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MINKOWSKI SYMMETRY SETS FOR 1-PARAMETER FAMILIES OF PLANE CURVES

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Abstract. In this paper the generic bifurcations of the Minkowski symmetry set for 1-parameter families of plane curves are classified and the necessary and sufficient geometric criteria for each type are given. The Minkowski symmetry set is an analogue of the standard Euclidean symmetry set, and is defined to be the locus of centres of all of its bitangent pseudo-circles. It is shown that the list of possible bifurcation types is different to that of the list of possible types for the Euclidean symmetry set.

1. Introduction

Symmetry sets and related constructions have provided useful representations of shapes for object recognition as well as attracted interest in their own right and in the geometric properties of curves that they reveal. In the standard Euclidean plane, the *(Euclidean) symmetry set* of a curve γ is defined as the locus of the centres of circles that are tangent to γ in at least two distinct points (bitangent), see for example [3, 4]. The medial axis of γ is a subset of its symmetry set, and is defined to be the locus of the centres of circles that are bitangent to γ and completely contained in γ. Introduced by Blum in 1967 [1], the medial axis (also referred to as the central set, the topological skeleton, and the shock set for grassfire flows) was originally designed as a tool for biological shape recognition and has found various applications in computer vision (see for example [5, 12]).

The Minkowski symmetry set of a curve γ was introduced in [13] as a Minkowski analogue of the (Euclidean) symmetry set. It is defined to be the locus of the centres of pseudo-circles that are bitangent to γ. In [10] the singularities of the Minkowski symmetry set for a generic curve are classified and in [11] a Minkowski version of the medial axis was introduced.

In [3], the transitions that occur for (Euclidean) symmetry sets of 1-parameter families of curves are classified. Moreover, the complete list of full bifurcation sets for a generic family of functions are given, and it is demonstrated that certain transitions are excluded for geometrical reasons. Analogous to this, in the present paper the generic bifurcations of the Minkowski symmetry set for 1-parameter families of plane curves are classified and their criteria are determined.

Main Theorem 1.1. *The possible transition types of the Minkowski Symmetry set for a generic curve are* $A_1^4(a), A_1^4(b), A_2^2(a), A_2^2(b)$, $A_1A_3(a)$, $A_1A_3(b)$, $A_1^2A_2(a)$, $A_1^2A_2(b)$ *and* A_4.

Remark 1.2. *Note that the list of possible transition types for the Minkowski Symmetry Set differs from that of the Euclidean Symmetry Set where only types* $A_1^4(a)$, $A_2^2(a)$, $A_2^2(b)$, $A_1A_3(a)$, $A_1^2A_2(a)$, *and* A_4 *can occur (see* [3]*).*

Remark 1.3. *For the Euclidean Medial Axis, only types* $A_1^4(a)$ *and* $A_1A_3(a)$ *can have the centre on the medial axis (see for example* [7]*). The Minkowski Medial Axis was defined in* [11] *as the locus of centres of pseudo-circles that are bitangent to* γ *with one of its branches. It follows that*

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only $A_1^4(b)$ *and* $A_1A_3(a)$ *can have their centres on the Minkowski Medial Axis (see main text and the table below).*

Remark 1.4. *In* [8] *an affine version of the Symmetry Set called the Affine Distance Symmetry Set was considered. It was shown that* $A_1^4(a)$, $A_1^4(b)$,$A_2^2(a)$, $A_2^2(b)$, $A_1A_3(a)$, $A_1A_3(b)$, $A_1^2A_2(a)$, $A_1^2A_2(b)$ *and* A_4 *could occur generically. In the case where* γ *is an oval (a strictly convex, smooth and closed curve), it was also shown that* $A_1^4(b), A_1^2A_2(b)$ *and* $A_1A_3(b)$ *were prohibited.*

	Euclidean	**Minkowski**	**Affine**
$A_1^4(a)$	✓	Odd # points per branch	✓
$A_1^4(b)$	×	Even # points per branch	Not for Ovals
$A_2^2(a)$	$\kappa_1'\kappa_2' > 0$	$\kappa_1'\kappa_2' > 0$ (M. Curvature)	✓
$A_2^2(b)$	$\kappa_1'\kappa_2' < 0$	$\kappa_1'\kappa_2' < 0$ (M. Curvature)	✓
$A_1A_3(a)$	✓	Points on different branches	✓
$A_1A_3(b)$	×	Points on the same branch	Not for Ovals
$A_1^2A_2(a)$	✓	A_1 points on same branch	✓
$A_1^2A_2(b)$	×	A_1 points on opposite branches	Not for Ovals
A_4	✓	✓	✓

2. THE MINKOWSKI PSEUDO-METRIC

The *Minkowski plane* $(\mathbb{R}^2_1, \langle , \rangle)$ is the vector space $\mathbb{R}^2$ endowed with the pseudo-scalar product $\langle u, v\rangle = -u_0v_0 + u_1v_1$, for any $u = (u_0, u_1)$ and $v = (v_0, v_1)$. A vector $u \in \mathbb{R}^2_1$ is called *timelike* if $\langle u, u\rangle < 0$, *spacelike* if $\langle u, u\rangle > 0$, and *lightlike* if $\langle u, u\rangle = 0$.

The norm of u is defined by $||u|| = \sqrt{|\langle u, u\rangle|}$, and the perpendicular operator $\perp$ assigns $u^\perp = (u_1, u_0)$.

There are three distinct types of pseudo-circles in $\mathbb{R}^2_1$ with centre $c \in \mathbb{R}^2_1$ and radius r, $r > 0$, are defined as follows:

$$\begin{aligned} H^1(c, -r) &= \{p \in \mathbb{R}^2_1 \,|\, \langle p - c, p - c\rangle = -r^2\}, \\ S^1_1(c, r) &= \{p \in \mathbb{R}^2_1 \,|\, \langle p - c, p - c\rangle = r^2\}, \\ LC^*(c) &= \{p \in \mathbb{R}^2_1 \setminus \{c\} \,|\, \langle p - c, p - c\rangle = 0\}. \end{aligned}$$

Observe that $LC^*(c)$ is the union of the two lines through c with tangent directions $(1, 1)$ and $(1, -1)$, with the point c removed. The pseudo-circle $H^1(c, -r)$ has two branches which can be parametrised by $c + (\pm r\cosh(t), r\sinh(t))$, $t \in \mathbb{R}$. The pseudo-circle $S^1(c, r)$ is also composed of two branches and these can be parametrised by $c + (r\sinh(t), \pm r\cosh(t))$, $t \in \mathbb{R}$.

Let $\gamma : S^1 \to \mathbb{R}^2_1$ be an immersion, where S^1 is the unit Euclidean circle. Call the curve γ the image of the map γ and say that it is a closed smooth curve (that is, γ is a regular closed curve and may have points of self-intersection).

The curve γ at t_0 is said to be spacelike if $\gamma'(t_0)$ is spacelike and is said to be timelike if $\gamma'(t_0)$ is timelike. These are open properties so there is a neighbourhood of t_0 where the curve is either spacelike or timelike. If $\gamma'(t_0)$ is lightlike then $\gamma(t_0)$ is said to be a lightlike point. It is shown in [13] that the set of lightlike points of γ is the union of at least four disjoint non-empty and closed subsets of γ. The complement of these sets are disjoint connected spacelike or timelike pieces of the curve γ.

The spacelike and timelike components of γ can be parametrised by arc length. Suppose that $\gamma(s)$, $s \in (\lambda, \mu)$, is an arc length parametrisation of a component of γ. Then $\boldsymbol{t}(s) = \gamma'(s)$ is a unit tangent vector and $\boldsymbol{t}'(s) = \kappa(s)\boldsymbol{n}(s)$, where $\kappa(s)$ is the Minkowski curvature of γ at s and $\boldsymbol{n}$ is the unit normal vector at s. The tangent and unit normal vectors are pseudo-orthogonal so they are of different types, that is, one is spacelike and the other is timelike.

When γ is not necessarily parametrised by arclength, the unit tangent is given by

$$T(t) = \frac{\gamma'(t)}{|\langle\gamma'(t), \gamma'(t)\rangle|^{\frac{1}{2}}},$$

the unit normal by

$$N(t) = (-1)^{\beta} T(t),$$

where $\beta = 1$ if γ is spacelike and $\beta = 2$ if γ is timelike, and the Minkowski curvature (dropping the parameter t) is given by

$$\kappa = \frac{\langle\gamma', \gamma''^{\perp}\rangle}{|\langle\gamma', \gamma'\rangle|^{\frac{3}{2}}}.$$

3. The Minkowski Symmetry Set

The evolute of a spacelike or timelike component of $\gamma(s)$, $s \in (\lambda, \mu)$ is the image of the map

$$e(t) = \gamma(t) - \frac{1}{\kappa(t)} N(t).$$

In general, the curvature tends to infinity as t tends to λ or μ and the evolute of the curve γ is not defined at these lightlike points. However, the caustic of γ is defined everywhere and contains the evolute of γ (see for example [13]). The caustic can be defined via the *the family of distance-squared functions* $f : S^1 \times \mathbb{R}^2_1 \to \mathbb{R}$ on γ given by

$$f(t, c) = \langle\gamma(t) - c, \gamma(t) - c\rangle.$$

Denote by $f_c : S^1 \to \mathbb{R}$ the function given by $f_c(t) = f(t, c)$. We say that f_c has an A_k-singularity at t_0 if $f'_c(t_0) = f''_c(t_0) = \ldots = f^{(k)}_c(t_0) = 0$ and $f^{(k+1)}_c(t_0) \neq 0$. This is equivalent to the existence of a local re-parametrisation h of γ at t_0 such that $(f \circ h)(t) = \pm t^{k+1}$. Geometrically, f_c has an A_k-singularity if and only if the curve γ has contact of order $k+1$ at $\gamma(t_0)$ with the pseudo-circle of centre c and radius r, with $r = \langle\gamma(t_0) - c, \gamma(t_0) - c\rangle$. Thus, the curve γ has point contact of order 1 with a pseudo-circle at t_0 if it transversally intersects the pseudo-circle at $\gamma(t_0)$. The order of contact is 2 if the circle and the curve have ordinary tangency at $\gamma(t_0)$.

The caustic of γ is the local component $\mathcal{B}_1$ of the bifurcation set of the family f, given by

$$\mathcal{B}_1 = \{c \in \mathbb{R}^2_1 \,|\, \exists t \in S^1 \text{ such that } f'_c(t) = f''_c(t) = 0\}.$$

This is the set of points $c \in \mathbb{R}^2_1$ such that the germ f_c has a degenerate singularity at some point t. In [13] it was shown that the caustic of γ is defined at all points on γ including its lightlike points where it is a smooth curve and has ordinary tangency with γ.

The multi-local component of the bifurcation set of the family f is defined as

$$\mathcal{B}_2 = \{c \in \mathbb{R}^2_1 \,|\, \exists t_1, t_2 \text{ such that } t_1 \neq t_2,\ f_c(t_1) = f_c(t_2),\ f'_c(t_1) = f'_c(t_2) = 0\}.$$

The full-bifurcation set of f is defined as

$$\mathrm{Bif}(f) = \mathcal{B}_1 \cup \mathcal{B}_2.$$

Definition 3.1. *The Minkowski Symmetry Set (MSS) of γ is the locus of centres of pseudo-circles which are tangent to γ in at least two distinct points p and q. The pairs of points p, q are called bitangent pairs.*

The MSS is precisely the multi-local component $\mathcal{B}_2$ of the bifurcation set of the family of distance-squared function f on γ.

In [10] it is shown that the singularities which can occur on the MSS for a generic plane curve are A_1, A_2, A_1^3, A_1A_2 and A_3, and that they are all versally unfolded. It follows that these singularities are also versally unfolded for a 1-parameter family of plane curves. It can happen for a generic 1-parameter family of plane curves that at isolated points one of the above singularities occurs at lightlike points and this case is also dealt with in [10]. It only remains now to show the versality and the transition type for the other generically occurring singularities for a 1-parameter family of plane curves, namely $A_1^4, A_1^2A_2, A_1A_3, A_2^2$ and A_4. For these singularities, which only occur generically for a family of curves depending on a parameter, the bifurcation sets undergo a (sudden) structural change as we vary the parameter, so for this reason (following [7]) we refer to these as 'transitions'.

In [3] it was shown that for general functions some of these singularities occur in two distinct transition types. For example, in the A_1^4 case there exist two types referred to as $A_1^4(a)$ and $A_1^4(b)$. It was shown in that paper that only types $A_1^4(a), A_2^2(a), A_2^2(b), A_1A_3(a), A_1^2A_2(a)$ and A_4 could occur for (Euclidean) symmetry sets (see Table on page 362). In the present paper a similar analysis is carried out for the Minkowski symmetry set and the geometric conditions for the possible types are determined. In particular, the following theorem is proven:

Theorem 3.2. *The possible transition types of the Minkowski Symmetry set for a generic curve are* $A_1^4(a), A_1^4(b), A_2^2(a), A_2^2(b)$*,* $A_1A_3(a)$*,* $A_1A_3(b)$*,* $A_1^2A_2(a)$*,* $A_1^2A_2(b)$ *and* A_4*.*

Each generically occurring singularity type is considered in turn. Considering the reduction of the distance-squared family to its normal form, the necessary geometrical criteria for each transition type (e.g. a or b) is determined.

4. The A_1^4 singularity

Consider the standard multi-versal unfolding of an A_1^4 singularity given by

$$G : \mathbb{R}^{(4)} \times \mathbb{R}^3 \to \mathbb{R},$$

where $\mathbb{R}^{(4)}$ denotes the set of parameters t_1, t_2, t_3, t_4, $\mathbb{R}^3$ denotes the $\boldsymbol{y}$-space of unfolding parameters (y_1, y_2, y_3) and the multi-versal unfolding G is given by

$$\begin{aligned} G_i &: (t_i, \boldsymbol{y}) \mapsto t_i^2 + y_i, i = 1, 2 \text{ and } 3 \\ G_4 &: (t_4, \boldsymbol{y}) \mapsto t_4^2. \end{aligned}$$

Consider now four families of curve segments $\gamma_1, \gamma_2, \gamma_3$ and γ_4 each being close to one of the tangency points. With family parameter u, denote these segments as

$$\gamma_{i,u}(s_i) = (X_{i,u}(s_i), Y_{i,u}(s_i)),$$

where the arclength parameters s_i are close to zero. Take $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}_1^2$, and denote by $\boldsymbol{x}_0$ the A_1^4-point on the MSS. Then the family of Minkowski distance functions on the family of curve segments consists of four germs

$$F_i : \mathbb{R} \times \mathbb{R} \times \mathbb{R}_1^2, (0, 0, \boldsymbol{x}_0) \to \mathbb{R},$$

given by

$$F_i(s_i, u, \boldsymbol{x}) = \langle \boldsymbol{x} - \gamma_{i,u}, \boldsymbol{x} - \gamma_{i,u} \rangle.$$

Using standard techniques, as outlined in [3], and used for example in [8] and [9], the aim is to reduce the family F_i to a standard family G_i. The big bifurcation set (BBS), which sits in $\boldsymbol{y}$-space and comprises of subsets which correspond to A_1^2 sets of G, contains all the possible types bifurcations of A_1^4, and the individual bifurcation sets can be recovered locally by slicing the BBS with non-singular families of surfaces passing through the origin in $\boldsymbol{y}$-space. Firstly, the possible generic transition types and their criteria are found, and then through keeping track of the geometric properties in reducing the family to the standard type, the relevant bifurcation type can be determined.

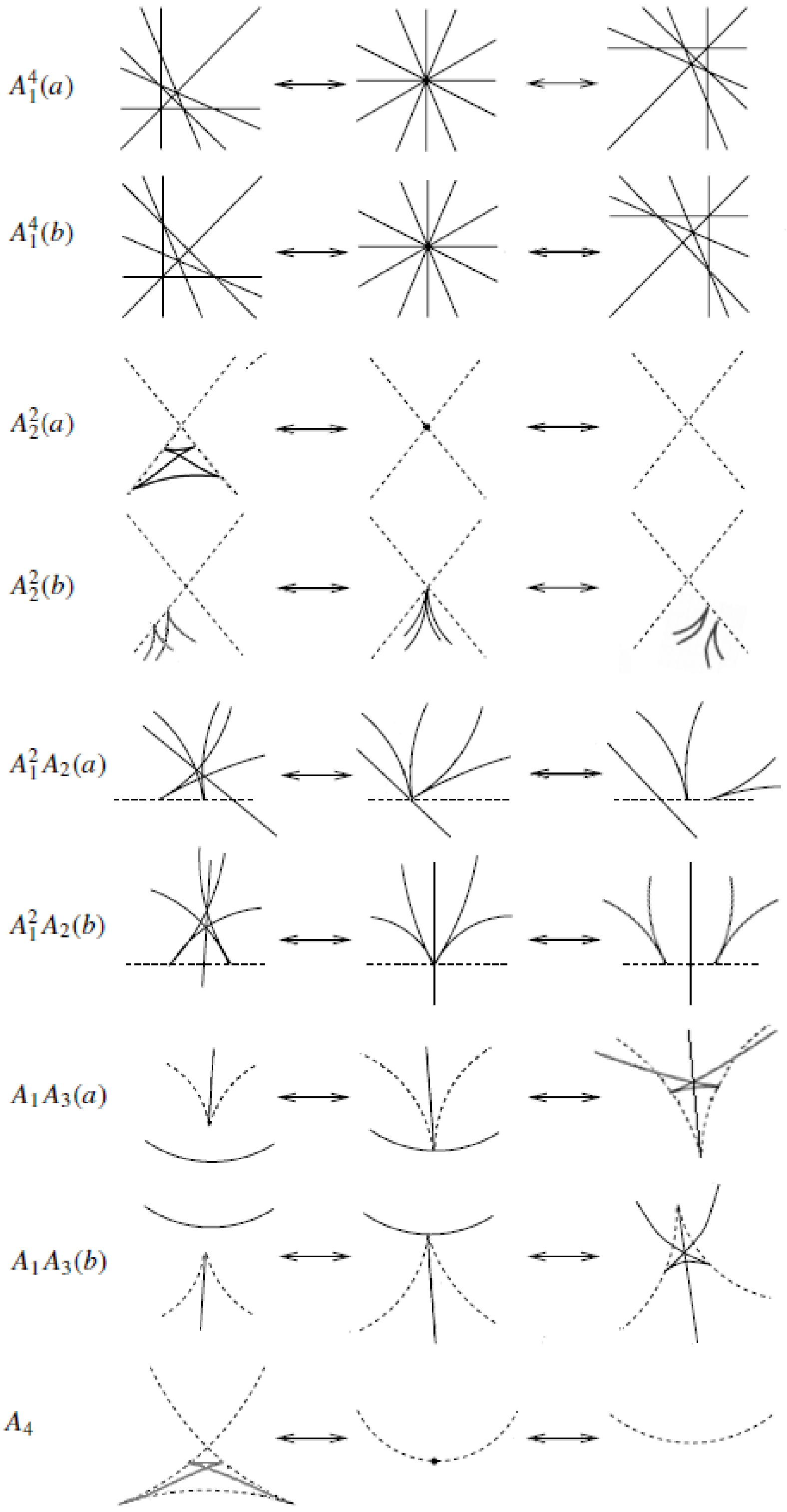

FIGURE 1. The transitions that can occur on Minkowski Symmetry Sets.

4.1. **Bad planes.** Following [3], a plane containing the origin given by the equation

$$a_1y_1 + a_2y_2 + a_3y_3 = 0$$

is called a *bad plane* if it contains any of the limiting tangent vectors to the strata of the big bifurcation set of G. Non-generic transitions occur when these slicing surfaces are themselves tangent to the limiting tangent vectors to the strata of the big bifurcation set tending to the origin. A plane can be represented by a point with homogeneous coordinates $(a_1 : a_2 : a_3)$ in the real projective plane $\mathbb{R}P^2$ and the pencils of bad planes therefore correspond to lines in $\mathbb{R}P^2$.

If Δ represents the set of bad planes each component of $\mathbb{R}P^2 - \Delta$ represent collections of normals, which as kernels of $dh(0)$ give C^0-stratified equivalent functions of h. (For remarks on stratified equivalence see for example [3] and [2].) Each connected component of $\mathbb{R}P^2 - \Delta$ can potentially give a different type of transition. By considering each region in turn and identifying the type of transition it is possible to determine the criteria for realising each one.

The one-dimensional strata adjacent to the BBS for the standard A_1^4 are

$$\begin{aligned} A_1^3 &\;:\; \{(a_1, a_2, a_3) = (t_1, t_1, t_1) \cup (t_1, 0, 0) \cup (0, 0, t_2) \cup (0, 0, t_3)\} \\ A_1^2/A_1^2 &\;:\; \{(a_1, a_2, a_3) = (t_1, t_1, 0) \cup (0, t_2, t_2) \cup (t_3, 0, t_3)\}. \end{aligned}$$

The limiting tangent vectors to these one-dimensional strata are therefore given by $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 1)$, $(1, 1, 0)$, $(0, 1, 1)$, and $(1, 0, 1)$ so the bad planes are given by $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, and $a_1 + a_2 + a_3 = 0$, $a_1 + a_2 = 0$, $a_2 + a_3 = 0$, and $a_1 + a_3 = 0$.

It is determined that the shaded regions of Figure 3 (right) correspond to one type of transition and the non-shaded regions give another from which the following proposition can be deduced.

Proposition 4.1. *If $a_1a_2a_3(a_1 + a_2 + a_3)$ is negative the point $(a_1 : a_2 : a_3)$ lies in the shaded region of Figure 3 (right) and the corresponding full bifurcation set has type $A_1A_3(a)$. If however $a_1a_2a_3(a_1 + a_2 + a_3)$ is positive, then the point lies in the unshaded region and the corresponding full bifurcation set is of type $A_1A_3(b)$.*

Since it is assumed that each F_i is a multi-versal unfolding, then by the uniqueness of multi-versal unfoldings each of the unfoldings G_i in the standard multi-versal unfolding G can be induced from the affine distance functions F_i by

$$G_i(t_i, \boldsymbol{y}) = F_i(A_i(t_i, \boldsymbol{y}), B(\boldsymbol{y})) + C(\boldsymbol{y}), \text{for } i = 1, 2, 3 \text{ and } 4, \tag{1}$$

where each $A_i : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ is a germ at $(0, \mathbf{0})$ and B, C denote the germs

$$B : (\mathbb{R}^3, \mathbf{0}) \to (\mathbb{R} \times \mathbb{R}^2, (0, \boldsymbol{x}_0)) \quad \text{and} \quad C : (\mathbb{R}^3, \mathbf{0}) \to (\mathbb{R}, d_0).$$

$$\begin{tikzcd}
\mathbb{R} \times \mathbb{R}^3 \arrow[r, "G"] \arrow[d, "(A_i \times B)"] & \mathbb{R} \times \mathbb{R}^3 \arrow[r] \arrow[d, "(-C \times B)"] & \mathbb{R}^3 \arrow[r, "h"] \arrow[d, "B"] & \mathbb{R} \arrow[d, "\text{identity}"] \\
C \arrow[r, "F"] & D \arrow[r] & D \arrow[r, "\pi_1"] & \mathbb{R}
\end{tikzcd}$$

From the commutative diagram it can be seen that $h = \pi_1 \circ B$, where π_1 denotes projection onto the first coordinate. Thus, B_1 (where B_i denotes the i^{th} component of B) is the map h on the standard A_1^4 set (the BBS), which corresponds to the plane through the origin in $\boldsymbol{y}$-space representing the tangent plane to the surface with which we are slicing the BBS. This tangent plane thus corresponds to the kernel of the map h on the BBS, i.e.

$$\ker dB_1 : \mathbb{R}^3 \to \mathbb{R}, \text{ with matrix } \left.\left(\frac{\partial B_1}{\partial y_1}, \frac{\partial B_2}{\partial y_2}, \frac{\partial B_3}{\partial y_3}\right)\right|_{\boldsymbol{y}=\mathbf{0}}.$$

Hence the kernel plane has equation

$$\left.\frac{\partial B_1}{\partial y_1}\right|_{\boldsymbol{y}=\mathbf{0}} y_1 + \left.\frac{\partial B_2}{\partial y_2}\right|_{\boldsymbol{y}=\mathbf{0}} y_2 + \left.\frac{\partial B_3}{\partial y_3} y_3\right|_{\boldsymbol{y}=\mathbf{0}} = 0.$$

Proposition 4.2. *The MSS has a transition of type $A_1^4(a)$ if there are an odd number of points on each branch and is of type $A_1^4(b)$ if there are an even number of points on each branch.*

Proof. Consider the case $i = 1$:

$$\left.\begin{pmatrix} \frac{\partial G_1}{\partial t_1} & \frac{\partial G_1}{\partial y_1} & \frac{\partial G_1}{\partial y_2} & \frac{\partial G_1}{\partial y_3} \end{pmatrix}\right|_{\boldsymbol{y}=\mathbf{0}} = (2t_1 \ 1 \ 0 \ 0).$$

Using relation (1) and applying the chain rule for derivatives gives the left-hand side of this as:

$$\left.\begin{pmatrix} \frac{\partial F_1}{\partial s_1} & \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \frac{\partial F_1}{\partial x_3} \end{pmatrix}\right|_{(A_1(t_1,\mathbf{0}),\boldsymbol{x}_0)} \times \left.\begin{pmatrix} \frac{\partial A_1}{\partial t_1} & \frac{\partial A_1}{\partial y_1} & \frac{\partial A_1}{\partial y_2} & \frac{\partial A_1}{\partial y_3} \\ 0 & \frac{\partial B_1}{\partial y_1} & \frac{\partial B_1}{\partial y_2} & \frac{\partial B_1}{\partial y_3} \\ 0 & \frac{\partial B_2}{\partial y_1} & \frac{\partial B_2}{\partial y_2} & \frac{\partial B_2}{\partial y_3} \\ 0 & \frac{\partial B_3}{\partial y_1} & \frac{\partial B_3}{\partial y_2} & \frac{\partial B_3}{\partial y_3} \end{pmatrix}\right|_{(t_1,\mathbf{0})} + \left.\begin{pmatrix} 0 & \frac{\partial C}{\partial y_1} & \frac{\partial C}{\partial y_2} & \frac{\partial C}{\partial y_3} \end{pmatrix}\right|_{\boldsymbol{y}=\mathbf{0}}.$$

The same can be done for G_2, G_3 and G_4, which have the right side of the first line as $(2t_2 \ 0 \ 1 \ 0)$, $(2t_3 \ 0 \ 0 \ 1)$ and $(2t_4 \ 0 \ 0 \ 0)$ respectively. Now $\frac{\partial F_i}{\partial s_i}(0, \boldsymbol{x}_0) \equiv 0$ because F_i has an A_1 singularity at $(0, \boldsymbol{x}_0)$. Also, $\frac{\partial F_i}{\partial x_1} = -2x_1 + 2X_{u,i}(s_i)$, $\frac{\partial F_i}{\partial x_2} = 2x_2 - 2Y_{u,i}(s_i)$. The substitution $t_i = 0$ can be made since only the 0-jets are required.

Taking all the G_i together gives the system:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \left.\begin{pmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \\ \frac{\partial F_3}{\partial u} & \frac{\partial F_3}{\partial x_1} & \frac{\partial F_3}{\partial x_2} \\ \frac{\partial F_4}{\partial u} & \frac{\partial F_4}{\partial x_1} & \frac{\partial F_4}{\partial x_2} \end{pmatrix}\right|_{(A(t_i,\mathbf{0}),\boldsymbol{x}_0)} \times JB + \begin{pmatrix} JC \\ JC \\ JC \\ JC \end{pmatrix} \tag{2}$$

where, for conciseness, JB and JC denote the matrices

$$JB = \left.\begin{pmatrix} \frac{\partial B_1}{\partial y_1} & \frac{\partial B_1}{\partial y_2} & \frac{\partial B_1}{\partial y_3} \\ \frac{\partial B_2}{\partial y_1} & \frac{\partial B_2}{\partial y_2} & \frac{\partial B_2}{\partial y_3} \\ \frac{\partial B_3}{\partial y_1} & \frac{\partial B_3}{\partial y_2} & \frac{\partial B_3}{\partial y_3} \end{pmatrix}\right|_{\boldsymbol{y}=\mathbf{0}}, \quad JC = \left.\begin{pmatrix} \frac{\partial C}{\partial y_1} & \frac{\partial C}{\partial y_2} & \frac{\partial C}{\partial y_3} \end{pmatrix}\right|_{\boldsymbol{y}=\mathbf{0}}.$$

Subtracting the bottom row from the other rows in equation (2) gives

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \left.\begin{pmatrix} \frac{\partial F_1}{\partial u} - \frac{\partial F_4}{\partial u} & \frac{\partial F_1}{\partial x_1} - \frac{\partial F_4}{\partial x_1} & \frac{\partial F_1}{\partial x_2} - \frac{\partial F_4}{\partial x_2} \\ \frac{\partial F_2}{\partial u} - \frac{\partial F_4}{\partial u} & \frac{\partial F_2}{\partial x_1} - \frac{\partial F_4}{\partial x_1} & \frac{\partial F_2}{\partial x_2} - \frac{\partial F_4}{\partial x_2} \\ \frac{\partial F_3}{\partial u} - \frac{\partial F_4}{\partial u} & \frac{\partial F_3}{\partial x_1} - \frac{\partial F_4}{\partial x_1} & \frac{\partial F_3}{\partial x_2} - \frac{\partial F_4}{\partial x_2} \\ \frac{\partial F_4}{\partial u} & \frac{\partial F_4}{\partial x_1} & \frac{\partial F_4}{\partial x_2} \end{pmatrix}\right|_{(A(t_i,\mathbf{0}),\boldsymbol{x}_0)} \times JB + \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ JC \end{pmatrix}.$$

Substituting $\frac{\partial F_i}{\partial x_1}$ and $\frac{\partial F_i}{\partial x_2}$ and ignoring the last row yields the following system:

$$I_3 = \begin{pmatrix} \frac{\partial F_1}{\partial u} - \frac{\partial F_4}{\partial u} & X_1 - X_4 & -Y_1 + Y_4 \\ \frac{\partial F_2}{\partial u} - \frac{\partial F_4}{\partial u} & X_2 - X_4 & -Y_2 + Y_4 \\ \frac{\partial F_3}{\partial u} - \frac{\partial F_4}{\partial u} & X_3 - X_4 & -Y_3 + Y_4 \end{pmatrix} \times \begin{pmatrix} \frac{\partial B_1}{\partial y_1} & \frac{\partial B_1}{\partial y_2} & \frac{\partial B_1}{\partial y_3} \\ \frac{\partial B_2}{\partial y_1} & \frac{\partial B_2}{\partial y_2} & \frac{\partial B_2}{\partial y_3} \\ \frac{\partial B_3}{\partial y_1} & \frac{\partial B_3}{\partial y_2} & \frac{\partial B_3}{\partial y_3} \end{pmatrix}$$

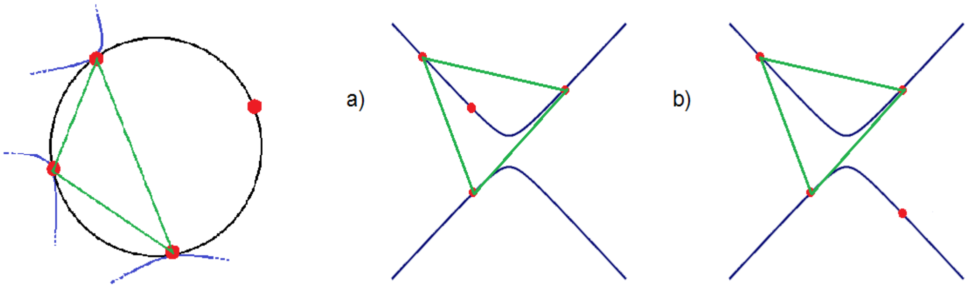

FIGURE 2. Left: Given three points on a circle, a fourth point necessarily lies outside the triangle formed by the other three. Right: Given three points on a pseudo-circle, a fourth point can either lie inside (resulting in singularity $A_1^4(a)$), or outside the triangle formed by the other three (resulting in the singularity $A_1^4(b)$).

where I_3 represents the (3×3) identity matrix.

The derivatives of B_1 can now be evaluated. Since the product of the two matrices is the identity, they must be inverse to each other. Now, the inverse of the first matrix can be used to calculate the required entries of the second matrix. So,

$$\frac{\partial B_1}{\partial y_1} = \beta \det \begin{pmatrix} X_2 - X_4 & -Y_2 + Y_4 \\ X_3 - X_4 & -Y_3 + Y_4 \end{pmatrix},$$

where $\beta = 1$ if γ is spacelike and $\beta = 2$ if γ is timelike.

Multiplying the second column by -1 gives

$$\frac{\partial B_1}{\partial y_1} = -\beta \det \begin{pmatrix} X_2 - X_4 & Y_2 - Y_4 \\ X_3 - X_4 & Y_3 - Y_4 \end{pmatrix}.$$

Similarly,

$$\frac{\partial B_1}{\partial y_2} = -\beta \det \begin{pmatrix} X_1 - X_4 & Y_1 - Y_4 \\ X_3 - X_4 & Y_3 - Y_4 \end{pmatrix},$$

$$\frac{\partial B_1}{\partial y_3} = -\beta \det \begin{pmatrix} X_1 - X_4 & Y_1 - Y_4 \\ X_2 - X_4 & Y_2 - Y_4 \end{pmatrix}.$$

Let $q_1 = \gamma_2 - \gamma_3$, $q_2 = \gamma_3 - \gamma_4$, $q_3 = \gamma_4 - \gamma_1$ and $q_4 = \gamma_1 - \gamma_2$. Now, $\frac{\partial B_1}{\partial y_1} = -\beta \det \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$, $\frac{\partial B_1}{\partial y_2} = \beta \det \begin{pmatrix} q_2 \\ q_3 \end{pmatrix}$, $\frac{\partial B_1}{\partial y_3} = -\beta \det \begin{pmatrix} q_3 \\ q_4 \end{pmatrix}$, and $\frac{\partial B_1}{\partial y_1} + \frac{\partial B_1}{\partial y_2} + \frac{\partial B_1}{\partial y_3} = -\beta \det \begin{pmatrix} q_3 \\ q_4 \end{pmatrix}$.

Now $\det(q_i, q_j) > 0$ if and only if the anticlockwise (Euclidean) angle from q_i to q_j is less than π. It then follows that $\frac{\partial B_1}{\partial y_1} \frac{\partial B_1}{\partial y_2} \frac{\partial B_1}{\partial y_3} \left(\frac{\partial B_1}{\partial y_1} + \frac{\partial B_1}{\partial y_2} + \frac{\partial B_1}{\partial y_3} \right) > 0$ if and only if no point p_i is inside the triangle formed by the other three p_j. This condition fails if and only if there are an even number of points on each branch and the resulting singularity is of type $A_1^4(b)$. On the other hand, if one of the branches contains only one point, and the other branch contains three, then the triangle formed by the point on the first branch and the 'outer' two points of the branch of three will necessarily contain the fourth point (see figure 2) and the singularity will be of type $A_1^4(a)$.

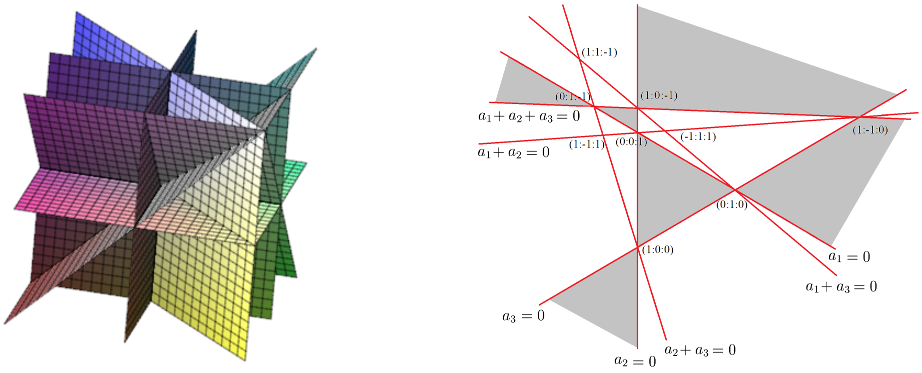

FIGURE 3. Left: The BBS for A_1^4. Right: The regions determining the different types for A_1^4.

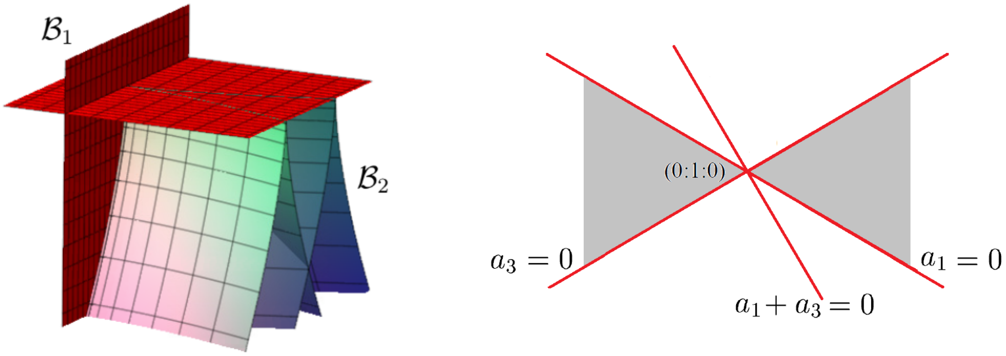

FIGURE 4. Left: The BBS for A_2^2. Right: The regions determining the different types for A_2^2.

5. THE A_2^2 SINGULARITY

Consider the following standard multi-versal unfolding of an $A_1^2A_2$ singularity given by

$$G:\mathbb{R}^{(2)}\times\mathbb{R}^3\to\mathbb{R}$$

where $\mathbb{R}^{(2)}$ denotes the parameters t_1, t_2 and $\mathbb{R}^3$ denotes the unfolding parameters $\boldsymbol{a}=(a_1,a_2,a_3)$ and the multi-versal unfolding is given by the two unfoldings:

$$\begin{aligned} G_1(t_1,\boldsymbol{a}) &= t_1^3+a_1t_1+a_2, \\ G_2(t_2,\boldsymbol{a}) &= t_2^3+a_3t_2. \end{aligned}$$

5.1. **The bad planes.** The one-dimensional strata adjacent to A_2^2 are

$$\begin{aligned} A_1A_2 \quad &: \quad \{(a_1,a_2,a_3)=(-3t_1^2,2t_1^3,0)\cup(0,-2t_2^3,-3t_2^2)\} \\ A_1^2/A_1^2 \quad &: \quad \{(a_1,a_2,a_3)=(-3t_2^2,0,-3t_2^2)\}. \end{aligned}$$

The limiting tangent vectors to these one-dimensional strata are given by $(1,0,0)$, $(0,1,0)$ and $(1,1,0)$ so the bad planes are given b $a_1=0$, $a_3=0$ and $a_1+a_3=0$.

Similarly to the previous case, the following proposition can be deduced.

Proposition 5.1. *If a_1a_2 is negative the point $(a_1 : a_2 : a_3)$ lies in the unshaded region of Figure 4 (right) and the corresponding full bifurcation set has type $A_2^2(a)$. If however a_1a_3 is positive, then the point lies in the shaded region and the corresponding full bifurcation set is of type $A_2^2(b)$.*

The Minkowski distance function on the two curve segments near the A_2^2 points consists of the two germs

$$F_1(t_1, u, x) = \langle \gamma_1(t_1, u) - x, \gamma_1(t_1, u) - x\rangle$$

$$F_2(t_2, u, x) = \langle \gamma_2(t_2, u) - x, \gamma_2(t_2, u) - x\rangle.$$

To reduce to G_1 and G_2, as in the A_1^4 case, using (1) and applying the chain rule gives the system:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \left.\begin{pmatrix} \frac{\partial^2 F_1}{\partial t_1 \partial u} & \frac{\partial^2 F_1}{\partial t_1 \partial x_1} & \frac{\partial^2 F_1}{\partial t_1 \partial x_2} \\ \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial^2 F_2}{\partial t_2 \partial u} & \frac{\partial^2 F_2}{\partial t_2 \partial x_1} & \frac{\partial^2 F_2}{\partial t_2 \partial x_2} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{pmatrix}\right|_{(A(t_i,\mathbf{0}),\boldsymbol{x}_0)} \times JB + \begin{pmatrix} \mathbf{0} \\ JC \\ \mathbf{0} \\ JC \end{pmatrix}.$$

Subtracting the bottom row from the second and then ignoring the bottom yields

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \left.\begin{pmatrix} \frac{\partial^2 F_1}{\partial t_1 \partial u} & \frac{\partial^2 F_1}{\partial t_1 \partial x_1} & \frac{\partial^2 F_1}{\partial t_1 \partial x_2} \\ \frac{\partial F_1}{\partial u} - \frac{\partial F_2}{\partial u} & \frac{\partial F_1}{\partial x_1} - \frac{\partial F_2}{\partial x_1} & \frac{\partial F_1}{\partial x_2} - \frac{\partial F_2}{\partial x_2} \\ \frac{\partial^2 F_2}{\partial t_2 \partial u} & \frac{\partial^2 F_2}{\partial t_2 \partial x_1} & \frac{\partial^2 F_2}{\partial t_2 \partial x_2} \end{pmatrix}\right|_{(A(t_i,\mathbf{0}),\boldsymbol{x}_0)} \times JB.$$

We can write $A_i(t_i, 0) = \alpha_i t_i +$ higher terms where

$$\alpha_i = (-\kappa/\kappa_i)^{\frac{1}{3}}$$

and here κ is the Minkowski curvature of γ at the two points of contact and κ_i' is the derivative of Minkowski curvature with respect to arclength on γ.

Differentiating F_1 (for example, though the same applies for F_2) gives

$$\frac{1}{2}\frac{\partial F_1(A_1(t_1, u), x)}{\partial t_1} = \alpha_1\langle(\gamma_1(t_1, u) - x), T_1\rangle$$

and differentiating this with respect to x gives

$$\left(\frac{1}{2}\frac{\partial^2 F_1(A_1(t_1, u), x)}{\partial t_1 \partial x_1}, \frac{1}{2}\frac{\partial^2 F_1(A_1(t_1, u), x)}{\partial t_1 \partial x_2}\right) = \alpha_1(X_1', -Y_1').$$

For the middle row we have

$$\left(\frac{1}{2}\frac{\partial F_i(A_i(t_i, u), x)}{\partial x_1}, \frac{1}{2}\frac{\partial F_i(A_i(t_i, u), x)}{\partial x_2}\right) = \langle(\gamma_1(t, u) - x), (-1, -1)\rangle.$$

Since F_1 has an A_2 singularity, $(\gamma(t, u) - x)$ can be written as $\frac{1}{\kappa_M}N_M$ and substituting this yields

$$\left(\frac{1}{2}\frac{\partial F_i(A_i(t_i, u), x)}{\partial x_1}, \frac{1}{2}\frac{\partial F_i(A_i(t_i, u), x)}{\partial x_2}\right) = 2\frac{1}{\kappa_M}(Y_i', -X_1').$$

Substituting these derivatives into the matrix equation gives:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \left.\begin{pmatrix} * & 2\alpha_1 X_1' & -2\alpha_1 Y_1' \\ * & \frac{2}{\kappa}(Y_1' - Y_2') & \frac{2}{\kappa}(X_2' - X_1') \\ * & 2\alpha_2 X_2' & -2\alpha_2 Y_2' \end{pmatrix}\right|_{(A(t_i,\mathbf{0}),\boldsymbol{x}_0)} \times JB.$$

Evaluating the cofactors gives

$$\frac{\partial B_1}{\partial a_1} = \frac{4}{\kappa}\alpha_2(\langle T_1, T_2\rangle \pm 1) \text{ and } \frac{\partial B_1}{\partial a_3} = \frac{4}{\kappa}\alpha_1(\langle T_1, T_2\rangle \pm 1),$$

where the sign of ± is the same for both derivatives and depends on whether the curves are spacelike or timelike.

The type of transition that occurs depends on the sign of $\frac{\partial B_1}{\partial a_1}\frac{\partial B_1}{\partial a_3}$. Now

$$\frac{\partial B_1}{\partial a_1}\frac{\partial B_1}{\partial a_3} = \frac{8}{\kappa_M^2}\alpha_1\alpha_2(\langle T_1, T_2\rangle \pm 1)^2$$

so the sign, and hence the transition type, depends on whether $\kappa_1'\kappa_2'$ is positive or negative.

Proposition 5.2. *In the multi-versal A_2^2 situation, assume in addition to $\kappa_i' \neq 0$, that $\kappa_1'+\kappa_2' \neq 0$ (κ_i' = the derivative of curvature on γ_0 with respect to arclength at the two contact points). Then the $A_2^2(a)$ or "moth transition" occurs when $\kappa_1'\kappa_2' > 0$ and the $A_2^2(b)$ or "nib transition" occurs when $\kappa_1'\kappa_2' < 0$.*

6. THE $A_1^2A_2$ SINGULARITY

Consider the following standard multi-versal unfolding of an $A_1^2A_2$ singularity given by

$$G : \mathbb{R}^{(3)} \times \mathbb{R}^3 \to \mathbb{R}$$

where $\mathbb{R}^{(3)}$ denotes the parameters t_1, t_2, t_3 and $\mathbb{R}^3$ denotes the unfolding parameters $\boldsymbol{a} = (a_1, a_2, a_3)$ and the multi-versal unfolding is given by the two unfoldings:

$$\begin{aligned} G_1(t_1, \boldsymbol{a}) &= t_1^3 + a_1 t_1, \\ G_2(t_2, \boldsymbol{a}) &= t_2^2 + a_2, \\ G_3(t_3, \boldsymbol{a}) &= t_3^2 + a_3. \end{aligned}$$

6.1. **The big bifurcation set.** At an $A_1^2A_2$ point the $\mathcal{B}_2$ set consists of three parts: The first is given as the solution of $G_1 = G_2$ and $G_1' = G_2' = 0$ and is a semi-cubic cylinder with the parametrisation $(-3t_1^2, 2t_1^3, a_3)$. The second is given as the solution of $G_1 = G_3$ and $G_1' = G_3' = 0$ and is a semi-cubic cylinder with the parametrisation $(-3t_1^2, a_2, 2t_1^3)$. The third component is a smooth surface which is the solution set of $G_2 = G_3$ and $G_2' = G_3' = 0$ and can be parametrised as (a_1, a_2, a_2). The $\mathcal{B}_1$ component given by $G_1' = G_1'' = 0$ is the smooth surface $(0, a_2, a_3)$. See Figure 5 (Left).

6.2. **The bad planes.** The one-dimensional strata adjacent to $A_1^2A_2$ are

$$\begin{aligned} A_1A_2 &: \{(a_1, a_2, a_3) = (0, a_2, 0) \cup (0, 0, a_3)\} \\ A_1^3 &: \{(a_1, a_2, a_3) = (-3t_1^2, -2t_1^3, -2t_1^3)\} \\ A_1^2/A_1^2 &: \{(a_1, a_2, a_3) = (3t_1^2, 2t_1^3, -2t_1^3)\}. \end{aligned}$$

The limiting tangent vectors to these one-dimensional strata are given by $(0, 1, 0)$, $(0, 0, 1)$ and $(1, 0, 0)$ so the bad planes are given by $a_2 = 0$, $a_3 = 0$ and $a_1 = 0$.

Proposition 6.1. *If a_1a_3 is positive the point $(a_1 : a_2 : a_3)$ lies in the shaded region of Figure 5 (right) and the corresponding full bifurcation set has type $A_1A_3(a)$. If however a_1a_3 is negative, then the point lies in the unshaded region and the corresponding full bifurcation set is of type $A_1A_3(b)$.*

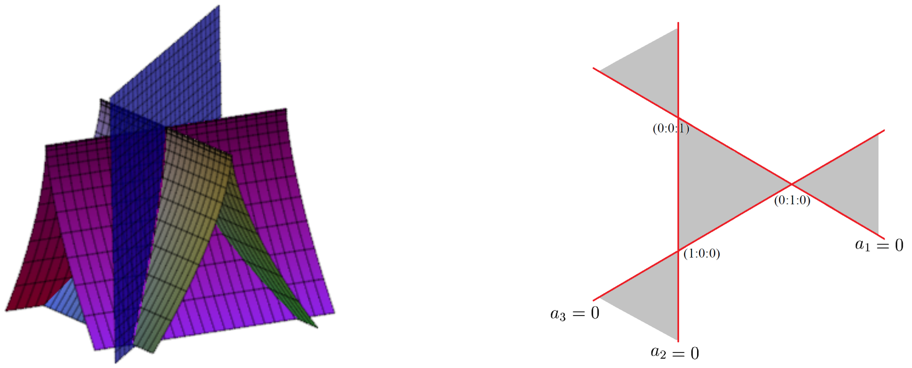

FIGURE 5. Left: The set $\mathcal{B}_2$ for $A_1^2A_2$. The $\mathcal{B}_1$ set (not shown) is the plane that contains both cuspidal edges of $\mathcal{B}_2$. Right: The regions determining the different types for $A_1^2A_2$.

Applying the chain rule to (1) in this case gives the system:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \left.\begin{pmatrix} \frac{\partial^2 F_1}{\partial t_1 \partial u} & \frac{\partial^2 F_1}{\partial t_1 \partial x_1} & \frac{\partial^2 F_1}{\partial t_1 \partial x_2} \\ \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \\ \frac{\partial F_3}{\partial u} & \frac{\partial F_3}{\partial x_1} & \frac{\partial F_3}{\partial x_2} \end{pmatrix}\right|_{(A(t_i,\mathbf{0}),\boldsymbol{x}_0)} \times JB + \begin{pmatrix} \mathbf{0} \\ JC \\ JC \\ JC \end{pmatrix}.$$

Subtracting the second row from the third and fourth rows gives:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \left.\begin{pmatrix} \frac{\partial^2 F_1}{\partial t_1 \partial u} & \frac{\partial^2 F_1}{\partial t_1 \partial v_1} & \frac{\partial^2 F_1}{\partial t_1 \partial v_2} \\ \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial u} - \frac{\partial F_1}{\partial u} & \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial v_1} & \frac{\partial F_2}{\partial x_2} - \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_3}{\partial u} - \frac{\partial F_3}{\partial u} & \frac{\partial F_3}{\partial x_1} - \frac{\partial F_1}{\partial x_1} & \frac{\partial F_3}{\partial x_2} - \frac{\partial F_1}{\partial x_2} \end{pmatrix}\right|_{(A(t_i,\mathbf{0}),\boldsymbol{x}_0)} \times JB + \begin{pmatrix} \mathbf{0} \\ JC \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}.$$

Ignoring the second row and substituting the derivatives gives

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \left.\begin{pmatrix} * & 2\alpha_1 X_1' & -2\alpha_1 Y_1' \\ * & \frac{2}{\kappa}(Y_2' - Y_1') & \frac{2}{\kappa}(X_1' - X_2') \\ * & \frac{2}{\kappa}(Y_3' - Y_1') & \frac{2}{\kappa}(X_1' - X_3') \end{pmatrix}\right|_{(A(t_i,\mathbf{0}),\boldsymbol{x}_0)} \times JB.$$

Since the bifurcation type depends on whether $\frac{\partial B_1}{\partial a_2}\frac{\partial B_1}{\partial a_3}$ is positive or negative, evaluating these terms using the cofactors of the matrix gives

$$\frac{\partial B_1}{\partial a_2}\frac{\partial B_1}{\partial a_3} = \frac{16\alpha_1^2}{\kappa^2}(X_1'^2 - Y_1'^2 - X_1'X_2' + Y_1'Y_2')(X_1'X_3' - X_1'^2 + Y_1'^2 - Y_1'Y_3')$$

and denoting by T_i the unit tangent vectors to γ at γ_i, this becomes

$$= -\frac{16\alpha_1^2}{\kappa^2}(\langle T_1, T_1\rangle - \langle T_1, T_2\rangle)(\langle T_1, T_1\rangle - \langle T_1, T_3\rangle).$$

$$= -\frac{16\alpha_1^2}{\kappa^2}((-1)^{\beta+1} - \langle T_1, T_2\rangle)((-1)^{\beta+1} - \langle T_1, T_3\rangle). \tag{3}$$

If the curves corresponding to the $A_1^2A_2$ point are spacelike, then the pseudo-circle is of type $S_1^1(c,r)$ (radius r and centred at c) and can be parametrised as $S_1^1(\theta) = c + r(\cosh(\theta), \pm\sinh(\theta))$, where the ± allows for the covering of both branches. The unit tangent vectors at γ_i are then given by $T_i = (\sinh(\theta_i), \pm\cosh(\theta_i))$. If both γ_1 and γ_i ($i = 2$ or 3) lie on the same branch, then

$$\langle T_1, T_i\rangle = -\sinh(\theta_1)\sinh(\theta_i) + \cosh(\theta_1)\cosh(\theta_i) = \cosh(\theta_1 - \theta_i)$$

so is greater than 1. If however γ_1 and γ_i lie on opposite branches then

$$\langle T_1, T_i\rangle = -\sinh(\theta_1)\sinh(\theta_i) - \cosh(\theta_1)\cosh(\theta_i) = -\cosh(\theta_1 + \theta_i)$$

so is less than -1. Since the curves γ_i are locally spacelike, $\beta = 1$ and the expression (3) is positive if γ_2 and γ_3, that is the two A_1 points, lie on the same branch and negative if they lie on opposite branches. It can be shown that the same result holds if the points are timelike. It follows that the point is of type $A_1^2A_2$ if of type (a) if the two A_1 points lie on the same branch, and of type (b) if they lie on opposite branches of the pseudo-circle.

7. The A_1A_3 singularity

Consider the following standard multi-versal unfolding of an A_1A_3 singularity given by

$$G : \mathbb{R}^{(2)} \times \mathbb{R}^3 \to \mathbb{R}$$

where $\mathbb{R}^{(2)}$ denotes the parameters t_1, t_2 and $\mathbb{R}^3$ denotes the unfolding parameters $\boldsymbol{a} = (a_1, a_2, a_3)$ and the multi-versal unfolding is given by the two unfoldings:

$$\begin{aligned} G_1(t_1, \boldsymbol{a}) &= t_1^4 + a_1t_1^2 + a_2t_1 + a_3, \\ G_2(t_2, \boldsymbol{a}) &= t_2^2. \end{aligned}$$

7.1. **The big bifurcation set.** At an A_1A_3 point the $\mathcal{B}_2$ set itself consists of two parts: The first is given as the solution to both $G_1 = G_2$ and $G_1' = G_2' = 0$ and is the swallowtail surface parametrised by $(a_1, -4t_1^3 - 2a_1t_1, 3t_1^4 + 2t_1^2a_1)$. The second component occurs locally near the A_3 point and is given by $G_1(t_1) = G_1(-t_1)$ and $G_1(t_1)' = G_1(-t_1)' = 0$. This second component is the half plane $(-2t_1^2, 0, y_3)$. The $\mathcal{B}_1$ component given by $G_1' = G_1'' = 0$ is the semi-cubic cylinder $(-6t_1^2, 8t_1^3, a_3)$, (see Figure 6 (left)).

7.2. **The bad planes.** The adjacent singularities of codimension 1 are as follows:

$$\begin{aligned} A_3 &: \{(a_1, a_2, a_3) = (0, 0, a_3)\} \\ A_1A_2 &: \{(a_1, a_2, a_3) = (-6t_1^2, 8t_1^3, -3t_1^4)\} \\ A_1^3 &: \{(a_1, a_2, a_3) = (-2t_1^2, 0, t_1^4)\} \\ A_1^2/A_1^2 &: \{(a_1, a_2, a_3) = (a_1, 0, 0)\}. \end{aligned}$$

The limiting tangent vectors to these one-dimensional strata are given by $(1, 0, 0)$, and $(0, 0, 1)$ so the bad planes are given by $a_1 = 0$ and $a_3 = 0$.

Proposition 7.1. *If a_1a_3 is positive the point $(a_1 : a_2 : a_3)$ lies in the shaded region of Figure 6 (right) and the corresponding full bifurcation set has type $A_1A_3(a)$. If however a_1a_3 is negative, then the point lies in the unshaded region and the corresponding full bifurcation set is of type $A_1A_3(b)$.*

Applying the chain rule to 1 gives the system:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \left.\begin{pmatrix} \frac{\partial^3 F_1}{\partial^2 t_1 \partial u} & \frac{\partial^3 F_1}{\partial^2 t_1 \partial x_1} & \frac{\partial^3 F_1}{\partial^2 t_1 \partial x_2} \\ \frac{\partial^2 F_1}{\partial t_1 \partial u} & \frac{\partial^2 F_1}{\partial t_1 \partial x_1} & \frac{\partial^2 F_1}{\partial t_1 \partial x_2} \\ \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{pmatrix}\right|_{(A(t_i,\mathbf{0}),\boldsymbol{x}_0)} \times JB + \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ JC \\ JC \end{pmatrix}.$$

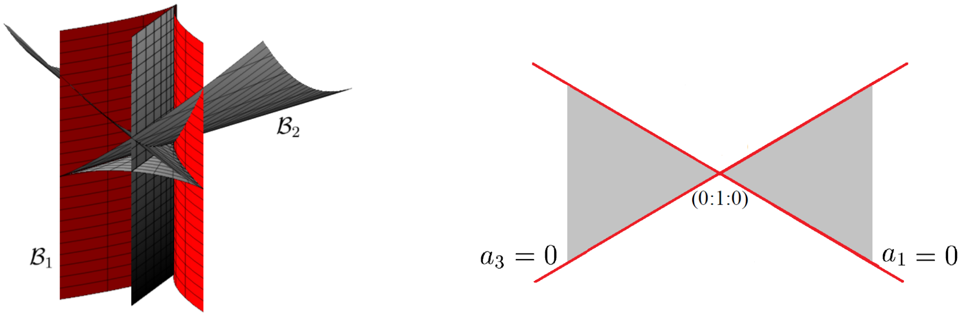

FIGURE 6. Left: The BBS for A_1A_3. Right: The regions determining the different types for A_1A_3.

Subtracting the last row from the third, and then ignoring the last gives:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \left.\begin{pmatrix} \frac{\partial^3 F_1}{\partial^2 t_1 \partial u} & \frac{\partial^3 F_1}{\partial^2 t_1 \partial x_1} & \frac{\partial^3 F_1}{\partial^2 t_1 \partial x_2} \\ \frac{\partial^2 F_1}{\partial t_1 \partial u} & \frac{\partial^2 F_1}{\partial t_1 \partial x_1} & \frac{\partial^2 F_1}{\partial t_1 \partial x_2} \\ \frac{\partial F_1}{\partial u} - \frac{\partial F_2}{\partial u} & \frac{\partial F_1}{\partial x_1} - \frac{\partial F_2}{\partial x_1} & \frac{\partial F_1}{\partial x_2} - \frac{\partial F_2}{\partial x_2} \end{pmatrix}\right|_{(A(t_i,\mathbf{0}),\boldsymbol{x}_0)} \times JB$$

Now,

$$\frac{\partial F_i}{\partial x_1} = 2X_i - 2x_1, \frac{\partial F_i}{\partial x_2} = -2Y_i + 2x_2$$

and $\gamma_i - \boldsymbol{x} = (X - x_1, Y - x_2) = \frac{1}{\kappa}N$ where $N = (-1)^\beta(Y_1', X_1')$. Hence, $\frac{\partial F_i}{\partial x_1} = 2Y_1(-1)^\beta$ and $\frac{\partial F_i}{\partial x_2} = -2X_1'(-1^\beta)$. Substituting these derivatives into the matrix equation gives:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \left.\begin{pmatrix} * & 2\alpha_1^2 X_1'' + 4\alpha_2 X_1' & -2\alpha_1^2 Y_1'' - 4\alpha_2 Y_1' \\ * & 2\alpha_1 X_1' & -2\alpha_1 Y_1' \\ * & \frac{2}{\kappa}(-1)^\beta(Y_1' - Y_2') & \frac{2}{\kappa}(-1)^\beta(X_2' - X_1') \end{pmatrix}\right|_{(A(t_i,\mathbf{0}),\boldsymbol{x}_0)} \times JB.$$

Recall that the type of bifurcation depends upon whether $\frac{\partial B_1}{\partial a_1}\frac{\partial B_1}{\partial a_3}$ is positive or negative.

$$\begin{aligned} \frac{\partial B_1}{\partial a_1} &= \det \begin{vmatrix} 2\alpha_1 X_1' & -2\alpha_1 Y_1' \\ \frac{2}{\kappa}(-1)^\beta(Y_1' - Y_2') & \frac{2}{\kappa}(-1)^\beta(X_2' - X_1') \end{vmatrix} \\ &= 2\alpha_1 X_1' \frac{2}{\kappa}(X_2' - X_1') + 2\alpha_1 Y_1' \frac{2}{\kappa}(Y_1' - Y_2') \\ &= \frac{4\alpha_1}{\kappa}(-X_1'^2 + Y_1'^2 + X_1'X_2' - Y_1'Y_2') \\ &= \frac{4\alpha_1}{\kappa}(\langle T_1, T_1\rangle - \langle T_1, T_2\rangle). \end{aligned}$$

and $\frac{\partial B_1}{\partial a_3} = -(2\alpha_1^2 X_1'' + 4\alpha_2 X_1')2\alpha_1 Y_1' + 2\alpha_1 X_1'(2\alpha_1^2 Y_1'' + 4\alpha_2 Y_1') = 4\alpha_1^3(X_1'Y_1'' - X_1''Y_1') = 4\alpha_1^3\kappa$.

$$\frac{\partial B_1}{\partial a_1}\frac{\partial B_1}{\partial a_3} = 16\alpha_1^4(-1)^\beta(\langle T_1, T_1\rangle - \langle T_1, T_2\rangle).$$

So if γ_1 and γ_2 are both spacelike, this gives $16\alpha_1^4(1 - \langle T_1, T_2\rangle)$ which is negative if γ_1 and γ_2 lie on the same branch and positive if they lie on opposite branches (see Section 6). On the other hand, if they are both timelike this gives $-16\alpha_1^4(-1 - \langle T_1, T_2\rangle)$. Parametrising the pseudo-circle of type $H^1(c, -r)$ as

$$H^1(\theta) = c + r(\pm\sinh(\theta), \cosh(\theta)),$$

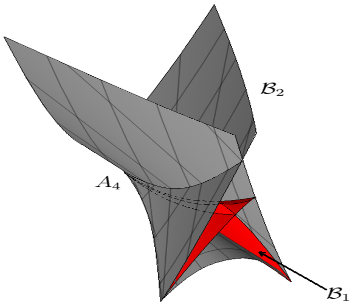

FIGURE 7. The BBS for A_4.

the unit tangent vector is given by $T = (\pm\cosh(\theta), \sinh(\theta))$. Now if γ_1 and γ_2 lie on the same branch $\langle T_1, T_2\rangle = -\cosh(\theta_1)\cosh(\theta_2) + \sinh(\theta_1)\sinh(\theta_2) = -\cosh(\theta_1 - \theta_2)$ which is less than -1. However if γ_1 and γ_2 lie on opposite branches

$$\langle T_1, T_2\rangle = \cosh(\theta_1)\cosh(\theta_2) + \sinh(\theta_1)\sinh(\theta_2) = \cosh(\theta_1 + \theta_2)$$

which is greater than 1. Hence the expression $-16\alpha_1^4(-1 - \langle T_1, T_2\rangle)$ is negative if γ_1 and γ_2 lie on the same branch and positive if they lie on opposite branches (the same conditions as for spacelike). It follows that the type is $A_1A_3(a)$ if both contact points lie on opposite branches and type $A_1A_3(b)$ occur on the same branch of the pseudo-circle.

8. THE A_4 SINGULARITY

Consider the following standard versal unfolding of an A_4 singularity given by

$$G : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$$

where $\mathbb{R}$ denotes the parameters t and $\mathbb{R}^3$ denotes the unfolding parameters $\boldsymbol{a} = (a_1, a_2, a_3)$ and the versal unfolding is given by

$$G(t, \boldsymbol{a}) \quad = \quad t^5 + a_1t^3 + a_2t^2 + a_3t.$$

8.1. **The big bifurcation set.** The bifurcation set $\mathcal{B}_1$ of the standard A_4 singularity G is the swallowtail surface which can be parametrised by $(a_1, -10t^3 - 3a_1t)$, and its bifurcation set $\mathcal{B}_2$ is another swallowtail, which sits inside the swallowtail $\mathcal{B}_1$ and can be parametrised by $(-3s^2 - 4st - 3t^2, 2s^3 + 8s^2t + 8st^2 + 2t^3, -4s^3t - 7s^2t^2 - 4st^3)$. See Figure 7. The adjacent 1-dimensional strata are found to be

$$\begin{aligned}
A_3 \quad &: \quad \{(a_1, a_2, a_3) = (-10t^2, 20t^3, -15t^4)\} \\
A_1A_2 \quad &: \quad \{(a_1, a_2, a_3) = (-60t^2, -80t^3, 960t^4)\} \\
A_2/A_2 \quad &: \quad \{(a_1, a_2, a_3) = (-\frac{10}{3}t^2, 0, 5t^4)\} \\
A_1^2/A_1^2 \quad &: \quad \{(a_1, a_2, a_3) = (-4t^2, 0, \frac{16}{5}t^4)\}.
\end{aligned}$$

The limiting tangent vectors to these one-dimensional strata are all given by $(1, 0, 0)$, so the only bad planes is given by $a_1 = 0$. Examining representations from both components show that only one transition type exists for A_4.

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