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CLASSIFICATION AT INFINITY OF POLYNOMIALS OF DEGREE 3 IN 3 VARIABLES

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Abstract. We classify singularities at infinity of polynomials of degree 3 in 3 variables. Based on this classification, we calculate the jump in the Milnor number of an isolated singularity at infinity, when we pass from the special fiber to a generic fiber. As an application of the results, we investigate the existence of global fibrations of degree 3 polynomials in complex affine 3-space and search for information about the topology of the fibers in each equivalence class.

1. Introduction

The study of natural fibrations of a polynomial $f : \mathbb{C}^n \to \mathbb{C}$ was introduced by Broughton [5] long ago. At the same time, Pham [10] studied the conditions for a polynomial to have a good behavior at infinity, and Hà and Lê [8] obtained a criterion of regularity at infinity for complex polynomials in two variables. Since then the global theory of singularities of polynomials has been developed from the point of view of this article, with contributions by [16], [18], [14], [7], [9], [17], and others.

In the local case, the presence of singularities is a natural obstruction for the existence of a trivial fibration associated to the germ f. In the global context, however, the fibers of a polynomial can be topologically distinct, even without the presence of singularities. The values of f for which the topology of the fiber changes are denominated *atypical* values. The determination of these special values depends on the behavior of f at infinity.

In the case of polynomials in two variables, different characterizations of atypical values are known, whereas in higher dimensions this is still an open problem.

As in the local case, the Milnor number at infinity, and the sum of (boundary) Milnor numbers of the generic fiber are useful invariants for studying the topology of the fiber.

In [12], Siersma and Smeltink classified the singularities at infinity of polynomials of degree 4 in two variables, obtaining conditions for the equivalence of polynomials whose homogeneous part of degree 4 are equivalent.

In this work we use Siersma and Smeltink's method to classify singularities at infinity of polynomials $f : \mathbb{C}^3 \to \mathbb{C}$ of degree 3 in 3 variables,

$$f(x_0, x_1, x_2) = f_1(x_0, x_1, x_2) + f_2(x_0, x_1, x_2) + f_3(x_0, x_1, x_2),$$

where f_i homogeneous polynomial of degree i for $i = 1, 2, 3$. We restrict the classification to the case that all compactified fibers have only isolated singularities. Based on this classification, we study the equisingularity at infinity of the family $f = t$. We say that a polynomial f is of Broughton type if f has no affine singularities and the set of atypical values is non-empty. In each equivalence class of a degree 3 polynomial in $\mathbb{C}^3$, we give conditions for the existence of examples of Broughton type.

2. General setting

Consider a polynomial $f : \mathbb{C}^n \to \mathbb{C}$, $X = \{x \in \mathbb{C}^n; f(x) = 0\}$. The zero set X is an affine variety embedded in $\mathbb{C}^n$. Let $\mathrm{Sing}(f)$ be the singular locus of f, that is,

$$\mathrm{Sing}(f) = \{x \in \mathbb{C}^n; \mathrm{grad}(f) = 0\}.$$

We consider $\mathbb{P}^n$ as the standard compactification of $\mathbb{C}^n$ for some fixed affine coordinates. We use the following notations: let $\overline{X}$ be the compactification of X in $\mathbb{P}^n$, $X^\infty = \overline{X} \cap H^\infty$ its intersection with the hyperplane at infinity $H^\infty = \mathbb{P}^{n-1}$ and $X_t = f^{-1}(t)$. We write $f = f_d + f_{d-1} + \ldots + f_0$, f_i a homogeneous polynomial of degree $i = 0, \ldots, d$, and $F = f_d + x_{n+1} f_{d-1} + \ldots + x_{n+1}^d f_0$ the homogenization of f. Then we can associate to F the hypersurface

$$\mathbb{X} := \{((x : x_{n+1}), t)| \in \mathbb{P}^n \times \mathbb{C} : F(x, x_{n+1}) - t x_{n+1}^d = 0\}.$$

The map $\tau : \mathbb{X} \to \mathbb{C}$ is the projection to the t-coordinate and $\tau^{-1}(t) = \overline{X_t}$. As above, $X_t^\infty = \overline{X}_t \cap H^\infty$

At a point $p \in H^\infty$ we consider the boundary pair $< \overline{X_t}, \overline{X_t} \cap H^\infty >_p$ which is a family of germs depending on $t \in \mathbb{C}$. We say that X_t has a singularity at infinity if at least one of the members of this pair is singular. Singular points of $\overline{X_t}$ at infinity are solutions of $\mathrm{grad}(f_d = 0)$. We can distinguish between two types:

(i) Singular points of X_t^∞ where $\overline{X}_t$ is smooth. These are given by the conditions $\mathrm{grad}(f_d) = 0$ and $f_{d-1} \neq 0$.

(ii) Singular points of X_t^∞, where $\overline{X}_t$ is singular. These are given by the conditions $\mathrm{grad}(f_d) = 0$ and $f_{d-1} = 0$.

Definition 2.1. The polynomial f is general at infinity at a point Q if $\overline{X} \pitchfork H^\infty$ at Q. We say f is general at infinity if this condition holds for all $Q \in \overline{X} \cap H^\infty$.

Definition 2.2. The polynomial $f : \mathbb{C}^n \to \mathbb{C}$ is topologically trivial at infinity if f is locally topologically trivial for all $t_0 \in \mathbb{C}$.

The following definition was given in [13].

Definition 2.3. (Definition 4.1 in [13]) We define the following classes of polynomials.

- **(i)** We say f is a $\mathcal{F}$-*type* polynomial if its compactified fibers and their restrictions to the hyperplane at infinity have at most isolated singularities.
- **(ii)** We say f is a $\mathcal{B}$-*type* polynomial if its compactified fibers have at most isolated singularities.

The $\mathcal{F}$-class is contained in the $\mathcal{B}$-class. Moreover, both are contained in the $\mathcal{W}$-class, consisting of polynomials for which the proper extension $\tau : \mathbb{X} \to \mathbb{C}$ has only isolated singularities with respect to some Whitney stratification of $\mathbb{X}$ such that $\mathbb{X}^\infty = \mathbb{X} \cap H^\infty$ is an union of strata, see [14].

Based on results of [14] and [13] we can get information about the topology of the generic fiber X_t. The following theorem ([14],Theorem 3.1) is known as the Bouquet Theorem.

Theorem 2.4. *([14],Theorem 3.1) Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial with isolated $\mathcal{W}$-singularities at infinity. Then the general fiber of f is homotopy equivalent to a bouquet of spheres of real dimension $n - 1$.*

Let now $(p, t) \in \mathbb{P}^{n-1} \times \mathbb{C}$ be a singular point of X_t^∞. This may be a singular point of $\overline{X}_t$, or a point where $\overline{X}_t$ is non-singular but tangent to H^∞ at p. If (p, t) is an isolated singularity of X_t^∞, then we denote its Milnor number by μ_p^∞. Notice that the singularity $(p, t) \in X_t^\infty$ does not depend on t. In contrast, the Milnor number of the fiber $\overline{X}_t$ at the point p, that we denote by

$\mu_p(\overline{X}_t)$, may jump at a finite number of values of t. Let us denote by $\mu_{p,gen}$ the value of $\mu_p(\overline{X}_t)$ for generic t.

For a finite number of bifurcation values t this type can change and the Milnor number can drop with a value $\lambda_p^t = \mu_p(\overline{X}_t) - \mu_p^\infty$. We denote by

$$\lambda = \text{the sum of all jumps at infinity of the family } f = t.$$

In the case f has only isolated singularities (in the affine space), we denote by

$$\mu = \text{the total Milnor number of the affine singularities.}$$

Notice that this invariant can be computed by the following formula

$$\mu = \dim_{\mathbb{C}} \frac{\mathbb{C}[x_1, \ldots, x_n]}{Jf},$$

where the ideal $Jf \subset \mathbb{C}[x_1, \ldots, x_n]$ is the Jacobian ideal of f (see for instance, [15], pg 1).

In this paper we consider degree 3 polynomials in $\mathbb{C}^3$, so that the fibers X_t are in general singular surfaces in $\mathbb{C}^3$, and X_t^∞ are singular curves in $\mathbb{P}^2$.

It follows from Theorem 2.4 that $b_0(X_t) = 1$ and $b_1(X_t) = 0$, where $b_i(X_t), i = 0, 1$ are Betti numbers of the generic fiber X_t.

For the $\mathcal{F}$-class one can combine two formulas from [14] and [13], to compute the second Betti number b_2 of the generic fiber:

$$b_2 = \lambda + \mu = (d-1)^3 - \sum_i (\mu_{p_i} gen + \mu_{p_i}^\infty) \tag{1}$$

The right hand side can be computed via boundary data. In the left hand side λ is the sum of all jumps in the family $f = t$. This makes it possible to compute not only b_2 but also μ. A similar formula exists for $\mathcal{B}$-type, and we refer to [13], pg 663-664.

$$b_2(G) = \lambda + \mu = (\chi^{3,d} - 1) - \sum_{x \in \sum} \mu_{x,gen} - \chi^\infty \tag{2}$$

where G is the generic fiber, and $\chi^{3,d}$ is the Euler characteristic of the smooth hypersurface $V_{gen}^{3,d}$ of degree d in $\mathbb{P}^3$ of the generic fiber G of f and $\chi^\infty = \chi(\{f_d(x) = 0\})$. In general, the following formula holds ([15], pg 8)

$$\chi^{n,d} = \chi V_{gen}^{n,d} = n + 1 - \frac{1}{d}\{1 + (-1)^n (d-1)^{n+1}\}.$$

We shall denote by $\operatorname{Atyp}(f)$ the set of atypical fibers of f. It is known that

$$\operatorname{Atyp}(f) = f(\operatorname{Sing}(f)) \cup B_\infty(f),$$

where $B_\infty(f)$ comes from the contribution of singularities at infinity.

3. Classification of polynomials of degree 3

The purpose of this section is to classify singularities at infinity of polynomials $\mathbb{C}^3 \to \mathbb{C}$ of degree 3 of the form

$$f(x_0, x_1, x_2) = f_1(x_0, x_1, x_2) + f_2(x_0, x_1, x_2) + f_3(x_0, x_1, x_2),$$

f_i is homogeneous polynomial of degree i. We write $f_1(x_0, x_1, x_2) = a_0 x_0 + a_1 x_1 + a_2 x_2$, $f_2(x_0, x_1, x_2) = a_3 x_0^2 + a_4 x_0 x_1 + a_5 x_0 x_2 + a_6 x_1^2 + a_7 x_1 x_2 + a_8 x_2^2$. Let $t \in \mathbb{C}$, the homogenization F of $f - t = 0$ is given by

$$F(x_0, x_1, x_2, x_3) = x_3^2 f_1(x_0, x_1, x_2) + x_3 f_2(x_0, x_1, x_2) + f_3(x_0, x_1, x_2) - t x_3^3.$$

Definition 3.1. We say that f is affine equivalent to g ($f \approx g$) if there exist linear affine transformations $T : \mathbb{C}^3 \to \mathbb{C}^3$, $L : \mathbb{C} \to \mathbb{C}$, such that $g = L \circ f \circ T^{-1}$.

Notice that if f and g are equivalent, the linear transformation L sends the fibers $f = t$ of f to the fibers $g = L(t)$ of g.

Our aim in this section is to classify the singularities at infinity of the fibers $f = t$. We give a complete classification of polynomials of type $\mathcal{F}$ in Theorem 3.4 and the polynomials of type $\mathcal{B}$ are classified in Proposition 3.6.

We start the classification by making linear changes of coordinates to reduce the homogeneous polynomial f_3 to one of the following normal forms (see [4] or [6]).

(a) general: $x_0^3 + x_1^3 + x_2^3 - 3\lambda x_0 x_1 x_2, \lambda^3 - 1 \neq 0$.
(b) nodal: $x_0^3 + x_1^3 + x_0 x_1 x_2$.
(c) cuspidal: $-x_0^3 + x_2 x_1^2$.
(d) conic plus tangent: $(x_0^2 + x_1 x_2)x_1$.
(e) conic plus chord: $(x_0^2 + x_1 x_2)x_0$.
(f) three concurrent lines: $x_0^3 + x_1^3$
(g) triangle: $x_0 x_1 x_2$.
(h) double line plus simple line: $x_0 x_1^2$.
(i) triple line: x_1^3.

Let $\overline{X}_t \subset \mathbb{P}^3$ be the cubic surface defined by $F - t x_3^3 = 0$.
Note that affine equivalences extend to the projective space sending H^∞ to H^∞. Translations act as the identity on H^∞. The compatification $\overline{X}_t$ is sent to $\overline{g^{-1}(t)}$ biholomorphically. The types of local singularities do not change by affine equivalences.

Our classification is based on changes of coordinates and the recognition principles of singularities of function germs that we review in section 3.1.

3.1. **Recognition of simple singularities.** Set $g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be a holomorphic function germ at the origin.

We recall the normal forms of simple singularities of germs of functions
$g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$, due to Arnol'd [2].

$A_k : x^{k+1} + y^2 + z^2; k \geq 1$
$D_k : x^{k-1} + xy^2 + z^2; k \geq 4$
$E_6 : x^3 + y^4 + z^2$
$E_7 : x^3 + xy^3 + z^2$
$E_8 : x^3 + y^5 + z^2$

The results in this section are from Bruce and Wall [6].

The map-germ $g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ is quasihomogeneous of type $(w_1, w_2, w_3; d)$ if

$$f(\lambda^{w_1} x, \lambda^{w_2} y, \lambda^{w_3} z) = \lambda^d f(x, y, z).$$

The normal forms of simple singularities are quasi homogeneous of the following types:

$A_k : (2, k + 1, k + 1; 2k + 2)(k \geq 1)$
$D_k : (2, k - 2, k - 1; 2k - 2)(k \geq 4)$
$E_6 : (4, 3, 6; 12)$
$E_7 : (6, 4, 8; 18)$
$E_8 : (10, 6, 15; 30)$

A function f is semiquasihomogeneous with respect to the weights $(w_1, w_2, w_3; d)$ if all terms of weight $< d$ in its Taylor expansion vanish and those of weight d define a function with an isolated singularity.

Lemma 3.2 ([6], Lemma 1(a))**.** *If $f(x,y,z)$ is semiquasihomogeneous with respect to one of the sets of weights above we can, by change of coordinates, reduce the terms of weight d to the normal forms for A_k, D_k, E_6, E_7 or E_8 given above,and the resulting function will remain semiquasihomogeneous.*

We also quote the following Lemma [6].

Lemma 3.3 ([6], Lemma 4)**.** *Let $f = x_0^2 + f_3(x_0, x_1, x_2)$. If $x_0 = 0$ cuts $f_3 = 0$ in 3 distinct lines, a double and a simple line, respectively a triple line, then $f = 0$ has a D_4, D_5 respectively E_6 singularity at 0 and no others. Also f has two possible normal forms for the D_4 case, and a unique normal form for D_5 and E_6.*

3.2. **The classification theorem of polynomials of type $\mathcal{F}$.** In Theorem 3.4 we classify the fibers $f = t$ of polynomials f for which the degree three homogeneous part f_3 has respectively 1, 2 or 3 singularities at infinity. The results of this theorem along with the results of the next section, are listed in the tables 1-6, we use the notation ∞ to indicate that the singularity is non-isolated.

Theorem 3.4. (a) *Let f_3 the unimodular family, $f_3 \approx x_0^3 + x_1^3 + x_2^3 - 3\lambda x_0 x_1 x_2, \lambda^3 - 1 \neq 0$, then f is general at infinity.*

(b) *Let f_3 be nodal, $f_3 \approx x_0^3 + x_1^3 + x_0 x_1 x_2$. Then, f is affine equivalent to*

$$a_0x_0 + a_1x_1 + a_2x_2 + a_5x_0x_2 + a_7x_1x_2 + a_8x_2^2 + x_0^3 + x_1^3 + x_0x_1x_2.$$

In this case, $\overline{X}_t$ is smooth at infinity or $Q = (0:0:1:0)$ is a singular point at infinity of type $A_k, 1 \le k \le 5$ or Q is a non-isolated singularity. The conditions for each singularity type are given in Table 1.

(c) *Let f_3 be cuspidal, $f_3 \approx -x_0^3 + x_2x_1^2$. Then f is affine equivalent to*

$$a_0x_0 + a_1x_1 + a_2x_2 + a_4x_0x_1 + a_5x_0x_2 + a_8x_2^2 - x_0^3 + x_2x_1^2.$$

The following conditions hold: $\overline{X}_t$ is smooth at infinity or $Q = (0:0:1:0)$ is a singular point at infinity of type A_1, A_2, D_4, D_5, E_6 or Q is a non-isolated singularity. The conditions for each singularity type are given in Table 2.

(d) *Let f_3 be conic plus tangent, $f_3 \approx x_2x_1^2 + x_0^2x_1$. Then f is affine equivalent to*

$$a_0x_0 + a_1x_1 + a_2x_2 + a_5x_0x_2 + a_7x_1x_2 + a_8x_2^2 + x_2x_1^2 + x_0^2x_1.$$

It follows that $\overline{X}_t$ is smooth at infinity or $Q = (0:0:1:0)$ is a singular point at infinity of type A_1, A_3, D_4, D_5, E_6 or Q is a non-isolated singularity. The conditions for each singularity type are given in Table 3.

(e) *Let f_3 be three concurrent lines, $f_3 \approx x_0^3 + x_1^3$. Then f is affine equivalent to*

$$a_0x_0 + a_1x_1 + a_2x_2 + a_4x_0x_1 + a_5x_0x_2 + a_7x_1x_2 + a_8x_2^2 + x_0^3 + x_1^3.$$

It follows that $\overline{X}_t$ is smooth at infinity or $Q = (0:0:1:0)$ is a singular point at infinity of type $A_k; 2 \le k \le 5$, D_4 or Q is a non-isolated singularity. The conditions for each singularity are given in Table 4.

(f) *Let f_3 be conic plus chord, $f_3 \approx x_0^3 + x_0x_1x_2$. Then f is affine equivalent to*

$$a_0x_0 + a_1x_1 + a_2x_2 + a_3x_0^2 + a_6x_1^2 + a_8x_2^2 + x_0^3 + x_0x_1x_2.$$

Then $\overline{X}_t$ is smooth at infinity or $Q = (0:0:1:0)$ and $R = (0:1:0:0)$ are singular points at infinity. It follows that Q and R is a singularity of type A_1A_0, A_1A_1, A_2A_0, A_2A_1, A_3A_0, A_3A_1, A_4A_0, A_4A_1, A_5A_0, A_5A_1 or a non-isolated singularity. In Table 5 we give the conditions for each singularity type of the points Q and R.

(g) *Let f_3 is triangle, $f_3 \approx x_0x_1x_2$. Then f is affine equivalent to*

$$a_0x_0 + a_1x_1 + a_2x_2 + a_3x_0^2 + a_6x_1^2 + a_8x_2^2 + x_0x_1x_2.$$

It follows that $\overline{X}_t$ is smooth at infinity or $Q = (0:0:1:0)$, $R = (0:1:0:0)$ or $S = (1:0:0:0)$ are singular points at infinity. The singularities of Q, R and S are of type $A_1A_0A_0$, $A_2A_0A_0$, $A_3A_0A_0$, $A_4A_0A_0$, $A_1A_1A_0$, $A_1A_1A_1$, $A_2A_1A_0$, $A_2A_1A_1$, $A_3A_0A_0$, $A_4A_0A_0$, $A_3A_1A_0$, $A_3A_1A_1$, or else a non-isolated singularity. In Table 6 we give the conditions for each singularity type of the point Q.

Remark 3.5. In the proof below we always keep the same notation x_0, x_1, x_2, x_3 for the variables, even after changing coordinates.

Proof. **(a)** Is clear from definition.

(b) If f_3 is nodal, making changes of coordinates $X_0 = x_0+h_0, X_1 = x_1+h_1$ and $X_2 = x_2+h_2$, we can eliminate the quadratic terms $a_3x_0^2, a_4x_0x_1$ and $a_6x_1^2$. The homogenization F of f is given by:

$$F = a_0x_0x_3^2 + a_1x_1x_3^2 + a_2x_2x_3^2 + x_3(a_5x_0x_2 + a_7x_1x_2 + a_8x_2^2) + x_0^3 + x_1^3 + x_0x_1x_2 - tx_3^3.$$

It is easy to verify that if $a_8 \neq 0$, $\overline{X}_t$ is smooth at infinity. If $a_8 = 0$, $Q = (0:0:1:0)$ is the only singular point of $\overline{X}_t$ at infinity. So if $a_8 = 0$ and $x_2 = 1$, we have:

$$F(x_0, x_1, 1, x_3) = a_0x_0x_3^2 + a_1x_1x_3^2 + a_2x_3^2 + a_5x_0x_3 + a_7x_1x_3 + x_0^3 + x_1^3 + x_0x_1 - tx_3^3.$$

Let $a_2 \neq a_5a_7$. Then $Q = (0:0:1:0)$ is a singularity of type A_1.
Let $a_2 = a_5a_7$. Then

$$F(x_0, x_1, 1, x_3) = (a_5x_3 + x_1)(x_0 + a_7x_3) + a_0x_0x_3^2 + a_1x_1x_3^2 + x_0^3 + x_1^3 - tx_3^3.$$

We make the change of coordinates $X_0 = x_0 + a_7x_3$, $X_1 = x_1 + a_5x_3$, $X_3 = x_3$, and keeping the same notation $x_i, i = 0, 1, 2, 3$ for the new coordinates, we get

$$F = x_0x_1 + a_0(x_0 - a_7x_3)x_3^2 + a_1(x_1 - a_5x_3)x_3^2 + (x_0 - a_7x_3)^3 + (x_1 - a_5x_3)^3 - tx_3^3 = 0.$$

If $\gamma \neq t$, where

$$\gamma = -a_0a_7 - a_1a_5 - a_7^3 - a_5^3,$$

then Q is a singularity of type A_2. If $\gamma = t$, giving weights $(4, 4, 2; 8)$ it follows that if $a_0 \neq -3a_7^2$ and $a_1 \neq -3a_5^2$, then Q is a singularity of type A_3. If $\gamma = t$, $a_0 = -3a_7^2, a_7 \neq 0$ and $a_1 \neq -3a_5^2$ (similarly $\gamma = t$, $a_0 \neq -3a_7^2$, $a_5 \neq 0$ and $a_1 = -3a_5^2$), giving weights $(5, 5, 2; 10)$ it follows that Q is a singularity of type A_4. If $\gamma = t$, $a_0 = -3a_7^2, a_7 = 0$ and $a_1 \neq -3a_5^2$ (similarly $\gamma = t$, $a_0 \neq -3a_7^2$, $a_5 = 0$ and $a_1 = -3a_5^2$), giving weights $(6, 6, 2; 12)$ we get that Q is a singularity of type A_5. If $\gamma = t$, $a_0 = -3a_7^2$ and $a_1 = -3a_5^2$, Q is a non-isolated singularity. See Table 1.

(c) If f_3 is cuspidal, making changes of coordinates $X_0 = x_0 + h_0, X_1 = x_1 + h_1$ and $X_2 = x_2+h_2$, we can eliminate the quadratic terms $a_3x_0^2, a_6x_1^2$ and $a_7x_1x_2$. The homogenization F of f is given by:

$$F = a_0x_0x_3^2 + a_1x_1x_3^2 + a_2x_2x_3^2 + x_3(a_4x_0x_1 + a_5x_0x_2 + a_8x_2^2) - x_0^3 + x_1^2x_2 - tx_3^3.$$

It is easy to verify that if $a_8 \neq 0$, $\overline{X}_t$ is smooth at infinity. If $a_8 = 0$, $Q = (0:0:1:0)$ is the only singular point of $\overline{X}_t$ at infinity. So if $a_8 = 0$ and $x_2 = 1$, we have:

$$F(x_0, x_1, 1, x_3) = a_0x_0x_3^2 + a_1x_1x_3^2 + a_2x_3^2 + a_4x_0x_1x_3 + a_5x_0x_3 - x_0^3 + x_1^2 - tx_3^3.$$

Let $a_5 \neq 0$. Then $Q = (0:0:1:0)$ is a singularity of type A_1.

Let $a_5 = 0$ and $a_2 \neq 0$, then Q is a singularity of type A_2. If $a_2 = a_5 = 0$, then

$$F = x_1^2 + x_1 q_2(x_0 x_3) + q_3(x_0, x_3),$$

where $q_2(x_0, x_3) = x_3^2 + a_4 x_0 x_3$ and $q_3(x_0, x_3) = a_0 x_0 x_3^2 - x_0^3 - t x_3^3$. The discriminant of the cubic q_3 is $D(q_3) = 27t^2 - 4a_0^3$. If $D \neq 0$, then $q_3 = 0$ factors into 3 different lines and Q has type D_4. When $D = 0$ and $a_0 \neq 0$, the cubic q_3 has a double line and a simple line. Let $\delta = 27a_1^6 - a_0^3 a_4^6$. In this case, we have the following possibilities:
(i) $D = 0, a_0 \neq 0$ and $\delta \neq 0$, then Q has type D_5 for 2 different values of t;
(ii) $D = 0$, $a_0 \neq 0$ and $\delta = 0$, then the singularity is non isolated.
When $D = 0$ and $a_0 = 0$, the cubic q_3 has a triple line. In this case, if $a_1 \neq 0$, Q has type E_6, and if $a_1 = 0$, Q is a non-isolated singularity. See Table 2.

(d) If f_3 is conic plus tangent, making changes of coordinates $X_0 = x_0 + h_0, X_1 = x_1 + h_1$ and $X_2 = x_2 + h_2$, we can eliminate the quadratic terms $a_3 x_0^2, a_4 x_0 x_1$ and $a_6 x_1^2$. The homogenization F of f is given by:

$$F = a_0 x_0 x_3^2 + a_1 x_1 x_3^2 + a_2 x_2 x_3^2 + x_3(a_5 x_0 x_2 + a_7 x_1 x_2 + a_8 x_2^2) + x_0^2 x_1 + x_1^2 x_2 - t x_3^3.$$

It is easy to verify that if $a_8 \neq 0$, $\overline{X}_t$ is smooth at infinity. If $a_8 = 0$, $Q = (0:0:1:0)$ is the only singular point of $\overline{X}_t$ at infinity. So if $a_8 = 0$ and $x_2 = 1$, we have:

$$F(x_0, x_1, 1, x_3) = a_0 x_0 x_3^2 + a_1 x_1 x_3^2 + a_2 x_3^2 + a_5 x_0 x_3 + a_7 x_1 x_3 + x_0^2 x_1 + x_1^2 x_2 - t x_3^3.$$

Let $a_5 \neq 0$. Then $Q = (0:0:1:0)$ is a singularity of type A_1.
Let $a_5 = 0$, completing square and making the change $X_1 = x_1 + \frac{a_7}{2} x_3$, we get that if $a_2 - \frac{a_7^2}{4} \neq 0$, then Q is a singularity of type A_3. If $a_2 - \frac{a_7^2}{4} = 0$, making changes of coordinates $X_1 = x_1 - \frac{a_7}{2} x_3$ and giving weights $(2,3,2;6)$, then $F = x_1^2 + q_3(x_0, x_3)$, where

$$q_3(x_0, x_3) = a_0 x_0 x_3^2 - \frac{a_7}{2} x_0^2 x_3 - \frac{a_1 a_7}{2} x_3^3 - t x_3^3.$$

Analyzing the discriminant $D(q_3)$ of q_3 we have

$$D(q_3) = a_7^2(a_0^2 - 2t a_7 - a_1 a_7^2) = 0 \Rightarrow t = \gamma,$$

where

$$\gamma = \frac{a_0^2 - a_1 a_7^2}{2a_7} \qquad if \qquad a_7 \neq 0 \qquad or \qquad a_7 = 0.$$

If $a_7 \neq 0$ and $t \neq \gamma$, giving weights $(2,3,2;6)$, then Q is a singularity of type D_4. If $a_7 \neq 0$ and $t = \gamma$, giving weights $(2,4,3;8)$, then Q is a singularity of type D_5. If $a_7 = 0$ and $a_0 \neq 0$, then Q is a singularity of type D_5 for all values of t. If $a_0 = 0$ and $t \neq 0$, Q is a singularity of type E_6. Now if $t = 0$, then Q is a non-isolated singularity. See Table 3.

(e) If f_3 is three concurrent lines, making changes of coordinates $X_0 = x_0 + h_0, X_1 = x_1 + h_1$ and $X_2 = x_2 + h_2$, we can eliminate the quadratic terms $a_3 x_0^2$ and $a_6 x_1^2$. Note the symmetry in x_0, x_1. The computations below and Table 4 are up to this symmetry. The homogenization F of f is given by:

$$F = a_0 x_0 x_3^2 + a_1 x_1 x_3^2 + a_2 x_2 x_3^2 + x_3(a_4 x_0 x_1 + a_5 x_0 x_2 + a_7 x_1 x_2 + a_8 x_2^2) + x_0^3 + x_1^3 - t x_3^3.$$

It is easy to verify that if $a_8 \neq 0$, $\overline{X}_t$ is smooth at infinity. If $a_8 = 0$, $Q = (0:0:1:0)$ is the only singular point of $\overline{X}_t$ at infinity. So if $a_8 = 0$ and $x_2 = 1$, we have:

$$F(x_0, x_1, 1, x_3) = a_0 x_0 x_3^2 + a_1 x_1 x_3^2 + a_2 x_3^2 + a_4 x_0 x_1 x_3 + a_5 x_0 x_3 + a_7 x_1 x_3 + x_0^3 + x_1^3 - t x_3^3.$$

If $a_5 \neq 0$ (similarly $a_7 \neq 0$), making changes of coordinates $X_0 = a_2 x_3 + a_5 x_0 + a_7 x_1$, to get

$$F = x_0x_3 + Ax_0x_3^2 + Bx_3^3 + Cx_1^2x_3 + Dx_0x_1x_3 + Ex_1x_3^2 + Fx_1^3 - Gx_0^2x_1 - Hx_0^2x_3 + Ix_0x_1^2 + x_0^3 - tx_3^3,$$

where

$$A = \frac{a_0}{a_5} + \frac{3a_2^2}{a_5^3}, B = \frac{-a_0a_2}{a_5} - \frac{a_2^3}{a_5^3}, C = -\frac{a_4a_7}{a_5} - \frac{3a_2a_7^2}{a_5^3}, D = \frac{a_4}{a_5} + \frac{6a_2a_7}{a_5^3},$$

$$E = \frac{-a_0a_7}{a_5} + a_1 - \frac{a_2a_4}{a_5} - \frac{3a_2^2a_7}{a_5^3}, F = 1 - \frac{a_7^3}{a_5^3}, G = \frac{3a_7}{a_5^3}, H = \frac{3a_2}{a_5^3}, I = \frac{3a_7^2}{a_5^3}$$

Taking weights $(3,2,3;6)$, we have that if $F \neq 0$, Q is a singularity of type A_2. If $a_5 \neq 0$ and $F = 0$, such as $a_7 \neq 0$, giving weights $(4,2,4;8)$ it follows that if $IC \neq 0$, Q is a singularity of type A_3. If $a_5 \neq 0$, $a_5^3 = a_7^3$ and $C = 0$, making changes of coordinates $x_0 = X_0 - Ex_1x_3$ and $x_3 = X_3 - Ix_1^2$, that $E \neq 0$, Q is a singularity of type A_4. If $E = 0$ and $B \neq t$, Q is a singularity of type A_5. If $B = t$, then Q is a non-isolated singularity.

If $a_5 = a_7 = 0$ and $a_2 \neq 0$, giving weights $(2,2,3;6)$, then Q is a singularity of type D_4. Now if $a_2 = 0$, then Q is a non-isolated singularity. See Table 4.

(f) If f_3 is conic plus chord, making changes of coordinates $X_0 = x_0 + h_0, X_1 = x_1 + h_1$ and $X_2 = x_2 + h_2$, we can eliminate the quadratic terms $a_4x_0x_1, a_5x_0x_2$ and $a_7x_1x_2$. The homogenization F of f is given by:

$$F = a_0x_0x_3^2 + a_1x_1x_3^2 + a_2x_2x_3^2 + x_3(a_3x_0^2 + a_6x_1^2 + a_8x_2^2) + x_0^3 + x_0x_1x_2 - tx_3^3.$$

It is easy to verify that if $a_6.a_8 \neq 0$, $\overline{X}_t$ is smooth at infinity. If $a_8 = 0$, $Q = (0:0:1:0)$ is the singular point of $\overline{X}_t$ at infinity. If $a_6 = 0$, $R = (0:1:0:0)$ is the singular point of $\overline{X}_t$ at infinity. Let $a_8 = 0$ and $x_2 = 1$, we have:

$$F(x_0, x_1, 1, x_3) = a_0x_0x_3^2 + a_1x_1x_3^2 + a_2x_3^2 + a_3x_0^2x_3 + a_6x_1^2x_3 + x_0^3 + x_0x_1x_2 - tx_3^3.$$

If $a_2, a_6 \neq 0$ and $a_8 = 0$, Q is a singularity of type A_1 and R is of type A_0. If $a_1, a_2 \neq 0$ and $a_6 = a_8 = 0$, both singularities Q and R of type A_1. If $a_2 = a_8 = 0$ and $t \neq 0$, Q is a singularity of type A_2, with this information if $a_1 = a_6 = 0$, R is a singularity of type A_2. If $a_2 = a_8 = t = 0$, giving weights $(4,4,2;8)$ it follows that if $a_0a_1 \neq 0$, Q is a singularity of type A_3. If $a_1 = a_2 = a_8 = t = 0$, giving weights $(5,5,2;10)$ it follows that if $a_0a_6 \neq 0$, Q is a singularity of type A_4. If $a_0 = a_1 = a_2 = a_8 = t = 0$, then Q is a singularity of type non-isolated. If $a_1 = a_2 = a_6 = a_8 = t = 0$, then Q is a singularity of type non-isolated. If $a_0 = a_2 = a_8 = t = 0$, giving weights $(5,5,2;10)$ it follows that if $a_1a_3 \neq 0$, Q is a singularity of type A_4. If $a_0 = a_2 = a_3 = a_8 = t = 0$, giving weights $(5,5,2;10)$ it follows that if $a_1 \neq 0$, Q is a singularity of type A_5. If $a_0 = a_1 = a_2 = a_8 = t = 0$, then Q is a non-isolated singularity. Q and R points are symmetric, see Table 5.

(g) If f_3 is triangle, making changes of coordinates $X_0 = x_0 + h_0, X_1 = x_1 + h_1$ and $X_2 = x_2 + h_2$, we can eliminate the quadratic terms $a_4x_0x_1, a_5x_0x_2$ and $a_7x_1x_2$. The homogenization F of f is given by:

$$F = a_0x_0x_3^2 + a_1x_1x_3^2 + a_2x_2x_3^2 + x_3(a_3x_0^2 + a_6x_1^2 + a_8x_2^2) + x_0x_1x_2 - tx_3^3.$$

It is easy to verify that if $a_3 = a_6.a_8 \neq 0$, $\overline{X}_t$ is smooth at infinity. If $a_8 = 0$, $Q = (0:0:1:0)$ is the singular point of $\overline{X}_t$ at infinity. If $a_6 = 0$, $R = (0:1:0:0)$ is the singular point of $\overline{X}_t$ at infinity. If $a_3 = 0$, $S = (1:0:0:0)$ is the singular point of $\overline{X}_t$ at infinity. Let $a_8 = 0$ and $x_2 = 1$, we have:

$$F(x_0, x_1, 1, x_3) = a_0x_0x_3^2 + a_1x_1x_3^2 + a_2x_3^2 + a_3x_0^2x_3 + a_6x_1^2x_3 + x_0x_1x_2 - tx_3^3.$$

If $a_2, a_3, a_6 \neq 0$ and $a_8 = 0$, Q is a singularity of type A_1, R is a singularity of type A_0 and $S(1:0:0:0)$ is a singularity of type A_0. If $a_0, a_1, a_2 \neq 0$ and $a_3 = a_6 = a_8 = 0$, all singularities Q, R and S are of type A_1. If $a_2 = a_8 = 0$ and $t \neq 0$, Q is a singularity of type A_2, with this information if $a_1 = a_6 = 0$, R is a singularity of type A_2 and $a_0 = a_3 = 0$, S is a singularity of type A_2. If $a_2 = a_8 = t = 0$, giving weights $(4,4,2;8)$ it follows that if $a_0 a_1 \neq 0$, Q is a singularity of type A_3. If $a_1 = a_2 = a_8 = t = 0$, giving weights $(5,5,2;10)$ it follows that if $a_0 a_6 \neq 0$, Q is a singularity of type A_4. If $a_0 = a_1 = a_2 = a_8 = t = 0$, then Q is a non-isolated singularity. If $a_1 = a_2 = a_6 = a_8 = t = 0$, then Q is a non-isolated singularity. If $a_0 = a_2 = a_8 = t = 0$, giving weights $(5,5,2;10)$ it follows that if $a_1 a_3 \neq 0$, Q is a singularity of type A_4. If $a_0 = a_2 = a_3 = a_8 = t = 0$, giving weights $(5,5,2;10)$ it follows that if $a_1 \neq 0$, Q is a singularity of type A_5. If $a_0 = a_1 = a_2 = a_8 = t = 0$, then Q is a non-isolated singularity. Q, R and S points are symmetric, see Table 6. □

3.3. **Classification of polynomials of type $\mathcal{B}$.** In Proposition 3.6 we give the classification of polynomials of degree 3, classifying the isolated singularities at infinity of $f = f_1 + f_2 + f_3$, in the cases in which f_3 has non isolated singularities. We denote this class of polynomials by $\mathcal{B} \setminus \mathcal{F}$.

Proposition 3.6. **(a)** *Let $f_3 = x_0 x_1^2$ (double line plus simple line). After a change of coordinates which leaves invariant the cubic f_3, we get the following possibilities for the singular points at infinity:*

(1) *Two points $Q = (1:0:0:0)$ and $R = (-a_8 : 0 : a_5 : 0)$, where Q has type A_1, R has type A_k, $2 \leq k \leq 5$, or non-isolated singularity.*
(2) *One point $Q = (1:0:0:0)$ with type A_3, D_4, D_5 or non-isolated singularity.*
(3) *One point $R = (0:0:1:0)$ with type A_4 or D_5.*

(b) *Let $f_3 = x_1^3$ (triple line). After a change of coordinates which leaves invariant the cubic f_3, we get that the singular points at infinity are:*

(1) *If $Q = (1:0:0:0)$ with type type A_2, $R = (0:0:1:0)$ with A_2, or non-isolated singularity.*
(2) *If $Q = (1:0:0:0)$, with type A_5 or non-isolated singularity.*

Proof. **(a)** Let $f_3 = x_0 x_1^2$, making changes of coordinates $X_0 = x_0 + h_0, X_1 = x_1 + h_1$ and $X_2 = x_2 + h_2$, we can eliminate the quadratic terms $a_4 x_0 x_1$ and $a_6 x_1^2$. In this case, the set $\operatorname{grad}(f_3) = 0$ gives a $\mathbb{P}^1$ at infinity. That is, $Sing f_3 = \mathbb{P}^1 = \{(x_0 : 0 : x_2 : 0), (x_0, x_2) \in \mathbb{C}^2\}$. The singularities at infinity are the points of the intersection $Sing f_3 \cap \{f_2 = 0\}$. Hence, they are the solutions $f_2(x_0, 0, x_2) = a_3 x_0^2 + a_5 x_0 x_2 + a_8 x_2^2 = 0$. We assume that $((a_3, a_5, a_8) \neq (0,0,0))$. We distinguish two cases:

(i) $a_5^2 - 4a_3 a_8 \neq 0$ and **(ii)** $a_5^2 - 4a_3 a_8 = 0$.

(i) When $a_5^2 - 4a_3 a_8 \neq 0$, the polynomial $f_2(x_0, 0, x_2) = 0$ has two distinct roots $(\alpha_1 x_0 + \beta_1 x_2)$, $(\alpha_2 x_0 + \beta_2 x_2)$.

If $a_8 \neq 0$ then we can make $x_2 = \alpha_1 x_0 + \beta_1 x_2$ to eliminate a_3. In this case $a_5 \neq 0$. Then the solutions are $Q = (1:0:0:0)$ and $R = (-a_8 : 0 : a_5 : 0)$. So we have

$$F = a_0 x_0 x_3^2 + a_1 x_1 x_3^2 + a_2 x_2 x_3^2 + a_5 x_0 x_2 x_3 + a_7 x_1 x_2 x_3 + a_8 x_2^2 x_3 + x_0 x_1^2 - t x_3^3.$$

The Hessian of F at the point $Q = (1:0:0:0)$, $\operatorname{Hess}(F)(1:0:0:0) = \frac{a_5^2}{4}$. Since $a_5 \neq 0$, then $Q = (1:0:0:0)$ is always A_1. The Hessian of F at the point $R = (-a_8 : 0 : a_5 : 0)$, $\operatorname{Hess}(F)(-a_8 : 0 : a_5 : 0) = 2a_5 a_8$, R is a singularity of type A_1.

If $a_8 = 0$, the solutions are $Q = (1:0:0:0)$ and $R = (0:0:1:0)$, since

$$\operatorname{Hess}(F)(1:0:0:0) = \frac{a_5^2}{4} \neq 0,$$

then $Q = (1:0:0:0)$ is always A_1. Now let's analyze the point R. Making changes of coordinates $x_0 = \frac{1}{a_5}(X_0 - a_2x_3 - a_7x_1)$ we have on the chart $x_2 = 1$:

$$F = X_0x_3 + \frac{a_0}{a_5}(X_0 - a_2x_3 - a_7x_1)x_3^2 + a_1x_1x_3^2$$
$$+ \frac{1}{a_5}(X_0 - a_2x_3 - a_7x_1)x_1^2 + \frac{a_3}{a_5^2}(X_0 - a_2x_3 - a_7x_1)^2x_3 - tx_3^3$$

If $a_7 \neq 0$, R is a singularity of type A_2, if $a_7 = 0$ and $a_2 \neq 0$ R is a singularity of type A_3. If $a_2 = 0$ and $a_1 \neq 0$ R is a singularity of type A_4, if $a_1 = 0$ and $t \neq 0$, R is a singularity of type A_5 and finally $t = 0$, then R is a non-isolated singularity.

(ii) When $a_5^2 - 4a_3a_8 = 0$, the polynomial $f_2(x_0, 0, x_2) = 0$ has only one root, $\alpha x_0 + \beta x_2$.

If $a_8 \neq 0$ then we can make $x_2 = \alpha x_0 + \beta x_2$ to eliminate a_3 and a_5. In this case, the only solution is $Q = (1:0:0:0)$ and on the chart $x_0 = 1$:

$$F = a_0x_3^2 + a_1x_1x_3^2 + a_2x_2x_3^2 + a_8x_2^2x_3 + x_0x_1^2 - tx_3^3$$

Giving weights $(4, 2, 4; 8)$, if $a_0 \neq 0$, Q is a singularity of type A_3, otherwise if $a_0 = 0$ completing square we have

$$F = (x_1 + \frac{a_1}{2}x_3^2) - \frac{a_1^2}{4}x_3^4 + a_2x_2x_3^2 + a_8x_2^2x_3 - tx_3^3 =$$
$$x_1^2 - \frac{a_1^2}{4}x_3^4 + x_3q(x_2, x_3).$$

Discriminant of q is

$$D(q) = -4a_8t - a_2^2.$$

If $D(q) \neq 0$ what is $t \neq \frac{-a_2^2}{4a_8}$ and $a_1 \neq 0$, Q is a singularity of type D_4. If $D(q) = 0$ what is $t = \frac{-a_2^2}{4a_8}$ and $a_1 \neq 0$, Q is a singularity of type D_5 and finally $a_1 = 0$, then Q is a non-isolated singularity.

If $a_8 = 0$, $a_3 \neq 0$, then the solution is $R = (1:0:0:0)$. The calculations are similar to the first case, we get A_4 if $a_7 \neq 0$ and D_5 when $a_7 = 0$. If $a_3 = a_5 = a_8 = 0$ the function is no longer of $\mathcal{B}$-type.

(b) Let $f_3 = x_1^3$, making changes of coordinates $X_0 = x_0 + h_0, X_1 = x_1 + h_1$ and $X_2 = x_2 + h_2$, we can eliminate the quadratic term $a_6x_1^2$. In this case, the set $\text{grad}(f_3) = 0$ gives a $\mathbb{P}^1$ at infinity. That is, $Sing f_3 = \mathbb{P}^1 = \{(x_0 : 0 : x_2 : 0), (x_0, x_2) \in \mathbb{C}^2\}$. The singularities at infinity are the points of the intersection $Sing f_3 \cap \{f_2 = 0\}$. Hence, they are the solutions

$$f_2(x_0, 0, x_2) = a_3x_0^2 + a_5x_0x_2 + a_8x_2^2 = 0.$$

We assume that $((a_3, a_5, a_8) \neq (0, 0, 0))$. We distinguish two cases:

(i) $a_5^2 - 4a_3a_8 \neq 0$ and **(ii)** $a_5^2 - 4a_3a_8 = 0$.

(i) When $a_5^2 - 4a_3a_8 \neq 0$, the polynomial $f_2(x_0, 0, x_2) = 0$ has two distinct roots Q and R. By a projective transformation, leaving invariant $x_1 = 0$ are can arrange $Q = (1:0:0:0)$ and $R = (0:0:1:0)$. Therefore we can assume $a_3 = a_8 = 0$. So we have

$$F = a_0x_0x_3^2 + a_1x_1x_3^2 + a_2x_2x_3^2 + a_4x_0x_1x_3 + a_5x_0x_2x_3 + a_7x_1x_2x_3 + x_1^3 - tx_3^3$$

The Hessian of F at the point Q, $\text{Hess}(F)(1:0:0:0) = 0$. Then when $a_4a_5 \neq 0$, $Q = (1:0:0:0)$ is always A_2. The Hessian of F at the point R, $\text{Hess}(F)(0:0:1:0) = 0$, if $a_4a_5 \neq 0$, R is a singularity of type A_2.

If $a_4a_5 = 0$ than f is a non-isolated singularity.

(ii) When $a_5^2 - 4a_3a_8 = 0$, the polynomial $f_2(x_0, 0, x_2) = 0$ has only one root, Q.

By a projective transformation, leaving invariant $x_1 = 0$ are can arrange $Q = (1 : 0 : 0 : 0)$, $a_3 = 0$ and therefore also $a_5 = 0$, $a_8 \neq 0$. On chart $x_0 = 1$:

$$F = a_0x_3^2 + a_1x_1x_3^2 + a_2x_2x_3^2 + a_4x_0x_1x_3 + a_7x_1x_2x_3 + a_8x_2^2x_3 + x_1^3 - tx_3^3.$$

If $a_4 \neq 0$, making changes of coordinates $x_1 = \frac{1}{a_4}(X_1 - a_0x_3 - a_8x_2^2)$ and giving weights $(6, 2, 6; 12)$, Q is a singularity of type A_5, if $a_4 = 0$ Q is a non-isolated singularity. If $a_8 = 0$, then f is not of $\mathcal{B}$-type. □

4. Equisingularity at infinity

In this section we compute the invariants of the singularities in order to study the topology of the Milnor fiber. The jump λ on the Milnor number at infinity will play an important role in the description of the topology of the regular fiber.

A careful description of regularity conditions, equisingularity and topological triviality at infinity has given by M. Tibăr in [17] (see also [14], [13] and [15]).

As usual, the notation $A_k \to A_{k+1}$ means that the singularity at infinity jumped from A_k to A_{k+1} for some value of the atypical set. For non-isolated singularities we replace λ by $*$.

Using the formulas (1) and (2) of the section 2 it is possible to calculate the Betti number b_2 and the Milnor number μ of the generic fiber.

Definition 4.1. Let f be a polynomial of types $\mathcal{F}$ or $\mathcal{B}$. We say that $f = t_0$ has no Milnor-jumps at infinity at the point Q if there is a neighborhood D of t_0 in $\mathbb{C}$, such that the jump $\lambda = \mu_{t_0}^Q - \mu_t^Q$ is equal to zero, $\forall t \in D$, where μ_t is the Milnor number of F at the point Q.

Applying the results of Theorem 3.4 and Proposition 3.6 we can calculate λ. Knowing λ and using the formulas (1) and (2), we can calculate b_2 and μ.

For example if f_3 is nodal and the singularity of Q is of type A_3 for $f = t_0$ and A_2 for $f = t$, $t \neq t_0$, it follows that $\lambda = 1$. From (1) we get that $b_2 = \mu + \lambda = 8 - (2 + 1) = 5$. As $\lambda = 1$, we get $\mu = 4$.

In the case where f has more than one singularity, we need to check the possible combinations of all singularities.

For example, if f_3 is conic plus chord, let's say R is a singularity of type A_1 and Q is a singularity of type A_2 for $f \neq t_0$ and A_3 for $f = t_0$. Then, for A_3A_1 singularities, we have $\lambda = 1$, $b_2 = 3$ and $\mu = 2$.

In Theorem 4.2 and 4.3 we apply the classification given in Theorem 3.4 and Proposition 3.5 to get information about the topology of the generic fiber $f = t$ for polynomials of type $\mathcal{F}$ and $\mathcal{B}$.

Theorem 4.2. *Let f be of $\mathcal{F}$-type. We consider the family $f = t$. The following Tables 1 to 6 give all possibilities for the singularities of $\overline{X}_t$ at the point Q respectively R and S at infinity.*

(i) *In all cases with singularities of types A_0 and A_1 only there are no jumps.*

(ii) *All jumps ($\lambda \neq 0$) are indicated in the tables. The t- values are indicated in the proof.*

(iii) *If there are no jumps ($\lambda = 0$) then the family $f = t$ is equisingular at infinity.*

Proof. First, notice that (i) and (iii) follow easily. In fact, in all cases with singularities of type A_0 and A_1 only, there are no jumps, that is, $\lambda = 0$. In these cases the family $f = t$ is equisingular at infinity.

To prove (ii) we follow the proofs in Theorem 3.4. Especially the places where t appears in the (in)equalities gives rise to the jumps. The tables contain all necessary information. Special care is needed for combinations of several critical points at infinity P, Q or S.

The invariants λ, μ and b_2 of the generic fiber X_t take into account the combinations of singularities Q, R and S. □

Nodal : $f_3 = x_0^3 + x_1^3 + x_0x_1x_2$					
$f = a_0x_0 + a_1x_1 + a_2x_2 + a_5x_0x_2 + a_7x_1x_2 + a_8x_2^2 + x_0^3 + x_1^3 + x_0x_1x_2$					
$Q(0:0:1:0)$	$\neq 0$	$= 0$	λ	μ	b_2
A_0	a_8		0	7	7
A_1	$a_2 - a_5a_7$	a_8	0	6	6
$A_2 \to A_3$	$\gamma, a_0 + 3a_7^2, a_1 + 3a_5^2$	$a_8, a_2 - a_5a_7$	1	4	5
$A_2 \to A_4$	$\gamma, a_1 + 3a_5^2, a_7$	$a_8, a_2 - a_5a_7, a_0 + 3a_7^2$	2	3	5
$A_2 \to A_5$	$\gamma, a_1 + 3a_5^2$	$a_8, a_2 - a_5a_7, a_0 + 3a_7^2, a_7$	3	2	5
$A_2 \to A_\infty$	γ	$a_8, a_2 - a_5a_7, a_0 + 3a_7^2, a_1 + 3a_5^2$	*	–	–
All jumps occur if $t = \gamma$, where $\gamma = -a_0a_7 - a_1a_5 - a_7^3 - a_5^3$					

TABLE 1.

Cuspidal: $f_3 = -x_0^3 + x_1^2x_2$					
$f = a_0x_0 + a_1x_1 + a_2x_2 + a_4x_0x_1 + a_5x_0x_2 + a_8x_2^2 - x_0^3 + x_1^2x_2$					
$Q(0:0:1:0)$	$\neq 0$	$= 0$	λ	μ	b_2
A_0	a_8		0	6	6
A_1	a_5	a_8	0	5	5
A_2	a_2	a_8, a_5	0	4	4
$D_4 \to D_5$	D, δ	a_8, a_2, a_5	2	0	2
$D_4 \to \infty$	D	a_8, a_2, a_5, δ	–	–	–
$D_4 \to E_6$	a_1	a_8, a_2, a_5, a_0	2	0	2
All jumps occur if $D = 27t^2 - 4a_0^3, \delta = 27a_1^6 - a_0^3a_4^6$					

TABLE 2.

In the following tables, each line corresponds to a class of polynomial (up to affine equivalence) with the same behavior near the boundary H^∞. We list only cases with isolated singularities, but in some cases we also list the "next" non-isolated class.

The notation $X \to Y$ means that X is the generic type, which jumps to Y nongeneric.

The expression $\gamma \neq t$ in each table expresses the condition that the fibers $t \neq \gamma$ are generic and $t = \gamma$ is the exceptional fiber. The last line of each table characterizes the values of t for which the jump occurs.

In Theorem 4.3, we discuss the topology of the generic fiber of $\mathcal{B}-$ type polynomials $f = f_1 + f_2 + f_3$. The results are consequence of the formula (2) for b_2 in Section 2, and the following formulas for the top Betti defect, $\Delta_{n-1}(f)$, given by Siersma and Tibar in [14], [13] and [15].

$$\Delta_{n-1}(f) = (d-1)^n - b_{n-1}(f)$$

$$\Delta_{n-1}(f) = \sum_{p \in \sum_f^\infty \cap \{f_{d-1}=0\}} \mu_p(\overline{X_0}) + (-1)^n \Delta\chi^\infty,$$

where

$$\Delta\chi^\infty := \chi^{n-1,d} - \chi(\{f_d = 0\}),$$

and

$$\chi^{n-1,d} = n - \frac{1}{d}\{1 + (-1)^{n-1}(d-1)^n\}$$

Conic plus tangent: $f_3 = x_0^2x_1 + x_1^2x_2$					
$f = a_0x_0 + a_1x_1 + a_2x_2 + a_5x_0x_2 + a_7x_1x_2 + a_8x_2^2 + x_0^2x_1 + x_1^2x_2$					
$Q(0:0:1:0)$	$\neq 0$	$= 0$	λ	μ	b_2
A_0	a_8		0	5	5
A_1	a_5	a_8	0	4	4
A_3	$a_2 - \frac{a_7^2}{4}$	a_8, a_5	0	2	2
$D_4 \to D_5$	γ, a_7	$a_8, a_2 - \frac{a_7^2}{4}, a_5$	1	0	1
D_5	a_0	a_8, a_2, a_5, a_7	0	0	0
$E_6 \to \infty$		a_8, a_2, a_5, a_0, a_7	$*$	–	–
The jumps occur if $t = \gamma$, where $\gamma = \frac{a_0^2 - a_1a_7^2}{2a_7}$ except if Q is non-isolated, when $t = 0$					

TABLE 3.

Three concurrent lines: $f_3 = x_0^3 + x_1^3$					
$f = a_0x_0 + a_1x_1 + a_2x_2 + a_4x_0x_1 + a_5x_0x_2 + a_7x_1x_2 + a_8x_2^2 + x_0^3 + x_1^3$					
$Q(0:0:1:0)$	$\neq 0$	$= 0$	λ	μ	b_2
A_0	a_8		0	4	4
A_2	$a_5, a_5^3 - a_7^3$	a_8	0	2	2
A_3	a_5, γ	$a_8, a_5^3 - a_7^3$	0	1	1
A_4	a_5, E	$a_8, a_5^3 - a_7^3, \gamma$	0	0	0
$A_5 \to \infty$	a_5, B	$a_8, a_5^3 - a_7^3, \gamma, E$	0	1	1
D_4	a_2	a_8, a_5, a_7	0	0	0
∞		a_8, a_2, a_5, a_7	–	–	–
$\gamma = \frac{-a_4a_7}{a-5} - \frac{3a_2a_7^2}{a_5^3}$, $E = -\frac{a_4a_7}{a_5} - \frac{3a_2a_7^2}{a_5^3}$, $B = \frac{-a_0a_2}{a_5} - \frac{a_2^3}{a_5^3}$. There are no jumps in this case					

TABLE 4.

Conic plus chord: $f_3 = x_0^3 + x_0x_1x_2$						
$f = a_0x_0 + a_1x_1 + a_2x_2 + a_3x_0^2 + a_6x_1^2 + a_8x_2^2 + x_0^3 + x_0x_1x_2$						
$Q(0:0:1:0)$	$R(0:1:0:0)$	$\neq 0$	$= 0$	λ	μ	b_2
A_0	A_0	a_6, a_8		0	6	6
A_1	A_0	a_6, a_2	a_8	0	5	5
A_1	A_1	a_1, a_2	a_6, a_8	0	4	4
$A_2 \to A_3$	A_0	a_6, a_0, a_1	a_2, a_8	1	3	4
$A_2 \to A_3$	A_1	a_1, a_0	a_2, a_6, a_8	1	2	3
$A_2 \to A_4$	A_0	a_0, a_6	a_1, a_2, a_8	2	2	4
$A_2 \to A_4$	A_0	a_1, a_3, a_6	a_0, a_2, a_8	2	2	4
$A_2 \to A_4$	A_1	a_1, a_3	a_0, a_2, a_6, a_8	2	1	3
$A_2 \to A_5$	A_0	a_1, a_6	a_0, a_2, a_3, a_8	3	1	4
$A_2 \to A_5$	A_1	a_1	a_2, a_3, a_6, a_8	3	0	3
$A_2 \to \infty$	∞		a_1, a_2, a_3, a_6, a_8	$*$	–	–
All jumps occur if $t = 0$.						

TABLE 5.

denotes the Euler characteristic of the smooth hypersurface $V_{gen}^{n-1,d}$ of degree d in $\mathbb{P}^{n-1}$.

Theorem 4.3. *Let f be of type $\mathcal{B}\setminus\mathcal{F}$-type. We consider the family $f=t$. The following Tables 7 and 8 give all possibilities for the singularities of $\overline{X}_t$ at a point Q at infinity.*

Proof. **(a)** We first consider $f=f_1+f_2+f_3$, were $f_3(x_0,x_1,x_2)=x_0x_1^2$ and the singularities at infinity are $Q=(0:0:1:0)$ and $R=(-a_8:0:a_5:0)$. If $a_5^2\neq 4a_3a_8$, $Q\neq R$, it follows from Proposition 3.5 that Q is a singular point of type A_1 and R is A_k, $1\leq k\leq 5$.

To compute $b_2(f)$, note that $\Delta\chi^\infty=-3$. When (Q,R) is A_1A_k we have

$$\Delta_2(f)=1+k+3=k+4\Rightarrow b_2(f)=8-(k+4).$$

If $Q=R$, a singularity is of type A_3, D_4 or D_5, in these cases we have

$$\Delta_2(f)=\mu_p(\overline{X})+3\Rightarrow b_2(f)=8-(\mu_p(\overline{X})+3).$$

Triangle: $f_3=x_0x_1x_2$							
$f=a_0x_0+a_1x_1+a_2x_2+a_3x_0^2+a_6x_1^2+a_8x_2^2+x_0x_1x_2$							
$Q(0:0:1:0)$	$R(0:1:0:0)$	$S(0:0:1:0)$	$\neq 0$	$=0$	λ	μ	b_2
A_0	A_0	A_0	a_3,a_6,a_8		0	5	5
A_1	A_0	A_0	a_6,a_2,a_3	a_8	0	4	4
A_1	A_1	A_0	a_0,a_2,a_6	a_3,a_8	0	3	3
A_1	A_1	A_1	a_0,a_1,a_2	a_3,a_6,a_8	0	2	2
$A_2\to A_3$	A_0	A_0	a_3,a_6,a_0,a_1	a_2,a_8	1	2	3
$A_2\to A_3$	A_1	A_0	a_0,a_1,a_6	a_2,a_3,a_8	1	1	2
$A_2\to A_3$	A_1	A_1	a_0,a_1	a_2,a_3,a_6,a_8	1	0	1
$A_2\to A_4$	A_0	A_0	a_0,a_3,a_6	a_1,a_2,a_8	2	1	3
$A_2\to A_4$	A_0	A_0	a_1,a_3,a_6	a_0,a_2,a_8	2	1	3
All jumps occur if $t=0$.							

TABLE 6.

double line plus simple line: $f_3=x_0x_1^2$						
$f=a_0x_0+a_1x_1+a_2x_2+a_3x_0^2+a_5x_0x_2+a_7x_1x_2+a_8x_2^2+x_0x_1^2$						
$Q(1:0:0:0)$	$R(-a_8:0:a_5:0)$	$\neq 0$	$=0$	λ	μ	b_2
A_1	A_1	γ,a_8		0	3	3
A_1	A_2	γ,a_7	a_8	0	2	2
A_1	A_3	γ,a_2	a_7,a_8	0	1	1
A_1	A_4	γ,a_1	a_2,a_7,a_8	0	0	0
A_1	$A_5\to\infty$	γ	a_1,a_2,a_7,a_8	–	–	–
A_3		a_0,a_8	γ	0	2	2
$D_4\to D_5$		a_1,a_8	γ	1	0	1
$D_4\to\infty$			a_1,γ	–	–	–
	A_4	a_3,a_7	a_8,a_5	0	1	1
	D_5	a_3	a_5,a_7,a_8	0	0	0
$\gamma=a_5^2-4a_3a_8$.All jumps occur if $t=\frac{-a_2^2}{16a_8}$.						

TABLE 7.

The results appear in Table 7.

(b) We first consider $f = f_1+f_2+f_3$, where $f_3(x_0, x_1, x_2) = x_1^3$ and the singularities at infinity are $Q = (0 : 0 : 1 : 0)$ and $R = (-a_8 : 0 : a_5 : 0)$. If $a_5^2 \neq 4a_3a_8$, $Q \neq R$, follow Proposition 3.5 Q is a singular point of type A_2 and R is A_2. To compute $b_2(f)$, note that $\Delta\chi^\infty = -2$. When (Q, R) is A_2A_2 we have $\Delta_2(f) = 2 + 2 + 2 = 6 \Rightarrow b_2(f) = 8 - 6 = 2$. If $Q = R$, a singularity is of type A_5, in these case we have $\Delta_2(f) = \mu_p(\overline{X}) + 2 \Rightarrow b_2(f) = 8 - (\mu_p(\overline{X}) + 2)$. The results appear in Table 8. □

5. Examples of Broughton type and global fibrations

In this section we assume that the singularities of all polynomials are of type $\mathcal{F}$ or $\mathcal{B}$.

Definition 5.1. A polynomial $f : \mathbb{C}^n \to \mathbb{C}$ is of Broughton type if f has no affine singularities, but the set of atypical values $\mathrm{Atyp}(f)$ is non empty.

According to our notations, if f has no singularities in $\mathbb{C}^3$, then $(a_0, a_1, a_2) \neq (0, 0, 0)$.

Theorem 5.2. *Let $f = f_1 + f_2 + f_3$, where f is polynomial of degree 3 of type $\mathcal{F}$ or $\mathcal{B}$ on $\mathbb{C}^3$. If f is a polynomial of Broughton type then $\lambda \neq 0, \mu = 0$ and the following conditions hold.*

- **(i)** *f_3 is cuspidal and the only singularity at infinity of two special fibers is of type D_5 or a single fiber of type E_6.*
- **(ii)** *f_3 is conic plus tangent and the only singularity at infinity of the special fibers is of type D_5.*
- **(iii)** *f_3 is conic plus chord and the combination of the singularities at infinity of the special fiber is of type A_1A_5.*
- **(iv)** *f_3 is triangle and the combination of the singularities at infinity of the special fiber is of type $A_1A_1A_3$.*
- **(v)** *f_3 is double line plus simple line and the singularities at infinity of the special fiber is of type D_5.*

Proof. The proof follows directly from the Tables. □

Example 5.3. Let f_3 be three concurrent lines or f_3 is nodal. Then

$$f(x_0, x_1, x_2) = f_1(x_0, x_1, x_2) + f_2(x_0, x_1, x_2) + f_3(x_0, x_1, x_2)$$

is not a polynomial of Broughton type.

Proof. See the tables. □

Theorem 5.4. *Let $f = f_1 + f_2 + f_3 : \mathbb{C}^3 \to \mathbb{C}$ a polynomial of degree 3 of type $\mathcal{F}$ or $\mathcal{B}$. Then f is a global fibration iff $\lambda = \mu = 0$, which is one of the following cases:*

triple line: $f_3 = x_1^3$						
$f = a_0x_0 + a_1x_1 + a_2x_2 + a_3x_0^2 + a_4x_0x_1 + a_5x_0x_2 + a_7x_1x_2 + a_8x_2^2 + x_1^3$						
$Q(1:0:0:0)$	$R(0:0:1:0)$	$\neq 0$	$= 0$	λ	μ	b_2
A_2	A_2	γ	a_8	0	2	2
A_5		a_4, a_8	γ	0	1	1
∞		a_0, a_8	γ, a_4	–	–	–
$\gamma = a_5^2 - 4a_3a_8$. There are no jumps in this case						

Table 8.

(i) *f_3 is conic plus tangent and the only singularity at infinity of the special fiber is of type D_5.*
(ii) *f_3 is three concurrent lines and the only singularity at infinity of the special fiber is of type D_4 or A_4.*
(iii) *f_3 is double line plus simple line and the only singularity at infinity of the special fiber is of type A_4 or D_5.*

Proof. $\lambda = \mu = 0$ for $\mathcal{F}$, $\mathcal{B}$- class $\Leftrightarrow$ global fibration follows from ([14], Corollary 5.8). □

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References

[1] V.I. Arnold, *Critical points of functions on a manifold with boundary the simple Lie groups B_k, C_k, F_4 and singularities of evolutes*, Uspekhi Mat. Nauk 33 (1978) , no 5(203), 91-105, 237. DOI: 10.1017/cbo9780511662713.007
[2] V.I. Arnold, *Singularities of fractions and behaviour of polynomials at infinity*,Tr. Mat. Inst. Steklova 221 (1998), 48-68.
[3] A. Bodin, M. Tibar, *Topological Triviality of Families of Complex Polynomials*, Adv. Math. 199 2006, 136-150.
[4] E. Brieskorn and H. Knorrer, *Plane algebraic curves*, Birkhauser Verlag Basel (1986).
[5] S.A. Broughton, *On the topology of polynomial hypersurfaces*, Proceedings A.M.S Symp. in Pure Math.,vol. 40, I (1983), 165-178.
[6] J.W. Bruce and C. T. C. Wall, *On The Classification of Cubic Surfaces*, J.London Math. Soc(2) 19 (1979), 245-256.
[7] A. H. Durfee, *Five definitions of critical point at infinity*, Progress in Math. vol. 162, (1998).
[8] H. Hamm and D. T. Lê, *Sur la topologie de polinômes complexes*, Acta Math. Vietnamica 9 (1984), 21-32.
[9] D. T. Lê, *Complex analytic functions with isolated singularities*, J. Algebraic Geometry, 1 (1992), 83-100.
[10] F. Pham, *Vanishing homologies and the n variable saddle point method*, Arcata Proc. of Symp. in Pure Math. vol. 40, II (1983), 319-333.
[11] N. R. Ribeiro, *Singularidades no infinito de funções polinomiais*, Tese de Doutorado, ICMC-USP (2012). DOI: 10.11606/t.55.2012.tde-21022013-085310
[12] D. Siersma and J. Smeltink, *Classification of singularities at infinity of polynomials of degree 4 in two variables*, Georgian Mathematical Journal, Vol. 7 , Number 1, (2000), 179-198. DOI: 10.1515/gmj.2000.179
[13] D. Siersma, M. Tibăr, *Deformations of Polynomials, Boundary Singularities and Monodromy*, Mosc. Math. J. 3 (2003), 1-19.
[14] D. Siersma, M. Tibăr, *Singularities at infinity and their vanishing cycles*, Duke Math. Journal 80 (3) (1995), 771-783. DOI: 10.1215/s0012-7094-95-08026-0
[15] D. Siersma, M. Tibăr, *Singularity Exchange at the Frontier of the Space, Real and Complex Singularities*, (São Carlos Workshop 2004). Trends in Mathematics, pp 327-342, Birkhauser Verlag 2006. DOI: 10.1007/978-3-7643-7776-2_23
[16] M. Tibăr, *On the monodromy fibration of polynomial functions with singularities at infinity*, C.R. Acad. Sci. Paris, 324 (1997), 1031-1035. DOI: 10.1016/s0764-4442(97)87881-0
[17] M. Tibăr, *Polynomials and vanishing cycles*, Cambridge Tracts in Mathematics, 170. Cambridge University Presss, (2007).
[18] M. Tibăr, *Regularity at infinity of real and complex polynomial maps*, Singularity Theory, the C.T.C. Wall Anniversary Volume, LMS Lecture Notes Series 263 (1999), 249-264. Cambridge University Press.

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