) \longrightarrow \Omega^\bullet_{X,x}(\log D_I) \stackrel{\mathrm{Res}_{C^I}}{\longrightarrow} \omega^\bullet_{C^I,x}[-n] \longrightarrow 0.$$

From basic properties of regular meromorphic differential forms it follows that $\omega^\bullet_{\widetilde{C}^{(n)}}$ is isomorphic to the direct sum $\omega^\bullet_{\widetilde{C}^I}$ taking through all permissible collections $I = (i_1 \cdots i_n)$. Further, any differential form $\omega \in W_n(\Omega^\bullet_X(\log D))$ is decomposed into the sum of elements $\omega_I \in \Omega^\bullet_X(\log D_I)$. Let us denote the sum of $\mathrm{Res}_{C^I}(\omega_I)$ by $\mathrm{Res}_{C^{(n)}}(\omega)$. One then obtains an exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow W_{n-1}(\Omega^\bullet_X(\log D)) \longrightarrow W_n(\Omega^\bullet_X(\log D)) \stackrel{\mathrm{Res}_{C^{(n)}}}{\longrightarrow} \iota_*\omega^\bullet_{\widetilde{C}^{(n)}}[-n] \longrightarrow 0.$$

This yields the existence of required isomorphisms. QED.

Corollary 5. *Under conditions of Proposition* 3 *there are natural isomorphisms of cohomology spaces*

$$\mathcal{H}^i(\mathrm{Gr}^W_n(\Omega^\bullet_X(\log D))) \cong \mathcal{H}^i(\omega^\bullet_{\widetilde{C}^{(n)}})[-n],$$

where $i \geq 1$ and $1 \leq n \leq k$.

Proof. Since the normalization π is finite and, therefore it is an *affine* morphism then

$$\mathcal{H}^i(\pi_*\omega^\bullet_{\widetilde{C}^{(n)}}) \cong \mathcal{H}^i(\omega^\bullet_{\widetilde{C}^{(n)}}),\ i \geq 1,$$

and the desired assertion follows from Proposition above. QED.

Remark 7. Analyzing a more general situation where complexes $\omega^\bullet_{\widetilde{C}^{(n)}}$ and $\omega^\bullet_{C^{(n)}}$ are non-isomorphic, the corresponding isomorphisms in the formulation of Proposition 3 should be replaced by epimorphisms.

Remark 8. Suppose that a (finite) group G acts on a manifold X. Then it is not difficult to verify that the residue mapping Res_C is *compatible* with the action of this group in the usual sense. In this case the complex of regular meromorphic forms $\omega^\bullet_{X/G}$ on the quotient variety X/G is a *resolution* of constant sheaf [11]. Making use of simplest properties of sheaves $\Omega^p_X(\log D)$, Ω^q_C and the corresponding subsheaves invariant relative to action of G, one obtains the isomorphism of Lemma (1.19) from [22] for divisors with normal crossings on a V-variety.

Let us examine a simple application. The canonical decreasing Hodge filtration F on the logarithmic de Rham complex $\Omega^p_X(\log D)$ is defined as follows:

$$F^n(\Omega^p_X(\log D)) = \begin{cases} \Omega^p_X(\log D), & n \leq p, \\ 0, & n > p. \end{cases}$$

Suppose now that D is a reduced divisor as before and the natural inclusions

$$\sum_{\#(I)=p} \Omega^p_{X,x}(\log D_I) \longrightarrow \Omega^p_{X,x}(\log D) \tag{12}$$

are isomorphisms for all $1 \le p < k_x$. Then $W_n(\Omega^p_X(\log D)) \cong \Omega^p_X(\log D)$ for all $n \ge p$ similarly to the classical case of divisors with normal crossings (see (3.1.8) in [7]). Hence, under assumptions of Proposition 3 one can define a natural morphism α from the complex $\Omega^\bullet_X(\log D)$ endowed by Hodge filtration F into the same complex with *decreasing* filtration W given as $W^n = W_{-n}$.

Corollary 6. *Under the same assumptions the above morphism α is a filtered quasi-isomorphism if $\mathcal{H}^i(\omega^\bullet_{\widetilde{C}^{(n)}}) = 0$ for $i \neq 0$.*

Proof. Analogously to the proof of (3.1.8.2) in [7]. QED.

Remark 9. Of course, for divisors with normal crossings inclusions (12) are isomorphisms for all $p \ge 1$. A special class of divisors with $\sum_{i=1}^k \Omega^1_{X,x}(\log D_i) \cong \Omega^1_{X,x}(\log D)$ is considered in Theorem 2.9 by [20] (see Section 2.)

Remark 10. The vanishing condition of Corollary 6 means that the complex of regular meromorphic forms on the normalization $\widetilde{C}^{(n)}$ is *acyclic* in positive dimensions. Besides the case of divisors with normal crossings examined in [7], another types of varieties satisfied this condition are known. Among them there are rational normal complete intersections, quotient singularities of smooth varieties with action of a finite group (see [11]), V-varieties (see [22]), and so on.

Remark 11. If two complexes $\Omega^\bullet_X(\log D)$ and $\Omega^\bullet_X(\star D)$ endowed with standard Hodge filtration are quasi-isomorphic (see, for example, [10]), then the morphism β from Proposition (3.1.8) of [7] is a quasi-isomorphism. If additionally the condition of the previous Corollary 6 is satisfied, then α is also a quasi-isomorphism. This means that in all cases mentioned in Remark above there are isomorphisms (3.1.8.2) of [7]:

$$R^n j_* \mathbf{C} \cong \mathcal{H}^n(j_*\Omega^*_{X^*}) \cong \mathcal{H}^n(\Omega^\bullet_X(\log D)),$$

where X is a manifold, $X^* = X \setminus D$, $j\colon X^* \to X$ is the canonical inclusion.

Further analysis shows that under standard assumptions on the ambient manifold X (smooth, Kähler, complete) the bifiltered complex $(\Omega^\bullet_X(\log D), F, W)$ can be used (similarly to [22], p.532) for computation of the canonical mixed Hodge structure on the cohomology of complements $H^*(X \setminus D, \mathbb{C})$ as well as on the local cohomology $H^*_C(X, \Omega^\bullet_X(\log D))$ without the using of resolution theorems or a standard reduction to the case of divisors with normal crossings.

In conclusion we note that the differential d is *strictly compatible* ([7], (1.1.5)) with filtration W at degree $k+1$, that is,

$$d\Omega^k_X(\log D) \cap W_n(\Omega^{k+1}_X(\log D)) = d(W_n(\Omega^k_X(\log D)), \ n \in \mathbb{Z}.$$

Consequently, the weight filtration on the *canonically truncated* logarithmic de Rham complex

$$\tau_{\ge k}\Omega^\bullet_X(\log D)$$

is also well-defined; in its turn, it induces the weight filtration on the complex of regular meromorphic differential forms on a complete intersection with the help of the multiple residue map.

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CHERN CLASSES OF FREE HYPERSURFACE ARRANGEMENTS

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Abstract. The Chern class of the sheaf of logarithmic derivations along a simple normal crossing divisor equals the Chern-Schwartz-MacPherson class of the complement of the divisor. We extend this equality to more general divisors, which are locally analytically isomorphic to free hyperplane arrangements.

1. Introduction

For us, an *arrangement* in a nonsingular variety V is a reduced divisor D consisting of a union of nonsingular hypersurfaces, such that at each point D is locally analytically isomorphic to a hyperplane arrangement. We say that the arrangement is *free* if all these local models may be chosen to be free hyperplane arrangements. It follows that D is itself a *free divisor* on V: the sheaf of logarithmic differentials $\Omega^1_V(\log D)$ along D is locally free. Equivalently, its dual sheaf of *logarithmic derivations,* $\mathrm{Der}_V(-\log D) := \Omega^1_V(\log D)^\vee$, is locally free. Free hyperplane arrangements in $\mathbb{P}^n$ and divisors with simple normal crossings in a nonsingular variety give examples of free hypersurface arrangements.

In this note we extend to free hypersurface arrangements a result that is known to hold for these examples.

Theorem 1.1. *Let V be a nonsingular complex variety, and let $D \subseteq V$ be a free hypersurface arrangement. Then*

$$c(\mathrm{Der}_V(-\log D)) \cap [V] = c_{SM}(1\!\!1_{V \smallsetminus D}) \quad .$$

Here, $c_{\mathrm{SM}}(1\!\!1_{V\smallsetminus D})$ is the *Chern-Schwartz-MacPherson* class of the constructible function $1\!\!1_{V\smallsetminus D}$, in the sense of [13], see also [9], Example 19.1.7.

For simple normal crossing divisors, the equality of Theorem 1.1 was verified in [10] (Proposition 15.3) and [2]. For free projective hyperplane arrangements, it is Theorem 4.1 in [5], where it is obtained as a simple corollary of a result of Mustaţă and Schenck ([14]). Theorem 1.1 will be obtained here by considering the blow-ups giving an embedded resolution of D. Each blow-up will be analyzed by using MacPherson's *graph construction,* showing (Claim 2.4) that the Chern class of the corresponding sheaf of logarithmic derivations is preserved by push-forward. The theorem will then follow from the corresponding behavior of the Chern-Schwartz-MacPherson class and from the case of normal crossing divisors.

In particular, this will give an independent proof (and a substantial generalization) of the case of free hyperplane arrangements treated in [5].

The term 'hypersurface arrangement' is often used in the literature to simply mean a union of hypersurfaces (nonsingular or otherwise). This is a substantially more general notion than the one used in this note. The statement of Theorem 1.1 is *not* true in this generality, even for free divisors. For example, if V is a surface (so that every reduced divisor is free in V), a condition of local homogeneity is necessary for this result to hold, as observed by Xia Liao (cf. [12]).

The paper is organized as follows: in §2 we recall the basic definitions and reduce the main theorem to showing that Chern classes of sheaves of logarithmic derivations are preserved through certain types of blow-ups. This is proven in §3, using the graph construction. In §4 we offer a simple example, and show that the theorem is equivalent to a projection formula for Chern classes of certain coherent sheaves.

A word on the hypotheses: the freeness of the divisor is used crucially in the application of the graph construction; its local analytic structure is less essential, but convenient in some coordinate arguments. It is conceivable that the proof given here may be generalized to divisors satisfying a less restrictive local homogeneity requirement.

The result in this note generalizes Theorem 4.1 in [5]. I presented the results of [5] in my talk at the Hefei conference on Singularity Theory, and I take this opportunity to thank Xiuxiong Chen and Laurentiu Maxim for the invitation to speak at the conference and for organizing a very successful and thoroughly enjoyable meeting. I also thank the referee for valuable suggestions.

2. Set-up

2.1. We work over an algebraically closed field of characteristic 0; the reader is welcome to assume the ground field is $\mathbb{C}$. (Characteristic 0 is required in the theory of Chern-Schwartz-MacPherson classes. See [11] or [3] for a discussion of the theory over algebraically closed fields of characteristic 0.)

Chern-Schwartz-MacPherson classes are classes in the Chow group of a variety V defined for constructible functions on V, and are characterized by the normalization requirement that $c_{\mathrm{SM}}(1\!\!1_V) \in A_*V$ equals $c(TV) \cap [V]$ if V is nonsingular and the covariance property

$$\alpha_* c_{\mathrm{SM}}(\varphi) = c_{\mathrm{SM}}(\alpha_*\varphi)$$

for all proper morphisms $\alpha : V \to V'$ and all constructible functions φ on V. Over $\mathbb{C}$, the push-forward of a constructible function is defined by taking weighted Euler characteristics of fibers: for a subvariety $Z \subseteq V$, $\alpha_*(1\!\!1_Z)(p) = \chi(Z \cap \alpha^{-1}(p))$. Thus, c_{SM} determines a natural transformation from the functor of constructible functions to the Chow functor. The existence of this natural transformation was conjectured by Deligne and Grothendieck, and proved by MacPherson ([13]). Interest in these classes has resurged in the past few years; comparison with other classes for singular varieties gives an intersection theoretic invariant of singularities generalizing directly the Milnor number. A recent survey may be found in [16].

The interested reader may consult Example 19.1.7 in [9] for an efficient summary of MacPherson's definition; an alternative construction is presented in [3]. In any case, the details of the definition of these classes are not needed for this paper: only the key covariance property recalled above will be used. Note that if V_1 and V_2 are constructible subsets of V, then

$$c_{\mathrm{SM}}(1\!\!1_{V_1\cup V_2}) = c_{\mathrm{SM}}(1\!\!1_{V_1} + 1\!\!1_{V_2} - 1\!\!1_{V_1\cap V_2}) = c_{\mathrm{SM}}(1\!\!1_{V_1}) + c_{\mathrm{SM}}(1\!\!1_{V_2}) - c_{\mathrm{SM}}(1\!\!1_{V_1\cap V_2}) \quad :$$

the Chern-Schwartz-MacPherson classes satisfy 'inclusion-exclusion'; for example, they are additive on disjoint unions. Also, if V is complete, so that the constant map $\kappa : V \to \mathrm{pt}$ is proper, then by covariance

$$\kappa_* c_{\mathrm{SM}}(1\!\!1_U) = c_{\mathrm{SM}}(\kappa_* 1\!\!1_U) = c_{\mathrm{SM}}(\chi(U) 1\!\!1_{\mathrm{pt}}) = \chi(U)[\mathrm{pt}]$$

for any constructible U in V. This says that the degree of $c_{\mathrm{SM}}(1\!\!1_U)$ equals the Euler characteristic $\chi(U)$, generalizing the Poincaré-Hopf theorem to singular and/or noncomplete varieties. (This was one of the motivations for the original definition of these classes by M.-H. Schwartz, cf. [17, 18].)

2.2. The covariance property of Chern-Schwartz-MacPherson classes has the following immediate consequence. Let V be a variety, and let $X \subseteq V$ be a subscheme. Let $\rho : \widetilde{V} \to V$ be a proper map, and let $X' \subseteq \widetilde{V}$ be any subscheme such that ρ restricts to an isomorphism $\widetilde{V} \smallsetminus X' \to V \smallsetminus X$. Then

$$\rho_* c_{\mathrm{SM}}(1\!\!1_{\widetilde{V}\smallsetminus X'}) = c_{\mathrm{SM}}(1\!\!1_{V\smallsetminus X}) \quad .$$

Indeed, $\rho_*(1\!\!1_{\widetilde{V}\smallsetminus X'}) = 1\!\!1_{V\smallsetminus X}$.

In particular:

Lemma 2.1. *Let V be a nonsingular variety, and let $D \subseteq V$ be a subscheme. Let $\rho : \widetilde{V} \to V$ be a proper morphism such that $\widetilde{V}$ is nonsingular, and the support D' of $\rho^{-1}(D)$ is a divisor with normal crossings and nonsingular components. Then*

$$c_{SM}(1\!\!1_{V\smallsetminus D}) = \rho_*(c(\mathrm{Der}_{\widetilde{V}}(-\log D')) \cap [\widetilde{V}]) \quad .$$

Proof. As recalled in §1, since D' is a simple normal crossing divisor in $\widetilde{V}$, then

$$c_{\rm SM}(1\!\!1_{\widetilde{V}\smallsetminus D'}) = c(\Omega^1_{\widetilde{V}}(\log D')^\vee)\cap[\widetilde{V}] = c(\mathrm{Der}_{\widetilde{V}}(-\log D'))\cap[\widetilde{V}] \quad .$$

This is proved in e.g., [2], Theorem 1; we quickly recall the argument, for the convenience of the reader. Let D'_i, $i=1,\dots,N$ be the components of D'. Since D' is a divisor with normal crossings, $c(\Omega^1_{\widetilde{V}}(\log D')^\vee)$ equals $c(T\widetilde{V})/\prod_i(1+D'_i)$ (as is well-known, and easily verified). Now the stated equality is clear if $N=0$. For $N>0$:

$$\frac{c(T\widetilde{V})}{\prod_i(1+D'_i)} = \frac{c(T\widetilde{V})}{\prod_{i<N}(1+D'_i)}\left(1-\frac{D'_N}{1+D'_N}\right) = \frac{c(T\widetilde{V})}{\prod_{i<N}(1+D'_i)} - \frac{c(TD'_N)\cdot D'_N}{\prod_{i<N}(1+D'_i)}$$

and therefore

$$\frac{c(T\widetilde{V})}{\prod_i(1+D'_i)}\cap[\widetilde{V}] = \frac{c(T\widetilde{V})}{\prod_{i<N}(1+D'_i)}\cap[\widetilde{V}] - \frac{c(TD'_N)}{\prod_{i<N}(1+D'_i)}\cap[D'_N] \quad .$$

Arguing by induction on N, the first summand equals the $c_{\rm SM}$ class of the complement of the union of the first $N-1$ components, and the second equals the $c_{\rm SM}$ class of the trace of this complement on the N-th component. The equality follows then by the additivity of Chern-Schwartz-MacPherson classes on disjoint unions.

The equality implies the formula stated in the lemma, by covariance:

$$c_{\rm SM}(1\!\!1_{V\smallsetminus D}) = \rho_* c_{\rm SM}(1\!\!1_{\widetilde{V}\smallsetminus D'}) = \rho_*(c(\mathrm{Der}_{\widetilde{V}}(-\log D'))\cap[\widetilde{V}]).$$

□

2.3. Now let V be a nonsingular variety, and let D be a hypersurface arrangement, as in §1. In particular: at every $p\in D$, there is a choice of analytic coordinates $x_1,\dots,x_n$ such that the ideal of D in the completion $k[[x_1,\dots,x_n]]$ is generated by a product of linear polynomials $\sum\lambda_i x_i$, defining a central hyperplane arrangement $\mathscr{A}_p$.

Lemma 2.2. *The divisor D is free on V if and only if each $\mathscr{A}_p$ is a free central hyperplane arrangement.*

Proof. Recall that a divisor in a nonsingular variety V is free if and only if its singularity subscheme is empty or Cohen-Macaulay of codimension 2 in V at each $p\in D$. It follows that a central hyperplane arrangement is free if and only if its singularity subscheme is empty or Cohen-Macaulay of codimension 2 at the origin. (Cf. [20], Proposition 2.4.)

The statement then follows from the fact that a local ring is Cohen-Macaulay if and only if its completion is ([7], Corollary 2.1.8). □

Under the hypotheses of Theorem 1.1, $\mathrm{Der}_V(-\log D)$ is locally free. With $\rho:\widetilde{V}\to V$ as in the statement of Lemma 2.1, $\mathrm{Der}_{\widetilde{V}}(-\log D')$ is also locally free, as D' is a divisor with simple normal crossings. Lemma 2.1 reduces Theorem 1.1 to proving that if D is a free hypersurface arrangement in V, and $\rho:\widetilde{V}\to V$ is as in the statement of Lemma 2.1, then

$$(*)\qquad \rho_*(c(\mathrm{Der}_{\widetilde{V}}(-\log D'))\cap[\widetilde{V}]) = c(\mathrm{Der}_V(-\log D))\cap[V] \quad .$$

2.4. Next, we observe that an embedded resolution ρ of a hypersurface arrangement D may be obtained by blowing up along the intersections of the components of the arrangement, in order of increasing dimension, and that these intersections are all nonsingular. In order to verify (*), it suffices to verify that the stated equality holds for each of these blow-ups. More precisely: Given a hypersurface arrangement D in a nonsingular variety V, let Z be a component of lowest dimension among the intersections of components of D; let $\pi:\hat{V}\to V$ be the blow-up of V along Z; let E be the exceptional divisor of this blow-up; and let D' be the divisor in $\hat{V}$ consisting of E and the proper transforms of the components of D.

Lemma 2.3. *With notation as above, if D is a free hypersurface arrangement, then so is D'.*

Proof. We can work analytically at a point $p \in V$, so we may assume that D is given by a product of linear forms cutting out the center Z at p. We may in fact assume that there are analytic coordinates $x_1, \dots, x_n$ at p so that Z is given by $x_1 = \cdots = x_r = 0$, and the generator of the ideal of D is a homogeneous polynomial $F(x_1, \dots, x_r) = \prod L_i(\underline{x})$, with L_i linear.

Let $q \in \hat{V}$ be a point over p. We may choose analytic coordinates $(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)$ at q so that $\hat{x}_1 = 0$ is the exceptional divisor, and the blow-up map is given by

$$(x_1, \dots, x_n) = (\hat{x}_1, \hat{x}_1\hat{x}_2, \dots, \hat{x}_1\hat{x}_r, \hat{x}_{r+1}, \dots, \hat{x}_n) \quad .$$

The ideal for D' at q is then generated by

$$\hat{x}_1\, F(1, \hat{x}_2, \dots, \hat{x}_r) \quad ;$$

omitting the factors in $F(1, \hat{x}_2, \dots, \hat{x}_r)$ that do not vanish at q, we write the generator for D' at q as

$$\hat{x}_1\, Q(\hat{x}_2, \dots, \hat{x}_r) \quad ,$$

where Q is a product of linear forms. In particular, D' is a hypersurface arrangement in $\hat{V}$. We have to verify that it is free.

Note that the divisor defined by $Q(\hat{x}_2, \dots, \hat{x}_r)$ is free at q: indeed, the hyperplane arrangement defined by $F(x_1, \dots, x_n)$ is free by assumption, and $Q(x_2, \dots, x_r)$ generates the ideal of this arrangement at points $(t, 0, \dots, 0)$ with $t \neq 0$. By Saito's criterion ([15], Theorem 4.19), $Q(\hat{x}_2, \dots, \hat{x}_r)$ is the determinant of a set of $n-1$ logarithmic derivations $\theta_2, \dots, \theta_n$ at q. Since $\theta_2(\hat{x}_1) = \cdots = \theta_n(\hat{x}_1) = 0$, these derivations are logarithmic with respect to $\hat{x}_1\, Q(\hat{x}_2, \dots, \hat{x}_r)$.

On the other hand the Euler derivation $\theta_1 = \hat{x}_1\, \partial/\partial\hat{x}_1 + \hat{x}_2\, \partial/\partial\hat{x}_2 + \cdots + \hat{x}_n\, \partial/\partial\hat{x}_n$ is logarithmic with respect to $\hat{x}_1 Q$ as this is homogeneous (cf. [15], Definition 4.7), and $\det(\theta_1, \dots, \theta_n)$ is a unit multiple of $\hat{x}_1\, Q(\hat{x}_2, \dots, \hat{x}_r)$. This shows that D' is free, again by Saito's criterion. □

2.5. By Lemma 2.3, $\mathrm{Der}_{\hat{V}}(-\log D')$ is locally free if $\mathrm{Der}_V(-\log D)$ is, and we may consider its ordinary Chern classes. We have reduced the proof of Theorem 1.1 to the following statement.

Claim 2.4. *Let D be a free hypersurface arrangement on a nonsingular variety V; let $\pi : \hat{V} \to V$ be the blow-up of V along a component of lowest dimension of the intersection of components of D, and let $D' = (\pi^{-1}(D))_{red}$, as above. Then*

$$\pi_*(c(\mathrm{Der}_{\hat{V}}(-\log D')) \cap [\hat{V}]) = c(\mathrm{Der}_V(-\log D)) \cap [V] \quad .$$

The next section is devoted to the proof of this claim, and this will complete the proof of Theorem 1.1.

3. Proof of Theorem 1.1

3.1. We will prove Claim 2.4 as an application of MacPherson's *graph construction.* Given a homomorphism $\sigma : \mathscr{E} \to \mathscr{F}$ of vector bundles on a variety Y, consider the graph of $\lambda\sigma$ for $\lambda \in k$, as a subbundle of $\mathscr{E} \oplus \mathscr{F}$. For all λ, this defines an embedding of Y in the Grassmannian $G = \mathrm{Grass}_{\mathrm{rk}\,\mathscr{E}}(\mathscr{E} \oplus \mathscr{F})$, such that the pull-back of the universal subbundle ζ of G is isomorphic to $\mathscr{E}$. The graph construction describes the limit 'as $\lambda \to \infty$' of this embedding as a cycle in G, using which one may compare the Chern classes of $\mathscr{E}$ and $\mathscr{F}$. We refer the reader to Example 18.1.6 in [9] for the details and key properties of this useful construction. We will use the fact that if σ restricts to an isomorphism on a subbundle $\mathscr{K}$ of $\mathscr{E}$ over a subvariety E of Y, then $\mathscr{K}$ embeds as a subbundle of $\zeta|_E$ over the cycle at infinity; and an analogous dual statement concerning epimorphisms. These facts are straightforward consequences of the construction.

As in §2, we denote by Z the center of the blow-up, E the exceptional divisors; and the natural morphisms as in this diagram:

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "p"'] & \hat{V} \arrow[d, "\pi"] \\ Z \arrow[r, "\iota"'] & V \end{tikzcd}$$

By assumption Z is a nonsingular subvariety of V; we let r be its codimension. In a neighborhood of Z, Z is the transversal intersection of r components of D: indeed, if $D_1, \dots, D_r$ cut out Z at a point, then Z is contained in a connected component of $D_1 \cap \cdots \cap D_r$, so it must be equal to it as $D_1 \cap \cdots \cap D_r$ is nonsingular by our hypothesis on D. The key lemma will be the following:

Lemma 3.1. *Under the hypotheses of Claim 2.4:*

- *There is a vector bundle homomorphism* $\sigma : \pi^* \operatorname{Der}_V(-\log D) \to \operatorname{Der}_{\hat{V}}(-\log D')$ *that is an isomorphism in the complement of* E.
- *The restriction of* σ *to* E *induces a morphism of complexes of vector bundles*

$$\begin{tikzcd} \mathscr{O}_E \arrow[r, hook] \arrow[d, "="'] & \pi^* \operatorname{Der}_V(-\log D)|_E \arrow[r, two heads] \arrow[d, "\sigma|_E"] & p^* \operatorname{Der}_Z \arrow[d, "="'] \\ \mathscr{O}_E \arrow[r, hook] & \operatorname{Der}_{\hat{V}}(-\log D')|_E \arrow[r, two heads] & p^* \operatorname{Der}_Z \end{tikzcd}$$

The monomorphisms and epimorphisms shown in this diagram will be defined in the course of the proof of Lemma 3.1, in §3.3; the monomorphisms will be monomorphisms of vector bundles.

Claim 2.4 follows from Lemma 3.1, as we now show. Applying the graph construction to σ yields a cycle $\sum_i a_i[W_i]$ of dimension $n = \dim \hat{V}$ in the Grassmannian

$$G = \operatorname{Grass}_n(\pi^* \operatorname{Der}_V(-\log D) \oplus \operatorname{Der}_{\hat{V}}(-\log D'))$$

over $\hat{V}$. The details of the construction of the subvarieties W_i are immaterial here; the key property of this cycle is that since σ is an isomorphism off E,

$$c(\pi^* \operatorname{Der}_V(-\log D)) \cap [\hat{V}] - c(\operatorname{Der}_{\hat{V}}(-\log D')) \cap [\hat{V}] = \sum_{W_i \to E} a_i \eta_{i*}(c(\zeta) \cap [W_i]) \quad ,$$

where $\eta_i : W_i \to \hat{V}$ are the maps induced by projection, and ζ is the rank-n universal bundle on G. (See (c) in Example 18.1.6 of [9].) Pushing forward to V, and since $\operatorname{Der}_V(-\log D)$ is assumed to be locally free,

$$c(\operatorname{Der}_V(-\log D)) \cap [V] - \pi_*(c(\operatorname{Der}_{\hat{V}}(-\log D')) \cap [\hat{V}]) = \sum_{W_i \to E} a_i \pi_* \eta_{i*}(c(\zeta) \cap [W_i]) \quad .$$

Therefore, in order to verify Claim 2.4 it suffices to prove that $\pi_* \eta_*(c(\zeta) \cap [W]) = 0$ for every component $W = W_i$ projecting into E via $\eta = \eta_i$. We let $\underline{\eta}$ be the morphism $W \to E$:

$$\begin{tikzcd} W \arrow[d, "\underline{\eta}"'] \arrow[dr, "\eta"] & \\ E \arrow[r, "j"'] \arrow[d, "p"'] & \hat{V} \arrow[d, "\pi"] \\ Z \arrow[r, "\iota"'] & V \end{tikzcd}$$

We have $\pi \circ \eta = \iota \circ p \circ \underline{\eta}$. Thus it suffices to show that

$$p_* \underline{\eta}_*(c(\zeta) \cap [W]) = 0 \quad .$$

The component W lies in $G|_E = \mathrm{Grass}_n(\pi^* \mathrm{Der}_V(-\log D)|_E \oplus \mathrm{Der}_{\hat V}(-\log D')|_E)$, and ζ restricts to the universal bundle $\zeta|_E$ on $G|_E$; $c(\zeta) \cap [W] = c(\zeta|_E) \cap [W]$ by functoriality of Chern classes. By the second part of Lemma 3.1, the restriction of $\zeta|_E$ to each component W is the middle term in a complex of vector bundles

$$\mathscr{O}_E \hookrightarrow \zeta|_E \twoheadrightarrow p^* \mathrm{Der}_Z \quad ;$$

this follows from the facts recalled at the beginning of this section. We obtain then that $c(\zeta|_E) = c(p^* \mathrm{Der}_Z)\, c(\xi)$, where $\xi = \ker(\zeta|_E \to p^* \mathrm{Der}_Z)/\mathscr{O}_E$ is the homology of this complex. By the projection formula,

$$p_* \underline{\eta}_*(c(\zeta) \cap [W]) = c(\mathrm{Der}_Z) \cap p_* \underline{\eta}_*(c(\xi) \cap [W]) \quad .$$

Since $\dim W = \dim V$ and ξ has rank $= \mathrm{codim}_Z V - 1$, the nonzero components of $c(\xi) \cap [W]$ have dimension $> \dim Z$, and therefore $p_* \underline{\eta}_*(c(\xi) \cap [W]) = 0$. It follows that $\pi_* \eta_*(c(\zeta) \cap [W]) = 0$ as needed. □

3.2. We are thus reduced to proving Lemma 3.1. Recall that Z denotes the codimension r, nonsingular center of the blow-up. We will use the following notation:

- By assumption, there exist r components $D_1, \dots, D_r$ of D such that Z is a connected component of $D_1 \cap \cdots \cap D_r$. We will denote by D^+ the union of $D_1, \dots, D_r$. Note that D^+ is a divisor with normal crossings in a neighborhood of Z.
- $\hat D$ will denote $\pi^{-1}(D)$, so that $D' = \hat D_{\mathrm{red}}$.
- Similarly, $\hat D^+$ will be $\pi^{-1}(D^+)$.

Remark 3.2. The difference between a divisor and its reduction is immaterial here (in characteristic zero). For a divisor A in a nonsingular variety V, the sections of the sheaf $\mathrm{Der}_V(-\log A)$ may be defined as those derivation which send a section F corresponding to A to a multiple of F: in other words, there is an exact sequence

$$0 \longrightarrow \mathrm{Der}_V(-\log A) \longrightarrow \mathrm{Der}_V \longrightarrow \mathscr{O}_A(A)$$

where (locally) the last map applies a given derivation to F (see e.g. [8], §2). It is straightforward to verify that if ∂ is a derivation, and F_{red} consists of the factors of F taken with multiplicity 1, then $\partial(F) \in (F)$ if and only if $\partial(F_{\mathrm{red}}) \in (F_{\mathrm{red}})$. Thus $\mathrm{Der}_V(-\log A)$ and $\mathrm{Der}_V(-\log A_{\mathrm{red}})$ coincide as subsheaves of Der_V. Therefore, we may use $\hat D$ in place of D', and we don't need to make a distinction between $\hat D^+$ and its reduction. ⌟

Remark 3.3. We recall the following useful description of $\mathrm{Der}_V(-\log D)$ (cf. [15], Proposition 4.8): if D is the union of distinct components D_i, then $\mathrm{Der}_V(-\log D) = \cap_i \mathrm{Der}_V(-\log D_i)$ within Der_V. Indeed, it suffices to prove that

$$\mathrm{Der}_V(-\log(A \cup B)) = \mathrm{Der}_V(-\log A) \cap \mathrm{Der}_V(-\log B)$$

if A and B have no components in common. This amounts to the statement that if F and G have no common factors, then $\partial(FG) \in (FG)$ if and only if $\partial(F) \in (F)$ and $\partial(G) \in (G)$ for all derivations ∂, which is immediate. ⌟

Remark 3.4. In particular, if a divisor A consists of a selection of the components of D, then $\mathrm{Der}_V(-\log D) \subseteq \mathrm{Der}_V(-\log A)$. Therefore, we have inclusions

$$\mathrm{Der}_V(-\log D) \subseteq \mathrm{Der}_V(-\log D^+) \quad , \quad \mathrm{Der}_{\hat V}(-\log \hat D) \subseteq \mathrm{Der}_{\hat V}(-\log \hat D^+) \quad .$$

Further, the monomorphism $\mathrm{Der}_V(-\log D) \hookrightarrow \mathrm{Der}_V(-\log D^+)$ of locally free sheaves remains a monomorphism after pull-back via π: the determinant of this morphism is nonzero on V, and it remains nonzero on the blow-up $\hat V$. ⌟

Lemma 3.5. *The (reduction of the) divisor $\hat D^+$ is a divisor with normal crossings in a neighborhood of E, and $\pi^* \mathrm{Der}_V(-\log D^+) \cong \mathrm{Der}_{\hat V}(-\log \hat D^+)$.*

Proof. The first assertion is a simple verification in local coordinates (cf. Lemma 2.3). The second assertion only need be verified in a neighborhood of E, so it reduces to the case of normal crossings, where it is straightforward. More details may be found in Theorem 4.1 of [6]. (Also cf. Lemma 1.3 in [4].) $\square$

We may use the isomorphism obtained in Lemma 3.5 to identify $\pi^* \operatorname{Der}_V(-\log D^+)$ and $\operatorname{Der}_{\hat V}(-\log \hat D^+)$. Via this identification, we will verify that $\pi^* \operatorname{Der}_V(-\log D)$ is contained in $\operatorname{Der}_{\hat V}(-\log \hat D)$. The corresponding monomorphism of locally free sheaves $\pi^* \operatorname{Der}_V(-\log D) \hookrightarrow \operatorname{Der}_{\hat V}(-\log \hat D) = \operatorname{Der}_{\hat V}(-\log D')$ will give the homomorphism σ whose existence is claimed in Lemma 3.1.

Note that the sought-for σ appears to go in the wrong direction. The differential of π maps $\operatorname{Der}_{\hat V}$ to $\pi^* \operatorname{Der}_V$, and restricts to a homomorphism $\operatorname{Der}_{\hat V}(-\log D^+) \to \pi^* \operatorname{Der}_V(-\log D^+)$. This is an isomorphism as observed in Lemma 3.5, and the claim here is that its *inverse* restricts to a morphism

$$\sigma : \pi^* \operatorname{Der}_V(-\log D) \longrightarrow \operatorname{Der}_{\hat V}(-\log \hat D) \quad ,$$

which will then clearly be an isomorphism off E as needed in §3.1.

Lemma 3.6. *Via the isomorphism $\pi^* \operatorname{Der}_V(-\log D^+) \cong \operatorname{Der}_{\hat V}(-\log \hat D^+)$, we have the inclusion*

$$\pi^* \operatorname{Der}_V(-\log D) \subseteq \operatorname{Der}_{\hat V}(-\log \hat D).$$

Proof. By definition of $\operatorname{Der}_V(-\log D)$ there is an exact sequence

$$(\dagger) \qquad \operatorname{Der}_V(-\log D) \longrightarrow \operatorname{Der}_V(-\log D^+) \longrightarrow \mathscr{O}_D(D)$$

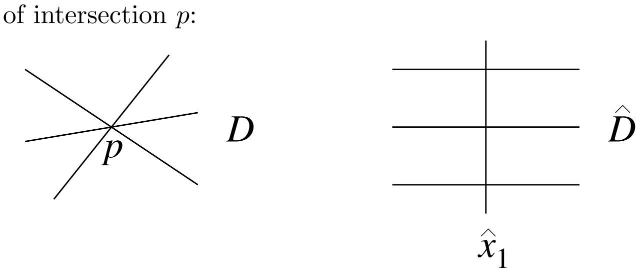
where the first map is a monomorphism, and the second applies a given logarithmic derivation to a section F defining D. Pulling back to $\hat V$ gives a complex

$$(\ddagger) \qquad \pi^* \operatorname{Der}_V(-\log D) \longrightarrow \pi^* \operatorname{Der}_V(-\log D^+) \cong \operatorname{Der}_{\hat V}(-\log \hat D^+) \longrightarrow \pi^* \mathscr{O}_D(D) \cong \mathscr{O}_{\hat D}(\hat D)$$

The first map remains a monomorphism (Remark 3.4), and maps $\pi^* \operatorname{Der}_V(-\log D)$ into the kernel of the second map, which is $\operatorname{Der}_{\hat V}(-\log \hat D)$ by definition of the latter. $\square$

This completes the proof of the first part of Lemma 3.1. Note that σ is a monomorphism of sheaves, not of vector bundles.

Example 3.7. Let $V = \mathbb{P}^2$, and let D be the divisor consisting of three distinct concurrent lines. We blow-up at the point of intersection p:



In affine coordinates centered at p, we may assume D has equation $F = x_1 x_2 (x_1 + x_2) = 0$. We choose coordinates $\hat x_1, \hat x_2$ in an affine chart in the blow-up $\hat V$ so that the blow-up map is given by

$$x_1 = \hat x_1 \quad , \quad x_2 = \hat x_1 \hat x_2 \quad ;$$

the exceptional divisor E has equation $\hat x_1 = 0$, and $\hat D$ is given by the vanishing of $\hat F = \hat x_1^3 \hat x_2 (1 + \hat x_2)$ (the fourth component is at ∞ in this chart); it is a divisor with normal crossings.

We work in the local rings R, $\hat R$ at $(0,0)$ in both V and $\hat V$. We can let D^+ be the divisor $x_1 x_2 = 0$, so that $\hat D^+$ has ideal $(\hat x_1^2 \hat x_2)$. Bases for $\operatorname{Der}_V(-\log D^+)$, $\operatorname{Der}_{\hat V}(-\log \hat D^+)$ are

$$\langle x_1 \partial_1, x_2 \partial_2 \rangle \quad , \quad \langle \hat x_1 \hat\partial_1, \hat x_2 \hat\partial_2 \rangle$$

where $\partial_i = \partial/\partial x_i$, $\hat\partial_i = \partial/\partial \hat x_i$, and, as the reader may verify, the isomorphism

$$\pi^* \operatorname{Der}_V(-\log D^+) \xrightarrow{\sim} \operatorname{Der}_{\hat V}(-\log \hat D^+)$$

maps $\pi^*(x_1\partial_1)$ to $\hat x_1\hat\partial_1 - \hat x_2\hat\partial_2$ and $\pi^*(x_2\partial_2)$ to $\hat x_2\hat\partial_2$. A derivation $a_1(x)x_1\partial_1 + a_2(x)x_2\partial_2$ is in $\operatorname{Der}_V(-\log D)$ iff

$$(a_1(x)x_1\partial_1 + a_2(x)\partial_2)(x_1x_2(x_1+x_2)) = a_1(x)F + a_1(x)x_1^2x_2 + a_2(x)F + a_2(x)x_1x_2^2 \in (F),$$

that is, iff

$$a_1(x)x_1 + a_2(x)x_2 \in (x_1+x_2) \quad .$$

It follows that a basis for $\operatorname{Der}_V(-\log D)$ is

$$\langle x_1\partial_1 + x_2\partial_2, (x_1+x_2)x_2\partial_2\rangle \quad ,$$

and we may represent sequence (†) at $(0,0)$ as

$$R \oplus R \xrightarrow{\left(\begin{smallmatrix}1 & 0\\ 1 & x_1+x_2\end{smallmatrix}\right)} R\oplus R \xrightarrow{\left(\begin{smallmatrix}x_1^2x_2 & x_1x_2^2\end{smallmatrix}\right)} R/(F)$$

Tensoring by $\hat R$ gives the corresponding sequence (‡):

$$\hat R \oplus \hat R \xrightarrow{\left(\begin{smallmatrix}1 & 0\\ 1 & \hat x_1(1+\hat x_2)\end{smallmatrix}\right)} \hat R\oplus \hat R \xrightarrow{\left(\begin{smallmatrix}\hat x_1^3\hat x_2 & \hat x_1^3\hat x_2^2\end{smallmatrix}\right)} \hat R/(\hat F)$$

which realizes $\pi^*\operatorname{Der}_V(-\log D)$ as a submodule of

$$\operatorname{Der}_{\hat V}(-\log \hat D) = \ker(\begin{pmatrix}\hat x_1^3\hat x_2 & \hat x_1^3\hat x_2^2\end{pmatrix}) = \operatorname{im}\left(\begin{pmatrix}1 & 0\\ 1 & 1+\hat x_2\end{pmatrix}\right)$$

In these coordinates, a matrix representation for $\pi^*\operatorname{Der}_V(-\log D) \hookrightarrow \operatorname{Der}_{\hat V}(-\log \hat D)$ is evidently $\left(\begin{smallmatrix}1 & 0\\ 0 & \hat x_1\end{smallmatrix}\right)$. ⌟

Remark 3.8. Example 3.7 illustrates the general local situation: dualizing Proposition 4.5 in [19], one sees that one may always choose local coordinates and bases for $\operatorname{Der}_V(-\log D)$, $\operatorname{Der}_{\hat V}(-\log \hat D)$ so that the matrix of $\pi^*\operatorname{Der}_V(-\log D) \hookrightarrow \operatorname{Der}_{\hat V}(-\log \hat D)$ is diagonal, with entries given by powers of the equation for the exceptional divisor. This will not be needed in the following, but it is a useful model to keep in mind in reading what follows. ⌟

3.3. We are left with the task of proving the second part of Lemma 3.1, which amounts to the existence of a certain trivial subbundle and an epimorphism to $p^*\operatorname{Der}_Z$ for both $\pi^*\operatorname{Der}_V(-\log D)$ and $\operatorname{Der}_{\hat V}(-\log \hat D)$. We will prove that there is a commutative diagram of locally free sheaves on E:

$$\begin{tikzcd}
\mathscr{O}_E \arrow[rr, hook] \arrow[dr] & & \operatorname{Der}_{\hat V}(-\log \hat D)|_E \arrow[dr] & \\
& \pi^*\operatorname{Der}_V(-\log D)|_E \arrow[ur, "\sigma|_E"'] \arrow[rr, two heads] & & p^*\operatorname{Der}_Z
\end{tikzcd}$$

such that the composition $\mathscr{O}_E \to p^*\operatorname{Der}_Z$ is the zero morphism. The top horizontal morphism will be a monomorphism of vector bundles, and it follows from the commutativity of the diagram that so is the leftmost slanted morphism. Similarly, the bottom horizontal morphism will be an epimorphism, and it follows that so is the rightmost slanted morphism. Thus, the full statement of Lemma 3.1 follows from the existence of this diagram.

3.3.1. We deal with the epimorphism side first. According to our hypotheses, the center Z of the blow-up is the transversal intersection (in a neighborhood of Z) of the r components of D^+, and is contained in the other components of D. As $Z \subseteq V$, we have a natural embedding of $Der_Z \cong TZ$ as the kernel of the natural map from $\mathrm{Der}_V|_Z \cong TV|_Z$ to the normal bundle $N_Z V$.

Lemma 3.9. *There is an exact sequence of vector bundles*

$$0 \longrightarrow \mathscr{O}_Z^{\oplus r} \longrightarrow \mathrm{Der}_V(-\log D^+)|_Z \longrightarrow \mathrm{Der}_V|_Z \longrightarrow N_Z V \longrightarrow 0 \quad .$$

In particular, there is an epimorphism $\mathrm{Der}_V(-\log D^+)|_Z \twoheadrightarrow \mathrm{Der}_Z$.

Proof. We have (see Remark 3.2) an exact sequence

$$0 \longrightarrow \mathrm{Der}_V(-\log D^+) \longrightarrow \mathrm{Der}_V \longrightarrow \mathscr{O}_{D^+}(D^+)$$

The image of the rightmost map is the ideal of $\mathscr{O}_{D^+}(D^+)$ defined locally by the partials of a generator for the ideal of D^+. Near Z, where Z is the complete intersection of $D_1, \dots, D_r$, it is easy to verify that this ideal is isomorphic to $\oplus_{i=1}^r \mathscr{O}_{D_i}(D_i)$. Thus, tensoring by $\mathscr{O}_Z$ gives an exact sequence

$$0 \to \mathrm{Tor}_1(\mathscr{O}_Z, \oplus_{i=1}^r \mathscr{O}_{D_i}(D_i)) \to \mathrm{Der}_V(-\log D^+)|_Z \to \mathrm{Der}_V|_Z \to \oplus_{i=1}^r \mathscr{O}_{D_i}(D_i)|_Z \to 0$$

(The leftmost term is 0 as Der_V is locally free.) The term $\oplus_{i=1}^r \mathscr{O}_{D_i}(D_i)|_Z$ is $N_Z V$, and the map from $\mathrm{Der}_V|_Z$ is the standard projection $TV|_Z \to N_Z V$. The Tor on the left is the direct sum of $\mathrm{Tor}_1(\mathscr{O}_Z, \mathscr{O}_{D_i}(D_i))$, and it is easy to verify that each such term is $\cong \mathscr{O}_Z$, as claimed. □

Remark 3.10. We can choose local parameters $x_1, \dots, x_n$ for V at a point of Z such that x_i is a generator for the ideal of D_i for $i = 1, \dots, r$. Then $\mathrm{Der}_V(-\log D^+)$ has a basis given by derivations

$$x_1\partial_1\,,\, \dots, x_r\partial_r\,,\, \partial_{r+1}\,,\, \dots\,,\, \partial_n$$

where $\partial_i = \partial/\partial x_i$. With the same coordinates, $\partial_{r+1}, \dots \partial_n$ restrict to a basis for Der_Z, and the epimorphism found in Lemma 3.9 acts in the evident way. The kernel is spanned by the restrictions of $x_i\partial_i$, $i = 1, \dots, r$; these are the r trivial factors appearing on the left in the sequence in Lemma 3.9.

Also note that the 'Euler derivation' $x_1\partial_1 + \dots + x_r\partial_r$ spans a trivial subbundle $\mathscr{O}_Z \hookrightarrow \mathscr{O}_Z^{\oplus r}$ of the kernel. Thus, we have a complex of vector bundles

$$\mathscr{O}_Z \hookrightarrow \mathrm{Der}_V(-\log D^+)|_Z \twoheadrightarrow \mathrm{Der}_Z$$

on Z. Pulling back to E, this gives a complex of vector bundles on E:

$$\mathscr{O}_E \hookrightarrow \pi^* \mathrm{Der}_V(-\log D^+)|_E \twoheadrightarrow \pi^* \mathrm{Der}_Z \quad .$$

We have to verify that the same occurs for $\pi^* \mathrm{Der}_V(-\log D)$ and $\mathrm{Der}_{\hat{V}}(-\log \hat{D})$. ⌟

Consider $\mathrm{Der}_V(-\log D)$. We have (Remark 3.4) inclusions

$$\mathrm{Der}_V(-\log D) \subseteq \mathrm{Der}_V(-\log D^+) \subseteq \mathrm{Der}_V \quad .$$

Restricting to Z, and in view of Lemma 3.9, we get morphisms

$$\mathrm{Der}_V(-\log D)|_Z \longrightarrow \mathrm{Der}_V(-\log D^+)|_Z \twoheadrightarrow \mathrm{Der}_Z \quad .$$

Claim 3.11. *The composition* $\mathrm{Der}_V(-\log D)|_Z \to \mathrm{Der}_Z$ *is an epimorphism.*

Proof. Working with local parameters as in Remark 3.10, it suffices to note that the derivations $\partial_{r+1}, \dots, \partial_n$ are in $\mathrm{Der}_V(-\log D)$: this is clear, since by assumption D admits a local generator of the form

$$x_1 \cdots x_r\, G(x_1, \dots, x_r).$$

□

Pulling back to E and using Lemma 3.6 we get morphisms

$$\begin{tikzcd} \pi^* \operatorname{Der}_V(-\log D)|_E \arrow[r, "\sigma|_E"] \arrow[rrr, bend right, two heads] & \operatorname{Der}_{\hat{V}}(-\log \hat{D})|_E \arrow[r] & \pi^* \operatorname{Der}_V(-\log D^+)|_E \arrow[r, two heads] & p^* \operatorname{Der}_Z \end{tikzcd}$$

and this yields the commutative triangle on the right in the diagram at the beginning of the section.

3.3.2. Finally, we have to deal with the triangle on the left.

Lemma 3.12. *Let A be a nonsingular hypersurface of a nonsingular variety V. Then there is an exact sequence of vector bundles*

$$0 \longrightarrow \mathscr{O}_A \longrightarrow \operatorname{Der}_V(-\log A)|_A \longrightarrow \operatorname{Der}_A \longrightarrow 0 \quad .$$

Proof. This is a particular case of Lemma 3.9. □

Remark 3.13. Applying this lemma to $E \subseteq \hat{V}$ gives a distinguished copy of $\mathscr{O}_E$ in $\operatorname{Der}_{\hat{V}}(-\log E)|_E$. Adopting local parameters at a point of Z as in Remark 3.10, we can choose coordinates

$$\hat{x}_1 \,,\, \hat{x}_2 \,,\, \dots \,,\, \hat{x}_r \,,\, \hat{x}_{r+1} \,,\, \dots \,,\, \hat{x}_n$$

at a point of E in a chart of the blow-up $\hat{V}$ so that the blow-up map is given by

$$\begin{cases} x_1 = \hat{x}_1 & \\ x_i = \hat{x}_1\hat{x}_i & i = 2, \dots, r \\ x_j = \hat{x}_j & j = r+1, \dots, n \end{cases}$$

The exceptional divisor is given by $\hat{x}_1 = 0$. Then a basis for $\operatorname{Der}_{\hat{V}}(-\log E)$ at this point is

$$\hat{x}_1\hat{\partial}_1 \,,\, \hat{\partial}_2 \,,\, \dots \,,\, \hat{\partial}_n$$

where $\hat{\partial}_i = \partial/\partial\hat{x}_i$. The distinguished copy $\mathscr{O}_E \subseteq \operatorname{Der}_{\hat{V}}(-\log E)$ found in Lemma 3.12 is spanned by $\hat{x}_1\hat{\partial}_1$. ⌟

Now recall (Remark 3.4) that $\operatorname{Der}_{\hat{V}}(-\log \hat{D}) \subseteq \operatorname{Der}_{\hat{V}}(-\log E)$.

Claim 3.14. *The distinguished $\mathscr{O}_E \subseteq \operatorname{Der}_{\hat{V}}(-\log E)|_E$ is contained in $\operatorname{Der}_{\hat{V}}(-\log \hat{D})|_E$.*

Proof. We work in coordinates as in Remark 3.10 and 3.13. By hypothesis, D is given analytically by the vanishing of $F = x_1 \cdots x_r \cdot G(x_1 \cdots x_r)$, where G is homogeneous. In the chart considered above in the blow-up, $\hat{D}$ is therefore given by the vanishing of

$$\hat{F} = \hat{x}_1^m \hat{x}_2 \cdots \hat{x}_r \, G(1, \hat{x}_2, \dots, \hat{x}_r)$$

where m is the multiplicity of D along Z. We then see that

$$(\hat{x}_1\hat{\partial}_1)\hat{F} = m\, \hat{x}_1^m \hat{x}_2 \cdots \hat{x}_r \, G(1, \hat{x}_2, \dots, \hat{x}_r) = m\hat{F} \in (\hat{F}) \quad :$$

this shows that $\hat{x}_1\hat{\partial}_1 \in \operatorname{Der}_{\hat{V}}(-\log \hat{D})$, as claimed. □

Since $\operatorname{Der}_{\hat{V}}(-\log \hat{D}) \subseteq \operatorname{Der}_{\hat{V}}(-\log \hat{D}^+)$, and the latter is $\cong \pi^* \operatorname{Der}_V(-\log D^+)$, we can view the distinguished $\mathscr{O}_E$ as a subsheaf of $\pi^* \operatorname{Der}_V(-\log D^+)|_E$. Chasing coordinates, it is straightforward to check that

$$\hat{x}_1\hat{\partial}_1 \mapsto x_1\partial_1 + \cdots + x_r\partial_r \quad :$$

that is, this copy of $\mathscr{O}_E$ corresponds to the 'Euler derivation' identified in Remark 3.10. Further, we see that it is also contained in $\pi^* \operatorname{Der}_V(-\log D)|_E$: indeed, since in the chosen analytic coordinates F is homogeneous (up to factors not vanishing along Z), the Euler derivation acts on F by multiplying it by its degree.

At this point we have the following situation:

$$\begin{tikzcd} \mathscr{O}_E \arrow[r, hook] \arrow[rrr, bend left] & \pi^* \operatorname{Der}_V(-\log D)|_E \arrow[r, "\sigma|_E"] & \operatorname{Der}_{\hat V}(-\log \hat D)|_E \arrow[r] & \pi^* \operatorname{Der}_V(-\log \hat D^+)|_E \end{tikzcd}$$

This yields the commutative triangle on the left in the diagram at the beginning of the section. (The above construction shows that the monomorphisms from $\mathscr{O}_E$ to $\operatorname{Der}_{\hat V}(-\log \hat D)|_E$ and $\pi^* \operatorname{Der}_V(-\log \hat D^+)|_E$ are monomorphisms of vector bundles, as needed.) The composition with the projection to $p^* \operatorname{Der}_Z$ is 0 as noted in Remark 3.10, so this completes the proof of Lemma 3.1. Claim 2.4 follows from Lemma 3.1 as shown in §3.1, so this concludes the proof of Theorem 1.1.

4. Further remarks and examples

4.1. **An example.** We illustrate Theorem 1.1 by computing the Chern class of a sheaf of logarithmic derivations in a simple case. Any value Theorem 1.1 may have lies in the contrast between the standard computation, by means of the basic sequence defining the sheaf, and the computation using Chern-Schwartz-MacPherson classes, which has a very different, 'combinatorial' flavor.

We assume D consists of $m \geq 2$ nonsingular components D_i, each of class X, meeting pairwise transversally along a codimension-2 nonsingular complete subvariety Z.

—Computation using $c_{\rm SM}$-classes. As $D = \cup_i D_i$, and since all components meet along Z, we have

$$1\!\!1_D = 1\!\!1_Z + \sum_i 1\!\!1_{D_i \smallsetminus Z} = (\sum_i 1\!\!1_{D_i}) - (m-1)1\!\!1_Z \quad ,$$

and hence

$$1\!\!1_{V \smallsetminus D} = 1\!\!1_V - \sum_i 1\!\!1_{D_i} + (m-1)1\!\!1_Z \quad .$$

Since V, all D_i, and Z are nonsingular, the basic normalization property of $c_{\rm SM}$ classes (§2.1) gives

$$c_{\rm SM}(1\!\!1_{V\smallsetminus D}) = c(TV)\cap[V] - \sum_i c(TD_i)\cap[D_i] + (m-1)c(TZ)\cap[Z] \quad .$$

We are assuming that all components have the same class X, and hence Z has class $X \cdot X$. Thus, this gives

$$c_{\rm SM}(1\!\!1_{V\smallsetminus D}) = c(TV)\left(1 - \sum_{i=1}^m \frac{X}{1+X} + (m-1)\frac{X^2}{(1+X)^2}\right)\cap[V]$$

According to Theorem 1.1, this class equals $c(\operatorname{Der}_V(-\log D))\cap[V]$. That is,

$$c(\operatorname{Der}_V(-\log D)) = \frac{c(TV)\,(1-(m-2)X)}{(1+X)^2} \quad .$$

Standard computation. The basic sequence recalled in Remark 3.2 may be completed to

$$0 \longrightarrow \operatorname{Der}_V(-\log D) \longrightarrow \operatorname{Der}_V \longrightarrow \mathscr{O}_D(D) \longrightarrow \mathscr{O}_{JD}(D) \longrightarrow 0 \quad ,$$

where JD is the *singularity subscheme* (or *Jacobian subscheme*) of D. Therefore,

$$c(\operatorname{Der}_V(-\log D)) = \frac{c(\operatorname{Der}_V)}{c(\mathscr{O}_D(D))}\, c(\mathscr{O}_{JD}(D)) = \frac{c(TV)}{1+D}\, c(\mathscr{O}_{JD}(D)) \quad .$$

In the case at hand, we are assuming that D is defined by a section $f_1 \cdots f_m$ of $\mathscr{O}(mX)$; Z is defined by (say) $f_1 = f_2 = 0$, meeting transversally at every point of Z; and $f_i = (a_i f_1 + b_i f_2)$ for $i \geq 3$, without multiple components. Thus, $f_1 \cdots f_m = P(f_1, f_2)$ for a homogeneous polynomial $P(s,t)$ with constant coefficients. As the differentials df_1 and df_2 are assumed to be linearly independent everywhere along Z, the ideal of JD is generated by $\frac{\partial P}{\partial s}(f_1, f_2)$ and $\frac{\partial P}{\partial t}(f_1, f_2)$, and these have no component in common. It follows that JD is a complete intersection of two sections of $\mathscr{O}((m-1)X)$, so $\mathscr{O}_{JD}$ is resolved by a Koszul complex:

$$0 \to \mathscr{O}_V(-2(m-1)X) \to \mathscr{O}_V(-(m-1)X) \oplus \mathscr{O}_V(-(m-1)X) \to \mathscr{O}_V \to \mathscr{O}_{JD} \to 0$$

and twisting by $\mathscr{O}_V(D) = \mathscr{O}_V(mX)$ gives the exact sequence

$$0 \longrightarrow \mathscr{O}_V((-m+2)X) \longrightarrow \mathscr{O}_V(X) \oplus \mathscr{O}_V(X) \longrightarrow \mathscr{O}_V(D) \longrightarrow \mathscr{O}_{JD}(D) \longrightarrow 0 \quad .$$

Thus

$$c(\mathscr{O}_{JD}(D)) = \frac{c(\mathscr{O}_V(D))\, c(\mathscr{O}(-(m-2))X)}{c(\mathscr{O}_V(X))^2} = \frac{(1+D)(1-(m-2)X)}{(1+X)^2} \quad .$$

Taking this into account in the expression for $c(\mathrm{Der}_V(-\log D))$ given above, we recover the result of the c_{SM} computation. ⌟

While this may be largely a matter of taste, the standard computation appears to us to involve subtler information than the alternative combinatorial computation via c_{SM} classes afforded by applying Theorem 1.1. The point is that the c_{SM} class already includes information on the singularity subscheme JD: see [1] for the precise relation. Computing the c_{SM} class, which is straightforward for a hypersurface arrangement, takes automatically care of accounting for the total Chern class of $\mathscr{O}_{JD}(D)$.

4.2. **A projection formula.** If $\mathscr{E}$ is a vector bundle on a scheme X, and $\alpha : Y \to X$ is a proper morphism, then for any class A in the Chow group of Y we have

$$\alpha_*(c(\alpha^*\mathscr{E}) \cap A) = c(\mathscr{E}) \cap \alpha_*(A) \quad .$$

This is a basic result on Chern classes, see Theorem 3.2 (c) in [9]. On a nonsingular variety, a notion of total Chern class is available for all coherent sheaves: this follows from the isomorphism $K_0(V) \cong K^0(V)$ for V nonsingular ([9], §15.1) and the Whitney formula. However, a straightforward projection formula as in the case of vector bundles does not hold for arbitrary coherent sheaves.

Example 4.1. Let V be nonsingular, let $X, Y \hookrightarrow V$ be irreducible hypersurfaces, and let $i : X \hookrightarrow V$ be the inclusion. From the exact sequence

$$0 \longrightarrow \mathscr{O}_V(-X) \longrightarrow \mathscr{O}_V \longrightarrow \mathscr{O}_X \longrightarrow 0$$

it follows that $c(\mathscr{O}_X) = \frac{1}{1-X}$, and similarly $c(\mathscr{O}_Y) = \frac{1}{1-Y}$. As $i^*(\mathscr{O}_X) = \mathscr{O}_X$, we see that

$$i_*(c(i^*\mathscr{O}_X) \cap [X]) = i_*([X]) \neq c(\mathscr{O}_X) \cap i_*([X]) = \frac{[X]}{1-X} \quad :$$

the projection formula does not hold in this case. On the other hand, $i^*(\mathscr{O}_Y) = \mathscr{O}_{X \cap Y}$, and $X \cap Y$ is a divisor in X with bundle $\mathscr{O}_X(X \cap Y) = i^*\mathscr{O}_V(Y)$; therefore,

$$i_*(c(i^*\mathscr{O}_Y) \cap [X]) = i_*\left(\frac{[X]}{1 - i^*Y}\right) = \frac{[X]}{1-Y} = c(\mathscr{O}_Y) \cap i_*([X]) \quad :$$

the projection formula does hold in this case. ⌟

The difference between the two cases considered in this example is a matter of Tor functors:

$$\mathrm{Tor}_1^{\mathscr{O}_V}(\mathscr{O}_X, \mathscr{O}_X) \cong \mathscr{O}_X(-X)$$

is not trivial, while $\mathrm{Tor}_1^{\mathscr{O}_V}(\mathscr{O}_X, \mathscr{O}_Y)$ vanishes. It is essentially evident from the definitions that for a coherent sheaf $\mathscr{F}$ on V, and a morphism $\alpha : W \to V$,

$$\alpha^* c(\mathscr{F}) = \prod_{i \geq 0} c(\mathrm{Tor}_i^{\mathscr{O}_V}(\mathscr{O}_W, \mathscr{F}))^{(-1)^i} = c(\alpha^*\mathscr{F}) \cdot \prod_{i \geq 1} c(\mathrm{Tor}_i^{\mathscr{O}_V}(\mathscr{O}_W, \mathscr{F}))^{(-1)^i} \quad ,$$

and in particular $c(\alpha^*\mathscr{F}) = \alpha^* c(\mathscr{F})$ if the higher Tors vanish. In this case (for example, in the case of vector bundles) the projection formula holds if α is proper. More generally, the projection formula holds if α_* maps to 1 the total Chern classes of the higher Tors.

4.3. Now recall the situation of this paper, and particularly the blow-up considered in Claim 2.4 and §3: D is a hypersurface arrangement in a nonsingular variety V, and Z is an intersection of minimal dimension of components of D. In fact, $Z = D_1 \cap \cdots \cap D_r$, where $D_1, \dots, D_r$ are components of D meeting with normal crossings in a neighborhood of Z. We denote by D^+ the union of these components, and we have observed (Remark 3.4) that $\mathrm{Der}_V(-\log D) \subseteq \mathrm{Der}_V(-\log D^+)$.

The sections obtained by applying the derivations in $\mathrm{Der}_V(-\log D^+)$ to a section F defining D, together with F, define a subscheme J^+D of $\mathscr{O}_D$, which should be viewed as a 'modified Jacobian subscheme' of the hypersurface arrangement D (depending on the choice of the subdivisor D^+). We consider the coherent sheaf $\mathscr{O}_{J^+D}(D)$ in V.

Finally, recall that $\pi : \hat{V} \to V$ denotes the blow-up of V along Z.

Claim 4.2. *The formula in Theorem 1.1 is implied by the statement that, for all blow-ups as above, $\mathscr{O}_{J^+D}(D)$ satisfies the projection formula with respect to the blow-up map π:*

$$\pi_*(c(\pi^*\mathscr{O}_{J^+D}(D)) \cap [\hat{V}]) = c(\mathscr{O}_{J^+D}(D)) \cap [V] \quad .$$

Proof. Arguing as in §2, we only need to deal with the case of a single blow-up; we will show that the given formula is equivalent to the formula in Claim 2.4.

Restricting the basic sequence recalled in Remark 3.2 to D^+ gives an exact sequence

$$0 \longrightarrow \mathrm{Der}_V(-\log D) \longrightarrow \mathrm{Der}_V(-\log D^+) \longrightarrow \mathscr{O}_D(D)$$

and the image of the last morphism is the ideal generated by applying the derivations from $\mathrm{Der}_V(-\log D^+)$ to F; this ideal defines J^+D, so we have an exact sequence

$$0 \longrightarrow \mathrm{Der}_V(-\log D) \longrightarrow \mathrm{Der}_V(-\log D^+) \longrightarrow \mathscr{O}_D(D) \longrightarrow \mathscr{O}_{J^+D}(D) \longrightarrow 0$$

on V. Notice that this implies that

$$c(\mathrm{Der}_V(-\log D)) = \frac{c(\mathrm{Der}_V(-\log D^+))}{1+D}\, c(\mathscr{O}_{J^+D}(D)) \quad .$$

Now we claim that (with notation as in §3) there is an exact sequence

$$(\diamond) \quad 0 \longrightarrow \mathrm{Der}_{\hat{V}}(-\log \hat{D}) \longrightarrow \pi^*\mathrm{Der}_V(-\log D^+) \longrightarrow \pi^*\mathscr{O}_D(D) \longrightarrow \pi^*\mathscr{O}_{J^+D}(D) \longrightarrow 0 \quad .$$

Indeed, pulling back the last terms of the previous sequence to $\hat{V}$ gives the last terms of $(\diamond)$, by right-exactness of $-\otimes_{\mathscr{O}_V}\mathscr{O}_{\hat{V}}$; via the isomorphisms $\pi^*\mathrm{Der}_V(-\log D^+) \cong \mathrm{Der}_{\hat{V}}(-\log \hat{D}^+)$ (Lemma 3.5) and $\pi^*\mathscr{O}_D(D) \cong \mathscr{O}_{\hat{D}}(\hat{D})$, the morphism in the middle is seen to act by applying derivations from $\mathrm{Der}_{\hat{V}}(-\log \hat{D})$ to a section defining $\hat{D}$. Hence its kernel is $\mathrm{Der}_{\hat{V}}(-\log \hat{D})$, as needed for $(\diamond)$. From $(\diamond)$, it follows that

$$c(\mathrm{Der}_{\hat{V}}(-\log \hat{D})) = \frac{c(\pi^*\mathrm{Der}_V(-\log D^+))}{1+\pi^*D}\, c(\pi^*\mathscr{O}_{J^+D}(D)) \quad ,$$

and hence, applying the ordinary projection formula (as $\mathrm{Der}_V(-\log D^+)$ is locally free)

$$\pi_*(c(\mathrm{Der}_{\hat{V}}(-\log \hat{D})) \cap [\hat{V}]) = \frac{c(\mathrm{Der}_V(-\log D^+))}{1+D}\, \pi_*(c(\pi^*\mathscr{O}_{J^+D}(D)) \cap [\hat{V}]) \quad .$$

Comparing with the previous equality of Chern classes, we see that the projection formula for $\mathscr{O}_{J^+D}(D)$,

$$\pi_*(c(\pi^*\mathscr{O}_{J^+D}(D)) \cap [\hat{V}]) = c(\mathscr{O}_{J^+D}(D)) \cap [V]$$

is equivalent to

$$\pi_*(c(\mathrm{Der}_{\hat{V}}(-\log \hat{D})) \cap [\hat{V}]) = c(\mathrm{Der}_{\hat{V}}(-\log \hat{D})) \cap [V] \quad ,$$

that is the formula in Claim 2.4, as claimed. (As pointed out in Remark 3.2, we can replace $\hat{D}$ for D' in Claim 2.4.) □

By Claim 4.2, an independent proof of the projection formula for $\mathscr{O}_{J+D}(D)$ would give an alternative proof of Theorem 1.1. Note that the relevant Tor does not vanish in general; the task amounts to showing that its Chern class pushes forward to 1. We were not able to construct a more direct proof of this fact.

Remark 4.3. Xia Liao has shown ([12]) that the equality in Theorem 1.1, for any divisor D, is equivalent to a projection formula involving the blow-up along the (ordinary) Jacobian subscheme of D. ⌟

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ORBIFOLD GROUPS, QUASI-PROJECTIVITY AND COVERS

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Abstract. We discuss properties of complex algebraic orbifold groups, their characteristic varieties, and their abelian covers. In particular, we deal with the question of (quasi)-projectivity of orbifold groups. We also prove a structure theorem for the variety of characters of normal-crossing quasi-projective orbifold groups. Finally, we extend Sakuma's formula for the first Betti number of abelian covers of orbifold fundamental groups. Several examples are presented, including a compact orbifold group which is not projective and a Zariski pair of plane curves in $\mathbb{P}^2$ that can be told by considering an unbranched cover of $\mathbb{P}^2$ with an orbifold structure.

Introduction

Any finitely presented group G is the fundamental group of a closed oriented 4-manifold. If we ask these manifolds to have extra-properties, some restrictions may apply. For example, such a group is said to be *Kähler* (resp. *projective*) if it is the fundamental group of a compact Kähler manifold (resp. a projective manifold). Note that projective groups are Kähler groups, but the converse is still an open question posed by Serre. In this direction, it is worth mentioning that there exist compact Kähler manifolds whose homotopy type cannot be realized by a smooth projective manifold (cf. [25]).

The family of projective groups is a subfamily of *quasi-projective* groups. Recall that a quasi-projective manifold is the difference of two projective varieties. The study of Kähler, projective and quasi-projective groups is closely related to orbifold groups, or more precisely to orbicurve groups, i.e. orbifold fundamental groups of complex 1-dimensional orbifolds. Recently, orbifold groups (in any complex dimension) have been considered (cf. [24, 9] also [17] for real orbifolds).

The first purpose of this paper is to define and study the properties of the different classes of complex orbifold fundamental groups such as compact, locally finite, and normal crossing. In particular, we prove that orbifold fundamental groups are quasi-projective, but compact orbifold groups in general are not projective (see §1). In this context, we develop in §2 the concept of *saturated* orbifolds, which will allow one to transform orbifolds without altering their fundamental group.

Our second purpose (see §3) is to extend two classical results regarding the variety of characters on smooth quasi-projective fundamental groups (due to Arapura [1] and the authors [4]) and normal-crossing compact Kähler orbiface groups (due to Campana [9]) to the general case of normal-crossing quasi-projective orbifold groups.

Finally in §5, we extend Sakuma's formula (cf. [21, 15]) to orbifold fundamental groups and their abelian covers in terms of their orbifold characteristic varieties. In order to do so, in §4 we present the concepts of unbranched and branched coverings as well as the possible uniformizations (Galois, regular, and virtually regular). Such formulas are illustrated with examples in dimensions one and two.

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1. Orbifold groups

Definition 1.1. Let $\bar{X}$ be a projective Riemann surface and let $\varphi : \bar{X} \to \mathbb{Z}_{\geq 0}$ be a function such that $S_\varphi := \{p \in X \mid \varphi(p) \neq 1\}$ is finite. The pair $(\bar{X}, \varphi)$ is said to be a 1-dimensional *orbifold* or an *orbicurve*. The *positive part* of the orbicurve is $X^+_\varphi := \bar{X} \setminus \varphi^{-1}(0)$ and we say that the orbifold is *compact* if $X^+_\varphi = \bar{X}$. The set $S^{>1}_\varphi := X^+_\varphi \cap S_\varphi$ is called the *singular part* and $\varphi(p)$ is the *orbifold index* of $p \in X^+_\varphi$.

The geometrical interpretation is the following. The source of the charts centered at $p \in X^+_\varphi$ are of the type $\Delta/\mu_{\varphi(p)}$ where $\mu_n := \{z \in \mathbb{C} \mid z^n = 1\}$, Δ is an open disk centered at 0 and $\mu_{\varphi(p)}$ acts on Δ by multiplication. This interpretation suggests the following definition.

Definition 1.2. Let $(\bar{X}, \varphi)$ be an orbicurve. Let $X_\varphi := \bar{X} \setminus S_\varphi$ and $G := \pi_1(X_\varphi; p_0)$ for some $p_0 \in X_\varphi$. For each $p \in S_\varphi$ choose a meridian $x_p \in G$ (its conjugacy class is well defined). Then, the *orbifold fundamental group* of (X, φ) is defined as

$$\pi_1^{\text{orb}}(\bar{X}, \varphi; p_0) := G/\langle x_p^{\varphi(p)} \rangle.$$

A group is said to be an *orbicurve group* if it is the orbifold fundamental group of an orbicurve.

Remark 1.3. If the group can be described as the orbifold group of a compact orbicurve then we will refer to it as a *compact orbicurve group.*

Proposition 1.4. *Any orbicurve group is quasi-projective.*

In order to prove this result we introduce the following concept.

Definition 1.5. Let X be a smooth quasi-projective surface, let $\mathcal{D}$ be a divisor in X and let $D \subset X$ an irreducible component of $\mathcal{D}$. An *n-fold blow-up* ρ of $(X, \mathcal{D})$ on D is a composition of blowing-ups $\rho_j : X_j \to X_{j-1}$, $1 \leq j \leq n$, such that $X_0 := X$, the center of ρ_1 is a point of D which is smooth in $\mathcal{D}$ and if $E_j \subset X_j$ is the exceptional component of ρ_j then the center of ρ_{j+1} is the intersection of E_j with the strict transform of D. The component E_j is called the *j-th exceptional component* of ρ.

Remark 1.6. As a general comment, consider a double point on a smooth surface, i.e. a point of local equations $\mathcal{D} := \{x^2 - y^2 = 0\}$ in a small ball $\mathbb{B}$ around $P = (0, 0)$. Perform a blow-up $\rho : \tilde{\mathbb{B}} \to \mathbb{B}$ centered at P and consider $\gamma(t) = (e^{2\pi\sqrt{-1}t}, 0)$ which is a product of the meridians around the two components of $\mathcal{D}$. Note that $G_P := \pi_1(\mathbb{B} \setminus \mathcal{D}) = \pi_1(\tilde{\mathbb{B}} \setminus (E \cup \tilde{\mathcal{D}}))$ where E is the exceptional divisor and $\tilde{\mathcal{D}}$ is the strict transform of $\mathcal{D}$. Using both affine charts and using $\gamma(t) = (e^{2\pi\sqrt{-1}t}, 1)$ it is easy to see that $\gamma = \gamma_E$ as elements of G_P, where γ_E is a meridian around E.

Analogously, if we consider $G := \pi_1(X \setminus \mathcal{D}) = \pi_1(X \setminus (E_1 \cup \mathcal{D}))$, ρ an n-fold blow-up ρ of $(X, \mathcal{D})$ on an irreducible divisor D and μ is a meridian around D, then μ is also a meridian around E_1. Using the property discussed in the previous paragraph, μ^2 is a meridian around E_2 and by induction μ^j is a meridian around the j-th exceptional component of ρ in G.

Proof of Proposition 1.4. Let $(\bar{X}, \varphi)$ be an orbicurve. A quasi-projective surface Z will be constructed satisfying $\pi_1(Z) \cong \pi_1^{\text{orb}}(\bar{X}, \varphi)$.

Let $Y := X^+_\varphi \times \mathbb{P}^1$ be a surface, and let $\mathcal{D} := S^{>1}_\varphi \times \mathbb{P}^1 \subset Y$. For each $p \in S^{>1}_\varphi$ consider a $\varphi(p)$-fold blow-up $\rho_p : \tilde{Y} \to Y$ on the divisor $F_p := \{p\} \times \mathbb{P}^1$. Let E^p_j be the j-th exceptional component of ρ_p. Let x_p be a meridian of $\{p\} \times \mathbb{P}^1$ in $\pi_1(\tilde{Y})$. Following the previous remark,

x_p^j is a meridian of E_j^p in $\pi_1(\tilde{Y})$. The surface

$$Z := \tilde{Y} \setminus \bigcup_{p \in S_\varphi^{>1}} \left(F_p \cup \bigcup_{j=1}^{\varphi(p)-1} E_j^p \right)$$

is quasi-projective. The groups $\pi_1(X_\varphi^+)$, $\pi_1(Y)$ and $\pi_1(\tilde{Y})$ are naturally isomorphic. The kernel of the epimorphism $\pi_1(\tilde{Y}) \twoheadrightarrow \pi_1(Z)$ is normally generated by the meridians $x_p^{\varphi(p)}$ of $E_{\varphi(p)}^p$. Then, $\pi_1(Z)$ is isomorphic to $\pi_1(\tilde{Y})/\langle x_p^{\varphi(p)} \rangle$ which, by the definition of orbicurve group, is nothing but $\pi_1^{\mathrm{orb}}(\bar{X}, \varphi)$. □

Remark 1.7. As shown in [14, Theorem II.2.3], compact orbicurve groups are projective groups.

We will define orbifolds and orbifold groups following Campana (cf. [9] and bibliography therein). Since we are mostly interested in quasi-projective groups, after using Zariski-Lefschetz theory we can restrict our attention to the curve and surface case. However, since we will deal with orbifold covers (see §4), orbifolds with abelian quotient singularities will also be allowed.

Definition 1.8. Let $\bar{X}$ be a projective variety with only abelian quotient singularities and let $\mathcal{D} = \bigcup_{j=1}^r D_i$ be the decomposition of a hypersurface in irreducible components. Let us consider a function $\varphi : \{D_1, \dots, D_r\} \to \mathbb{Z}_{\geq 0}$, $n_i := \varphi(D_i)$. An *orbifold* is simply a pair $(\bar{X}, \varphi)$. The *positive part* of the orbifold is defined as $X_\varphi^+ := \bar{X} \setminus \varphi^{-1}(0)$. The orbifold is said to be *compact* if $\bar{X} = X_\varphi^+$. The orbifold will be a *normal-crossing orbifold* (NC for short) if $\mathcal{D}$ is a normal crossing divisor with smooth components.

Remark 1.9. Note that, for technical reasons, the components of $\mathcal{D}$ are allowed to have index one (that is, $n_j = 1$). However, this plays no important role in the definition of an orbifold. Hence, if no ambiguity seems likely to arise, we denote by the same symbols an orbifold and its analogous where $\varphi^{-1}(1)$ is disregarded. What is really important in the definition is the quasi-projective variety X_φ^+ and the components D_j with $n_j > 1$. Following the definitions for the orbicurve case we also define

$$S_\varphi := \{D_j \mid n_j \neq 1\},\ S_\varphi^{>1} := \{D_j \mid n_j > 1\},\ X_\varphi := \bar{X} \setminus \left(\bigcup S_\varphi\right),\ \mathring{X}_\varphi := \bar{X} \setminus \mathcal{D}.$$

Note that $\mathring{X}_\varphi \subset X_\varphi \subset X_\varphi^+$. In $\pi_1(X_\varphi)$ and $\pi_1(\mathring{X}_\varphi)$ one has special conjugacy classes: for each D_i we consider the meridians of D_i in either $\pi_1(X_\varphi)$ or $\pi_1(\mathring{X}_\varphi)$. Note that the kernel of the epimorphism $\pi_1(X_\varphi) \twoheadrightarrow \pi_1(\bar{X})$ is the subgroup generated by the meridians of D_j, $n_j \neq 1$ whereas the kernel of the epimorphism $\pi_1(\mathring{X}_\varphi) \twoheadrightarrow \pi_1(\bar{X})$ is the subgroup generated by the meridians of $D_1, \dots, D_r$.

Definition 1.10. Under the notation above, given an orbifold $(\bar{X}, \varphi)$ we define its *orbifold fundamental group* as the group $\pi_1^{\mathrm{orb}}(\bar{X}, \varphi; p_0)$, $p_0 \in \mathrm{Reg}(\mathring{X}_\varphi) := \mathring{X}_\varphi \setminus \mathrm{Sing}(\mathring{X}_\varphi)$ obtained as the quotient of $\pi_1(\mathring{X}_\varphi; p_0)$ by the subgroup normally generated by $\{\mu_j^{n_j}\}_{1 \leq j \leq r}$, where μ_j is a meridian of D_j. Note that $\pi_1(\mathring{X}_\varphi)$ can also be replaced by $\pi_1(X_\varphi)$ in this definition.

For $p \in X_\varphi^+$ one can define the *local orbifold fundamental group* $\pi_1^{\mathrm{orb}}(\bar{X}, \varphi)_p$ as the quotient of $\pi_1(\mathrm{Reg}(\mathring{X}_\varphi))_p$ by the subgroup normally generated by the appropriate powers $\mu_j^{n_j}$ of the meridians μ_j of $\mathcal{D}$ in a small ball around p. The orbifold $(\bar{X}, \varphi)$ shall be called *locally finite at* p if $\pi_1^{\mathrm{orb}}(\bar{X}, \varphi)_p$ is a finite group, and *locally finite* (or simply LF) if it is locally finite at p, $\forall p \in X_\varphi^+$.

We need to extend the notion of the orbifold index of a point in an orbifold as we did for orbicurves in Definition 1.1.

Definition 1.11. Let $(\bar{X},\varphi)$ be an NC-orbifold and let $p\in\bar{X}$. We define the *orbifold index* $\nu(p)=\nu_{(\bar{X},\varphi)}(p)$ of p as follows:

$$\nu(p):=\begin{cases}\iota_p & \text{if } p\in\bar{X}\setminus\mathcal{D},\\ n_j\cdot\iota_p & \text{if } p\in D_j\setminus\bigcup_{i\neq j}D_i,\\ n_i\cdot n_j\cdot\iota_p & \text{if } p\in D_i\cap D_j, i\neq j,\end{cases}$$

where $\iota_p=|A|$ if $(\bar{X},p)\cong(\mathbb{C}^2/A,0)$, the quotient by the linear action of a small abelian subgroup $A\subset\mathrm{GL}(2;\mathbb{C})$ (note that $\iota_p=1$ iff $p\in\mathrm{Reg}(\bar{X})$).

Remark 1.12. If $p\in\mathrm{Reg}(X_\varphi)$ (or $\mathrm{Reg}(\mathring{X}_\varphi)$) then $\pi_1^{\text{orb}}(X,\varphi)_p$ is a trivial group.

Proposition 1.13. *If $p\in X_\varphi^+$ then $\nu(p)=\#\pi_1^{\text{orb}}(X,\varphi)_p$.*

Proof. We distinguish several cases for p such that $(\bar{X},p)\cong(\mathbb{C}^2/A,0)$ where A is a small abelian group (hence cyclic). Let $\mathbb{B}_p$ be a small neighborhood of p (a quotient of a ball $\mathbb{B}_0$ in $\mathbb{C}^2$).

Let us suppose that $p\in\mathring{X}_\varphi$. In this case $\pi_1^{\text{orb}}(\bar{X},\varphi)_p$ is isomorphic to $\pi_1(K_p)$, where K_p is the link of the singularity $(\bar{X},p)$ which is a lens space with fundamental group A and the result follows.

Let us assume now that p belongs only to one irreducible component $D_i\subset\mathcal{D}$ where D_i is the image of $Y:=\{y=0\}\subset\mathbb{C}^2$. We have a short exact sequence

$$0\to\pi_1(\mathbb{B}_0\setminus Y)\to\pi_1(\mathbb{B}_p\setminus D_i)\to A\to 0.$$

Both $\pi_1(\mathbb{B}_0\setminus Y)$ and $\pi_1(\mathbb{B}_p\setminus D_i)$ are isomorphic to $\mathbb{Z}$ (written with multiplicative notation), which is generated by an element t which projects to a generator of A. By the definition of the action, the image of a generator of $\pi_1(\mathbb{B}_0\setminus Y)$ is a meridian μ_i of D_i which equals t^{ι_p}. Hence, we obtain $\pi_1^{\text{orb}}(X,\varphi)_p$ from $\pi_1(\mathbb{B}_p\setminus D_i)$ by killing $x_i^{n_i}=t^{i_p n_i}=t^{\nu(p)}$ and the result follows.

Finally, let us assume that p belongs to two irreducible components $D_i,D_j\subset\mathcal{D}$ where D_i is the image of $Y:=\{y=0\}\subset\mathbb{C}^2$ and D_j is the image of $X:=\{x=0\}$. The covering induces the following short exact sequence:

$$0\to\pi_1(\mathbb{B}_0\setminus(X\cup Y))\to\pi_1(\mathbb{B}_p\setminus(D_i\cup D_j))\to A\to 0. \tag{1.1}$$

Both $\pi_1(\mathbb{B}_0\setminus Y)$ and $\pi_1(\mathbb{B}_p\setminus D_i)$ are isomorphic to $\mathbb{Z}^2$ (written with multiplicative notation as above). The group $\pi_1(\mathbb{B}_0\setminus(X\cup Y))$ is generated by commuting meridians of X and Y whose images are x_i and x_j. We can choose an element $t\in\pi_1(\mathbb{B}_p\setminus(D_i\cup D_j))$ which projects to a generator of A. With a suitable choice of t, we have $t^{\iota_p}=x_i x_j^k$ (k depends on the specific action and is coprime with ι_p). Hence (1.1) induces the following short exact sequence

$$0\to\langle x,y\mid[x,y]=1,x^{n_j}=x^{n_i}=1\rangle\to\pi_1^{\text{orb}}(X,\varphi)_p\to A\to 0$$

and the result follows. □

Remark 1.14. Note that if p is an orbifold point of index m, then $\pi_1^{\text{orb}}(\bar{X},\varphi)_p$ is cyclic of order m. If p is an ordinary double point of $\mathcal{D}$ belonging to two components D_i,D_j with $n_i,n_j>1$, then $\pi_1^{\text{orb}}(X,\varphi)_p$ is the product of two finite cyclic groups. As a consequence, if $(\bar{X},\varphi)$ is a normal crossing orbifold then it is in particular a locally finite orbifold.

Definition 1.15. A group G is said to be an *orbifold group* if it is isomorphic to $\pi_1^{\text{orb}}(\bar{X},\varphi;p_0)$ for some orbifold $(\bar{X},\varphi)$. If one can choose $(\bar{X},\varphi)$ to be such that $n_i>0$, $\forall i$, then we say that G is a *compact orbifold group.* If, moreover $(\bar{X},\varphi)$ is a locally finite (resp. normal crossing orbifold), we say that G is an LF (resp. NC) *compact orbifold group.*

Remark 1.16. Note that an orbifold group as defined below is also the fundamental group of an orbifold $(\bar{X},\varphi)$ where $\bar{X}$ is smooth.

Remark 1.17. We do not define the more general concept of LF *or* NC *orbifold groups* since they coincide immediately with the concept of *orbifold group* by the following fact. If we blow up a point in $p \in \mathcal{D}$, we obtain a new surface $\bar{Y}$ and a new divisor $\hat{\mathcal{D}}$ with $r+1$ irreducible components (the strict transforms of the components D_i, with the same notation, and the exceptional component D_{r+1}). We can define a map $\hat{\varphi}$ such that $\hat{\varphi}(D_i) = n_i$, $1 \leq i \leq r$, and $\hat{\varphi}(D_{r+1}) = 0$ and the orbifold fundamental group does not change. An iterated application of this procedure will give us a normal crossing divisor.

Proposition 1.18. *Let us consider in $\mathbb{P}^2$ the arrangement of lines $\mathcal{L}$ given by the equation $xyz(x^2-z^2)(y^2-z^2)$ and consider the orbifold structure $\varphi_{\mathcal{L}}$ given by assigning 2 to each line in $\mathcal{L}$. Let $G := \pi_1^{\text{orb}}(\mathbb{P}^2, \varphi_{\mathcal{L}})$. The meridians in G of the exceptional components of the blowing-ups of the quadruple points of $\mathcal{L}$ are of infinite order.*

Proof. It is easy to see that

$$G := \langle x_i, y_j, \gamma_z : x_i^2 = y_j^2 = \gamma_z^2 = [x_i, y_j] = 1, \gamma_z = (XY)^{-1} \rangle_{i,j=1,2,3},$$

where $X = x_1x_2x_3$, $Y = y_1y_2y_3$ and x_i (resp. y_j) $i, j = 1,2,3$ are meridians around the vertical (resp. horizontal) lines and γ_z is a meridian around the line at infinity $\{z=0\}$. Denote by γ_{E_x} (resp. γ_{E_y}) the meridian in G around the exceptional divisor E_x (resp. E_y) after blowing up the point $[0:1:0]$ (resp. $[1:0:0]$). Note that $\gamma_{E_x} = \gamma_z X = Y^{-1}$, $\gamma_{E_y} = \gamma_z Y = X^{-1}$. By symmetry, it is enough to show that X has infinite order in G or equivalently $c := X^2 \in G'$ has infinite order. Using Reidemeister-Schreier method it is easily seen that

$$G' = \langle a_i, b_j, c \mid \ [a_i, b_j] = 1, [a_1, a_2] = [b_1, b_2] = c^4, \ c \text{ central} \rangle_{i,j=1,2}. \tag{1.2}$$

It is straightforward that c has infinite order. □

Using the same ideas as in Proposition 1.4 we obtain the following result.

Proposition 1.19. *Any orbifold group is a quasi-projective group.*

Proof. As above, consider $(\bar{X}, \varphi)$ an orbifold for which $\mathcal{D} = D_1 \cup \cdots \cup D_r$. For each divisor $D_j \in S_\varphi^{>1}$ let ρ_j be the n_j-fold blow-up on D_j and denote by $\rho : \bar{Y} \to \bar{X}$ the composition of all of them. Let us denote by $E_{k,j}$, $1 \leq k \leq n_j$, $1 \leq j \leq r$ the k-th exceptional component of ρ_j. Define $Y := \bar{Y} \setminus \bigcup_{D_j \in S_\varphi^{>1}} \left(D_j \cup \bigcup_{k=1}^{n_j-1} E_{k,j} \right)$, where D_j here denotes the strict transform of D_j by ρ and similarly with $E_{k,j}$. Note that $\bar{Y}$ is the result of a finite process of blow-ups of a projective variety $\bar{X}$, hence Y is quasi-projective variety. Moreover, using Remark 1.6 it is straightforward to check that Y satisfies the required property $\pi_1(Y) \cong \pi_1^{\text{orb}}(\bar{X}, \varphi)$. □

In light of Remark 1.7 and Proposition 1.19, the following question arises:

Question 1.20. Is any compact orbifold group (or NC-compact orbifold group) a projective group?

A negative answer to the first part is provided by the ideas given in Example 2.5 and Proposition 1.18. This seems to suggest that NC-compact orbifolds are a reasonable class of orbifolds to work with for our purposes.

Proposition 1.21. *Compact orbifold groups are not necessarily* NC*-compact orbifold groups, and thus not projective groups.*

Proof. We are going to prove that the compact orbifold group G presented in Proposition 1.18 is not an NC-compact orbifold group. We will proceed by contradiction. Assume that G is an NC-compact orbifold group. Since the subgroup G' is of finite index, it is also an NC-compact orbifold group. The group G' is described as a central extension of $\mathbb{Z}^4$ by $\mathbb{Z}$ as it is deduced from the presentation (1.2). Since G' is torsion free, the group G' is in fact projective and thus Kähler. The group

$$H = \langle a_i, b_j, d \mid\ [a_i, b_j] = 1, [a_1, a_2] = [b_1, b_2] = d,\ d \text{ central}\rangle_{i,j=1,2}. \tag{1.3}$$

is an index-four subgroup of G' ($d = c^4$) and hence, it is also projective. Moreover, H is the Heisenberg group $H(2)$ (following the notation in [10]). This group is nilpotent, but not *almost abelian* (i.e. no finite-index subgroup is abelian). Since the rank of its abelianization is 4, it cannot be Kähler using [10, Corollary 4.5] (one can also use [7, Corollary 3.8] to obtain this statement). This contradicts the original asumption and thus G cannot be an NC-compact orbifold group and thus not a projective group. □

Remark 1.22. From another point of view, Proposition 1.18 implies that the local fundamental group $\pi_1^{\text{orb}}(\mathbb{P}^2, \varphi_{\mathcal{L}})_{[0:1:0]}$ is infinite and thus the orbifold $(\mathbb{P}^2, \varphi_{\mathcal{L}})$ has no uniformization in the sense of [24, Theorem 2.4].

Remark 1.23. Note that Propositions 1.19 and 1.21 partially answer questions posed by Simpson [22, §8].

2. Saturated orbifolds

Since we are mainly interested in orbifold groups it is sometimes useful to replace in $(\bar{X}, \varphi)$ the function φ by another function $\tilde{\varphi}$ where $\tilde{\varphi}(D_i)$ is defined by the actual order of μ_i in $\pi_1^{\text{orb}}(\bar{X}, \varphi; p_0)$; we may perform this operation only when $n_i > 0$ in order to have $X_\varphi^+ = X_{\tilde{\varphi}}^+$. This notion is somehow related with [20, Condition (1.3.3)].

Definition 2.1. Given an orbifold $(\bar{X}, \varphi)$ (for a fixed $\mathcal{D}$), we say that φ is a *saturated orbifold structure* if for any meridian μ_i of D_i (with $n_i > 0$), the order of μ_i in $\pi_1^{\text{orb}}(\bar{X}, \varphi; p_0)$ is exactly n_i.

There is a natural way to saturate an orbifold. Unless otherwise stated we will consider only saturated orbifolds in the sequel. Sometimes an extra saturation can be performed; even if $n_i = 0$, it may happen that μ_i is of finite order in $\pi_1^{\text{orb}}(\bar{X}, \varphi; p_0)$. Note that in that case if we define $\tilde{\varphi}(n_i)$ to be this order, then $X_\varphi^+ \subsetneq X_{\tilde{\varphi}}^+$.

We are going to study different kinds of saturation and their relationship with the concept of NC-orbifolds. Let $(\bar{X}, \varphi)$ be an orbifold; if $\mathcal{D}$ is not a normal crossing divisor there is a sequence $\pi : \bar{Y} \to \bar{X}$ of blowing-ups (which is an isomorphism outside $\mathring{X}$) such that $\pi^{-1}(\mathcal{D})$ becomes a normal crossing divisor. An orbifold structure ψ can be endowed to $\bar{Y}$ as in Remark 1.17, i.e. ψ vanishes on any exceptional component of π. This procedure does not change the orbifold fundamental group but in general X_φ^+ and Y_ψ^+ are not isomorphic; in particular, when $(\bar{X}, \varphi)$ is not NC, $(\bar{Y}, \psi)$ is not a compact orbifold even if $(\bar{X}, \varphi)$ is.

We are going to consider now a more general class of saturations where X_φ^+ may change without modifying $\pi_1^{\text{orb}}(\bar{X}, \varphi)$.

Definition 2.2. Let $(\bar{X}, \varphi)$ be an orbifold and let $p \in \bigcup S_\varphi^+$. Let $\pi : \bar{Y} \to \bar{X}$ be the blowing-up of p and keep the notation of Remark 1.17. We say that p is an LF-*point at first level* if the order of the meridian μ_{r+1} is finite in $\pi_1(\bar{X}, \varphi)_p$.

Let $\pi : \bar{Y} \to \bar{X}$ be the blowing-up of an LF-point at first level; let $\hat{\mathcal{D}} := \pi^{-1}(\mathcal{D})$; with the notation of Remark 1.17, we consider a saturation ψ such that $\psi(D_i) := n_i$, $1 \leq i \leq r$ and $\psi(D_{r+1})$ is the order of the meridian μ_{r+1} in $\pi_1^{\text{orb}}(\bar{X}, \varphi)_p$.

Definition 2.3. A point p is an LF-*point* if all of its infinitely near points are LF-points at first level (in particular, if an orbifold is locally finite at a point p then this point is an LF-point).

Example 2.4. By the very construction $\pi_1^{\text{orb}}(\bar{X},\varphi)\cong\pi_1^{\text{orb}}(\bar{Y},\psi)$. Hence if p is an LF-point we can obtain a sequence of blow-ups such that the divisor $\mathcal{D}$ becomes a normal crossing divisor *over* p and such that all the exceptional divisors have non-zero orbifold indices.

Example 2.5. If $p\in X^+_\varphi$ is an ordinary double point $p\in D_i\cap D_j$ of $\mathcal{D}$ then $\psi(D_{r+1})=\operatorname{lcm}(n_i,n_j)$; if $p\in D_i$ is a smooth point of $\mathcal{D}$ then $\psi(D_{r+1})=n_i$.

Examples 2.5 and 2.4 show that LF and NC compact orbifold fundamental groups are the same class of groups.

Example 2.6. Locally finiteness may not happen for more complicated singular points. As a simple example if $p\in\mathcal{D}$ is an ordinary triple singular point with orbifold indices for each branch $u,v,w\in\mathbb{N}$ such that $\frac{1}{u}+\frac{1}{v}+\frac{1}{w}\leq 1$, then p is not an LF-point. In the same way if p is an ordinary cusp and the orbifold index is ≥ 6 then p is not an LF-point.

We set the global version of Definitions 2.2 and 2.3.

Definition 2.7. Let $(\bar{X},\varphi)$ be an orbifold and let $p\in\bigcup S^+_\varphi$. Let $\pi:\bar{Y}\to\bar{X}$ be the blowing-up of p (keeping again the notation of Remark 1.17). We say that p is a *finite-type point at first level* if the order of the meridian μ_{r+1} is finite in $\pi_1(\bar{X},\varphi)$. A point p is a finite-type point if all of its infinitely near points are finite-type points at first level.

Remark 2.8. Let us consider $\bar{X}=\mathbb{P}^2$, $\mathcal{D}$ the union of three lines through a point p with orbifold indices $u,v,w\in\mathbb{N}$ such that $\frac{1}{u}+\frac{1}{v}+\frac{1}{w}\leq 1$. It is clear that p is not an LF-point at first level and it is easy to see that it is a finite-type point, since the meridian around the exceptional component is in fact trivial. The quadruple points of the arrangement in Proposition 1.18 are not of finite type. Hence the classes of *compact orbifold* and NC-*compact orbifold* groups do not coincide.

Let us start from a saturated orbifold structure. Hence, if all the points of $\mathcal{D}$ (it is enough to check it for singular points of $\mathcal{D}$ worse than nodal points) are LF-points (or finite-type point) we can replace $(\bar{X},\varphi)$ by an NC-orbifold structure in a surface after successive blowing-ups without changing the fundamental group. In the first case we call this structure *locally saturated*; in the second case it is called *globally saturated*. Moreover, this can be done respecting the compactness.

We finish this section with a new saturation procedure which modifies $\pi_1^{\text{orb}}(\bar{X},\varphi)$. An interesting object of study associated with $\pi_1^{\text{orb}}(\bar{X},\varphi)$ is the set of its characteristic varieties, see §3, which is a stratification of the space of characters. Since $H_1^{\text{orb}}(\bar{X},\varphi;\mathbb{Z})_p$ is generated by the meridians of the components of $\mathcal{D}$ passing through p, we can associate to D_{r+1} the order of μ_{r+1} in $H_1^{\text{orb}}(\bar{X},\varphi;\mathbb{Z})_p$ (or in $H_1^{\text{orb}}(\bar{X},\varphi;\mathbb{Z})$). The orbifold structure is called *locally homologically saturated* or *globally homologically saturated*.

Example 2.9. If we consider an ordinary triple point where all the components have index 2, the local homological saturation is given by assigning 2 to the exceptional component. It is easily seen that the local saturation assigns index 4.

3. Orbifolds and characteristic varieties

The relationship between orbifolds and characteristic varieties (or similar invariants) appear implicitly in the works of Beauville [6] and Arapura [1] and explicitly in the works of Campana, e.g. [9], Simpson-Corlette [12], Delzant [13] and ourselves [4], among others. Except in Campana's work, the relationship comes from the following fact: given a smooth variety (projective, quasi-projective or Kähler) the positive-dimensional components of the characteristic varieties can be obtained as pull-back by mappings whose targets are orbifolds. Campana's work focuses on the study of characteristic varieties of compact Kähler orbifolds (more precisely, NC-projective orbifolds in the language of §1). In this section we will study the characteristic varieties of quasi-projective orbifolds. For a detailed exposition of the concept of characteristic varieties (or Green-Lazarsfeld invariant), the reader can check any of the above references. Some definitions will also be given in §5.

Before we state the aforementioned results in the context and language of orbifolds we need to recall the concept of orbifold morphism, which as it occurs in the classical case, allows one to define a morphism of fundamental groups.

Definition 3.1. Let $(\bar{X},\varphi),(\bar{Y},\psi)$ be orbifolds with divisors $\mathcal{D}:=\bigcup_{y=1}^{r} D_j\subset\bar{X}$, $n_j:=\varphi(D_j)$, $\mathcal{E}:=\bigcup_{k=1}^{s} E_j\subset\bar{Y}$, $m_k:=\psi(E_k)$. A dominant holomorphic map $\Phi: X^+_\varphi\to Y^+_\psi$ defines an *orbifold map* $\Phi^{\text{orb}}:(\bar{X},\varphi)\to(\bar{Y},\psi)$ if for each $k\in\{1,\dots,s\}$, the divisor $\Phi^*(E_k)$ can be written as $\sum_{j=1}^{r} h_{j,k}D_j+m_kH_k$ where m_k divides n_jh_{jk} and H_k is a divisor in X^+_φ.

Proposition 3.2 ([11, 3])**.** *Let $\Phi^{\text{orb}}:(\bar{X},\varphi)\to(\bar{Y},\psi)$ be an orbifold map. This map induces (in a functorial way) a morphism $\Phi^{\text{orb}}_*:\pi_1^{\text{orb}}(\bar{X},\varphi)\to\pi_1^{\text{orb}}(\bar{Y},\psi)$. Moreover, if $(\bar{Y},\psi)$ is an orbicurve and the generic fiber of Φ^{orb} is irreducible then Φ^{orb}_* is surjective.*

There are two main examples of orbifold morphisms: either the target is an orbicurve or the orbifolds have the same dimension. The last case (étale or branched covers) is specially interesting when all the fibers are finite.

Let us compare the following results. We use the language of §1 if needed. The natural definition of $\mathcal{V}^{\text{orb}}_k$ (which is the orbifold analogue of $\mathcal{V}_k$, the k-th characteristic variety) can be found in §5.

Theorem 3.3 ([4, Theorem 1])**.** *Let X be a smooth quasi-projective variety and let $\mathcal{V}_k(X)$ be the k-th characteristic variety of X. Let V be an irreducible component of $\mathcal{V}_k(X)$. Then one of the two following statements holds:*

(1) *There exists an orbicurve $(\bar{C},\psi)$, an orbifold morphism $\Phi^{\text{orb}}:X\to(\bar{C},\psi)$ and an irreducible component W of $\mathcal{V}^{\text{orb}}_k(\bar{C},\psi)$ such that $V=(\Phi^{\text{orb}})^*(W)$.*
(2) *V is an isolated torsion point not of type* (1).

Theorem 3.4 ([8, Théorème 3.1])**.** *Let $(\bar{X},\varphi)$ be an* NC*-compact Kähler orbifold surface. Let V be an irreducible component of $\mathcal{V}^{\text{orb}}_k(\bar{X},\varphi)$. Then, one of the following statements holds:*

(1) V is an isolated torsion point.
(2) There exists a compact hyperbolic orbicurve $(\bar{C},\psi)$, where the genus of $\bar{C}$ is at least 1*, an orbifold map $\Phi^{\text{orb}}:(\bar{X},\varphi)\to(\bar{C},\psi)$ and an irreducible component W of $\mathcal{V}^{\text{orb}}_k(\bar{C},\psi)$ such that $V=(\Phi^{\text{orb}})^*(W)$.*

The goal of this section is to state and prove a combination of the above theorems.

Theorem 3.5. *Let $(\bar{X},\varphi)$ be an* NC*-quasi-projective orbifold surface. Let V be an irreducible component of $\mathcal{V}^{\text{orb}}_k(\bar{X},\varphi)$. Then, one of the following statements holds:*

(1) *There exists an orbicurve $(\bar{C},\psi)$, an orbifold map $\Phi^{\text{orb}}:(\bar{X},\varphi)\to(\bar{C},\psi)$ and an irreducible component W of $\mathcal{V}^{\text{orb}}_k(\bar{C},\psi)$ such that $V=(\Phi^{\text{orb}})^*(W)$.*
(2) *V is an isolated torsion point.*

Proof. Let $(\bar{X},\varphi)$ be an NC-quasi-projective orbifold surface. Let $\mathcal{D}$ be the hypersurface defining the orbifold structure where we assume that $\mathcal{D}=\bigcup_{j=1}^{r+s} D_j$, where $n_j\geq 2$ if $1\leq j\leq r$ and $n_{r+k}=0$ if $1\leq k\leq s$. We may assume the orbifold structure is saturated.

We proceed as in the proof of Proposition 1.19. Let $\pi:\bar{Y}\to\bar{X}$ the composition of the $\sum_{j=1}^{r} n_j$ blow-ups indicated in that proof. We denote by D_i the strict transforms of D_i and by $E_{k,j}$,

$1 \leq k \leq n_j$, $1 \leq j \leq r$, the exceptional components of π. Let $Y := \bar{Y} \setminus \bigcup_{j=1}^{r} \left(D_j \cup \bigcup_{k=1}^{n_j-1} E_{k,j} \right)$. Recall that $\pi_1^{\text{orb}}(\bar{X}, \varphi) \cong \pi_1(Y)$.

We can apply Theorem 3.3 to Y. Let us consider a component V of $\mathcal{V}_k(Y)$ of type (1) and consider the orbifold map given in the statement. Let us write this orbifold map in the language of §1. We consider in $\bar{Y}$ the hypersurface

$$\hat{\mathcal{D}} = \bigcup_{j=1}^{r} \left(D_j \cup \bigcup_{k=1}^{n_j} E_{k,j} \right) \cup \bigcup_{\ell=1}^{s} D_{r+\ell},$$

and the map $\hat{\varphi}$ given by:

$$\underset{1 \leq j \leq r+s}{\hat{\varphi}(D_j) = 0}, \qquad \underset{1 \leq k < n_j, 1 \leq j \leq r}{\hat{\varphi}(E_{k,j}) = 0}, \quad \underset{1 \leq j \leq r}{\hat{\varphi}(E_{n_j,j}) = 1}.$$

Since $Y^+_{\hat{\varphi}} = Y$, the map given by Theorem 3.3 can be written as $\hat{\Phi}^{\text{orb}} : (\bar{Y}, \hat{\varphi}) \to (\bar{C}, \psi)$. Let us consider $\hat{\Phi} : Y \to \bar{C}$ the underlying dominant holomorphic mapping.

Note that $\check{E}_j := E_{n_j,j} \cap Y$ is isomorphic to $\mathbb{C}^*$. Let us assume that $\hat{\Phi}_{\check{E}_j}$ is not constant and hence dominant on $\bar{C}$; in particular, it determines an orbifold morphism $\hat{\Phi}^{\text{orb}} : (E_{n_j,j}, \varphi_j) \to (\bar{C}, \psi)$ where φ_j is the induced orbifold structure, which is the trivial one. The only possible choices for $(\bar{C}, \psi)$ are either $\mathbb{C}^*$ (with smooth structure) or $\mathbb{C}_{2,2}$; the characteristic varieties of these orbifolds are finite and we are led to a contradiction.

Then, we have proven that $\hat{\Phi}_{\check{E}_j}$ is constant and denote by $p_j \in \bar{C}$ its image. Let us consider a small neighborhood $\mathcal{U}_j$ of $\bigcup_{k=1}^{n_j-1} E_{k,j}$; this curve is a linear chain of rational smooth curves with self-intersection -2 and the space $\tilde{\mathcal{U}}_j$ obtained from $\mathcal{U}_j$ by contracting the curves is isomorphic to the quotient of a neighborhood $\tilde{\mathcal{U}}_j$ of the origin in $\mathbb{C}^2$ by the action of a cyclic group of order k_j. We may lift $\hat{\Phi}$ to a dominant morphism $\hat{\Phi}_j : \tilde{\mathcal{U}}_j \setminus \{0\} \to \bar{C}$; it is easily seen that if $\hat{\Phi}_j$ cannot be extended to the origin, then $\bar{C} \cong \mathbb{P}^1$ and the characteristic varieties of $(\bar{C}, \psi)$ are finite. Since this is not possible, $\hat{\Phi}_j$ can be extended and $\hat{\Phi}$ can be extended to $\bigcup_{k=1}^{n_j-1} E_{k,j}$ by sending the curve to p_j.

A similar argument allows us to extend $\hat{\Phi}$ to the regular part of $\hat{\mathcal{D}}$ in D_j; moreover it is also possible to extend it to $D_j \cap E_{n_j,j}$ (with image p_j). Finally we can extend it to the double points $D_i \cap D_j$, $1 \leq i < j \leq r$. Moreover, since this map is constant on $\bigcup_{k=1}^{n_j} E_{k,j}$, we can contract these divisors (the exceptional divisors of π) and we obtain a holomorphic map $\Phi : X^+_\varphi \to \bar{C}$.

All we are left to do is to check that Φ defines the required orbifold morphism. Before we prove this, note that Φ_* induces a morphism of orbifold fundamental groups. To see this, let μ_j be a meridian around D_j; note that $\mu_j^{n_j}$ is a meridian of $E_{n_j,j}$ whose image by Φ_* is trivial and hence the map induces a morphism of the orbifold fundamental groups.

Let us assume that D_j is contained in the preimage of p_j and let us compute its multiplicity in $\Phi^*(p_j)$, say a_j. If we compose Φ and π the multiplicity of $E_{k,j}$ in the divisor defined by p_j equals ka_j. Let b_j the multiplicity of p_j by ψ; the condition of Definition 3.1 for orbifold morphism implies that $n_j a_j$ divides b_j which is exactly the needed condition for Φ. Hence, the required $\Phi^{\text{orb}} : (\bar{X}, \varphi) \to (\bar{C}, \psi)$ is constructed. □

4. Unbranched and branched orbifold covers

One of the advantages of using orbifold fundamental groups is that we can study standard ramified covers as unbranched orbifold covers. For technical reasons, we restrict our attention to NC-orbifolds.

Definition 4.1. We call an orbifold morphism $\pi : (\bar{Y}, \varphi_Y) \to (\bar{X}, \varphi_X)$ an *orbifold unbranched covering* if the fibers of π are finite and the following equality holds

$$\nu_{(\bar{Y},\varphi_Y)}(y) \cdot \deg \pi_y = \nu_{(\bar{X},\varphi_X)}(x)$$

$\forall x \in X^+_\varphi$, $\forall y \in \pi^{-1}(x)$ (see Definition 1.11).

Remark 4.2. For the shake of simplicity we will often refer to *orbifold unbranched covering* as *unbranched covering*. Note that usual *unbranched covering* are actually *orbifold unbranched covering*.

The main point in Definition 4.1 is that *orbifold unbranched coverings* behave for orbifold fundamental groups as unbranched coverings behave for fundamental groups. In particular, the monodromy action completely determines the orbifold unbranched coverings.

Proposition 4.3. *An orbifold unbranched covering induces an injective morphism on orbifold fundamental groups. Moreover, let $(\bar{X}, \varphi_X)$ be an orbifold and let us denote $G := \pi_1^{\text{orb}}(\bar{X}, \varphi_X)$. Let $H \subset G$ be a finite-index subgroup; then there is an orbifold unbranched covering $\pi : (\bar{Y}, \varphi_Y) \to (\bar{X}, \varphi_X)$ such that $\pi_*(\pi_1^{\text{orb}}(\bar{Y}, \varphi_Y)) = H$. Moreover, $(\bar{Y}, \varphi_Y)$ is essentially unique (i.e., both $Y^+_{\varphi_Y}$ and $S^+_{\varphi_Y}$ are unique up to isomorphism).*

As in the standard case, the cover is said to be *regular* or *Galois* if $H \trianglelefteq G$; in that case the group G/H acts on $Y^+_{\varphi_Y}$ with quotient $X^+_{\varphi_X}$.

Proposition 4.4. *An orbifold unbranched covering satisfies both the path and homotopy lifting properties and are determined by the monodromy representation $\rho : \pi_1^{\text{orb}}(X, \varphi_X) \to \Sigma_n$, $n := \#G/H$.*

A proof of these results can be found basically rewriting [20, Theorem 1.3.9] in the language of orbifolds instead of in the language of branched coverings.

Example 4.5. Consider $\mathbb{P}^1_{2,3,5}$ and the subgroup of $G := \pi_1^{\text{orb}}(\mathbb{P}^1_{2,3,5}) = \langle \mu_2, \mu_3, \mu_5 : \mu_2^2 = \mu_3^3 = \mu_5^5 = (\mu_2\mu_3\mu_5) = 1, \rangle$ given by the kernel of

$$\begin{array}{rccl} \rho : & G & \to & \Sigma_5 \\ & \mu_2 & \mapsto & (1,5)(2,3) \\ & \mu_3 & \mapsto & (1,4,3) \\ & \mu_5 & \mapsto & (1,2,3,4,5). \end{array} \tag{4.1}$$

Note that the preimage of the orbifold point of order 2 has three points. For two of them, the local degree of the map is 2 (and hence their index is 1) whereas on the remaining point the local degree of ρ is 1 (and hence it should become a point of index 2). Analogously, around the orbifold point of order 3, the preimage has three points: two of which will have orbifold index 3 and one with orbifold index 1. Finally, around the orbifold point of index 5, the preimage is a local uniformization. Hence the local conditions on the orbifold points of the covering are given to satisfy Definition 4.1. A simple Euler characteristic computation shows that ρ induces in fact a (non-regular) unbranched covering from $\mathbb{P}^1_{2,3,3}$ to $\mathbb{P}^1_{2,3,5}$ of order 5.

Example 4.6. Consider the following morphism:

$$\begin{array}{rccl} \pi : & \mathbb{P}^1 & \to & \mathbb{P}^1 \\ & [x:y] & \mapsto & [(x^3-y^3)^2 : (x^2+y^2)^3] \end{array} \tag{4.2}$$

Generically, fibers have 6 different preimages. The special fibers are at $[1:0]$ (the roots of $(x^2+y^2)^3$), $[0:1]$ (the roots of $(x^3-y^3)^2$), $[1:1]$ (the roots of $y^2x^2(2xy+3x^2+3y^2)$), and $[2:1]$ (the roots of $(x^4-2yx^3-2xy^3+y^4)(y+x)^2$). Therefore this induces a non-regular unbranched covering from $\mathbb{P}^1_{6(2),2(3)}$ to $\mathbb{P}^1_{3(2),3}$ of order 6 (where the subindex $k(m)$ stands for k points of index m).

Definition 4.7. An orbifold unbranched cover $\pi : (\bar{Y}, \varphi_Y) \to (\bar{X}, \varphi_X)$ is a *uniformization* of $(\bar{X}, \varphi_X)$ if $\bar{Y}$ does not contain points of orbifold index greater than 1. The uniformization will be *Galois* or *regular* if π realizes a quotient of Y^+_φ by the action of a finite group (which may not act freely).

Remark 4.8. As in the standard case, a uniformization (or more generally an unbranched cover) is Galois if and only if the image G_Y of $\pi_1^{\text{orb}}(\bar{Y}, \varphi_Y)$ in $G_X := \pi_1^{\text{orb}}(\bar{X}, \varphi_X)$ is a normal subgroup (the group action is carried by G_X/G_Y). Recall that if π is a finite Galois uniformization, then the image of a meridian of a component with orbifold index $n_i > 1$ by the monodromy action is a product of cycles of the same length n_i (with no fixed points). This is not a characterization of Galois uniformization as Example 4.9 shows. This condition is called *virtual regularity* in [19].

Also note that saturation (see Definition 2.1) is trivially a necessary condition for the existence of a uniformization.

Example 4.9. Let us consider an orbicurve $(\bar{C}, \varphi)$ where $\bar{C}$ is an elliptic curve and the divisor contains two points of index 2. Recall that

$$\pi_1^{\text{orb}}(\bar{C}, \varphi) = \langle a, b, x, y \mid a^2 = b^2 = 1, ab = [x, y] \rangle$$

Consider the morphism:

$$\begin{array}{rrcl} \rho : & \pi_1^{\text{orb}}(\bar{C}, \varphi) & \to & \Sigma_4 \\ & a, x & \mapsto & (1,2)(3,4) \\ & b & \mapsto & (1,3)(2,4) \\ & y & \mapsto & (1,2,3). \end{array} \tag{4.3}$$

This morphism defines an unbranched orbifold cover; using Riemann-Hurwitz formula the source of this cover is a Riemann surface of genus 3 (with no point of orbifold index greater than 1). This is an example of a uniformization which is virtually regular, but not regular.

For our purposes, a more global and regular definition of unbranched covering will be enough.

Definition 4.10 ([17, 24])**.** Let $(\bar{X}, \varphi)$ be an orbifold. We say $(\bar{X}, \varphi')$ is a *suborbifold* of $(\bar{X}, \varphi)$ (or equivalently $(\bar{X}, \varphi)$ is a *superorbifold* of $(\bar{X}, \varphi')$) if $\varphi'(D_i)|\varphi(D_i)$ (meaning there exists $k \in \mathbb{Z} \setminus \{0\}$ such that $\varphi(D_i) = k\varphi'(D_i)$ in particular, if $\varphi(D_i) = 0$, then $\varphi'(D_i) = 0$).

On the other side branched orbifold coverings can also be defined. The definitions will be straightforward for the orbicurve case.

Definition 4.11. A Galois covering $\pi : \bar{Y} \to \bar{X}$ between two orbifolds $(\bar{X}, \varphi_X)$ and $(\bar{Y}, \varphi_Y)$ is a *branched orbifold covering* if there exists a superorbifold structure $(\bar{X}, \varphi_s)$ for which π defines an unbranched orbifold covering.

5. Sakuma's formulæ

Given an orbifold $(\bar{X}, \varphi_X)$, we will define $b_1^{\text{orb}}(\bar{X}, \varphi_X)$ as the rank of the abelianization of $G_{\varphi_X} := \pi_1^{\text{orb}}(\bar{X}, \varphi_X)$, that is, $\operatorname{rank}(G_{\varphi_X}/G'_{\varphi_X})$. After taking a superorbifold, all branched orbifold coverings can be assumed to be unbranched. Consider $\pi : (\bar{Y}, \varphi_Y) \to (\bar{X}, \varphi_X)$ an unbranched covering.

Note that, any unbranched covering $\pi : (\bar{Y}, \varphi_Y) \to (\bar{X}, \varphi_X)$ produces the action of the group of deck transformations G_φ over $H_1^{\text{orb}}(\bar{Y}, \varphi_Y)$ by conjugation, that is, consider $\bar{g} \in G_\varphi$ the class of $g \in G_{\varphi_X}$ and $\bar{x} \in H_1^{\text{orb}}(\bar{Y}, \varphi_Y)$ the class of $x \in G_{\varphi_Y}$, then $\bar{g} \cdot \bar{x} = \overline{gxg^{-1}}$. Since

$\overline{ghaxh^{-1}g^{-1}} = \overline{g[h,ax]ag^{-1}gxg^{-1}} = \overline{gxg^{-1}}$, the action is well defined. This action endows $H_1^{\text{orb}}(\bar{Y},\varphi_Y)$ with a module structure over the group ring $\mathbb{Z}[G_\varphi]$. After tensoring by $\mathbb{C}$, the group $H_1^{\text{orb}}(\bar{Y},\varphi_Y)$ acquires a $\mathbb{C}[G_\varphi]$-module structure.

Recall the definition of the characteristic variety of a finitely presented group G. Consider a free resolution of a $\mathbb{C}[H_1(G)]$-module M

$$\mathbb{C}[H_1(G)]^m \xrightarrow{\phi} \mathbb{C}[H_1(G)]^n \to M,$$

then $\mathcal{V}_k(M) := V(F_k(M))$, where $F_k(M)$ is the k-th Fitting ideal (or elementary ideal) of M and $V(I)$ denotes the zero set of the ideal I. Recall that F_k is defined as 0 if $k \leq \max\{0, n-m\}$, 1 if $k > n$. Otherwise F_k is the set of minors of order $(n-k+1)\times(n-k+1)$ of a presentation matrix A_ϕ, which is an $n\times m$ matrix with coefficients in $\mathbb{C}[H_1(G)]$. Note that $\mathcal{V}_{k+1}(M) \subseteq \mathcal{V}_k(M)$ and $\mathcal{V}_{n+1}(M) = \emptyset$. For any $\xi \in \mathbb{C}[H_1(G)]$, it is common to define as $\operatorname{null}(M,\xi)$ (nullity of ξ) or $d_\xi(M)$ (depth of ξ) as the maximum $k \in \mathbb{Z}$ such that $\xi \in \mathcal{V}_k(M)$.

We will denote by $\mathcal{V}_k^{\text{orb}}(\bar{X},\varphi_X)$ and $\operatorname{null}^{\text{orb}}(\bar{X},\varphi_X)$, the invariants of the $\mathbb{C}[H_1(G)]$-module M described above where G is the orbifold fundamental group $G := \pi_1^{\text{orb}}(\bar{X},\varphi_X)$.

Unless otherwise stated, all groups orbifold homology groups H_1^{orb} will be considered as $\mathbb{C}[G_\varphi]$-modules. Sakuma's formulæ [21, Theorem 7.3] (see also [15, Proposition 2.5.6]) can be combined and extended in the following result.

Theorem 5.1. *Under the above conditions, if $\pi : (\bar{Y},\varphi_Y) \to (\bar{X},\varphi_X)$ is a*

$$b_1^{\text{orb}}(\bar{Y},\varphi_Y) = b_1^{\text{orb}}(\bar{X},\varphi_X) + \sum_{\xi\in\operatorname{Hom}(G_\varphi,\mathbb{C}^*)\setminus\{1\}} \operatorname{null}^{\text{orb}}(\bar{X},\xi) \tag{5.1}$$

where $G_\varphi := G_{\varphi_X}/G_{\varphi_Y}$, $\operatorname{null}^{\text{orb}}(\bar{X},\xi)$ is the depth of ξ considered as a character in $\pi_1^{\text{orb}}(\bar{X},\varphi_X)$.

Remark 5.2. Note that there is a connection between b_1^{orb} and b_1, namely

$$b_1^{\text{orb}}(\bar{X},\varphi_X) = b_1(X^+_{\varphi_X}) = b_1(\bar{X}\setminus\varphi_X^{-1}(0)).$$

Proof of Theorem 5.1. The proof offered in [21] also works in this context. We will briefly outline the original proof.

Step 1. From representation theory one has

$$H_1^{\text{orb}}(\bar{Y},\varphi_Y) \cong \bigoplus_{\xi\in\operatorname{Hom}(G_\varphi,\mathbb{C}^*)} \left[H_1^{\text{orb}}(\bar{Y},\varphi_Y)\right]_\xi,$$

where

$$\left[H_1^{\text{orb}}(\bar{Y},\varphi_Y)\right]_\xi = \{x\in H_1^{\text{orb}}(\bar{Y},\varphi_Y) \mid g(x) = \xi(g)\cdot x \ \forall g\in G_\varphi\}.$$

Step 2. Using Proposition 4.3 there exists an orbifold $(\bar{X}_\xi,\varphi_\xi)$ such that

$$\left[H_1^{\text{orb}}(\bar{Y},\varphi_Y)\right]_\xi \cong \left[H_1^{\text{orb}}(\bar{X}_\xi,\varphi_\xi)\right]_\xi.$$

The orbifold $(\bar{X}_\xi,\varphi_\xi)$ corresponds to the kernel of $G_{\varphi_X} \to G_\varphi \xrightarrow{\xi} \mathbb{C}^*$, which is of finite index in G_{φ_X}.

Step 3. For $\xi \neq 1$ one has

$$\dim_{\mathbb{C}}\left[H_1^{\text{orb}}(\bar{X}_\xi,\varphi_\xi)\right]_\xi = \operatorname{null}^{\text{orb}}(\bar{X},\xi). \qquad \square$$

Remark 5.3. Note that even if $(\bar{X},\varphi_X)$ is not an NC-orbifold, the notion of unbranched covers may easily be defined as long as orbifolds with (normal) arbitrary singularities are allowed. In that case we may also consider the orbifold $(\hat{X},\hat{\varphi}_X)$ obtained after a sequence of blow-ups such that the transform of $\mathcal{D}$ by this sequence of blow-ups is a normal crossing divisor and $\hat{\varphi}_X$ is defined by homological saturation. The pull-back of π defines another orbifold $(\hat{Y},\hat{\varphi}_Y)$. Note that $\hat{Y} = \hat{Y}^+_\varphi$ and it has only abelian quotient singularities; it is a resolution of $\bar{Y}$ which may have more complicated singularities. There is a natural surjection $\pi_1^{\text{orb}}(\hat{Y}, \hat{\varphi}_Y) \twoheadrightarrow \pi_1^{\text{orb}}(\bar{Y}, \varphi_Y)$ which is not in general an isomorphism. Nevertheless, generalizing Libgober's arguments in [18], it can be proved that the first Betti numbers coincide.

To illustrate Theorem 5.1, we can compute the genus of the uniformization of $\bar{X} := \mathbb{P}^1_{d_1,\dots,d_{n+1}}$ in some cases where for instance the abelianization map $\pi : (\bar{X}_{\text{ab}}, \varphi_{\text{ab}}) \to (\bar{X}, \varphi_X)$ is a uniformization. According to [20, Theorem 1.3.43] this is the case whenever d_i divides

$$\operatorname{lcm}(d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_{n+1}).$$

On the one hand one can directly use the Riemann-Hurwitz formula to obtain

$$\chi(\bar{X}_{\text{ab}}) = 2 - 2g(\bar{X}_{\text{ab}}) = (1-n)\frac{d_1 \cdot \dots \cdot d_{n+1}}{d} + \sum_{k=1}^{n+1} \frac{d_1 \cdot \dots \cdot d_{n+1}}{d d_k},$$

where $d := \operatorname{lcm}(d_1, \dots, d_{n+1})$. This implies

$$g(\bar{X}_{\text{ab}}) = \frac{d_1 \cdot \dots \cdot d_{n+1}}{2d} \left[-1 + \sum_{k=1}^{n+1} \left(1 - \frac{1}{d_k}\right) \right] + 1. \tag{5.2}$$

Example 5.4. Consider the case $1 \leq d_1 \leq \dots \leq d_n \leq d_{n+1} = d = \operatorname{lcm}(d_1, \dots, d_n)$. This is a particular case of the result mentioned above and hence the universal abelian covering $\pi : \bar{X}_{\text{ab}} \to \bar{X} := \mathbb{P}^1_{d_1,\dots,d_{n+1}}$ is in fact a uniformization. Using Theorem 5.1 one can obtain $b_1(\bar{X}_{\text{ab}}) = b_1^{\text{orb}}(\bar{X}_{\text{ab}})$ by counting the characters in the orbifold characteristic variety of $\bar{X}$. Note that the space of characters on $\pi_1^{\text{orb}}(\bar{X})$ is a union of $(n+1)$-tuples

$$\mathbb{T} := \{(\xi_1, \dots, \xi_n, \xi_{n+1}) \mid \xi_j \in \mu_{d_j}, \prod_{k=1}^{n+1} \xi_k = 1\} \subset (\mathbb{C}^*)^{n+1},$$

where $\mu_n \subset \mathbb{C}^*$ is the subgroup of n-th roots of unity. Since the equation in the definition of $\mathbb{T}$ can always be solved for ξ_{n+1} one has that $\mathbb{T} \cong \mu_{d_1} \times \dots \times \mu_{d_n}$. Denote by $\ell(\xi)$ the length of $\xi \in \mathbb{T}$, that is, the number of non-trivial coordinates of ξ. From [4, Proposition 3.11] one deduces that $\operatorname{depth}(\xi) = \ell(\xi) - 2$. Denote by $\ell'(\xi)$ the length of ξ in the first n coordinates, that is, its length as an element of $\mu_{d_1} \times \dots \times \mu_{d_n}$. Note that $\ell(\xi) = \ell'(\xi) + 1$ unless its last coordinate is 1, in which case $\ell(\xi) = \ell'(\xi)$. Therefore if we define

$$b_1'(\bar{X}_{\text{ab}}) := \sum_{\xi \in \mathbb{T}} \ell'(\xi).$$

Then $b_1'(\bar{X}_{\text{ab}}) - b_1(\bar{X}_{\text{ab}}) = \frac{D}{d}$, where $D := d_1 \cdot \dots \cdot d_n$ which is the order of the kernel of the map $\mu_{d_1} \times \dots \times \mu_{d_n} \to \mu_d$ given by multiplication. Hence,

$$b_1'(\bar{X}_{\text{ab}}) = \sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} (\#I - 1) \prod_{i \in I} (d_i - 1)$$

and thus,

$$b_1(\bar{X}_{\text{ab}}) = \left[\sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} (\#I - 1) \prod_{i \in I} (d_i - 1) \right] - \frac{D}{d}.$$

Using (5.2) this implies

$$\frac{D}{d^2} \left[-1 + \sum_k \left(1 - \frac{1}{d_k}\right) \right] + 2 = \left[\sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} (\#I - 1) \prod_{i \in I} (d_i - 1) \right] - \frac{D}{d}.$$

For computational purposes, the depth $\text{null}^{\text{orb}}(\bar{X},\xi)$ can also be obtained from $\mathring{X}_\varphi$.

Proposition 5.5. *Under the above conditions,*

$$\mathcal{V}_k^{\text{orb}}(X,\varphi)\setminus\{1\}=\mathcal{V}_k(\mathring{X}_\varphi)\cap\mathbb{T}_\varphi\setminus\{1\},$$

where $\mathbb{T}_\varphi$ *is the inclusion of* $\mathbb{T}(X,\varphi)$ *into* $\mathbb{T}(\mathring{X}_\varphi)$ *given by the surjection* $\pi_1(\mathring{X}_\varphi)\twoheadrightarrow\pi_1^{\text{orb}}(X,\varphi)$.

Proof. The proof is analogous to the one shown in [5, Proposition 2.26] for $k=1$. □

Example 5.6. Consider the space $\mathcal{M}$ of sextics with the following combinatorics:

(1) $\mathcal{C}$ is a union of a smooth conic $\mathcal{C}_2$ and a quartic $\mathcal{C}_4$.
(2) $\text{Sing}(\mathcal{C}_4)=\{P,Q\}$ where Q is a cusp of type $\mathbb{A}_4$ and P is a node of type $\mathbb{A}_1$.
(3) $\mathcal{C}_2\cap\mathcal{C}_4=\{Q,R\}$ where Q is a $\mathbb{D}_7$ on $\mathcal{C}$ and R is a $\mathbb{A}_{11}$ on $\mathcal{C}$.

The space $\mathcal{M}=\mathcal{M}^{(1)}\cup\mathcal{M}^{(2)}$ is a union of two connected components. Any such sextics $\mathcal{C}_6^{(i)}=\mathcal{C}_2^{(i)}\cup\mathcal{C}_4^{(i)}$ in $\mathcal{M}^{(i)}$ can be characterized by the fact that the conic q passing through R and Q such that $\text{mult}_R(q,\mathcal{C}_2^{(i)})=\text{mult}_R(q,\mathcal{C}_4^{(i)})=3$, and $\text{mult}_Q(q,\mathcal{C}_2^{(i)})=1$ satisfies $\text{mult}_Q(q,\mathcal{C}_4^{(i)})=3+i$. The following example is presented in [2], we refer to it for details. Consider the orbifolds $(\mathbb{P}^2,\varphi_i)$, where $\varphi_i(\mathcal{C}_4^{(i)})=0$ and $\varphi_i(\mathcal{C}_2^{(i)})=2$. Using Proposition 5.5 and [5, Proposition 3.1] it can be checked that

$$\mathcal{V}_1^{\text{orb}}(\mathbb{P}^2,\varphi_i)\setminus\{1\}=\begin{cases}\emptyset & \text{if } i=1\\ \{(1,-1)\} & \text{if } i=2,\end{cases}\quad \mathcal{V}_2^{\text{orb}}(\mathbb{P}^2,\varphi_i)\setminus\{1\}=\emptyset$$

and hence, using Sakuma's formula 5.1 one has

$$b_1^{\text{orb}}(Y_i,\varphi_{Y_i})=\begin{cases}0 & \text{if } i=1\\ 1 & \text{if } i=2,\end{cases}$$

where (Y_i,φ_{Y_i}) denotes the unramified covering of $(\mathbb{P}^2,\varphi_i)$, since $b_1^{\text{orb}}(\mathbb{P}^2,\varphi_i)=0$. This provides an alternative way to show that $\mathcal{C}_6^{(1)}$ and $\mathcal{C}_6^{(2)}$ form a Zariski pair, that is, two curves with the same combinatorics but different embedding in $\mathbb{P}^2$. In other words, we prove that $(\mathbb{P}^2,\mathcal{C}_6^{(1)})$ and $(\mathbb{P}^2,\mathcal{C}_6^{(2)})$ are not homeomorphic by showing that $\pi_1^{\text{orb}}(\mathbb{P}^2,\varphi_1)$ and $\pi_1^{\text{orb}}(\mathbb{P}^2,\varphi_2)$ are not isomorphic. Note that any homeomorphism of $\mathbb{P}^2$ sending $\mathcal{C}_6^{(1)}$ to $\mathcal{C}_6^{(2)}$ should send a meridian around $\mathcal{C}_2^{(1)}$ to a meridian around $\mathcal{C}_2^{(2)}$ and hence $\pi_1^{\text{orb}}(\mathbb{P}^2,\varphi_1)$ and $\pi_1^{\text{orb}}(\mathbb{P}^2,\varphi_2)$ should be isomorphic.

We can readily recover, using Theorem 5.1 and Proposition 5.5, known computations of the first Betti number of Hirzebruch congruence covers associated to line arrangements in $\mathbb{P}^2$, see [16, 23].

For example, consider the orbifold $X=(\mathbb{P}^2,\varphi)$ associated to the 6 lines Ceva arrangement, where φ takes value n for all lines. Then let Y be the orbifold cover associated to the abelianization $\pi_1^{\text{orb}}(X)\to(\mathbb{Z}/n\mathbb{Z})^5$. A straightforward counting argument shows that $b_1^{\text{orb}}(Y)=5(n-1)(n-2)$.

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INTERSECTION SPACES AND HYPERSURFACE SINGULARITIES

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Abstract. We give an elementary introduction to the first author's theory of intersection spaces associated to complex projective varieties with only isolated singularities. We also survey recent results on the deformation invariance of intersection space homology in the context of projective hypersurfaces with an isolated singularity.

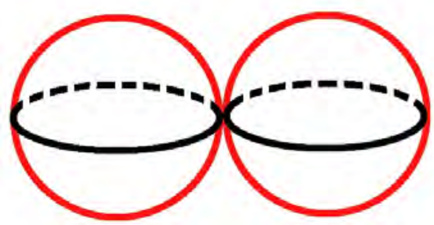
1. Introduction

Convention: By "manifold" we mean a "complex projective manifold", and by "singular space" we mean a "complex projective variety of pure complex dimension n". We are only interested in "middle-perversity" calculations, so any mentioning of other perversity functions will be ignored. Unless otherwise specified, all (intersection) (co)homology groups will be computed with rational coefficients.

Manifolds have an amazing hidden symmetry called Poincaré Duality, which ultimately is reflected in their (co)homology: ranks of (co)homology groups in complementary degrees are equal. Singular spaces, on the other hand, do not possess such symmetry. For example, consider the complex projective curve

$$X = \{(x : y : z) \in \mathbb{CP}^2 \mid xy = 0\}.$$

Then X is a union of two projective lines $\mathbb{CP}^1$ meeting at the point $(0 : 0 : 1)$. Topologically, X is just $S^2 \vee S^2$, a wedge of two 2-spheres.



As $H_0(X) = \mathbb{Q}$ and $H_2(X) = \mathbb{Q} \oplus \mathbb{Q}$, it follows that $\mathrm{rk} H_0(X) \neq \mathrm{rk} H_2(X)$.

Classically, much of the manifold theory, e.g., Morse theory, Lefschetz theorems, Hodge decompositions, and especially Poincaré Duality, is recovered in the singular context if, instead of the usual (co)homology, one uses Goresky-MacPherson's intersection homology groups $IH_*(X)$, see [8, 10]. These are homology groups of a complex of "allowed chains", defined by imposing restrictions on how chains meet the singular strata. For example, if X is a complex projective

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variety of complex dimension n with only isolated singularities, then $IH_i(X)$ is the i-th homology group of the chain complex $(IC_*(X), \partial)$ defined by the following allowability conditions: if ξ is a PL i-chain on X with support $|\xi|$ (in a sufficiently fine triangulation of X compatible with the natural stratification $X \supset \mathrm{Sing}(X)$), then $\xi \in IC_i(X)$ if, and only if,

$$\dim_{\mathbb{R}} |\xi| \cap \mathrm{Sing}(X) < i - n$$

and

$$\dim_{\mathbb{R}} |\partial\xi| \cap \mathrm{Sing}(X) < i - 1 - n,$$

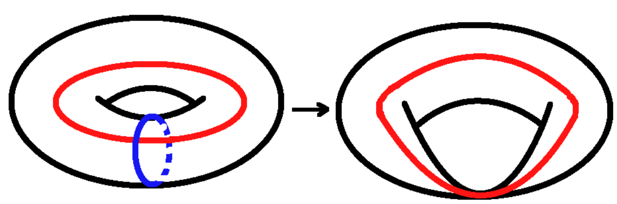
and with boundary operator induced from the usual boundary operator on chains of X. So low-dimensional allowable chains cannot meet the singularities of X.

It is known that the intersection (co)homology groups of a singular space coincide with the usual (co)homology groups of a "small" resolution (provided such a resolution exists), e.g., see [10]. This precludes the existence in general of a cup product (i.e., ring structure) on intersection cohomology: indeed, there are spaces having two small resolutions which have non-isomorphic cohomology rings. Intersection (co)homology is not a homotopy invariant (e.g., the intersection homology of a cone is not trivial, see [14, Section 4.7]), and is lacking functoriality in general (e.g., see [14, Section 4.8]). And, much like the usual homology theory, it is also rather unstable under deformation of singularities. We shall illustrate this last assertion by an example.

Example 1.1. Consider the equation

$$y^2 = x(x-1)(x-s)$$

or its homogeneous version $v^2w = u(u-w)(u-sw)$, defining a curve in $\mathbb{CP}^2$, where the complex parameter s is constrained to lie inside the unit disc, $|s| < 1$. For $s \neq 0$, the equation defines an elliptic curve V_s, homeomorphic to a 2-torus T^2. The curve $V := V_0$ corresponding to $s = 0$, has a nodal singularity. Thus V is homeomorphic to a pinched torus, that is, T^2 with a meridian collapsed to a point, or, equivalently, a cylinder $I \times S^1$ with coned-off boundary, where $I = [0, 1]$.



The ordinary homology group $H_1(V)$ has rank one, generated by the longitudinal circle (while the meridian circle bounds the cone with vertex at the singular point of V). The intersection homology group $IH_1(V)$ agrees with the intersection homology of the normalization S^2 of V (the longitude in V is not an "allowed" 1-cycle, while the meridian bounds an allowed 2-chain), so:

$$IH_1(V) = IH_1(S^2) = H_1(S^2) = 0.$$

Thus, as $H_1(V_s) = H_1(T^2)$ has rank 2, neither ordinary homology nor intersection homology remains invariant under the smoothing deformation $V \rightsquigarrow V_s$.

This raises the following questions: *Is there a homology theory for singular spaces, possessing Poincaré Duality, ring structure, homotopy invariance, functoriality, and which is also stable under deformations? Does this theory carry a Kähler package (i.e., Hodge decompositions, Hodge star duality, Lefschetz-type theorems; see* [13, Section 7.3]*)?*

Parts of this question were answered positively by the first author in [1]. Aspects of deformation invariance and Hodge theory were considered in [4]. The aim of this note is to present a quick account of these developments.

2. Intersection Spaces

The first author's approach [1] consists of a homotopy-theoretic method which associates to a certain singular space X a CW complex IX, called the "intersection space of X". Moreover, as shown in [1], IX is a rational Poincaré complex, i.e., its (reduced) homology groups satisfy Poincaré Duality over the rationals. One thing to notice immediately is that the "intersection space cohomology" $\widetilde{H}^*(IX)$ has an internal ring structure defined by the usual cup product in cohomology, so the cohomology of the intersection space is not generally isomorphic to the intersection cohomology of the space itself. Moreover, this theory has a DeRham interpretation (see [2]) and, as we will try to convince the reader, it is more stable under deformations of singularities.

Roughly speaking, the intersection space IX associated to a singular space X is defined by replacing links of singularities by their corresponding Moore approximations, a process the first author termed "spatial homology truncation". Let us say a few words about Moore approximations. Let L be a simply-connected CW complex, and fix an integer n. The Moore approximation construction guarantees the existence of a CW complex $L_{<n}$ together with a structural map $f : L_{<n} \to L$, so that $f_* : H_r(L_{<n}) \to H_r(L)$ is an isomorphism if $r < n$, and $H_r(L_{<n}) \cong 0$ for all $r \geq n$. (Moreover, these isomorphisms hold over the integers.) In more detail, let $C_*(L)$ be the cellular chain complex of L, and let $Z_n(L)$ denote the n-cycles. Suppose first that the following assumption holds: $Z_n(L)$ has a basis $\langle z_\alpha \rangle$ consisting of n-cells. Let $\langle y_\beta \rangle$ be the remaining n-cells, i.e., those n-cells that are not cycles. The y_β's generate a subgroup $Y \subset C_n(L)$ such that

$$C_n(L) = Z_n(L) \oplus Y,$$

and the n-skeleton of L can then be written as

$$L^{(n)} = L^{(n-1)} \cup \langle z_\alpha \rangle \cup \langle y_\beta \rangle.$$

We define:

$$L_{<n} := L^{(n-1)} \cup \langle y_\beta \rangle \overset{f=incl}{\hookrightarrow} L.$$

In the general case, any simply-connected CW complex L is homotopy equivalent relative to the $(n-1)$-skeleton to a CW complex L' which satisfies the above technical assumption. Then we set:

$$L_{<n} := (L')_{<n} \hookrightarrow L' \overset{h.e.}{\to} L,$$

i.e., the structural map $f : L_{<n} \to L$ is the composition of the inclusion followed by the homotopy equivalence. The process often works in the non-simply-connected case by ad-hoc considerations. For example, if $\partial_n = 0$, i.e., $C_n(L) = Z_n(L)$, we can choose $L_{<n} := L^{(n-1)}$ with structural map given by the inclusion map. In particular, if L is path-connected, then $L_{<1}$ is just a point. Note also that if $n > \dim L$, then $L_{<n} = L$.

Let us now discuss the construction of the intersection space in a very simple situation, namely when the variety X has only an isolated singularity x. Let L be the link of the singular point, and M the manifold with boundary $\partial M = L$ obtained from X by removing a small open conical neighborhood of x. Topologically,

$$X = M \cup_L cone(L).$$

Choose a CW structure on L, and let $g : L_{<n} \xrightarrow{f} L = \partial M \hookrightarrow M$ be the composition of inclusion of the boundary followed by the structural map of the Moore approximation of L at level $n = \dim_{\mathbb{C}} X$. The intersection space IX is then defined as the mapping cone of g, that is:

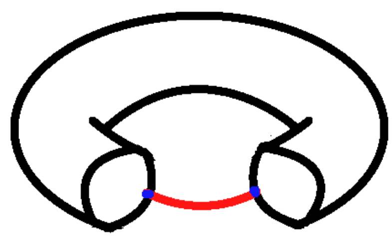
$$IX := cone(g) := M \cup_g cone(L_{<n}).$$

Therefore, in the case when the structural map of the Moore approximation of L is an inclusion, the intersection space IX is obtained from M by coning off a certain subset of the link.

If there are several isolated singularities, the intersection space is defined by performing spatial homology truncation on each of the links, simultaneously.

Note that the construction of the intersection space IX involves choices of subgroups $Y_i \subset C_n(L_i)$, where the L_i are the links of the singularities. Moreover, the chain complexes $C_*(L_i)$ depend on the CW structures on the links. However, as shown in [1, Theorem 2.18], the rational homology of IX is well-defined and independent of all choices.

Example 2.1. In the case of the nodal curve V of Example 1.1, the link of the singular point is $\partial I \times S^1$, two circles. The intersection space IV of V is a cylinder $I \times S^1$ together with an interval, whose one endpoint is attached to a point in $\{0\} \times S^1$ and whose other endpoint is attached to a point in $\{1\} \times S^1$.



Thus IV is homotopy equivalent to the figure eight and

$$H_1(IV) = \mathbb{Q} \oplus \mathbb{Q},$$

which does agree with $H_1(V_s)$.

Remark 2.2. As suggested by Example 2.1, the middle homology of the intersection space IX takes into account more cycles than the corresponding intersection homology group of X. More precisely, for X of complex dimension n with only isolated singularities, $IH_n(X)$ is generally smaller than both $H_n(X - Sing(X))$ and $H_n(X)$, being a quotient of the former and a subgroup of the latter, while $H_n(IX)$, is generally bigger than both $H_n(X - Sing(X))$ and $H_n(X)$, containing the former as a subgroup and mapping to the latter surjectively, see [1].

3. String Theory and Mirror Symmetry

The homology of intersection spaces addresses certain questions in type II string theory. Let us give a quick account of how this story unfolds.

In addition to the four dimensions that model our space-time classically, (super)string theory requires six dimensions for a string to vibrate. Supersymmetry considerations force these six (real) dimensions to be a Calabi-Yau space (i.e., a compact complex Kähler manifold with trivial first Chern class). However, given the multitude of known topologically distinct Calabi-Yau threefolds, the (super)string model remains undetermined. It is therefore important to have mechanisms that allow one to move from one Calabi-Yau space to another. Topologically speaking, since any two closed oriented (real) six-manifolds are bordant, and bordisms are obtained by performing a finite number of surgeries, surgery seems to be a good way to travel from one

Calabi-Yau manifold to another. In Physics, a solution to this problem was first proposed by Green-Hübsch [11, 12] who, motivated by Reid's fantasy [17], conjectured that topologically distinct Calabi-Yau's could be connected to each other by means of "conifold transitions", which should induce a phase transition between the corresponding (super)string models.

A conifold transition starts out with a nonsingular Calabi-Yau threefold, passes through a singular variety – the conifold – by a deformation of complex structure, and arrives at a topologically distinct nonsingular Calabi-Yau threefold by a small resolution of singularities. The deformation collapses embedded three-spheres (the "vanishing cycles") to isolated ordinary double points, while the resolution resolves the singular points by replacing each of them with a $\mathbb{CP}^1$. A conifold transition can be described locally by means of surgeries on the vanishing cycles (see [6, 18]). In Physics, the topological change was interpreted by Strominger as the condensation of massive black holes to massless ones. It is then desirable to record these massless particles as classes in good (co)homology theories. In type IIA string theory, there are charged two-branes that wrap around the $\mathbb{CP}^1$ 2-cycles, and which become massless when these 2-cycles are collapsed to points by the resolution map. As intersection homology is invariant under small resolutions, the intersection homology of the conifold accounts for all of these massless two-branes, so it is the physically correct homology theory for type IIA string theory. Similarly, in type IIB string theory there are charged three-branes wrapped around the vanishing cycles, and which become massless as these vanishing cycles are collapsed by the deformation of complex structure. Neither ordinary homology nor intersection homology of the conifold account for these massless three-branes, but the homology of the intersection space of the conifold yields the correct count. So it appears that the homology of intersection spaces is the physically correct homology theory in the IIB string theory; see [1, Section 3] for more details.

In relation to mirror symmetry, given a Calabi-Yau threefold X, the mirror map associates to it another Calabi-Yau threefold X° so that type IIB string theory on $\mathbb{R}^4 \times X$ corresponds to type IIA string theory on $\mathbb{R}^4 \times X^\circ$. If X and X° are nonsingular, their homology Betti numbers are related by precise algebraic identities, e.g., $\beta_3(X^\circ) = \beta_2(X) + \beta_4(X) + 2$, etc. A conjecture of Morrison [16] asserts that the mirror of a conifold transition is again a conifold transition, but performed in the reverse order (so, mirror symmetry is supposed to exchange resolutions and deformations). Thus, if in the above discussion, X and X° are mirrored conifolds (in mirrored conifold transitions), the intersection space homology of one space and the intersection homology of the mirror space form a mirror-pair, in the sense that

$$\beta_3(IX^\circ) = I\beta_2(X) + I\beta_4(X) + 2, \tag{1}$$

etc., where $I\beta_i$ denotes the i-th intersection homology Betti number (see [1] for details).

The above mirror symmetry considerations suggest that one could expect to be able to compute the intersection space homology $H_*(IX)$ of a variety X in terms of the topology of a smoothing family X_s, by "mirroring" known results relating the intersection homology groups $IH_*(X)$ of X to the topology of a resolution of singularities $\widetilde{X}$. This point of view will be exploited in the next section, see [4] for complete details. Moreover, equation (1) can serve as a beacon in constructing a mirror X° for a given singular X, as it restricts the topology of those X° that can act as a mirror of X.

4. Deformations of Singularities

All results in this section can be formulated for projective hypersurfaces with several isolated singularities. For simplicity, we restrict our attention to the case of a single isolated singularity.

Let f be a homogeneous polynomial in $n+2$ variables with complex coefficients such that the complex projective hypersurface

$$V = \{f = 0\} \subset \mathbb{CP}^{n+1}$$

has only one isolated singularity x. Let L_x, F_x and $T_x : H_n(F_x) \to H_n(F_x)$ denote the link, Milnor fiber and local monodromy operator of the isolated hypersurface singularity germ (V, x), respectively. By [15], the link L_x is an $(n-2)$-connected closed oriented $(2n-1)$-dimensional manifold. Also, the Milnor fiber F_x is a parallelizable $(n-1)$-connected $2n$-dimensional manifold, which has the homotopy type of $\bigvee S^n$, a wedge of n-spheres. The number $\mu_x = \mathrm{rk} H_n(F_x)$ of these n-spheres (which are also called "vanishing cycles") is the local Milnor number at x. It is known that all eigenvalues of T_x are roots of unity. We say that the local monodromy operator T_x is trivial if all eigenvalues of T_x are equal to 1.

Theorem 4.1 ([4]). *Let $V \subset \mathbb{CP}^{n+1}$ be a complex projective hypersurface with only one isolated singular point x. Let V_s denote a nearby smoothing of V. Then:*

(a) $\widetilde{H}_i(V_s) \cong \widetilde{H}_i(IV)$, for all $i < 2n$, $i \neq n$;
(b) $\widetilde{H}_n(V_s) \cong \widetilde{H}_n(IV) \iff T_x$ is trivial.

Recall that intersection homology is invariant under small resolutions. Therefore, as inspired by the conifold transition picture of the previous section, we regard the local trivial monodromy condition of Theorem 4.1 as "mirroring" that of the existence of small resolutions.

In fact, the isomorphisms of Theorem 4.1 are often induced by a map. More precisely, by the construction of an intersection space, there is a canonical map $can : IV \to V$. Also, there is a specialization map $sp : V_s \to V$ which collapses the vanishing cycles to a point. In [4], we showed that under a mild technical assumption on the homology of the link (that is, if $n \neq 2$ and $H_{n-1}(L_x; \mathbb{Z})$ is torsion-free), one can define a map $\eta : IV \to V_s$ so that $can = sp \circ \eta$. Then the following holds:

Theorem 4.2 ([4]). *The isomorphisms in Theorem 4.1(a) are induced by the map η. Moreover,*

$$\eta_* : H_n(IV) \to H_n(V_s)$$

is a monomorphism, and it is an isomorphism if and only if the local monodromy operator T_x is trivial. In particular, in the latter case, the dual maps in cohomology are ring isomorphisms.

We regard the result of Theorem 4.2 as "mirroring" the fact that the intersection homology groups $IH_i(V)$ of V are vector subspaces of the corresponding homology groups $H_i(\widetilde{V})$ of any resolution $\widetilde{V}$ of V, the latter being an easy application of the Bernstein-Beilinson-Deligne-Gabber decomposition theorem, e.g., see [5, 7, 9].

Together with Remark 2.2, the result of Theorem 4.2 also yields the following bounds on the rank of the rational vector space $H_n(IV)$:

$$\mathrm{rk} IH_n(V) \leq \mathrm{rk} H_n(IV) \leq \mathrm{rk} H_n(V_s).$$

In the case when the map η above can be defined, it can be used to put a mixed Hodge structure on each of the cohomology groups $\widetilde{H}^i(IV)$. Here we also use the fact that the specialization map $sp : V_s \to V$ induces mixed Hodge structure homomorphisms in cohomology, provided the cohomology groups $H^*(V_s)$ of the smoothing are considered with their "limit" mixed Hodge structures, as defined by Schmid-Steenbrink. More precisely, under the assumptions of Theorem 4.3 below, η^* can be used to transfer this limit mixed Hodge structure to the groups $\widetilde{H}^i(IV)$. We therefore get the following:

Theorem 4.3 ([4]). *If V is a complex projective hypersurface with only one isolated singular point x at which the local monodromy operator is trivial, then each cohomology group $\widetilde{H}^i(IV)$ carries a mixed Hodge structure so that the map $can : IV \to V$ induces a mixed Hodge structure homomorphism in (reduced) cohomology.*

The existence of mixed Hodge structures on intersection space cohomology groups (though restricted by our context and hypotheses) is already very surprising, especially after noting that the intersection space associated to a complex projective variety is not itself an algebraic variety in general. Nevertheless, one can regard the statement of Theorem 4.3 as "mirroring" the classical fact stating that for an equidimensional complex projective variety V, each intersection cohomology group $IH^i(V)$ carries a weight i pure Hodge structure, so that the natural map $H^i(V) \to IH^i(V)$ is a homomorphism of mixed Hodge structures. As already predicted by Remark 2.2, one could not expect in general to get purity on intersection space cohomology groups. However, at least when the monodromy operator of the smoothing family $\{V_s\}_s$ is trivial one does get purity. To give a concrete example, this occurs when a family of smooth genus 2 curves V_s degenerates into a union of two smooth elliptic curves meeting transversally at one double point, see [4, Example 5.4].

Regarding invariants of intersection spaces, it follows from [1] that if $n = \dim_{\mathbb{C}} V$ is even, then the signature $\sigma(IV)$ of the intersection space (as defined via the Poincaré duality pairing on $H_n(IV)$) coincides with the Goresky-MacPherson (intersection homology) signature of V. The difference between the Euler characteristics of the two theories was computed in [4] as follows:

Theorem 4.4 ([4]). *For a complex projective hypersurface $V \subset \mathbb{CP}^{n+1}$ with only one isolated singular point x, we have that:*

$$\chi(\widetilde{H}_*(IV)) - \chi(IH_*(V)) = -2\chi_{<n}(L_x), \tag{2}$$

where L_x is the link of x, and $\chi_{<n} := \sum_{i<n}(-1)^i\beta_i$.

Of course, when x is actually a smooth point (so the link is a $(2n-1)$-dimensional sphere) the intersection space IV is homotopy equivalent to V with a small open ball about x excised, and the given formula is easily seen to hold. In fact, formula (2) holds more generally in the context of an even dimensional pseudomanifold X with only isolated singularities (see [1, Corollary 2.14]). We indicate here a very elementary proof of this formula (as suggested by the referee) in the case when X has only an isolated singular point x. Let (M, L) be the manifold, with boundary the link L of x, obtained by excising a small open neighborhood of x. Then considering the relevant Mayer-Vietoris sequence, and using the additivity of the Euler characteristic, we have on the one hand

$$\chi(IX) = \chi(M) + \chi(cone(L_{<n})) - \chi(L_{<n}) = \chi(M) + 1 - \chi_{<n}(L),$$

and on the other

$$I\chi(X) = \chi(M) + I\chi(cone(L)) - \chi(L) = \chi(M) + \chi_{<n}(L),$$

where we have used the standard cone calculation for $IH^*(cone(L))$ and the fact that L is an odd-dimensional manifold. The result follows immediately.

5. Open Questions

We conclude this survey with a list of open problems, some of which are motivated by the "mirror symmetry" analogy discussed above.

(1) Is there a sheaf-theoretic description for the intersection space (co)homology, similar to the one for intersection homology (by Deligne's intersection sheaf complex [10])? At least in the context of this note, the answer should be related to the complex of nearby cycles of a smoothing family.
(2) Is there a version of (weak and hard) Lefschetz theorems for the (co)homology of the intersection space of a complex projective variety?
(3) Is there a canonical mixed Hodge structure on the intersection space cohomology of a complex projective variety?
(4) How much of the above work can be extended in the context of more general singularities? For example, consider the case of a hypersurface V with a higher-dimensional smooth singular locus Σ so that (V, Σ) is a Whitney stratification of V. The intersection space IV was defined in [1] under certain assumptions on the link bundle. In the same vein, the intersection space associated to certain stratified pseudomanifolds with depth 2 stratifications was constructed in [3]. It would be interesting to study the associated homology groups for these intersection spaces.

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FIRST CASES OF INTERSECTION SPACES IN STRATIFICATION DEPTH 2

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Abstract. Previous constructions of intersection spaces for stratified pseudomanifolds all required the stratification depth to be at most 1. Here, we construct intersection spaces for certain simple stratifications of depth 2, involving different singularity links.

1. Introduction

In [1], we introduced a method that associates to certain classes of stratified pseudomanifolds X CW complexes
$$I^{\bar{p}}X,$$
the *intersection spaces of* X, where $\bar{p}$ is a perversity in the sense of Goresky and MacPherson's intersection homology, such that the *ordinary* (reduced) cohomology $\widetilde{H}^*(I^{\bar{p}}X;\mathbb{Q})$ satisfies generalized Poincaré duality when X is closed and oriented. The resulting cohomology theory $X \rightsquigarrow HI^*_{\bar{p}}(X) = H^*(I^{\bar{p}}X)$ is not isomorphic to intersection cohomology $IH^*_{\bar{p}}(X)$, since the former has a $\bar{p}$-internal cup product while the latter does not, in general. For example, the singular Calabi-Yau quintic
$$X = \{z \in \mathbb{P}^4 \mid z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5z_0z_1z_2z_3z_4 = 0\}$$
has intersection cohomology ranks
$$\operatorname{rk} IH^2(X) = 25,\ \operatorname{rk} IH^3(X) = 2,\ \operatorname{rk} IH^4(X) = 25,$$
whereas
$$\operatorname{rk} HI^2(X) = 1,\ \operatorname{rk} HI^3(X) = 204,\ \operatorname{rk} HI^4(X) = 1.$$
The expository article [5] of the present volume provides a gentle introduction to intersection spaces and surveys results obtained by Maxim and the author in [4] on the stability of HI^* under nearby smooth deformations of a singular projective hypersurface. Given a spectrum E in the sense of stable homotopy theory, one may form $EI^*_{\bar{p}}(X) = E^*(I^{\bar{p}}X)$. This, then, yields an approach to defining intersection versions of generalized cohomology theories such as K-theory. The theory HI^* also addresses questions in type II string theory related to the existence of massless D-branes arising in the course of a Calabi-Yau conifold transition. These questions are answered by IH^* for IIA theory, and by HI^* for IIB theory; see Chapter 3 of [1]. A de Rham-type description of HI^* has been developed in [2], which has been applied in [3] to obtain spectral sequence degeneration results for flat bundles and equivariant cohomology of isometric group actions.

Up to the present point, intersection spaces have only been constructed for singular spaces that possess a stratification of depth at most 1, although a construction method for greater depths has

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been proposed in [1], pp. 186 – 189. In the present note, we implement this method for certain stratifications of depth 2. We consider only the two middle perversities $\bar{m}$ and $\bar{n}$. Let X be an n-dimensional, compact, oriented PL pseudomanifold (without boundary) with $n = 2k > 0$ even. Suppose X can be endowed with a PL stratification of the form $X = X_n \supset X_1 \supset X_0$ with X_1 a circle and X_0 a point, such that the respective links L, L_0 of the two singular strata $X_1 - X_0, X_0$ are simply connected. For the link L of the odd-(co)dimensional stratum $X_1 - X_0$, we require the following strong version of the Witt condition: X satisfies the *strong Witt condition*, if L possesses a CW structure such that the cellular boundary operator $C_{k-1}(L) \to C_{k-2}(L)$ is injective. This condition implies of course that the middle homology $H_{k-1}(L)$ of the manifold L vanishes, which is the classical Witt condition of [9] when rational coefficients are used. The strong Witt condition is obviously satisfied if L has no middle-dimensional cells. Closely related versions of this condition have been considered in the literature before. Weinberger [13] calls an n-manifold *antisimple*, if its chain complex is chain homotopy equivalent to a complex of projective modules P_* with $P_i = 0$ for $i = [n/2]$. Hausmann considers manifolds that have a handlebody without middle-dimensional handles, which is stronger than our condition; see [7, p.334, p.336]. For X satisfying the strong Witt condition, we follow the method of [1] to construct the two middle-perversity intersection spaces $I^{\bar{m}}X$ and $I^{\bar{n}}X$. As expected, they turn out to be equal, and we put $IX = I^{\bar{m}}X = I^{\bar{n}}X$. The main theorem (Theorem 6.2) asserts that there exists a Poincaré duality isomorphism

$$D : \widetilde{H}^{n-r}(IX;\mathbb{Q}) \xrightarrow{\cong} \widetilde{H}_r(IX;\mathbb{Q})$$

that is compatible with Poincaré-Lefschetz duality for the exterior of the singular set.

The basic paradigm for the construction of intersection spaces is to replace links by their spatial homology truncations (Moore approximations), where the truncation degree is determined by the perversity function. We review spatial homology truncation in Section 3. The simple connectivity assumption on the links is adopted to ensure the existence of homology truncations, and is in practice not always necessary. Roughly, we proceed as follows: We first disassemble the boundary of a regular neighborhood of the singular set, so that we can build a nice homotopy theoretic model of it. This involves certain simple kinds of homotopy colimits, whose properties we collect in Section 2. In the disassembled state, the pieces are the link bundle over $X_1 - X_0$, the space obtained from the link L_0 of X_0 by removing cone neighborhoods of its two singular points, and a space $\ddot{L}$, PL homeomorphic to two copies of L, where the two other pieces are glued. The gluing involves maps from $\ddot{L}$ to the other two pieces. We then apply spatial homology truncation to truncate all these pieces (more precisely, the bundle over $X_1 - X_0$ is truncated in a fiberwise fashion), as well as the maps relating them to each other. Then the truncated pieces are reassembled again, using the truncated maps, and IX is the homotopy cofiber of the map from the reassembly to the complement of the open regular neighborhood of the singular set.

Notation and Conventions: If X and Y are topological spaces, $A \subset X$ a subspace, and $f : A \to Y$ a continuous map, then $Y \cup_f X$ denotes the space obtained from the disjoint union of X and Y by attaching X along A to Y using the map f, that is, $Y \cup_f X = (Y \sqcup X)/(a \sim f(a)$ for all $a \in A)$. Our convention for the mapping cylinder $Y \cup_f X \times I$ of a map $f : X \to Y$ is that the attaching is carried out at time 1, that is, the points of $X \times \{1\} \subset X \times I$ are attached to Y using f. The homology $H_*(f)$ of the map f is defined to be $H_*(f) = H_*(Y \cup_f X \times I, X \times \{0\})$. For products in cohomology and homology, we will use the conventions of Spanier's book [10]. In particular, for an inclusion $i : A \subset X$ of spaces and elements $\xi \in H^p(X)$, $x \in H_n(X, A)$, the formula $\partial_*(\xi \cap x) = i^*\xi \cap \partial_* x$ holds for the connecting homomorphism ∂_* (no sign). For the compatibility between cap- and cross-product, one has the sign

$$(\xi \times \eta) \cap (x \times y) = (-1)^{p(n-q)} (\xi \cap x) \times (\eta \cap y),$$

where $\xi \in H^p(X)$, $\eta \in H^q(Y)$, $x \in H_m(X)$, and $y \in H_n(Y)$.

2. Required Properties of Homotopy Pushouts

In order to form the intersection space of a given pseudomanifold, one has to glue together pieces obtained at various stages of homology (Moore) towers. The gluing is accomplished via homotopy pushouts, whose fundamentals we shall collect in the present section. It is not possible to glue through ordinary pushouts, since the output of spatial homology truncation is only well-defined up to homotopy.

A *3-diagram* Γ *of spaces* is a diagram of the form

$$X \xleftarrow{f} A \xrightarrow{g} Y,$$

where A, X, Y are topological spaces and f, g are continuous maps. If A, X, Y are CW complexes and f, g are cellular, then we call Γ a *CW-3-diagram.* The *realization* $|\Gamma|$ of Γ is the pushout of f and g, that is,

$$|\Gamma| = (X \sqcup Y)/(f(a) \sim g(a), \text{ for all } a \in A).$$

If Γ is a CW-3-diagram and g is the inclusion of a subcomplex, then $|\Gamma|$ is a CW complex, [8]. In particular, the mapping cylinder $\mathrm{cyl}(f)$ is a CW complex in a natural way. A *morphism* $\Gamma \to \Gamma'$ of 3-diagrams is a commutative diagram

$$\begin{tikzcd} X \arrow[d] & A \arrow[l, "f"'] \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ X' & A' \arrow[l, "f'"'] \arrow[r, "g'"] & Y' \end{tikzcd} \tag{1}$$

in the category of topological spaces. If Γ and Γ' are both CW-3-diagrams, then we call the morphism *cellular*, if all vertical arrows are cellular maps. The universal property of the pushout implies that a morphism $\Gamma \to \Gamma'$ induces a map $|\Gamma| \to |\Gamma'|$ between realizations. If $\Gamma \to \Gamma'$ is cellular, with g, g' subcomplex inclusions, then $|\Gamma| \to |\Gamma'|$ is cellular. A homotopy theoretic weakening of a morphism is the notion of an *h-morphism* $\Gamma \to_h \Gamma'$. This is again a diagram of the above form (1), but the two squares are required to commute only up to homotopy. An h-morphism does not induce a map between realizations. The remedy is to use the *homotopy pushout*, or double mapping cylinder. This is a special case of the notion of a *homotopy colimit.* To a 3-diagram Γ we associate another 3-diagram $H(\Gamma)$ given by

$$X \cup_f A \times I = \mathrm{cyl}(f) \overset{\text{at } 0}{\hookleftarrow} A \overset{\text{at } 0}{\hookrightarrow} \mathrm{cyl}(g) = Y \cup_g A \times I.$$

If Γ is a CW-3-diagram, then $\mathrm{cyl}(f)$ and $\mathrm{cyl}(g)$ are CW complexes and hence $H(\Gamma)$ is again a CW-3-diagram. We define the homotopy pushout, or homotopy colimit, of Γ to be

$$\mathrm{hocolim}(\Gamma) = |H(\Gamma)|.$$

If Γ is CW, then, as the two maps in $H(\Gamma)$ are subcomplex inclusions, $|H(\Gamma)|$ is a CW complex. Sometimes, especially in large diagrams, we will omit parentheses and briefly write $H\Gamma$ for

$H(\Gamma)$, $C_*|H\Gamma|$ for the chain groups $C_*(|H\Gamma|)$, and $H_*|H\Gamma|$ for the homology groups $H_*(|H\Gamma|)$. The morphism $H(\Gamma) \to \Gamma$ given by

$$\begin{tikzcd} X \cup_f A \times I \arrow[d, "r"] & A \arrow[l] \arrow[r] \arrow[d, "\mathrm{id}_A"] & Y \cup_g A \times I \arrow[d, "r"] \\ X & A \arrow[l, "f"'] \arrow[r, "g"] & Y, \end{tikzcd}$$

where the maps r are the canonical mapping cylinder retractions, induces a canonical map

$$\operatorname{hocolim}(\Gamma) \longrightarrow |\Gamma|.$$

A morphism $\Gamma \to \Gamma'$, given by

$$\begin{tikzcd} X \arrow[d, "\xi"] & A \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "\alpha"] & Y \arrow[d, "\eta"] \\ X' & A' \arrow[l, "f'"'] \arrow[r, "g'"] & Y', \end{tikzcd}$$

induces a morphism $H(\Gamma) \to H(\Gamma')$, given by

$$\begin{tikzcd} X \cup_f A \times I \arrow[d, "\xi\cup(\alpha\times\mathrm{id}_I)"] & A \arrow[l, hook', "\text{at } 0"'] \arrow[r, hook, "\text{at } 0"] \arrow[d, "\alpha"] & Y \cup_g A \times I \arrow[d, "\eta\cup(\alpha\times\mathrm{id}_I)"] \\ X' \cup_{f'} A' \times I & A' \arrow[l, hook', "\text{at } 0"'] \arrow[r, hook, "\text{at } 0"] & Y' \cup_{g'} A' \times I, \end{tikzcd}$$

which in turn induces a map $\operatorname{hocolim}(\Gamma) \to \operatorname{hocolim}(\Gamma')$. If $\Gamma \to \Gamma'$ is cellular, then $H(\Gamma) \to H(\Gamma')$ is cellular. Since the horizontal arrows are subcomplex inclusions, $\operatorname{hocolim}(\Gamma) \to \operatorname{hocolim}(\Gamma')$ is thus also cellular. If α, ξ and η are homeomorphisms, then $\xi \cup (\alpha \times \mathrm{id}_I)$ and $\eta \cup (\alpha \times \mathrm{id}_I)$ are homeomorphisms and hence $\operatorname{hocolim}(\Gamma) \to \operatorname{hocolim}(\Gamma')$ is a homeomorphism. An h-morphism $\Gamma \to_h \Gamma'$ together with a choice of homotopies between clockwise and counterclockwise compositions will induce a map on the homotopy pushout,

$$\operatorname{hocolim}(\Gamma) \longrightarrow |\Gamma'|.$$

Indeed, let

$$\begin{tikzcd} X \arrow[d, "\xi"] & A \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "\alpha"] & Y \arrow[d, "\eta"] \\ X' & A' \arrow[l, "f'"'] \arrow[r, "g'"] & Y' \end{tikzcd}$$

be the given h-morphism. Let $F : A \times I \to X'$ be a homotopy between $F_0 = f'\alpha$ and $F_1 = \xi f$. Let $G : A \times I \to Y'$ be a homotopy between $G_0 = g'\alpha$ and $G_1 = \eta g$. Then

$$\begin{tikzcd} X \cup_f A \times I \arrow[d, "\xi\cup_f F"] & A \arrow[l, hook', "\text{at } 0"'] \arrow[r, hook, "\text{at } 0"] \arrow[d, "\alpha"] & Y \cup_g A \times I \arrow[d, "\eta\cup_g G"] \\ X' & A' \arrow[l, "f'"'] \arrow[r, "g'"] & Y', \end{tikzcd}$$

commutes (on the nose) and thus defines a morphism $H(\Gamma) \to \Gamma'$. This morphism induces a continuous map on realizations $\operatorname{hocolim}(\Gamma) = |H(\Gamma)| \to |\Gamma'|$.

A pair (X, A) of (compactly generated) topological spaces is an *NDR pair*, if the inclusion $A \subset X$ is a closed cofibration. A relative CW complex, for instance, is an NDR pair.

Proposition 2.1. *If*

$$\begin{tikzcd} Y \arrow[d, "\simeq"', "\phi_Y"] & A \arrow[l, "f"'] \arrow[r, hook, "i"] \arrow[d, "\simeq"', "\phi_A"] & X \arrow[d, "\simeq"', "\phi_X"] \\ Y' & A' \arrow[l, "f'"'] \arrow[r, hook, "i'"] & X' \end{tikzcd} \tag{2}$$

is a commutative diagram of continuous maps such that i and i' are inclusions of NDR pairs and ϕ_Y, ϕ_A, ϕ_X are homotopy equivalences, then

$$\phi_Y \cup \phi_X : Y \cup_f X \longrightarrow Y' \cup_{f'} X'$$

is a homotopy equivalence.

This is Theorem 1.13 in Section 1 of [6], where a proof can be found. For our purposes, for example when cellular approximation is required, we need to weaken the assumptions of the above proposition by requiring the left square of the diagram to be merely homotopy commutative. A similar conclusion will then hold if the pushouts are replaced by homotopy pushouts.

Proposition 2.2. *If the right hand square of diagram (2) commutes and the left hand square commutes up to homotopy, i and i' are inclusions of NDR pairs and ϕ_Y, ϕ_A, ϕ_X are homotopy equivalences, then the homotopy pushouts of the first and second row are homotopy equivalent,*

$$Y \cup_f A \times I \cup_i X \simeq Y' \cup_{f'} A' \times I \cup_{i'} X'.$$

(In fact, both of these homotopy pushouts are equivalent to $Y' \cup_{f'} X'$.)

Proof. Let $H : A \times I \to Y'$ be a homotopy between $H_0 = f'\phi_A$ and $H_1 = \phi_Y f$. We claim that the map

$$\phi_Y \cup_f H : Y \cup_f A \times I \longrightarrow Y'$$

is a homotopy equivalence. To see this, consider the following homotopy $\{F_s\}_{0 \le s \le 1}$,

$$F_s : Y \cup_f A \times I \longrightarrow Y'.$$

For given s, F_s consists of ϕ_Y on Y. On $A \times [s, 1] \subset A \times I$, use $H(a, t)$, $a \in A$, $s \le t \le 1$. On the remaining part $A \times [0, s] \subset A \times I$, use $H(a, s)$ (constant in t). Then $F_0 = \phi_Y \cup_f H$ and F_1 is ϕ_Y on Y and $F_1(a, t) = H(a, 1) = \phi_Y f(a)$ for all t. We may think of F_1 as the composition of the mapping cylinder retraction

$$r : Y \cup_f A \times I \longrightarrow Y$$

induced by projection, and $\phi_Y : Y \to Y'$. Since both of these maps are homotopy equivalences, so is F_1. Thus $F_0 = \phi_Y \cup_f H$ is homotopic to a homotopy equivalence, thus itself a homotopy equivalence. Applying Proposition 2.1 to the (on the nose) commutative diagram

$$\begin{tikzcd} Y \cup_f A \times I \arrow[d, "\simeq"', "\phi_Y \cup_f H"] & A \arrow[l, hook', "\text{at } 0"'] \arrow[r, hook, "i"] \arrow[d, "\simeq"', "\phi_A"] & X \arrow[d, "\simeq"', "\phi_X"] \\ Y' & A' \arrow[l, "f'"'] \arrow[r, hook, "i'"] & X' \end{tikzcd}$$

yields the result that

$$\phi_Y \cup_f H \cup \phi_X : Y \cup_f A \times I \cup X \longrightarrow Y' \cup_{f'} X'$$

is a homotopy equivalence. Applying Proposition 2.1 to the commutative diagram

$$\begin{tikzcd} Y' \cup_{f'} A' \times I \arrow[d, "r'", "\simeq"'] & A' \arrow[l, hook', "\mathrm{at}\,0"'] \arrow[r, hook, "i'"] \arrow[d, equal] & X' \arrow[d, equal] \\ Y' & A' \arrow[l, "f'"'] \arrow[r, hook, "i'"] & X' \end{tikzcd}$$

where r' is the mapping cylinder retraction, yields an equivalence

$$Y' \cup_{f'} A' \times I \cup X' \xrightarrow{\simeq} Y' \cup_{f'} X'.$$

Both equivalences together show that

$$Y \cup_f A \times I \cup_i X \simeq Y' \cup_{f'} A' \times I \cup_{i'} X'.$$

□

Proposition 2.3. *If a manifold M is decomposed as $M = M_+ \cup_{M_0} M_-$, with M_0 a compact codimension one submanifold and $M_\pm$ codimension 0 submanifolds with common boundary $\partial M_+ = M_0 = M_+ \cap M_- = \partial M_-$ so that $M = |\Gamma|$ with $\Gamma = (M_+ \leftarrow M_0 \rightarrow M_-)$, then there is a homeomorphism $|\Gamma| \cong |H(\Gamma)|$.*

Proof. The codimension one submanifold $M_0 = \partial M_+$ has a collar in M_+ and a collar in M_-, as $M_0 = \partial M_-$. Using this bicollar, a homeomorphism to the double mapping cylinder can be readily constructed. □

3. Spatial Homology Truncation

The goal of spatial homology truncation is to associate to a CW complex K and an integer k a complex $t_{<k}K$ together with a cellular map $t_{<k}K \to K$, which induces an isomorphism $H_r(t_{<k}K) \to H_r(K)$ in degrees $r < k$, whereas $H_r(t_{<k}K) = 0$ for $r \geq k$.

Definition 3.1. A CW complex K is called *k-segmented* if it contains a subcomplex $K_{<k} \subset K$ such that $H_r(K_{<k}) = 0$ for $r \geq k$ and

$$i_* : H_r(K_{<k}) \xrightarrow{\cong} H_r(K) \text{ for } r < k,$$

where i is the inclusion of $K_{<k}$ into K.

Not every k-dimensional complex is k-segmented, but if K is simply connected, then K is homotopy equivalent to a k-segmented complex by [1, Prop. 1.6, p. 12]. If the group of k-cycles of a k-dimensional CW complex K has a basis of cells, then K is k-segmented. Spatial homology truncation should also apply to maps $f : K \to L$. However, counterexamples in [1] show that in general there need not exist a truncated map $t_{<k}f : t_{<k}K \to t_{<k}L$, which fits with the structural maps into a homotopy commutative square, see pages 3–5 and p. 39 of *loc. cit.* This problem can be addressed by introducing the following category.

Definition 3.2. The category $\mathbf{CW}_{k\supset\partial}$ of *k-boundary-split CW complexes* consists of the following objects and morphisms: Objects are pairs (K, Y), where K is a simply connected CW complex and $Y \subset C_k(K;\mathbb{Z})$ is a subgroup of the k-th cellular chain group of K that arises as the image $Y = s(\operatorname{im}\partial)$ of some splitting $s : \operatorname{im}\partial \to C_k(K;\mathbb{Z})$ of the boundary map $\partial : C_k(K;\mathbb{Z}) \to \operatorname{im}\partial (\subset C_{k-1}(K;\mathbb{Z}))$. (Given K, such a splitting always exists, since $\operatorname{im}\partial$ is free abelian.) A morphism $(K, Y_K) \to (L, Y_L)$ is a cellular map $f : K \to L$ such that $f_*(Y_K) \subset Y_L$.

Let $\mathbf{HoCW}_{k-1}$ denote the category whose objects are CW complexes and whose morphisms are rel $(k-1)$-skeleton homotopy classes of cellular maps. Let

$$t_{<\infty} : \mathbf{CW}_{k\supset\partial} \longrightarrow \mathbf{HoCW}_{k-1}$$

be the natural projection functor, that is, $t_{<\infty}(K, Y_K) = K$ for an object (K, Y_K) in $\mathbf{CW}_{k\supset\partial}$, and $t_{<\infty}(f) = [f]$ for a morphism $f : (K, Y_K) \to (L, Y_L)$ in $\mathbf{CW}_{k\supset\partial}$. The following theorem is part of Theorem 1.41 in [1].

Theorem 3.3. *Let $k \geq 3$ be an integer. There is a covariant assignment $t_{<k} : \mathbf{CW}_{k\supset\partial} \longrightarrow \mathbf{HoCW}_{k-1}$ of objects and morphisms together with a natural transformation $\mathrm{emb}_k : t_{<k} \to t_{<\infty}$ such that for an object (K, Y) of $\mathbf{CW}_{k\supset\partial}$, one has $H_r(t_{<k}(K, Y)) = 0$ for $r \geq k$, and*

$$\mathrm{emb}_k(K, Y)_* : H_r(t_{<k}(K, Y)) \xrightarrow{\cong} H_r(K)$$

is an isomorphism for $r < k$.

This means in particular that given a morphism f, one has squares

$$\begin{tikzcd}
t_{<k}(K, Y_K) \arrow[r, "\mathrm{emb}_k(K,Y_K)"] \arrow[d, "t_{<k}(f)"'] & t_{<\infty}(K, Y_K) \arrow[d, "t_{<\infty}(f)"] \\
t_{<k}(L, Y_L) \arrow[r, "\mathrm{emb}_k(L,Y_L)"] & t_{<\infty}(L, Y_L)
\end{tikzcd}$$

that commute in $\mathbf{HoCW}_{k-1}$. If $k \leq 2$, then the situation is much simpler and the category $\mathbf{CW}_{k\supset\partial}$ is not needed at all. For $k = 1$, there is a covariant truncation functor $t_{<1} : \mathbf{CW}^0 \to \mathbf{HoCW}$, where $\mathbf{CW}^0$ is the category of path-connected CW complexes and cellular maps. For $k = 2$, there is a covariant truncation functor $t_{<2} : \mathbf{CW}^1 \to \mathbf{HoCW}$, where $\mathbf{CW}^1$ is the category of simply connected CW complexes and cellular maps. See [1, Section 1.1.5]. We call a space T together with a structural map $e : T \to K$ a *cohomological k-truncation* of K, if $H^r(T) = 0$ for $r \geq k$, and $e^* : H^r(K) \to H^r(T)$ is an isomorphism for $r < k$.

4. Homological Tools

Let j be a positive integer.

Definition 4.1. A CW complex K satisfies *condition* (INJ_j) if and only if the cellular chain boundary operator $\partial_j : C_j(K) \to C_{j-1}(K)$ is injective.

The condition is in particular satisfied if K has no j-cells. It implies of course that $H_j(K) = 0$. Let $Z_j(K) \subset C_j(K)$ denote the subgroup of j-cycles.

Lemma 4.2. *If K satisfies condition* (INJ_j)*, then the following statements hold (for (1) and (2) assume that K is simply connected):*
(1) There is a unique completion of K to an object $(K, Y_j) \in \mathbf{CW}_{j\supset\partial}$, namely $Y_j = C_j(K)$.
(2) There is a unique completion of K to an object $(K, Y_{j+1}) \in \mathbf{CW}_{j+1\supset\partial}$, namely $Y_{j+1} = 0$.
(3) K is j-segmented and $(j+1)$-segmented.
(4) $t_{<j}(K, Y_j) = t_{<j+1}(K, Y_{j+1}) = K^j$.
(5) $t_{<j+1}(K, Y_{j+1})$ is an (integral) cohomological $(j+1)$-truncation.

Proof. The injectivity of $\partial_j : C_j(K) \to C_{j-1}(K)$ means that $Z_j(K) = 0$. Hence, for the decomposition $Z_j(K) \oplus Y_j = C_j(K)$ to hold, we must take $Y_j = C_j(K)$. The injectivity of ∂_j also implies that $\partial_{j+1} = 0 : C_{j+1}(K) \to C_j(K)$ and thus $Z_{j+1}(K) = C_{j+1}(K)$. Hence, for the decomposition $Z_{j+1}(K) \oplus Y_{j+1} = C_{j+1}(K)$ to hold, we must take $Y_{j+1} = 0$. This proves (1) and (2). The $(j+1)$-skeleton of K has the form

$$K^{j+1} = K^{j-1} \cup \bigcup_{\alpha} y_\alpha^j \cup \bigcup_{\beta} z_\beta^{j+1},$$

where the y_α^j are the j-cells and the z_β^{j+1} the $(j+1)$-cells of K. Since $\{z_\beta^{j+1}\}$ is a basis for $Z_{j+1}(K)$, Lemma 1.2 of [1] implies that K^{j+1}, and thus K, is $(j+1)$-segmented. Furthermore, Proposition 1.3 of *loc. cit.* shows that the truncating subcomplex $t_{<j+1}(K, Y_{j+1} = 0) \subset K^{j+1}$ is unique (if we insist on not changing the j-skeleton) and given by $t_{<j+1}(K, Y_{j+1}) = K^j$ because K has no $(j+1)$-cells that are not cycles. Similarly, the empty set is a basis for $Z_j(K) = 0$, so we may apply Lemma 1.2 of [1] to conclude that K^j, and thus K, is j-segmented, proving (3). By Proposition 1.3 *loc. cit.*, the truncating subcomplex $t_{<j}(K, Y_j) \subset K^j$ is unique (if we insist on not changing the $(j-1)$-skeleton) and given by

$$t_{<j}(K, Y_j) = K^{j-1} \cup \bigcup_{\alpha} y_\alpha^j = K^j,$$

since $\{y_\alpha^j\}$ is the set of j-cells of K that are not cycles. This proves statement (4). Statement (5) follows from Remark 1.42 of [1], observing that $\operatorname{Ext}(H_j(K), \mathbb{Z}) = 0$ is a consequence of (INJ_j). □

To a CW-3-diagram we wish to associate certain Mayer-Vietoris type sequences that compute the homology of their homotopy pushouts. Furthermore, to cellular morphisms of such diagrams we wish to associate long exact sequences of these Mayer-Vietoris sequences. This is carried out in the rest of this section through a progression of ever more general statements culminating in Proposition 4.5. The reader may want to consult [12, Chapter 0] for a general setup of n-ads of CW complexes, but we only need $n = 3$, i.e. triads.

Lemma 4.3. *Let $(Q; Q_+, Q_-)$ be a CW-triad so that $Q = Q_+ \cup Q_-$ and $Q_0 = Q_+ \cap Q_-$ is a subcomplex of Q_+ and of Q_-. Let $i : Q_0 \hookrightarrow Q$ be the inclusion map and $q_*^- : C_*(Q) \to C_*(Q)/C_*(Q_+)$, $q_*^+ : C_*(Q) \to C_*(Q)/C_*(Q_-)$ the natural projections. Then:*
(1) The inclusions $Q_- \subset Q$, $Q_+ \subset Q$ induce isomorphisms

$$\frac{C_*(Q_-)}{C_*(Q_0)} \xrightarrow{\cong} \frac{C_*(Q)}{C_*(Q_+)}, \quad \frac{C_*(Q_+)}{C_*(Q_0)} \xrightarrow{\cong} \frac{C_*(Q)}{C_*(Q_-)}.$$

(2) The sequence

$$0 \to C_*(Q_0) \xrightarrow{i_*} C_*(Q) \xrightarrow{(q_*^-, -q_*^+)} \frac{C_*(Q)}{C_*(Q_+)} \oplus \frac{C_*(Q)}{C_*(Q_-)} \to 0$$

is exact.

Proof. (1) Since $Q_+ \cup Q_- = Q$, the claim follows from the short exact sequences

$$0 \to \frac{C_*(Q_+)}{C_*(Q_+ \cap Q_-)} \longrightarrow \frac{C_*(Q)}{C_*(Q_-)} \longrightarrow \frac{C_*(Q)}{C_*(Q_+ \cup Q_-)} \to 0,$$

$$0 \to \frac{C_*(Q_-)}{C_*(Q_+ \cap Q_-)} \longrightarrow \frac{C_*(Q)}{C_*(Q_+)} \longrightarrow \frac{C_*(Q)}{C_*(Q_+ \cup Q_-)} \to 0$$

of the triad $(Q; Q_+, Q_-)$; see [12, p. 5] for these sequences.

(2) The injectivity of i_* is clear. Let $[a] \in C_*(Q)/C_*(Q_+)$, $[b] \in C_*(Q)/C_*(Q_-)$. By (1), there exist chains $\alpha \in C_*(Q_-)$, $\beta \in C_*(Q_+)$ with $q_*^-(\alpha) = [a]$, $q_*^+(\beta) = [b]$. Since $q_*^-(\beta) = 0$ and $q_*^+(\alpha) = 0$, we have

$$\begin{aligned}(q_*^-, -q_*^+)(\alpha - \beta) &= (q_*^-(\alpha - \beta), -q_*^+(\alpha - \beta)) \\ &= (q_*^-(\alpha), -q_*^+(-\beta)) \\ &= ([a], [b]).\end{aligned}$$

Thus $(q_*^-, -q_*^+)$ is surjective. The composition $(q_*^-, -q_*^+) \circ i_*$ is zero because $Q_0 \subset Q_+$, $Q_0 \subset Q_-$. Let $q \in C_*(Q)$ be a chain such that $q_*^-(q) = 0$, $q_*^+(q) = 0$. This implies that $q \in C_*(Q_+) \cap C_*(Q_-) = C_*(Q_+ \cap Q_-) = C_*(Q_0)$, proving exactness at $C_*(Q)$. $\square$

Let $(Q; Q_+, Q_-)$ be a CW-triad with $Q = Q_+ \cup Q_-$ and set $Q_0 = Q_- \cap Q_+$. Let $(R; R_+, R_-)$ be a CW-triad with $R = R_+ \cup R_-$ and set $R_0 = R_- \cap R_+$. Let Γ be the CW-3-diagram

$$Q_+ \hookleftarrow Q_0 \hookrightarrow Q_-$$

and let Θ be the CW-3-diagram

$$R_+ \hookleftarrow R_0 \hookrightarrow R_-.$$

Suppose that Γ is a CW sub-3-diagram of Θ, that is, there is a commutative diagram

$$\begin{tikzcd} Q_+ \arrow[d] & Q_0 \arrow[l] \arrow[r] \arrow[d] & Q_- \arrow[d] \\ R_+ & R_0 \arrow[l] \arrow[r] & R_- \end{tikzcd}$$

of subcomplex inclusions. Assume furthermore that the equations

$$R_+ \cap Q = Q_+, \;\; R_- \cap Q = Q_-$$

hold. These equations imply

$$R_0 \cap Q = (R_+ \cap R_-) \cap Q = (R_+ \cap Q) \cap (R_- \cap Q) = Q_+ \cap Q_- = Q_0.$$

Since $Q_+ = Q \cap R_+$, the triad $(R; R_+, Q)$ has an associated exact sequence

$$0 \to \frac{C_*(Q)}{C_*(Q_+)} \longrightarrow \frac{C_*(R)}{C_*(R_+)} \longrightarrow \frac{C_*(R)}{C_*(Q \cup R_+)} \to 0.$$

Similarly, since $Q_- = Q \cap R_-$, the triad $(R; R_-, Q)$ has an associated exact sequence

$$0 \to \frac{C_*(Q)}{C_*(Q_-)} \longrightarrow \frac{C_*(R)}{C_*(R_-)} \longrightarrow \frac{C_*(R)}{C_*(Q \cup R_-)} \to 0.$$

These two sequences add to give an exact sequence

$$0 \to \frac{C_*(Q)}{C_*(Q_+)} \oplus \frac{C_*(Q)}{C_*(Q_-)} \longrightarrow \frac{C_*(R)}{C_*(R_+)} \oplus \frac{C_*(R)}{C_*(R_-)} \longrightarrow \frac{C_*(R)}{C_*(Q \cup R_+)} \oplus \frac{C_*(R)}{C_*(Q \cup R_-)} \to 0.$$

Lemma 4.3 applied to the triads $(Q; Q_+, Q_-)$ and $(R; R_+, R_-)$ delivers exact sequences

$$0 \to C_*(Q_0) \longrightarrow C_*(Q) \longrightarrow \frac{C_*(Q)}{C_*(Q_+)} \oplus \frac{C_*(Q)}{C_*(Q_-)} \to 0,$$

$$0 \to C_*(R_0) \longrightarrow C_*(R) \longrightarrow \frac{C_*(R)}{C_*(R_+)} \oplus \frac{C_*(R)}{C_*(R_-)} \to 0.$$

We obtain the following commutative 3×3-diagram with exact columns and exact rows:

$$
\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C_*(Q_0) \arrow[r] \arrow[d] & C_*(Q) \arrow[r] \arrow[d] & \frac{C_*(Q)}{C_*(Q_+)} \oplus \frac{C_*(Q)}{C_*(Q_-)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_*(R_0) \arrow[r] \arrow[d] & C_*(R) \arrow[r] \arrow[d] & \frac{C_*(R)}{C_*(R_+)} \oplus \frac{C_*(R)}{C_*(R_-)} \arrow[r] \arrow[d] & 0 \\
& \frac{C_*(R_0)}{C_*(Q_0)} \arrow[d] & \frac{C_*(R)}{C_*(Q)} \arrow[d] & \frac{C_*(R)}{C_*(Q \cup R_+)} \oplus \frac{C_*(R)}{C_*(Q \cup R_-)} \arrow[d] & \\
& 0 & 0 & 0 &
\end{tikzcd}
$$

The inclusion $C_*(R_0) \to C_*(R)$ induces a map $C_*(R_0)/C_*(Q_0) \to C_*(R)/C_*(Q)$. The identity on $C_*(R)$ induces quotient maps

$$
\frac{C_*(R)}{C_*(Q)} \longrightarrow \frac{C_*(R)}{C_*(Q \cup R_+)}, \; \frac{C_*(R)}{C_*(Q)} \longrightarrow \frac{C_*(R)}{C_*(Q \cup R_-)}.
$$

We use these to complete the above diagram to a commutative diagram

$$
\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C_*(Q_0) \arrow[r] \arrow[d] & C_*(Q) \arrow[r] \arrow[d] & \frac{C_*(Q)}{C_*(Q_+)} \oplus \frac{C_*(Q)}{C_*(Q_-)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_*(R_0) \arrow[r] \arrow[d] & C_*(R) \arrow[r] \arrow[d] & \frac{C_*(R)}{C_*(R_+)} \oplus \frac{C_*(R)}{C_*(R_-)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \frac{C_*(R_0)}{C_*(Q_0)} \arrow[r] \arrow[d] & \frac{C_*(R)}{C_*(Q)} \arrow[r] \arrow[d] & \frac{C_*(R)}{C_*(Q \cup R_+)} \oplus \frac{C_*(R)}{C_*(Q \cup R_-)} \arrow[r] \arrow[d] & 0, \\
& 0 & 0 & 0 &
\end{tikzcd}
$$

which has exact top and middle row, as well as exact columns. By the 3×3-lemma, the bottom row is exact as well. Using the isomorphisms of Lemma 4.3(1), this diagram can be rewritten as

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C_*(Q_0) \arrow[r] \arrow[d] & C_*(Q) \arrow[r] \arrow[d] & \frac{C_*(Q_-)}{C_*(Q_0)} \oplus \frac{C_*(Q_+)}{C_*(Q_0)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_*(R_0) \arrow[r] \arrow[d] & C_*(R) \arrow[r] \arrow[d] & \frac{C_*(R_-)}{C_*(R_0)} \oplus \frac{C_*(R_+)}{C_*(R_0)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \frac{C_*(R_0)}{C_*(Q_0)} \arrow[r] \arrow[d] & \frac{C_*(R)}{C_*(Q)} \arrow[r] \arrow[d] & \frac{C_*(R)}{C_*(Q \cup R_+)} \oplus \frac{C_*(R)}{C_*(Q \cup R_-)} \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Given a map $a : (X, Y) \to (X', Y')$ of pairs, we write $H_*(a, a|)$ for

$$H_*(X' \cup_a X \times I, (Y' \cup_{a|} Y \times I) \cup (X \times \{0\})).$$

Lemma 4.4. *Let* Θ *be* any *CW-3-diagram*

$$S_+ \xleftarrow{f} S_0 \xrightarrow{g} S_-$$

and let Γ, *given by*

$$P_+ \xleftarrow{f|} P_0 \xrightarrow{g|} P_-,$$

be a cellular subdiagram of Θ. *Then the inclusion morphism*

(3)
$$\begin{tikzcd}
P_+ \arrow[d, hook] & P_0 \arrow[l, "f|"'] \arrow[r, "g|"] \arrow[d, hook] & P_- \arrow[d, hook] \\
S_+ & S_0 \arrow[l, "f"'] \arrow[r, "g"] & S_-
\end{tikzcd}$$

induces on homology the following commutative diagram with exact Mayer-Vietoris-type rows and exact columns:

(4)
$$\begin{tikzcd}
H_r(P_0) \arrow[r] \arrow[d] & H_r|H\Gamma| \arrow[r] \arrow[d] & H_r(g|) \oplus H_r(f|) \arrow[r, "\partial_*"] \arrow[d] & H_{r-1}(P_0) \arrow[d] \\
H_r(S_0) \arrow[r] \arrow[d] & H_r|H\Theta| \arrow[r] \arrow[d] & H_r(g) \oplus H_r(f) \arrow[r, "\partial_*"] \arrow[d] & H_{r-1}(S_0) \arrow[d] \\
H_r(S_0, P_0) \arrow[r] \arrow[d, "\partial_*"] & H_r(|H\Theta|, |H\Gamma|) \arrow[r] \arrow[d, "\partial_*"] & H_r(g, g|) \oplus H_r(f, f|) \arrow[r, "\partial_*"] \arrow[d, "\partial_*"] & H_{r-1}(S_0, P_0) \\
H_{r-1}(P_0) \arrow[r] & H_{r-1}|H\Gamma| \arrow[r] & H_{r-1}(g|) \oplus H_{r-1}(f|). &
\end{tikzcd}$$

Proof. The inclusion $\Gamma \subset \Theta$ induces an inclusion $H(\Gamma) \subset H(\Theta)$:

$$\begin{tikzcd} Q_+ := P_+ \cup_{f|} P_0 \times I \arrow[d, hook] & Q_0 := P_0 \arrow[l, hook', "\text{at } 0"'] \arrow[r, hook, "\text{at } 0"] \arrow[d, hook] & P_- \cup_{g|} P_0 \times I =: Q_- \arrow[d, hook] \\ R_+ := S_+ \cup_f S_0 \times I & R_0 := S_0 \arrow[l, hook', "\text{at } 0"'] \arrow[r, hook, "\text{at } 0"] & S_- \cup_g S_0 \times I =: R_-. \end{tikzcd}$$

Both $H(\Gamma)$ and $H(\Theta)$ are CW-3-diagrams. With

$$Q = |H(\Gamma)| = Q_+ \cup_{Q_0} Q_-, \ R = |H(\Theta)| = R_+ \cup_{R_0} R_-,$$

the equations

$$Q_+ \cap Q_- = Q_0, \ R_+ \cap R_- = R_0, \ R_+ \cap Q = Q_+, \ R_- \cap Q = Q_-$$

hold. Thus the previous considerations yield a commutative 3×3-diagram with exact rows and columns:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & C_*(P_0) \arrow[r] \arrow[d] & C_*|H(\Gamma)| \arrow[r] \arrow[d] & \frac{C_*(P_- \cup_{g|} P_0 \times I)}{C_*(P_0)} \oplus \frac{C_*(P_+ \cup_{f|} P_0 \times I)}{C_*(P_0)} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C_*(S_0) \arrow[r] \arrow[d] & C_*|H(\Theta)| \arrow[r] \arrow[d] & \frac{C_*(S_- \cup_g S_0 \times I)}{C_*(S_0)} \oplus \frac{C_*(S_+ \cup_f S_0 \times I)}{C_*(S_0)} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \frac{C_*(S_0)}{C_*(P_0)} \arrow[r] \arrow[d] & \frac{C_*|H(\Theta)|}{C_*|H(\Gamma)|} \arrow[r] \arrow[d] & \frac{C_*|H(\Theta)|}{C_*(|H\Gamma| \cup S_+ \cup_f S_0 \times I)} \oplus \frac{C_*|H(\Theta)|}{C_*(|H\Gamma| \cup S_- \cup_g S_0 \times I)} \arrow[r] \arrow[d] & 0. \\ & 0 & 0 & 0 & \end{tikzcd}$$

Let $(A; A_+, A_-)$ be the CW-triad

$$A = |H\Theta|, \ A_+ = |H\Gamma| \cup R_+, \ A_- = R_-,$$

which satisfies $A_+ \cup A_- = A$. With $A_0 = A_+ \cap A_-$, we have

$$A_0 = (R_+ \cap R_-) \cup (|H\Gamma| \cap R_-) = S_0 \cup Q_-.$$

The isomorphism

$$\frac{C_*(A)}{C_*(A_+)} \cong \frac{C_*(A_-)}{C_*(A_0)}$$

of Lemma 4.3(1) thus identifies

$$\frac{C_*|H(\Theta)|}{C_*(|H\Gamma| \cup S_+ \cup_f S_0 \times I)} \cong \frac{C_*(S_- \cup_g S_0 \times I)}{C_*(P_- \cup_{g|} P_0 \times I \cup S_0)}.$$

In particular,

$$H_*(|H(\Theta)|, |H\Gamma| \cup S_+ \cup_f S_0 \times I) \cong H_*(g, g|),$$

and similarly

$$H_*(|H(\Theta)|, |H\Gamma| \cup S_- \cup_g S_0 \times I) \cong H_*(f, f|).$$

The above chain-level 3×3-diagram then induces the desired diagram of exact Mayer-Vietoris-type sequences on homology. $\square$

Proposition 4.5. *Let Γ be any CW-3-diagram*

$$X_+ \xleftarrow{\xi_+} X_0 \xrightarrow{\xi_-} X_-$$

and let Γ' be any CW-3-diagram

$$Y_+ \xleftarrow{\eta_+} Y_0 \xrightarrow{\eta_-} Y_-.$$

Any cellular morphism $\epsilon : \Gamma \to \Gamma'$ given by

$$\begin{tikzcd}
X_+ \arrow[d, "\epsilon_+"] & X_0 \arrow[l, "\xi_+"'] \arrow[r, "\xi_-"] \arrow[d, "\epsilon_0"] & X_- \arrow[d, "\epsilon_-"] \\
Y_+ & Y_0 \arrow[l, "\eta_+"'] \arrow[r, "\eta_-"] & Y_-
\end{tikzcd}$$

induces on homology the following commutative diagram with exact Mayer-Vietoris-type rows and exact columns:

(5)
$$\begin{tikzcd}
H_r(X_0) \arrow[r] \arrow[d, "\epsilon_{0*}"'] & H_r|H\Gamma| \arrow[r] \arrow[d, "|H\epsilon|_*"'] & H_r(\xi_-) \oplus H_r(\xi_+) \arrow[r, "\partial_*"] \arrow[d, "\epsilon_{-*} \oplus \epsilon_{+*}"'] & H_{r-1}(X_0) \arrow[d, "\epsilon_{0*}"'] \\
H_r(Y_0) \arrow[r] \arrow[d] & H_r|H\Gamma'| \arrow[r] \arrow[d] & H_r(\eta_-) \oplus H_r(\eta_+) \arrow[r, "\partial_*"] \arrow[d] & H_{r-1}(Y_0) \arrow[d] \\
H_r(\epsilon_0) \arrow[r] \arrow[d, "\partial_*"] & H_r(|H\epsilon|) \arrow[r] \arrow[d, "\partial_*"] & H_r(\eta_-, \xi_-) \oplus H_r(\eta_+, \xi_+) \arrow[r, "\partial_*"] \arrow[d, "\partial_*"] & H_{r-1}(\epsilon_0) \\
H_{r-1}(X_0) \arrow[r] & H_{r-1}|H\Gamma| \arrow[r] & H_{r-1}(\xi_-) \oplus H_{r-1}(\xi_+). &
\end{tikzcd}$$

Here, the map $|H\epsilon| : |H\Gamma| \to |H\Gamma'|$ is induced by ϵ as explained in Section 2.

Proof. Set $S_+ = Y_+ \cup_{\epsilon_+} X_+ \times I$, $S_0 = Y_0 \cup_{\epsilon_0} X_0 \times I$, $S_- = Y_- \cup_{\epsilon_-} X_- \times I$, and define the CW-3-diagram $\Theta = \Gamma' \cup_\epsilon \Gamma \times I$ to be

$$S_+ \xleftarrow{\sigma_+ = \eta_+ \cup \xi_+ \times \mathrm{id}_I} S_0 \xrightarrow{\sigma_- = \eta_- \cup \xi_- \times \mathrm{id}_I} S_-.$$

Then Γ is a cellular subdiagram of Θ by including Γ at the free end of the cylinders:

(6)
$$\begin{tikzcd}
X_+ \arrow[d, hook, "\text{at } 0"] & X_0 \arrow[l, "\xi_+"'] \arrow[r, "\xi_-"] \arrow[d, hook, "\text{at } 0"] & X_- \arrow[d, hook, "\text{at } 0"] \\
Y_+ \cup_{\epsilon_+} X_+ \times I & Y_0 \cup_{\epsilon_0} X_0 \times I \arrow[l, "\sigma_+"'] \arrow[r, "\sigma_-"] & Y_- \cup_{\epsilon_-} X_- \times I.
\end{tikzcd}$$

The canonical cellular inclusion $\iota : \Gamma' \hookrightarrow \Theta$ given by

$$\begin{tikzcd}
Y_+ \arrow[d, hook, "\iota_+"] & Y_0 \arrow[l, "\eta_+"'] \arrow[r, "\eta_-"] \arrow[d, hook, "\iota_0"] & Y_- \arrow[d, hook, "\iota_-"] \\
Y_+ \cup_{\epsilon_+} X_+ \times I & Y_0 \cup_{\epsilon_0} X_0 \times I \arrow[l, "\sigma_+"'] \arrow[r, "\sigma_-"] & Y_- \cup_{\epsilon_-} X_- \times I
\end{tikzcd}$$

induces a cellular morphism $H(\iota) : H(\Gamma') \to H(\Theta)$ given by

$$\begin{tikzcd}
Y_+ \cup_{\eta_+} Y_0 \times J \arrow[d, hook, "\iota_+ \cup \iota_0 \times \mathrm{id}_J"] & Y_0 \arrow[l, hook', "\mathrm{at}\,0"'] \arrow[r, hook, "\mathrm{at}\,0"] \arrow[d, hook, "\iota_0"] & Y_- \cup_{\eta_-} Y_0 \times J \arrow[d, hook, "\iota_- \cup \iota_0 \times \mathrm{id}_J"] \\
S_+ \cup_{\sigma_+} S_0 \times J & S_0 \arrow[l, hook', "\mathrm{at}\,0"'] \arrow[r, hook, "\mathrm{at}\,0"] & S_- \cup_{\sigma_-} S_0 \times J,
\end{tikzcd} \tag{7}$$

where we have written J for the second copy of the unit interval in order to distinguish it from the first copy, I. The realization $|H(\iota)| : |H(\Gamma')| \to |H(\Theta)|$ is a cellular map and a homotopy equivalence, since $|H(\Theta)|$ deformation retracts to $|H(\Gamma')|$ by deformation retracting $I = [0,1]$ to 1. More formally, applying Proposition 2.1 to

$$\begin{tikzcd}
Y_\pm \arrow[d, hook, "\iota_\pm", "\simeq"'] & Y_0 \times \{1\} \arrow[l, "\eta_\pm"'] \arrow[r, hook, "\mathrm{incl}"] \arrow[d, hook, "\iota_0", "\simeq"'] & Y_0 \times J \arrow[d, hook, "\iota_0 \times \mathrm{id}_J", "\simeq"'] \\
S_\pm & S_0 \times \{1\} \arrow[l, "\sigma_\pm"'] \arrow[r, hook, "\mathrm{incl}"] & S_0 \times J,
\end{tikzcd}$$

we see that $\iota_\pm \cup \iota_0 \times \mathrm{id}_J$ are homotopy equivalences. Then applying Proposition 2.1 to (7), we deduce that $|H(\iota)|$ is an equivalence. Diagram (6) is of type (3), so that by Lemma 4.4, we have a commutative diagram

$$\begin{tikzcd}
H_r(X_0) \arrow[r] \arrow[d] & H_r|H\Gamma| \arrow[r] \arrow[d] & H_r(\xi_-) \oplus H_r(\xi_+) \arrow[r, "\partial_*"] \arrow[d] & H_{r-1}(X_0) \arrow[d, "\epsilon_{0*}"'] \\
H_r(S_0) \arrow[r] \arrow[d] & H_r|H\Theta| \arrow[r] \arrow[d] & H_r(\sigma_-) \oplus H_r(\sigma_+) \arrow[r, "\partial_*"] \arrow[d] & H_{r-1}(S_0) \arrow[d] \\
H_r(S_0, X_0) \arrow[r] \arrow[d, "\partial_*"] & H_r(|H\Theta|, |H\Gamma|) \arrow[r] \arrow[d, "\partial_*"] & H_r(\sigma_-, \xi_-) \oplus H_r(\sigma_+, \xi_+) \arrow[r, "\partial_*"] \arrow[d, "\partial_*"] & H_{r-1}(S_0, X_0) \\
H_{r-1}(X_0) \arrow[r] & H_{r-1}|H\Gamma| \arrow[r] & H_{r-1}(\xi_-) \oplus H_{r-1}(\xi_+) &
\end{tikzcd}$$

with exact rows and columns. Using the deformation retraction $I = [0,1] \mapsto 1$ throughout the diagram, we obtain the desired diagram (5). $\square$

In constructing the duality isomorphism D of Theorem 6.2, we shall make use of Lemma 4.6 below, a standard result from linear algebra. The lemma is ultimately really only relevant in the middle dimension, see Remark 6.4 following the proof of the theorem.

Lemma 4.6. *(*[1, Lemma 2.46]*) Let*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] & D \arrow[r] \arrow[d] & E \arrow[d] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

be a commutative diagram of rational vector spaces with exact rows. Then there exists a map $C \to C'$ completing the diagram commutatively.

5. Construction of the Intersection Spaces

Let X^n be an oriented, compact, PL stratified pseudomanifold of even dimension $n = 2k$ with a PL stratification of the form $X_n = X^n \supset X_1 \supset X_0$, $X_1 \cong S^1$, $X_0 = \{x_0\}$. There are thus three strata. (The case of a depth 1 stratified space X^n with stratification $\hat{X}_n = X^n \supset \hat{X}_1 \cong S^1$, $\hat{X}_0 = \varnothing$, and possibly twisted link bundle (mapping torus) can be treated within the present framework by inserting a point $x_0 \in \hat{X}_1$ as a new stratum $\hat{X}_0 = \{x_0\}$, whose link is the suspension of the link of $\hat{X}_1$.) Let N_0 be a regular neighborhood of x_0 in X. Then $N_0 = \mathrm{cone}(L_0)$, where L_0 is a compact PL stratified pseudomanifold of dimension $n-1$, the link of x_0. Set $X' = X - \mathrm{int}(N_0)$, a compact pseudomanifold with boundary. This X' has one singular stratum, $X_1' = X_1 \cap X' \cong \Delta^1$, where Δ^1 is a closed interval. Let L be the link of X_1', a closed manifold of dimension $n-2$. To be able to carry out spatial homology truncation, we assume that the links L and L_0 are simply connected. (In specific cases this assumption is not always necessary, since a space may very well have a Moore approximation even if it is not simply connected.) The space L_0 may be singular with singular stratum $L_0 \cap X_1 = L_0 \cap X_1' = \partial\Delta^1 = \{\Delta^0_0, \Delta^0_1\}$ (two points). The link L, being triangulable, certainly has some CW structure.

Assumption: The space L possesses a CW structure such that condition (INJ_{k-1}) is satisfied.

(This is the strong Witt condition from the introduction.) Fix such a CW structure on L from now on. A regular neighborhood of Δ^0_i, $i = 0, 1$, in L_0 is PL homeomorphic to $\mathrm{cone}(L)$. If we remove the interiors of these two cones from L_0, we obtain a compact $(n-1)$-manifold W, which is a bordism between L at Δ^0_0 and L at Δ^0_1. Choose any CW structure on W so that ∂W is a subcomplex (This is possible, since W can be triangulated with ∂W as a simplicial subcomplex.) A normal regular neighborhood of X_1' in X' is PL homeomorphic to a product $\Delta^1 \times \mathrm{cone}(L)$. In more detail, this can be seen as follows: By Theorem 2.1 of [11], a normal regular neighborhood N of X_1' in X' is the total space $N = |\xi|$ of a cone block bundle ξ, with fiber $\mathrm{cone}(L)$ over X_1'. As the base X_1' is PL homeomorphic to Δ^1, Theorem I, 1.1 of [11, Appendix] applies to show that ξ is trivial, that is, there is a cone block bundle isomorphism $\xi \cong X_1' \times \mathrm{cone}(L)$. Thus $N = |\xi| \cong X_1' \times \mathrm{cone}(L) \cong \Delta^1 \times \mathrm{cone}(L)$. Removing from X' the preimage of $X_1' \times \mathring{\mathrm{cone}}(L)$, where $\mathring{\mathrm{cone}}(L)$ denotes the open cone, under the trivialization, we get a compact n-manifold M with boundary ∂M. In order to describe ∂M as the realization of a 3-diagram, set $N_L = \mathrm{cl}\,\partial(N - (N \cap \partial X'))$, where cl is closure in X'. Then N_L is the total space of the link bundle of $X_1 - X_0$, restricted to X_1'. In the terminology of [11], N_L is the *rim* of the cone block bundle ξ. This rim is a compact manifold with boundary ∂N_L which is equal to the boundary of W. Let us denote this common boundary by Λ. Then $\partial M = |\Theta|$, where Θ is the 3-diagram

$$W \xleftarrow{\mathrm{incl}} \Lambda \xhookrightarrow{\mathrm{incl}} N_L.$$

Let

$$\phi : (N, N_L, N \cap \partial X') \xrightarrow{\cong} (\Delta^1 \times \mathrm{cone}(L), \Delta^1 \times L, (\partial\Delta^1) \times \mathrm{cone}(L))$$

denote the above trivialization of the regular neighborhood and let Γ_∂ be the 3-diagram

$$W \xleftarrow{f'} \ddot{L} \xhookrightarrow{\mathrm{incl} \times \mathrm{id}} \overline{L},$$

where we wrote $\overline{L} = \Delta^1 \times L$, $\ddot{L} = \partial\Delta^1 \times L$, and the map f' is the composition

$$\ddot{L} \xrightarrow{\phi|^{-1}} \Lambda \hookrightarrow W.$$

Then ϕ induces a morphism $\Theta \to \Gamma_\partial$ given by

$$\begin{tikzcd} W \arrow[d, equal] & \Lambda \arrow[l, "\mathrm{incl}"'] \arrow[r, hook, "\mathrm{incl}"] \arrow[d, "\cong"', "\phi|"] & N_L \arrow[d, "\cong"', "\phi|"] \\ W & \ddot{L} \arrow[l, "f'"'] \arrow[r, hook, "\mathrm{incl}\times\mathrm{id}"] & \overline{L}. \end{tikzcd}$$

This morphism induces a homeomorphism $\partial M = |\Theta| \cong |\Gamma_\partial|$, and a homeomorphism $|H(\Theta)| \cong |H(\Gamma_\partial)|$. For example, if the link-type does not change running along $X_1 - X_0$ into x_0, then L_0 is the suspension of L and W is the cylinder $W \cong I \times L$. The boundary of M is a mapping torus with fiber L and we may think of f' as the monodromy of the mapping torus. In the diagram Γ_∂, $\overline{L}$ is equipped with the product CW structure. The map f' is in general not cellular.

We shall proceed to define the middle perversity intersection spaces $I^{\bar{m}}X$ and $I^{\bar{n}}X$. It will turn out that the above strong Witt assumption (INJ_{k-1}) on L implies that $I^{\bar{m}}X = I^{\bar{n}}X$. Roughly, the construction paradigm of intersection spaces says that in order to obtain $I^{\bar{p}}X$, for a given perversity $\bar{p}$, every link $\mathcal{L}$ of a stratum of codimension c must be replaced by its spatial homology $k_{\mathcal{L}}(\bar{p})$-truncation (Moore approximation), where

$$k_{\mathcal{L}}(\bar{p}) = c - 1 - \bar{p}(c).$$

The first step is to replace Γ_∂ by a CW-3-diagram Γ in which f' is replaced by a cellular approximation. Thus, let Γ be the CW-3-diagram

$$W \xleftarrow{f} \ddot{L} \hookrightarrow \overline{L},$$

where f is a cellular approximation of f'. In the h-morphism $\Gamma_\partial \to \Gamma$ defined by

$$\begin{tikzcd} W \arrow[d, equal] & \ddot{L} \arrow[l, "f'"'] \arrow[r, hook] \arrow[d, equal] & \overline{L} \arrow[d, equal] \\ W & \ddot{L} \arrow[l, "f"'] \arrow[r, hook] & \overline{L}, \end{tikzcd}$$

the left hand square commutes up to homotopy and the right hand square commutes. Hence, we may apply Proposition 2.2 to obtain a homotopy equivalence

$$|H(\Gamma_\partial)| \simeq |H(\Gamma)|.$$

By Proposition 2.3,

$$|\Theta| \cong |H(\Theta)|.$$

Composing, we get a homotopy equivalence

$$\partial M = |\Theta| \cong |H(\Theta)| \cong |H(\Gamma_\partial)| \simeq |H(\Gamma)|.$$

The space $|H(\Gamma)|$ will be the homotopy theoretic model of the boundary of M that we will subsequently work with.

Let us first discuss the intersection space for the lower middle perversity $\bar{p} = \bar{m}$. For our X, we must truncate L and W. The truncation degrees are

$$\begin{aligned} k_L(\bar{m}) &= n - 2 - \bar{m}(n-1) = k, \\ k_W(\bar{m}) &= n - 1 - \bar{m}(n) = k. \end{aligned}$$

Thus there is one common cut-off degree for both L and W, namely k.

By Lemma 4.2(2), $(L, Y_L = 0)$ is the unique completion of L to an object in $\mathbf{CW}_{k\supset\partial}$. Note that W is simply connected: Write W' for the space obtained from L_0 by deleting one of the two points in $L_0 \cap X_1$. A neighborhood in L_0 of such a point is PL homeomorphic to the cone on L. By the Seifert-van Kampen theorem,

$$1 = \pi_1(L_0) \cong \pi_1(W') *_{\pi_1(L)} \pi_1(\mathrm{cone}(L)) = \pi_1(W')$$

and so

$$1 = \pi_1(W') \cong \pi_1(W) *_{\pi_1(L)} \pi_1(\mathrm{cone}(L)) = \pi_1(W),$$

using the simple connectivity of L. Let (W, Y_W) be any completion of W to an object in $\mathbf{CW}_{k\supset\partial}$. Let $f_i : L = \Delta^0_i \times L \to W$ be the restriction of f to $\Delta^0_i \times L \subset \partial\Delta^1 \times L = \ddot{L}$, $i = 0, 1$. Since the cellular maps f_i satisfy $f_{i*}(Y_L) \subset Y_W$, they both define morphisms $f_i : (L, Y_L) \to (W, Y_W)$ in $\mathbf{CW}_{k\supset\partial}$. Thus there exist truncation cellular maps $t_{<k}(f_i) : t_{<k}(L, Y_L) \to t_{<k}(W, Y_W)$ such that

$$\begin{tikzcd} L_{<k} \arrow[r, "t_{<k}(f_i)"] \arrow[d, "e_L"'] & W_{<k} \arrow[d, "e_W"] \\ L \arrow[r, "f_i"] & W \end{tikzcd}$$

commutes (a priori) up to homotopy rel $(k-1)$-skeleton, where we have written $L_{<k} = t_{<k}(L, Y_L)$, $W_{<k} = t_{<k}(W, Y_W)$, e_L is a cellular rel $(k-1)$-skeleton representative of the homotopy class $\mathrm{emb}_k(L, Y_L)$, and e_W is a cellular rel $(k-1)$-skeleton representative of $\mathrm{emb}_k(W, Y_W)$. We set $\ddot{L}_{<k} = (\Delta^0_0 \times L_{<k}) \sqcup (\Delta^0_1 \times L_{<k})$,

$$t_{<k}(f) = t_{<k}(f_0) \sqcup t_{<k}(f_1) : \ddot{L}_{<k} \longrightarrow W_{<k},$$

and $e_{\ddot{L}} = e_L \sqcup e_L : \ddot{L}_{<k} \to \ddot{L}$. The diagram

$$\begin{tikzcd} \ddot{L}_{<k} \arrow[r, "t_{<k}(f)"] \arrow[d, "e_{\ddot{L}}"'] & W_{<k} \arrow[d, "e_W"] \\ \ddot{L} \arrow[r, "f"] & W \end{tikzcd} \tag{8}$$

commutes (a priori) up to homotopy rel $(k-1)$-skeleton. By Lemma 4.2(4), $L_{<k} = L^{k-1}$ and thus $\ddot{L}_{<k} = \ddot{L}^{k-1}$. The map $t_{<k}(f)$ factors as

$$\ddot{L}^{k-1} \xrightarrow{f|} W^{k-1} \subset W_{<k}.$$

The map $e_{\ddot{L}}$ is the skeletal inclusion $\ddot{L}^{k-1} \hookrightarrow \ddot{L}$. Since the restriction of e_W to W^{k-1} is the skeletal inclusion $W^{k-1} \hookrightarrow W$, we deduce that the diagram (8) commutes on the nose, not just up to homotopy.

Applying Proposition 4.5 to the cellular morphism

$$\begin{tikzcd} W_{<k} \arrow[d, "e_W"'] & \ddot{L}_{<k} \arrow[l, "t_{<k}f"'] \arrow[r, equal] \arrow[d, hook] & \ddot{L}_{<k} \arrow[d, hook] \\ W & \ddot{L} \arrow[l, "f"'] \arrow[r, equal] & \ddot{L} \end{tikzcd}$$

yields the commutative diagram

$$
\begin{tikzcd}
H_r(\ddot{L}_{<k}) \arrow[r, "(t_{<k}f)_*"] \arrow[d] & H_r(W_{<k}) \arrow[r] \arrow[d, "e_{W*}"'] & H_r(t_{<k}f) \arrow[r, "\partial_*"] \arrow[d] & H_{r-1}(\ddot{L}_{<k}) \arrow[d] \\
H_r(\ddot{L}) \arrow[r, "f_*"] \arrow[d] & H_r(W) \arrow[r] \arrow[d] & H_r(f) \arrow[r, "\partial_*"] \arrow[d] & H_{r-1}(\ddot{L}) \arrow[d] \\
H_r(\ddot{L}, \ddot{L}_{<k}) \arrow[r] \arrow[d, "\partial_*"] & H_r(e_W) \arrow[r] \arrow[d, "\partial_*"] & H_r(f, t_{<k}f) \arrow[r, "\partial_*"] \arrow[d, "\partial_*"] & H_{r-1}(\ddot{L}, \ddot{L}_{<k}) \\
H_{r-1}(\ddot{L}_{<k}) \arrow[r] & H_{r-1}(W_{<k}) \arrow[r] & H_{r-1}(t_{<k}f) &
\end{tikzcd}
\tag{9}
$$

with exact rows and columns.

Lemma 5.1. *The map*

$$H_r(f) \longrightarrow H_r(f, t_{<k}f)$$

is an isomorphism for $r \geq k$, while

$$H_r(f, t_{<k}f) = 0$$

for $r < k$.

Proof. The proof is based on an examination of the above diagram (9) in the three cases $r < k$, $r = k$, and $r > k$. Suppose $r < k$. Then $H_r(W_{<k}) \to H_r(W)$ and $H_{r-1}(W_{<k}) \to H_{r-1}(W)$ are isomorphisms. By exactness of the second column of the diagram, $H_r(e_W) = 0$. Similarly, the exactness of the long sequence of the last column implies that $H_{r-1}(\ddot{L}, \ddot{L}_{<k}) = 0$. By the exactness of the third row, $H_r(f, t_{<k}f) = 0$.

Suppose next that $r = k$. Since L satisfies condition (INJ$_{k-1}$) and $\ddot{L} \cong L \sqcup L$, we have $H_{k-1}(\ddot{L}_{<k}) \cong H_{k-1}(\ddot{L}) = 0$. Together with $H_k(W_{<k}) = 0$, the exactness of the top row shows that $H_k(t_{<k}f) = 0$. An application of the 5-lemma to the ladder

$$
\begin{tikzcd}
H_{k-1}(\ddot{L}_{<k}) \arrow[r] \arrow[d, "\cong"'] & H_{k-1}(W_{<k}) \arrow[r] \arrow[d, "\cong"'] & H_{k-1}(t_{<k}f) \arrow[r, "\partial_*"] \arrow[d] & H_{k-2}(\ddot{L}_{<k}) \arrow[r] \arrow[d, "\cong"'] & H_{k-2}(W_{<k}) \arrow[d, "\cong"'] \\
H_{k-1}(\ddot{L}) \arrow[r] & H_{k-1}(W) \arrow[r] & H_{k-1}(f) \arrow[r, "\partial_*"] & H_{k-2}(\ddot{L}) \arrow[r] & H_{k-2}(W)
\end{tikzcd}
$$

yields that

$$H_{k-1}(t_{<k}f) \longrightarrow H_{k-1}(f)$$

is an isomorphism. The exact sequence

$$0 = H_k(t_{<k}f) \to H_k(f) \to H_k(f, t_{<k}f) \xrightarrow{\partial_* = 0} H_{k-1}(t_{<k}f) \xrightarrow{\cong} H_{k-1}(f)$$

shows that

$$H_k(f) \longrightarrow H_k(f, t_{<k}f)$$

is an isomorphism.

If $r > k$, then using the exact sequences

$$0 = H_r(W_{<k}) \longrightarrow H_r(t_{<k}f) \longrightarrow H_{r-1}(\ddot{L}_{<k}) = 0$$

and

$$0 = H_{r-1}(W_{<k}) \longrightarrow H_{r-1}(t_{<k}f) \longrightarrow H_{r-2}(\ddot{L}_{<k}) = 0,$$

we obtain

$$H_r(t_{<k}f) = 0,\ H_{r-1}(t_{<k}f) = 0.$$

(If $r = k+1$, then $H_{r-2}(\ddot{L}_{<k}) = 0$ is implied by (INJ_{k-1}).) From the exactness of the sequence

$$0 = H_r(t_{<k}f) \to H_r(f) \to H_r(f, t_{<k}f) \xrightarrow{\partial_*} H_{r-1}(t_{<k}f) = 0$$

we deduce that the middle map is an isomorphism. □

Set $\overline{L}_{<k} = \Delta^1 \times L_{<k}$. The notation $\overline{L}_{<k}$ is potentially ambiguous because it could also be construed to indicate a spatial homology truncation $t_{<k}$ of $\overline{L}$. This ambiguity is deliberate, for $\Delta^1 \times L_{<k}$ is indeed a valid homology truncation of $\overline{L}$: The map $\mathrm{id}_{\Delta^1} \times \mathrm{incl} : \overline{L}_{<k} = \Delta^1 \times L_{<k} \to \Delta^1 \times L = \overline{L}$ induces an isomorphism $H_r(\overline{L}_{<k}) \cong H_r(L_{<k}) \cong H_r(L) \cong H_r(\overline{L})$ for $r < k$, and $H_r(\overline{L}_{<k}) \cong H_r(L_{<k})$ vanishes in degrees $r \geq k$. Let $\Gamma^{\bar{m}}$ be the 3-diagram

$$W_{<k} \xleftarrow{t_{<k}(f)} \ddot{L}_{<k} \xhookrightarrow{\mathrm{incl}\times\mathrm{id}} \overline{L}_{<k}.$$

The diagram of commutative squares

$$\begin{tikzcd}
W_{<k} \arrow[d, "e_W"'] & \ddot{L}_{<k} \arrow[l, "t_{<k}(f)"'] \arrow[r, hook, "\mathrm{incl}\times\mathrm{id}"] \arrow[d, hook, "e_{\ddot{L}}=\mathrm{incl}"'] & \overline{L}_{<k} \arrow[d, hook, "\mathrm{id}_{\Delta^1}\times\mathrm{incl}"] \\
W & \ddot{L} \arrow[l, "f"'] \arrow[r, hook, "\mathrm{incl}\times\mathrm{id}"] & \overline{L}
\end{tikzcd}$$

defines a cellular morphism $\epsilon : \Gamma^{\bar{m}} \to \Gamma$, which induces a cellular map $|H(\epsilon)| : |H(\Gamma^{\bar{m}})| \to |H(\Gamma)|$.

Definition 5.2. The *lower middle perversity intersection space* $I^{\bar{m}}X$ of X is the homotopy cofiber of the composition

$$|H(\Gamma^{\bar{m}})| \xrightarrow{|H(\epsilon)|} |H(\Gamma)| \simeq \partial M \hookrightarrow M.$$

For the upper middle perversity $\bar{p} = \bar{n}$, we have the cut-off values

$$\begin{aligned} k_L(\bar{n}) &= n - 2 - \bar{n}(n-1) = k - 1, \\ k_W(\bar{n}) &= n - 1 - \bar{n}(n) = k. \end{aligned}$$

The intersection space $I^{\bar{n}}X$ is defined using the construction principle of Definition 5.2, employing an appropriate diagram $\Gamma^{\bar{n}}$ instead of $\Gamma^{\bar{m}}$. Let us construct this $\Gamma^{\bar{n}}$. Since L satisfies condition (INJ_{k-1}), Lemma 4.2(1) asserts that $(L, Y'_L = C_{k-1}(L))$ is the unique completion of L to an object in $\mathbf{CW}_{k-1\supset\partial}$. Furthermore, by Lemma 4.2(4),

$$t_{<k_L(\bar{n})}(L, Y'_L) = t_{<k-1}(L, Y'_L) = t_{<k}(L, Y_L = 0) = L^{k-1}.$$

Therefore, a CW-3-diagram $\Gamma^{\bar{n}}$ of the required type

$$t_{<k_W(\bar{n})}(W, Y_W) \leftarrow \partial\Delta^1 \times t_{<k_L(\bar{n})}(L, Y'_L) \hookrightarrow \Delta^1 \times t_{<k_L(\bar{n})}(L, Y'_L)$$

can be defined by

$$\begin{aligned} \Gamma^{\bar{n}} &= \Big(t_{<k}(W, Y_W) \leftarrow \partial\Delta^1 \times t_{<k-1}(L, Y'_L) \hookrightarrow \Delta^1 \times t_{<k-1}(L, Y'_L)\Big) \\ &= \Big(W_{<k} \leftarrow \partial\Delta^1 \times t_{<k}(L, Y_L) \hookrightarrow \Delta^1 \times t_{<k}(L, Y_L)\Big) \\ &= \Big(W_{<k} \xleftarrow{t_{<k}(f)} \ddot{L}_{<k} \hookrightarrow \overline{L}_{<k}\Big) \\ &= \Gamma^{\bar{m}}. \end{aligned}$$

Thus, as expected,

$$I^{\bar{m}}X = I^{\bar{n}}X$$

due to the strong Witt assumption on L. We shall denote this space by IX.

6. The Duality Theorem

Rational homology and cohomology will be used throughout this section. Let $e : |H\Gamma^{\bar m}| \to \partial M$ be the composition of $|H\epsilon|$ with the homotopy equivalence $|H\Gamma| \simeq \partial M$.

Proposition 6.1. *Cap product with the fundamental class $[\partial M] \in H_{n-1}(\partial M)$ induces an isomorpism*

$$H^{n-r}|H\Gamma^{\bar m}| \xrightarrow{\cong} H_{r-1}(e)$$

such that

$$\begin{tikzcd}
H^{n-r}(\partial M) \arrow[r, "e^*"] \arrow[d, "-\cap[\partial M]"', "\cong"] & H^{n-r}|H\Gamma^{\bar m}| \arrow[d, "\cong"] \\
H_{r-1}(\partial M) \arrow[r] & H_{r-1}(e)
\end{tikzcd}$$

commutes. This isomorphism is determined uniquely by the above commutativity requirement.

Proof. The morphism $\epsilon : \Gamma^{\bar m} \to \Gamma$ induces a map of standard Mayer-Vietoris sequences for double mapping cylinders:

$$\begin{tikzcd}
H^{n-r-1}(\ddot L) \arrow[r, "\delta^*"] \arrow[d, "\mathrm{restr}"'] & H^{n-r}|H\Gamma| \arrow[r] \arrow[d, "|H\epsilon|^*"'] & H^{n-r}(W) \oplus H^{n-r}(\overline{L}) \arrow[r, "f^* + \mathrm{restr}"] \arrow[d, "e_W^* \oplus \mathrm{restr}"'] & H^{n-r}(\ddot L) \arrow[d, "\mathrm{restr}"'] \\
H^{n-r-1}(\ddot L_{<k}) \arrow[r, "\delta^*"] & H^{n-r}|H\Gamma^{\bar m}| \arrow[r] & H^{n-r}(W_{<k}) \oplus H^{n-r}(\overline{L}_{<k}) \arrow[r] & H^{n-r}(\ddot L_{<k})
\end{tikzcd}$$

(The last arrow in the bottom row is $(t_{<k}f)^* + \mathrm{restr}$.) Using the homotopy equivalence $|H\Gamma| \simeq \partial M$, this diagram may be rewritten as

$$\begin{tikzcd}
H^{n-r-1}(\ddot L) \arrow[r, "\delta^*"] \arrow[d, "\mathrm{restr}"'] & H^{n-r}(\partial M) \arrow[r] \arrow[d, "e^*"'] & H^{n-r}(W) \oplus H^{n-r}(\overline{L}) \arrow[r] \arrow[d, "e_W^* \oplus \mathrm{restr}"'] & H^{n-r}(\ddot L) \arrow[d, "\mathrm{restr}"'] \\
H^{n-r-1}(\ddot L_{<k}) \arrow[r, "\delta^*"] & H^{n-r}|H\Gamma^{\bar m}| \arrow[r] & H^{n-r}(W_{<k}) \oplus H^{n-r}(\overline{L}_{<k}) \arrow[r] & H^{n-r}(\ddot L_{<k}).
\end{tikzcd} \tag{10}$$

An application of Proposition 4.5 to $\epsilon : \Gamma^{\bar m} \to \Gamma$ yields a commutative diagram

$$\begin{tikzcd}
H_{r-1}(\ddot L_{<k}) \arrow[r] \arrow[d] & H_{r-1}|H\Gamma^{\bar m}| \arrow[r] \arrow[d, "|H\epsilon|_*"'] & H_{r-1}(\overline{L}_{<k}, \ddot L_{<k}) \oplus H_{r-1}(t_{<k}f) \arrow[r] \arrow[d, "\mathrm{incl}_* \oplus e_{W*}"'] & H_{r-2}(\ddot L_{<k}) \arrow[d] \\
H_{r-1}(\ddot L) \arrow[r] \arrow[d] & H_{r-1}|H\Gamma| \arrow[r] \arrow[d] & H_{r-1}(\overline{L}, \ddot L) \oplus H_{r-1}(f) \arrow[r] \arrow[d] & H_{r-2}(\ddot L) \arrow[d] \\
H_{r-1}(\ddot L, \ddot L_{<k}) \arrow[r] \arrow[d] & H_{r-1}(|H\epsilon|) \arrow[r] \arrow[d] & H_{r-1}(\overline{L}, \overline{L}_{<k} \cup \ddot L) \oplus H_{r-1}(f, t_{<k}f) \arrow[r] \arrow[d] & H_{r-2}(\ddot L, \ddot L_{<k}) \\
H_{r-2}(\ddot L_{<k}) \arrow[r] & H_{r-2}|H\Gamma^{\bar m}| \arrow[r] & H_{r-2}(\overline{L}_{<k}, \ddot L_{<k}) \oplus H_{r-2}(t_{<k}f) &
\end{tikzcd}$$

with exact rows and columns. Again using $|H\Gamma| \simeq \partial M$, we can in particular extract the following map of Mayer-Vietoris sequences:

$$\begin{array}{ccccccc} H_{r-1}(\ddot{L}) & \longrightarrow & H_{r-1}(\partial M) & \longrightarrow & H_{r-1}(\overline{L}, \ddot{L}) \oplus H_{r-1}(f) & \longrightarrow & H_{r-2}(\ddot{L}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{r-1}(\ddot{L}, \ddot{L}_{<k}) & \longrightarrow & H_{r-1}(e) & \longrightarrow & H_{r-1}(\overline{L}, \overline{L}_{<k} \cup \ddot{L}) \oplus H_{r-1}(f, t_{<k}f) & \longrightarrow & H_{r-2}(\ddot{L}, \ddot{L}_{<k}). \end{array} \tag{11}$$

We shall distinguish the cases $r > k$ and $r \leq k$. Suppose that $r > k$. Then, since $n = 2k$, $n - r < k$ and the maps restr and e_W^* in diagram (10) are isomorphisms. By the 5-lemma, $e^* : H^{n-r}(\partial M) \to H^{n-r}|H\Gamma^{\bar{m}}|$ is an isomorphism as well. Let us prove next that $H_{r-1}(\partial M) \to H_{r-1}(e)$ is an isomorphism. Since $r - 1 \geq k$, the maps

$$H_{r-1}(f) \longrightarrow H_{r-1}(f, t_{<k}f) \text{ and } H_r(f) \longrightarrow H_r(f, t_{<k}f)$$

are isomorphisms by Lemma 5.1. Applying the same lemma to

$$\begin{array}{ccc} \ddot{L}_{<k} & \longrightarrow & \overline{L}_{<k} \\ \downarrow & & \downarrow \\ \ddot{L} & \longrightarrow & \overline{L} \end{array}$$

instead of

$$\begin{array}{ccc} \ddot{L}_{<k} & \longrightarrow & W_{<k} \\ \downarrow & & \downarrow \\ \ddot{L} & \xrightarrow{f} & W, \end{array}$$

we see that the maps

$$H_{r-1}(\overline{L}, \ddot{L}) \longrightarrow H_{r-1}(\overline{L}, \overline{L}_{<k} \cup \ddot{L})$$

and

$$H_r(\overline{L}, \ddot{L}) \longrightarrow H_r(\overline{L}, \overline{L}_{<k} \cup \ddot{L})$$

are isomorphisms. The map

$$H_{r-1}(\ddot{L}) \longrightarrow H_{r-1}(\ddot{L}, \ddot{L}_{<k})$$

is an isomorphism, as follows from the exact sequence

$$H_{r-1}(\ddot{L}_{<k}) \longrightarrow H_{r-1}(\ddot{L}) \longrightarrow H_{r-1}(\ddot{L}, \ddot{L}_{<k}) \xrightarrow{\partial_*} H_{r-2}(\ddot{L}_{<k})$$

by observing that $H_{r-1}(\ddot{L}_{<k}) = 0$ and even $H_{r-2}(\ddot{L}_{<k}) = 0$, since in the worst case (when $r - 2 = k - 1$),

$$H_{k-1}(\ddot{L}_{<k}) = H_{k-1}(L^{k-1}) \oplus H_{k-1}(L^{k-1})$$

and

$$H_{k-1}(L^{k-1}) = \ker(\partial_{k-1} : C_{k-1}(L) \to C_{k-2}(L)) = 0$$

by condition (INJ_{k-1}). The map

$$H_{r-2}(\ddot{L}) \longrightarrow H_{r-2}(\ddot{L}, \ddot{L}_{<k})$$

is injective by $H_{r-2}(\ddot{L}_{<k}) = 0$. In summary, the diagram (11) has the form

$$\begin{tikzcd}
H_r(\overline{L}, \ddot{L}) \oplus H_r(f) \arrow[r, "\cong"] \arrow[d, "\partial_*"'] & H_r(\overline{L}, \overline{L}_{<k} \cup \ddot{L}) \oplus H_r(f, t_{<k}f) \arrow[d, "\partial_*"] \\
H_{r-1}(\ddot{L}) \arrow[r, "\cong"] \arrow[d] & H_{r-1}(\ddot{L}, \ddot{L}_{<k}) \arrow[d] \\
H_{r-1}(\partial M) \arrow[r] \arrow[d] & H_{r-1}(e) \arrow[d] \\
H_{r-1}(\overline{L}, \ddot{L}) \oplus H_{r-1}(f) \arrow[r, "\cong"] \arrow[d, "\partial_*"'] & H_{r-1}(\overline{L}, \overline{L}_{<k} \cup \ddot{L}) \oplus H_{r-1}(f, t_{<k}f) \arrow[d, "\partial_*"] \\
H_{r-2}(\ddot{L}) \arrow[r, hook] & H_{r-2}(\ddot{L}, \ddot{L}_{<k}).
\end{tikzcd}$$

This is enough to deduce from a sharp version of the 5-lemma that $H_{r-1}(\partial M) \to H_{r-1}(e)$ is an isomorphism, as claimed. Let

$$H^{n-r}|H\Gamma^{\bar{m}}| \xrightarrow{\cong} H_{r-1}(e)$$

be the unique isomorphism such that the square

$$\begin{tikzcd}
H^{n-r}(\partial M) \arrow[r, "e^*", "\cong"'] \arrow[d, "-\cap[\partial M]"', "\cong"] & H^{n-r}|H\Gamma^{\bar{m}}| \arrow[d, dotted] \\
H_{r-1}(\partial M) \arrow[r, "\cong"] & H_{r-1}(e)
\end{tikzcd}$$

commutes.

Suppose that $r \leq k$. If $r < k$, then $n - r \geq k + 1$ and $H^{n-r-1}(\ddot{L}_{<k}) = 0$. If $r = k$, then

$$H^{n-r-1}(\ddot{L}_{<k}) = \mathrm{Hom}(H_{k-1}(\ddot{L}^{k-1}), \mathbb{Q}) = 0$$

by condition (INJ_{k-1}). Since $H^{n-r}(W_{<k}) = 0$ and $H^{n-r}(\overline{L}_{<k}) = 0$, the exactness of

$$H^{n-r-1}(\ddot{L}_{<k}) \xrightarrow{\delta^*} H^{n-r}|H\Gamma^{\bar{m}}| \longrightarrow H^{n-r}(W_{<k}) \oplus H^{n-r}(\overline{L}_{<k})$$

shows that

$$H^{n-r}|H\Gamma^{\bar{m}}| = 0.$$

We shall show that $H_{r-1}(e) = 0$ also. The exactness of

$$H_{r-1}(\ddot{L}_{<k}) \xrightarrow{\cong} H_{r-1}(\ddot{L}) \xrightarrow{0} H_{r-1}(\ddot{L}, \ddot{L}_{<k}) \xrightarrow{0} H_{r-2}(\ddot{L}_{<k}) \xrightarrow{\cong} H_{r-2}(\ddot{L})$$

implies that $H_{r-1}(\ddot{L}, \ddot{L}_{<k}) = 0$. Since $r-1 < k$, we infer from Lemma 5.1 that $H_{r-1}(f, t_{<k}f) = 0$. Similarly, $H_{r-1}(\overline{L}, \overline{L}_{<k} \cup \ddot{L}) = 0$, which can either also be deduced from Lemma 5.1 by taking $W = \overline{L}$, $f = \mathrm{incl} : \ddot{L} \hookrightarrow \overline{L}$, $W_{<k} = \overline{L}_{<k} = I \times L^{k-1}$, $t_{<k}(f) = \mathrm{incl} : (\partial I) \times L^{k-1} \to I \times L^{k-1}$, or directly from the exact sequence

$$0 = H_{r-1}(I \times L, I \times L_{<k}) \longrightarrow H_{r-1}(\overline{L}, \overline{L}_{<k} \cup \ddot{L}) \longrightarrow H_{r-2}((\partial I) \times L, (\partial I) \times L_{<k}) = 0.$$

The vanishing of $H_{r-1}(e)$ follows from the exactness of

$$H_{r-1}(\ddot{L}, \ddot{L}_{<k}) \longrightarrow H_{r-1}(e) \longrightarrow H_{r-1}(f, t_{<k}f) \oplus H_{r-1}(\overline{L}, \overline{L}_{<k} \cup \ddot{L}).$$

Thus for $r \leq k$, the zero map is the unique isomorphism

$$H^{n-r}|H\Gamma^{\bar{m}}| \xrightarrow{\cong} H_{r-1}(e)$$

such that the commutativity requirement is met. □

To a triple of continuous maps

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[dr, "\psi\phi"'] & B \arrow[d, "\psi"] \\ & C \end{tikzcd}$$

one can associate the 3-diagrams

$$\Gamma(\phi) = \Big(B \xleftarrow{\phi} A \times \{1\} \hookrightarrow \operatorname{cone}(A)\Big),\ \Gamma(\psi) = \Big(C \xleftarrow{\psi} B \times \{1\} \hookrightarrow \operatorname{cone}(B)\Big),$$

$$\Gamma(\psi\phi) = \Big(C \xleftarrow{\psi\phi} A \times \{1\} \hookrightarrow \operatorname{cone}(A)\Big),$$

and the morphisms $\Gamma(\phi) \to \Gamma(\psi\phi) \to \Gamma(\psi)$ given by

$$\begin{tikzcd} B \arrow[d, "\psi"'] & A \times \{1\} \arrow[l, "\phi"'] \arrow[r, hook] \arrow[d, "\mathrm{id}"'] & \operatorname{cone}(A) \arrow[d, "\mathrm{id}"'] \\ C \arrow[d, "\mathrm{id}"'] & A \times \{1\} \arrow[l, "\psi\phi"'] \arrow[r, hook] \arrow[d, "\phi"'] & \operatorname{cone}(A) \arrow[d, "\operatorname{cone}(\phi)"'] \\ C & B \times \{1\} \arrow[l, "\psi"'] \arrow[r, hook] & \operatorname{cone}(B). \end{tikzcd}$$

These morphisms induce maps $|\Gamma(\phi)| \to |\Gamma(\psi\phi)| \to |\Gamma(\psi)|$. Applying this to the triple

(12)
$$\begin{tikzcd} {|H\Gamma^{\bar{m}}|} \arrow[r, "e"] \arrow[dr, "g"'] & \partial M \arrow[d, hook, "j"] \\ & M, \end{tikzcd}$$

and observing $|\Gamma(g)| = IX$ and $|\Gamma(j)| = M/\partial M$, we obtain a map $\gamma : IX \to M/\partial M$. Let $\mu : M \to IX$ denote the canonical inclusion of the target of the map g into the mapping cone IX of this map.

Theorem 6.2. *Let X be an n-dimensional, compact, oriented PL pseudomanifold with n even. Suppose X can be endowed with a PL stratification of the form $X = X_n \supset X_1 \supset X_0$ with $X_1 \cong S^1$ and X_0 a point, such that the links of the two strata are simply connected and X satisfies the strong Witt condition. Then there exists a Poincaré duality isomorphism*

$$D : \widetilde{H}^{n-r}(IX) \xrightarrow{\cong} \widetilde{H}_r(IX)$$

for the reduced (co)homology of the middle perversity intersection space IX of X that extends Poincaré duality for the exterior $(M, \partial M)$ of the singular set, that is, D makes

$$\begin{tikzcd} \widetilde{H}^{n-r}(IX) \arrow[r, "\mu^*"] \arrow[d, "\cong"', "D"] & H^{n-r}(M) \arrow[d, "\cong"', "-\cap[M,\partial M]"] \\ \widetilde{H}_r(IX) \arrow[r, "\gamma_*"] & H_r(M, \partial M) \end{tikzcd}$$

commute.

Proof. The isomorphism D will be fitted into an isomorphism between the cohomology exact sequence of the pair $|H\Gamma^{\bar{m}}| \to M$ and the complementary homology exact sequence of the triple (12). Proposition 6.1 provides a commutative square

$$\begin{tikzcd} H^{n-r}(\partial M) \arrow[r, "e^*"] \arrow[d, "-\cap[\partial M]"', "\cong"] & H^{n-r}|H\Gamma^{\bar{m}}| \arrow[d, "\cong"] \\ H_{r-1}(\partial M) \arrow[r] & H_{r-1}(e). \end{tikzcd} \tag{13}$$

The connecting homomorphism $\partial_* : H_n(M, \partial M) \to H_{n-1}(\partial M)$ sends the fundamental class $[M, \partial M]$ to $\partial_*[M, \partial M] = [\partial M]$. Since for $j^* : H^{n-r}(M) \to H^{n-r}(\partial M)$ and $\xi \in H^{n-r}(M)$ we have

$$\partial_*(\xi \cap [M, \partial M]) = j^*\xi \cap \partial_*[M, \partial M]$$

(see [10], Chapter 5, Section 6, 20, page 255), the square

$$\begin{tikzcd} H^{n-r}(M) \arrow[r, "j^*"] \arrow[d, "-\cap[M,\partial M]"', "\cong"] & H^{n-r}(\partial M) \arrow[d, "\cong"', "-\cap[\partial M]"] \\ H_r(M, \partial M) \arrow[r, "\partial_*"] & H_{r-1}(\partial M) \end{tikzcd} \tag{14}$$

commutes. Since $g^* = e^* \circ j^*$ and the connecting homomorphism

$$\partial_* : H_r(M, \partial M) \longrightarrow H_{r-1}(e)$$

of the triple factors as

$$H_r(M, \partial M) \xrightarrow{\partial_*} H_{r-1}(\partial M) \longrightarrow H_{r-1}(e),$$

composing diagram (14) and diagram (13) yields a commutative square

$$\begin{tikzcd} H^{n-r}(M) \arrow[r, "g^*"] \arrow[d, "-\cap[M,\partial M]"', "\cong"] & H^{n-r}|H\Gamma^{\bar{m}}| \arrow[d, "\cong"] \\ H_r(M, \partial M) \arrow[r, "\partial_*"] & H_{r-1}(e). \end{tikzcd}$$

We use these squares in the diagram

$$\begin{tikzcd} H^{n-r-1}(M) \arrow[r, "g^*"] \arrow[d, "-\cap[M,\partial M]"', "\cong"] & H^{n-r-1}|H\Gamma^{\bar{m}}| \arrow[r, "\delta^*"] \arrow[d, "\cong"] & H^{n-r}(g) \arrow[r] & H^{n-r}(M) \arrow[r, "g^*"] \arrow[d, "-\cap[M,\partial M]"', "\cong"] & H^{n-r}|H\Gamma^{\bar{m}}| \arrow[d, "\cong"] \\ H_{r+1}(M, \partial M) \arrow[r, "\partial_*"] & H_r(e) \arrow[r, "j_*"] & H_r(g) \arrow[r] & H_r(M, \partial M) \arrow[r, "\partial_*"] & H_{r-1}(e), \end{tikzcd}$$

By Lemma 4.6, there exists a map

$$D : \widetilde{H}^{n-r}(IX) = H^{n-r}(g) \longrightarrow H_r(g) = \widetilde{H}_r(IX)$$

filling in the diagram commutatively. By the 5-lemma, D is an isomorphism. $\square$

Remark 6.3. The simple connectivity conditions on the links L, L_0 only enter in so far as to ensure that the homological truncations $L_{<k}$, $W_{<k}$ exist. Actually, regardless of simple connectivity, the strong Witt condition on L alone guarantees that $L_{<k}$ exists (because then we may take $L_{<k} = L^{k-1}$). The simple connectivity of both L and L_0 is a sufficient condition for the existence of $W_{<k}$, but certainly not a necessary condition. Example 6.5 below illustrates this by considering

nonsimply connected links such that $L_{<k}, W_{<k}$ exist with the correct properties. The simple connectivity assumption never enters otherwise in the proof of Proposition 6.1 and Theorem 6.2, so that these results remain true for nonsimply connected L, L_0, provided the truncations exist.

Remark 6.4. The construction of the duality isomorphism D in the proof of Theorem 6.2 uses Lemma 4.6 and thus involves an element of choice. A canonical construction of an isomorphism $D : \widetilde{H}^{n-r}(IX) \to \widetilde{H}_r(IX)$ in all degrees r except the middle, avoiding that lemma, runs as follows: Suppose $r > k$. Then, as was shown in the proof of Proposition 6.1, $e^* : H^{n-r}(\partial M) \to H^{n-r}|H\Gamma^{\bar{m}}|$ and $H_{r-1}(\partial M) \to H_{r-1}(e)$ are isomorphisms. From the exact sequences

$$H^{n-(r+1+i)}(\partial M) \xrightarrow[e^*]{\cong} H^{n-(r+1+i)}|H\Gamma^{\bar{m}}| \xrightarrow{\delta^*} H^{n-(r+i)}(e)$$
$$\longrightarrow H^{n-(r+i)}(\partial M) \xrightarrow[e^*]{\cong} H^{n-(r+i)}|H\Gamma^{\bar{m}}|, \; i = 0, 1,$$

we deduce that $H^{n-r-1}(e) = 0$ and $H^{n-r}(e) = 0$. The exact triple sequence

$$H^{n-r-1}(e) \xrightarrow{\delta^*} H^{n-r}(M, \partial M) \longrightarrow H^{n-r}(g) \longrightarrow H^{n-r}(e)$$

implies that $H^{n-r}(M, \partial M) \to H^{n-r}(g)$ is an isomorphism. From the sequences

$$H_{r+i}(\partial M) \xrightarrow{\cong} H_{r+i}(e) \xrightarrow{\partial_*} H_{r+i-1}|H\Gamma^{\bar{m}}|$$
$$\longrightarrow H_{r+i-1}(\partial M) \xrightarrow{\cong} H_{r+i-1}(e), \; i = 0, 1,$$

we infer that $H_r|H\Gamma^{\bar{m}}| = 0$ and $H_{r-1}|H\Gamma^{\bar{m}}| = 0$. Hence $H_r(M) \to H_r(g)$ is an isomorphism by the exactness of

$$H_r|H\Gamma^{\bar{m}}| \xrightarrow{g_*} H_r(M) \longrightarrow H_r(g) \xrightarrow{\partial_*} H_{r-1}|H\Gamma^{\bar{m}}|.$$

Define D to be the unique isomorphism such that the square

$$\begin{tikzcd} H^{n-r}(M,\partial M) \arrow[r, "\cong"] \arrow[d, "-\cap[M,\partial M]"', "\cong"] & H^{n-r}(g) \arrow[d, dotted, "D"] \\ H_r(M) \arrow[r, "\cong"] & H_r(g) \end{tikzcd}$$

commutes. Suppose $r < k$. Then, as was established in the proof of Proposition 6.1,

$$H^{n-(r+1)}|H\Gamma^{\bar{m}}| = 0$$

and $H_r(e) = 0$. Therefore, by the exactness of

$$0 = H^{n-(r+1)}|H\Gamma^{\bar{m}}| \xrightarrow{\delta^*} H^{n-r}(g) \longrightarrow H^{n-r}(M) \xrightarrow{g^*} H^{n-r}|H\Gamma^{\bar{m}}| = 0,$$

the map $H^{n-r}(g) \to H^{n-r}(M)$ is an isomorphism. The exact sequence

$$0 = H_r(e) \longrightarrow H_r(g) \longrightarrow H_r(M, \partial M) \xrightarrow{\partial_*} H_{r-1}(e) = 0$$

shows that $H_r(g) \to H_r(M, \partial M)$ is an isomorphism. Define D to be the unique isomorphism such that

$$\begin{tikzcd} H^{n-r}(g) \arrow[r, "\cong"] \arrow[d, dotted, "D"'] & H^{n-r}(M) \arrow[d, "\cong"', "-\cap[M,\partial M]"] \\ H_r(g) \arrow[r, "\cong"] & H_r(M,\partial M) \end{tikzcd}$$

commutes. In the middle dimension $r = k$, we have $H_{k-1}(e) = 0$ and $H_k(\partial M) \to H_k(e)$ is an isomorphism. Hence the commutative diagram with exact rows

$$\begin{tikzcd} H_{k+1}(M,\partial M) \arrow[r, "\partial_*"] \arrow[d, equal] & H_k(\partial M) \arrow[r] \arrow[d, "\cong"'] & H_k(M) \arrow[r] \arrow[d] & H_k(M,\partial M) \arrow[r, "\partial_*"] \arrow[d, equal] & H_{k-1}(\partial M) \arrow[d] \\ H_{k+1}(M,\partial M) \arrow[r, "\partial_*"] & H_k(e) \arrow[r] & H_k(g) \arrow[r] & H_k(M,\partial M) \arrow[r, "\partial_*"] & H_{k-1}(e) = 0 \end{tikzcd}$$

shows that $H_k(g) \to H_k(M, \partial M)$ is a surjection and $H_k(M) \to H_k(g)$ is an injection.

Not every stratified pseudomanifold possesses an intersection space. This is not unexpected in view of the rich internal algebraic structure of HI^*, as opposed to intersection cohomology IH^*. For a given perversity, obstructions to constructing an intersection space arise from certain differentials in the Serre spectral sequences of the link bundles. The techniques introduced in the present paper seem to be useful in studying other depth 2 stratifications as well, or perhaps even higher depth, but will not in general by themselves suffice to construct intersection spaces in more general situations. For example, one might consider a stratification $X_n \supset X_2 \supset X_0$ with X_2 a 2-sphere and X_0 a point. Then, as $X_2 - X_0$ is contractible, the link bundle over $X_2 - X_0$ is trivializable (as it is in this paper) and its total space looks like $\operatorname{int}(D^2) \times L$. The link L_0 of X_0 looks like $L_0 = W \cup_{\partial W} S^1 \times \operatorname{cone}(L)$, where W is a manifold with boundary $\partial W = S^1 \times L$. The exterior ∂M of the singular set looks like $|\Gamma_\partial|$, with Γ_∂ the 3-diagram

$$W \xleftarrow{f} S^1 \times L \hookrightarrow D^2 \times L.$$

The map f is a homeomorphism onto its image ∂W. To form the intersection space, one would have to produce the broken arrow in the 3-diagram

$$\Gamma^{\bar m} = \left(W_{<k_W} \dashleftarrow S^1 \times L_{<k_L} \hookrightarrow D^2 \times L_{<k_L} \right)$$

by suitably truncating f. However, as mentioned above, fiberwise truncation of maps is generally obstructed. Appropriate assumptions on links and/or structure groups of the involved bundles will imply that these obstructions vanish. Thus, particular features of the geometry of a given pseudomanifold X enter in an interesting, nontrivial way to enable or disable the existence of intersection spaces for X.

We conclude with a simple 6-dimensional example, illustrating in particular Proposition 6.1.

Example 6.5. Suppose that X is a 6-dimensional pseudomanifold with stratification

$$X_6 \supset X_1 \supset X_0,$$

X_1 a circle and X_0 a point. Suppose the link L of $X_1 - X_0$ is the 4-manifold $L = S^1 \times S^3$. Let T be a 2-torus with two disjoint small open discs removed. The boundary of T consists of two circles, $\partial T = S^1_c \sqcup S^1_d$. The 5-manifold $W = T \times S^3$ has boundary $\partial W = S^1_c \times S^3 \sqcup S^1_d \times S^3$. Suppose that the link L_0 of X_0 is

$$L_0 = W \cup_{\partial W} (\operatorname{cone}(S^1_c \times S^3) \sqcup \operatorname{cone}(S^1_d \times S^3))$$

and that the link bundle $\Delta^1 \times L$ is attached to W by the identity maps $\{0\} \times L \xrightarrow{\mathrm{id}} S^1_c \times S^3$ and $\{1\} \times L \xrightarrow{\mathrm{id}} S^1_d \times S^3$. We equip the circle factor of L with the CW structure $S^1 = e^0 \cup e^1$ and the 3-sphere factor with the structure $S^3 = e^0_S \cup e^3_S$. Then L receives the product cell structure. We endow T with the CW structure

$$T = (e^0_0 \cup e^0_1) \cup (a \cup b \cup c \cup d \cup e^1_d) \cup e^2,$$

where a, b, c, d and e^1_d are 1-cells such that a, b, c are all attached as loops to e^0_0, whereas d is attached as a loop to e^0_1 and e^1_d joins the two 0-cells e^0_0 and e^0_1. The 2-cell e^2 is attached by the word $abe^1_d d(e^1_d)^{-1}a^{-1}b^{-1}c^{-1}$. Then $S^1_c \subset T$ and $S^1_d \subset T$ are the subcomplexes $S^1_c = e^0_0 \cup c$, $S^1_d = e^0_1 \cup d$. The space W receives the product cell structure. As $\partial_2 : C_2(T) \to C_1(T)$ maps e^2 to $d - c$, we have $[c] = [d] \in H_1(T)$. This group $H_1(T)$ has rank 3 generated by $[a], [b]$ and $[c] = [d]$. Consequently, the homology of W is given by the following generators:

$H_0(W)$	$[e^0_0 \times e^0_S]$
$H_1(W)$	$[a \times e^0_S], [b \times e^0_S], [c \times e^0_S]$
$H_2(W)$	0
$H_3(W)$	$[e^0_0 \times e^3_S]$
$H_4(W)$	$[a \times e^3_S], [b \times e^3_S], [c \times e^3_S]$
$H_5(W)$	0

Note that the strong Witt condition on L is satisfied, as L has no 2-dimensional cells. The link L_0 is not homeomorphic to the suspension of L, since $H_1(L_0)$ has rank 2, generated by $[a \times e^0_S]$ and $[b \times e^0_S]$, while the suspension has trivial first homology. Thus X cannot be restratified with depth 1. Note also that L is not simply connected, but this presents no problem, since the required spatial homology truncation does exist and is given by the 1-skeleton:

$$L_{<k} = L_{<3} = L_{<2} = L^1 = (e^0 \cup e^1) \times e^0_S = S^1 \times \mathrm{pt}\,.$$

The structural map e_L is the inclusion $e_L : L_{<k} = S^1 \times \mathrm{pt} \hookrightarrow S^1 \times S^3 = L$. The spatial homology truncation of W is

$$W_{<k} = W_{<3} = W_{<2} = T \times e^0_S.$$

Thus $W_{<3}$ is precisely the 2-skeleton W^2 of W and the structural map $e_W : W_{<3} \to W$ is the skeletal inclusion $W^2 \hookrightarrow W$. The map $f : \ddot{L} \to W$ is the inclusion given on the component $\{0\} \times L$ by $\{0\} \times L \xrightarrow{\mathrm{id}} S^1_c \times S^3 \hookrightarrow \partial W \hookrightarrow W$ and on the component $\{1\} \times L$ by $\{1\} \times L \xrightarrow{\mathrm{id}} S^1_d \times S^3 \hookrightarrow \partial W \hookrightarrow W$. Its homological truncation $t_{<k}f = t_{<3}f : \ddot{L}_{<3} \to W_{<3}$ is the inclusion given on the two components of $\ddot{L}_{<3}$ by

$$\{0\} \times (e^0 \cup e^1) \times e^0_S \xrightarrow{\mathrm{id}} (e^0_0 \cup c) \times e^0_S \hookrightarrow T \times e^0_S$$

and

$$\{1\} \times (e^0 \cup e^1) \times e^0_S \xrightarrow{\mathrm{id}} (e^0_1 \cup d) \times e^0_S \hookrightarrow T \times e^0_S.$$

The diagram of inclusions

$$\begin{tikzcd}
\ddot{L}_{<3} \arrow[r, hook, "t_{<3}f"] \arrow[d, hook, "e_{\ddot{L}}"'] & W_{<3} \arrow[d, hook, "e_W"] \\
\ddot{L} \arrow[r, hook, "f"] & W
\end{tikzcd}$$

commutes. The $\bar{m}$-perverse 3-diagram $\Gamma^{\bar{m}}$ is given by

$$\begin{aligned}
\Gamma^{\bar{m}} &= (\, W_{<3} \xleftarrow{t_{<3}f} \ddot{L}_{<3} \xhookrightarrow{\mathrm{incl} \times \mathrm{id}} \overline{L}_{<3} \,) \\
&= (\, T \times e^0_S \hookleftarrow (\partial I) \times S^1 \times e^0_S \hookrightarrow I \times S^1 \times e^0_S \,),
\end{aligned}$$

that is, a handle $I \times S^1$ is attached to T along the two boundary circles of the surface T. Hence $|H\Gamma^{\bar{m}}|$ is the orientable closed surface Σ_2 of genus 2. The map

$$e = |H(\epsilon)| : |H\Gamma^{\bar{m}}| \longrightarrow |H\Gamma| = \partial M$$

is given by

$$\mathrm{id}_{\Sigma_2} \times \mathrm{incl} : \Sigma_2 \times e^0_S \hookrightarrow \Sigma_2 \times S^3.$$

A straightforward calculation yields the following table of generators, illustrating the Poincaré duality isomorphism

$$H^{n-r}|H\Gamma^{\bar{m}}| \xrightarrow{\cong} H_{r-1}(e)$$

of Proposition 6.1.

	$H^{n-r}\|H\Gamma^{\bar{m}}\|$	$H_{r-1}(e)$
$r=0$	0	0
$r=1$	0	0
$r=2$	0	0
$r=3$	0	0
$r=4$	$[\Sigma_2 \times \mathrm{pt}]^*$	$[\mathrm{pt} \times S^3]$
$r=5$	$[a \times \mathrm{pt}]^*$	$[b \times S^3]$
	$[b \times \mathrm{pt}]^*$	$[a \times S^3]$
	$[c \times \mathrm{pt}]^*$	$[z \times S^3]$
	$[z \times \mathrm{pt}]^*$	$[c \times S^3]$
$r=6$	$1 = [\mathrm{pt}]^*$	$[\Sigma_2 \times S^3]$

Here, $[\cdot]$ denotes the homology class of a cycle and $[\cdot]^*$ the image in cohomology of the linear dual of a homology class under the universal coefficient isomorphism. Poincaré duals are listed next to each other in the same row. The cycle z is $z = I \cup_{\partial I} e^1_d$.

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HOMOTOPICALLY TRIVIAL DEFORMATIONS

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The aim of this note is to call attention to a question about deformations that are homotopically trivial. First we need a definition.

Definition 1. A proper morphism of complex spaces $f : X \to Y$ is called a *homotopy fiber bundle* if Y has an open cover $Y = \cup U_i$ such that for every i and for every $y \in U_i$ the inclusion

$$f^{-1}(y) \hookrightarrow f^{-1}(U_i) \quad \text{is a homotopy equivalence.}$$

For every $y \in Y$ there is an open neighborhood $y \in U_y$ such that $f^{-1}(y)$ is a deformation retract of $f^{-1}(U_y)$. Choose a retraction $r_y : f^{-1}(U_y) \to f^{-1}(y)$. Thus $f^{-1}(y) \hookrightarrow f^{-1}(U_y)$ is a homotopy equivalence and so f is a homotopy fiber bundle iff for every $y \in Y$ and $y' \in U_y$ the induced map

$$r_{y' \to y} : f^{-1}(y') \to f^{-1}(y) \quad \text{is a homotopy equivalence.}$$

Similarly, if R a commutative ring, then $f : X \to Y$ is called an *R-homology fiber bundle* if

$$H_*\big(f^{-1}(y), R\big) \to H_*\big(f^{-1}(U_i), R\big) \quad \text{is an isomorphism.}$$

As above, these conditions hold iff the retraction maps $r_{y' \to y}$ induce isomorphisms

$$\big(r_{y' \to y}\big)_* : H_*\big(f^{-1}(y'), R\big) \to H_*\big(f^{-1}(y), R\big).$$

We are mostly interested in cases when the fibers of f are irreducible.

If the fibers are reducible, some pathological cases are shown by Example 9. To avoid these, one should also assume that the image of the fundamental class $\big[f^{-1}(y)\big]$ in $H_*\big(f^{-1}(U_i), R\big)$ is independent of y. Equivalently, the retraction $r_{y' \to y}$ maps $\big[f^{-1}(y')\big]$ to $\big[f^{-1}(y)\big]$.

All the examples of $\mathbb{Z}$-homology fiber bundles that we know are also homotopy fiber bundles but being a $\mathbb{Q}$-homology fiber bundle is a much weaker property.

The main problem we want to consider is the following.

Question 2. Let $f : X \to Y$ be a homotopy or $\mathbb{Z}$-homology fiber bundle. Under what conditions is it a topological or differentiable fiber bundle?

If X is non-normal, it is easy to give examples of homotopy fiber bundles $f : X \to Y$ where not all fibers $f^{-1}(y)$ are homeomorphic to each other, see Example 8. In the first version of [14] it was asked whether the answer was positive for normal spaces. We show in Example 19 that this is not the case. However, we still hope that for smooth varieties the situation is as nice as possible.

Conjecture 3. *Let $f : X \to Y$ be a homotopy or $\mathbb{Z}$-homology fiber bundle such that X is smooth. Then f is smooth hence $f : X \to Y$ is a differentiable fiber bundle.*

A stronger version, more closely related to deformation theory is formulated as Conjecture 11.

4 (Origin of the conjecture)**.** We were led to this question by the study of universal covers of projective varieties. Their modern study was initiated by Shafarevich [19, Sec.IX.4]; see [11, 2, 18, 4, 3] and the references there for recent results and surveys. One aim of these investigations is to understand projective varieties whose universal cover is "simple." There are several ways to define what "simple" should mean; here we focus on a topological variant considered in [14].

Question 4.1. Describe projective varieties X whose universal cover $\tilde{X}$ is homotopic to a finite CW complex.

This seems to be a rather difficult problem in general, so here we consider a series of special cases that seem especially important for applications.

Let X be a smooth projective variety and $f : X \to Y$ a surjective morphism. Let $\tilde{Y} \to Y$ denote the universal cover. By pull-back we obtain $\tilde{f} : \tilde{X} \to \tilde{Y}$. In light of [14] the following seems quite plausible.

Question 4.2. Assume that $\tilde{Y}$ is contractible and $\tilde{X}$ is homotopic to a finite CW complex. Does this imply that f is a homotopy fiber bundle?

Conversely, if f is a homotopy fiber bundle and $\tilde{Y}$ is homotopic to a finite CW complex then most likely $\tilde{X}$ is homotopic to a finite CW complex. Thus if Conjecture 3 is true then we would have a rather complete understanding of when a variety X with a nontrivial morphism $X \to Y$ has a "simple" universal cover.

5 (First properties)**.** If $f : X \to Y$ is a $\mathbb{Q}$-homology fiber bundle then all fibers $f^{-1}(y)$ have the same dimension and the same number of irreducible components. Thus if X is normal then, by taking the Stein factorization, we may assume that all fibers are irreducible.

Assume that $g : S \to C$ is an elliptic surface such that all reduced fibers are smooth. Then g is a $\mathbb{Q}$-homology fiber bundle but it is a $\mathbb{Z}$-homology fiber bundle only if there are no multiple fibers.

We see below that there are many $\mathbb{Q}$-homology fiber bundles that are not $\mathbb{Z}$-homology fiber bundles (16, 17).

It is much harder to get nontrivial examples of $\mathbb{Z}$-homology fiber bundles. For now we note two basic results.

Proposition 6. *Let X, Y be normal spaces and $f : X \to Y$ a $\mathbb{Z}$-homology fiber bundle. Then every fiber of f is generically reduced and f is smooth at every smooth point of* $\operatorname{red} f^{-1}(y)$ *for every $y \in Y$.*

Proof. As we noted, we may assume that all fibers are irreducible. In the terminology of [12, I.3.9–10], f is a well defined family of proper algebraic cycles. Moreover, all fibers have multiplicity 1. Thus the scheme theoretic fibers are generically reduced and f is smooth at every smooth point of $\operatorname{red} f^{-1}(y)$ for every $y \in Y$ by [12, I.6.5]. □

Corollary 7. *Let $f : X \to Y$ be a $\mathbb{Z}$-homology fiber bundle where X is smooth and Y is normal. Then Y is smooth and f is flat with local complete intersection fibers.*

Proof. Pick $y \in Y$ and let $x \in \operatorname{red} f^{-1}(y)$ be a smooth point. Then f is smooth at x by Proposition 6. Since X is smooth at x, this implies that Y is smooth at y.

Thus Y is smooth, f is equidimensional and X is smooth, hence Cohen-Macaulay. These imply that f is flat [10, Exrc.III.10.9]. □

Families of curves.

We start with two examples of families of reducible curves.

Example 8. $X \subset \mathbb{P}^3_{\mathbf{x}} \times \mathbb{C}_s$ is a reducible surface and $f : X \to \mathbb{C}_s$ is the coordinate projection. The fiber X_0 is the projective closure of the 3 coordinate axes in $\mathbb{C}^3$ and X_s is obtained by sliding the x_3-axis along the x_2 axis. In concrete equations

$$X := \big(x_1 = x_3 = 0\big) \cup \big(x_2 = x_3 = 0\big) \cup \big(x_1 = x_2 - sx_0 = 0\big) \subset \mathbb{P}^3_{\mathbf{x}} \times \mathbb{C}_s.$$

It is easy to check that $f : X \to \mathbb{C}_s$ is a homotopy fiber bundle, all the fibers are reduced, the fibers X_s are isomorphic to each other for $s \neq 0$ but X_0 is not homeomorphic to X_s for $s \neq 0$.

It is straightforward to modify this example and obtain an irreducible (but still non-normal) surface S with a proper morphism $f : S \to \mathbb{C}$ which is a homotopy fiber bundle such that not all fibers are homeomorphic to each other.

Example 9. Here again $X \subset \mathbb{P}^3_{\mathbf{x}} \times \mathbb{C}_t$ is a reducible surface and $f : X \to \mathbb{C}_t$ is the coordinate projection. The general fiber is a line L_t and a conic C_t intersecting at a point $p \in \mathbb{P}^3$. As we approach the special fiber, the conic degenerates to a pair of lines $L_0 + L_0'$ and the line L_0 is also the limit of the family L_t. In concrete equations

$$X := \big(x_2 = x_3 - tx_1 = 0\big) \cup \big(x_3 = x_0x_2 - tx_2^2 = 0\big) \subset \mathbb{P}^3_{\mathbf{x}} \times \mathbb{C}_s.$$

In this example the retraction map induces isomorphisms

$$\big(r_{y' \to y}\big)_* : H_*\big(L_t \cup C_t, R\big) \to H_*\big(L_0 \cup L_0', R\big)$$

but the fundamental class $\big[L_t \cup C_t\big]$ is mapped to $2\big[L_0\big] + \big[L_0'\big]$ which is different from the fundamental class of the central fiber $\big[L_0 \cup L_0'\big]$.

For curves the following result completes the picture.

Proposition 10. *Let Y be a normal complex space and $f : X \to Y$ a $\mathbb{Z}$-homology fiber bundle of relative dimension 1 with smooth general fibers. Let $\pi : X^n \to X$ be the normalization of X. Then*

(1) $\pi : X^n \to X$ is a homeomorphism and
(2) $f \circ \pi : X^n \to Y$ is smooth hence a differentiable fiber bundle.

Proof. As we noted before, we may assume that f has irreducible fibers.

Let us start with the case when Y is a smooth curve. Let B be a general fiber and B_0 any fiber. Let B_0^n be the corresponding fiber in $X^n \to Y$. Note that the retraction $r : B \to B_0$ factors through B_0^n. It is easy to see that $H_1(B_0^n, \mathbb{Z}) \to H_1(B_0, \mathbb{Z})$ is surjective iff $B_0^n \to B_0$ is a homeomorphism. Thus if $X \to Y$ is a $\mathbb{Z}$-homology fiber bundle then $B_0^n \to B_0$ and $X^n \to X$ are homeomorphisms. We may thus assume that X is normal. In particular the fibers are reduced and $p_a(B_0) = p_a(B) = p_g(B)$ since B is smooth.

Let $B_0' \to B_0$ denote the seminormalization and $B_0'' \to B_0'$ the normalization. If m_i are the multiplicities of the points of B_0' then

$$p_a(B_0) \geq p_a(B_0') = p_a(B_0'') + \textstyle\sum(m_i - 1) = p_g(B_0'') + \sum(m_i - 1).$$

We can thus estimate the topological Euler characteristic as

$$\begin{array}{rcl} \chi^{top}(B_0) & = & \chi^{top}(B_0') = \chi^{top}(B_0'') - \sum(m_i - 1) = 2 - 2p_g(B_0'') - \sum(m_i - 1) \\ & \geq & 2 - 2p_a(B_0) + \sum(m_i - 1). \end{array}$$

On the other hand, if f is a homotopy fiber bundle then

$$\chi^{top}(B_0) = \chi^{top}(B) = 2 - 2p_g(B) = 2 - 2p_a(B_0).$$

Comparing these two we see that $\sum(m_i - 1) = 0$ and $p_a(B_0) = p_a(B_0')$ hence $B_0 \cong B_0' \cong B_0''$. Thus every fiber of f is smooth.

This implies the general case by applying Proposition 12 to the class of all smooth projective curves as $\mathcal{V}$. $\square$

Reduction to 1-parameter families.

Here we show that a variant of Conjecture 3 can be reduced to the case when $\dim Y = 1$. To make this precise, fix a class of projective varieties $\mathcal{V}$ and consider the following.

Conjecture 11. *Let $f : X \to Y$ be a projective morphism of complex spaces, Y normal. Assume that*

(1) there is a Zariski dense open subset $Y^0 \subset Y$ such that the fibers of f over Y^0 are all in $\mathcal{V}$ and

(2) f is a homotopy (resp. $\mathbb{Z}$-homology) fiber bundle.

Let $\pi : X^n \to X$ be the normalization of X. Then

(3) $\pi : X^n \to X$ is a homeomorphism and

(4) $f \circ \pi : X^n \to Y$ is smooth hence a differentiable fiber bundle.

We can now state the precise form of the dimension reduction.

Proposition 12. *Fix a class of smooth projective varieties $\mathcal{V}$ and assume that Conjecture 11 holds for $\mathcal{V}$ whenever $\dim Y = 1$.*

Then Conjecture 11 holds for $\mathcal{V}$ in general.

Proof. Let $\Delta \subset \mathbb{C}$ be the unit disc and $\phi : \Delta \to Y$ any holomorphic map whose image is not contained in $Y \setminus Y^0$. Let X^n_ϕ denote the normalization of $X \times_Y \Delta$. By assumption, $X^n_\phi \to \Delta$ is smooth hence it is the simultaneous normalization of $\phi^* f : X \times_Y \Delta \to \Delta$. In particular, the normalization of the fibers are all smooth and the normalization map is a homeomorphism.

By [13, Thm.1], there is a monomorphism $Y' \to Y$ such that $\phi^* f : X \times_Y \Delta \to \Delta$ has a simultaneous normalization iff ϕ factors through $Y' \to Y$. Thus $Y' = Y$, the composite $f \circ \pi : X^n \to Y$ is smooth and $\pi : X^n \to X$ is a homeomorphism. $\square$

Localization.

Motivated by Proposition 12, from now on we concentrate on 1-parameter families. That is, X is a normal analytic space and $f : X \to \Delta$ a proper morphism with central fiber $X_0 = f^{-1}(0)$. By shrinking Δ we may assume that $X \setminus X^0 \to \Delta^*$ is a topological fiber bundle.

We show that if X_0 has isolated singularities then $\mathbb{Z}$-homology fiber bundles can be characterized in terms of the Milnor fibers of the singular points of X_0. Subsequent examples show that there are global issues for non-isolated singularities.

Proposition 13. *Let X be a normal analytic space and $f : X \to \Delta$ a proper morphism with central fiber $X_0 = f^{-1}(0)$. Assume that X_0 has only isolated singularities $p_i \in X_0$. For each i, let B_i be a small ball around p_i and set $M_{i,t} := X_t \cap B_i$. (If X_t is smooth, this is the Milnor fiber.) The following are equivalent.*

(1) For $0 < |t| \ll 1$, the retraction map $r_t : X_t \to X_0$ is an R-homology equivalence.

(2) For $0 < |t| \ll 1$ every $M_{i,t}$ is an R-homology ball.

Proof. Choose $\Delta_\epsilon \subset \Delta$ small enough so that X_t meets ∂B_i transversely for any i and any $t \in \Delta_\epsilon$. One can choose the retraction such that r_t induces a homeomorphism

$$r_t : X_t \setminus \cup_i M_{i,t} \cong X_0 \setminus \cup_i M_{i,0}.$$

Comparing the long exact homology sequences of the pairs

$$r_t : \big(X_t, \cup_i M_{i,t}\big) \to \big(X_0, \cup_i M_{i,0}\big)$$

we see that $r_t : X_t \to X_0$ is an R-homology equivalence iff the restrictions $r_{i,t} : M_{i,t} \to M_{i,0}$ are R-homology equivalences. Since the $M_{i,0}$ are contractible, the latter holds iff the $M_{i,t}$ are R-homology balls. $\square$

When the source X of the mapping f is smooth, the following local result for non-isolated singularities is a corollary of the work of A'Campo on monodromy of singularities. We thank A'Campo for pointing this out.

Proposition 14. *Let X be smooth and $p \in X$ a point. Let $f : (X,p) \to \Delta$ be a germ of analytic mapping. Let B be a Milnor ball around p and D a Milnor disc around $f(p)$. Set $M_{i,t} := X_t \cap B_i$ for any $t \in D$ (for $t \neq f(p)$ this is the Milnor fiber.) Let R be any ring. The following are equivalent.*

(1) For $0 < |t| \ll 1$, the retraction map $r_t : X_t \to X_0$ is an R-homology equivalence.
(2) The morphism f is smooth at p.

Proof. In [1] it is proved that under the given hypothesis the Lefschetz number of the monodromy of the Milnor fibration equals 0 if f is not smooth at 0 and it is obvious that it equals 1 if f is smooth. If the retraction map is a R-homology equivalence, then the Lefschetz number of the monodromy of the Milnor fibration equals 1. $\square$

Milnor fibers of isolated singularities have been extensively studied. For surfaces the following result seems to have been known but not explicitly stated; see [20, 15] for closely related results. The argument below was shown to us by A. Némethi.

Proposition 15. *Let X be a normal threefold and $f : X \to \Delta$ a $\mathbb{Z}$-homology fiber bundle whose general fiber is smooth and whose central fiber X_0 is normal. Then f is smooth, X is smooth and f is a differentiable fiber bundle.*

Proof. Using Proposition 13, we need to consider the Milnor fibers of the singular points of X_0.

In general, let $(s \in S)$ be an isolated surface singularity and M the Milnor fiber of a smoothing. The link L of S is diffeomorphic to the boundary ∂M of M. Let μ_0, μ_+, μ_- denote the number of zero (resp. positive, negative) eigenvalues of the intersection form on the middle cohomology of M.

If M is a $\mathbb{Q}$-homology ball then these are all 0. By [20, 2.24], $\mu_0 + \mu_+ = 2p_g(s \in S)$ where p_g denotes the geometric genus of the singularity $(s \in S)$. For a normal surface singularity $p_g(s \in S) = \dim_s R^1 g_* \mathcal{O}_{S'}$ where $g : S' \to S$ is a resolution of singularities. Thus if M is a $\mathbb{Q}$-homology ball then $(s \in S)$ is a rational singularity.

(If $(s \in S)$ is an isolated non-normal surface singularity, then $p_g(s \in S) = \dim_s R^1 g_* \mathcal{O}_{S'} - \dim \mathcal{O}_{\bar{S}}/\mathcal{O}_S$ where $\bar{S} \to S$ is the normalization. There are many examples where M is a $\mathbb{Q}$-homology ball yet $(\bar{s} \in \bar{S})$ is not a rational singularity.)

If M is a $\mathbb{Z}$-homology ball, then $L \sim \partial M$ is a $\mathbb{Z}$-homology sphere, hence $\mathrm{Cl}(S) \cong H^2(L, \mathbb{Z})$ is trivial [17, p.240]. Thus S is rational and K_S is Cartier; this happens only if S is a Du Val singularity. For smoothings of isolated hypersurface singularities there are vanishing cycles. $\square$

Remark 16. This suggests that Conjecture 11 may hold for $\mathcal{V} = \{\text{smooth surfaces}\}$, but there are many more cases to check. We do not even know what happens when the special fiber has isolated (but non-normal) singularities.

By contrast, there are many normal surface singularities whose Milnor fiber is a $\mathbb{Q}$-homology ball. See, for instance, [15, 5.9].

Example 17. Let $X^n \subset \mathbb{P}^N$ be a smooth variety and $Y \subset X$ a hyperplane section such that $X \setminus Y \cong \mathbb{C}^n$ The simplest examples are smooth quadrics $\mathbb{Q}^n \subset \mathbb{P}^{n+1}$ where Y is a tangent plane; for more complicated examples with $\dim X = 3$ see [7, 9, 8].

One gets a family of n–folds $f : \mathbf{X} \to \mathbb{C}$ whose general fibers X_t are isomorphic to X and whose special fiber X_0 is isomorphic to the cone over Y (possibly with some embedded points at the vertex). For quadrics an explicit example is the family

$$\left(x_0^2 + \cdots x_{n-1}^2 + t x_n^2 + t x_{n+1}^2 = 0\right) \subset \mathbb{P}^n_{\mathbf{x}} \times \mathbb{C}_t.$$

Note that the rank drops by 2 at the origin.

If n is odd, this is a $\mathbb{Q}$-homology fiber bundle but the retraction map

$$\mathbb{Z} \cong H_{n+1}\big(X_t, \mathbb{Z}\big) \to H_{n+1}\big(X_0, \mathbb{Z}\big) \cong \mathbb{Z}$$

is multiplication by 2. In all the other 3-fold examples the retraction induces

$$\mathbb{Z} \cong H_4\big(X_t, \mathbb{Z}\big) \to H_4\big(X_0, \mathbb{Z}\big) \cong \mathbb{Z}$$

which is multiplication by $\deg X > 1$.

The following lemma shows that this construction never gives interesting $\mathbb{Z}$-homology equivalences.

Proposition 18. *Let $X \subset \mathbb{P}^N$ be a smooth projective variety and $Y = H \cap X \subset X$ a hyperplane section. Let $C(Y)$ denote the cone over Y and $r_Y : X \to C(Y)$ the retraction. Assume that $X \setminus Y$ is a $\mathbb{Z}$-homology ball and r_Y is a $\mathbb{Z}$-homology equivalence. Then X is a linear subspace.*

Proof. Let $L \in H^2\big(\mathbb{P}^N, \mathbb{Z}\big)$ denote the hyperplane class. We will prove that cap product with L gives isomorphisms

$$\cap L : H_{i+2}(X, \mathbb{Z}) \cong H_i(X, \mathbb{Z}) \quad \text{for } 0 \le i \le 2\dim Y. \tag{18.1}$$

Composing the even ones gives an isomorphism

$$\big(\cap L\big)^{\dim X} : H_{2\dim X}(X, \mathbb{Z}) \cong H_0(X, \mathbb{Z}).$$

Thus $\deg X = 1$ and so X is a linear subspace.

Since r_Y is a $\mathbb{Z}$-homology equivalence, (18.1) is equivalent to

$$\cap L : H_{i+2}\big(C(Y), \mathbb{Z}\big) \cong H_i\big(C(Y), \mathbb{Z}\big) \quad \text{for } 0 \le i \le 2\dim Y. \tag{18.2}$$

This map can be factored as the Gysin map $H_{i+2}\big(C(Y), \mathbb{Z}\big) \to H_i\big(Y, \mathbb{Z}\big)$ followed by the inclusion map $H_i\big(Y, \mathbb{Z}\big) \to H_i\big(C(Y), \mathbb{Z}\big)$.

Taking the cone over a cycle gives a natural isomorphism $H_i(Y, \mathbb{Z}) \cong H_{i+2}\big(C(Y), \mathbb{Z}\big)$ and the Gysin map is its inverse. Again using that r_Y is a $\mathbb{Z}$-homology equivalence, $H_i\big(Y, \mathbb{Z}\big) \to H_i\big(C(Y), \mathbb{Z}\big)$ is isomorphic to the inclusion map $H_i\big(Y, \mathbb{Z}\big) \to H_i\big(X, \mathbb{Z}\big)$. The latter is an isomorphism for $i \le 2\dim Y$ since $X \setminus Y$ is a $\mathbb{Z}$-homology ball. This shows (18.1). $\square$

Families of cubic hypersurfaces.

In [5] several families with constant Lê numbers and non-constant topology are produced. One of them is a family of homogeneous polynomials, giving examples of homotopy fiber bundles which are not locally trivial topologically. All the examples in [5] are non-normal but here we construct a normal variant. Notice that all these examples belong to a class of non-isolated singularities that has been studied systematically in [6].

Example 19. Consider the family of homogeneous cubic polynomials

$$f_t(x_1,x_2,x_3,y_1,y_2,y_3) := (y_1,y_2,y_3)\cdot\begin{pmatrix} tx_1 & x_2 & x_3 \\ x_2 & tx_3 & x_1 \\ x_3 & x_1 & tx_2 \end{pmatrix}\cdot\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

Set $F(t,\mathbf{x},\mathbf{y}) = f_t(\mathbf{x},\mathbf{y})$ and $C_0 := \mathbb{C}_t \setminus \{0,-2,-2\xi,-2\xi^2\}$ where ξ is a third root of unity. Consider the family of cubic hypersurfaces

$$X := \big(F(t,\mathbf{x},\mathbf{y}) = 0\big) \subset \mathbb{P}^5_{\mathbf{x},\mathbf{y}} \times C_0$$

and let $\pi : X \to C_0$ be the second projection. For $t \in C_0$ the fiber $\pi^{-1}(t)$ is denoted by

$$X_t = \big(f_t(\mathbf{x},\mathbf{y}) = 0\big) \subset \mathbb{P}^5_{\mathbf{x},\mathbf{y}}.$$

We claim that $\pi : X \to C_0$ has the following properties.

(1) The singular set of X_t is the 2-plane $(y_1 = y_2 = y_3 = 0)$ for every $t \in C_0$. Furthermore, X_t is normal and has only canonical singularities.
(2) $\pi : X \to C_0$ is a homotopy fiber bundle.
(3) $\pi : X \to C_0$ is not topologically locally trivial in any neighborhood of t if $\xi' t^3 - 3t + 2\xi' = 0$ for some third root of unity ξ'. (For example $t = 1$ is one such value.)

Proof. The 2-plane $P := (y_1 = y_2 = y_3 = 0)$ is clearly contained in $\operatorname{Sing} X_t$. If we project X_t from P, the fibers are linear spaces. By an explicit computation we see that C_0 was chosen such that the fibers are all 2-dimensional. So $X_t \setminus P$ is a rank 2 vector bundle over $\mathbb{P}^2$, hence smooth. This implies that X_t is smooth in codimension 1, hence normal.

The projection shows that X_t has a resolution $r_t : \bar{X}_t \to X_t$ where $\bar{X}_t$ is a $\mathbb{P}^2$-bundle over $\mathbb{P}^2$. The exceptional divisor $E_t \subset \bar{X}_t$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ but the restriction of r_t gives a conic bundle structure $r_t|_{E_t} : E_t \to P$. Corresponding to the fibers of this conic bundle, $P = \operatorname{Sing} X_t$ is stratified according to the rank of the matrix

$$\begin{pmatrix} tx_1 & x_2 & x_3 \\ x_2 & tx_3 & x_1 \\ x_3 & x_1 & tx_2 \end{pmatrix}.$$

The third assertion follows from this and from the proof of [5, Prop.7] almost word by word. It is not worth to reproduce it, but the key idea is that any homeomorphism between X_s and X_t carries the singular set of X_s to the singular set of X_t and preserves the stratification. For generic t the locus of non-maximal rank is a smooth cubic curve but for $t = 1$ it is a singular cubic curve.

For the second assertion we check, by a direct computation, the conditions of Lemmas 20 and 21. Alternatively, comparing the homology sequences of the pairs $\big(\bar{X}_t, E_t\big)$ and $\big(X_t, P\big)$ shows that $\pi : X \to C_0$ is a $\mathbb{Z}$-homology fiber bundle. □

We follow the ideas of [5]. Let $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a family of holomorphic function germs depending holomorphically on a parameter. Define $F : \mathbb{C}^n \times \mathbb{C}$ by $F(x,t) := f_t(x)$. Consider the projection $\pi : \mathbb{C}^n \times \mathbb{C}$ to the second factor. Let B_ϵ be the closed ball of radius ϵ centered at the origin of $\mathbb{C}^n$, let $\mathbb{S}_\epsilon$ be its boundary sphere and let D_δ be the disk of radius δ centered at 0. Denote the punctured disk by D^*_δ.

Lemma 20. *Let ϵ, δ and η be radii such that for any $t \in D_\eta$ the restriction*

$$f_t : B_\epsilon \cap f_t^{-1}(D^*_\delta) \to D^*_\delta$$

is a locally trivial fibration. Then the following restrictions of the projection mapping are homotopy fiber bundles:

$$\pi : B_\epsilon \times D_\eta \cap F^{-1}(0 \times D_\eta) \to D_\eta,$$

$$\pi : \mathbb{S}_\epsilon \times D_\eta \cap F^{-1}(0 \times D_\eta) \to D_\eta.$$

Proof. The condition implies that for any $t \in D_\eta$ the inclusions of $f_t^{-1}(0) \cap B_\epsilon$ in $f_t^{-1}(D_\delta) \cap B_\epsilon$ and of $f_t^{-1}(0) \cap \mathbb{S}_\epsilon$ in $f_t^{-1}(D_\delta) \cap \mathbb{S}_\epsilon$ are homotopy equivalences. The condition also implies that for any $\xi \leq \eta$ the inclusions of $F^{-1}(0 \times D_\eta) \cap (B_\epsilon \times D_\xi)$ in $F^{-1}(D_\delta \times D_\eta) \cap (\mathbb{S}_\epsilon \times D_\xi)$ and of $F^{-1}(0 \times D_\eta) \cap (\mathbb{S}_\epsilon \times D_\xi)$ in $F^{-1}(D_\delta \times D_\eta) \cap (B_\epsilon \times D_\xi)$ are homotopy equivalences.

The condition and Ehresmann Fibration Theorem implies that the following restrictions of the projection mapping are differentiable locally trivial fibrations:

$$\pi : B_\epsilon \times D_\eta \cap F^{-1}(D_\delta \times D_\eta) \to D_\eta,$$

$$\pi : \mathbb{S}_\epsilon \times D_\eta \cap F^{-1}(D_\delta \times D_\eta) \to D_\eta. \quad \square$$

Usually one checks the condition of the previous Lemma by showing, for any $t \in D_\eta$, that in the ball B_ϵ the function f_t has no critical points outside $f_t^{-1}(0)$ and that the fibers $f_t^{-1}(s)$ are transverse to ∂B_ϵ for any $s \in D_\delta \setminus \{0\}$.

The Lemma above helps in the local case. From it one can deduce that certain projective morphisms are homotopy fiber bundles. Suppose that f_t is a family of homogeneous polynomials. Let $V(F) \subset \mathbb{P}^{n-1} \times D_\eta$ be the family of projective varieties defined by the zeros of F. Denote by π the projection of $\mathbb{P}^{n-1} \times D_\eta$ to the second factor.

Lemma 21. *Suppose that the condition of the previous lemma is satisfied, and in addition that f_t is a family of homogeneous polynomials. Then the restriction of the projection*

$$\pi : V(F) \to D_\eta$$

is a homotopy fiber bundle.

Proof. It is enough to prove that for any $\xi \leq \eta$ the inclusion of

$$V(f_0) \hookrightarrow V(F) \cap \pi^{-1}(D_\xi) \tag{1}$$

is a homotopy equivalence. By the previous lemma we know that

$$\pi : \mathbb{S}_\epsilon \times D_\eta \cap F^{-1}(0 \times D_\eta) \to D_\eta$$

is a homotopy fiber bundle. Therefore the inclusion

$$F^{-1}(0,0) \cap (\mathbb{S}_\epsilon \times \{0\}) \hookrightarrow F^{-1}(0 \times D_\eta) \cap (\mathbb{S}_\epsilon \times D_\xi)$$

is a homotopy equivalence for any $\xi \leq \eta$. Thus the induced homomorphisms of homotopy groups are isomorphisms.

There is an free action of the sphere $\mathbb{S}^1$ of complex numbers of modulus 1 which is equivariant with respect to the inclusion whose quotient is the inclusion (1). Applying the long exact sequence of homotopy groups associated to the fibrations given by the quotients of the free action we conclude that the inclusion (1) induces isomorphisms of homotopy groups. Whitehead's Theorem implies that then it is a homotopy equivalence. $\square$

Remark 22. The proof of [5, Thm.10] gives that if B_ϵ is a Milnor ball of f_0 and the Lê numbers of f_t with respect to a prepolar coordinate system (a sufficiently generic coordinate system, see [16, p.26] for a precise definition), then the condition in Lemma 20 is satisfied.

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HOMOLOGICAL REPRESENTATIONS OF BRAID GROUPS AND KZ CONNECTIONS

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Abstract. We give a relation between the homological representations of the braid groups studied by Lawrence, Krammer and Bigelow and the monodromy of the KZ equation with values in the space of null vectors in the tensor product of Verma modules of $sl_2(\mathbf{C})$ when the parameters are generic. We also discuss the case of special parameters in relation with integral representations of the space of conformal blocks by hypergeometric integrals.

1. Introduction

The purpose of this article is to review recent developments concerning the relation between homological representations of the braid groups and the monodromy representations of the Knizhnik-Zamolodchikov (KZ) connections.

Homological representations appeared in the work of Lawrence [15] in relation with Hecke algebra representations of the braid groups. These representations were extensively investigated by Bigelow and Krammer and are called the Lawrence-Krammer-Bigelow (LKB) representations. In particular, they independently proved in [2] and [14] that the LKB representations of the braid groups are faithful.

On the other hand, it was shown by Schechtman-Varchenko [19] and others that the solutions of the KZ equation are expressed by hypergeometric integrals. By means of these integral representations we describe a relation between the LKB representations and the monodromy representations of the KZ connections. There are two parameters λ and κ, which are related to the highest weight and the KZ connection respectively. We consider the KZ equation with values in the space of null vectors in the tensor product of Verma modules of $sl_2(\mathbf{C})$ and show that a specialization of the LKB representation is equivalent to the monodromy representation of such KZ equation for generic parameters λ and κ. A more detailed treatment of this result is given in [13]. A relation between the Krammer representations of Artin groups and the monodromy of KZ connections is also described by an infinitesimal method by Marin [17].

The case of special parameters are important from the viewpoint of conformal field theory (see [6], [20] and [22]). We discuss the problem of expressing the space of conformal blocks by hypergeometric integrals. It is important to understand the KZ connection for the conformal block bundle as a Gauss-Manin connection. We refer the reader to the work of Looijenga [16] in recent advances concerning this subject.

The paper is organized in the following way. In Section 2 we recall the definition of the LKB representations. In Section 3 we describe the KZ connection. In Section 4 we review basic properties of Verma modules for the Lie algebra $sl_2(\mathbf{C})$ and define the space of null vectors. In Section 5 we state a comparison theorem between the LKB representations and the monodromy representations of the KZ connections with values in the space of null vectors. In Section 6 we describe the horizontal sections for the KZ connections by means of hypergeometric integrals. Section 7 is devoted to a proof of the comparison theorem stated in Section 5. We discuss the case of special parameters in relation with the space of conformal blocks in Section 8.

2. Lawrence-Krammer-Bigelow representations

For a space X we denote by $\mathcal{F}_n(X)$ the configuration space of ordered distinct n points in X. Namely,

$$\mathcal{F}_n(X) = \{(x_1, \cdots, x_n) \in X^n \ ; \ x_i \neq x_j \text{ if } i \neq j\},$$

where X^n denotes the n-fold Cartesian product of X.

We denote by B_n the braid group with n strands. We fix a set of distinct n points in $\mathbf{R}^2$ as

$$Q = \{(1,0), \cdots, (n,0)\},$$

where we set $p_\ell = (\ell, 0)$, $\ell = 1, \cdots, n$. We take a 2-dimensional disk in $\mathbf{R}^2$ containing Q in the interior. We fix a positive integer m and consider the configuration space $\mathcal{F}_m(\Sigma)$ of ordered distinct m points in $\Sigma = D \setminus Q$, which is also denoted by $\mathcal{F}_{n,m}(D)$. The symmetric group $\mathfrak{S}_m$ acts freely on $\mathcal{F}_m(\Sigma)$ by the permutations of distinct m points. The quotient space of $\mathcal{F}_m(\Sigma)$ by this action is by definition the configuration space of unordered distinct m points in Σ and is denoted by $\mathcal{C}_m(\Sigma)$. We also denote this configuration space by $\mathcal{C}_{n,m}(D)$.

In the papers by Bigelow [2], [3] and by Krammer [14] the case $m = 2$ was extensively studied, but for our purpose it is convenient to consider the case when m is an arbitrary positive integer such that $m \geq 2$.

We identify $\mathbf{R}^2$ with the complex plane $\mathbf{C}$. The quotient space $\mathbf{C}^m/\mathfrak{S}_m$ defined by the action of $\mathfrak{S}_m$ by the permutations of coordinates is analytically isomorphic to $\mathbf{C}^m$ by means of the elementary symmetric polynomials. Now the image of the hyperplanes defined by $t_i = p_\ell$, $\ell = 1, \cdots, n$, and the diagonal hyperplanes $t_i = t_j$, $1 \leq i \leq j \leq m$, are complex codimension one irreducible subvarieties of the quotient space $D^m/\mathfrak{S}_m$. This allows us to give a description of the first homology group of $\mathcal{C}_{n,m}(D)$ as

$$H_1(\mathcal{C}_{n,m}(D); \mathbf{Z}) \cong \mathbf{Z}^{\oplus n} \oplus \mathbf{Z} \tag{2.1}$$

where the first n components correspond to the nomal loops of the images of hyperplanes $t_i = p_\ell$, $\ell = 1, \cdots, n$, and the last component corresponds to the normal loop of the image of the diagonal hyperplanes $t_i = t_j$, $1 \leq i \leq j \leq m$, namely, the discriminant set. We consider the homomorphism

$$\alpha : H_1(\mathcal{C}_{n,m}(D); \mathbf{Z}) \longrightarrow \mathbf{Z} \oplus \mathbf{Z} \tag{2.2}$$

defined by $\alpha(x_1, \cdots, x_n, y) = (x_1 + \cdots + x_n, y)$. Composing with the abelianization map $\pi_1(\mathcal{C}_{n,m}(D), x_0) \to H_1(\mathcal{C}_{n,m}(D); \mathbf{Z})$, we obtain the homomorphism

$$\beta : \pi_1(\mathcal{C}_{n,m}(D), x_0) \longrightarrow \mathbf{Z} \oplus \mathbf{Z}. \tag{2.3}$$

Let $\pi : \widetilde{\mathcal{C}}_{n,m}(D) \to \mathcal{C}_{n,m}(D)$ be the covering corresponding to $\operatorname{Ker} \beta$. Now the group $\mathbf{Z} \oplus \mathbf{Z}$ acts as the deck transformations of the covering π and the homology group $H_*(\widetilde{\mathcal{C}}_{n,m}(D); \mathbf{Z})$ is considered to be a $\mathbf{Z}[\mathbf{Z} \oplus \mathbf{Z}]$-module. Here $\mathbf{Z}[\mathbf{Z} \oplus \mathbf{Z}]$ stands for the group ring of $\mathbf{Z} \oplus \mathbf{Z}$. We express $\mathbf{Z}[\mathbf{Z} \oplus \mathbf{Z}]$ as the ring of Laurent polynomials $R = \mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$. We consider the homology group

$$H_{n,m} = H_m(\widetilde{\mathcal{C}}_{n,m}(D); \mathbf{Z})$$

as an R-module by the action of the deck transformations.

In a similar way as is described in the case of $m = 2$ in [2], it can be shown that $H_{n,m}$ is a free R-module of rank

$$d_{n,m} = \begin{pmatrix} m + n - 2 \\ m \end{pmatrix}. \tag{2.4}$$

A basis of $H_{n,m}$ as a free R-module is discussed in relation with the homology of local systems in the next sections. Let $\mathcal{M}(D, Q)$ denote the mapping class group of the pair (D, Q), which consists of the isotopy classes of homeomorphisms of D which fix Q setwise and fix the boundary ∂D pointwise. The braid group B_n is naturally isomorphic to the mapping class group $\mathcal{M}(D, Q)$. Now a homeomorphism f representing a class in $\mathcal{M}(D, Q)$ induces a homeomorphism $\tilde{f} : \mathcal{C}_{n,m}(D) \to \mathcal{C}_{n,m}(D)$, which is uniquely lifted to a homeomorphism of $\widetilde{\mathcal{C}}_{n,m}(D)$. This homeomorpshim commutes with the deck transformations.

Therefore, for $m \geq 2$ we obtain a representation of the braid group

$$\rho_{n,m} : B_n \longrightarrow \mathrm{Aut}_R\, H_{n,m} \tag{2.5}$$

which is called the homological representation of the braid group or the Lawrence-Krammer-Bigelow (LKB) representation. Let us remark that in the case $m = 1$ the above construction gives the reduced Burau representation over $\mathbf{Z}[q^{\pm 1}]$.

3. KZ connection

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra and $\{I_\mu\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to the Cartan-Killing form. Although we deal with the case $\mathfrak{g} = sl_2(\mathbf{C})$, we formulate the KZ connection in the case of a general complex semi-simple Lie algebra. We set $\Omega = \sum_\mu I_\mu \otimes I_\mu$. Let $r_i : \mathfrak{g} \to \mathrm{End}(V_i)$, $1 \leq i \leq n$, be representations of the Lie algebra $\mathfrak{g}$. We denote by Ω_{ij} the action of Ω on the i-th and j-th components of the tensor product $V_1 \otimes \cdots \otimes V_n$. It is known that the Casimir element $c = \sum_\mu I_\mu \cdot I_\mu$ lies in the center of the universal enveloping algebra $U\mathfrak{g}$. Let us denote by $\Delta : U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}$ the coproduct, which is defined to be the algebra homomorphism determined by $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in \mathfrak{g}$. Since Ω is expressed as $\Omega = \frac{1}{2}\left(\Delta(c) - c \otimes 1 - 1 \otimes c\right)$ we have the relation

$$[\Omega, x \otimes 1 + 1 \otimes x] = 0 \tag{3.1}$$

for any $x \in \mathfrak{g}$ in the tensor product $U\mathfrak{g} \otimes U\mathfrak{g}$. By means of the above relation we obtain the quadratic relations:

$$[\Omega_{ik}, \Omega_{ij} + \Omega_{jk}] = 0, \quad (i, j, k \ \text{ distinct}), \tag{3.2}$$

$$[\Omega_{ij}, \Omega_{k\ell}], \quad (i, j, k, \ell \ \text{ distinct}), \tag{3.3}$$

which are call the infinitesimal pure braid relations.

We denote by X_n the configuration space $\mathcal{F}_n(\mathbf{C})$. Namely, X_n is defined as

$$X_n = \{(z_1, \cdots, z_n) \in \mathbf{C}^n \ ; \ z_i \neq z_j \text{ if } i \neq j\}.$$

We define the Knizhnik-Zamolodchikov (KZ) connection as the 1-form

$$\omega = \frac{1}{\kappa} \sum_{1 \leq i < j \leq n} \Omega_{ij} d\log(z_i - z_j) \tag{3.4}$$

with values in $\mathrm{End}(V_1 \otimes \cdots \otimes V_n)$ for a non-zero complex parameter κ. We have $d\omega = 0$ on X_n.

We set $\omega_{ij} = d\log(z_i - z_j)$, $1 \leq i \neq j \leq n$. It follows from the above infinitesimal pure braid relations among Ω_{ij} together with Arnold's relation

$$\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{k\ell} + \omega_{k\ell} \wedge \omega_{ij} = 0 \tag{3.5}$$

that $\omega \wedge \omega = 0$ holds. It follows that ω defines a flat connection for a trivial vector bundle over the configuration space X_n with fiber $V_1 \otimes \cdots \otimes V_n$. A horizontal section of the above flat bundle is a solution of the total differential equation

$$d\varphi = \omega\varphi \tag{3.6}$$

for a function $\varphi(z_1, \cdots, z_n)$ on X_n with values in $V_1 \otimes \cdots \otimes V_n$. This total differential equation can be expressed as a system of partial differential equations

$$\frac{\partial \varphi}{\partial z_i} = \frac{1}{\kappa} \sum_{j, j \neq i} \frac{\Omega_{ij}}{z_i - z_j} \varphi, \quad 1 \leq i \leq n, \tag{3.7}$$

which is called the KZ equation. The KZ equation was first introduced in [7] as the differential equation satisfied by n-point functions in Wess-Zumino-Witten conformal field theory on the Riemann sphere with the gauge symmetry of affine Lie algebras.

Let $\phi(z_1, \cdots, z_n)$ be the matrix whose columns are linearly independent solutions of the KZ equation. By considering the analytic continuation of the solutions with respect to a loop γ in X_n with a base point x_0 we obtain the matrix $\theta(\gamma)$ defined by

$$\phi(z_1, \cdots, z_n) \mapsto \phi(z_1, \cdots, z_n)\theta(\gamma). \tag{3.8}$$

Since the KZ connection ω is flat the matrix $\theta(\gamma)$ depends only on the homotopy class of γ. The fundamental group $\pi_1(X_n, x_0)$ is the pure braid group P_n. As the above holonomy of the connection ω we have linear representations of the pure braid group

$$\theta : P_n \to \mathrm{GL}(V_1 \otimes \cdots \otimes V_n) \tag{3.9}$$

depending on the parameter κ.

The symmetric group $\mathfrak{S}_n$ acts on X_n by the permutations of coordinates. We denote the quotient space $X_n/\mathfrak{S}_n$ by Y_n. The fundamental group of Y_n is the braid group B_n. In the case $V_1 = \cdots = V_n = V$, the symmetric group $\mathfrak{S}_n$ acts diagonally on the trivial vector bundle over X_n with fiber $V^{\otimes n}$ and the connection ω is invariant by this action. Thus we have a one-parameter family of linear representations of the braid group

$$\theta : B_n \to \mathrm{GL}(V^{\otimes n}). \tag{3.10}$$

It is known by [5] and [10] that this representation is described by means of quantum groups.

4. Space of null vectors

Let us recall basic definitions about the Lie algebra $sl_2(\mathbf{C})$ and its Verma modules. As a complex vector space the Lie algebra $sl_2(\mathbf{C})$ has a basis H, E and F satisfying the relations:

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H. \tag{4.1}$$

For a complex number λ we denote by M_λ the Verma module of $sl_2(\mathbf{C})$ with highest weight λ. Namely, there is a non-zero vector $v_\lambda \in M_\lambda$ called the highest weight vector satisfying

$$Hv_\lambda = \lambda v_\lambda, \ Ev_\lambda = 0 \tag{4.2}$$

and M_λ is spanned by $F^j v_\lambda$, $j \geq 0$. The elements H, E and F act on this basis as

$$\begin{cases} H \cdot F^j v_\lambda = (\lambda - 2j) F^j v_\lambda \\ E \cdot F^j v_\lambda = j(\lambda - j + 1) F^{j-1} v_\lambda \\ F \cdot F^j v_\lambda = F^{j+1} v_\lambda. \end{cases} \tag{4.3}$$

It is known that if $\lambda \in \mathbf{C}$ is not a non-negative integer, then the Verma module M_λ is irreducible.

The Shapovalov form is the symmetric bilinear form

$$S : M_\lambda \times M_\lambda \longrightarrow \mathbf{C}$$

characterized by the conditions:

$$\begin{aligned} &S(v, v) = 1, \\ &S(Fx, y) = S(x, Ey) \ \text{ for any } x, y \ \in M_\lambda. \end{aligned}$$

It follow from the above conditions that

$$S(F^i v, F^j v) = 0 \text{ if } i \neq j,$$
$$S(F^i v, F^i v) = i!\lambda(\lambda - 1) \cdots (\lambda - i + 1).$$

For $\Lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbf{C}^n$ we put $|\Lambda| = \lambda_1 + \cdots + \lambda_n$ and consider the tensor product $M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n}$. For a non-negative integer m we define the space of weight vectors with weight $|\Lambda| - 2m$ by

$$W[|\Lambda| - 2m] = \{x \in M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n} \ ; \ Hx = (|\Lambda| - 2m)x\} \tag{4.4}$$

and consider the space of null vectors defined by

$$N[|\Lambda| - 2m] = \{x \in W[|\Lambda| - 2m] \ ; \ Ex = 0\}. \tag{4.5}$$

The KZ connection ω defined in the previous section commutes with the diagonal action of $\mathfrak{g}$ on $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$, hence it acts on the space of null vectors $N[|\Lambda| - 2m]$. This means that it is sensible to investigate the horizontal sections of the KZ connection with valued in $N[|\Lambda| - 2m]$.

We also consider the space of coinvariant tensors defined by

$$L[|\Lambda| - 2m] = W[|\Lambda| - 2m]/(F \cdot W[|\Lambda| - 2m + 2]).$$

The dual space $L[|\Lambda| - 2m]^*$ is identified with the space of linear forms $f : W[|\Lambda| - 2m] \longrightarrow \mathbf{C}$ such that $f(x) = 0$ for $x \in F \cdot W[|\Lambda| - 2m + 2]$. The isomorphism $M_\lambda \cong M_\lambda^*$ given by the Shappvalov form induces an isomorphism

$$N[|\Lambda| - 2m] \cong L[|\Lambda| - 2m]^*.$$

5. Relation between LKB representation and KZ connection

We fix a complex number λ and consider the case $\lambda_1 = \cdots = \lambda_n = \lambda$ and deal with the space of null vectors

$$N[n\lambda - 2m] \subset M_\lambda^{\otimes n}.$$

As the monodromy of the KZ connection

$$\omega = \frac{1}{\kappa} \sum_{1 \leq i < j \leq n} \Omega_{ij} d \log(z_i - z_j)$$

with values in $N[n\lambda - 2m]$ we obtain the linear representation of the braid group

$$\theta_{\lambda,\kappa} : B_n \longrightarrow \operatorname{Aut} N[n\lambda - 2m].$$

The next theorem describes a relationship between a specialization of the Lawrence-Krammer-Bigelow representation $\rho_{n,m}$ and the representation $\theta_{\lambda,\kappa}$.

Theorem 5.1. *There exists an open dense subset U in $(\mathbf{C}^*)^2$ such that for $(\lambda, \kappa) \in U$ the Lawrence-Krammer-Bigelow representation $\rho_{n,m}$ with the specialization*

$$q = e^{-2\pi\sqrt{-1}\lambda/\kappa}, \quad t = e^{2\pi\sqrt{-1}/\kappa}$$

is equivalent to the monodromy representation of the KZ connection $\theta_{\lambda,\kappa}$ with values in the space of null vectors

$$N[n\lambda - 2m] \subset M_\lambda^{\otimes n}.$$

In the rest of this section we recall some basic notions needed to prove Theorem 5.1. First, we recall some basic definition for local systems. Let M be a smooth manifold and V a complex vector space. Given a linear representation of the fundamental group

$$r : \pi_1(M, x_0) \longrightarrow GL(V)$$

there is an associated flat vector bundle E over M. The local system $\mathcal{L}$ associated to the representation r is the sheaf of horizontal sections of the flat bundle E. Let $\pi : \widetilde{M} \to M$ be the universal covering. We denote by $\mathbf{Z}\pi_1$ the group ring of the fundamental group $\pi_1(M, x_0)$. We consider the chain complex

$$C_*(\widetilde{M}) \otimes_{\mathbf{Z}\pi_1} V$$

with the boundary map defined by $\partial(c \otimes v) = \partial c \otimes v$. Here $\mathbf{Z}\pi_1$ acts on $C_*(\widetilde{M})$ via the deck transformations and on V via the representation r. The homology of this chain complex is called the homology of M with coefficients in the local system $\mathcal{L}$ and is denoted by $H_*(M, \mathcal{L})$.

The open dense set in Theorem 5.1 is closely related to the vanishing theorem of the homology of generic local system over the complement of hyperplane arrangements. Let us briefly recall this aspect. Let $\mathcal{A} = \{H_1, \cdots, H_N\}$ be a set of affine hyperplanes in the complex vector space $\mathbf{C}^n$. We call the set $\mathcal{A}$ a complex hyperplane arrangement. We consider the complement

$$M(\mathcal{A}) = \mathbf{C}^n \setminus \bigcup_{H \in \mathcal{A}} H.$$

Let $\mathcal{L}$ be a complex rank one local system over $M(\mathcal{A})$ associated with a representation of the fundamental group

$$r : \pi_1(M(\mathcal{A}), x_0) \longrightarrow \mathbf{C}^*.$$

We investigate the homology of $M(\mathcal{A})$ with coefficients in the local system $\mathcal{L}$. For our purpose the homology of locally finite chains $H_*^{lf}(M(\mathcal{A}), \mathcal{L})$ also plays an important role.

We summarize basic properties of the above homology groups. For a complex hyperplane arrangement $\mathcal{A}$ we choose a smooth compactification $i : M(\mathcal{A}) \longrightarrow X$ with normal crossing divisors. We shall say that the local system $\mathcal{L}$ is generic if and only if an isomorphism

$$i_*\mathcal{L} \cong i_!\mathcal{L} \tag{5.1}$$

holds, where $i_*\mathcal{L}$ is the direct image and $i_!\mathcal{L}$ is the extension of $\mathcal{L}$ by 0 outside of $M(\mathcal{A})$ as a sheaf. The condition $i_*\mathcal{L} \cong i_!\mathcal{L}$ means that the monodromy of $\mathcal{L}$ along any divisor at infinity is not equal to 1. The following theorem was shown in [9].

Theorem 5.2. *If the local system $\mathcal{L}$ is generic in the above sense, then there is an isomorphism*

$$H_*(M(\mathcal{A}), \mathcal{L}) \cong H_*^{lf}(M(\mathcal{A}), \mathcal{L}).$$

Moreover, we have $H_k(M(\mathcal{A}), \mathcal{L}) = 0$ for any $k \neq n$.

Proof. In general we have isomorphisms

$$H^*(X, i_*\mathcal{L}) \cong H^*(M(\mathcal{A}), \mathcal{L}), \quad H^*(X, i_!\mathcal{L}) \cong H_c^*(M(\mathcal{A}), \mathcal{L})$$

where H_c denotes cohomology with compact supports.

There are Poincaré duality isomorphisms:

$$H_k^{lf}(M(\mathcal{A}), \mathcal{L}) \cong H^{2n-k}(M(\mathcal{A}), \mathcal{L}),$$
$$H_k(M(\mathcal{A}), \mathcal{L}) \cong H_c^{2n-k}(M(\mathcal{A}), \mathcal{L}).$$

By the hypothesis $i_*\mathcal{L} \cong i_!\mathcal{L}$ we obtain an isomorphism

$$H_k^{lf}(M(\mathcal{A}), \mathcal{L}) \cong H_k(M(\mathcal{A}), \mathcal{L}).$$

It follows from the above Poincaré duality isomorpshims and the fact that $M(\mathcal{A})$ has a homotopy type of a CW complex of dimension at most n we have

$$H_k^{lf}(M(\mathcal{A}),\mathcal{L}) \cong 0, \quad k < n$$
$$H_k(M(\mathcal{A}),\mathcal{L}) \cong 0, \quad k > n.$$

Therefore we obtain $H_k(M(\mathcal{A}),\mathcal{L}) = 0$ for any $k \neq n$. □

For a positive integer m we consider the projection map

$$\pi_{n,m} : X_{n+m} \longrightarrow X_n \tag{5.2}$$

given by $\pi_{n,m}(z_1,\cdots,z_n,t_1,\cdots,t_m) = (z_1,\cdots,z_n)$, which defines a fiber bundle over X_n. For $p \in X_n$ the fiber $\pi_{n,m}^{-1}(p)$ is denoted by $X_{n,m}$. Let $(p_1,\cdots,p_n)$ be the coordinates for p. Then, $X_{n,m}$ is the complement of hyperplanes defined by

$$t_i = p_\ell, \quad 1 \le i \le m,\ 1 \le \ell \le n, \quad t_i = t_j, \quad 1 \le i < j \le m. \tag{5.3}$$

Such arrangement of hyperplanes is called a discriminantal arrangement. The symmetric group $\mathfrak{S}_m$ acts on $X_{n,m}$ by the permutations of the coordinates functions $t_1,\cdots,t_m$. We put $Y_{n,m} = X_{n,m}/\mathfrak{S}_m$.

Identifying $\mathbf{R}^2$ with the complex plane $\mathbf{C}$, we have the inclusion map

$$\iota : \mathcal{F}_{n,m}(D) \longrightarrow X_{n,m}, \tag{5.4}$$

which is a homotopy equivalence. By taking the quotient by the action of the symmetric group $\mathfrak{S}_m$, we have the inclusion map

$$\bar{\iota} : \mathcal{C}_{n,m}(D) \longrightarrow Y_{n,m}, \tag{5.5}$$

which is also a homotopy equivalence.

We take $p = (1,2,\cdots,n)$ as a base point. We consider a local system over $X_{n,m}$ defined in the following way. Let $\xi_{i\ell}$ and η_{ij} be normal loops around the hyperplanes $t_i = p_\ell$ and $t_i = t_j$ respectively. We fix complex numbers α_ℓ, $1 \le \ell \le n$, and γ and by the correspondence

$$\xi_{i\ell} \mapsto e^{2\pi\sqrt{-1}\alpha_\ell}, \quad \eta_{ij} \mapsto e^{4\pi\sqrt{-1}\gamma}$$

we obtain the representation

$$r : \pi_1(X_{n,m},x_0) \longrightarrow \mathbf{C}^*.$$

We denote by $\mathcal{L}$ the associated rank one local system on $X_{n,m}$.

Let us consider the embedding

$$i_0 : X_{n,m} \longrightarrow (\mathbf{C}P^1)^m = \underbrace{\mathbf{C}P^1 \times \cdots \times \mathbf{C}P^1}_{m}. \tag{5.6}$$

Then we take blowing-ups at multiple points $\pi : \widehat{(\mathbf{C}P^1)^m} \longrightarrow (\mathbf{C}P^1)^m$ and obtain a smooth compactification $i : X_{n,m} \to \widehat{(\mathbf{C}P^1)^m}$ with normal crossing divisors. We can write down the condition $i_*\mathcal{L} \cong i_!\mathcal{L}$ explicitly by computing the monodromy of the local system $\mathcal{L}$ along divisors at infinity.

The local system $\mathcal{L}$ on $X_{n,m}$ is invariant under the action of the symmetric group $\mathfrak{S}_m$ and induces the local system $\overline{\mathcal{L}}$ on $Y_{n,m}$. We will deal with the case $\alpha_1 = \cdots = \alpha_\ell = \alpha$. By a similar argument using a smooth compactification of $Y_{n,m}$ we have the following proposition.

Proposition 5.1. *There is an open dense subset V in $\mathbf{C}^2$ such that for $(\alpha,\gamma) \in V$ the associated local system $\overline{\mathcal{L}}$ on $Y_{n,m}$ satisfies*

$$H_*(Y_{n,m},\overline{\mathcal{L}}) \cong H_*^{lf}(Y_{n,m},\overline{\mathcal{L}})$$

and $H_k(Y_{n,m}, \overline{\mathcal{L}}) = 0$ *for any* $k \neq m$*. Moreover, we have*

$$\dim H_m(Y_{n,m}, \overline{\mathcal{L}}^*) = d_{n,m}, \tag{5.7}$$

where we use the same notation as in equation (2.4) for $d_{n,m}$*.*

For the purpose of describing the homology groups $H_m^{lf}(X_{n,m}, \mathcal{L})$ and $H_m^{lf}(Y_{n,m}, \overline{\mathcal{L}})$ we introduce the following notation. We fix the base point $p = (1, \cdots, n)$. For non-negative integers $m_1, \cdots, m_{n-1}$ satisfying

$$m_1 + \cdots + m_{n-1} = m \tag{5.8}$$

we define a bounded chamber $\Delta_{m_1, \cdots, m_{n-1}}$ in $\mathbf{R}^m$ by

$$\begin{aligned}
&1 < t_1 < \cdots < t_{m_1} < 2 \\
&2 < t_{m_1+1} < \cdots < t_{m_1+m_2} < 3 \\
&\cdots \\
&n-1 < t_{m_1+\cdots+m_{n-2}+1} + \cdots + t_m < n.
\end{aligned}$$

We put $M = (m_1, \cdots, m_{n-1})$ and we write Δ_M for $\Delta_{m_1,\cdots,m_{n-1}}$. We denote by $\overline{\Delta}_M$ the image of Δ_M by the projection map $\pi_{n,m}$. The bounded chamber Δ_M defines a homology class $[\Delta_M] \in H_m^{lf}(X_{n,m}, \mathcal{L})$ and its image $\overline{\Delta}_M$ defines a homology class $[\overline{\Delta}_M] \in H_m^{lf}(Y_{n,m}, \overline{\mathcal{L}})$. We shall observe that under certain generic conditions $[\overline{\Delta}_M]$ for $M = (m_1, \cdots, m_{n-1})$ with $m_1 + \cdots + m_{n-1} = m$ form a basis of $H_m^{lf}(Y_{n,m}, \overline{\mathcal{L}})$.

As we have shown in Theorem 5.2 there is an isomorphism $H_m(X_{n,m}, \mathcal{L}) \cong H_m^{lf}(X_{n,m}, \mathcal{L})$ if the condition $i_*\mathcal{L} \cong i_!\mathcal{L}$ is satisfied. In this situation we denote by $[\widetilde{\Delta}_M]$ the homology class in $H_m(X_{n,m}, \mathcal{L})$ corresponding to $[\Delta_M]$ in the above isomorphism and call $[\widetilde{\Delta}_M]$ the regularized cycle for $[\Delta_M]$.

6. Hypergeometric integrals

In this section we describe solutions of the KZ equation for the case $\mathfrak{g} = sl_2(\mathbf{C})$ by means of hypergeometric integrals following Schechtman and Varchenko [19]. A description of the solutions of the KZ equation was also given by Date, Jimbo, Matsuo and Miwa [4]. We refer the reader to [1] and [18] for general treatments of hypergeometric integrals.

For parameters κ and λ we consider the multi-valued function

$$\Phi_{n,m} = \prod_{1 \le i < j \le n} (z_i - z_j)^{\frac{\lambda_i \lambda_j}{\kappa}} \prod_{1 \le i \le m, 1 \le \ell \le n} (t_i - z_\ell)^{-\frac{\lambda_\ell}{\kappa}} \prod_{1 \le i < j \le m} (t_i - t_j)^{\frac{2}{\kappa}} \tag{6.1}$$

defined over X_{n+m}. Let $\mathcal{L}$ denote the local system associated to the multi-valued function Φ. The restriction of $\mathcal{L}$ on the fiber $X_{n,m}$ is the local system associated with the parameters

$$\alpha_\ell = -\frac{\lambda_\ell}{\kappa}, \quad 1 \le \ell \le n, \quad \gamma = \frac{1}{\kappa}. \tag{6.2}$$

The symmetric group $\mathfrak{S}_m$ acts on $X_{n,m}$ by the permutations of the coordinate functions $t_1, \cdots, t_m$. The function $\Phi_{n,m}$ is invariant by the action of $\mathfrak{S}_m$. The local system $\mathcal{L}$ over $X_{n,m}$ defines a local system on $Y_{n,m}$, which we denote by $\overline{\mathcal{L}}$. The local system dual to $\mathcal{L}$ is denoted by $\mathcal{L}^*$.

We put $v = v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_n}$ and for $J = (j_1, \cdots, j_n)$ set $F^J v = F^{j_1} v_{\lambda_1} \otimes \cdots \otimes F^{j_n} v_{\lambda_n}$, where $j_1, \cdots, j_n$ are non-negative integers. The weight space $W[|\Lambda| - 2m]$ has a basis $F^J v$ for each J

with $|J| = j_1 + \cdots + j_n = m$. For the sequence of integers $(i_1, \cdots, i_m) = (\underbrace{1, \cdots, 1}_{j_1}, \cdots, \underbrace{n, \cdots, n}_{j_n})$ we set

$$S_J(z,t) = \frac{1}{(t_1 - z_{i_1}) \cdots (t_m - z_{i_m})} \tag{6.3}$$

and define the rational function $R_J(z,t)$ by

$$R_J(z,t) = \frac{1}{j_1! \cdots j_n!} \sum_{\sigma \in \mathfrak{S}_m} S_J(z_1, \cdots, z_n, t_{\sigma(1)}, \cdots, t_{\sigma(m)}). \tag{6.4}$$

For example, we have

$$R_{(1,0,\cdots,0)}(z,t) = \frac{1}{t_1 - z_1}, \quad R_{(2,0,\cdots,0)}(z,t) = \frac{1}{(t_1 - z_1)(t_2 - z_1)}$$
$$R_{(1,1,0,\cdots,0)}(z,t) = \frac{1}{(t_1 - z_1)(t_2 - z_2)} + \frac{1}{(t_2 - z_1)(t_1 - z_2)}$$

and so on.

Since $\pi_{n,m} : X_{m+n} \to X_m$ is a fiber bundle with fiber $X_{n,m}$ the fundamental group of the base space X_n acts naturally on the homology group $H_m(X_{n,m}, \mathcal{L}^*)$. Thus we obtain a representation of the pure braid group

$$r_{n,m} : P_n \longrightarrow \mathrm{Aut}\, H_m(X_{n,m}, \mathcal{L}^*) \tag{6.5}$$

which defines a local system on X_n denoted by $\mathcal{H}_{n,m}$. In the case $\lambda_1 = \cdots = \lambda_n$ there is a representation of the braid group

$$r_{n,m} : B_n \longrightarrow \mathrm{Aut}\, H_m(Y_{n,m}, \overline{\mathcal{L}}^*) \tag{6.6}$$

which defines a local system $\overline{\mathcal{H}}_{n,m}$ on $Y_{n,m}$. For any horizontal section $c(z)$ of the local system $\mathcal{H}_{n,m}$ we consider the hypergeometric type integral

$$\int_{c(z)} \Phi_{n,m} R_J(z,t)\; dt_1 \wedge \cdots \wedge dt_m \tag{6.7}$$

for the above rational function $R_J(z,t)$.

According to Schechtman and Varchenko, solutions of the KZ equation are described in the following way.

Theorem 6.1 (Schechtman and Varchenko [19])**.** *The integral*

$$\sum_{|J|=m} \left(\int_{c(z)} \Phi_{n,m} R_J(z,t)\; dt_1 \wedge \cdots \wedge dt_m \right) F^J v$$

lies in the space of null vectors $N[|\Lambda| - 2m]$ and is a solution of the KZ equation.

We assume the conditions $i_* \mathcal{L} \cong i_! \mathcal{L}$ and $i_* \overline{\mathcal{L}} \cong i_! \overline{\mathcal{L}}$ in the following. These conditions are satisfied for (λ, κ) in an open dense subset in $(\mathbf{C}^*)^2$. By the assumption we have an isomorphism $H_m(X_{n,m}, \mathcal{L}) \cong H_m^{lf}(X_{n,m}, \mathcal{L})$ and we can take the regularized cycles $[\widetilde{\Delta}_M] \in H_m(X_{n,m}, \mathcal{L})$ for the bounded chamber Δ_M.

We will consider the integral

$$\sum_{|J|=m} \left(\int_{\Delta_M} \Phi_{n,m} R_J(z,t)\; dt_1 \wedge \cdots \wedge dt_m \right) F^J v$$

in the space of null vectors $N[|\Lambda| - 2m]$. In general the above integral is divergent. We replace the integration cycle by the regularized cycle $[\widetilde{\Delta}_M]$ to obtain the convergent integral. This is called the regularized integral. We refer the reader to [1] for details on this aspect.

Let us denote by $\Omega^p(X_{n,m})$ the space of smooth p-forms on $X_{n,m}$. The twisted de Rham complex $(\Omega^*(X_{n,m}), \nabla)$ is a complex with the differential

$$\nabla : \Omega^p(X_{n,m}) \longrightarrow \Omega^{p+1}(X_{n,m})$$

defined by

$$\nabla\varphi = d\varphi + d\log\Phi \wedge \varphi.$$

There is a non-degenerate pairing between the homology with local coefficients and the cohomology of the twisted de Rham complex

$$H_m(X_{n,m}, \mathcal{L}^*) \times H^m(\Omega^*(X_{n,m}), \nabla) \longrightarrow \mathbf{C}$$

given by

$$(c, \varphi) \mapsto \int_c \Phi\varphi.$$

We define the map $\rho : W[|\lambda| - 2m] \longrightarrow \Omega^*(X_{n,m})$ by

$$\rho(F^J v) = R_J(z,t) dt_1 \wedge \cdots \wedge dt_m$$

for J with $|J| = m$. It is shown in [6] such that ρ induces a map

$$L[|\lambda| - 2m] \longrightarrow H^m(\Omega^*(X_{n,m}), \nabla).$$

This means that $\rho(w)$ is a closed form for any $w \in W[|\lambda| - 2m]$ and that the image of $F \cdot W[|\lambda| - 2m+2]$ is contained in the space of exact forms in the twisted de Rham complex $(\Omega^*(X_{n,m}), \nabla)$.

7. Proof of Theorem 5.1

In this section we give a proof of Theorem 5.1. We have the following proposition.

Proposition 7.1. *There exists an open dense subset U in $(\mathbf{C}^*)^2$ such that for $(\lambda, \kappa) \in U$ the following properties (1) and (2) are satisfied.*

(1) The integrals in Theorem 6.1 over $[\widetilde{\Delta}_M]$ for $M = (m_1, \cdots, m_{n-1})$ with $m_1 + \cdots + m_{n-1} = m$ are linearly independent.

(2) The homology classes $[\overline{\Delta}_M]$ for $M = (m_1, \cdots, m_{n-1})$ with $m_1 + \cdots + m_{n-1} = m$ form a basis of $H^{lf}_m(Y_{n,m}, \overline{\mathcal{L}}^*) \cong H_m(Y_{n,m}, \overline{\mathcal{L}}^*)$.

Here $m_1, \cdots, m_{n-1}$ are non-negative integers.

Proof. We suppose that λ_1 is not a non-negative integer. Let us observe that for $\Lambda = (\lambda_1, \cdots, \lambda_n)$ the space of null vectors $N[|\Lambda| - 2m]$ has dimension $d_{n,m}$. This can be shown as follows. First, let us consider the weight space

$$\begin{aligned}&(M_{\lambda_2} \otimes \cdots \otimes M_{\lambda_n})[\lambda_2 + \cdots + \lambda_n - 2m] \\ &= \{x \in M_{\lambda_2} \otimes \cdots \otimes M_{\lambda_n} \ ; \ Hx = (\lambda_2 + \cdots + \lambda_n - 2m)x\}.\end{aligned}$$

There is an isomorphism

$$\xi : (M_{\lambda_2} \otimes \cdots \otimes M_{\lambda_n})[\lambda_2 + \cdots + \lambda_n - 2m] \longrightarrow N[|\Lambda| - 2m]$$

defined by

$$u \mapsto v_{\lambda_1} \otimes u - \frac{1}{\lambda_1} F v_{\lambda_1} \otimes Eu + \frac{1}{\lambda_1(\lambda_1 - 1)} F^2 v_{\lambda_1} \otimes E^2 u - \cdots$$

This shows that $N[|\Lambda| - 2m]$ has a basis indexed by $J' = (j_1, j_2, \cdots, j_n)$ with $j_1 = 0$ and $j_2 + \cdots + j_n = m$, where $j_2, \cdots, j_n$ are non-negative integers. Let us denote by $S_{n,m}$ the set of such indices J'. The above weight space has a basis $u_{J'}$ indexed by $J' \in S_{n,m}$. We have the corresponding basis $\xi(u_{J'})$ of $N[|\Lambda|-2m]$.

We set $\alpha_1, \cdots, \alpha_n$ and γ as in (6.2). We put

$$\widetilde{\Phi}_{n,m} = \prod_{1\le i\le m, 1\le \ell\le n} (t_i - z_\ell)^{\alpha_\ell} \prod_{1\le i<j\le m} (t_i - t_j)^{2\gamma} \tag{7.1}$$

and for $J' \in S_{n,m}$ put

$$\alpha'_J = \prod_{k=2}^{n} (j_k)!\alpha_k(\alpha_k+\gamma)\cdots(\alpha_k+(j_k-1)\gamma). \tag{7.2}$$

We assume that $\alpha_1, \cdots, \alpha_n$ and γ are positive. We express the integral in Theorem 5.1 over the cycle Δ_M in the linear combination for the basis $\xi(u_{J'})$ of $N[|\Lambda|-2m]$ and we donote by $\widetilde{R}_{J'}(z,t)$ the corresponding rational function. In [21] Varchenko gave a formula for the determinant

$$\det_{M,J'} \left(\alpha_{J'} \int_{\Delta_M} \widetilde{\Phi}_{n,m} \widetilde{R}_{J'}(z,t) dt_1 \wedge \cdots \wedge dt_m \right), \tag{7.3}$$

where $M = (m_1, \cdots, m_{n-1})$ with $m_1 + \cdots + m_{n-1} = m$ and $J' \in S_{n,m}$. According to Varchenko's formula the above determinant is expressed as a non-zero constant times the gamma factor given by

$$\prod_{i=0}^{m-1} \left(\frac{\Gamma((i+1)\gamma+1)^{n-1}}{\Gamma(\gamma+1)^{n-1}} \ \frac{\Gamma(\alpha_1+i\gamma+1)\cdots(\alpha_n+i\gamma+1)}{\Gamma(\alpha_1+\cdots+\alpha_n+(2m-2-i)\gamma+1)} \right)^{\nu_i} \tag{7.4}$$

where ν_i is defined by

$$\nu_i = \begin{pmatrix} m+n-i-3 \\ m-i-1 \end{pmatrix}. \tag{7.5}$$

Since the gamma function does not has zeros and has only poles of order one at non-positive integers, it is clear that the determinant is zero only when the denominator of the gamma factor has a pole. Considering the regularized integrals over the cycles $[\widetilde{\Delta}_M]$ we can analytically continue the determinant formula to complex numbers $\alpha_1, \cdots, \alpha_n$ and γ.

Let us recall that we deal with the case

$$\alpha_\ell = -\frac{\lambda}{\kappa}, \quad 1 \le \ell \le n, \quad \gamma = \frac{1}{\kappa}.$$

From the determinant formula we observe that the linearly independence for the solutions of the KZ equation in (1) in the statement of the proposition is satisfied for (λ, κ) in an open dense subset in $(\mathbf{C}^*)^2$. Under the same condition we have the linear independence for the homology classes $[\overline{\Delta}_M]$ for $M = (m_1, \cdots, m_{n-1})$ with $m_1 + \cdots + m_{n-1} = m$. Since we have $\dim H_m^{lf}(Y_{n,m}, \overline{\mathcal{L}}^*) = d_{m,n}$ we obtain the property (2). This completes the proof of our proposition. $\square$

Let us consider the specialization map

$$s : R = \mathbf{Z}[q^{\pm 1}, t^{\pm 1}] \longrightarrow \mathbf{C} \tag{7.6}$$

defined by the substitutions $q \mapsto e^{-2\pi\sqrt{-1}\lambda/\kappa}$ and $t \mapsto e^{2\pi\sqrt{-1}/\kappa}$. This induces in a natural way a homomorphism

$$H_m(\widetilde{\mathcal{C}}_{n,m}(D); \mathbf{Z}) \longrightarrow H_m(Y_{n,m}, \overline{\mathcal{L}}^*). \tag{7.7}$$

We take a basis $[c_M]$ of $H_m(\widetilde{\mathcal{C}}_{n,m}(D); \mathbf{Z})$ as the R-module for $M = (m_1, \cdots, m_{n-1})$ with $m_1 + \cdots + m_{n-1} = m$ in such a way that $[c_M]$ maps to the regularized cycle for $[\overline{\Delta}_M]$ by the above specialization map. We observe that the LKB representation specialized at $q \mapsto e^{-2\pi\sqrt{-1}\lambda/\kappa}$ and $t \mapsto e^{2\pi\sqrt{-1}/\kappa}$ is identified with the linear representation of the braid group $r_{n,m} : B_n \to \operatorname{Aut} H_m(Y_{n,m}, \overline{\mathcal{L}}^*)$.

Now the fundamental solutions of the KZ equation with values in $N[n\lambda - 2m]$ is given by the matrix of the form

$$\left(\int_{\widetilde{\Delta}_M} \omega_{M'} \right)_{M,M'}$$

with $M = (m_1, \cdots, m_{n-1})$ and $M' = (m'_1, \cdots, m'_{n-1})$ such that $m_1 + \cdots + m_{n-1} = m$ and $m'_1 + \cdots + m'_{n-1} = m$. Here $\omega_{M'}$ is a multivalued m-form on $X_{n,m}$. The determinant of this matrix is computed in [22]. We observe that this determinant is non-zero for a generic (λ, κ). This fact is used to show Proposition 7.1. The column vectors of the above matrix form a basis of the solutions of the KZ equation with values in $N[n\lambda - 2m]$. Thus the representation $r_{n,m} : B_n \to \operatorname{Aut} H_m(Y_{n,m}, \overline{\mathcal{L}}^*)$ is equivalent to the action of B_n on the solutions of the KZ equation with values in $N[n\lambda - 2m]$. This completes the proof of Theorem 5.1.

As a consequence of the above argument we have an isomorphism

$$H_m(Y_{n,m}, \overline{\mathcal{L}}^*) \cong N[n\lambda - 2m],$$

which is equivariant under the action of the braid group B_n.

8. Integral representation of the space of conformal blocks

Let us recall some basic definitions concerning the affine Lie algebras and their representations. We deal with the case $\mathfrak{g} = sl_2(\mathbf{C})$. Let us denote by $\mathbf{C}((\xi))$ the ring of formal Laurent series consisting of the power series $\sum_{n=-m}^{\infty} a_n \xi^n$, $a_n \in \mathbf{C}$ for some positive integer m. The loop algebra $L\mathfrak{g}$ is the tensor product $\mathfrak{g} \otimes \mathbf{C}((\xi))$ equipped with a Lie algebra structure by

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg.$$

Now the affine Lie algebra $\widehat{\mathfrak{g}}$ is the central extension

$$\widehat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbf{C}c$$

with the Lie bracket defined by

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + \operatorname{Res}_{\xi=0}(df \cdot g)\langle X, Y\rangle\, c$$

where $\langle X, Y\rangle$ stands for the Cartan-Killing form for $\mathfrak{g}$.

Let A_+ denote the subring of $\mathbf{C}((\xi))$ consisting of the power series of the form $\sum_{n>0} a_n \xi^n$. In a similar way, we denote by A_- the subring consisting of $\sum_{n<0} a_n \xi^n$. We put

$$\begin{aligned} N_+ &= [\mathfrak{g} \otimes A_+] \oplus \mathbf{C}E \\ N_0 &= \mathbf{C}H \oplus \mathbf{C}c \\ N_- &= [\mathfrak{g} \otimes A_-] \oplus \mathbf{C}F, \end{aligned}$$

which gives a direct sum decomposition of Lie algebras

$$\widehat{\mathfrak{g}} = N_+ \oplus N_0 \oplus N_-.$$

Let k and λ be complex numbers. We consider the representation $M_{k,\lambda}$ of $\widehat{\mathfrak{g}}$ with the non-zero vector v satisfying the following properties :

(1) $N_+ v = 0,\ Hv = \lambda v,\ cv = kv$
(2) $M_{k,\lambda}$ is freely generated by v over $U(N_-)$,

where $U(N_-)$ is the universal enveloping algebra of $N_=$. The representation $M_{k,\lambda}$ is called the Verma module with highest weight λ and level k. It is known that for generic values k and λ the Verma module $M_{k,\lambda}$ is irreducible.

Now let us consider the case when k is a positive integer and λ is an integer such that $0 \leq \lambda \leq k$. In this case the Verma module $M_{k,\lambda}$ is not irreducible. In fact the vector $\chi = (E \otimes \xi)^{k-\lambda+1} v$ satisfies $N_+ \chi = 0$ and $U(N_-)\chi$ is a proper submodule of $M_{k,\lambda}$. The quotient module $H_{k,\lambda}$ is irreducible and is called the integrable highest weight module with highest weight λ and level k. When we fix the level k we will write H_λ for $H_{k,\lambda}$. We refer the reader to Kac [8] for details.

We denote by V_λ the irreducible $\mathfrak{g}$-module with highest weight λ. Namely, there exists a non-zero vector $v \in V_\lambda$ such that $Hv = \lambda v$ and $Ev = 0$. The vector space V_λ has a basis $v, Fv, \cdots, F^\lambda v$. The Verma module $M_{k,\lambda}$ is written as $U(N_-)V_\lambda$.

We define the space of conformal blocks for a Riemann sphere $\mathbf{C}P^1$ with marked points as follows. See [11] for a more detailed exposition of the subject. We take distinct points $p_1, \cdots, p_n, p_{n+1}$ such that $p_{n+1} = \infty$. We fix an affine coordinate function z for $\mathbf{C}P^1 \setminus \{\infty\}$. Let z_j be the coordinate for p_j, $1 \leq j \leq n$, and we set $\xi_j = z - z_j$. We take $\xi_{n+1} = 1/z$ as a local coordinate around ∞. We assign integers $\lambda_1, \cdots, \lambda_n, \lambda_{n+1}$ satisfying $0 \leq \lambda_j \leq k$, $1 \leq j \leq n+1$ to the points $p_1, \cdots, p_n, p_{n+1}$. We set $p = (p_1, \cdots, p_n, p_{n+1})$ and $\lambda = (\lambda_1, \cdots, \lambda_n, \lambda_{n+1})$. Let us denote by $\mathcal{M}_p$ the set of meromorphic functions on $\mathbf{C}P^1$ with poles at most at $p_1, \cdots, p_n, p_{n+1}$.

The space of conformal blocks $\mathcal{H}(p, \lambda)$ is by definition the space of coinvariant tensors

$$(H_{\lambda_1} \otimes \cdots \otimes H_{\lambda_{n+1}})/(\mathfrak{g} \otimes \mathcal{M}_p)$$

where $\mathfrak{g} \otimes \mathcal{M}_p$ acts diagonally on the tensor product $H_{\lambda_1} \otimes \cdots \otimes H_{\lambda_{n+1}}$ by means of the Laurent expansion of a meromorphic function at $p_1, \cdots, p_{n+1}$ with respect to the local coordinates $\xi_1, \cdots, \xi_{n+1}$. We define its dual space $\mathcal{H}(p, \lambda)^*$ as the space of multilinear forms

$$H_{\lambda_1} \otimes \cdots \otimes H_{\lambda_{n+1}} \longrightarrow \mathbf{C}$$

invariant under the diagonal action of $\mathfrak{g} \otimes \mathcal{M}_p$ defined in the above way.

It turns out that the space of conformal blocks $\mathcal{H}(p, \lambda)$ is a finite dimensional complex vector space. In fact $\mathcal{H}(p, \lambda)$ is isomorphic to the quotient space of the space of coinvariant tensors defined by

$$(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_{n+1}})/\mathfrak{g}$$

for the diagonal action of $\mathfrak{g}$. The kernel of the surjective homomorphism $(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_{n+1}})/\mathfrak{g} \to \mathcal{H}(p, \lambda)$ is described by algebraic relations depending on z coming from the existence of the null vector χ in the definition of the integrable highest weight module H_λ. The dual construction gives an injective map

$$\mathcal{H}(p, \lambda)^* \longrightarrow \mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_{n+1}}, \mathbf{C}).$$

We fix λ and consider the disjoint union

$$\mathcal{E}_\lambda = \bigcup_{(p_1, \cdots, p_n) \in X_n} \mathcal{H}(p, \lambda),$$

which has a structure of a vector bundle over X_n. It turns out that the KZ connection with the parameter $\kappa = K + 2$ induces a flat connection on the vector bundle $\mathcal{E}_\lambda$. We call $\mathcal{E}_\lambda$ the conformal block bundle. In a similar way, we define the dual conformal block bundle $\mathcal{E}_\lambda^*$ over X_n whose fiber is $\mathcal{H}(p, \lambda)^*$.

In our case the space of coinvariant tensors $L[|\lambda| - 2m]$ with $|\lambda| = \lambda_1 + \cdots + \lambda_n$ and $m = \frac{1}{2}(|\lambda| - \lambda_{n+1})$ is isomorphic to $(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_{n+1}})/\mathfrak{g}$. We have the following theorem.

Theorem 8.1 (Feigin, Schechtman and Varchenko [6])**.** *The map*

$$\rho : L[|\lambda| - 2m] \to H^m(\Omega^m(X_{n,m}), \nabla)$$

factors through the projection map $L[|\lambda| - 2m] \to \mathcal{H}(p, \lambda)$.

Furthermore, as is stated in [6], it is shown in [22] that the induced map

$$\overline{\rho} : \mathcal{H}(p, \lambda) \longrightarrow H^m(\Omega^m(X_{n,m}), \nabla)$$

is injective. Let us consider the dual map $\overline{\phi} : H_m(X_{n,m}, \mathcal{L}^*) \longrightarrow \mathcal{H}(p, \lambda)^*$ defined by

$$\langle \overline{\phi}(c), w \rangle = \int_c \Phi \overline{\rho}(w)$$

for $w \in V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_{n+1}}$. It follows from the above construction that the map $\overline{\phi}$ is surjective.

There is a flat bundle $\mathcal{H}_{n,m}$ over the configuration space X_n whose fiber is the homology $H_m(X_{n,m}, \mathcal{L}^*)$. We have a surjective bundle map

$$\mathcal{H}_{n,m} \longrightarrow \mathcal{E}_\lambda^*$$

which is compatible with the flat structures. It turns out that any horizontal section of the dual conformal block bundle $\mathcal{E}_\lambda^*$ is expressed as

$$\langle \overline{\phi}(c(z)), w \rangle = \int_{c(z)} \Phi \overline{\rho}(w)$$

with a horizontal section $c(z)$ of the flat bundle $\mathcal{H}_{n,m}$. This gives a representation of horizontal sections of the conformal block bundle as hypergeometric integrals. A difficulty here is that a generic condition described in the previous sections does not hold here in general and there is a phenomena of resonance at infinity (see [6] and [20]). We refer the reader to recent work of Looijenga [16] as an interpretation of KZ connections as Gauss-Manin connections and variations of Hodge structures.

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CHERN CLASSES OF LOGARITHMIC VECTOR FIELDS

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Abstract. Let X be a nonsingular complex variety and D a reduced effective divisor in X. In this paper we study the conditions under which the formula $c_{SM}(1_U) = c(\mathrm{Der}_X(-\log D)) \cap [X]$ is true. We prove that this formula is equivalent to a Riemann-Roch type of formula. As a corollary, we show that over a surface, the formula is true if and only if the Milnor number equals the Tjurina number at each singularity of D. We also show the Rimann-Roch type of formula is true if the Jacobian scheme of D is nonsingular or a complete intersection.

1. Introduction

Let X be a nonsingular variety over $\mathbb{C}$, and U be the complement of a free divisor D in X. In this paper, we study the conditions under which the formula

$$c_{SM}(1_U) = c(\mathrm{Der}_X(-\log D)) \cap [X] \tag{1}$$

is true. The left hand side of the formula is the Chern-Schwartz-MacPherson class of the open subvariety U, and the right hand side is the total Chern class of the sheaf of logarithmic vector fields along D. Throughout this paper we are working over the Chow homology theory A_*. The Chern classes of vector bundles and coherent sheaves are treated as operators on the Chow ring as described in [5].

We show that this formula is equivalent to an analogue of a Riemann-Roch type of formula. As a corollary, we show that on a surface, formula (1) is true if and only if the Tjurina number and the Milnor number are the same for all singularities of D.

The question is motivated by two previous results of Aluffi [2] [4], which state that if D is a normal crossing divisor, or a free hyperplane arrangement of a projective space, then formula (1) is true. Because normal crossing divisors are free divisors (for this fact and a definition of free divisors, see [7]), It is a natural question to ask if formula (1) holds for any free divisor D. The result of this paper implies the freeness of the divisor is in general insufficient to guarantee the validity of formula (1).

Replacing X by a complex manifold, we get a complex analytic version of formula (1). In this case, it is well known that the local quasi homogeneity of the divisor at an isolated singularity is characterized by the equality of the Tjurina number and the Milnor number[6]. So the corollary about formula (1) over surfaces can be restated in a slight different manner: formula (1) is true for surfaces if and only if the divisor D is locally quasi homogeneous.

2. Technical preparations

2.1. **Chern-Schwartz-MacPherson class of a hypersurface.** Continue with the notations introduced in the previous section. Recall there is a unique natural transformation from the functor of constructible functions to the Chow functor, associating the characteristic function of a nonsingular variety to the total Chern class of its tangent bundle. Then $c_{SM}(1_U)$ is the image of $1_X - 1_D$ in the Chow group of X. For the purpose of the calculation in this paper, we don't need a detailed description of the MacPherson transformation. What we need is a formula of the CSM classes of singular hypersurfaces proved in [4].

Lemma 2.1 ([4])**.** *Let D be an effective divisor in a nonsingular variety X, and let J_D be its Jacobian scheme. Then*

$$c_{SM}(1_D) = c(\mathrm{Der}_X) \cap (s(D,X) + c(\mathscr{O}(D))^{-1} \cap (s(J_D,X)^\vee \otimes_X D)) \tag{2}$$

Remark 2.2. The following is a brief clarification of several notations in the formula and some properties we will use in the next section. Proofs and more detailed explanations can be found in [3]

(1) The Jacobian scheme J_D, also called the singular subscheme of the divisor D, is the subscheme locally defined by the equation of D and all its partial derivatives.
(2) For a subscheme Y of X, $s(Y,X)$ is the *Segre class* of Y in X in the sense of [5], Chapter 4.
(3) The Segre class $s(D,X)$ is easy to compute for an effective divisor D in X. It equals the class $\frac{[D]}{c(\mathscr{O}(D))}$.
(4) If $A = \oplus_i a^i$ is a rational equivalence class on a scheme X, indexed by codimension, we let
$$A^\vee = \sum_{i \geq 0} (-1)^i a^i$$
the *dual* of A; and for a divisor D we let
$$A \otimes_X D = \sum_{i\geq 0} \frac{a^i}{c(\mathscr{O}(D))^i}$$
the *tensor* of A by D. The subscript of the tensor tells the ambient variety where the codimension of A is calculated.
(5) This tensor product of a rational equivalence class and a divisor satisfies the "associative law".
$$(A \otimes_X D_1) \otimes_X D_2 = A \otimes_X (D_1 \otimes D_2)$$
(6) if $\mathscr{F}$ is a coherent sheaf of rank 0 on a nonsingular variety X, then
$$c(\mathscr{F} \otimes D) \cap [X] = (c(\mathscr{F}) \cap [X]) \otimes_X D$$

2.2. **sheaves of logarithmic vector fields.** Let X be a nonsingular variety over a field k, and D a reduced effective divisor on X. The sheaf of logarithmic vector fields along the divisor D is a subsheaf of the sheaf of regular derivations. Over an open subset where the divisor D has a local equation f, it is given by $\mathrm{Der}_X(-\log D)(U) = \{\theta \in \mathrm{Der}_X(U) \,|\, \theta f \in (f)\}$. Saito was the first to study this sheaf in full generality[7]. He showed that $\mathrm{Der}_X(-\log D)$ is a reflexive sheaf. Its dual sheaf is called the sheaf of logarithmic 1-forms. Saito also gave a criterion when $\mathrm{Der}_X(-\log D)$ is locally free. A divisor is free if $\mathrm{Der}_X(-\log D)$ is locally free. Normal crossing divisors (in particular nonsingular divisors) are always free.

From the definition of the $\mathrm{Der}_X(-\log D)$, we obtain an exact sequence:

$$0 \to \mathrm{Der}_X(-\log D) \to \mathrm{Der}_X \to \mathscr{J}_D(D) \to 0$$

where $\mathscr{J}_D$ is called the Jacobian ideal of D. It is an ideal sheaf of $\mathscr{O}_D$ locally generated by θf for all $\theta \in \mathrm{Der}_X(U)$. From this description, we see the second arrow in the above exact sequence takes each $\theta \in \mathrm{Der}_X(U)$ to θf over an open subset U.

Combining this sequence with the sequence defining the Jacobian scheme J_D:

$$0 \to \mathscr{J}_D \to \mathscr{O}_D \to \mathscr{O}_{J_D} \to 0$$

we get a long exact sequence

$$0 \to \mathrm{Der}_X(-\log D) \to \mathrm{Der}_X \to \mathscr{O}_D(D) \to \mathscr{O}_{J_D}(D) \to 0 \tag{3}$$

Exact sequence (3) implies that a singular divisor D is free if and only if its Jacobian scheme J_D is Cohen-Macaulay of codimension 2 [1] [8]. In fact this can be seen from the following argument. To say J_D is Cohen-Macaulay of codimension 2 is equivalent to say the $\mathscr{O}_{J_D}$ is a Cohen-Macaulay $\mathscr{O}_X$-module of dimension $n-2$, where $n = dimX$. This condition is true if and only if the depths of all stalks of $\mathscr{O}_{J_D}$ are $n-2$. The Auslander-Buschbaum formula tells us the projective dimension of $\mathscr{O}_{J_D}$ at each stalk is thus 2, which in turn is equivalent to the syzygy sheaf $\mathrm{Der}_X(-\log D)$ being locally free. This simple observation allows us to deduce the fact that any reduced effective divisor in a surface is free. In fact, the Jacobian scheme J_D is 0-dimensional in this case. A 0-dimensional module is always Cohen-Macaulay because the depth is smaller or equal to the dimension for a finitely generated module over a local ring.

3. A Riemann-Roch type formula

Theorem 3.1. *Formula* (1) *is true if and only if a Riemann-Roch type formula* $\pi_*(c(\mathscr{O}_E) \cap [\tilde{X}]) = c(\mathscr{O}_{J_D}) \cap [X]$ *holds. Here* J_D *is the Jacobian scheme of* D*,* $\tilde{X}$ *is the blowup of* X *along* J_D*,* E *is the exceptional divisor, and* π *is the morphism* $\tilde{X} \to X$*. The same result can also be stated in terms of a comparison between a Segre class and a Chern class: Formula* (1) *is true if and only if* $[X] - s(J_D, X)^\vee = c(\mathscr{O}_{J_D}) \cap [X]$*.*

Proof. The proof of the theorem is based on equation (2) and exact sequence (3) in the previous section. Taking the total Chern class for exact sequence (3), we have:

$$c(\mathrm{Der}_X(-\log D)) = \frac{c(\mathrm{Der}_X) \cdot c(\mathscr{O}_{J_D}(D))}{c(\mathscr{O}_D(D))} = \frac{c(\mathrm{Der}_X) \cdot c(\mathscr{O}_{J_D}(D))}{c(\mathscr{O}(D))}$$

In the second equality we use the fact that $c(\mathscr{O}_D(D)) = c(\mathscr{O}(D))$. This can be seen by tensoring $\mathscr{O}(D)$ to the exact sequence $0 \to \mathscr{O}(-D) \to \mathscr{O}_X \to \mathscr{O}_D \to 0$ and then taking Chern classes. Then

$$c_{SM}(1_U) = c(\mathrm{Der}_X(-\log D)) \cap [X] \iff$$

$$c(\mathrm{Der}_X) \cap \left([X] - \frac{[D]}{c(\mathscr{O}(D))} - \frac{s(J_D, X)^\vee \otimes_X D}{c(\mathscr{O}(D))}\right) = \frac{c(\mathrm{Der}_X) \cdot c(\mathscr{O}_{J_D}(D))}{c(\mathscr{O}(D))} \cap [X] \iff$$

$$\frac{c(\mathrm{Der}_X)}{c(\mathscr{O}(D))} \cap ([X] - s(J_D, X)^\vee \otimes_X D) = \frac{c(\mathrm{Der}_X) \cdot c(\mathscr{O}_{J_D}(D))}{c(\mathscr{O}(D))} \cap [X] \iff$$

$$[X] - s(J_D, X)^\vee \otimes_X D = c(\mathscr{O}_{J_D}(D)) \cap [X] \iff$$

$$[X] - s(J_D, X)^\vee = c(\mathscr{O}_{J_D}) \cap [X]$$

In the last step, we "tensor" the classes on each side of the equation by the divisor $-D$. The tensor product of $[X]$ and $-D$ is $[X]$ itself because the codimension of the class $[X]$ is 0. The tensor product of $S(J_D, X)^\vee \otimes_X D$ and $-D$ is $S(J_D, X)^\vee$ according to property (5) of 2.2. The right hand side term of the equation is taken care of by property (6) of 2.2, because $\mathscr{O}_{J_D}$ is a rank 0 $\mathscr{O}_X$-module.

Next we want to prove the Riemann-Roch type formula. Recall the Segre class is preserved by proper morphisms of schemes: $\pi_*(s(E, \tilde{X})) = s(J_D, X)$[5]. It is also easy to see the dual notation of classes is compatible with the push forward of classes. Thus we have:

$$\begin{aligned}
[X] - S(J_D, X)^\vee &= \pi_*([\tilde{X}] - s(E, \tilde{X})^\vee) \\
&= \pi_* \left([\tilde{X}] - (\frac{[E]}{1+E})^\vee \right) \\
&= \pi_* \left([\tilde{X}] - \frac{-[E]}{1-E} \right) \\
&= \pi_* \left((1 + \frac{E}{1-E}) \cap [\tilde{X}] \right) \\
&= \pi_* \left(\frac{1}{1-E} \cap [\tilde{X}] \right) \\
&= \pi_*(c(\mathscr{O}_E) \cap [\tilde{X}])
\end{aligned}$$

In this computation, the notations is chosen so that $[E] \in A_{n-1}(\tilde{X})$ and E as a divisor is an abbreviation of $c_1(\mathscr{O}(E))$. □

Corollary 3.2. *Let D be a reduced effective divisor on a nonsingular complex surface X. Then formula* (1) *is true if and only if the Tjurina number equals the Milnor number for all singularities of D.*

Proof. We compare the degree zero components of $c(\mathscr{O}_P) \cap [X]$ and $s(J_D, X)$. The corollary is based on the following results:

1. $s(J_D, X) = \sum \mu(P)[P]$ where the sum is taken over all singular points P of the divisor D and $\mu(P)$ is the milnor number of P.
2. $c(\mathscr{O}_{J_D}) = \prod c(\mathscr{O}_P)^{l(P)}$ where the product is taken over all singular points P of the divisor D. $l(P)$ is the length of the stalk of $\mathscr{O}_{J_D}$ at P (because we have isolated singularities, each such stalk is an Artinian ring). These numbers are also called Tjurina numbers in literatures. $\mathscr{O}_P$ is the structure sheaf of the nonsingular subscheme supported at P.
3. $c(\mathscr{O}_P) \cap [X] = [X] - [P]$

The proof of property (1) can be found in section 7.1 of [5] and example 10.14 from [9]. The ideas in the proof is that $s(J_D^{'}, X)$ is closely related to the Milnor class, and is equal to this class when $s(J_D^{'}, X)$ is a 0-dimensional cycle class. Here $J_D^{'}$ is the subscheme having the same support as J_D but defined by a (in priori) smaller ideal generated by all partial derivatives of a local equation of D (the ideal J_D contains all partial derivatives of a local equation of D as well as a local equation of D). On the other hand, $s(J_D^{'}, X) = s(J_D, X)$ because the ideal J_D is integral over the smaller ideal $J_D^{'}$. As a result, $s(J_D, X)$ computes the Milnor numbers of the singular points.

Property (2) is obtained by considering the sequences

$$0 \to \mathscr{I}^j / \mathscr{I}^{j+1} \to \mathscr{I}^i \to \mathscr{I}^{i+1} \to 0$$

where $\mathscr{I}$ is the ideal sheaf of the Jacobian subscheme J_D supported at a singular point P, giving P the nonsingular scheme structure. In another word, $\mathscr{I}$ corresponds to the maximal ideal of the local ring of $\mathscr{O}_{J_D}$ at P. Because all stalks of $\mathscr{O}_{J_D}$ are Artinian rings, some big powers of $\mathscr{I}$ become 0, so we only have finite sequences to consider. Also notice the sheaves $\mathscr{I}^j / \mathscr{I}^{j+1}$ are free $\mathscr{O}_P$-modules of rank $l(\mathscr{I}^j / \mathscr{I}^{j+1})$. Taking Chern classes for these sequences, we get

$$c(\mathscr{I}^j)/c(\mathscr{I}^{j+1}) = c(\mathscr{I}^j / \mathscr{I}^{j+1}) = c(\mathscr{O}_P)^{l(\mathscr{I}^j / \mathscr{I}^{j+1})}$$

Multiplying all such equations together, we get the desired result.

Another way to understand this result by K-thoery of coherent sheaves can be found in example 15.1.5 and 15.3.6 of [5].

Property (3) is a rather trivial case of the Riemann-Roch without denominators. A general discussion can be found in section 15.3 of [5]. □

As mentioning in the introduction, the Milnor number being equal to Tjurina number characterizes local quasi homogeneity for isolated singularities [6]. The previous corollary can be stated in the following manner:

Corollary 3.3. *Let D be a reduced effective divisor on a nonsingular complex surface X. Then formula* (1) *is true if and only if the divisor D is locally quasi homogeneous.*

4. A further discussion of the Riemann-Roch type formula

In previous section, we showed the original formula (1) is true if and only if a formula concerning a Segre class is true, and it is moreover equivalent to a Riemann-Roch type formula. Although it is a byproduct of the original study, the last formula can be studied independent of the context of Chern classes of logarithmic vector fields. We ask the question: For what type of the subscheme Y of a nonsingular scheme X is the formula:

$$[X] - s(Y,X)^\vee = c(\mathscr{O}_Y) \cap [X]$$

true?

The formula can be easily tested true if $Y = D$ is a divisor. In fact, both sides equal to $\frac{[X]}{c(\mathscr{O}(-D))}$ in this case. the Segre class $s(Y,X) = \frac{[D]}{c(\mathscr{O}(D))}$ according to item (3) of 2.2 so the dual $s(Y,X)^\vee = \frac{-[D]}{c(\mathscr{O}(-D))}$. On the other hand $c(\mathscr{O}_D) = \frac{[X]}{c(\mathscr{O}(-D))}$ by taking Chern classes on the exact sequence $0 \to \mathscr{O}(-D) \to \mathscr{O}_X \to \mathscr{O}_D \to 0$.

Moreover, we have a little deeper result:

Theorem 4.1. *The above formula is true if Y is regularly embedded in X of codimension 2.*

Proof. Notice in these cases, the subscheme Y has the normal bundle in X. Let N be the normal bundle. We have an easy expression of the Segre class in terms of the normal bundle:

$$s(Y,X) = c(N)^{-1} \cap [Y]$$

Then

$$[X] - s(Y,X)^\vee = [X] - c(N^\vee)^{-1} \cap (-1)^2[Y] = [X] - c(N^\vee)^{-1} \cap [Y]$$

By an application of Riemann-Roch without denominators, we can also show

$$c(\mathscr{O}_Y) \cap [X] = [X] - c(N^\vee)^{-1} \cap [Y].$$

More details can be found in example 15.3.5 and example 18.2.1 from [5]. □

Remark 4.2. By the same reference [5] example 15.3.5, we can also show that this theorem is not true for regular embedding of codimension 3 or higher. For example in codimension 3, we have:

$$[X] - s(Y,X)^\vee = [X] - c(N^\vee)^{-1} \cap (-1)^3[Y] = [X] + c(N^\vee)^{-1} \cap [Y]$$

and

$$c(\mathscr{O}_Y) \cap [X] = [X] + c(N^\vee)^{-1}(2 - c_1(N))(1 - c_1(N))^{-1} \cap [Y]$$

so apparently these two expressions are not the same. If the embedding is of higher codimension, the Riemann-Roch formula for computing $c(\mathscr{O}_Y) \cap [X]$ is even more complicated.

Corollary 4.3. *Formula* (1) *is true if the Jacobian scheme of the divisor D is regularly embedded in X of codimension 2. (The freeness of the divisor D is automatic by the conditions in this case.)*

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FACTORS OF JACOBIANS AND ISOTRIVIAL ELLIPTIC SURFACES

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Abstract. We show that the rank of the Mordell-Weil group of an isotrivial elliptic surface over $\mathbb{C}(t)$ can be calculated as the number of isogeny factors which are elliptic curves in the jacobian of the cyclic cover of a projective line associated to the elliptic surface. We illustrate this method by calculating the ranks in several examples, some of which recover already known results, and discuss relation between open questions on factors of jacobians and elliptic surfaces.

1. Introduction

In papers [3] and [15] we developed a method for calculation of the ranks of Mordell-Weil groups of isotrivial complex elliptic threefolds which yields an expression for these ranks in terms of the Albanese variety of cyclic covers of the base of the elliptic fibration. In many cases this leads to explicit values of the rank (cf. [3],[15]) since the structure of Albanese variety, always depending on the singularities of the discriminant of elliptic fibration, is often rather simple even for discriminants with quite complicated singularities. In present note we illustrate a similar approach to the study of Mordell-Weil ranks of isotrivial *elliptic surfaces*. The upshot is a relation between the Mordell Weil rank of an elliptic surface with generic fiber isomorphic to an elliptic curve E and the isogeny factors isomorphic to E in the Jacobian of the appropriate cyclic cover of the base of elliptic fibration. More precisely we have the following.

Theorem 1.1. *Let $\mathcal{E} \to \mathbb{P}^1$ be an isotrivial elliptic surface over $\mathbb{C}$. Denote by E a generic fiber of this fibration and let $\Gamma = AutE$. Denote by C_Γ the cyclic cover of $\mathbb{P}^1$ branched over the zero set of the discriminant of $\mathcal{E}$ over which the pullback of $\mathcal{E}$ is biholomorphic to a direct product. Let $Jac(C_\Gamma)$ be the Jacobian of C_Γ and let*

$$r = \{max\ k | Jac(C_\Gamma) \sim_\Gamma E^k \times A\} \tag{1}$$

(here $\sim_\Gamma$ denotes equivariant isogeny of abelian varieties with Γ-action). If E has a complex multiplication then the rank of Mordell-Weil of $\mathcal{E}$ satisfies:

$$rkMW(\mathcal{E}) = 2r \tag{2}$$

Otherwise this rank is r.

Approach to the study of isotrivial families via covering space over which family trivializes in the case of surfaces was used in the past (e.g. [8] [1]). The advantage of the case of surfaces over high dimensional elliptic fibrations is that the ranks of elliptic surfaces were the objects of intense scrutiny for a long time (cf. [17]). The theorem 1.1 allows to understand the values of the Mordell-Weil ranks from a different perspective. As examples we recover several results of Usui (cf. [24]), Shioda (cf. [21]) and others, in particular a calculation of the maximal known at the moment rank of elliptic surfaces (i.e.68). Our calculations of Mordell-Weil ranks in these

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[1] I am grateful to R. Kloosterman for pointing out this reference; see also his thesis www.math.hu-berlin.de/ klooster/publ.php

examples depends on the description of the Jacobian of Fermat curves due to Koblitz (cf. [10] cf. also [1]) [2] and hyperelliptic curves given in [6] (cf. also [22]). Understanding products of elliptic curves which appear as factors of Jacobians is an interesting problem (cf. [4]) and in fact one can use the theorem 1.1 to obtain for some cyclic covers the multiplicity of a curve in the isogeny decomposition of the Jacobian (cf. 4.8) using available information on the Mordell-Weil ranks.

The content of the paper is as follows. In section 2 we recall definitions and introduce notations. Section 3 contains a proof the theorem 1.1 and in section 4 we discuss examples illustrating approach to the ranks of elliptic surfaces using Jacobians. Concluding section 5 contains a discussion of related problems.

2. Preliminaries

By elliptic surface we mean a smooth projective surface $\mathcal{E}$ together with a morphism $\pi : \mathcal{E} \to C$ where C is a smooth curve whose generic fiber is a genus one curve and which moreover is endowed with a section $s_0 : C \to \mathcal{E}$. Section s_0 allows to give to fibers of π the structure of elliptic curve. An elliptic surface is called isotrivial (resp. trivial) if the j-invariant of a generic fiber over $c \in C$ is a constant function of c (resp. $\mathcal{E}$ is birational to the surface $E \times C$ for some elliptic curve E over $\mathbb{C}$). Below $E_{\mathcal{E}}$ denotes the elliptic curve which is a smooth fiber of an isotrivial surface $\mathcal{E}$ (subscript will be omitted when the choice of $\mathcal{E}$ is clear from context). We refer for the basics of the theory of elliptic surfaces to the surveys [17] or [5, Ch.1]. For additional material related to this discussion see [15].

Recall that the Mordell-Weil group of $\mathcal{E}$ (denoted $MW(\mathcal{E})$) is the group of sections $s : C \to \mathcal{E}$ of π with the group structure given by addition of $s_1(c), s_2(c) \in E_c$ where E_c is the fiber of $\mathcal{E}$ over c with the group structure existing on any smooth curve of genus one after choice of $s_0(c)$ as the zero. This group of sections is finitely generated, *unless* $\mathcal{E}$ is trivial but in latter case the group of sections modulo the subgroup of constant sections of $C \times E$ given by $s_e : c \to e, e \in E$ (the Chow trace) is still finitely generated (cf. [12], [17]). The Mordell-Weil group in the case when $\mathcal{E}$ is trivial, is the quotient of the group of sections by the subgroup of constant sections. The morphism π gives to $\mathcal{E}$ the structure of elliptic curve over C and from this view point $MW(\mathcal{E})$ is just the group of points of elliptic curve over the function field $\mathbb{C}(C)$ (again if $\mathcal{E} \neq E \times C$ and the quotient by the subgroup $E(\mathbb{C})$ otherwise).

Since $MW(\mathcal{E})$ is a finitely generated abelian group, it is isomorphic to $\mathbb{Z}^r \oplus Tor$ where $r \in \mathbb{Z}^{\geq 0}$ and Tor is a finite abelian group. The integer r is called the rank of the elliptic surface. The rank of $\mathcal{E}$ has the following expression: (Shioda-Tate formula)

$$r = rkNS(E) - 2 - \sum_{v \in \Delta_\pi} (m(F_v) - 1) \tag{3}$$

where $NS(\mathcal{E})$ is the Neron-Severi group of $\mathcal{E}$, Δ_π is the set of points in C over which the fibers of π are singular and $m_v(F)$ is the number of irreducible component in $\pi^{-1}(v)$. Most calculations of the ranks are based on a use of (3). Note that set $\Delta_\pi \subset C$ consists of the points at which the discriminant vanishes (the latter is an element of $H^0(C, \mathcal{L}^{12})$ for some line bundle on C cf. [5, th.1.4.1].

In many cases, the ranks and Mordell-Weil groups of elliptic surfaces are known. However it seems is unknown if there is a universal bound (cf. [17]). The largest known rank 68 of elliptic surfaces is achieved by $y^2 = x^3 + t^{360k} - 1$ (cf. [21] and the section 4.5 below). Over

[2] interestingly, Shioda's calculation in these example depends on properties of Delsarte surfaces closely related to Fermat surfaces.

function fields of characteristic $p > 0$, the ranks are unbounded for both isotrivial (cf. [18]) and non-isotrivial cases (cf. [23]).

We shall need the following description of isotrivial surfaces in terms of trivial ones which we shall briefly sketch (cf. [5, 1.4.2] and references there).

Proposition 2.1. *Let $\pi : \mathcal{E} \to C$ be an isotrivial fibration with generic fiber E. Let $\Gamma = AutE$ be the automorphism group of E (i.e. a cyclic group of order $2, 4$ or 6). Then there is a curve C_Γ and a covering map $\pi_\Gamma : C_\Gamma \to C$ with the covering group Γ and ramification set supported at Δ_π such that one has birational isomorphism:*

$$\mathcal{E} \times_C C_\Gamma = C_\Gamma \times_{\mathbb{C}} E_{\mathcal{E}} \tag{4}$$

Proof. As in [15] one can use the results in [13] and [11] to deduce that there is a Γ-covering $C' - S \to C - \Delta_\pi$ such that C' is a smooth projective curve, S is a finite subset of C' and $\pi^{-1}(C - \Delta_\pi) = E \times (C' - S)/\Gamma$ where the quotient on the right is taken for the diagonal action of Γ. Using the identification $E \times (C' - S)/\Gamma \times_{C-\Delta_\pi} C_\Gamma - S = E \times (C' - S)$ the birational equivalence (4) is clear.

Alternatively, since the j-invariant i.e. the map $C - \Delta_\pi \to H/PSL_2(\mathbb{Z})$ is constant (here H is the upper half-plane) the monodromy representation $\pi_1(C - \Delta_\pi) \to Aut^+ H_1(E, \mathbb{Z})$ factors through $\pi_1(C - \Delta_\pi) \to Aut(E)$ (cf. [5, p.40]). Hence the pullback of $\mathcal{E}$ on the covering of $C - \Delta_\pi$ corresponding to the latter homomorphism of the fundamental group yields a family with constant j-invariant and trivial holonomy i.e. the direct product. $\square$

Finally recall that the Jacobian of a curve C can be characterized as an abelian variety universal with respect to holomorphic maps into abelian varieties A i.e.

$$Mor(C, A) = Hom(Jac(C), A) \tag{5}$$

(on the left is the group of maps up to a translation by a point in A). Moreover this correspondence is compatible with holomorphic maps of C and in particular if Γ is a subgroup of $Aut(C)$ acting on A then for the group of Γ-maps one has:

$$Mor_\Gamma(C, A) = Hom_\Gamma(Jac(C), A) \tag{6}$$

where the subscript indicates equivariant maps.

3. Proof of theorem 1.1

In this section we shall prove the theorem 1.1. Let $\mathcal{E}$ be isotrivial but non trivial elliptic surface. Let $s : C \to \mathcal{E}$ be a point of $\mathcal{E}$ over $\mathbb{C}(C)$. The map $C_\Gamma \to \mathcal{E} \times_C C_\Gamma$ given by $\tilde{c} \to (s(\pi_\Gamma(\tilde{c})), \tilde{c})$, ($\tilde{c} \in C_\Gamma$) yields the lift $\tilde{s}$ of s which is a section of the trivial (cf. (4)) elliptic surface $C_\Gamma \times E_{\mathcal{E}}$. Unless s has order 2 in $MW(\mathcal{E})$, $\tilde{s}$ is not constant since otherwise $\tilde{s}(\tilde{c}) = (\tilde{c}, e)$ and $e \in E_{\mathcal{E}}$ must be Γ-invariant i.e. is a 2-torsion point. Let $Sec'(C_\Gamma \times E)$ be a subgroup of the group of sections of $C_\Gamma \times E$ isomorphic to $MW(C_\Gamma \times E)$. [3] It can be defined using a splitting of the sequence

$$0 \to E \to Sec(C_\Gamma \times E) \to MW(C_\Gamma \times E) \to 0 \tag{7}$$

For example, a map $Sec(C_\Gamma \times E) \to E$ given by sending $\tilde{s}$ to $\tilde{s}(\tilde{c})$ for a point $\tilde{c} \in C_\Gamma$ yields such a split. One has the following isomorphisms of groups in which the first one is obvious while the

[3] or, if $\Gamma = \mathbb{Z}_2$ or $\mathbb{Z}_4$, rather a maximal 2-torsion free subgroup among the subgroups with the following property: the only constant Γ-invariant (for diagonal action of Γ) element in $Sec'(C_\Gamma \times E)$ is the zero section.

second is a consequence of the universal property of the Jacobian with respect to the maps from the curve to abelian varieties (cf.(5)):

$$Sec'(C_\Gamma \times E) = Mor(C_\Gamma, E) = Hom(Jac(C_\Gamma), E) \tag{8}$$

The lift of s induces the equivariant map $Jac(C_\Gamma) \to E$ with respect to the natural action of $\Gamma \subset AutE$ on E. Vice versa, equivariant map from Jacobian of C_Γ to E induces the Γ-equivariant map of C_Γ which viewed as a section of $C_\Gamma \times E$ descents to a section of $\mathcal{E}$. Hence

$$MW(\mathcal{E}) = Hom_\Gamma(Jac(C_\Gamma), E) \tag{9}$$

Next let $Jac(C_\Gamma) \sim_\Gamma E^r \times A$ and A is not Γ-isogenous to $E \times A'$. By Poincare reducibility theorem (cf. [16]) the latter is equivalent to $Hom_\Gamma(A, E) = 0$ and hence $Hom_\Gamma(Jac(C_\Gamma), E) = Hom(E^r, E) = End(E)^r$. The latter has rank $2r$ if E has complex multiplication since $rkEnd(E) = 2$ in this case. Otherwise $rkHom(E^r, E) = r$.

Note that the above argument shows that calculation of the rank in theorem 1.1 also holds if $\mathcal{E}$ is trivial.

4. Elliptic surfaces related to Fermat curves

4.1. **Cyclic covers of $\mathbb{P}^1$.** In most examples considered below the curves over which the elliptic surfaces becomes trivial and Jacobians of which according to the theorem 1.1 determine the Mordell Weil groups are quotients of Fermat curve. In particular the Jacobians of these curves are subvarieties of Jacobians of Fermat curves. We recall results from [10] describing the factors of Jacobians of Fermat curves.

Let F_N be the curve given by the equation:

$$x^N + y^N + z^N = 0 \tag{10}$$

The one dimensional components of $Jac(F_N) = H^0(F_N, \Omega^1_{F_N})^*/H_1(F_N, \mathbb{Z})$ correspond to one dimensional subspaces of $H^0(F_N, \Omega^1_{F_N})$ generated by forms

$$\omega_{r,s,t} = \frac{x^{Nr-1}y^{Ns-1}dx}{y^{N-1}} \quad (Nr, Ns, Nt \in \mathbb{Z}^+, r+s+t=1) \tag{11}$$

Note that $\omega_{r,s,t}$ spans an eigenspace for transformation induced by $(x, y, z) \to (\zeta_N x, y, z)$ (resp. transformation induced by $(x, y, z) \to (x, \zeta_N y, z)$) where $\zeta_N = exp(\frac{2\pi\sqrt{-1}}{N})$ corresponding to the eigenvalue: $exp(2\pi\sqrt{-1}r)$ (resp. $exp(2\pi\sqrt{-1}s)$).

The curves which are the isogeny components of $Jac(F_N)$ all appear as the factors of the abelian varieties denoted as $J_{[r,s,t]}$ where $[r, s, t]$ is the orbit of the triple for the action defined below. $J_{[r,s,t]}$ are all of CM type and as such correspond to the cyclotomic fields $\mathbb{Q}(\zeta_M)$ (cf. [19], [14]) with $M|N$. The CM type of $J_{[r,s,t]}$ is given by the subset of $Gal(\mathbb{Q}(\zeta_M)/\mathbb{Q}) = (\mathbb{Z}/M\mathbb{Z})^*$ defined as follows:

$$H_{r,s,t} = \{u \in (\mathbb{Z}/M\mathbb{Z})^* | < ur > + < us > + < ut >= 1\} \tag{12}$$

(here $< \cdot >$ denotes the least non negative rational residue modulo 1). The abelian varieties $J_{[r,s,t]}$ are labeled by the orbits of the following action of $H_{r,s,t}$ on triples (r, s, t): $u(r, s, t) = (< ur >, < us >, < ut >)$. Each of $J_{[r,s,t]}$ is isogenous to the product $E^{CardH_{r,s,t}}$ for an appropriate CM curve E.

Proposition 4.1. *The abelian varieties $J_{[r,s,t]}$ having the curve E_0 with j-invariant zero as an isogeny component are given in the following table:*

(13)

M	Mr	Ms	Mt	$CardH_{r,s,t}$	*label*
3	1	1	1	1	$3(i)$
6	1	1	4	1	$6(i)$
6	1	2	3	1	$6(ii)$
12	1	1	10	2	$12(i)$
12	1	2	9	2	$12(ii)$
12	1	3	8	2	$12(iii)$
12	1	4	7	2	$12(iv)$
12	1	5	6	2	$12(v)$
15	1	2	12	4	$15(i)$
15	1	4	10	4	$15(ii)$
18	1	3	14	3	$18(i)$
21	1	4	16	6	$21(i)$
21	1	8	12	6	$21(ii)$
24	1	1	22	4	$24(i)$
24	1	4	19	4	$24(ii)$
24	1	5	18	4	$24(iii)$
24	1	6	17	4	$24(iv)$
24	1	7	16	4	$24(v)$
24	1	10	13	4	$24(vi)$
24	1	11	12	4	$24(vii)$
30	1	5	24	4	$30(i)$
30	1	10	19	4	$30(ii)$
39	1	16	22	12	$39(i)$
48	1	22	25	8	$48(i)$
60	1	10	49	8	$60(i)$

Proof. Abelian varieties $J_{[r,s,t]}$ admitting the curve E_0 as isogeny component admit the automorphism of order 6 and hence $\mathbb{Q}(\zeta_6)$ is a subfield of $End(J_{[r,s,t]}) \otimes \mathbb{Q} = \mathbb{Q}(\zeta_M)$. Therefore $3|M$. The table (13) is the part of the table from [10] corresponding to M with this divisibility condition. $\square$

Proposition 4.2. *Let C_{6m} be the cyclic cover of $\mathbb{P}^1$ which is a compactification of the curve $s^6 = t^{6m} - 1$. Denote by T the automorphism of Jacobian induced by the automorphism of C_{6m} given by $(s,t) \to (\zeta_6 s, t)$. Let $\mathcal{S}(E_0)$ be the set of ordered triples $\frac{6m}{M}(r,s,t)$ (where $M|6m$) such that $\frac{r}{M} = \frac{1}{6}$ and such that no two triples belong to the same orbit of $H_{r,s,t}$. Then the maximal number of E_0-factors in $Jac(C_{6m})$ on the tangent space at identity of which T acts as multiplication by ζ_6 is equal to*

$$\sum_{(r,s,t)\in\mathcal{S}(\mathcal{E}_0)} CardH_{r,s,t} \tag{14}$$

Proof. C_{6m} is the quotient of F_{6m} by the action of the group of roots of unity generated by $(x,y,z) \to (\zeta_m x, y, z)$. Hence $Jac(C_{6m})$ is component of the product of varieties $J_{[r,s,t]}$ such that $6m(r,s,t) \in (\mathbb{Z}^+)^3$ and such that $6r \in \mathbb{Z}^+$. The equivariance condition restricts r further to $r = \frac{1}{6}$. The claim follows. $\square$

The following examples were obtained by Shioda and Usui (cf. [24]) using a different method which we shall derive from Prop. 4.2.

Example 4.3. Elliptic surface $\mathcal{E}_6 : y^2 = x^3 + t^6 - 1$ The curve C_6 is the Fermat curve F_6 and we count the number of E_0-components in the isogeny class of $Jac(F_6)$. This is the number of triples in the table (13) with $M|6$, and $r = \frac{1}{6}$. We obtain the following triples (Mr, Ms, Mt):

$$(1,1,4),(1,4,1),(1,2,3),(1,3,2). \tag{15}$$

Hence $Jac(C_6) \sim E_0^4 \times A$ where A does not have E_0 isogeny components and hence $rkMW(\mathcal{E}_6) = 8$ (cf. [24] where the corresponding lattice given by the height pairing is identified with the lattice E_8).

Example 4.4. Consider the elliptic surface $\mathcal{E}_9 : y^2 = x^3 + t^9 - 1$. $rkMW(\mathcal{E}_9)$ depends on the number of E_0 components of $u^6 = t^9 - 1$. This curve is the quotient of F_{18} by the action of $\mu_3 \times \mu_2$ given by multiplication of coordinates. Hence the E_0 components of $Jac(F_{18}/\mu_3 \times \mu_2)$ correspond to triples (a, b, c) such that,

a) $a + b + c = 18$ and

b) $3|a, 2|b$

c)$M|18$

The triples satisfying these conditions are:

$$3(i):(6,6,6),6(i):(3,12,3),6(ii):(3,6,9),6(ii)*:(9,6,3),18(i):(3,14,1). \tag{16}$$

The equivariance conditions yield that $a = 3$. This is satisfied by: $6(i), 6(ii), 18(i)$. $CardH_{r,s,t}$ in the first two cases is 1 and in the case $18(i)$ it is 3. Hence Jacobian contains E_0^5 and hence $rkMW(\mathcal{E}_9) = 10$ (cf. [24]).

Example 4.5. Consider the surface $\mathcal{E}_{360} : y^2 = x^3 + t^{360} - 1$. Shioda's calculation yields $rkMW(\mathcal{E}_{360}) = 68$. To see this from the viewpoint of the theorem 1.1 we need to calculate the number of (equivariant) E_0-factors of $Jac(C_{360})$. They correspond to the triples (r, s, t) such that $M|360$ and one of components (r, s, t) is $\frac{1}{6}$. Note that this implies that $6|M$. The triples satisfying these condition are given by the following list. Below $(R, S, T) = 360(r, s, t)$; each triple in the list comes from the triple $\frac{1}{(g.c.d(R,S,T))}(R, S, T)$ appearing in the table with $M = \frac{R+S+T}{g.c.d(R,S,T)}$; we indicate its label in table (13) in front and add asterisk if it is obtained by a permutation of a triple in (13).

$$6(i):(60,60,240),6(i)*:(60,240,60),6(ii):(60,120,180),6(ii)*:(60,180,120), \tag{17}$$

$$12(ii):(60,30,270),12(ii)*:(60,270,30),18(i):(60,20,280),18(ii)*:(60,280,20), \tag{18}$$

$$24(ii):(60,15,285),24(ii)*:(60,285,15),30(i):(20,60,288),30(i)*:(20,288,60), \tag{19}$$

$$60(i):(60,6,294),60(i)*:(60,294,6) \tag{20}$$

However $60(i), 60(i)*$ give the same abelian 4-fold since $(60, 6, 294) \cdot 7^2 = (60, 294, 6)$ ($H_{[1,10,49]} = \{7^i 31^j | i, j \in \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\}$ cf. [10]). Similarly $(60, 15, 285) \cdot (7 \cdot 13) = (60, 285, 15)$ (($H_{[1,4,19]} = \{7^i 13^j | i, j \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\}$) i.e. $J_{[60,285,15]}$ coincides with $J_{[60,15,285]}$. There are no other repetitions in abelian varieties corresponding to triples (17)-(20) as follows by direct calculation using data on $H_{r,s,t}$ from [10]. The dimensions of $J_{[r,s,t]}$ above are as follows. For $6(i), 6(i)*, 6(ii), 6(ii)*$ the dimension is equal to 1, for $12(ii), 12(**)$ it is 2, for $18(i), 18(ii)*$ it is 3, for $24(ii), 24(ii)*$ it is 4, for $30(i), 30(i)*$ it is 4 and for $60(i), 60(i)*$ it is 8. Taking into account that, as was mentioned, $J_{[1,10,49]} = J_{[1,49,10]}$ and $J_{[4,19,1]} = J_{[4,1,19]}$ we obtain:

$$\sum CardH_{r,s,t} = 1 + 1 + 1 + 1 + 2 + 2 + 3 + 3 + 4 + 4 + 4 + 8 = 34 \tag{21}$$

Hence the total rank is $2 \times 34 = 68$

Example 4.6. The case of surfaces $y^2 = x^3 + t^{360k} - 1$ can be analyzed similarly. We need to know the number of factors of the Jacobian of $s^6 = t^{360k} - 1$ which are μ_6-equivalent to the curve with j-invariant zero with given automorphism of order 6. Those are all the factors of the Jacobian of Fermat curve (10) with $N = 360k$. The equivariance conditions on (r, s, t) to appear as the factor in the Jacobian of the curve $s^6 = t^{360k} - 1$ is $Mu \cdot (\frac{N}{M}) = \frac{N}{6}$ i.e. $Mu = \frac{M}{6}$ (where $u = r, s, t$) which is the same as in the case $k = 1$. Hence the collection of the varieties $J_{[r,s,t]}$ which are the product of the curves E_0 is independent of k i.e. $rkMW$ by theorem 1.1 is independent of k as well.

Example 4.7. In [6] the authors show that the hyperelliptic curve:

$$H : y^2 = x(x-1)(x-\lambda)(x-\mu)(x-\nu) \tag{22}$$

where $\mu = \nu\frac{1-\lambda}{1-\nu}$ and λ, μ, ν are pairwise distinct, different from 0 and 1 has the Jacobian isogenous to the product of two copies of the curve:

$$E : y^2 = x(x-1)(x-\Lambda) \tag{23}$$

where Λ is a solution of

$$\nu^2\lambda^2\Lambda^2 + 2\nu\mu(-2\nu + \Lambda)\Lambda + \mu^2 = 0. \tag{24}$$

Then the elliptic surface

$$E \times H/\mu_2 \tag{25}$$

where the diagonal action of μ_2 is via multiplication by -1 on the first factor and via hyperelliptic involution on the second has the following ranks:

$$rkMW = \begin{cases} 2 & \text{if } E \text{ is without CM} \\ 4 & \text{if } E \text{ has CM} \end{cases} \tag{26}$$

This provides isotrivial elliptic surface with arbitrary j-invariant and positive Mordell-Weil rank. Many examples with large $rkMW$ can be constructed using examples of hyperelliptic curves with split Jacobian given in [22]

Example 4.8. In work [2] the authors calculated the rank of elliptic surface $y^2 = x^3 - 27(t^{12} - 11t^6 - 1)$ is equal to 18. For the curve $u^6 = t^{12} - 11t^6 - 1$ this translates into $Jac = E_0^9 \times A$ with $A \nsim E_0 \times A'$ for any A'.

5. Abelian varieties, families with higher dimensional bases and some questions

5.1. **Isotrivial abelian varieties.** Using results from [15] theorem 1.1 can be extended to the case of abelian varieties over $\mathbb{C}(t)$:

Theorem 5.1. *Let $\mathcal{A} \to \mathbb{P}^1$ be isotrivial family of abelian varieties over $\mathbb{C}$ with a simple generic fiber. Fix a projective embedding of $\mathcal{A}$, denote by A a generic fiber of this fibration and let Γ be the automorphism group of A preserving the polarization. Denote by C_Γ the cover of $\mathbb{P}^1$ branched over the zero set of the discriminant of $\mathcal{A}$ over which the pullback of $\mathcal{A}$ is biholomorphic to a direct product (cf.* [15]*, section 2.1 references there). Let $Jac(C_\Gamma)$ denote the Jacobian of C_Γ and let*

$$r = \{max\ k | Jac(C_\Gamma) \sim_\Gamma A^k \times B\} \tag{27}$$

(here $\sim_\Gamma$ denotes equivariant isogeny of abelian varieties with Γ-action). Then

$$rkMW(\mathcal{A}) = r\dim_{\mathbb{Q}} End(A) \otimes \mathbb{Q} \tag{28}$$

Example 5.2. Let $\mathcal{E}$ be an isotrivial elliptic surface with fiber being the curve E_0 and such that $rkMW(\mathcal{E}) > 0$. Let $\mathcal{A}_n = \mathcal{E} \times_{\mathbb{P}^1} ... \times_{\mathbb{P}^1} \mathcal{E}$ (n-fold product). Then $rkMW(\mathcal{A}_n) = \gamma \cdot n \ \ \gamma \in \mathbb{Z}$.

Indeed, $\mathcal{A}_n$ becomes trivial over the same cover C of $\mathbb{P}^1$ as A_1, and the argument used in the proof of 1.1 yields that $rkME(\mathcal{A}_n) = rdimHom(E_0, E_0^n) = 2nr$ where r is the multiplicity of E_0 in $Jac(C)$.

5.2. **Remarks on isotrivial elliptic threefolds.** The following relation, shown for example in [7], between the Mordell-Weil ranks of elliptic surfaces and ranks of Mordell-Weil groups of threefolds was used in [3] in order to obtain restrictions on the fundamental groups of the complements to discriminants.

Proposition 5.3. *Let $\mathcal{E} \to \mathbb{P}^2$ be an elliptic threefold, Let L be a generic line in $\mathbb{P}^2$ and $\mathcal{E}|_L$ is the restriction of $\mathcal{E}$ on L. Then $rkMW(\mathcal{E}) \le rkMW(\mathcal{E}|_L)$.*

In particular universal bounds on ranks of elliptic surfaces over $\mathbb{C}(t)$ yield bounds on ranks of n-folds (over $\mathbb{P}^{n-1}$). The relation between the fundamental groups and ranks from [3] is the following:

Theorem 5.4. *Let D be the zero set of the discriminant of elliptic threefold $\mathcal{E} \to \mathbb{P}^2$. Assume that D is irreducible and that the only singularities of C are either ordinary nodes or ordinary cusps. Then*

$$rkMW(\mathcal{E}) = rk\pi_1(\mathbb{P}^2 - D)'/\pi_1''(\mathbb{P}^2 - D) \tag{29}$$

(right hand side is the quotient of the commutator of the fundamental group by the second commutator)

Known bounds on the right hand side in (29), coming from various interpretations, (cf. [3], [9]) are linear in degree d of D. This leads to the following question:

Question 5.5. Let $f(t)$ be a polynomial of degree $d \equiv 0(mod\ 6)$ and $C_d(f)$ be the cyclic cover of $\mathbb{P}^1$ given by equation the $u^6 = f(t)$. Consider the elliptic surface $\mathcal{E}_f$ given by

$$y^2 = x^3 + f(t) \tag{30}$$

Does exist $\epsilon > 0$ and a positive constant α such that for the rank of $\mathcal{E}_f$ one has

$$rkMW(\mathcal{E}_f) \le \alpha d^{1-\epsilon} \tag{31}$$

By the theorem 1.1 this is equivalent to the following:

Question 5.6. Does there exist a bound on the number of isogeny component of $Jac(C_d(f))$ isomorphic to E_0 of the form $\alpha d^{1-\epsilon}$?

Note that for the curve $C_d(f)$ in (5.5) one has $g(C_d(f)) = \frac{5}{2}d - 5$ i.e. $\epsilon \ge 0$ in (31). As was mentioned known examples in characteristic zero obeys bound (31) with $\epsilon = 1$. It would be interesting to know what is ϵ in positive characteristic.

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MOTIVIC BIVARIANT CHARACTERISTIC CLASSES AND RELATED TOPICS

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ABSTRACT. We have recently constructed a bivariant analogue of the motivic Hirzebruch classes. A key idea is the construction of a suitable universal bivariant theory in the algebraic-geometric (or compact complex analytic) context, together with a corresponding "bivariant blow-up relation" generalizing Bittner's presentation of the Grothendieck group of varieties. Before we already introduced a corresponding universal "oriented" bivariant theory as an intermediate step on the way to a bivariant analogue of Levine–Morel's algebraic cobordism. Switching to the differential topological context of smooth manifolds, we similarly get a new geometric bivariant bordism theory based on the notion of a "fiberwise bordism". In this paper we make a survey on these theories.

1. INTRODUCTION

For the category of finite sets, the number of elements of the set is a basic invariant and the natural numbers $\mathbb{N}$ is the collection of such invariants. The number of elements of a finite set F is called the *cardinality*, denoted by $c(F)$ or $|F|$. The cardinality satisfies the following properties:

(1) if $X \cong X'$ (set-isomorphism), then $c(X) = c(X')$,
(2) $c(X) = c(Y) + c(X \setminus Y)$ for a subset $Y \subset X$ (a *scissor formula*),
(3) $c(X \times Y) = c(X) \times c(Y)$,
(4) $c(pt) = 1$.

The above property (1) is a crucial requirement for counting elements of finite sets. Now, when we consider a similar "cardinality" or invariant on a suitable subcategory of topological spaces, we modify the above requirements (1) and (2) as follows:

(1)' If $X \cong X'$ ($\mathcal{TOP}$-isomorphism), then $c(X) = c(X')$,
(2)' $c(X) = c(Y) + c(X \setminus Y)$ for a closed subset $Y \subset X$,
(3) $c(X \times Y) = c(X) \times c(Y)$,
(4) $c(pt) = 1$.

If such a topological cardinality exists, then it follows that

$$c(\mathbb{R}^1) = c\Big((-\infty, 0) \sqcup \{0\} \sqcup (0, \infty)\Big) = c(\mathbb{R}^1) + 1 + c(\mathbb{R}^1)\,,$$

so that $c(\mathbb{R}^1) = -1$ and $c(\mathbb{R}^n) = (-1)^n$. Thus, for a finite CW-complex X, $c(X)$ is equal to the Euler–Poincaré characteristic $\chi(X)$. The existence of such a topological cardinality is guaranteed by homology theory. To be more precise

$$c(X) := \chi_c(X) = \sum (-1)^i \dim_{\mathbb{R}} H^i_c(X;\mathbb{R}) = \sum (-1)^i \dim_{\mathbb{R}} H_i^{BM}(X;\mathbb{R}) \in \mathbb{Z}\,.$$

Here $H_*^{BM}(X)$ is the Borel–Moore homology group of a locally compact X. Of course to make sense of this, we have to assume that $H_*^{BM}(-)$ is finite dimensional for all spaces considered. Such a very nice

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context is for example the semi-algebraic (or more generally o-minimal) context (e.g., see [35, Chapter 2]).

Let us consider now a similar "cardinality" or invariant on the category $\mathcal{V}$ of complex algebraic varieties, say "$\mathcal{V}$-cardinality", by modifying (1)' and (2)' as

(1)" If $X \cong X'$ ($\mathcal{V}$-isomorphism), then $c(X) = c(X')$,
(2)" $c(X) = c(Y) + c(X \setminus Y)$ for a closed subvariety $Y \subset X$,
(3) $c(X \times Y) = c(X) \times c(Y)$,
(4) $c(pt) = 1$.

If such an "algebraic-geometric" cardinality exists, then we have

$$c(\mathbb{P}^n) = c(\mathbb{C}^0 \sqcup \mathbb{C}^1 \sqcup \cdots \sqcup \mathbb{C}^n) = 1 + c(\mathbb{C}^1) + \cdots + c(\mathbb{C}^1)^n \, .$$

Note that we cannot do the same trick as above for $c(\mathbb{R}^1) = -1$. The existence of such an algebraic cardinality is guaranteed by Deligne's theory of mixed Hodge structures. Let u, v be two variables, then the *Deligne–Hodge polynomial* $\chi_{u,v}$ is defined by

$$\chi_{u,v}(X) = \sum (-1)^i (-1)^{p+q} \dim_{\mathbb{C}} Gr^p_{\mathcal{F}} Gr^{\mathcal{W}}_{p+q}(H^i_c(X;\mathbb{C})) u^p v^q \in \mathbb{Z}[u,v] \, .$$

Here $\mathcal{W}$ is the weight filtration and $\mathcal{F}$ the Hodge filtration of the corresponding mixed Hodge structure. Then $\chi_{u,v}$ is such an algebraic-geometric cardinality with $\chi_{u,v}(\mathbb{C}^1) = uv$. Let us consider the specialization $u = y, v = -1$. Then we have

$$\chi_y(X) := \chi_{y,-1}(X) = \sum (-1)^i (-1)^q \dim_{\mathbb{C}} Gr^p_{\mathcal{F}}(H^i_c(X;\mathbb{C})) y^p \, ,$$

i.e., only the Hodge (but not the weight) filtration is used. This is called χ_y*-genus* of X.

Let $\mathrm{Iso}(\mathcal{V})$ be the free abelian group generated by the isomorphism classes of complex algebraic varieties. Then the above χ_y can be considered as the homomorphism $\chi_y : \mathrm{Iso}(\mathcal{V}) \to \mathbb{Z}[y]$ defined by $\chi_y([X]) := \chi_y(X)$. Because of the condition (2)" we get

$$\chi_y : K_0(\mathcal{V}) := \frac{\mathrm{Iso}(\mathcal{V})}{\{[X] - [Y] - [X \setminus Y] \mid Y \subset X\}} \to \mathbb{Z}[y] \hookrightarrow \mathbb{Q}[y] \, ,$$

where Y is a closed algebraic subset of X and $\{[X] - [Y] - [X \setminus Y] \mid Y \subset X\}$ is the abelian subgroup generated by the elements of the form $[X] - [Y] - [X \setminus Y]$. $K_0(\mathcal{V})$ is called the Grothendieck group (or ring) of complex algebraic varieties, with $c(X) = [X]$ the universal motivic "algebraic-geometric" cardinality. $K_0(\mathcal{V}) = K_0(\mathcal{V}/pt)$ can be extended to a covariant (and also contravariant) functor $K_0(\mathcal{V}/-)$ by

$$K_0(\mathcal{V}/X) := \frac{\{[V \to X]\}}{\left\{ [W \xrightarrow{h} X] - [Z \xrightarrow{h|_Z} X] - [W \setminus Z \xrightarrow{h|_{W \setminus Z}} X] \;\middle|\; Z \subset W \right\}} \, ,$$

where Z is a closed subvariety of W. Here and in the following $\{\cdots\}$ always denotes the corresponding free abelian group (or its subgroup) generated by the listed elements. $K_0(\mathcal{V}/-)$ is covariantly functorial by composition of arrows, whereas for the contravariance one takes the corresponding fiber products. Moreover, these functorialities are compatible with the cross product $\times$ coming from the product of varieties. Note that the same construction works for the category $\mathcal{V} = \mathcal{V}_k^{(qp)}$ of (quasi-projective) algebraic varieties (i.e., reduced separated schemes of finite type) over a base field k, and with a little bit more care (see [7]), also for the "compactifiable" complex analytic context.

Another sort of "algebraic-geometric" cardinality is given by a characteristic number

$$c(M) := \pi_{M*}\left(c\ell(TM) \cap [M]\right) \in H_*(pt) \otimes R$$

of the tangent bundle TM of a compact complex algebraic or analytic manifold (or complete smooth algebraic variety) M, with the constant map $\pi_M : M \to pt$ proper and $[M] \in H_*(M)$ the fundamental class of M. Here $H_*(X) = H_{2*}^{BM}(X)$ the even degree Borel-Moore homology or the Chow homology $H_*(X) = CH_*(X)$ (as in [19]). Of course the condition (2)" above does not make sense then. But if

$$c\ell : Vect(-) \to H^*(-) \otimes R$$

is a contravariantly functorial characteristic class from the isomorphism classes of algebraic (or analytic) vector bundles to the appropriate cohomology $H^*(-)$ tensorized with the ring R (i.e., the usual even degree cohomology or the operational Chow cohomology of [19]), then (1)" follows from the projection formula. And if $c\ell$ is also multiplicative (resp., normalized), then this implies (3) (resp., (4)) above. Moreover, as a substitute for (2)", the characteristic number $c(M)$ depends in this case only on the (co)bordism class of M in the algebraic or complex cobordism group of a point (as explained later):

$(2)^{bor}$ If $[M] = [M'] \in \Omega_*^{LM}(pt)$ or $[M] = [M'] \in \Omega_*^{U}(pt)$, then $c(M) = c(M')$.

An important example of such a functorial multiplicative and normalized characteristic class of a complex or algebraic vector bundle E is the Hirzebruch or generalized Todd class of E defined by

$$td_y(E) := \prod_{i=1}^{\operatorname{rank} E} \left(\frac{\alpha_i(1+y)}{1 - e^{-\alpha_i(1+y)}} - \alpha_i y \right) \in H^*(-) \otimes \mathbb{Q}[y]$$

where $\alpha_i \in H^1(-)$ are the Chern roots of E, i.e., the total Chern class of E is given by

$$c(E) = \prod_{i=1}^{\operatorname{rank} E} (1 + \alpha_i) \in H^*(-) \,.$$

The corresponding characteristic number $c(M) =: \chi_y(M)$ is the Hirzebruch χ_y-genus of the manifold M. Note that for a compact complex algebraic manifold M this also agrees with the earlier definition given above in terms of Hodge numbers. And as explained in [7], it is the most general characteristic number having an "additive" extension to singular varieties (over any base field of characteristic zero, and for compactifiable complex analytic varieties), i.e., satisfying the "scissor formula" (2)". Note that the Deligne–Hodge polynomial $\chi_{u,v}(M)$ for a compact complex algebraic manifold M is not a characteristic number in this sense.

Remark 1.1. The Hirzebruch class unifies the following three classes, which are important in geometry and topology:

- $y = -1$: $td_{-1}(E) = \prod_{i=1}^{\operatorname{rank} E} (1 + \alpha_i) = c(E)$, the total Chern class,
- $y = 0$: $td_0(E) = \prod_{i=1}^{\operatorname{rank} E} \frac{\alpha_i}{1 - e^{-\alpha_i}} = td(E)$, the total (original) Todd class,
- $y = 1$: $td_1(E) = \prod_{i=1}^{\operatorname{rank} E} \frac{\alpha_i}{\tanh \alpha_i} = L(E)$, the total Thom–Hirzebruch L-class.

A Grothendieck–Riemann–Roch-type theorem for the χ_y-genus is the following:

Theorem 1.2 ([7] (cf. [36], [46])). *Consider the compact complex analytic or the algebraic context over a base field k of characteristic zero.*

(1) There exists a unique natural transformation (functorial for proper morphisms)

$$T_{y_*} : K_0(\mathcal{V}/-) \to H_*(-) \otimes \mathbb{Q}[y]$$

such that for a smooth variety X

$$T_{y_*}([X \xrightarrow{\mathrm{id}_X} X]) = td_y(TX) \cap [X] \,.$$

Whether X is singular or not, $T_{y_*}(X) := T_{y_*}([X \xrightarrow{id_X} X])$ is called the motivic Hirzebruch class of X.

(2) *When $X = pt$ is a point, $T_{y_*} : K_0(\mathcal{V}/pt) = K_0(\mathcal{V}) \to \mathbb{Q}[y]$ equals χ_y.*

The above Hirzebruch class transformation $T_{y_*} : K_0(\mathcal{V}/-) \to H_*(-) \otimes \mathbb{Q}[y]$ "unifies" the following three well-known characteristic classes of singular varieties. Here we work either in the category $\mathcal{V} = \mathcal{V}_k^{(qp)}$ of (quasi-projective) algebraic varieties over a base field k, with $H_*(X) = CH_*(X)$ the Chow homology groups, or in the category $\mathcal{V} = \mathcal{V}_c^{an}$ of compact reduced complex analytic spaces, with $H_*(X) = H_{2*}^{BM}(X)$ the even degree Borel-Moore homology in the complex algebraic or analytic context:

- MacPherson's Chern class transformation [7, 25, 31]:

$$c_* : F(X) \to H_*(X),$$

defined on the group $F(X)$ of constructible functions in the algebraic context for k of characteritic zero or in the compact complex analytic context. The transformation $c_* : F(-) \to H_*(-)$ is the unique one satisying the smooth condition that for a smooth M, $c_*(1\!\!1_M) = c(TM) \cap [M]$ where TM is the tangent bundle of M.

- Baum–Fulton–MacPherson's Todd class or Riemann–Roch transformation [4, 19]:

$$td_* : G_0(X) \to H_*(X) \otimes \mathbb{Q},$$

defined on the Grothendieck group $G_0(X)$ of coherent sheaves in the algebraic context in any characteristic. In the compact complex analytic context such a transformation can be deduced (compare with [7]) from Levy's K-theoretical Riemann-Roch transformation [30]. The transformation $td_* : G_0(-) \to H_*(-) \otimes \mathbb{Q}$ is the unique one satisying the smooth condition that for a smooth M, $td_*(\mathcal{O}_M) = td(TM) \cap [M]$.

- Goresky– MacPherson's homology L-class [21], which is extended as a natural transformation by Cappell–Shaneson [12] (see also [7, 42, 41]):

$$L_* : \Omega(X) \to H_*(X) \otimes \mathbb{Q}$$

defined on the cobordism group $\Omega(X)$ of selfdual constructible sheaf complexes (for the Verdier duality).. This transformation is only defined for compact spaces in the complex algebraic or analytic context, with H_* the usual homology, since its definition is based on a corresponding signature invariant together with the Thom–Pontrjagin construction. The transformation $L_* : \Omega(-) \to H_*(-) \otimes \mathbb{Q}$ satisfies the smooth condition that for a smooth M, $L_*(\mathbb{Q}_M[\dim M]) = L(TM) \cap [M]$.

The unification means that there are natural transformations ϵ, mC_0 and sd so that the following diagrams of transformations commute:

$$\begin{tikzcd}
 & K_0(\mathcal{V}/X) \arrow[dl, "\epsilon"'] \arrow[dr, "T_{-1_*}"] & \\
F(X) \arrow[rr, "c_* \otimes \mathbb{Q}"'] & & H_*(X) \otimes \mathbb{Q}.
\end{tikzcd}$$

$$\begin{tikzcd} & K_0(\mathcal{V}/X) \arrow[dl, "mC_0"'] \arrow[dr, "T_{0*}"] & \\ G_0(X) \arrow[rr, "td_*"'] & & H_*(X)\otimes\mathbb{Q}. \end{tikzcd}$$

$$\begin{tikzcd} & K_0(\mathcal{V}/X) \arrow[dl, "sd"'] \arrow[dr, "T_{1*}"] & \\ \Omega(X) \arrow[rr, "L_*"'] & & H_*(X)\otimes\mathbb{Q}. \end{tikzcd}$$

This "unification" could be considered as a positive answer to MacPherson's question posed in [32]. Here the corresponding uniqueness result follows from the surjectivity of ϵ and mC_0, whereas for the L-class transformation this uniqueness only holds on the image of the transformation sd (which is not surjective).

Moreover, in [7] we also constructed in the algebraic context for k of characteristic zero and in the compact complex analytic context, a motivic Chern class transformation (functorial for proper morphisms)

$$mC_y : K_0(\mathcal{V}/X) \to G_0(X)\otimes\mathbb{Z}[y]\ .$$

This satisfies the normalization condition

$$mC_y(M) := mC_y([id_M]) = \sum_{i=0}^{dim(M)} [\Lambda^i T^*M]\cdot y^i =: \lambda_y([T^*M])\cap[\mathcal{O}_M]$$

for M smooth, with λ_y the total λ-class. Then the Hirzebruch class transformation T_{y*} could also be defined as the composition $td_*\circ mC_y$, renormalized by the multiplication $\times(1+y)^{-i}$ on $H_i(X)\otimes\mathbb{Q}[y]$ to fit with the normalization condition above (see [7]). So mC_y could be considered as a K-theoretical refinement of T_{y*}.

W. Fulton and R. MacPherson have introduced Bivariant Theory [20] (see also [19]). As reviewed very quickly in §2, a bivariant theory is defined on morphisms, instead of objects, and "unifies" both a covariant functor and a contravariant functor. Important topics in Bivariant Theories are what they call *Grothendieck transformations* between given two bivariant theories. A Grothendieck transformation is a bivariant natural transformation. The main objectives of [20] are bivariant-theoretic Riemann–Roch transformations or bivariant analogues of various theorems of Grothendieck–Riemann–Roch type. A key example of [20, Part II] is the bivariant Riemann–Roch transformation $\tau : \mathbb{K}_{alg} \to \mathbb{H}\otimes\mathbb{Q}$ on the category $\mathcal{V} = \mathcal{V}^{qp}_{\mathbb{C}}$ of complex quasi-projective varieties, with $\mathbb{K}_{alg}(f)$ the bivariant algebraic K-theory of f-perfect complexes and $\mathbb{H}$ the even degree bivariant homology. It unifies the covariant Todd class transformation td_* and the contravariant Chern character ch. An algebraic version on the category $\mathcal{V} = \mathcal{V}^{qp}_k$ of quasi-projective varieties over a base field k of any characteristic was constructed later on in [19, Example 18.3.19], with $\mathbb{H} = CH$ the bivariant operational Chow groups.

As another example, in [20, Part I, §6] Fulton and MacPherson constructed a bivariant Whitney class transformation. And they asked in the complex algebraic context for a corresponding bivariant Chern class transformation $\gamma : \mathbb{F} \to \mathbb{H}$ on their bivariant theory $\mathbb{F}$ of constructible functions satisfying a suitable local Euler condition, which generalizes the covariant MacPherson Chern class transformation $c_* : F \to H_*(-)$. For $\mathbb{H}$ the even degree bivariant homology, this problem was solved by Brasselet [6] in a suitable context (even for compact analytic spaces), whereas Ernström–Yokura [17] solved it for $\mathbb{H} = A^{PI} \supset CH$ another bivariant operational Chow group theory (for the notation A^{PI} see [17]). In [18], by introducing another bivariant theory $\tilde{\mathbb{F}}$ of constructible functions, they also introduced a bivariant

Chern class transformation $\gamma : \tilde{\mathbb{F}} \to CH$. Their approach is based on the usual calculus of constructible functions and the surjectivity of $c_* : F(X) \to CH_*(X)$. Therefore it works in the algebraic context over any base field k of characteristic zero, even though it was stated in [18] only in the complex algebraic context. Here $\tilde{\mathbb{F}}(X \to pt) = F(X)$ follows from the multiplicativity of c_* with respect to cross products $\times$.

In [38] we obtain in the quasi-projective context (over a base field k of any characteristic) two bivariant analogues

$$mC_y = \Lambda_y^{mot} : \mathbb{K}_0(\mathcal{V}_k^{qp}/X \to Y) \to \mathbb{K}_{alg}(X \to Y) \otimes \mathbb{Z}[y]$$

and

$$T_y : \mathbb{K}_0(\mathcal{V}^{qp}/X \to Y) \to \mathbb{H}(X \to Y) \otimes \mathbb{Q}[y]$$

of the motivic Chern and Hirzebruch class transformations mC_y and T_y, with T_y defined as the composition $\tau \circ mC_y$, renormalized by the multiplication $\times(1+y)^i$ on $\mathbb{H}^i(-) \otimes \mathbb{Q}[y]$. Moreover, T_y unifies the bivariant Riemann–Roch transformation $\tau : \mathbb{K}_{alg} \to \mathbb{H} \otimes \mathbb{Q}$ (for $y = 0$) and the bivariant Chern class transformation $\gamma : \tilde{\mathbb{F}} \to CH$ (for $y = -1$). Note that a bivariant L-class transformation (corresponding to $y = 1$) is still missing. In [9, 10] we considered a kind of general construction of a bivariant analogue of a given natural transformation between two covariant functors, but our approach presented in this paper is quite different from it. The former is more "operational", but the latter is more "direct" and very "motivic".

In this paper we make a survey on the above results [38] as well as on a corresponding universal "oriented" bivariant theory [44], which is a first step on the way to a bivariant-theoretic analogue of Levine–Morel's or Levine–Pandharipande's algebraic cobordism [28, 29]. Finally we switch to a differential topological context of smooth manifolds and make a remark on a new geometric bivariant bordism theory based on the notion of a "fiberwise bordism" ([3], [45]).

2. Fulton–MacPherson's bivariant theory

We quickly recall some basic ingredients of Fulton–MacPherson's bivariant theory [20].

Let $\mathcal{V}$ be a category which has a final object pt and on which the fiber product or fiber square is well-defined, e.g. the category $\mathcal{V}_k^{(qp)}$ of (quasi-projective) algebraic varieties (i.e., reduced separated schemes of finite type) over a base field k, or $\mathcal{V}_{(c)}^{an}$ the category of (compact) reduced complex analytic spaces. We also consider a class of maps, called "confined maps" (e.g., proper maps in this algebraic or analytic geometric context), which are closed under composition and base change and contain all the identity maps. Finally, one fixes a class of fiber squares, called "independent squares" (or "confined squares", e.g., "Tor-independent" in algebraic geometry, a fiber square with some extra conditions required on morphisms of the square), which satisfy the following properties:

(i) if the two inside squares in

$$\begin{tikzcd} X'' \arrow[r, "h'"] \arrow[d, "f''"] & X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y'' \arrow[r, "h"'] & Y' \arrow[r, "g"'] & Y \end{tikzcd} \qquad \text{or} \qquad \begin{tikzcd} X' \arrow[r, "h''"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "h'"'] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ Z' \arrow[r, "h"'] & Z \end{tikzcd}$$

are independent, then the outside square is also independent.

(ii) for any morphism $f : X \to Y$, the following squares are independent:

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "\mathrm{id}_X"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{id}_X"'] & Y \arrow[d, "\mathrm{id}_Y"] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

A bivariant theory $\mathbb{B}$ on a category $\mathcal{V}$ with values in the category of (graded) abelian groups is an assignment to each morphism $X \xrightarrow{f} Y$ in the category $\mathcal{V}$ a (graded) abelian group (in most cases we can ignore a possible grading) $\mathbb{B}(X \xrightarrow{f} Y)$, which is equipped with the following three basic operations. The i-th component of $\mathbb{B}(X \xrightarrow{f} Y)$, $i \in \mathbb{Z}$, is denoted by $\mathbb{B}^i(X \xrightarrow{f} Y)$ (with $\mathbb{B}(X \xrightarrow{f} Y) =: \mathbb{B}^0(X \xrightarrow{f} Y)$ in the ungraded context).

Product operations: For morphisms $f : X \to Y$ and $g : Y \to Z$, the ($\mathbb{Z}$-bilinear) product operation

$$\bullet : \mathbb{B}^i(X \xrightarrow{f} Y) \otimes \mathbb{B}^j(Y \xrightarrow{g} Z) \to \mathbb{B}^{i+j}(X \xrightarrow{gf} Z)$$

is defined.

Pushforward operations: For morphisms $f : X \to Y$ and $g : Y \to Z$ with f *confined*, the ($\mathbb{Z}$-linear) pushforward operation

$$f_* : \mathbb{B}^i(X \xrightarrow{gf} Z) \to \mathbb{B}^i(Y \xrightarrow{g} Z)$$

is defined.

Pullback operations: For an *independent* square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y, \end{tikzcd}$$

the ($\mathbb{Z}$-linear) pullback operation

$$g^* : \mathbb{B}^i(X \xrightarrow{f} Y) \to \mathbb{B}^i(X' \xrightarrow{f'} Y')$$

is defined.

And these three operations are required to satisfy the seven compatibility axioms (see [20, Part I, §2.2] for details):

(B-1) product is associative,
(B-2) pushforward is functorial,
(B-3) pullback is functorial,
(B-4) product and pushforward commute,
(B-5) product and pullback commute,

(B-6) pushforward and pullback commute, and
(B-7) projection formula.

We also assume that $\mathbb{B}$ has *units*, i.e., there is an element $1_X \in \mathbb{B}^0(X \xrightarrow{\mathrm{id}_X} X)$ such that $\alpha \bullet 1_X = \alpha$ for all morphisms $W \to X$ and $\alpha \in \mathbb{B}(W \to X)$; such that $1_X \bullet \beta = \beta$ for all morphisms $X \to Y$ and $\beta \in \mathbb{B}(X \to Y)$; and such that $g^* 1_X = 1_{X'}$ for all $g : X' \to X$.

Let $\mathbb{B}, \mathbb{B}'$ be two bivariant theories on the category $\mathcal{V}$. A *Grothendieck transformation* from $\mathbb{B}$ to $\mathbb{B}'$

$$\gamma : \mathbb{B} \to \mathbb{B}'$$

is a collection of group homomorphisms

$$\mathbb{B}(X \to Y) \to \mathbb{B}'(X \to Y)$$

for all morphisms $X \to Y$ in the category $\mathcal{V}$, which preserves the above three basic operations (as well as the units, but not necessarily possible gradings):

(i) $\gamma(\alpha \bullet_{\mathbb{B}} \beta) = \gamma(\alpha) \bullet_{\mathbb{B}'} \gamma(\beta)$,
(ii) $\gamma(f_* \alpha) = f_* \gamma(\alpha)$, and
(iii) $\gamma(g^* \alpha) = g^* \gamma(\alpha)$.

Most of our bivariant theories in this paper are *commutative* (see [20, §2.2]), i.e., if whenever both

$$\begin{tikzcd} W \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd} \qquad \begin{tikzcd} W \arrow[r, "f'"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

are independent squares, then for $\alpha \in \mathbb{B}(X \xrightarrow{f} Z)$ and $\beta \in \mathbb{B}(Y \xrightarrow{g} Z)$

$$g^*(\alpha) \bullet \beta = f^*(\beta) \bullet \alpha.$$

$\mathbb{B}_*(X) := \mathbb{B}(X \to pt)$ becomes a covariant functor for *confined* morphisms and $\mathbb{B}^*(X) := \mathbb{B}(X \xrightarrow{id} X)$ becomes a contravariant ring valued functor for *any* morphisms, with $\mathbb{B}_*(X)$ a left $\mathbb{B}^*(X)$-module under the product $\cap := \bullet : \mathbb{B}^*(X) \otimes \mathbb{B}_*(X) \to \mathbb{B}_*(X)$. As to a possible grading, one sets

$$\mathbb{B}_i(X) := \mathbb{B}^{-i}(X \to pt) \quad \text{and} \quad \mathbb{B}^j(X) := \mathbb{B}^j(X \xrightarrow{id} X)$$

so that $\mathbb{B}^*(X)$ becomes a graded ring with $\cap : \mathbb{B}^j(X) \otimes \mathbb{B}_i(X) \to \mathbb{B}_{i-j}(X)$.

The following notion of a *canonical orientation* makes $\mathbb{B}_*$ a contravariant functor and $\mathbb{B}^*$ a covariant functor with the corresponding Gysin (or transfer) homomorphisms:

Definition 2.1. ([20, Part I, Definition 2.6.2]) Let $\mathcal{S}$ be a class of maps in $\mathcal{V}$, which is closed under compositions and contains all identity maps. Suppose that to each $f : X \to Y$ in $\mathcal{S}$ there is assigned an element $\theta(f) \in \mathbb{B}(X \xrightarrow{f} Y)$ satisfying that

(i) $\theta(g \circ f) = \theta(f) \bullet \theta(g)$ for all $f : X \to Y, g : Y \to Z \in \mathcal{S}$ and
(ii) $\theta(\mathrm{id}_X) = 1_X$ for all X with $1_X \in \mathbb{B}^*(X) := \mathbb{B}(X \xrightarrow{\mathrm{id}_X} X)$ the unit element.

Then $\theta(f)$ is called a *canonical orientation* of f. If we need to refer to which bivariant theory we consider, we denote $\theta_{\mathbb{B}}(f)$ instead of the simple notation $\theta(f)$.

For example the class $\mathcal{S}$ of *smooth* morphisms in the algebraic or analytic geometric context has canonical orientations for all the bivariant theories mentioned in the introduction, with all Cartesian squares independent.

Proposition 2.2. *For the composite $X \xrightarrow{f} Y \xrightarrow{g} Z$, if $f \in \mathcal{S}$ has a canonical orientation $\theta_{\mathbb{B}}(f)$, then we have the Gysin homomorphism (or transfer) defined by $f^!(\alpha) := \theta(f) \bullet \alpha$:*

$$f^! : \mathbb{B}(Y \xrightarrow{g} Z) \to \mathbb{B}(X \xrightarrow{gf} Z),$$

which is functorial. In particular, when $Z = pt$, we have the Gysin homomorphism: $f^! : \mathbb{B}_*(Y) \to \mathbb{B}_*(X)$.

Proposition 2.3. *For an independent square*
$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$
, if $g \in \mathcal{C} \cap \mathcal{S}$ and g has a canonical orientation $\theta_{\mathbb{B}}(g)$, then we have the Gysin homomorphism defined by $g_!(\alpha) := g'_*(\alpha \bullet \theta(g))$:

$$g_! : \mathbb{B}(X' \xrightarrow{f'} Y') \to \mathbb{B}(X \xrightarrow{f} Y),$$

which is functorial. In particular, for an independent square
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{id}_X"'] & Y \arrow[d, "\mathrm{id}_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$
with $f \in \mathcal{C} \cap \mathcal{S}$, we have the Gysin homomorphism: $f_! : \mathbb{B}^*(X) \to \mathbb{B}^*(Y)$.

The symbols $f^!$ and $g_!$ should carry the information of $\mathcal{S}$ and the canonical orientation θ, but we omit them for the sake of simplicity.

A Grothendieck transformation $\gamma : \mathbb{B} \to \mathbb{B}'$ of two bivariant theories $\mathbb{B}$ and $\mathbb{B}'$ induces natural transformations $\gamma_* : \mathbb{B}_* \to \mathbb{B}'_*$ and $\gamma^* : \mathbb{B}^* \to {\mathbb{B}'}^*$, i.e., we have the following commutative diagrams:
For any morphism $f : X \to Y$ we have the commutative diagram

$$\begin{tikzcd} \mathbb{B}^*(X) \arrow[r, "\gamma^*"] \arrow[d, "f^*"'] & \mathbb{B}'^*(X) \arrow[d, "f^*"] \\ \mathbb{B}^*(Y) \arrow[r, "\gamma^*"'] & \mathbb{B}'^*(Y). \end{tikzcd}$$

For a confined morphism $f : X \to Y$ we have the commutative diagram

$$\begin{tikzcd} \mathbb{B}_*(X) \arrow[r, "\gamma_*"] \arrow[d, "f_*"'] & \mathbb{B}'_*(X) \arrow[d, "f_*"] \\ \mathbb{B}_*(Y) \arrow[r, "\gamma_*"'] & \mathbb{B}'_*(Y). \end{tikzcd}$$

They are related by the *module property*

$$\gamma_*(\beta \cap \alpha) = \gamma^*(\beta) \cap \gamma_*(\alpha) \quad \text{for all} \quad \beta \in \mathbb{B}^*(X), \alpha \in \mathbb{B}_*(X).$$

Suppose that $f : X \to Y$ has a canonical orientation for both bivariant theories. A bivariant element $u_f \in \mathbb{B}'^*(X) = \mathbb{B}'(X \xrightarrow{\mathrm{id}_X} X)$ satisfying

$$\gamma(\theta_{\mathbb{B}}(f)) = u_f \bullet \theta_{\mathbb{B}'}(f)$$

is called a *Riemann–Roch formula* (see [20]). Such a Riemann–Roch formula gives rise to the following (wrong-way) commutative diagrams :

$$\begin{array}{ccc} \mathbb{B}^*(X) & \xrightarrow{\gamma^*} & \mathbb{B}'^*(X) \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle f_!(\,-\,\bullet u_f)} \\ \mathbb{B}^*(Y) & \xrightarrow[\gamma^*]{} & \mathbb{B}'^*(Y). \end{array} \qquad \begin{array}{ccc} \mathbb{B}_*(Y) & \xrightarrow{\gamma_*} & \mathbb{B}'_*(Y) \\ {\scriptstyle f^!}\downarrow & & \downarrow{\scriptstyle u_f\bullet f^!} \\ \mathbb{B}_*(X) & \xrightarrow[\gamma_*]{} & \mathbb{B}'_*(X). \end{array}$$

The most important and motivating example of such a Grothendieck transformation is Baum–Fulton–MacPherson's bivariant *Riemann–Roch transformation* ([20, Part II]):

$$\tau : \mathbb{K}_{alg} \to \mathbb{H} \otimes \mathbb{Q}\,,$$

or its algebraic counterpart of [19, Example 18.3.19]. Here $\mathcal{V} = \mathcal{V}_k^{qp}$ is the category of quasi-projective varieties over a base field k of any characteristic, with $\mathbb{H} = CH$ the bivariant operational Chow groups, or $\mathbb{H}$ the even degree bivariant homology in case $k = \mathbb{C}$. The independent squares in this context are the *Tor-independent* fiber squares. $\mathbb{K}_{alg}$ is the bivariant algebraic K-theory of relative perfect complexes, so that ${\mathbb{K}_{alg}}_*(X) = G_0(X)$ is the Grothendieck group of coherent sheaves and ${\mathbb{K}_{alg}}^*(X) = K^0(X)$ is the Grothendieck group of algebraic vector bundles. The associated contravariant transformation is the *Chern character*

$$\tau^* = ch : K^0(X) \to H^*(X) \otimes \mathbb{Q},$$

and the associated covariant transformation is the *Todd class transformation*

$$\tau_* = td_* : G_0(X) \to H_*(X) \otimes \mathbb{Q},$$

which is functorial for proper morphisms $f : X \to Y$. Moreover, they are related by the *module property*

$$td_*(\beta \cap \alpha) = ch(\beta) \cap td_*(\alpha) \quad \text{for all} \quad \beta \in K^0(X), \alpha \in G_0(X).$$

This generalizes the original Grothendieck- and Hirzebruch–Riemann–Roch Theorem. Both bivariant theories $\mathbb{K}_{alg}$ and $H_*(-) \otimes \mathbb{Q}$ are canonically oriented for the class $\mathcal{S}$ of smooth (or more generally of local complete intersection) morphism, with $\theta_{\mathbb{K}}(f) = \mathcal{O}_f := [\mathcal{O}_X] \in \mathbb{K}_{alg}(X \xrightarrow{f} Y)$ the class of the structure sheaf, and $\theta_{\mathbb{H}}(f) = [f] \in \mathbb{H}(X \xrightarrow{f} Y)$ the corresponding "relative fundamental class". They are related by the *Riemann–Roch formula*

$$\tau(\mathcal{O}_f) = td(T_f) \bullet [f]\,,$$

where $u_f := td(T_f) \in H^*(X) \otimes \mathbb{Q}$ and T_f is the (virtual) tangent bundle of f. See [20, (*) on p.124] for $\mathbb{H}$ the bivariant homology in case $k = \mathbb{C}$. For $\mathbb{H} = CH$ the bivariant Chow group and k of any characteristic, the above Riemann–Roch formula follows from [19, Theorem 18.2] as explained in [38]. The Riemann–Roch formula implies the following two results:

SGA 6-Riemann–Roch Theorem: The following diagram commutes for a proper smooth morphism $f : X \to Y$:

$$\begin{array}{ccc} K(X) & \xrightarrow{ch} & H^*(X) \otimes \mathbb{Q} \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle f_!(td(T_f)\cup\,-\,)} \\ K(Y) & \xrightarrow[ch]{} & H^*(Y) \otimes \mathbb{Q}. \end{array}$$

Verdier–Riemann–Roch Theorem: The following diagram commutes for a smooth morphism $f : X \to Y$:

$$\begin{array}{ccc} G_0(Y) & \xrightarrow{td_*} & H_*(Y) \otimes \mathbb{Q} \\ {\scriptstyle f^!}\Big\downarrow & & \Big\downarrow{\scriptstyle td(T_f)\cap f^!} \\ G_0(X) & \xrightarrow[td_*]{} & H_*(X) \otimes \mathbb{Q}. \end{array}$$

Both formulae are more generally true for a local complete intersection morphism f, which is special to the Grothendieck transformation τ. In this paper only the case of a smooth morphism will be used, and then similar results are also true for the other considered Grothendieck transformations. It should also be remarked that *one motivation of Fulton–MacPherson's bivariant theory was to unify the above three Riemann–Roch theorems ...* (see [20, Part II, §0.1.4]).

Definition 2.4. (i) Let $\mathcal{S}$ be another class of maps in $\mathcal{V}$, called "specialized maps" (e.g., smooth maps in algebraic geometry), which is closed under composition and under base change and containing all identity maps. Let $\mathbb{B}$ be a bivariant theory. If $\mathcal{S}$ has canonical orientations in $\mathbb{B}$, then we say that $\mathcal{S}$ is *canonical* $\mathbb{B}$*-oriented* and an element of $\mathcal{S}$ is called a *canonical* $\mathbb{B}$*-oriented* morphism.

(ii) Assume furthermore, that the orientation θ on $\mathcal{S}$ satisfies $\theta(f') = g^*\theta(f)$ for any independent square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

with $f \in \mathcal{S}$ (which means that the orientation θ is preserved under the pullback operation). Then we call θ a *nice canonical orientation* and say that $\mathcal{S}$ is *nice canonical* $\mathbb{B}$*-oriented*. Similarly an element of $\mathcal{S}$ is called a *nice canonical* $\mathbb{B}$*-oriented* morphism.

Consider for example the class $\mathcal{S}$ of all *smooth* morphisms for $\mathcal{V} = \mathcal{V}_k^{(qp)}$ the category of (quasi-projective) varieties over a base field k of any characteristic, with all fiber squares as the independent squares. Then this class has a nice canonical orientation θ with respect to $\mathbb{K}_{alg}$ or CH in any characteristic (with $\theta(f) = \mathcal{O}_f$ or $[f]$), to $\tilde{\mathbb{F}}$ in characteristic zero (with $\theta(f) = \mathbb{1}_f$) and to $\mathbb{F}$ or bivariant homology $\mathbb{H}$ for $k = \mathbb{C}$ (with $\theta(f) = \mathbb{1}_f$ or $[f]$).

3. A UNIVERSAL BIVARIANT THEORY ON THE CATEGORY OF VARIETIES

Let $\mathcal{V}$ be the category $\mathcal{V} = \mathcal{V}_k^{(qp)}$ of (quasi-projective) varieties over a base field k of any characteristic, or the category $\mathcal{V} = \mathcal{V}_c^{an}$ of compact reduced complex analytic spaces, with all fiber squares as the independent squares. As the "confined" and "specialized" maps we take the class $\mathcal{P}rop$ of *proper* and $\mathcal{S}m$ of *smooth* morphisms, respectively.

Theorem 3.1 ([44], [38])**.** *We define*

$$\mathbb{M}(\mathcal{V}/X \xrightarrow{f} Y)$$

to be the free abelian group generated by the set of isomorphism classes of proper morphisms $h : W \to X$ *such that the composite of* h *and* f *is a smooth morphism:*

$$h \in \mathcal{P}rop \quad \textit{and} \quad f \circ h : W \to Y \in \mathcal{S}m.$$

Then the association $\mathbb{M}$ *is a* bivariant theory *if the three operations are defined as follows:*
Product operation: For morphisms $f : X \to Y$ *and* $g : Y \to Z$*, the product operation*

$$\bullet : \mathbb{M}(\mathcal{V}/X \xrightarrow{f} Y) \otimes \mathbb{M}(\mathcal{V}/Y \xrightarrow{g} Z) \to \mathbb{M}(\mathcal{V}/X \xrightarrow{gf} Z)$$

is defined for $[V \xrightarrow{p} X] \in \mathbb{M}(\mathcal{V}/X \xrightarrow{f} Y)$ *and* $[W \xrightarrow{k} Y] \in \mathbb{M}(\mathcal{V}/Y \xrightarrow{g} Z)$ *by*

$$[V \xrightarrow{p} X] \bullet [W \xrightarrow{k} Y] := [V' \xrightarrow{p \circ k''} X],$$

and bilinearly extended. Here we consider the following fiber squares

$$\begin{tikzcd} V' \arrow[r, "p'"] \arrow[d, "k''"'] & X' \arrow[r, "f'"] \arrow[d, "k'"'] & W \arrow[d, "k"'] & \\ V \arrow[r, "p"'] & X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z. \end{tikzcd}$$

Pushforward operation: For morphisms $f : X \to Y$ *and* $g : Y \to Z$ *with* $f \in \mathcal{P}rop$*, the pushforward operation* $f_* : \mathbb{M}(\mathcal{V}/X \xrightarrow{gf} Z) \to \mathbb{M}(\mathcal{V}/Y \xrightarrow{g} Z)$ *is defined by* $f_*([V \xrightarrow{p} X]) := [V \xrightarrow{f \circ p} Y]$ *and linearly extended.*

Pullback operation: For an independent square $\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$ *the pullback operation*

$g^* : \mathbb{M}(\mathcal{V}/X \xrightarrow{f} Y) \to \mathbb{M}(\mathcal{V}/X' \xrightarrow{f'} Y')$ *is defined by* $g^*([V \xrightarrow{p} X]) := [V' \xrightarrow{p'} X']$ *and linearly extended. Here we consider the following fiber squares:*

$$\begin{tikzcd} V' \arrow[r, "g''"] \arrow[d, "p'"'] & V \arrow[d, "p"] \\ X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y. \end{tikzcd}$$

Remark 3.2. (1) The above bivariant theory $\mathbb{M}(\mathcal{V}/-)$ shall be called a *pre-motivic bivariant Grothendieck group* on the category $\mathcal{V}$ of varieties. $\theta(f) := [X \xrightarrow{\mathrm{id}_X} X]$ for the smooth morphism $f : X \to Y$ defines a nice canonical orientation on $\mathbb{M}(\mathcal{V}/-)$.
(2) $\mathbb{M}_*(\mathcal{V}/X) = \mathbb{M}(\mathcal{V}/X \to pt)$ is the free abelian group generated by the isomorphism classes $[V \xrightarrow{h} X]$, where h is proper and V is smooth. $\mathbb{M}_*(\mathcal{V}/-)$ is a covariant functor for proper morphisms and $\mathbb{M}_*(\mathcal{V}/-)$ is a contravariant functor for smooth morphisms.
(3) $\mathbb{M}^*(\mathcal{V}/X) = \mathbb{M}(\mathcal{V}/X \xrightarrow{\mathrm{id}_X} X)$ is the free abelian group generated by the isomorphism classes $[V \xrightarrow{h} X]$, where h is *proper and smooth*. It gets a ring structure $\cup$ by fiber products, with unit $1_X = [X \xrightarrow{\mathrm{id}_X} X]$. Then $\mathbb{M}^*(\mathcal{V}/-)$ is a contravariant functor for any morphisms and $\mathbb{M}^*(\mathcal{V}/-)$ is a covariant functor for morphisms which are smooth and proper.
(4) The bivariant product induces the following "cap product": $\cap : \mathbb{M}^*(\mathcal{V}/X) \times \mathbb{M}_*(\mathcal{V}/X) \to \mathbb{M}_*(\mathcal{V}/X)$. In particular, when X itself is a *smooth* variety, with $[X] := [X \xrightarrow{\mathrm{id}_X} X] \in \mathbb{M}_*(\mathcal{V}/X)$, we have the "Poincaré duality" homomorphism $\cap[X] : \mathbb{M}^*(\mathcal{V}/X) \to \mathbb{M}_*(\mathcal{V}/X)$, which is nothing but

$$[W \xrightarrow{k} X] \cap [X] = [W \xrightarrow{k} X].$$

More generally, the isomorphism class $[V \xrightarrow{h} X] \in \mathbb{M}_*(\mathcal{V}/X)$ of any proper morphism $h : V \to X$ from a *smooth* variety V to X gives rise to the homomorphism

$$\cap[V \xrightarrow{h} X] : \mathbb{M}^*(\mathcal{V}/X) \to \mathbb{M}_*(\mathcal{V}/X)$$

defined by $[W \xrightarrow{k} X] \cap [V \xrightarrow{h} X] = [W \times_X V \to X]$.

The pre-motivic bivariant Grothendieck group $\mathbb{M}(\mathcal{V}/-)$ has the following universal property:

Theorem 3.3 ([44], [38]). *Let $\mathbb{B}$ be a bivariant theory on $\mathcal{V}$ such that a smooth morphism f has a* nice canonical orientation $\theta(f) \in \mathbb{B}(f)$, *and let $c\ell : Vect(-) \to \mathbb{B}^*(-)$ be a contravariantly functorial characteristic class of algebraic (or analytic) vector bundles with values in the associated cohomology theory, which is multiplicative in the sense that $c\ell(V) = c\ell(V')c\ell(V'')$ for any short exact sequence of vector bundles $0 \to V' \to V \to V'' \to 0$. Assume $c\ell$ commutes with the canonical orientation θ, i.e. $\theta(f) \bullet cl(V) = f^*cl(V) \bullet \theta(f)$ for all smooth morphism $f : X \to Y$ and $V \in Vect(Y)$ (e.g. $\mathbb{B}$ is commutative).*

Then there exists a unique Grothendieck transformation $\gamma_{c\ell} : \mathbb{M}(\mathcal{V}/-) \to \mathbb{B}(-)$ satisfying the normalization condition that $\gamma_{c\ell}([X \xrightarrow{\mathrm{id}_X} X]) = c\ell(T_f) \bullet \theta(f)$ for a smooth morphism $f : X \to Y$. Here T_f is the relative tangent bundle of the smooth morphism f.

Remark 3.4. The above Grothendieck transformation $\gamma_{c\ell} : \mathbb{M}(\mathcal{V}/-) \to \mathbb{B}(-)$ satisfies the normalization condition $\gamma_{c\ell}([X \xrightarrow{\mathrm{id}_X} X]) = c\ell(T_f) \bullet \theta(f)$, which is nothing but the *Riemann-Roch formula* with $u_f = c\ell(T_f)$ for a smooth morphism $f : X \to Y$. So by the general theory we get the following Riemann–Roch theorems:

<u>SGA 6 -type Riemann–Roch Theorem</u>: The following diagram commutes for $f : X \to Y$ proper and smooth:

$$\begin{tikzcd} \mathbb{M}^*(\mathcal{V}/X) \arrow[r, "{\gamma_{c\ell}}^*"] \arrow[d, "f_!"'] & \mathbb{B}^*(X) \arrow[d, "f_!(c\ell(T_f)\cup -)"] \\ \mathbb{M}^*(\mathcal{V}/Y) \arrow[r, "{\gamma_{c\ell}}^*"'] & \mathbb{B}^*(Y). \end{tikzcd}$$

<u>Verdier-type Riemann–Roch Theorem</u>: The following diagram commutes for a smooth morphism $f : X \to Y$:

$$\begin{tikzcd} \mathbb{M}_*(\mathcal{V}/X) \arrow[r, "\gamma_{c\ell_*}"] \arrow[d, "f^!"'] & \mathbb{B}_*(X) \arrow[d, "c\ell(T_f)\cap f^!"] \\ \mathbb{M}_*(\mathcal{V}/Y) \arrow[r, "\gamma_{c\ell_*}"'] & \mathbb{B}_*(Y). \end{tikzcd}$$

Remark 3.5. (1) $\gamma_{c\ell} : \mathbb{M}(\mathcal{V}/X \xrightarrow{f} Y) \to \mathbb{B}(X \xrightarrow{f} Y)$ can be called a *bivariant pre-motivic characteristic class transformation*. When Y is a point pt,

$$\gamma_{c\ell *} : \mathbb{M}(\mathcal{V}/X \to pt) \to \mathbb{B}(X \to pt) = \mathbb{B}_*(X)$$

is the *unique natural transformation* satisfying that $\gamma_{c\ell *}([X \xrightarrow{id_X} X]) = c\ell(TX) \cap [X]$ for a smooth variety X. In other words, this gives rise to a *pre-motivic characteristic class transformation for singular varieties*. In a sense, this could be also a very general answer to MacPherson's question about the existence of a unified theory of characteristic classes for singular varieties. We emphasize that for the corresponding universal property of $\mathbb{M}(\mathcal{V}/X)$, we do not have to require the characteristic class $c\ell$ to be

multiplicative or to commute with the canonical orientation θ (since these properties are not used in the proof of Theorem 3.3 (iii), that $\gamma_{c\ell *}$ preserves the pushforward operation).
(2) In particular, we have the following commutative diagrams:

$$\begin{tikzcd} & \mathbb{M}_*(\mathcal{V}/X) \arrow[dl, "\epsilon"'] \arrow[dr, "\gamma_{c*}"] & \\ F(X) \arrow[rr, "c_*"'] & & H_*(X), \end{tikzcd}$$

with $H_*(X) = CH_*(X)$ in the algebraic context over a base field of characteristic zero, or $H_*(X) = H_{2*}^{BM}(X)$ in the complex algebraic or compact complex analytic context. Here $\epsilon([V \xrightarrow{h} X]) := h_* \mathbb{1}_V$.

$$\begin{tikzcd} & \mathbb{M}_*(\mathcal{V}/X) \arrow[dl, "mC_0"'] \arrow[dr, "\gamma_{td*}"] & \\ G_0(X) \arrow[rr, "td_*"'] & & H_*(X) \otimes \mathbb{Q}, \end{tikzcd}$$

with $H_*(X) = CH_*(X)$ in the algebraic context over a base field of any characteristic, or $H_*(X) = H_{2*}^{BM}(X)$ in the complex algebraic or compact complex analytic context. Here $mC_0([V \xrightarrow{h} X]) := [h_* \mathcal{O}_V] = h_*[\mathcal{O}_V]$.

$$\begin{tikzcd} & \mathbb{M}_*(\mathcal{V}/X) \arrow[dl, "sd"'] \arrow[dr, "\gamma_{L*}"] & \\ \Omega(X) \arrow[rr, "L_*"'] & & H_{2*}^{BM}(X) \otimes \mathbb{Q}. \end{tikzcd}$$

Here X has to be a compact complex algebraic or analytic variety, with

$$sd([V \xrightarrow{h} X]) := [h_* \mathbb{Q}_V[\dim V]] = h_*[\mathbb{Q}_V[\dim V]] \,.$$

(3) It follows from Hironaka's resolution of singularities ([23]) that there exists a surjection

$$\mathbb{M}_*(\mathcal{V}/X) \to K_0(\mathcal{V}/X)$$

in the algebraic context over a base field of characteristic zero, or in the compact complex analytic context. It turns out that if the natural transformation $\gamma_{c\ell *} : \mathbb{M}_*(\mathcal{V}/X) \to H_*(X) \otimes R$ (with R a $\mathbb{Q}$-algebra) can be pushed down to the relative Grothendieck group $K_0(\mathcal{V}/X)$, then it has to be a specialization of the Hirzebruch class transformation under a ring homomophism $\mathbb{Q}[y] \to R$, i.e., the following diagram commutes (see [7]):

$$\begin{tikzcd} \mathbb{M}_*(\mathcal{V}/X) \arrow[r, "\gamma_{c\ell *}"] \arrow[d, "q"'] & H_*(X) \otimes R \\ K_0(\mathcal{V}/X) \arrow[r, "T_{y*}"'] & H_*(X) \otimes \mathbb{Q}[y] \,. \arrow[u] \end{tikzcd}$$

And one of the main results of our previous paper [7] claims that in this context the above three diagrams also commute with $\mathbb{M}_*(\mathcal{V}/X)$ being replaced by the smaller group $K_0(\mathcal{V}/X)$ (fitting with T_{y*} for $y = -1, 0$ or 1).

Now it is natural to pose the following

Problem 3.6. *Formulate a reasonable bivariant analogue $\mathbb{K}_0(\mathcal{V}/X \xrightarrow{f} Y)$ of the relative Grothendieck group $K_0(\mathcal{V}/X)$ so that the following hold:*
(1) There is a natural group homomorphism $q : \mathbb{K}_0(\mathcal{V}/X \to pt) \to K_0(\mathcal{V}/X)$, which is an isomorphism in the algebraic context over a base field of characteristic zero, or in the compact complex analytic context.
(2) $\mathbb{B}q : \mathbb{M}(\mathcal{V}/X \xrightarrow{f} Y) \to \mathbb{K}_0(\mathcal{V}/X \xrightarrow{f} Y)$ is a certain quotient map, which specializes for Y a point to the quotient map $q : \mathbb{M}_*(\mathcal{V}/X) \to K_0(\mathcal{V}/X)$.
(3) $T_y : \mathbb{K}_0(\mathcal{V}/X \xrightarrow{f} Y) \to \mathbb{H}(X \xrightarrow{f} Y) \otimes \mathbb{Q}[y]$ is a Grothendieck transformation, which specializes for Y a point (in the algebraic context over a base field of characteristic zero, or in the compact complex analytic context) to the motivic Hirzebruch class transformation $T_{y_*} : K_0(\mathcal{V}/X) \to H_*(X) \otimes \mathbb{Q}[y]$.
(4) The following diagram commutes:

$$\begin{tikzcd} & \mathbb{M}(\mathcal{V}/X \xrightarrow{f} Y) \arrow[dl, "\mathbb{B}q"'] \arrow[dr, "\gamma_{T^*_{y*}}"] & \\ \mathbb{K}_0(\mathcal{V}/X \xrightarrow{f} Y) \arrow[rr, "T_y"'] & & \mathbb{H}(X \xrightarrow{f} Y) \otimes \mathbb{Q}[y]. \end{tikzcd}$$

Remark 3.7. The associated contravariant functor of such a bivariant theory $\mathbb{K}_0(\mathcal{V}/X \xrightarrow{f} Y)$, namely $K^0(\mathcal{V}/X) := \mathbb{K}_0(\mathcal{V}/X \xrightarrow{\mathrm{id}_X} X)$, can be considered as a contravariant counterpart of the relative Grothen-dieck group $K_0(\mathcal{V}/X)$. The natural transformation $T^*_y : K^0(\mathcal{V}/-) \to H^*(-) \otimes \mathbb{Q}[y]$ can be considered as a contravariant counterpart of the Hirzebruch class transformations T_{y*} satisfying the *module property*.

4. A BIVARIANT GROTHENDIECK GROUP $\mathbb{K}_0(\mathcal{V}/X \xrightarrow{f} Y)$

The following theorem is proved using the "Weak Factorisation Theorem" of [1, 40]:

Theorem 4.1 (Franziska Bittner [5]). *Let $K_0(\mathcal{V}/X)$ be the relative Grothendieck group of varieties over $X \in obj(\mathcal{V})$, with $\mathcal{V} = \mathcal{V}_k^{(qp)}$ (resp., $\mathcal{V} = \mathcal{V}_c^{an}$) the category of (quasi-projective) algebraic (resp., compact complex analytic) varieties over a base field k of characteristic zero. Then $K_0(\mathcal{V}/X)$ is isomorphic to $\mathbb{M}_*(X)$ modulo the "blow-up" relation*

$$[\emptyset \to X] = 0 \quad \text{and} \quad [Bl_Y X' \to X] - [E \to X] = [X' \to X] - [Y \to X]\,,$$

for any cartesian diagram (which shall be called the "blow-up diagram" from here on)

$$\begin{tikzcd} E \arrow[r, "i'"] \arrow[d, "q'"] & Bl_Y X' \arrow[d, "q"] & \\ Y \arrow[r, "i"] & X' \arrow[r, "f"] & X\,, \end{tikzcd}$$

with i a closed embedding of smooth spaces and $f : X' \to X$ proper. Here $Bl_Y X' \to X'$ is the blow-up of X' along Y with exceptional divisor E. Note that all these spaces other than X are also smooth (and quasi-projective in case $X', Y \in ob(\mathcal{V}_k^{qp})$).

The kernel of the canonical quotient map $q : \mathbb{M}_*(\mathcal{V}/X) \to K_0(\mathcal{V}/X)$ is the subgroup $BL(\mathcal{V}/X)$ of $\mathbb{M}_*(\mathcal{V}/X)$ generated by $[Bl_Y X' \to X] - [E \to X] - [X' \to X] + [Y \to X]$ for any blow-up diagram as above.

To obtain a bivariant analogue of the subgroup $BL(\mathcal{V}/X)$, we first observe the following result:

Lemma 4.2. *Let $h : X' \to X$ be a* smooth *morphism, with $i : S \to X'$ a closed embedding such that the composite $h \circ i : Z \to X$ is also* smooth *morphism. Consider the cartesian diagram*

$$\begin{array}{ccccc} E & \xrightarrow{i'} & Bl_S X' & & \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} & & \\ S & \xrightarrow[i]{} & X' & \xrightarrow[h]{} & X\,, \end{array}$$

with $q : Bl_S X' \to X'$ the blow-up of X' along S and $q' : E \to S$ the exceptional divisor map. Then:

(1) $h \circ q : Bl_S X' \to X$ and $h \circ q \circ i' : E \to X$ are also smooth *morphisms, with $Bl_S X', E$ quasi-projective in case $X', Y \in ob(\mathcal{V}_k^{qp})$.*
(2) This blow-up diagram commutes with any base change *in X, i.e. the corresponding fiber-square induced by pullback along a morphism $\tilde{X} \to X$ is isomorphic to the corresponding blow-up diagram of $\tilde{S} \to \tilde{X}'$.*
(3) The closed embeddings i, i' are regular *embeddings, and the projection map q as well as i, i' are of* finite Tor-dimension.

Definition 4.3. For a morphism $f : X \to Y$ in the category $\mathcal{V} = \mathcal{V}_k^{(qp)}$ or $\mathcal{V} = \mathcal{V}_c^{an}$, we consider a blow-up diagram

$$\begin{array}{ccccccc} E & \xrightarrow{i'} & Bl_S X' & & & & \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} & & & & \\ S & \xrightarrow{i} & X' & \xrightarrow{h} & X & \xrightarrow{f} & Y\,, \end{array}$$

with h proper and i a closed embedding such that $f \circ h$ *as well as* $f \circ h \circ i$ *are smooth.*

Let $\mathbb{BL}(\mathcal{V}/X \xrightarrow{f} Y)$ be the abelian subgroup of $\mathbb{M}(\mathcal{V}/X \xrightarrow{f} Y)$ generated by

$$[Bl_S X' \xrightarrow{hq} X] - [E \xrightarrow{hiq'} X] - [X' \xrightarrow{h} X] + [S \xrightarrow{hi} X]$$

for any such diagram, and we define

$$\mathbb{K}_0(\mathcal{V}/X \xrightarrow{f} Y) := \frac{\mathbb{M}(\mathcal{V}/X \xrightarrow{f} Y)}{\mathbb{BL}(\mathcal{V}/X \xrightarrow{f} Y)}\,.$$

The corresponding equivalence class of $[V \xrightarrow{p} X]$ shall be denoted by $\Big[[V \xrightarrow{p} X]\Big]$.

Theorem 4.4 ([38])**.** *Let $\mathcal{V} = \mathcal{V}_k^{(qp)}$ be the category of (quasi-projective) algebraic varieties (i.e., reduced separated schemes of finite type) over a base field k of any characteristic, or let $\mathcal{V} = \mathcal{V}_c^{an}$ be the category of compact reduced complex analytic spaces. Then $\mathbb{K}_0(\mathcal{V}/X \xrightarrow{f} Y)$ becomes a bivariant theory with the following three operations, so that the canonical projection $\mathbb{B}q : \mathbb{M}(\mathcal{V}/-) \to \mathbb{K}_0(\mathcal{V}/-)$ is a Grothendieck transformation.*
Product operation: For morphisms $f : X \to Y$ and $g : Y \to Z$ the product operation

$$\star : \mathbb{K}_0(\mathcal{V}/X \xrightarrow{f} Y) \otimes \mathbb{K}_0(\mathcal{V}/Y \xrightarrow{g} Z) \to \mathbb{K}_0(\mathcal{V}/X \xrightarrow{gf} Z)$$

is defined by $\Big[[V \xrightarrow{h} X]\Big] \star \Big[[W \xrightarrow{k} Y]\Big] := \Big[[V \xrightarrow{h} X] \bullet [W \xrightarrow{k} Y]\Big]$ and bilinearly extended.
Pushforward operation: For morphisms $f : X \to Y$ and $g : Y \to Z$ with $f \in \mathcal{P}rop$ the pushforward operation

$$f_* : \mathbb{K}_0(\mathcal{V}/X \xrightarrow{gf} Z) \to \mathbb{K}_0(\mathcal{V}/Y \xrightarrow{g} Z)$$

is defined by $f_*\left(\left[[V \xrightarrow{p} X]\right]\right) := \left[f_*([V \xrightarrow{p} X])\right]$ *and linearly extended.*

<u>Pullback operation</u>: *For an independent square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

the pullback operation

$$g^* : \mathbb{K}_0(\mathcal{V}/X \xrightarrow{f} Y) \to \mathbb{K}_0(\mathcal{V}/X' \xrightarrow{f'} Y')$$

is defined by $g^*\left(\left[[V \xrightarrow{p} X]\right]\right) := \left[g^*([V \xrightarrow{p} X])\right]$ *and linearly extended.*

For the proof of this theorem one only has to show that the three bivariant operations are well defined, i.e. that the subgroup $\mathbb{BL}(\mathcal{V}/X \xrightarrow{f} Y)$ is stable under the bivariant operations. For the pushforward this is clear, but for pullback and product this uses Lemma 4.2(2), as well as the fact that blowing up commutes with smooth (or more generally flat) pullback.

Remark 4.5. In the case when Y is a point, the blow-up diagram defining $\mathbb{BL}(\mathcal{V}/X \xrightarrow{f} pt)$ is nothing but the following:

$$\begin{array}{ccccc} E & \xrightarrow{i'} & Bl_S X' & & \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} & & \\ S & \xrightarrow{i} & X' & \xrightarrow{h} & X\,, \end{array}$$

such that $h : X' \to X$ is proper, X' and S are nonsingular, and $q : Bl_S X' \to X'$ is the blow-up of X' along S with $q' : E \to S$ the exceptional divisor map. Hence $\mathbb{BL}(\mathcal{V}/X \xrightarrow{f} pt)$ is nothing but $BL(\mathcal{V}/X)$, i.e., we have by Bittner's theorem $\mathbb{K}_0(\mathcal{V}/X \to pt) \simeq K_0(\mathcal{V}/X)$ in the compact complex analytic context, as well as in the algebraic context over a base field of characteristic zero. Finally note that we always have a group homomorphism $\mathbb{K}_0(\mathcal{V}/X \to pt) \to K_0(\mathcal{V}/X)$, since $Bl_S X' \backslash E \simeq X' \backslash S$ in the diagram above so that

$$[Bl_S X' \to X] - [E \to X] = [X' \to X] - [S \to X] \in K_0(\mathcal{V}/X)\,.$$

5. Motivic bivariant Chern and Hirzebruch class transformations

Now we are ready to state the following main theorem, which is about the *motivic bivariant Chern and Hirzebruch class transformations*.

Theorem 5.1 ([38]). *Let* $\mathcal{V} = \mathcal{V}_k^{qp}$ *be the category of quasi-projective algebraic varieties over a base field* k *of any characteristic.*
(1) There exists a unique Grothendieck transformation

$$mC_y = \Lambda_y^{mot} : \mathbb{K}_0(\mathcal{V}_k^{qp}/-) \to \mathbb{K}_{alg}(-) \otimes \mathbb{Z}[y]$$

satisfying the normalization condition that

$$\Lambda_y^{mot}\left(\left[[X \xrightarrow{\mathrm{id}_X} X]\right]\right) = \Lambda_y(T_f^*) \bullet \mathcal{O}_f$$

for a smooth morphism $f : X \to Y$.
(2) Let $T_y : \mathbb{K}_0(\mathcal{V}_k^{qp}/-) \to \mathbb{H}(-) \otimes \mathbb{Q}[y]$ *be defined as the composition* $\tau \circ \Lambda_y^{mot}$*, renormalized by* $\times(1+y)^i$ *on* $\mathbb{H}^i(-) \otimes \mathbb{Q}[y]$. *Here* $\mathbb{H}$ *is either the operational bivariant Chow group or the even degree*

bivariant homology theory for $k = \mathbb{C}$, with τ the corresponding Riemann-Roch transformation. Then T_y is the unique Grothendieck transformation satisfying the normalization condition that

$$T_y\Big(\Big[[X \xrightarrow{\mathrm{id}_X} X]\Big]\Big) = td_y(T_f) \bullet [f]$$

for a smooth morphism $f : X \to Y$.

Remark 5.2. (1) Let $c\ell : Vect(-) \to K^0(-) \otimes \mathbb{Z}[y] = \mathbb{K}^*_{alg}(-) \otimes \mathbb{Z}[y]$ be the characteristic class transformation $c\ell(V) := \lambda_y(V^*)$ given by the total λ-class of the dual vector bundle V^*. Then by Theorem 3.3 there is a unique Grothendieck transformation

$$\gamma_{c\ell} : \mathbb{M}(\mathcal{V}/-) \to \mathbb{K}_{alg}(-) \otimes \mathbb{Z}[y]$$

satisfying the normalization condition in (1) above. So one only has to show that this transformation $\gamma_{c\ell}$ vanishes on all subgroups $\mathbb{BL}(\mathcal{V}/X \xrightarrow{f} Y)$ generated by the "bivariant blow-up relations". The proof of this given in [38] is based on [22, Chapter IV, Theorem 1.2.1 and (1.2.6)].
(2) For the transformation

$$T_y := (1+y)^* \times \tau \circ \Lambda_y^{mot} : \mathbb{K}_0(\mathcal{V}_k^{qp}/-) \to \mathbb{H}(-) \otimes \mathbb{Q}[y, (1+y)^{-1}]$$

one only has to check the normalization condition of (2) above, which (as explained in [38]) follows from the Riemann–Roch formula

$$\tau(\mathcal{O}_f) = td(T_f) \bullet [f] ,$$

for the bivariant Riemann–Roch transformation τ (with f smooth).

Corollary 5.3 ([38])**.** *Let $\mathcal{V} = \mathcal{V}_k^{qp}$ be the category of quasi-projective algebraic varieties over a base field k of any characteristic. Then we have the following commutative diagrams of Grothendieck transformations:*
(1)

$$\begin{tikzcd} & \mathbb{K}_0(\mathcal{V}_k^{qp}/-) \arrow[dl, "mC_0"'] \arrow[dr, "T_0"] & \\ \mathbb{K}_{alg}(-) \arrow[rr, "\tau"'] & & \mathbb{H}(-) \otimes \mathbb{Q}. \end{tikzcd}$$

(2)

$$\begin{tikzcd} & \mathbb{K}_0(\mathcal{V}_k^{qp}/-) \arrow[dl, "\epsilon"'] \arrow[dr, "T_{-1}"] & \\ \tilde{\mathbb{F}}(-) \arrow[rr, "\gamma"'] & & CH(-) \otimes \mathbb{Q}, \end{tikzcd}$$

if k is of characteristic zero. Here ϵ is the unique Grothendieck transformation satisfying the normalization condition $\epsilon\Big(\Big[[X \xrightarrow{\mathrm{id}_X} X]\Big]\Big) = 1\!\!1_f$ for a smooth morphism $f : X \to Y$. Similarly for the bivariant Chern class transformation $\gamma : \mathbb{F}(-) \to A^{PI}(-) \otimes \mathbb{Q} \supset CH(-) \otimes \mathbb{Q}$ in case $k = \mathbb{C}$.
(3) Assume k is of characteristic zero. Then the associated covariant transformations in Theorem 5.1 (1) and (2) agree under the identification $\mathbb{K}_0(\mathcal{V}_k^{qp}/X \to pt) \simeq K_0(\mathcal{V}_k^{qp}/X)$ with the motivic Chern and Hirzebruch class transformations mC_y and T_{y*}.

Remark 5.4. We would speculate that Brasselet's bivariant Chern class transformation $\gamma : \mathbb{F}(-) \to \mathbb{H}(-)$ to Fulton–MacPherson's bivariant homology $\mathbb{H}(-)$ (see [6]) satisfies for a smooth morphism $f : X \to Y$ the "strong normalization condition" $\gamma(1\!\!1_f) = c(T_f) \bullet [f] \in \mathbb{H}(X \xrightarrow{f} Y)$ with $[f]$ the corresponding relative fundamental class. If this is the case, then Corollary 5.3 (2) would also be true for Brasselet's bivariant Chern class transformation $\gamma : \mathbb{F}(-) \to \mathbb{H}(-)$.

6. Oriented bivariant theories

In [28] Levine and Morel defined the algebraic cobordism (group) $\Omega^*_{LM}(X)$ for a *smooth* variety X, which is a contravariant functor. In fact they first constructed algebraic "bordism" theory $\Omega^{LM}_*(X)$ for *any* variety X as a covariant functor (for projective morphisms), with functorial Gysin maps for local complete intersection morphisms. Then the algebraic cobordism of a smooth (pure-dimensional) variety X is $\Omega^n_{LM}(X) := \Omega^{LM}_{\dim X - n}(X)$.

Our naïve question was

Question 6.1. What is a "real" cobordism theory on varieties or a contravariant version of Levine–Morel's algebraic "bordism" theory $\Omega^{LM}_*(X)$?

Their definition of $\Omega^{LM}_*(X)$ and their main theorem could be put as follows, omitting the things which we do not need in this paper.

Definition-Theorem 6.2. *We consider the category $\mathcal{V} = \mathcal{V}_k$ algebraic varieties over a base field k. For a variety X the "bordism group" $\Omega^{LM}_*(X)$ of algebraic cobordism cycles over X is defined by*

$$\Omega^{LM}_*(X) := \frac{\left\{[V \xrightarrow{h} X; L_1, L_2, \cdots, L_m] \,\Big|\, V \textit{ is smooth}, h \textit{ is projective}, \ L_i \textit{ are line bundles over } V\right\}^+}{\textit{three relations} \begin{cases} \textit{(Dim) Dimension Axiom,} \\ \textit{(Sect) Section Axiom,} \\ \textit{(FGL) Formal Group Law Axiom.}\end{cases}}.$$

Then for k of characteristic zero [1], *$\Omega^{LM}_*(X)$ is the universal oriented (graded) Borel–Moore functor with products of geometric type.*

The line bundles L_i in this definition are related to an "orientation" of this theory via the "first Chern class operator"

(1) $\widetilde{c_1}(L) : \Omega^{LM}_*(X) \to \Omega^{LM}_{*-1}(X); \ [V \xrightarrow{h} X; L_1, L_2, \cdots, L_m] \mapsto [V \xrightarrow{h} X; L_1, L_2, \cdots, L_m, h^*L]$

of a line bundle L over X, with $[-]$ denoting an isomorphism class. Here an isomorphism of algebraic cobordism cycles over X of (formal) dimension $dim(V) - m$

$$(V \xrightarrow{h} X; L_1, L_2, \cdots, L_m) \simeq (V' \xrightarrow{h'} X; L'_1, L'_2, \cdots, L'_m)$$

is given by an isomorphism $g : V \simeq V'$ with $h = h' \circ g$, such that $L_i \simeq g^* L'_{\sigma(i)}$ for all i and a permutation σ of $\{1, \ldots, m\}$. Then the set of such isomorphism classes

$$\left\{[V \xrightarrow{h} X; L_1, L_2, \cdots, L_m] \,\Big|\, V \text{ is smooth}, h \text{ is projective}, \ L_i \text{ are line bundles over } V\right\}$$

becomes a monoid with respect to disjoint union $\sqcup$, with unit the empty cobordism cycle. Then $\{\cdots\}^+$ denotes the corresponding group completion (of formal differences), which here is graded by the (formal) dimension of an algebraic cobordism cycle.

The three imposed relations, (D) Dimension Axiom, (S) Section Axiom, and (FGL) Formal Group Law Axiom, are related to the notion "of geometric type", which also involves the first Chern class operator, as shown in Definition 6.5 below. Hence, dropping all things related to first Chern class operators of line bundles, the group completion of the corresponding monoid (with respect to disjoint union $\sqcup$)

$$M_*(X) := \left\{[V \xrightarrow{h} X] \,\big|\, V \text{ is smooth}, h \text{ is projective}\right\}^+$$

[1] They use resolution of singularities, thus characteristic zero is necessary since the case of resolution of singularities in a positive characteristic is still unresolved.

is the universal (graded) "Borel–Moore functor with products" in the following sense:

Definition 6.3. A covariant functor H_* to the category of (graded) abelian groups is called a *(graded) Borel–Moore functor with products*, if it satisfies the following conditions:

(BM-1) it is covariantly functorial (preserving degrees) for pushforward of projective morphisms (called "projective pushforward").

(BM-2) it is additive, i.e. $H_*(X) \oplus H_*(Y) \simeq H_*(X \sqcup Y)$ via (BM-1) for the closed inclusions $X \to X \sqcup Y$ and $Y \to X \sqcup Y$.

(BM-3) it is contravariantly functorial (shifting degrees by the fiber dimension) for pullback of smooth morphisms (called "smooth pullback").

(BM-4) for any fiber square
$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$
with f projective and g smooth (hence f' projective and g' smooth), $g^* f_* = f'_*(g')^*$.

(BM-5) there exists a cross or external product $\times : H_*(X) \times H_*(Y) \to H_*(X \times Y)$ (adding degrees), which is commutative and associative together with a unit $1_{pt} \in H_0(pt)$, such that it commutes with projective pushforwards and smooth pullbacks.

Note that a smooth variety M has a "fundamental class" (with $\pi_M : M \to pt$ the constant smooth morphism):
$$[M] := 1_M := \pi_M^* 1_{pt} \in H_*(M) .$$

For example, the relative Grothendieck group of algebraic varieties $K_0(\mathcal{V}/-)$ is such a Borel–Moore functor with products. Similarly, $M_*(-)$ is covariantly functorial for projective morphisms by composition of arrows, as well as contravariantly functorial for smooth morphisms by taking fiber products, with the cross product given in the obvious way. It is graded by the dimension of V. Finally, also the Grothendieck group $G_0(-)$ of coherent sheaves, as well as for a base field k of characteristic zero the group of constructible functions $F(-)$, are Borel–Moore functors with products. Similarly for the cobordism group of selfdual constructible sheaf complexes $\Omega(X)$ in the complex algebraic or analytic context. Here we can even consider proper morphisms instead of projective morphisms in the definition above. Of course, $K_0(\mathcal{V}/-), G_0(-)$ and $F(-)$ are ungraded, i.e. $H_*(-) = H_0(-)$, whereas $\Omega(X)$ is $\mathbb{Z}_2$-graded.

Definition 6.4. Let H_* be a (graded) Borel–Moore functor with products. It is called *oriented* if for any line bundle L on X there exists a homomorphism (called "the first Chern class operator")
$$\widetilde{c_1}(L) : H_*(X) \to H_{*-1}(X)$$
such that

(OBM-6) for line bundles L, L' over X the two first Chern class operators commute; i.e.,
$$\widetilde{c_1}(L) \circ \widetilde{c_1}(L') = \widetilde{c_1}(L') \circ \widetilde{c_1}(L) .$$
Moreover $\widetilde{c_1}(L) = \widetilde{c_1}(L')$ for isomorphic line bundles L and L' .

(OBM-7) it is compatible with the projective pushforward, i.e., for a projective map $f : X \to Y$ with L a line bundle over Y the following diagram commutes (i.e., $f_* \circ \widetilde{c_1}(f^*L) = \widetilde{c_1}(L) \circ f_*$) :
$$\begin{array}{ccc} H_*(X) & \xrightarrow{f_*} & H_*(Y) \\ {\scriptstyle \widetilde{c_1}(f^*L)}\downarrow & & \downarrow{\scriptstyle \widetilde{c_1}(L)} \\ H_{*-1}(X) & \xrightarrow[f_*]{} & H_{*-1}(Y) . \end{array}$$

(OBM-8) it is compatible with the smooth pullback, i.e., for a smooth map $f : X \to Y$ with L a line bundle over Y the following diagram commutes (i.e., $f^* \circ \widetilde{c_1}(L) = \widetilde{c_1}(f^*L) \circ f^*$) :

$$\begin{tikzcd} H_*(X) \arrow[d, "\widetilde{c_1}(f^*L)"'] & H_*(Y) \arrow[l, "f^*"'] \arrow[d, "\widetilde{c_1}(L)"] \\ H_{*-1}(X) & H_{*-1}(Y) \arrow[l, "f^*"] \end{tikzcd} .$$

(OBM-9) the first Chern class operator commutes with the cross product, i.e., for a line bundle L over X, $\pi_1 : X \times Y \to X$ and $\alpha \in H_*(X)$ and $\beta \in H_*(Y)$ we have

$$\widetilde{c_1}(L)(\alpha) \times \beta = \widetilde{c_1}(\pi_1^* L)(\alpha \times \beta) .$$

For example, the universal *oriented* (graded) Borel-Moore functor with products is given by the group completion of the monoid of cobordism cycles (as before):

$$Z_*(X) := \left\{ [V \xrightarrow{h} X; L_1, L_2, \cdots, L_m] \middle| V \text{ is smooth}, h \text{ is projective}, L_i \text{ are line bundles over } V \right\}^+ .$$

Here we are using the obvious notion of a *natural transformation* of (oriented) Borel-Moore functors with product to make sense of the corresponding universal property. A simple example of such a transformation in the complex algebraic context is the "cycle class map" $[-] : CH_* \to H_{2*}^{BM}(-)$.

Definition 6.5. Let H_* be an oriented (graded) Borel–Moore functor with products. It is called *of geometric type* if the following three conditions holds:
(GT-10) (Dimension Axiom): For line bundles $L_1, L_2, \cdots, L_r$ on a smooth scheme M with $r > \dim M$,

$$\widetilde{c_1}(L_1) \circ \widetilde{c_1}(L_2) \circ \cdots \circ \widetilde{c_1}(L_r)([M]) = 0 \in H_*(M) .$$

(GT-11) (Section Axiom): Let L be a line bundle over a smooth scheme M with $s : M \to L$ a section transverse to the zero section of L. Let $Z := s^{-1}(0)$ and $i_Z : Z \to M$ be the inclusion of this submanifold. Then we have

$$\widetilde{c_1}(L)([M]) = (i_Z)_*([Z]) \in H_*(M) .$$

(GT-12) (Formal Group Law Axiom): Let L, L' be line bundles over a smooth scheme M. There is a formal group law $F_{H_*}(u, v) \in H_*(pt)[[u, v]]$ such that

$$F_{H_*}(\widetilde{c_1}(L), \widetilde{c_1}(L')) = \widetilde{c_1}(L \otimes L'),$$

and $F_{H_*}(u, v)$ is the image of the universal formal group law $F_{\mathbb{L}}(u, v) \in \mathbb{L}_*[[u, v]]$ under the homomorphism $\phi_{H_*} : \mathbb{L}_* \to H_*(pt)$ classifying the formal group law F_{H_*} on $H_*(pt)$. Here $\mathbb{L}_*$ is the Lazard ring.

For example the Chow group $CH_*(-)$, or in the complex algebraic context also the even degree Borel-Moore homology $H_{2*}^{BM}(-)$, have a canonical orientation with the *additive* formal group law F_a, whereas the Grothendieck group of coherent sheaves $G_0(-)$ has a canonical orientation with the *multiplicative* formal group law F_m:

$$F_a(u, v) := u + v \quad \text{and} \quad F_m(u, v) := u + v - u \cdot v .$$

Finally, a deep result of Levine–Morel [28] tells us that the formal group law of their algebraic cobordism $\Omega_*^{LM}(-)$ over a base field k of characteristic zero is given by the *universal* formal group of the Lazard ring $\mathbb{L}_*$.

In [44] we introduced (in greater generality) the following notion of an *oriented bivariant theory*.

Definition 6.6. Let $\mathbb{B}$ be a (graded) bivariant theory on the category $\mathcal{V} = \mathcal{V}_k$ of algebraic varieties over a base field k, with all fiber squares as the independent squares, and the projective (or more generally proper) morphisms as the confined maps.

Then $\mathbb{B}$ is called *oriented* if for any morphism $g : X \to Y$ and a line bundle L on X there exists a homomorphism (of degree one in the graded context, called "the first Chern class operator")

$$\widetilde{c_1}(L) : \mathbb{B}(X \xrightarrow{g} Y) \to \mathbb{B}(X \xrightarrow{g} Y)$$

such that

(OB-1) for line bundles L, L' over X the two first Chern class operators commute, i.e.,

$$\widetilde{c_1}(L) \circ \widetilde{c_1}(L') = \widetilde{c_1}(L') \circ \widetilde{c_1}(L) : \mathbb{B}(X \xrightarrow{g} Y) \to \mathbb{B}(X \xrightarrow{g} Y)\,.$$

Moreover $\widetilde{c_1}(L) = \widetilde{c_1}(L')$ for L, L' isomorphic line bundles.

(OB-2) it is compatible with the pushforward, i.e., for a confined map $f : X \to Y$ with L a line bundle over Y one has for all morphisms $g : Y \to Z$:

$$f_* \circ \widetilde{c_1}(f^*L) = \widetilde{c_1}(L) \circ f_* : \mathbb{B}(X \xrightarrow{g\circ f} Z) \to \mathbb{B}(Y \xrightarrow{g} Z)\,.$$

(OB-3) it is compatible with pullback, i.e., for any independent square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

and for L a line bundle over X one has:

$$g^* \circ \widetilde{c_1}(L) = \widetilde{c_1}(g'^*L) \circ g^* : \mathbb{B}(X \xrightarrow{f} Y) \to \mathbb{B}(X' \xrightarrow{f'} Y')\,.$$

(OB-4) the first Chern class operator commutes with the bivariant product, i.e., for all morphisms $f : X \to Y$, resp., $g : Y \to Z$, and a line bundle L over X, resp., L' over Y, one has:

$$\widetilde{c_1}(L)(\alpha \bullet \beta) = \widetilde{c_1}(L)(\alpha) \bullet \beta, \quad \text{resp.,} \quad \widetilde{c_1}(f^*L')(\alpha \bullet \beta) = \alpha \bullet \widetilde{c_1}(L')(\beta)$$

for all $\alpha \in \mathbb{B}(X \xrightarrow{f} Y)$ and $\beta \in \mathbb{B}(Y \xrightarrow{g} Z)$.

Assume, for example, that the oriented (graded) bivariant theory $\mathbb{B}$ on the category $\mathcal{V} = \mathcal{V}_k$ is also commutative, together with a nice canonical orientation for the class of smooth morphisms. Then it is easy to see that the associated covariant functor $\mathbb{B}_*$ is a (graded) oriented Borel-Moore functor with products, except maybe for the "additivity" property (BM-2) (required in Levine–Morel [28]), which in the bivariant context is often not needed. Assume in addition that the bivariant theory $\mathbb{B}$ is also "additive" in the sense that for all morphisms $g : X \sqcup Y \to Z$:

$$i_{X*} \oplus i_{Y*} : \mathbb{B}(X \xrightarrow{g|_X} Z) \oplus \mathbb{B}(Y \xrightarrow{g|_Y} Z) \xrightarrow{\sim} \mathbb{B}(X \sqcup Y \xrightarrow{g} Z)$$

for the closed (and also projective) inclusions $i_X : X \to X \sqcup Y$ and $i_Y : Y \to X \sqcup Y$ into the disjoint union. Then $\mathbb{B}_*$ also satisfies property (BM-2) so that it is a (graded) oriented Borel-Moore functor with products. Finally, the canonical orientation θ is called "additive" if

$$i_{X*}(\theta(g|_X)) + i_{Y*}(\theta(g|_Y)) = \theta(g)$$

for all smooth morphisms $g : X \sqcup Y \to Z$.

As an "oriented" analogue of the pre-motivic bivariant theory $\mathbb{M}(\mathcal{V}/X \xrightarrow{f} Y)$, in [44] we showed the following counterpart of Theorem 3.1, with (more or less) the same definition of the bivariant operations and the first Chern class operator similarly to (1) (given right after Definition-Theorem 6.2):

Theorem 6.7 (Universal oriented bivariant theory)**.** *Let*

$$\mathbb{OB}(X \xrightarrow{f} Y) := \left\{ [V \xrightarrow{h} X; L_1, \cdots L_m] \,\middle|\, h \textit{ is projective}, f \circ h \textit{ is smooth}, \ L_i \textit{ are line bundles over } V \right\}$$

be the free abelian group on isomorphism classes of f-relative cobordism cycles $[V \xrightarrow{h} X; L_1, \cdots L_m]$.

Then we have:

(1) $\mathbb{OB}(X \xrightarrow{f} Y)$ is a universal oriented bivariant theory (graded by m minus the fiber dimension of $f \circ h$).

(2) $\mathbb{OB}_*(X) := \mathbb{OB}(X \to pt)$ is a universal oriented Borel–Moore functor with products (graded by the dimension of V minus m), but without the "additivity" property (BM-2).

If one wants to get the corresponding "additive" counterparts, then one only has to work with the group completion

$$\mathbb{OB}^+(X \xrightarrow{f} Y) := \left\{ [V \xrightarrow{h} X; L_1, \cdots L_m] \,\middle|\, h \text{ is projective}, f \circ h \text{ is smooth}, \ L_i \text{ are line bundles over } V \right\}^+$$

of the monoid of isomorphism classes of f-relative cobordism cycles (with respect to disjoint union in V), so that the associated covariant theory

$$\mathbb{OB}^+_*(X) := \mathbb{OB}^+(X \to pt) = Z_*(X)$$

is nothing other than the corresponding cycle group of Levin–Morel [28].

It remains to see if one can impose suitable "bivariant-theoretic" relations of the geometric type

(b-Dim): "bivariant Dimension Axiom",
(b-Sect): "bivariant Section Axiom",
(b-FGL): "bivariant Formal Group Law Axiom",

on this oriented bivariant group $\mathbb{OB}^+(X \xrightarrow{f} Y)$, so that

$$\mathbb{B}\Omega(X \xrightarrow{f} Y) := \frac{\mathbb{OB}^+(X \xrightarrow{f} Y)}{\{\text{(b-Dim), (b-Sect), (b-FGL)}\}}$$

becomes an oriented bivariant theory, with $\mathbb{B}\Omega(X \to pt) = \Omega^{LM}_*(X)$. Then $\mathbb{B}\Omega(X \xrightarrow{f} Y)$ could be called a **bivariant algebraic cobordism** and $\mathbb{B}\Omega(X \xrightarrow{id} X)$ would be a contravariant analogue of $\Omega^{LM}_*(X)$.

Another possible way to a bivariant algebraic cobordism is the idea to adapt Levine–Pandharipande's new geometric construction of algebraic cobordism to a bivariant context, similarly to the way we extend Bittner's blow-up relation to a bivariant version.

In [29] Levine and Pandharipande gave a new geometric description to the algebraic cobordism (over a base field k of characteristic zero). Let Y be a smooth scheme. A morphism $\pi : Y \to \mathbb{P}^1$ is called a *double point degeneration* over $0 \in \mathbb{P}^1$ if $\pi^{-1}(0)$ can be written as

$$\pi^{-1}(0) = A \cup B$$

where A and B are smooth codimension one closed subschemes of Y, intersecting transversely. The intersection $D = A \cap B$ is the *double point locus* of π over $0 \in \mathbb{P}^1$. Let $N_{A/D}$ and $N_{A/D}$ be the normal bundles of D in A and B respectively. Then the projectivized bundles $\mathbb{P}(\mathcal{O}_D \oplus N_{A/D}) \to D$ and $\mathbb{P}(\mathcal{O}_D \oplus N_{B/D}) \to D$ are isomorphic (see [29]). Either one of these bundles is denoted by $\mathbb{P}(\pi) \to D$.

Definition 6.8. Let Y be smooth, with

$$g : Y \to X \times \mathbb{P}^1$$

a projective morphism such that the composite

$$\pi := \pi_2 \circ g : Y \to X \times \mathbb{P}^1 \to \mathbb{P}^1$$

is a double point degeneration over $0 \in \mathbb{P}^1$. Let $\xi \in \mathbb{P}^1$ be a regular value of π (which exists by "generic smoothness", since we are in characteristic zero). Then the map g is called a *double point cobordism* with degenerate fiber over 0 and smooth fiber over ξ with $Y_\xi := \pi^{-1}(\xi)$. The associated *double point relation* over X is defined by

$$[Y_\xi \to X] - [A \to X] - [B \to X] + [\mathbb{P}(\pi) \to X].$$

Note that one is allowed above to have $B = \emptyset$ (and therefore also $\mathbb{P}(\pi) = \emptyset$) with 0 and ξ both regular values of π. The corresponding relation

$$[Y_\xi \to X] - [A \to X] = 0$$

is called a "naive (co)bordism" between the smooth algebraic manifolds $A = \pi^{-1}(0)$ and $\pi^{-1}(\xi)$. It is just an "algebrization" of Quillen's definition of the complex cobordism relation [34] (and also see the next section). The "naive (co)bordism" relation holds in the algebraic cobordism group $\Omega_*^{LM}(X)$, but it is not enough to divide out the cycle group $M_*(X)$ only by this relation to get the algebraic cobordism group (see [28, Remark 1.2.9]). So this is different from the differential topological context of Quillen [34] in his study of complex cobordism. It is a beautiful and striking result of Levine–Pandharipande [29] that one only has to add the simplest kind of singularities (namely "double points") to get back the algebraic cobordism group $\Omega_*^{LM}(X)$:

Theorem 6.9 (Levine–Pandharipande [29])**.** *Let $\mathcal{R}_*(X) \subset M_*(X)$ be the subgroup generated by all the double point relations over X. One sets*

$$\omega_*^{LP}(X) = \left\{[V \xrightarrow{h} X] \middle| V \textit{is smooth}, h \textit{ is projective}\right\}^+ \Big/ \mathcal{R}_*(X) = M_*(X) / \mathcal{R}_*(X)\,.$$

Then $\omega_*^{LP}(X) \cong \Omega_*^{LM}(X)$.

Motivated by our previous constructions, the group completion (with respect to disjoint union in V)

$$\mathbb{M}(X \xrightarrow{f} Y) := \left\{[V \xrightarrow{h} X] \,\middle|\, h \text{ is projective}, f \circ h \text{ is smooth}\right\}^+$$

is a bivariant analogue of the above group $M_*(X) = \left\{[V \xrightarrow{h} X] \middle| V \text{is smooth}, h \text{ is projective}\right\}^+$. Thus it remains to be seen if one can construct a corresponding bivariant analogue $\mathcal{R}_*(X \xrightarrow{f} Y)$ of the above subgroup $\mathcal{R}_*(X)$ of all the double point relations, such that

(1) $\mathbb{B}\omega^{LP}(X \xrightarrow{f} Y) := \dfrac{\left\{[V \xrightarrow{h} X] \,\middle|\, h \text{ is projective}, f \circ h \text{ is smooth}\right\}^+}{\mathcal{R}_*(X \xrightarrow{f} Y)}$ is a bivariant theory,

(2) $\mathbb{B}\omega_*^{LP}(X) = \mathbb{B}\omega^{LP}(X \to pt) = \omega_*^{LP}(X)$.

If this can be done, it gives us a geometrically defined bivariant algebraic cobordism and thus the contravariant part $\mathbb{B}\omega^{LP}(X \xrightarrow{id_X} X)$ could be considered as a contravariant analogue of Levine–Morel's or Levine–Pandharipande's algebraic bordism.

7. A FINAL REMARK: CLASSICAL (CO)BORDISMS AND FIBERWISE BORDISM GROUPS

Thom's oriented bordism group Ω_n^{SO} is defined by

$$\Omega_n^{SO} := \{n\text{-dimensional closed oriented differentiable manifolds}\} \Big/ \text{bordant relation}.$$

Here two such n-dimensional manifolds M_1^n and M_2^n are called *bordant* if there exists an $(n+1)$-dimensonal compact oriented differentiable manifold W^{n+1} with boundary such that

$$\partial W = M_1 \sqcup (-1)M_2 \,,$$

where the signed one $(-1)M_2$ is the manfiold M_2 with the orientation reversed.

M. Atiyah extended the bordism group to a covariant functor on the category of topological spaces:

Definition 7.1 (Atiyah [2]). For a topological space X

$$\Omega_n^{SO}(X) := \left\{ M^n \xrightarrow{h} X \;\middle|\; M^n \in \mathcal{C}^\infty_{or,c},\ h \,\text{is continuous} \right\} \Big/ \text{bordant relation.}$$

Here $\mathcal{C}^\infty_{or,c}$ denotes the category of oriented closed $\mathcal{C}^\infty$-manifolds. Two continuous maps $h_1 : M_1^n \to X$ and $h_2 : M_2^n \to X$ are called *bordant* if

(1) there exists a compact $(n+1)$-dimensional oriented differentiable manifold W^{n+1} with boundary such that $\partial W = M_1 \sqcup (-1)M_2$,
(2) there exists a continuous map $H : W^{n+1} \to X$ such that $H|_{M_i} = h_i$ for $i = 1, 2$.

Furthermore, Conner and Floyd [13] extended it as a generalized homology theory so that for a pair (X, A) of CW-complexes, there exists a canonical isomorphism

$$\Phi : \Omega_n^{SO}(X, A) \xrightarrow{\cong} \mathcal{H}_n(X, A; \{MSO(i)\}) : = \lim_{i\to\infty} \left[S^{n+i}, (X/A) \wedge MSO(i)\right]_0$$

Here $\{MSO(i)\}$ is the corresponding Thom spectrum, i.e., the sequence of the Thom complexes $MSO(i)$, which are the Thom spaces of the universal oriented $\mathbb{R}^i$-bundle ξ_i over the classifying space $BSO(i)$. $[A, B]_0$ with the suffix 0 denotes the group of base-point preserving homotopy class of maps.

Definition 7.2. ([2]) For a topological pair (X, A) the *bordism cohomology group* $\Omega^n_{SO}(X, A)$ (called the *cobordism group*) is defined as *the generalized cohomology theory associated to the Thom spectrum* $\{MSO(i)\}$:

$$\Omega^n_{SO}(X, A) := \mathcal{H}^n(X, A; \{MSO(i)\}) : = \lim_{i\to\infty} \left[S^{i-n} \wedge (X/A), MSO(i)\right]_0 \,.$$

A *naïve and fundamental question* on the bordism cohomology group $\Omega^n_{SO}(X) := \Omega^n_{SO}(X, \emptyset)$ is the following:

Question 7.3. Can one give a geometric description of this cohomology group $\Omega^n_{SO}(X)$ like in the definition of the bordism homology group $\Omega_n^{SO}(X)$?

In fact, in the case when M is a closed oriented differentiable manifold, we have a simple solution for the above question:

$$\Omega^i_{SO}(M) = \left\{ N^{\dim M - i} \xrightarrow{h} M \;\middle|\; N^{\dim M - i} \in \mathcal{C}^\infty_{or,c}, h \text{ is continuous} \right\} \Big/ \text{bordant relation .}$$

This is thanks to the following *Atiyah–Thom–Poincaré duality theorem*:

Theorem 7.4. *([2]) For a closed oriented differentiable manifold M there exists canonical isomorphism*

$$\Omega^i_{SO}(M) \xrightarrow{\cong} \Omega^{SO}_{\dim M - i}(M) \,.$$

Remark 7.5. The above definitions together with this duality theorem also hold similarly, if SO is replaced by O or U, i.e., if one considers unoriented or complex (co)bordism (see [14, 34]).

Motivated by our construction of the bivariant Grothendieck group $\mathbb{K}_0(V/X \xrightarrow{f} Y)$ and the oriented bivariant theory $\mathbb{OB}(X \xrightarrow{f} Y)$, we can construct in a similar way a bivariant theory

$$\mathbb{B}\Omega^{SO}(X \xrightarrow{f} Y)$$

on the category $\mathcal{C}^\infty$ of differentiable manifolds (without boundary), such that $\mathbb{B}\Omega^{SO}(X \to pt) \simeq \Omega_*^{SO}(X)$ for X compact. Here the confined morphisms are the proper morphisms, whereas the independent squares are by definition the "transversal squares", i.e., with the differentiable maps f, g transversal so that the corresponding fiber product X' exists in this category $\mathcal{C}^\infty$ of differentiable manifolds. Note that a submersion is transversal to any morphism in the category $\mathcal{C}^\infty$. The corresponding contravariant theory $\mathbb{B}\Omega^{SO}(X \xrightarrow{id_X} X)$ can be seen as a new "cobordism group".

The basic idea for that is the following notion of *f-relative fiberwise bordism*. We set $FM^{-n}(X \xrightarrow{f} Y)$ to be the set of isomorphism classes

$$\left\{[V \xrightarrow{h} X] \,\middle|\, f \circ h \text{ is a proper submersion, whose tangent bundle to the fibers } T_{f\circ h} \text{ is oriented of rank } n\right\}.$$

Definition 7.6. Let $h_1 : V_1 \to X$ and $h_2 : V_2 \to X$ be two morphisms representing elements of the set $FM^{-n}(X \xrightarrow{f} Y)$. They are called *elementary f-relative fiberwise bordant* if there exists a differentiable manifold W with boundary ∂W and a morphism $H : W \to X$ such that

(1) $f \circ H : W \to Y$ is a proper submersion (i.e., also $f \circ H|_{\partial W}$ is a submersion),
(2) the tangent bundle to the fibers $T_{f\circ H}$ is oriented of rank $n+1$ (so that $T_{f\circ H|_{\partial W}}$ gets an induced orientation),
(3) $f \circ H : \partial W \to Y$ is isomorphic to $f \circ h_1 + f \circ h_2 : V_1 \sqcup (-1)V_2 \to Y$, where the signed one $(-1)V_2$ again indicates the reversed orientation of $T_{f\circ h_2}$.

By definition this notion only depends on the corresponding isomorphism classes. Moreover, the corresponding "elementary f-relative fiberwise bordism" relation is symmetric and reflexive. Let "f-relative fiberwise bordism" be the equivalence relation *generated by this*, i.e., $V_1 \xrightarrow{h_1} X$ and $V_2 \xrightarrow{h_2} X$ are "f-relative fiberwise bordant" if they can be *related by a finite string* of "elementary f-relative fiberwise bordant" morphisms.

This equivalence relation is also compatible with the monoid structure on $FM^{-n}(X \xrightarrow{f} Y)$ coming from the disjoint union with respect to V, so that

$$F\Omega^{-n}(X \xrightarrow{f} Y) := FM^{-n}(X \xrightarrow{f} Y)\Big/ f\text{-relative fiberwise bordism}$$

gets an induced monoid structure. The "trivial bordism" $W := V \times [0,1] \to V \xrightarrow{h} X$ shows that it is indeed an abelian group. By definition we have $F\Omega^{-n}(X \to pt) \simeq \Omega_n^{SO}(X)$ for X compact, since any continuous map $V \to X$ between smooth manifolds can be approximated by a differentiable map.

In the case of $F\Omega^{-n}(X \xrightarrow{id_X} X)$, we have

$$FM^{-n}(X \xrightarrow{id_X} X) = \left\{[V \xrightarrow{h} X] \,\middle|\, h \text{ is a proper submersion, with } T_h \text{ is oriented of rank } n\right\}.$$

Then id_X-relative fiberwise bordism is just fiberwise bordism, where two proper oriented submersions $V_1 \xrightarrow{h_1} X$ and $V_2 \xrightarrow{h_2} X$ are *fiberwise bordant*, if there exists a differentiable manifold W with boundary ∂W and a proper oriented submersion $H : W \to X$ of fiber dimension $n+1$ such that

$$H\big|_{\partial W} = h_1 + h_2 : V_1 \sqcup (-1)V_2 \to X .$$

Note that in this case "elementary id_X-relative fiberwise bordism" or fiberwise bordism is already an equivalence relation thanks to the "tubular neighborhood theorem". Thus we have

$$F\Omega^{-n}(X) := F\Omega^{-n}(X \xrightarrow{id_X} X) = FM^{-n}(X \xrightarrow{id_X} X)/\text{fiberwise bordant}\,.$$

Note that there is a tautological group homomorphism (commuting with cross products)

$$F\Omega^{-n}(X) \to \Omega^{-n}_{SO}(X)$$

to the usual *oriented cobordism* of the smooth manifold X, once we use the *geometric* definition of the later given by Quillen [34] and Dold [14] in the smooth context (just forget that $H : W \to X$ needs to be a submersion in our case).

Remark 7.7. (1) In the case when $X = S^1$ the 1-dimensional sphere, we have by the "mapping cylinder construction" $F\Omega^{-n}(S^1) \cong \Delta_n$, where Δ_n is the bordism group of orientation-preserving diffeomorphisms of closed oriented C^∞-manifolds of dimension n. It was introduced by Browder [11] and later on studied deeply by Kreck [26, 27].
(2) There is a surjection $F\Omega^{-n}(X \xrightarrow{id_X} X) \twoheadrightarrow \Omega^X_n(pt, \emptyset)$. Here $\Omega^X_n(Y, Y')$ is Weishu Shih's *fiber cobordism group* [39] for a topological pair $Y' \subset Y$.

Imitating the proof of Theorem 3.1, one finally gets the

Theorem 7.8. *Consider the category $\mathcal{C}^\infty$ of differentiable manifolds (without boundary), where the confined morphisms are the proper morphisms, whereas the independent squares are by definition the "transversal squares".*

Then the "f-relative fiberwise bordism group"

$$F\Omega(X \xrightarrow{f} Y) := \bigoplus_{n\geq 0} F\Omega^{-n}(X \xrightarrow{f} Y)$$

becomes a graded bivariant theory with the bivariant operations defined as in Theorem 3.1.

In [3] we will give more details about f-relative fiberwise bordism and the theorem above in the right context, namely in the category of differentiable spaces (e.g., see [33]), where instead of an orientation of $T_{f\circ h}$ we work more generally with a σ-structure in the sense of Dold [14] (e.g. with "structure groups" O or U instead of SO). Then it also becomes in close connection to the recent work of Emerson–Meyer [16] on a topological description of KK-theory.

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J. Schümann on "Characteristic classes of Hilbert schemes of points via symmetric products",

S. Yokura on "Fiberwise bordism groups".

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EQUIVARIANT CHERN CLASSES AND LOCALIZATION THEOREM

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ABSTRACT. For a complex variety with a torus action we propose a new method of computing Chern-Schwartz-MacPherson classes. The method does not apply resolution of singularities. It is based on the Localization Theorem in equivariant cohomology.

This is an extended version of the talk given in Hefei in July 2011.

Equivariant cohomology is a powerful tool for studying complex manifolds equipped with a torus action. The Localization Theorem of Atiyah and Bott and the resulting formula of Berline-Vergne allow to compute global invariants, for example invariants of singular subsets, in terms of some data attached to the fixed points of the action. We will concentrate on the equivariant Chern-Schwartz-MacPherson classes. The global class is determined by the local contributions coming from the fixed points. On the other hand, the sum of the local contributions divided by the Euler classes is equal to zero in an appropriate localization of equivariant cohomology. Especially for Grassmannians we obtain interesting formulas with nontrivial relations involving rational functions. We discuss the issue of positivity: the local equivariant Chern class may be presented in various ways, depending on the choice of generating circles of the torus. For some choices we find that the coefficients of the presentation are nonnegative. Also the coefficients in the Schur basis are nonnegative in many examples, but it turns out that not always.

We begin with §1 which contains a review of the results concerning the equivariant fundamental class of an invariant subvariety in a smooth G-manifold M. The first two chapters are valid for any algebraic group, but further we will consider only torus actions. The equivariant fundamental class lives in the equivariant cohomology $H^*_G(M)$. The invariant subvarieties contained in a vector space V, on which the torus $G = T = (\mathbb{C}^*)^n$ acts linearly, are of particular interest. If the weights of the torus action are nonnegative then the equivariant fundamental class is a nonnegative combination of monomials in $H^*_T(V) = \mathbb{Q}[t_1, t_2, \dots, t_n]$.

In §2 we discuss the equivariant version of the Chern-Schwartz-MacPherson class, denoted by c^T, which is a refinement of the equivariant fundamental class. To give the precise definition, following [33], one has to introduce equivariant homology. Eventually we will assume that the variety is contained in a smooth manifold. By Poincaré duality it is enough for our purpose to consider the equivariant Chern-Schwartz-MacPherson classes as the elements of equivariant cohomology of the ambient space.

The main tool of the equivariant cohomology for a torus action is the Localization Theorem of Atiyah-Bott or Berline-Vergne formula. It says that the equivariant cohomology class can be read from certain data concentrated at the fixed points of the action. The precise formulation of the Localization Theorem is recalled in §3.

The section §4 is a kind of interlude for fun. We give some examples of calculations based on the Localization Theorem for torus acting on projective spaces and Grassmannians. For example we show how the formula for Gysin map for the Grassmann bundle immediately follows from the Laplace determinant expansion.

Key words and phrases. Equivariant cohomology, Chern classes of singular varieties, localization.

Next, in §5 we discuss equivariant Chern-Schwartz-MacPherson classes of toric varieties. From the Localization Theorem we deduce that the equivariant class of an orbit $c^T(Tx)$ is equal to the fundamental class of its closure $[\overline{Tx}]$. Exactly the same formula holds in the nonequivariant setting, by [7].

The section §6 is important for whole inductive procedure of computations of equivariant Chern classes. The key statement is the following:

Theorem 1. *Suppose that X is a T-variety, not necessarily smooth, contained in a T-manifold M. Let $p \in X$ be an isolated fixed point. Then the zero degree of the class $c^T(X)$ restricted to $\{p\}$ is Poincaré dual of the product of weights appearing in the tangent representation $\mathbf{T}_pM$.*

It is also convenient to consider a version of the Localization Theorem in which we express the global cohomology class by its restriction to an arbitrary submanifold containing the fixed point set. The main example is the projective space $\mathbb{P}^n$. The class which we want to compute is the equivariant Chern-Schwartz-MacPherson class of the projective cone over a subvariety in $\mathbb{P}^{n-1}$. In §7 from partial localization we deduce the following:

Proposition 2. *Suppose that X is a T-invariant cone in a linear representation V of T. Let $h = c_1^T(\mathcal{O}_{\mathbb{P}(V)}(1)) \in H^2_T(\mathbb{P}(V))$ be the equivariant Chern class. If*

$$c^T(\mathbb{P}(X)) = \Big(\sum_{i=0}^{\dim(V)-1} b_i(t)h^i\Big) \cap [\mathbb{P}(V)] \in H_*^T(\mathbb{P}(V))$$

for $b_i(t) \in H^*_T(pt)$ then

$$c^T(X) = (b_0(t) + e_0) \cap [V] \in H^T_*(V)$$

where e_0 is the product of weights appearing in the representation V.

Even a seemingly trivial application of this result (discussed in §8) is meaningful. If $T = \mathbb{C}^*$ acts by scalar multiplication on $\mathbb{C}^n$, then T-invariant subvariety is just a cone in $\mathbb{C}^n$. The characteristic classes of cones were already considered by Aluffi and Marcolli. In their paper [4] there was given a formula for the Chern-Schwartz-MacPherson class of an open affine cone in $\mathbb{P}^n$. It is not a coincidence, that their computation agrees with our result about the equivariant class in $H^*_T(\mathbb{C}^n)$:

Proposition 3. *Suppose X a cone in $\mathbb{C}^n$. Let $x = c_1(\mathcal{O}_{\mathbb{P}^{n-1}}(1)) \in H^2(\mathbb{P}^{n-1})$ and let $t \in H^2_T(pt)$ be the generator corresponding to the identity character. If*

$$c_{SM}(\mathbb{P}(X)) = \Big(\sum_{i=0}^{n-1} a_i x^i\Big) \cap [\mathbb{P}^{n-1}] \in H_*(\mathbb{P}^{n-1})$$

then

$$c^T(X) = \Big(\sum_{i=0}^{n-1} a_i t^i + t^n\Big) \cap [\mathbb{C}^n] \in H^T_*(\mathbb{C}^n)\,.$$

The result follows from the previous one since the equivariant Chern class $h = c_1^T(\mathcal{O}_{\mathbb{P}^{n-1}}(1))$ is equal to

$$1 \otimes x - t \otimes 1 \in H^*_T(\mathbb{P}^{n-1}) = H^*_T(pt) \otimes H^*(\mathbb{P}^{n-1})\,.$$

By the product property of equivariant Chern-Schwartz-MacPherson classes we obtain for free „Feynman rule" for the polynomial G_X introduced [4, Lemma 3.10].

In the next section §9 we propose a new method of computing equivariant Chern-Schwartz-MacPherson classes which does not involve resolution of singularities. It is based on the fact, that the sum of the equivariant Chern-Schwartz-MacPherson classes localized at fixed points and divided by Euler classes is equal to zero, except from the zero degree. A similar observation was already made by Féher and Rimányi in [16, §8.1] for computation of Thom polynomials. On the other hand the zero degree of the equivariant Chern class is given by the result of §7 (stated before as Theorem 1). This way often we can compute local Chern classes by induction on the depth of the singularity.

Our main example in §10 is the determinant variety, the subset of square matrices $n \times n$ defined by the equation $\det = 0$. We study its compactification, the Schubert variety of codimension one in $\mathrm{Grass}_n(\mathbb{C}^{2n})$. We discuss computational problems appearing for that example. The concrete formula for the equivariant Chern class is a huge sum of fractions. Surprisingly all the difficulties lie in simplifying that expression. We compute the equivariant Chern class for the determinant variety for $n \leq 4$. It turns out that it is a nonnegative combination of monomials with suitable choice of generators of $H^2_T(pt)$. This supports the conjecture of Aluffi-Mihalcea that the Chern-MacPherson-Schwartz class of the Schubert varieties are effective. On the other hand for $n = 4$ the local equivariant Chern-Schwartz-MacPherson class expanded in the Schur basis has negative coefficients in few places. We present the result of calculations in §12.

The connection of our local formulas with the calculations of [5] and [23] is not clear. The formula for the global class can be read from the local contributions by Theorem 11. Nevertheless the shape of this relation seems to be combinatorially nontrivial due to presence of the denominators.

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1. Equivariant fundamental class

Let M be a complex manifold and $X \subset M$ a closed complex subvariety. The *fundamental class* of X, which is the Poincaré dual of the cycle defined by X is denoted by

$$[X] \in H^{2\,\mathrm{codim}(X)}(M)\,.$$

When the ambient manifold M is contractible, for example when M is an affine space, there is no use of $[X]$ since the cohomology of M is trivial. An interesting situation appears when an algebraic group G acts on M and X is preserved by the action. In that case there is an equivariant fundamental class of X which belongs to the equivariant cohomology of M

$$[X] \in H_G^{2\,\mathrm{codim}(X)}(M).$$

Now even if M is contractible we obtain a remarkable invariant of the pair (M, X). For contractible M its equivariant cohomology coincides with the equivariant cohomology of a point

$$H_G^{2\,\mathrm{codim}(X)}(M) \simeq H_G^{2\,\mathrm{codim}(X)}(pt)$$

and the cohomology of a point *is the ring of characteristic classes for* G. In particular

- if $G = (\mathbb{C}^*)^n$ then $H^*_G(pt) = \mathbb{Q}[t_1, t_2, \dots, t_n]$
- if $G = GL_n$ then $H^*_G(pt) = \mathbb{Q}[\sigma_1, \sigma_2, \dots, \sigma_n] = \mathbb{Q}[t_1, t_2, \dots, t_n]^{\Sigma_n}$
- in general the ring of characteristic classes coincides with the invariants of the Weyl group acting on the characteristic classes for the maximal torus

$$H^*_G(pt) = H^*_T(pt)^W.$$

(We consider here only cohomology with rational coefficients.)

For a torus $G = T$ we identify $H^*_T(pt)$ with the polynomial algebra spanned by characters of T, i.e.

$$H^*_T(pt) = \mathbb{Q}[T^\vee] = \bigoplus_{k=0}^{\infty} Sym^k[T^\vee \otimes \mathbb{Q}]\,.$$

A character $\lambda : T \to \mathbb{C}^*$ corresponds to an element of $H^2_T(pt)$.

We will briefly recall the construction of equivariant cohomology in §2. The reader can find its basic properties in [36]. For a review of equivariant cohomology in algebraic geometry see e.g. [18]. An extended discussions of different names for the equivariant fundamental class can be found in [9, §2.1].

For $G = GL_n$ equivariant cohomology and the equivariant fundamental classes $[X] \in H^*_{GL_n}(pt)$ has turned out to be an adequate tool for studying the Thom polynomials of singularities of maps. Here X is a set of singular jets in the space of all jets of maps. Its equivariant fundamental class $[X]$ is the universal characteristic class which describes cohomological properties of singular loci of maps. In the last decade there appeared a series of papers by Rimányi and his collaborators (starting from [37]) and Kazarian (see e.g. [25]). Powerful tools allowing effective computations were developed and some structure theorems were stated. The geometric approach to equivariant cohomology leads to positivity results [34, 31, 32]. The source of these results is the following principle:

Theorem 4. *If $X \subset \mathbb{C}^N$ is a cone in a polynomial representation of GL_n, then $[X]$ is a nonnegative combination of Schur functions.*

The examples of polynomial representations are the following: the natural representation, its tensor products, symmetric products, exterior products and in general quotients of the sums of tensor products. The Schur functions constitute a basis of the ring of characteristic classes

$$H^*_{GL_n}(pt) = H^*(\mathrm{Grass}_n(\mathbb{C}^\infty))$$

corresponding to the decomposition of the infinite Grassmannian into Schubert cells. For an algebraic treatment of Schur functions see [28].

A version of Theorem 4 holds for G being a product of the general linear groups. We will be interested in torus actions. Theorem 5 stated in [35] reduces to:

Theorem 5. *Let $T = (\mathbb{C}^*)^n$ and let $t_1, t_2, \dots, t_n \in \mathrm{Hom}(T, \mathbb{C}^*)$ be the characters corresponding to the decomposition of T into the product. Suppose $V = \bigoplus V_\lambda$ is a representation of T such that each weight λ appearing in V is a nonnegative combination of t_i's. Let $X \subset V$ be a variety preserved by T-action. Then the equivariant fundamental class $[X] \subset H^*_T(V) = \mathbb{Q}[t_1, t_2, \dots, t_n]$ is a polynomial with nonnegative coefficients.*

2. EQUIVARIANT CHERN CLASS

Our goal is to study more delicate invariants of subvarieties in representations of algebraic groups, the invariants which are refinements of the equivariant fundamental class. In most of the interesting cases the subvarieties to study are singular. Our first choice is the equivariant version of the Chern-Schwartz-MacPherson classes. We recall that the usual Chern-Schwartz-MacPherson classes, introduced in [29] and denoted by c_{SM}, live in homology, they are Poincaré duals of the Chern classes of the tangent bundle when the variety is smooth. These classes are functorial in a certain sense, and therefore usually they are computed via resolution of singularities.

The equivariant version of Chern-Schwartz-MacPherson classes was developed by Ohmoto [33]. To define these classes one has to recall the Borel construction of the equivariant cohomology. Let G be a topological group. Denote by $EG \to BG = EG/G$ the universal principal G-bundle. This bundle is defined up to G-equivariant homotopy. For a topological G-space the equivariant cohomology is by definition the cohomology of the associated X-bundle $EG \times^G X$. Now we apply this construction to G being an algebraic reductive group and X a complex algebraic G-variety. With the exclusion of the case of the trivial group, EG does not admit a finite dimensional model. Instead, EG always has an approximation by algebraic G-varieties, see [39]. For example if $G = \mathbb{C}^*$, then $EG = \mathbb{C}^\infty - \{0\}$ and $BG = \mathbb{P}^\infty$. It can be approximated by $U = \mathbb{C}^n - \{0\}$ with $U/G = \mathbb{P}^{n-1}$. In general as the approximation of EG we take an open set U in a linear representation V of G satisfying

- U is G-invariant,

- G acts freely on U and the action admits a geometric quotient,
- $V - U$ has a sufficiently large codimension in V.

If we are interested only in the cohomology classes of degrees bounded by d we take an approximation with $2\,\mathrm{codim}(V-U) > d+1$. Then

$$H^k_G(X) = H^k(U \times^G X)$$

for $k \leq 2d$. The equivariant Chern classes of a smooth G-variety coincides with the equivariant Chern class of the tangent bundle. By Borel construction it is the usual Chern class of the tangent bundle to fibers of the fibration $EG \times^G X \to BG$. Using an approximation it can be written as

$$c^G(X) = p^* c(U/G)^{-1} \cup c(U \times^G X) \in H^*(U \times^G X) \simeq H^*_G(X)\,,$$

where $p : U \times^G X \to U/G$ is the projection. If X is singular we apply the same formula with obvious modifications. First of all $U \times^G X$ is singular and we have the homology Chern-Schwartz-MacPherson class

$$c_{SM}(U \times^G X) \in H^{BM}_*(U \times^G X)\,.$$

(The superscript BM stands for Borel-Moore homology.) We are forced to use less known equivariant homology $H^G_*(X)$, [12, 14], which can be defined via approximation:

$$H^G_k(X) = H^{BM}_{k+2\dim(U/G)}(U \times^G X)$$

for $2n - k < 2\,\mathrm{codim}(V-U) - 1$, i.e. for $k > 2n - 2\,\mathrm{codim}(V-U) + 1$.

Definition 6. *The equivariant Chern-Schwartz-MacPherson class of X is defined by the formula*

$$c^G(X) = p^* c(U/G)^{-1} \cap c_{SM}(U \times^G X) \in H^{BM}_{*+2\dim(U/G)}(U \times^G X) \simeq H^G_*(X)\,.$$

The definition can be extended to the equivariant constructible functions on X.

Note that $H^G_*(X)$ can have nontrivial negative degrees, but the equivariant Chern-Schwartz-MacPherson class lives in $H^G_{\geq 0}(X)$.

We will not use the long name *equivariant Chern-Schwartz-MacPherson classes*. Hopefully saying just *equivariant Chern classes* in the context of possibly singular algebraic G-varieties or constructible functions will not lead to any confusion. Additionally we will always write $c^G(\mathbb{1}_X)$ instead of $c^G(X)$. Later, in §§9-10, where we compute the equivariant Chern classes of a subvariety in a smooth manifold M, for convenience we skip the cap-product $\cap[M]$ in the notation identifying $H^T_*(M)$ with $H^{2\dim(M)-*}_T(M)$.

The definition of equivariant Chern classes is in fact irrelevant. All what we need follows from the formal properties.

- **Normalization:** if X is smooth, then $c^G(\mathbb{1}_X)$ is Poincaré dual of the usual equivariant Chern class of the tangent bundle.
- **Functoriality:** for a G-constructible function α and a proper G-map $f : X \to Y$ we have $c^G(f_*\alpha) = f_* c^G(\alpha)$.
- **Product formula:** if X is G-variety and Y is G'-variety, then

  $$c^{G\times G'}(\mathbb{1}_{X\times Y}) = c^G(\mathbb{1}_X) \otimes c^{G'}(\mathbb{1}_Y)$$

  under the Küneth isomorphism $H^{G\times G'}_*(X \times Y) \simeq H^G_*(X) \otimes H^{G'}_*(Y)$. In particular when X is a trivial G-space, then

  $$c^G(\mathbb{1}_X) = 1 \otimes c_{SM}(\mathbb{1}_X) \in H^G_*(X) \simeq H^{-*}_G(pt) \otimes H^{BM}_*(X)\,.$$

- **Functoriality with respect to G:** Let $\phi : G' \to G$ be a group homomorphism and X a G-space. The induced map $\phi^* : H^G_*(X) \to H^{G'}_*(X)$ sends $c^G(\mathbb{1}_X)$ to $c^{G'}(\mathbb{1}_X)$.

All five properties easily follow from the corresponding properties of the usual Chern-Schwartz-MacPherson classes. The equivariant Chern class carries more information than nonequivariant Chern-Schwartz-MacPherson class. There is a natural map $H^G_*(X) \to H^{BM}_*(X)$ which is induced by the inclusion of the trivial group into G. It transports the equivariant Chern class to the nonequivariant one.

Let us focus on the case when $G = T$ is a torus and V is a complex linear representation. The equivariant homology $H^T_*(V)$ is a free rank one module over $H^*_T(pt)$ generated by $[V] \in H^T_{2\dim(V)}(V)$. The action of a character $\lambda \in H^2_T(pt)$ lowers the degree by 2. By Poincaré duality we have the isomorphisms

$$H^T_{2k}(V) \simeq H_T^{2(\dim(V)-k)}(V) \simeq H_T^{2(\dim(V)-k)}(pt) \simeq Sym^{\dim(V)-k}(T^\vee \otimes \mathbb{Q}) .$$

We start with the basic example.

Example 7. Let V be a complex linear representation of a torus T. Suppose that V decomposes as the sum of the weight spaces

$$V = \bigoplus_\lambda V_\lambda .$$

Then

$$c^T(\mathbb{1}_V) = \big(\prod_\lambda (1+\lambda)^{\dim(V_\lambda)}\big) \cap [V] \in H^T_*(V) \simeq Sym^{\dim(V)-*}(T^\vee \otimes \mathbb{Q})$$

and

$$c^T(\mathbb{1}_{\{0\}}) = [\{0\}] = \big(\prod_\lambda \lambda^{\dim(V_\lambda)}\big) \cap [V] \in H^T_0(V) \simeq Sym^{\dim(V)}(T^\vee \otimes \mathbb{Q}) .$$

The last formula follows from covariant functoriality.

Now let us see what the equivariant Chern class means for conical sets in affine spaces.

Example 8. Let $T = \mathbb{C}^*$ acts on $\mathbb{C}^n$ by scalar multiplication. Consider a nonempty cone $X \subset \mathbb{C}^n$. We will compute its equivariant Chern class with respect to the action of T. Denote by $\mathbb{P}(X) \subset \mathbb{P}^{n-1}$ the projectivization of X. Let $h = c_1(\mathcal{O}(1)) \in H^2(\mathbb{P}^{n-1})$ and let $t \in H^2_T(pt)$ be the element corresponding to the identity character. Suppose that

$$c_{SM}(\mathbb{1}_{\mathbb{P}(X)}) = (a_0 + a_1 h + \cdots + a_{n-1} h^{n-1}) \cap [\mathbb{P}^n] \in H_*(\mathbb{P}^n) .$$

We will show in §8 that the equivariant Chern class of the cone is equal to

$$c^T(\mathbb{1}_X) = (a_0 + a_1 t + \cdots + a_{n-1} t^{n-1} + t^n) \cap [\mathbb{C}^n] \in H^T_*(\mathbb{C}^n) .$$

This formula agrees with computation of Aluffi-Marcolli who calculated the invariant of conical sets defined as the Chern-Schwartz-MacPherson class of the constructible function $\mathbb{1}_X$ considered not in $\mathbb{C}^n$ but in $\mathbb{P}^n$.

Now suppose $G = T = (\mathbb{C}^*)^n$ is acting on a vector space V as in Theorem 5. We pose a question:

Question. *When does $c^T(\mathbb{1}_X) \in H^T_*(V) \simeq \mathbb{Q}[t_1, t_2, \ldots, t_n]$ have nonnegative coefficients?*

This is a special property of X since in general the answer is negative. If the equivariant Chern classes are effective, i.e. represented by an invariant cycle, then the answer is positive. Also it is easy to find a counterexample: if $T = \mathbb{C}^*$ acts on $V = \mathbb{C}^n$ by scalar multiplication and let X be a cone over a curve of genus $g > 1$ and of degree d. Then

$$c^T(\mathbb{1}_X) = ([X] + 2(1-g)t^{n-1} + t^n) \cap [\mathbb{C}^n] = d[\mathbb{C}^2] + 2(1-g)[\mathbb{C}^1] + [\mathbb{C}^0]$$

is a counterexample. On the other hand we have a bunch of positive examples: local equivariant Chern classes have nonnegative coefficients for

- toric singularities (see Corollary 18),
- generic hyperplane arrangements with a small number of hyperplanes [2],
- banana Feynman motives [3].

3. Localization theorem

For the moment we leave the question of positivity. Our current goal is to develop a calculus which would allow to compute equivariant Chern classes avoiding resolution of singularities. Our main tool is the Localization Theorem for torus action. The topological setup is the following: suppose the torus $T=(S^1)^n$ or $(\mathbb{C}^*)^n$ acts on a compact space M (decent enough, e.g. equivariant CW-complex). The equivariant cohomology $H^*_T(M)$ is a module over equivariant cohomology of the point

$$H^*_T(pt)=\mathbb{Q}[t_1,t_2,\dots t_n]\,.$$

The following theorem goes back to Borel.

Theorem 9 ([36], [6])**.** *The restriction to the fixed set*

$$\iota^*: H^*_T(M)\longrightarrow H^*_T(M^T)$$

becomes an isomorphism after localizing in the multiplicative set generated by the nontrivial characters

$$S=T^\vee-0\subset H^2_T(pt)\,.$$

If M is a manifold, then the inverse of the restriction map is given by the Atiyah-Bott/Berline-Vergne formula. To explain that let us fix a notation. We decompose the fixed point set into components $M^T=\bigsqcup_{\alpha\in A}M_\alpha$. Each M_α is a manifold and denote by $e_\alpha\in H^*_T(M_\alpha)=H^*(M_\alpha)\otimes\mathbb{Q}[t_1,t_2,\dots t_n]$ the equivariant Euler class of the normal bundle. The following map is the inverse of the restriction to the fixed points

$$\begin{array}{ccccc} S^{-1}H^*_T(M^T) & = & \bigoplus_{\alpha\in A}S^{-1}H^*_T(M_\alpha) & \xrightarrow{\simeq} & S^{-1}H^*_T(M) \\ & & \{x_\alpha\}_{\alpha\in A} & \mapsto & \sum_{\alpha\in A}\iota_{\alpha *}\left(\frac{x_\alpha}{e_\alpha}\right)\,, \end{array} \tag{1}$$

where $\iota_\alpha: M_\alpha\hookrightarrow M$ is the inclusion. The key point in the formula (1) is that the Euler class e_α is *invertible* in $S^{-1}H^*_T(M_\alpha)$.

Remark 10. Note that if M is a smooth compact algebraic variety and the action of the torus is algebraic, then $H^*_T(M)$ is a free module over $H^*_T(pt)$, so we do not kill any class inverting nontrivial characters.

I other words we can state the theorem:

Theorem 11 ([6, 15])**.** *Let M be an algebraic variety with algebraic torus action, then with the previous notation*

$$x=\sum_{\alpha\in A}\iota_{\alpha *}\left(\frac{x_{|M_\alpha}}{e_\alpha}\right)\in H^*_T(M)\,.$$

Therefore we can say that x is a sum of local contributions. Although one has to understand that this statement is a bit misleading. In fact it is not possible to extract individual summands in $H^*_T(M)$. This can be done only in the localized ring. How a single fixed point component contributes to the global class is obscured by the weights of the tangent representation.

Furthermore consider the push-forward, i.e. the integration along M

$$p_*=\int_M: H^*_T(M)\to H_T^{*-2\dim(M)}(pt)$$

where $p: M\to pt$ is the constant map. Another form of the Localization Theorem allows to express the integration along M by integrations along components of the fixed point set.

Theorem 12 (Berline-Vergne [10]). *For $x \in H_T^*(M)$ the integral can be computed by summation of local contributions*

$$\int_M x = \sum_{\alpha \in A} \int_{M_\alpha} \frac{x_{|M_\alpha}}{e_\alpha} \,. \tag{2}$$

In particular, when the fixed point set is discrete $M^T = \{p_0, p_1, \dots p_n\}$ then the Euler class is the product of weights

$$e_p = \prod_{\lambda \in \Lambda} \lambda^{\dim(V_\lambda)} \in H_T^*(\{p\}) \in \mathbb{Q}[t_1, t_2, \dots, t_n]\,,$$

provided that $\mathbf{T}_p M$, the tangent space at p is the sum of weight spaces

$$\mathbf{T}_p M = \bigoplus_{\lambda \in \Lambda} V_\lambda \,.$$

The integral along M is equal to the sum of fractions:

$$\int_M a = \sum_{p \in M^T} \frac{a_{|p}}{e_p} \,.$$

Remark 13. The Berline-Vergne formula (2) can be formulated for singular spaces embedded into a smooth manifold. The local factor $\frac{1}{e_\alpha}$ is replaced by $\frac{[X]_{|M_\alpha}}{e_\alpha}$, see [15, 9]. There is a generalization of the Theorem (11) for equivariant homology (or Chow groups) of singular spaces, but one needs an additional assumption allowing to define ι^*, [15, Proposition 6].

4. Some calculi of rational functions

Before examining equivariant Chern classes of Schubert varieties let us look closer at some computations based on the Localization Theorem for Grassmannians. Let us start with the projective space $M = \mathbb{P}^n$ with the standard torus $T = (\mathbb{C}^*)^{n+1}$ action. The fixed point set is discrete and consists of coordinate lines

$$M^T = \{p_0, p_1, \dots p_n\} \,.$$

The tangent space at the point p_k decomposes into one dimensional representations:

$$\mathbf{T}_{p_k} M = \bigoplus_{\ell \neq k} \mathbb{C}_{t_\ell - t_k} \,.$$

The Euler class is equal to

$$e_{p_k} = \prod_{\ell \neq k} (t_\ell - t_k) \,.$$

Let us integrate powers of $c_1 := c_1(\mathcal{O}(1))$. Of course

$$\int_{\mathbb{P}^n} c_1^m = \begin{cases} 0 & \text{for} \quad m < n \\ 1 & \text{for} \quad m = n \end{cases}$$

Applying Berline-Vergne (2) formula we get the identity

$$\sum_{k=0}^{n} \frac{(-t_k)^m}{\prod_{\ell \neq k}(t_\ell - t_k)} = \begin{cases} 0 & \text{for} \quad m < n \\ 1 & \text{for} \quad m = n \,, \end{cases} \tag{3}$$

which is not obvious at the first sight. For example we encourage the reader to compute by hand the sum

$$\frac{t_0^2}{(t_1-t_0)(t_2-t_0)(t_3-t_0)} + \frac{t_1^2}{(t_0-t_1)(t_2-t_1)(t_3-t_1)} + \frac{t_2^2}{(t_0-t_2)(t_1-t_2)(t_3-t_2)} + \frac{t_3^2}{(t_0-t_3)(t_1-t_3)(t_2-t_3)} .$$

This is exactly the expression (3) for $m = 2$, $n = 3$. Replacing

$$t_0 = 0\,, \quad t_1 = 1\,, \quad t_2 = 2\,,\ \dots\,,\ t_n = n$$

(i.e. specializing to a subtorus) the sum (3) is equal to

$$\sum_{k=0}^{n} \frac{(-1)^{m+k} k^m}{k!(n-k)!}$$

Multiplying by $n!$ we obtain

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{m+k} k^m = \begin{cases} 0 & \text{for} \quad m < n \\ n! & \text{for} \quad m = n\,. \end{cases}$$

which is a good exercise for students.

The integral of higher powers of c_1 is even more interesting: Let us see what we do get for $m > n$? For example $n = 2$, $m = 4$ we have

$$\frac{t_0^4}{(t_1 - t_0)(t_2 - t_0)} + \frac{t_1^4}{(t_0 - t_1)(t_2 - t_1)} + \frac{t_2^4}{(t_0 - t_2)(t_1 - t_2)}$$

It takes some time to check that the sum is equal to

$$t_0^2 + t_1^2 + t_2^2 + t_0 t_1 + t_0 t_2 + t_1 t_2, .$$

In terms of the elementary symmetric functions it is equal to

$$\sigma_1^2 - \sigma_2\,.$$

Proposition 14. *In general*

$$(-1)^k \int_{\mathbb{P}^n} c_1^{n+k}$$

is equal to the Schur function S_k (which corresponds to the Segre class of vector bundles).

Proof. By Jacobi-Trudy formula (which is the definition of the Schur function)

$$S_k(t_0, t_1, \dots, t_n) = \frac{\begin{vmatrix} t_0^{n+k} & t_0^{n-1} & t_0^{n-2} & \dots & t_0^1 & 1 \\ t_1^{n+k} & t_1^{n-1} & t_1^{n-2} & \dots & t_1^1 & 1 \\ t_2^{n+k} & t_2^{n-1} & t_2^{n-2} & \dots & t_2^1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ t_n^{n+k} & t_n^{n-1} & t_0^{n-2} & \dots & t_n^1 & 1 \end{vmatrix}}{\prod_{i<j}(t_i - t_j)}\,.$$

To prove the proposition it is enough to use Laplace expansion with respect to the first column and watch carefully the signs. □

Remark 15. It is wiser to use the dual Grassmannian of hyperplanes, then one gets rid of the factor $(-1)^k$. The general formula with positive signs for Grassmannians is given by Theorem 16.

We will have a look now at the calculus on $\mathrm{Grass}_m(\mathbb{C}^n)$. The fixed point set consists of coordinate subspaces:

$$\mathrm{Grass}_m(\mathbb{C}^n)^T = \{p_\lambda \,:\, \lambda = (\lambda_1 < \lambda_2 < \dots < \lambda_m),\ 1 \le \lambda_1,\ \lambda_m \le n\}$$

The tangent space at the fixed point p_λ decomposes into distinct line representations of T:

$$\mathbf{T}_{p_\lambda} \mathrm{Grass}_m(\mathbb{C}^n) = \bigoplus_{k \in \lambda, \ell \notin \lambda} \mathbb{C}_{t_\ell - t_k}\,.$$

The Euler class is equal to

$$e_{p_\lambda} = \prod_{k\in\lambda,\ \ell\notin\lambda} (t_\ell - t_k)\,.$$

Let us integrate a characteristic class of the tautological bundle $\mathcal{R}_m$. Suppose that the class $\phi(\mathcal{R}_m)$ is given by a symmetric polynomial in Chern roots $W(x_1, x_2, \dots, x_m)$. Then

$$\int_{\mathrm{Grass}_m(\mathbb{C}^n)} \phi(\mathcal{R}_m) = \sum_\lambda \frac{W(t_i : i\in\lambda)}{\prod_{k\in\lambda,\ \ell\notin\lambda}(t_\ell - t_k)}$$

It looks like a rational function, but we obtain a polynomial in t_i's of degree $deg(W) - \dim(\mathrm{Grass}_m(\mathbb{C}^n))$. This expression can be written as the iterated residue[1]

$$\frac{1}{m!} Res_{z_1=\infty} Res_{z_2=\infty} \dots Res_{z_m=\infty} \frac{W(z_1, z_2, \dots z_m)\prod_{i\neq j}(z_i - z_j)}{\prod_{i=1}^n\prod_{j=1}^m (t_i - z_j)}\,, \tag{4}$$

see [8]. Of course if $deg(W) < \dim(\mathrm{Grass}_m(\mathbb{C}^n)) = (n-m)m$, then

$$\sum_\lambda \frac{W(t_i : i\in\lambda)}{\prod_{k\in\lambda,\ \ell\notin\lambda}(t_\ell - t_k)} = 0$$

If $deg(W) = \dim(\mathrm{Grass}_m(\mathbb{C}^n))$, then we get a constant. For example for $W = c_1^{\dim(\mathrm{Grass}_m(\mathbb{C}^n))} = \big(-(x_1+x_2+\dots+x_m)\big)^{(n-m)m}$ we obtain the degree of the Plücker embedding $\mathrm{Grass}_m(\mathbb{C}^n) \subset \mathbb{P}(S^m(\mathbb{C}^n))$ (or the volume of $\mathrm{Grass}_m(\mathbb{C}^n)$). According to Hook Formula [19, §4.3]

$$deg(\mathrm{Grass}_m(\mathbb{C}^n)) = \frac{(m(n-m))!}{\prod_{(i,j)\in\lambda} h(i,j)}\,,$$

where $h(i,j)$ denotes the length of the hook with vertex at $(i,j)\in\lambda$ contained in the rectangle $m\times(n-m)$. For $\mathrm{Grass}_3(\mathbb{C}^7)$ the hook lengths are the following

6	5	4	3
5	4	3	2
4	3	2	1

Hence the degree is equal to

$$\frac{12!}{6\cdot 5\cdot 4\cdot 3\cdot 5\cdot 4\cdot 3\cdot 2\cdot 4\cdot 3\cdot 2\cdot 1} = 462.$$

It would be interesting to find an immediate connection of the Hook formula and the residue method given by the formula (4).

Let us now formulate a generalization of Proposition 14. For a partition $I = (i_1 \geq i_2 \geq i_n)$ the Schur function is defined by Jacobi-Trudy formula [28, §I.3]

$$S_I(t_1, t_2, \dots, t_n) = \frac{\begin{vmatrix} t_1^{n-1+i_1} & t_1^{n-2+i_2} & t_1^{n-3+i_3} & \dots & t_1^{1+i_{n-1}} & t_1^{i_n} \\ t_2^{n-1+i_1} & t_2^{n-2+i_2} & t_2^{n-3+i_3} & \dots & t_2^{1+i_{n-1}} & t_2^{i_n} \\ t_3^{n-1+i_1} & t_3^{n-2+i_2} & t_3^{n-3+i_3} & \dots & t_3^{1+i_{n-1}} & t_3^{i_n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ t_n^{n-1+i_1} & t_n^{n-2+i_2} & t_n^{n-3+i_3} & \dots & t_n^{1+i_{n-1}} & t_n^{i_n} \end{vmatrix}}{\prod_{1\leq i<j\leq n}(t_i - t_j)}\,.$$

[1]More general formulas were found recently by Magdalena Zielenkiewicz for Grassmannians of all classical groups.

The definition of Schur function is extended to characteristic classes of vector bundles. Expanding the determinant with respect to the first block column containing $m \times m$ minors we find the formula for push-forward:

Theorem 16. *Consider the quotient bundle $\mathcal{Q}$ and the tautological bundle $\mathcal{R}$ over $\mathrm{Grass}_m(\mathbb{C}^n)$. Let $J = (j_1 \geq j_2 \geq \cdots \geq j_{n-m})$ and $K = (k_1 \geq k_2 \geq \cdots \geq k_m)$ be partitions. Suppose $j_{n-m} - m \geq k_1$. Then*

$$\int_{\mathrm{Grass}_m(\mathbb{C}^n)} S_J(\mathcal{Q}) S_K(\mathcal{R}) = S_I(t_1, t_2, \ldots, t_n)\,,$$

where $I = (j_1 - m \geq j_2 - m \geq \cdots \geq j_{n-m} - m \geq k_1 \geq k_2 \geq \cdots \geq k_m)$.

A suitable modifications of Theorem 16 can be easily formulated for the partitions not satisfying the inequality $j_{n-m} - m \geq k_1$. The integral is equal up a sign to the Schur function for another partition or it is zero. By the splitting principle Theorem 16 implies the corresponding statement for Grassmannian bundles over any base, not necessarily over the classifying space BT. This way we obtain a proof of the Gysin homomorphism formula [24], [20, §4.1].

The equivariant Schubert calculus was studied by a number of authors: Knutson–Tao [26], Laksov–Thorup [27], Gatto–Santiago [21] and others. Some formulas can be obtained by taking residue at infinity [8, 9]. Concluding this section I would like to say that it seems that still the calculus of *rational* symmetric functions is not developed enough. In §10 we will present a method of computation of equivariant Chern classes of Schubert varieties. Unfortunately I do not know (maybe except Theorem 16) a tool which would allow us to simplify the expressions which appear in computation.

5. Toric varieties

We keep in mind that our purpose is to compute equivariant Chern classes. From the Localization Theorem it follows that equivariant Chern classes are determined by local equivariant Chern classes belonging to the homologies of the components of M^T. In the beginning let us consider the toric varieties, which are quite easy, but unfortunately not very general from our point of view.

Theorem 17. *Let X be a toric variety. Consider the cycle Ξ_X which is equal to the sum of the closures of orbits. Then Ξ_X represents the equivariant Chern class $c^T(1\!\!1_X) \in H_*^T(X)$.*

Proof. First we consider the case when X is a smooth toric variety. If $X = \mathbb{C}^1$ with the standard action of $T = \mathbb{C}^*$ then, indeed, the equivariant Chern class is equal $[\mathbb{C}] + [0] = [\Xi_X]$. By Whitney formula and the product property of sets the statement holds for $X = \mathbb{C}^n$ with the standard action of $T = (\mathbb{C}^*)^n$. Every smooth toric variety locally looks like $\mathbb{C}^n$ with the standard action of the torus, therefore the equation $c^T(1_X) = [\Xi_X]$ holds locally, i.e. after restriction to each fixed point. Let X be a complete smooth toric variety. Then $H_*^T(X)$ is free over $H_T^*(pt)$. By the Localization Theorem $c^T(1\!\!1_X) = \Xi_X$ holds globally. The noncomplete case follows since any smooth toric variety can be compactified equivariantly.

The singular case can be deduced as usual by functoriality. One sees that for smooth toric varieties the equivariant Chern class of the constructible function supported by a single orbit is exactly the fundamental class of the closure of that orbit without boundary cycles. The equality is preserved by the push-forward. □

Note that the theorem holds in equivariant homology of X and we do not have to use any embedding into a smooth manifold. The non-equivariant case was proven by Ehlers and Barthel-Brasselet-Fieseler [7] and it also follows immediately from [1].

The cycle representing the equivariant Chern class of a toric variety is effective. Therefore for the embedded case by Theorem 5 we have the corollary:

Corollary 18. *Let V be a representation of T. Suppose an affine T-variety X (possibly singular) is embedded equivariantly into V. If the weights of the torus acting on V are nonnegative then the coefficients of $c^T(1\!\!1_X) \in H_T^*(V) = \mathbb{Q}[t_1, t_2, \dots, t_n]$ are nonnegative.*

The situation described in the Corollary 18 appears when

$$X = X_\sigma = Spec(\mathbb{C}(\sigma^\vee \cap N))$$

is presented in the usual way: the embedding into

$$V = Spec(\mathbb{C}[x_1, x_2, \dots, x_n])$$

is given by a choice of the generators of the semigroup $\sigma^\vee \cap N$, see [17, §1.3].

Remark 19. All the singularities of the Schubert varieties in Grassmannians of planes $Grass_2(\mathbb{C}^n)$ are toric. Therefore the local equivariant Chern classes are nonnegative combinations of monomials for a suitable choice of a basis of $H^2_T(pt)$.

The global positivity of Chern classes of Schubert varieties in $Grass_2(\mathbb{C}^n)$ seems not to follow automatically. Except from the case of projective spaces only the Schubert varieties with isolated singularities (the partitions $(n-3,k)$ for $k \le n-4$, according to the standard convention) are toric. Nevertheless it was shown in [5, §4.3] that the nonequivariant Chern classes of Schubert varieties in $Grass_2(\mathbb{C}^n)$ are indeed effective.

6. EQUIVARIANT CHERN CLASS OF DEGREE ZERO

The following Theorem 20 is the key to the inductive procedure for computing equivariant Chern classes. The theorem says that the degree zero component of the equivariant Chern class localized at a fixed point does not depend seriously on the set itself, but only on wether the point belongs to the set or not.

Theorem 20. *Suppose that X is a T-variety, not necessarily smooth, contained in a T-manifold M. Let $p \in X$ be an isolated fixed point. Then the degree zero component of the class $c^T(1\!\!1_X)$ restricted to $\{p\}$ is Poincaré dual to the product of weights appearing in the tangent representation $\mathbf{T}_pM$*

$$\left(c^T(1\!\!1_X)_{(0)}\right)_{|p} = e_p \cap [p]\,.$$

By additivity of equivariant Chern classes it follows that if $p \notin X$ then $(c^T(1\!\!1_X)_{(0)})_{|p}) = 0$.

The core of the proof is the basic equation of Euler characteristics

$$\chi(X) = \chi(X^T)\,.$$

Nevertheless the argument demands some formal manipulations. First of all we note the following fact.

Proposition 21. *Let N be a complete manifold with a torus action. Let us decompose the fixed point set $N^T = \sqcup_{\alpha\in A} N_\alpha$ into connected components. Let $i_\alpha : N_\alpha \to N$ be the inclusion. The equivariant cohomology top Chern class of N is equal to the sum*

$$c^T_{top}(N) = \sum_{\alpha\in A} (i_\alpha)_* \left(c_{top}(N_\alpha)\right) \in H_T^{2\dim(N)}(N)\,. \tag{5}$$

Dually we have

$$c^T(1\!\!1_N)_{(0)} = \sum_{\alpha\in A} (i_\alpha)_* \left(c_{SM}(1\!\!1_{N_\alpha})_{(0)}\right) \in H_0^T(N)\,. \tag{6}$$

Proof. The proof is the straightforward application of the Theorem (11) since

$$i^*_\alpha(c^T_{top}(N)) = e_\alpha \cdot c_{top}(N_\alpha) \in H^*_T(N_\alpha) = H^*_T(pt) \otimes H^*(N_\alpha)\,.$$

□

Proof of Theorem 20. Denote by $\iota_p : \{p\} \to M$ the inclusion of the point. We will argue that for any equivariant constructible function $\alpha : M \to \mathbb{Z}$ the equality holds

$$\iota^*_p(c^T(\alpha)) = \alpha(p)\, e_p \cap [p] \in H^T_{-2\dim(M)}(\{p\})\,.$$

It is enough to show that statement for M complete and the constructible function of the shape $\alpha = f_*(1\!\!1_N)$ for an equivariant map $f : N \to M$ from a smooth complete variety N. (We can assume that N is smooth by the usual argument which is available thanks to equivariant completion [38] and equivariant resolution of singularities [11].) It remains to prove that

$$\iota^*_p f_*(c^T(1\!\!1_N)_{(0)}) = \chi(f^{-1}(p))\, e_p \cap [p] \in H^T_{-2\dim(M)}(\{p\})\,. \tag{7}$$

Let i_α be as in Proposition 21 and $f_\alpha = f\, i_\alpha : N_\alpha \to M$. We compute the push-forward of the zero degree component:

$$\begin{aligned} f_* c^T(1\!\!1_N)_{(0)} &= \sum_{\alpha\in A} f_*(i_\alpha)_*\big(c_{SM}(1\!\!1_{N_\alpha})_{(0)}\big) \\ &= \sum_{\alpha\in A} (f_\alpha)_*\left(c_{SM}(1\!\!1_{N_\alpha})_{(0)}\right) \in H^T_0(M)\,. \end{aligned} \tag{8}$$

Let $B \subset A$ be the set of components of N^T which are mapped to p. Then

$$\begin{aligned} \iota^*_p\big(f_* c^T(1\!\!1_N)_{(0)}\big) &= \iota^*_p\big(\sum_{\beta\in B} (f_\beta)_* c(1\!\!1_{N_\beta})_{(0)}\big) \\ &= \iota^*_p \sum_{\beta\in B} \chi(N_\beta)\,[p] \\ &= \sum_{\beta\in B} \chi(N_\beta)\, e_p \cap [p]\,. \end{aligned} \tag{9}$$

We conclude that the equation (7) holds because $\chi(f^{-1}(p)) = \chi(f^{-1}(p)^T)$ and $f^{-1}(p)^T = \bigsqcup_{\beta\in B} N_\beta$.
□

7. Partial localization

There exists the following modification of the localization formula: we can replace M^T by any invariant submanifold or even arbitrary invariant subset) Y containing the fixed point set M^T. Then the restriction map

$$H^*_T(M) \to H^*_T(Y)$$

becomes an isomorphism after inversion of nontrivial characters S. Also the Berline-Vergne formula holds, but it makes sense only for Y being a submanifold. Suppose that $Y = Y_1 \sqcup \{p\}$. It follows that for any $x \in H^*_T(M)$ we have

$$\frac{x_{|p}}{e_p} + \int_{Y_1} \frac{x_{|Y_1}}{e_{Y_1}} = 0 \tag{10}$$

for degree smaller than $\dim(M)$. We will apply this formula for Poincaré dual of $c^T(1\!\!1_X)$. The integral of the zero degree Chern-Schwartz-MacPherson class (which corresponds to the top degree of the cohomology class) is equal to the Euler characteristic and the same holds for the equivariant Chern class by the commutativity of the diagram:

$$\begin{array}{ccccccc} c^T(\mathbb{1}_X)_{(0)} & \in & H_0^T(M) & \longrightarrow & H_0(M) & \ni & c(X)_{(0)} \\ & & \downarrow & & \downarrow & & \\ \int_M c^T(\mathbb{1}_X)_{(0)} & \in & H_0^T(pt) & \xrightarrow{\simeq} & H_0(pt) & \ni & \chi(X)\,. \end{array}$$

We apply the partial localization and we find that

$$\frac{(c^T(X)_{(0)})_{|p}}{e_p} + \int_{Y_1} \frac{(c^T(X)_{(0)})_{|Y_1}}{e_{Y_1}} = \chi(X)\,. \tag{11}$$

Here e_{Y_1} is the equivariant Euler class of the normal bundle of Y_1. (Of course it may be of different degrees over distinct components of Y_1.)

Example 22. The partial localization allows us to compute the equivariant Chern class of the affine cone over a projective variety. Suppose T acts on $\mathbb{C}^n$ with nonzero weights

$$w_1, w_2, \dots, w_n\,.$$

First recall that the equivariant cohomology ring of $\mathbb{P}^{n-1}$ is the quotient of the polynomial algebra

$$H_T^*(pt)[h] = \mathbb{Z}[t_1, t_2, \dots, t_n, h]$$

by the relation

$$\prod_{i=1}^n (h + w_i) = 0.$$

Using the elementary symmetric functions σ_i the relation takes form

$$\sum_{i=0}^n \sigma_i(w_\bullet) h^{n-i} = 0\,. \tag{12}$$

Let $X \subset \mathbb{C}^n$ be a nonempty T-invariant cone and $\mathbb{P}(X) \subset \mathbb{P}^{n-1}$ its projectivization. We consider $X = \overline{X} - \mathbb{P}(X)$ as a constructible set in $\mathbb{P}^n$ and we will compute its equivariant Chern class in $H_T^*(\mathbb{P}^n)$. In this example we skip the Poincaré duals in the notation. Denote by $\iota : \mathbb{P}^{n-1} \to \mathbb{P}^n$ the inclusion. The equivariant Chern class of X restricted to $\mathbb{P}^{n-1}$ is equal to

$$\begin{aligned} \iota^* c^T(\mathbb{1}_X) &= \iota^* c^T(\mathbb{1}_{\overline{X}}) - \iota^*\iota_* c^T(\mathbb{1}_{\mathbb{P}(X)}) \\ &= (1+h)\cdot c^T(\mathbb{1}_{\mathbb{P}(X)}) - h\cdot c^T(\mathbb{1}_{\mathbb{P}(X)}) \\ &= c^T(\mathbb{1}_{\mathbb{P}(X)})\,. \end{aligned}$$

Suppose that the equivariant Chern class of $\mathbb{P}(X)$ is written as

$$c^T(\mathbb{1}_{\mathbb{P}(X)}) = \sum_{i=0}^{n-1} b_i(t) h^i \in H_T^*(\mathbb{P}^{n-1})$$

for some polynomials $b_i(t) \in H_T^*(pt)$ of degree $\leq n-1-i$. To compute the local equivariant Chern class at 0 we will apply the formulas (10) and (11) to $M = \mathbb{P}^n$, $Y = \{0\} \cup \mathbb{P}^{n-1}$ and $Y_1 = \mathbb{P}^{n-1}$. We compute

$$\int_{\mathbb{P}^{n-1}} \frac{c^T(\mathbb{1}_X)}{e_{Y_1}} = \int_{\mathbb{P}^{n-1}} \sum_{i=0}^{n-1} b_i(t) h^{i-1}\,.$$

Except from $i = 0$ the summands are integral (belong to $H_T^*(\mathbb{P}^{n-1})$) and they are of degree smaller than $n-1$. Therefore

$$\int_{\mathbb{P}^{n-1}} \frac{c^T(1\!\!1_X)}{e_{Y_1}} = \int_{\mathbb{P}^{n-1}} \frac{b_0(t)}{h}\,.$$

An easy calculation using (12) shows that the inverse Euler class of the normal bundle to $\mathbb{P}^{n-1}$ is equal to

$$h^{-1} = -\sum_{i=1}^{n} \frac{\sigma_{n-i}(w_\bullet)}{\sigma_n(w_\bullet)} h^{i-1}\,.$$

Hence

$$\int_{\mathbb{P}^{n-1}} \frac{c^T(1\!\!1_X)}{e_{Y_1}} = -\int_{\mathbb{P}^{n-1}} b_0(t) \sum_{i=1}^{n} \frac{\sigma_{n-i}(w_\bullet)}{\sigma_n(w_\bullet)} h^{i-1} = -\frac{b_0(t)}{\sigma_n(w_\bullet)}$$

By the formulas (10) and (11) and since $\sigma_n(w_\bullet) = e_p$ we find that

$$\frac{c^T(1\!\!1_X)_{|p}}{e_p} - \frac{b_0(t)}{e_p} = \chi(X) = 1\,.$$

Therefore

$$c^T(1\!\!1_X)_{|p} = b_0(t) + e_0 \in H_T^*(\{p\})$$

We obtain the following result:

Proposition 23. *Suppose that X is a nonempty T-invariant cone in a linear representation V of T. Let $h = c_1^T(\mathcal{O}_{\mathbb{P}(V)}(1))$ be the equivariant Chern class. If*

$$c^T(1\!\!1_{\mathbb{P}(X)}) = \Big(\sum_{i=0}^{\dim(V)-1} b_i(t)h^i\Big) \cap [\mathbb{P}(V)] \in H_*^T(\mathbb{P}(V)) \tag{13}$$

then

$$c^T(1\!\!1_X) = (b_0(t) + e_0) \cap [V] \in H_*^T(V)$$

where e_0 is the Euler class of the representation V.

Proof. First note, that restriction $H_T^*(V) \to H_T^*(pt)$ is an isomorphism. We apply the calculation of the previous example. The degree of b_0 is at most $\dim(V) - 1$, therefore it does not interfere with e_p, which is homogeneous of degree $\dim(V)$. □

8. Conical sets in an affine space

We come back to the Example 8 of §2 which was the starting point of our interest in equivariant Chern classes. In [4] there was defined an invariant of a conical set $X \subset \mathbb{C}^n$. It is equal to the Chern-Schwartz-MacPherson class of X considered as a constructible set in $\mathbb{P}^n$. This Chern class

$$c_{SM}(1\!\!1_X) \in H^*(\mathbb{P}^n)$$

is expressed via the Chern class of the projectivization. The calculation is based on the following formula:

Proposition 24 ([3, Prop 5.2])**.** *Let $X \subset \mathbb{C}^n$ be a nonempty conical set. Let $\overline{X} = X \cup \mathbb{P}(X)$ be the closure of X in $\mathbb{P}^n$. Let $x = c_1(\mathcal{O}_{\mathbb{P}^{n-1}}(1))$ and $\tilde{x} = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$. Suppose that*

$$c_{SM}(1\!\!1_{\mathbb{P}(X)}) = \Big(\sum_{i=0}^{n-1} a_i x^i\Big) \cap [\mathbb{P}^{n-1}] \in H_*(\mathbb{P}^{n-1}) \tag{14}$$

then

$$c_{SM}(1\!\!1_{\overline{X}}) = \Big((1+\tilde{x})\big(\sum_{i=0}^{n-1} a_i \tilde{x}^i\big) + \tilde{x}^n\Big) \cap [\mathbb{P}^n] \in H_*(\mathbb{P}^n) \tag{15}$$

It follows that

$$c_{SM}(1\!\!1_X) = \Big(\sum_{i=0}^{n-1} a_i \tilde{x}^i + \tilde{x}^n\Big) \cap [\mathbb{P}^n] \in H_*(\mathbb{P}^n) \tag{16}$$

It seems natural to look at the conical sets from the point of view of equivariant cohomology. Let $T = \mathbb{C}^*$ acts on $\mathbb{C}^n$ by scalar multiplication.

Proposition 25. *Under assumption of Proposition 24*

$$c^T(1\!\!1_X) = \Big(\sum_{i=0}^{n-1} a_i t^i + t^n\Big) \cap [\mathbb{C}^n] \in H_*^T(\mathbb{C}^n) \tag{17}$$

Proof. Assume that the usual, nonequivariant Chern class of $\mathbb{P}(X)$ satisfies the formula (14). To apply Proposition 23 we have to express the equivariant Chern class $c^T(1\!\!1_{\mathbb{P}(X)}) = 1 \otimes c_{SM}(1\!\!1_{\mathbb{P}(X)})$ by $h = c_1^T(\mathcal{O}_{\mathbb{P}(X)}(1))$. The point is that the torus T acts on the fibers of the tautological bundle $\mathcal{O}(-1)$ with weight equal to one, therefore the equivariant Chern class of $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ is equal to

$$h = 1 \otimes x - t \otimes 1 = x - t$$

under the identification

$$H_T^*(\mathbb{P}^{n-1}) = \mathbb{Q}[\,t\,] \otimes H^*(\mathbb{P}^{n-1}) = H^*(\mathbb{P}^{n-1})[\,t\,]\,.$$

Hence the equivariant Chern class of $1\!\!1_{\mathbb{P}(X)}$ can be written as

$$c^T(1\!\!1_{\mathbb{P}(X)}) = \Big(\sum_{i=0}^{n-1} a_i (h+t)^i\Big) \cap [\mathbb{P}^{n-1}] = \Big(\sum_{i=0}^{n-1} \sum_{j=0}^{i} \binom{i}{j} a_i t^{i-j} h^j\Big) \cap [\mathbb{P}^{n-1}]\,.$$

Here the coefficient $b_0(t)$ of the expression (13) is equal to

$$b_0(t) = \sum_{i=0}^{n-1} a_i t^i\,.$$

By Proposition 23 we obtain the claim. □

We see that formally the Chern-MacPherson-Schwartz class of $1\!\!1_X$ in $\mathbb{P}^n$ and the equivariant Chern class in $\mathbb{C}^n$ satisfy the same formula. The equivariant approach has the advantage that we have for free the Chern class of the product

$$c^{T\times T}(1\!\!1_{X\times Y}) = c^T(1\!\!1_X) \times c^T(1\!\!1_Y)\,.$$

Further we can restrict the Chern class of the product via diagonal inclusion $T \hookrightarrow T \times T$ to obtain $c^T(1\!\!1_{X\times Y})$. With the original approach the proof of the above property was a bit demanding, see [4, Lemma 3.10]

9. Computing equivariant Chern classes without resolution of singularities

Below we sketch a method of computing the equivariant Chern class of a T-invariant singular variety not using a resolution of singularities. The calculi will be done in equivariant cohomology and we will omit the Poincaré duality in the notation.

Assume that the fixed point set of the action of the torus on a complex manifold M is discrete. For a given class $x \in H_T^k(M)$ of degree $k < 2\dim(M)$ the integral $\int_M x$ vanishes. By the Localization

Theorem also the sum $\sum_{p\in M^T} \frac{x_{|p}}{e_p}$ has to vanish. In particular if $x = c^T(\mathbb{1}_X)$, then except from the zero degree

$$\sum_{p\in M^T} \frac{c^T(\mathbb{1}_X)_{|p}}{e_p} = 0\,. \tag{18}$$

This relation between local equivariant Chern classes allows in many cases to compute them inductively. Suppose $M^T = \{p_0, p_1, \ldots, p_N\}$ and assume that we know all local equivariant Chern classes for $p_1, p_2, \ldots, p_N$. Then

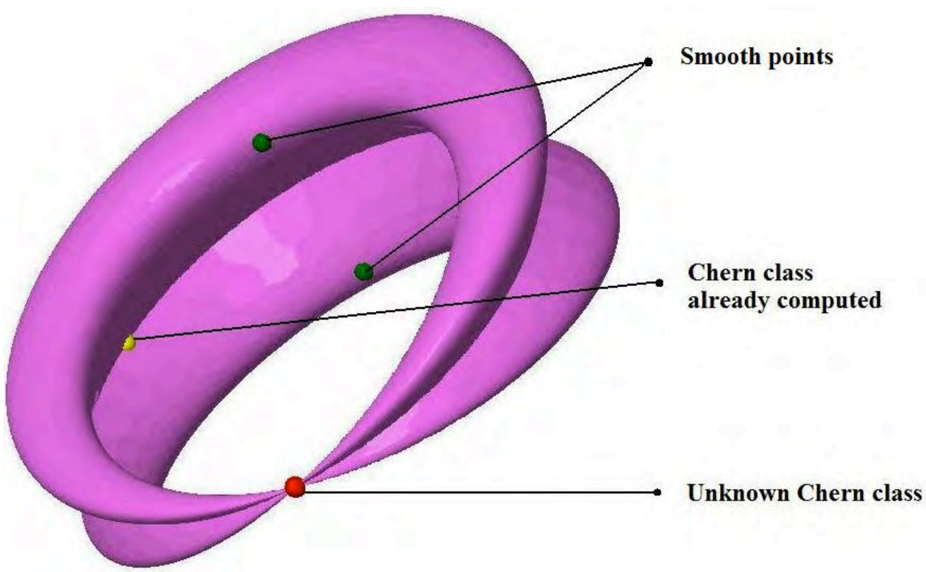
$$c^T(\mathbb{1}_X)_{p_0} = -\sum_{i=1}^{N} \frac{e_{p_0}}{e_{p_i}} c^T(\mathbb{1}_X)_{p_i} \tag{19}$$

except from the zero degree. For Grassmannians the quotient $\frac{e_{p_0}}{e_{p_i}}$ simplifies remarkably.

The zero component of the local equivariant Chern class is easy. If $p \in X^T$ then by Theorem 20 this class is equal to the Euler class at the point p

$$(c^T(\mathbb{1}_X)_{(0)})_p = e_p \in H_T^{2\,\dim(M)}(pt). \tag{20}$$

In fact this statement is the crucial point for computation. Any other equivariant characteristic class satisfies the relation 19. The condition fixing the zero equivariant Chern class and vanishing for the degrees higher than the dimension of the ambient space makes the equivariant Chern class unique.



Computation of the local equivariant Chern classes

Of course the inductive step of computation can be applied when for a given singularity one can find a compact variety for which this singularity is the only deepest one. If $X \subset \mathbb{C}^m$ is a cone then taking the closure of X in $\mathbb{P}^n$ will not introduce new singularities. In the next section we present another situation, when the compactifying variety is the Grassmannian.

10. Computation of local equivariant Chern class of the determinant variety

Let us compute the local equivariant Chern class of the variety

$$\Omega_1^o(n) = \{\phi \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n) \,:\, \det(\phi) = 0\}\,.$$

Its compactification in $Grass_n(\mathbb{C}^{2n})$ is the Schubert variety of codimension one

$$\Omega_1(n) = \{W \,:\, W \cap \langle \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \rangle \neq 0\}\,.$$

We will apply the method sketched above. Let us start with $n = 2$. The canonical neighborhood of the point $p_{1,2}$ in $\mathrm{Grass}_2(\mathbb{C}^4)$ is identified with

$$\mathrm{Hom}(span(\varepsilon_1, \varepsilon_2), span(\varepsilon_3, \varepsilon_4))$$

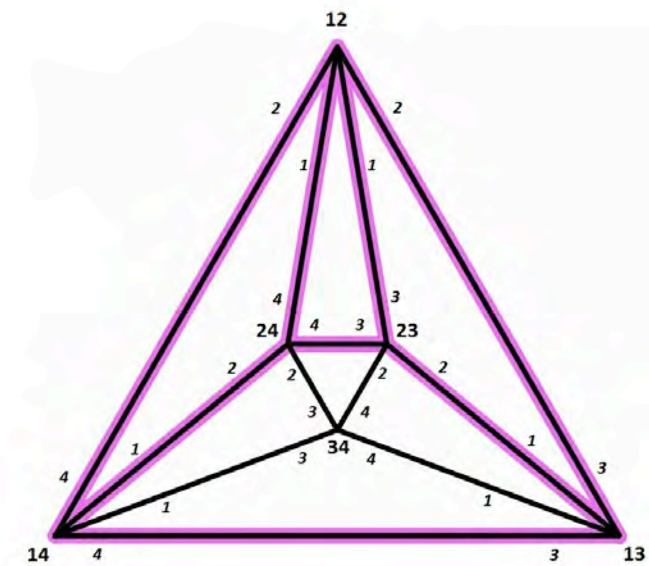
and the variety $\Omega_1(2)$ intersected with this neighbourhood is exactly $\Omega_1^o(2)$. The corresponding elements of $\Omega_1(2)$ are the planes spanned by the row-vectors of the matrix

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}\,.$$

The equation of $\Omega_1(2)$ is

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0\,.$$

Before performing computations let us draw the Goresky-Kottwitz-MacPherson graph ([22, Th. 7.2]) for $M = \mathrm{Grass}_2(\mathbb{C}^4)$ with the variety $\Omega_1(2)$ displayed.



Schubert variety Ω_1 in $\mathrm{Grass}_2(\mathbb{C}^4)$.

The numbers attached to the edges indicate the weights of the T actions along the one dimensional orbits. For example at the point $p_{1,3}$ in the direction towards $p_{1,2}$ the action is by the character $t_2 - t_3$. The variety $\Omega_1(2)$ is singular at the point $p_{1,2}$ and it is smooth at the remaining points. For example at the point $p_{1,3}$ the coordinates are

$$\begin{pmatrix} 1 & a & 0 & b \\ 0 & c & 1 & d \end{pmatrix}$$

and the equation of $\Omega_1(2)$ is $b=0$. For that point the local equivariant Chern class is equal to

$$(t_4-t_1)(1+t_2-t_1)(1+t_2-t_3)(1+t_4-t_3)\,.$$

The summand in the formula (18) is the following

$$\begin{aligned}&\frac{(t_4-t_1)(1+t_2-t_1)(1+t_2-t_3)(1+t_4-t_3)}{(t_4-t_1)(t_2-t_1)(t_2-t_3)(t_4-t_3)}=\\&=\left(1+\frac{1}{t_2-t_1}\right)\left(1+\frac{1}{t_2-t_3}\right)\left(1+\frac{1}{t_4-t_3}\right)\end{aligned}$$

We sum up the contribution coming from the fixed points $p_{1,3}$, $p_{1,4}$, $p_{2,3}$, $p_{2,4}$, simplify and multiply by $-(t_3-t_1)(t_4-t_1)(t_3-t_2)(t_4-t_2)$. We obtain

$$\begin{array}{ll}(t_3+t_4-t_1-t_2) & deg=1\\ (t_3+t_4-t_1-t_2)^2 & deg=2\\ (t_3+t_4-t_1-t_2)(2t_1t_2-t_1t_3-t_2t_3-t_1t_4-t_2t_4+2t_3t_4) & deg=3\\ -4(t_3-t_1)(t_4-t_1)(t_3-t_2)(t_4-t_2) & deg=4\end{array}$$

The terms of degree <4 coincide with the equivariant Chern class of $\Omega_1(2)$ localized at the point $p_{1,2}$. The result is symmetric in two groups of variables: $\{t_1,t_2\}$ and $\{t_3,t_4\}$. The coefficients of the expansion in the basis of the Schur functions

$$c^T(1\!\!1_{\Omega_1})=\sum a_{\mathrm{I,J}}S_{\mathrm{I}}(-t_1,-t_2)\cdot S_{\mathrm{J}}(t_3,t_4)$$

has the following coefficients:

	0	*1*	*11*	*2*	*21*	*22*
0		1	1	1	2	1
1	1	1	3	1	1	
11	1	3		1		
2	1	1	1			
21	2	1				
22	1					

Computations of the equivariant Chern class $\Omega_1(3)\subset \mathrm{Grass}_3(\mathbb{C}^6)$ can be continued without problems by the same method. At the points of the type p_I with $|I\cap\{1,2,3\}|=1$ the variety is smooth, while at the points p_I with $|I\cap\{1,2,3\}|=2$ the singularity is of the type $\Omega_1(2)_{p_{1,2}}\times\mathbb{C}^5$. We write the sum of fractions according to the rule (19) and simplify. For example the expression which has to be simplified to compute the degree one is the following:

$$\begin{aligned}&-\frac{(s_3-t_1)(s_3-t_2)(s_1-t_3)(s_2-t_3)}{(s_3-s_1)(s_3-s_2)(t_1-t_3)(t_2-t_3)}\,(s_3-t_3)\;+\;sym.+\\&\frac{(s_3-t_1)(s_3-t_2)(s_1-t_3)(s_2-t_3)}{(s_1-s_3)(s_2-s_3)(t_3-t_1)(t_3-t_2)}\,(s_1+s_2-t_1-t_2)\;+\;sym.\end{aligned}\tag{21}$$

(Here $s_1=t_4,\ s_2=t_5,\ s_3=t_6$.) The given summands are the contributions coming from the points $p_{3,4,5}$ and $p_{1,2,6}$. Of course the sum is equal to the fundamental class

$$[\Omega_1]=s_1+s_2+s_3-t_1-t_2-t_3$$

(which may be computed in another way). This example shows how a complicated rational functions may in fact lead to a simple result. The difficulty lies in simplifying that expression. Higher degree terms are much more complex. We write the final result in the Schur basis

$$c^T(1\!\!1_{\Omega_1})=\sum a_{\mathrm{I,J}}S_{\mathrm{I}}(-t_1,-t_2,-t_3)\cdot S_{\mathrm{J}}(s_1,s_2,s_3)\,.$$

The coefficients are the following:

	0	*1*	*11*	*2*	*111*	*21*	*3*	*211*	*31*	*22*	*311*	*221*	*32*	*321*	*222*	*33*	*331*	*322*	*332*	*333*
0		1	2	2	4	5	1	9	3	4	6	9	3	8	4	1	3	6	3	1
1	1	4	8	5	12	12	2	19	5	8	8	16	4	8	10	1	2	4	1	
11	2	8	12	9	16	16	3	18	6	8	6	10	4	4		1	1			
2	2	5	9	4	11	9	1	13	2	5	3	10	1	2	5			1		
111	4	12	16	11	8	16	4		6	10			4			1				
21	5	12	16	9	16	14	2	15	3	5	3	5	1	1						
3	1	2	3	1	4	2		3		1		2			1					
211	9	19	18	13		15	3		3	5			1							
31	3	5	6	2	6	3		3		1		1								
22	4	8	8	5	10	5	1	5	1		1									
311	6	8	6	3		3				1										
221	9	16	10	10		5	2		1											
32	3	4	4	1	4	1		1												
321	8	8	4	2		1														
222	4	10		5			1													
33	1	1	1		1															
331	3	2	1																	
322	6	4		1																
332	3	1																		
333	1																			

We note that all the coefficients are nonnegative.

While computing the equivariant Chern class of $\Omega_1(4) \subset \mathrm{Grass}_4(\mathbb{C}^8)$ appears a problem with the size of the expressions, since $\dim(\mathrm{Grass}_4(\mathbb{C}^8)) = 16$ and $\dim(T) = 8$. In a polynomial of degree 15 in 8 variables there are

$$490\,314 \text{ monomials.}$$

The expression is a sums of 68 fractions with factors $t_i - t_j$ in denominators. We might have used another compactification of $\mathbb{C}^{16}$, e.g. the projective space $\mathbb{P}^{16}$. There are less fixed points, but the denominators are more complicated. They are of the form $\prod[(t_i - t_j) - (t_k - t_\ell)]$.

One practical solution appears naturally. The fixed points can be divided into groups with $|I \cap \{1, 2, \ldots, n\}|$ fixed. Let $f_k(u_\bullet, v_\bullet)$ be the expression for the local equivariant Chern class of $\Omega_1(k)$ with $u_\bullet = (t_1, t_2, \ldots, t_k)$ and $v_\bullet = (t_{k+1}, t_{k+2}, \ldots, t_{2k})$. The local equivariant Chern $c^T(1\!\!1_{\Omega_1(n)})_{|p_{1,2,\ldots,n}}$ class can by computed by the formula (19), which becomes

$$-\sum_{k=1}^{n-1} \sum_{\substack{I \subset \{1,2,\ldots,2n\} \\ |I| = n,\ |I \cap \{1,2,\ldots,n\}| = k}} \frac{e_{p_{1,2,\ldots,n}}}{e_{p_I}} f_k(I)\, g_k(I)\,, \tag{22}$$

where $f_k(I)$ depends on the two group of variables

$$u_\bullet = t_{I \cap \{1,2,\ldots,n\}} \quad \text{and} \quad v_\bullet = t_{\{n+1,n+2,\ldots,2n\} \setminus I}$$

and $g_k(I)$ is the equivariant Chern class of the singular stratum of the type $\Omega_1(k)$. The factors in the quotients $\frac{e_{p_{1,2,\ldots,n}}}{e_{p_I}}$ cancel out partially and miraculously all the summands for a fixed k turn out to be integral. For $n = 3$ and degree one the summands are given by the formula (21).

Such a division of fixed points has a geometric meaning. In fact we deal with the partial localization (see §7). Consider the action of the subtorus $\mathbb{C}^*$ acting on $\mathbb{C}^{2n}$ with weight 1 on the first n coordinates and with the weight -1 on the remaining coordinates. Then the fixed point set decomposes into disjoint union of the products of the Grassmannians:

$$\mathrm{Grass}_n(\mathbb{C}^{2n})^{\mathbb{C}^*} = \bigsqcup_{k=0}^{n} \mathrm{Grass}_k(\mathbb{C}^n) \times \mathrm{Grass}_{n-k}(\mathbb{C}^n).$$

The summand for $k = 0$ consists of one point

$$\{0\} \oplus \langle \varepsilon_{n+1}, \varepsilon_{n+2}, \dots, \varepsilon_{2n} \rangle\,,$$

which does not belong to $\Omega_1(n)$, while for $k = n$ we have

$$\langle \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \rangle \oplus \{0\}\,,$$

the point which we are concerned with. Let $\mathcal{R}_k$ and $\mathcal{Q}_k$ be the tautological and the quotient bundles over $\mathrm{Grass}_k(\mathbb{C}^n)$. The result of the sum (22) is equal to

$$-(-1)^{n-k} \sum_{k=1}^{n-1} \int_{\mathrm{Grass}_k(\mathbb{C}^n)\times \mathrm{Grass}_{n-k}(\mathbb{C}^n)} \left[f_k(\mathcal{R}_k, \mathcal{Q}_{n-k}) \cdot \overline{g}_k \right]\,,$$

where $\overline{g}_k$ is (up to multiplication by a certain Euler class) the equivariant Chern class of the stratum of the singularity type $\Omega_1(k)$. Precisely

$$\overline{g}_k = e(\mathcal{Q}_k^* \otimes \mathcal{Q}_{n-k}) \cdot e(\mathcal{R}_k^* \otimes \mathcal{R}_{n-k}) \cdot$$
$$\cdot c(\mathcal{R}_k^* \otimes \mathcal{Q}_k) \cdot c(\mathcal{R}_{n-k}^* \otimes \mathcal{Q}_{n-k}) \cdot c(\mathcal{Q}_k \otimes \mathcal{R}_{n-k}^*)\,.$$

Using Fubini theorem we do not have to simplify a large expression in one step and we arrive to the result relatively quickly. Also knowing the Schur expansion of the functions f_k one can apply Theorem 16.

11. GKM-relations

Less time-consuming method of computation of the local equivariant Chern class is based on the relation discovered by Chen-Skjelbred [13], called GKM-relations after the rediscovery in [22]. These relation allow us to determine the local equivariant Chern class at the point p_I knowing only the local equivariant Chern classes at the neighbouring points in the GKM-graph. This is so since

$$c^T(\mathbb{1}_V)_{|p_{\mathrm{I}}} \equiv c^T(\mathbb{1}_V)_{|p_{\mathrm{J}}} \text{ modulo } (t_i - t_j)\,,$$

whenever

$$\mathrm{J} = (\mathrm{I} - \{i\}) \cup \{j\}\,.$$

Again the method works for all the degrees smaller then the dimension of the Grassmannian, since the intersection of the ideals $(t_i - t_j)$ is contained in the degree greater or equal to the dimension of the Grassmannian. That is so for any GKM-space. Now the problem of simplifying huge rational function is replaced by solving a relatively small system of linear equations.

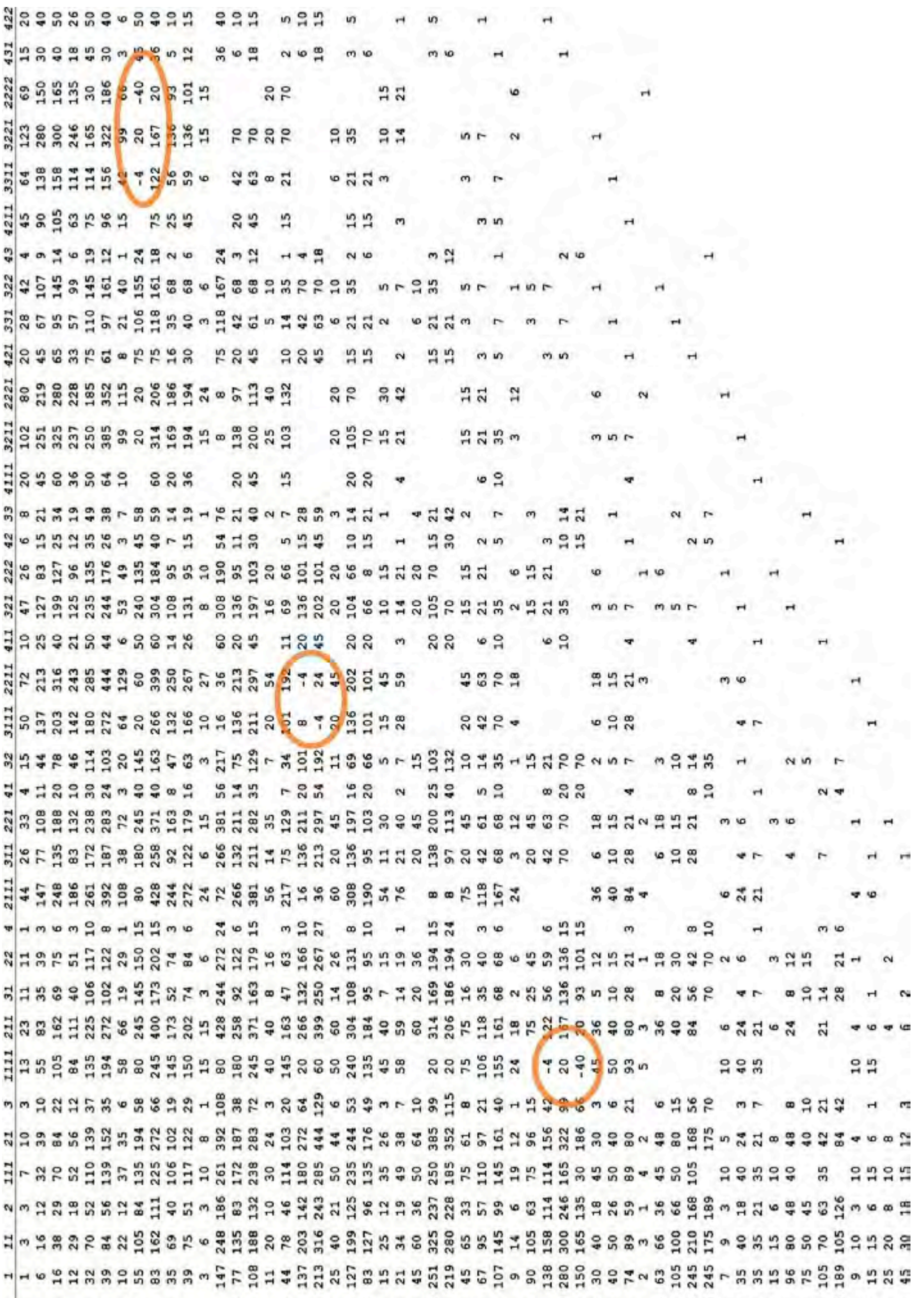
12. The result for $\mathrm{Grass}_4(\mathbb{C}^8)$

Let us write the local equivariant Chern class in the Schur basis

$$c^T(\mathbb{1}_{\Omega_1(4)})_{|p_{1,2,3,4}} = \sum a_{I,J} S_{\mathrm{I}}(-t_1, -t_2, -t_3, -t_4) \cdot S_{\mathrm{J}}(t_5, t_6, t_7, t_8)\,.$$

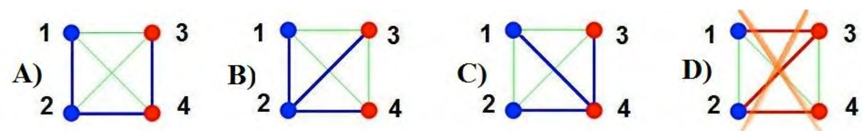
Just to quench readers curiosity we show here the most interesting fragment of the table of coefficients.

It is hard not to have impression that there should be a way of writing down this equivariant Chern class in a compact way. For example the equivariant Chern class of the tangent bundle written in the Schur basis is as much complicated as ours, but it is just $c(\mathrm{Hom}(\mathcal{R}_n, \mathcal{Q}_n))$.



It turns out that the local equivariant Chern class of $\Omega_1(4)$ is a positive combination of monomials in $-t_1, -t_2, -t_3, -t_4, t_5, t_6, t_7, t_8$. As one can see it is *not* a positive combinations of products of Schur functions. Fortunately we do not have a contradiction with the conjecture of Aluffi and Mihalcea [5] which says that the Chern-Schwartz-MacPherson classes are effective. Note that the Schubert varieties are only T-invariant, and the Theorem 4 does not apply. Instead we have a freedom with choosing the basis of weights. The local equivariant Chern class is a polynomial in $u_{i,j} = t_i - t_j$. To write $c^T(1\!\!1_X)$ in a unique way we chose a spanning tree of the full graph with vertices $1, 2, \ldots, 2n$. The edge between i and j (with the orientation forced by the partition) corresponds to the generator $t_j - t_i$. Some choices lead to an expression with nonnegative coefficients.

Positive monomial bases for $\mathrm{Grass}_2(\mathbb{C}^4)$



A) $t_2 - t_1, \quad t_4 - t_2, \quad t_3 - t_4$
B) $t_2 - t_1, \quad t_3 - t_2, \quad t_4 - t_2$
C) $t_4 - t_1, \quad t_4 - t_2, \quad t_4 - t_3$
D) $t_3 - t_1, \quad t_3 - t_2, \quad t_4 - t_2$ this is not a positive basis

The positivity condition for a graph is the following:

- Characters of the tangent representation are nonnegative sums of base elements.

That in fact supports the conjecture of Aluffi and Mihalcea in a stronger, equivariant version.

The original, nonequivariant version was checked by B. Jones [23] for cells in $\mathrm{Grass}_m(\mathbb{C}^n)$ for $m \leq 3$. In his computations equivariant cohomology and the Localization Theorem was used to compute the push-forward of classes from a resolution.

13. Further directions of work

Several goals have not been reached so far. The most obvious directions of further work would be:

- deduce positivity results,
- study global equivariant Chern classes of Schubert varieties and open cells,
- in particular relate our computations to the determinant formulas of [5] and the combinatorial interpretation of [30],
- develop a suitable calculus of symmetric rational functions to handle expressions appearing in the Berline-Vergne formula for Grassmannians.

We hope to realize this program in future.

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