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MONSTER TOWERS FROM DIFFERENTIAL AND ALGEBRAIC VIEWPOINTS

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Abstract. Monster Towers were fathered by J. G. Semple in 1954 and were frequently used in algebraic geometry since then. Those towers got a second youth with A. Kumpera's works in the 1980's and Bryant and Hsu's modern treatment (1993) of the Cartan prolongation in differential geometry. In consequence, there have emerged Goursat- and [special-multi]-flags living on the stages of the same Monster/Semple Towers, and featuring rich trees of singularities (albeit *not* of a wild functional type, omnipresent among more generic structures in the tangent bundles to manifolds).

Now a unification of the two approaches is in sight. In fact, after the works of Castro *et al* (2017), Mormul and Pelletier (2020), and recently of the present author (2020), there emerges a clear two-way dictionary allowing one to quickly interpret algebraically defined singularities in Semple Towers in differential terms, and also to make readable to algebraic geometers differential constructions done long since in Monster Towers.

1. Introduction

In differential geometry, in the early 2000s, there have been constructed various monster towers generalizing the by-then-already-classical Goursat Monster Tower (GMT) produced by Montgomery and Zhitomirskii in 1999 and published in [MonZ]. The GMT was produced with the systematic use of classical Cartan prolongation of vector distributions (in its modern version presented in the paper [BH]).

The stages of GMT are manifolds hosting Goursat distributions of all possible coranks. (Goursat distributions are particular rank-2 subbundles in the tangent bundle to a manifold having the flag of consecutive Lie squares regular and growing in ranks – slowly! – always only by one.) In fact, the r-th stage M^r (of dimension $r+2$) hosts a distribution D^r which is locally universal among all existing corank-r Goursat distributions E. Meaning that each such E considered anywhere locally on its home manifold of dimension $r+2$ is diffeomorphic to D^r around a certain properly chosen point in M^r.

This is a formidable property of the GMT – which reduces the study of otherwise unfathomable Goursat singularities to that of the model structures (D^r) living on the stages (M^r) of the GMT.

In a line of direct generalization, in the paper [M2], p. 159, generalized Cartan prolongations (gCp) were defined. These prolongations, when iterated many times, have proven instrumental in building, still in [M2], Remark 3, so-called Special m-Flag Monster Towers (SmFMT for short, $m \geq 2$). The stages of the SmFMT host distributions locally universal as concerns generating *special m-flags* of arbitrary lengths. So what are those flags?
They are nested sequences of subbundles in the tangent bundle to a manifold M, say, that originate from a rank-$(m+1)$ distribution D and that satisfy

$$D = D^r \subset D^{r-1} \subset D^{r-2} \subset \cdots \subset D^1 \subset TM\,, \tag{1}$$

where

$$\mathrm{rk}D^j = \dim M - jm = m + 1 + (r-j)m,$$

$D^j + [D^j,\, D^j] = D^{j-1}$ for $j = r,\, r-1, \dots, 1$, $D^0 = TM$, and such that D^1 possesses a corank one involutive subdistribution F ($[F,\, F] \subset F \subset D^1$).[1]

The present paper, of half comparative, half research character, arose out of author's getting hold of the work [CCKS]. An entirely new world has been unveiled, inhabited by generations of algebraic geometers investigating virtually the same towers, and that for the last 60+ years! (if under different names). A predominant name for them has been *Semple towers*, after the pioneering work [S] and a benchmark late follow-up [L-J]. (They are also sometimes called Semple-Demailly Towers.) Algebraic geometers were equally in acute need of glasses to watch (and to construct!) objects living on the stages of the Semple towers with base manifolds of various dimensions. In consequence, more or less standardized glasses for Semple Towers had gradually sprung into existence in Algebraic Geometry, the benchmark paper [L-J] included. (See also section 5.3 in the present paper for the discussion of an important example originally due to A. Campillo and analized – in certain glasses – in [L-J]).
In [CCKS] canonical charts in the stages of the Semple towers have been defined and given the name of C-charts (see also the extensive lectures [CK]).

1.1. **Paper's objective.** By now it is well-known that the local geometrical behaviours of special m-flags are enormously rich. [By way of comparison, Goursat flags are just 1-flags. The discrete parameter m — the constant increment in ranks of the members of the derived flag — says how much new room is obtained, over any point, when one Cartan-prolongs a flag. Traditionally (p. 7 in [M3]) m is called the *width* of a flag.]
It is already so for the width $m = 2$, and all the more so for $m > 2$. In order to be able to watch those local geometries, particular charts have been produced in the stages of SmFMT's – starting from the work [KuRub] having its conceptual roots in the year 1999 (cf., in particular, p. 5 in [M3]) – on open dense parts of the stages of each SmFMT. By an already-established tradition they are called Extended Kumpera–Ruiz (EKR for short, with, interestingly, 'R' not for Rubin but for – chronologically much earlier – Ruiz) coordinate charts.

Therefore, from the one side there are Monster Towers investigated in Differential Geometry for 20+ years. They are endowed with long lists of various EKR charts adjusted to the plethora of singularities hidden in the seemingly innocent definition (1) of special multi-flags recalled above.

From the other side there are Semple Towers investigated in Algebraic Geometry already for 60+ years. Currently, reiterating, they are equipped with the so-called C-type charts.

Until recently the comparison of the two approaches was hardly possible. Even the lecture of paper [CCKS] was hampered, on the differential geometry side, by the much different language being in use there. The objective of this work is to propose a clear translation procedure

$$\text{the EKR charts on the DG side} \longleftrightarrow \text{the } C\text{-charts on the AG side}$$

going in both directions: in Theorem 1 from the left to right, and in section 4.3 from the right to left. Furnished also are examples illustrating the translation. In section 5.3 there is given yet another example of a C-chart in action. Analyzed are prolongations of a particular 1-parameter family of singular 3D curves investigated earlier in [L-J].

1.2. **The definition of the prolongation.** In the following definition, excerpted from [BH], p. 454^{4-10}, one obtains the **definition of generalized Cartan prolongation** by simply replacing 'rank 2' by 'rank $m+1$', '2-dimensional' by '$(m+1)$-dimensional', '$\mathbb{PR}^1$' by '$\mathbb{PR}^m$', and '1-manifolds' by 'm-manifolds'

[1] for more explanations compare, for instance, page 165 in [M2]

'If $\mathcal{D}$ is a rank 2 distribution on a manifold M, then, regarding $\mathcal{D}$ as a vector bundle, we can certainly define its projectivization $\pi : \mathbb{P}\mathcal{D} \longrightarrow M$, which is a bundle over M whose typical fiber $\mathbb{P}\mathcal{D}_x$ is the space of 1-dimensional linear subspaces of the 2-dimensional vector space $\mathcal{D}_x$. Thus, the fibers of $\mathbb{P}\mathcal{D}$ are isomorphic to $\mathbb{P}\mathbb{R}^1$ as projective 1-manifolds. There is a canonical rank 2 distribution $\mathcal{D}^{(1)}$ on $\mathbb{P}\mathcal{D}$ defined by setting $\mathcal{D}^{(1)}_\xi = (\pi')^{-1}(\xi)$ for each linear subspace $\xi \subset \mathcal{D}_x$. The distribution $\mathcal{D}^{(1)}$ is called the *(first) prolongation* of $\mathcal{D}$.'

(Note that a formally equivalent definition appears also on page 1283 in [L-J]. Algebraic geometers had seemed unaware of some fields of interest of Élie Cartan.)

1.3. **Extended K-R pseudo-normal forms for special m-flags.** The aim of this section is to produce a huge variety of rank-$(m+1)$ distributions with polynomial vector field' generators on $\mathbb{R}^N$, N possibly very large and always being $1 \,(\mathrm{mod}\, m)$. For each $k \in \{1, 2, \ldots, m+1\}$ we are going to define an operation $\mathbf{k}$ producing new rank-$(m+1)$ distributions from older ones. The technical writing of its outcome, not the operation's formal definition, will depend on how many operations were done *before* $\mathbf{k}$. It also depends on whether $k = 1$ or not.

In fact, the outcome of $\mathbf{1}$ being performed as operation number l on a distribution

$$(Z_1, \ldots, Z_{m+1})$$

defined on $\mathbb{R}^s$ with coordinates $y_1, \ldots, y_s$, is a new rank-$(m+1)$ distribution defined on $\mathbb{R}^{s+m}$ with coordinates $y_1, \ldots, y_s, x^l_2, \ldots, x^l_{m+1}$, generated by the vector field

$$Z'_1 = Z_1 + x^l_2 Z_2 + \cdots + x^l_{m+1} Z_{m+1}$$

and by $Z'_2 = \frac{\partial}{\partial x^l_2}, \ldots, Z'_{m+1} = \frac{\partial}{\partial x^l_{m+1}}$.
While the outcome of $\mathbf{k}$, $2 \le k \le m+1$, is a rank-$(m+1)$ distribution generated by

$$Z'_1 = x^l_2 Z_1 + \cdots + x^l_k Z_{k-1} + Z_k + x^l_{k+1} Z_{k+1} + \cdots + x^l_{m+1} Z_{m+1}$$

and by $Z'_2 = \frac{\partial}{\partial x^l_2}, \ldots, Z'_{m+1} = \frac{\partial}{\partial x^l_{m+1}}$. (When $k = m+1$, the components in the expansion of Z'_1 end with Z_{m+1}.) In either case it is important that these local generators are written precisely in this order, yielding together a new more complex object $(Z'_1, Z'_2, \ldots, Z'_{m+1})$.

Extended K-R pseudo-normal forms (EKR for short) of length $r \ge 1$, denoted by $\mathbf{j}_1.\mathbf{j}_2 \ldots \mathbf{j}_r$, where $j_1, \ldots, j_r \in \{1, 2, \ldots, m+1\}$, are now defined inductively, starting from the empty label distribution

$$\big(\partial_1, \partial_2, \ldots, \partial_{m+1}\big) \;=\; \Big(\frac{\partial}{\partial x^0_1}, \frac{\partial}{\partial x^0_2}, \ldots, \frac{\partial}{\partial x^0_{m+1}}\Big)$$

on $\mathbb{R}^{m+1}(x^0_1, x^0_2, \ldots, x^0_{m+1})$. Then, assuming the distribution $\mathbf{j}_1 \ldots \mathbf{j}_{r-1}$ is already defined, $\mathbf{j}_1 \ldots \mathbf{j}_{r-1}.\mathbf{j}_r$ is the outcome of the operation $\mathbf{j}_r$ performed as the operation number r over $\mathbf{j}_1 \ldots \mathbf{j}_{r-1}$. (The adjective 'pseudo-normal' refers to numerous and often spurious real parameters featuring in those polynomial visualisations. Similarly, in the case of Goursat flags many a constant showing up in Kumpera-Ruiz visualisations are redundant – and because of that those visualisations are only *pseudo*-normal.)

By introducing, in the operation $\mathbf{j}_r$, the new affine coordinates $x^r_2, \ldots, x^r_{j_r}, \ldots, x^r_{m+1}$, all the line directions in $\big(Z_1, \ldots, Z_{m+1}\big)$ that are close to the direction of

$$Z_{j_r} + x^r_{j_r+1} Z_{j_r+1} + \cdots + x^r_{m+1} Z_{m+1} \tag{2}$$

at the reference point down in the $(r-1)$-st stage, are parametrized. These m free variables $x^r_2, \ldots, x^r_{m+1}$ float (or: dance) around their reference values in (2). Note that the first $j_r - 1$ of these reference values are 0 by the very definition of the operation $\mathbf{j}_r$.

For the reader feels already that the operations $\mathbf{1}, \mathbf{2}, \dots, \mathbf{m+1}$ are certain prolongations viewed locally. In fact, they are just different gCp's seen in possible different affine charts on the Grassmanians used in the gCp procedure. (When $m = 1$, the two operations **1** and **2** applied interweavingly lead to the well-known local Kumpera-Ruiz pseudo-normal forms in the theory of Goursat flags – compare the explanation an instant ago and p. 466 in [MonZ].)

Every EKR is a special m-flag of length equal to the number of operations used to produce it, and equal to the length of the relevant word encoding the sequence of operations. Moreover – important – that EKR can be taken such that $j_1 = 1$ and, for $l = 1, \dots, r-1$,

$$\text{if } j_{l+1} > \max(j_1, \dots, j_l),$$

then $j_{l+1} = 1 + \max(j_1, \dots, j_l)$ (the rule of the least possible new jumps upward in the words $\mathbf{j}_1 . \mathbf{j}_2 \dots \mathbf{j}_r$). More details about these constructions are given in [M2].

Definition 1. The set of all sequences (or: words) of length r over the alphabet

$$\{1, 2, \dots, m, m+1\}$$

satisfying the *least upward jumps rule* ('lujr' for short in all what follows) is denoted by Υ. The dependence of Υ on the length r (and, needless to say, on m) is understood implicitly.

2. Charts in the stages of Semple Towers

On the Algebraic Geometry side the construction of handy 'night glasses' in the stages of a Semple Tower with the base dimension $m+1$,[2] similar to the EKR charts on the DG side, starts from an otherwise unspecified set of coordinates $x_1, x_2, \dots, x_{m+1}$ mapping a part of the base manifold (say) M to $\mathbb{R}^{m+1}$. Following the exposition in [CCKS], these charts will be denoted — when speaking for the r-th stage $M(r)$ — by $C(p_1 p_2 \dots p_r)$ with the indices $1 \le p_1, p_2, \dots, p_r \le m+1$.

To proceed, we take an arbitrary fixed smooth immersed curve

$$\delta = (\delta_1, \delta_2, \dots, \delta_{m+1}), \quad 0 \neq \dot\delta = [\dot\delta_1, \dot\delta_2, \dots, \dot\delta_{m+1}].$$

Then, clearly, a certain p_1-th component of the velocity $\dot\delta$ is non-zero: $\dot\delta_{p_1} \neq 0$. This allows us to define a new group of m coordinates – components of the velocity vectors of curves C^1-close to δ, velocities computed – important – with respect to the chosen *parameter* coordinate x_{p_1}:

$$x_j(p_1) = \frac{dx_j}{dx_{p_1}}, \qquad \text{for } j \neq p_1,$$

and to put additionally $x_{p_1}(p_1) = x_{p_1}$. In this way we obtain a [local] system of coordinates $x_1, x_2, \dots, x_{m+1}$ extended by the newer ones $x_j(p_1)$ ($j \neq p_1$), on a domain lying in the 1st stage $M(1)$ of the Semple tower. (Recalling, $M(1)$ is of dimension $m + 1 + 1 \cdot m$.) The coordinate $x_{p_1}(p_1)$ having just served as a parameter is, taken plainly, redundant. Yet it is remembered and carried along in the construction, because it can still become useful. (In [CCKS] it is named 'retained'.)

The first prolongation $\delta^{(1)}$ of δ is (by definition) horizontal with respect to the *focal* distribution F^1 living on $M(1)$, while the $x_j(p_1)$ ($j \neq p_1$) are the angle-affine coordinates of the velocities of F^1-horizontal curves C^1-close to $\delta^{(1)}$. Their values float around the respective reference values being assumed on $\delta^{(1)}$. In a sharp distinction, the p_1-th component of the velocity of the reference curve $\delta^{(1)}$ is, naturally, 1.

[2] A certain disadvantage, in comparing the Differential and Algebraic approaches to the towers, is that in some contributions, including [CCKS], the symbol m stands for $m+1$ in [M3], [MP2] and throughout the entire present work.

To define the consecutive group of m local coordinates – now in the second stage $M(2)$ of Semple – that extends the previously obtained set of $2m+1$ coordinates, we consider a fixed curve γ traced in $M(1)$, horizontal with respect to F^1, immersed and passing at the reference moment, say $t=0$, by our preceding reference point $\delta^{(1)}(0)$. We take γ **not** necessarily C^1-close to $\delta^{(1)}$. This is central for the construction; its velocity $\dot\gamma$ may à priori have *any* one of its components non-zero! Let it be the $x_{p_2}(p_1)$ component. (The component $x_{p_1}(p_1)$, that is, $p_2 = p_1$ is not excluded.)

To the curves in $M(1)$ which are C^1-close to γ we associate the components of their velocities, mimicking the previous step. That is, by means of differentiating with respect to the coordinate $x_{p_2}(p_1)$ serving as a new parameter:

$$x_j(p_1p_2) = \frac{dx_j(p_1)}{dx_{p_2}(p_1)}\,, \qquad \text{for } j \neq p_2\,,$$

and putting additionally $x_{p_2}(p_1p_2) = x_{p_2}(p_1)$.

And so it goes recursively step after step. Around a point $p \in M(k-1)$ which already sits in the domain of a chart constructed on [a part of] $M(k-1)$, one considers arbitrary immersed curve $\mathit{\Gamma}$ in $M(k-1)$ horizontal with respect to the focal distribution F^{k-1} living on $M(k-1)$, $\mathit{\Gamma}(0) = p$, and picks a coordinate $x_j(p_1 \dots p_{k-1})$ $(1 \le j \le m+1)$ such that the j-th component of $\dot{\mathit{\Gamma}}(p)$ is non-zero. It is this index j that is declared to be p_k while $x_{p_k}(p_1 \dots p_{k-1})$ is a new variable parameter giving rise, by way of differentiation, to new coordinates $x_j\big(p_1 \dots p_{k-1}p_k\big)$, $j \neq p_k$:

$$x_j\big(p_1 \dots p_{k-1}p_k\big) = \frac{dx_j\big(p_1 \dots p_{k-1}\big)}{dx_{p_k}\big(p_1 \dots p_{k-1}\big)}$$

for $j \neq j_k$. As in the initial steps, the parameter of differentiation is carried along as a new retained variable

$$x_{p_k}\big(p_1 \dots\, p_{k-1}p_k\big) :\, = x_{p_k}\big(p_1 \dots\, p_{k-1}\big)\,.$$

After r such steps one arrives at a chart on an open part of $M(r)$ consisting, after eventually leaving out all the retained variables, of $m+1+rm$ coordinates. (Reiterating, our quantity m is $m-1$ in [CCKS].) Needless to say, this number coincides with the dimension of $M(r)$. The construction of the charts $C(p_1p_2 \dots p_r)$ on the AG side is now complete.

Attention. In [CCKS] this construction is originally put forward in a slightly different way which does not bring in the intermediate (if purely technical) horizontal curves δ, γ, $\mathit{\Gamma}$, and so on. Instead, the authors postulate for the non-vanishing of the differential of each variable that is chosen as retained, on the focal (i. e., horizontal) directions under consideration at each given step.

Example 1 (Example 5.1 in [CCKS]). Here is one of the systems of coordinates emerging in the above-recalled construction when $m=2$ and $r=5$. The retained variables are underlined.

$$\begin{array}{rcl}
x_1 & x_2 & x_3 \\
x_1(3) & x_2(3) & \underline{x_3(3)} \\
x_1(32) & \underline{x_2(32)} & x_3(32) \\
\underline{x_1(321)} & x_2(321) & x_3(321) \\
x_1(3212) & \underline{x_2(3212)} & x_3(3212) \\
x_1(32123) & x_2(32123) & \underline{x_3(32123)}
\end{array}$$

It is to be noted that the sequence of indices p_1, p_2, p_3, p_4, p_5 in this example does not satisfy the least upward jumps rule, in contrast to what is proclaimed (and sufficient) in Theorem 1 below.

3. Bringing the EKR charts into the C-format

The EKR charts serving any given r-th stage of the SmFMT consist of $m{+}1{+}r{\cdot}m$ coordinate functions. At the same time each C-chart serving [a part of] the stage $M(r)$ of the Semple Tower with $(m{+}1)$-dimensional base consists of $(r{+}1)(m{+}1)$ functions. Consequently, to allow for any comparison whatsoever, the EKR's are to be augmented by certain r functions of beforehand unspecified nature.

3.1. **Extending any given EKR system of coordinates by a set of redundant variables.** As already invoiced, those auxiliary (and ultimately ... redundant!) variables are needed for further comparison of the EKR's with the $C(p_1p_2\dots p_r)$ charts used on the AG side of the theory.

Recalling from section 1.3, our arbitrarily chosen EKR $\mathbf{j}_1.\mathbf{j}_2\dots\mathbf{j}_r$ satisfies the least upward jumps rule (cf. also [M2]). Arrange the coordinates that show up in $\mathbf{j}_1.\mathbf{j}_2\dots\mathbf{j}_r$ to appear in rows: $x^0_1,\, x^0_2,\dots,\, x^0_{m+1}$ in the 0-th row and $x^k_2,\, x^k_3,\dots,\, x^k_{m+1}$ in the k-th row, $1\le k\le r$. So the 0-th row stands out in that it is longer – it has the variable x^0_1 from the very beginning. Supposing that, for certain $k\ge 1$, the variables $x^0_1,\dots,x^{k-1}_1$ are already defined, we continue by adding in the k-th row the variable

$$x^k_1 :=\, x^{k-1}_{j_k}. \tag{3}$$

(Note that repetitions are not excluded in this process.) Upon arriving at $k=r$, all $r+1$ rows contain $m{+}1$ variables each. The variables $x^1_1,\, x^2_1,\dots,\, x^r_1$ are redundant in this $(r{+}1)\times(m{+}1)$ matrix of variables – they are repetitions of certain original EKR coordinates (and, possibly, are certain repetitions among themselves).

QUESTION. Are there some *differential* relations among the actual $(r{+}1)\times(m{+}1)$ variables? Before answering (in the affirmative), let us note for a future use

Observation 1. *In the stepwise production in section 1.3 of any EKR system $\mathbf{j}_1.\mathbf{j}_2\dots\mathbf{j}_r$, at each stage $k=0,\,1,\dots,\,r$ of its production, the only component, in the leading vector field generator Z_1, which is identically 1, is the $\partial_{x^k_1}$-component.*

Proof by induction on k.

• $k=0$. Yes, the initial starting generator $Z_1=\partial_{x^0_1}$ by definition.

•• $k\Rightarrow k+1$. If $j_{k+1}=1$ then, by (3), $x^{k+1}_1 \;=\; x^k_{j_{k+1}} \;=\; x^k_1$. On the other hand

$$Z_1' = Z_1 + \sum_{j=2}^{m+1} x^{k+1}_j Z_j$$

and it is clear that Z_1 and Z_1' have one and the same component 1. It is $\partial_{x^k_1}$ in Z_1, and it is $\partial_{x^{k+1}_1}$ in Z_1'.

If now $j_{k+1}>1$, then

$$Z_1' \;=\sum_{j=1}^{j_{k+1}} x^{k+1}_{j+1}Z_j \;+\; Z_{j_{k+1}} \;+\sum_{j=j_{k+1}+1}^{m+1} x^{k+1}_j Z_j$$

and so $Z_{j_{k+1}} \;=\; \partial_{x^k_{j_{k+1}}} \;=\; \partial_{x^{k+1}_1}$ (cf. (3)) is indeed the unique component in Z_1' being identically 1. Observation is proved. $\square$

3.2. **Differential relations among the superfluous variables.** The question presented in section 3.1 is central in this work. We will answer it in the present section. To that end we prolong an EKR system $(Z_1, Z_2, \dots, Z_{m+1})$ of the form $\mathbf{j}_1.\mathbf{j}_2 \dots \mathbf{j}_{r-1}$ to the EKR system $(Z_1', Z_2', \dots, Z_{m+1}')$ of the form $\mathbf{j}_1.\mathbf{j}_2 \dots \mathbf{j}_{r-1}.\mathbf{j}_r$.

By the very construction of the EKR's, it is the vector generator Z_1' which spans the running horizontal direction downstairs [that is lifted up to a point upstairs]. One can compute the rates of change of the coordinates x_j^{k-1}, $j = 2, 3, \dots, m+1$, that have appeared downstairs one step earlier, with respect to a variable well-parametrizing the integral curves of Z_1'. What remains is to identify such a variable, and then the answer will emerge. For the clarity of exposition we split the analysis in two parts: (a) $j_k = 1$ and (b) $j_k > 1$.

Case (a). $j_k = 1$. Let us watch carefully, with Observation 1 at hand, the vector field Z_1':

$$\begin{aligned} Z_1' &= Z_1 + x_2^k Z_2 + \cdots + x_{m+1}^k Z_{m+1} \\ &= \underbrace{\left(\cdots + \partial_{x_1^{k-1}} + \cdots\right)}_{Z_1} + x_2^k \partial_{x_2^{k-1}} + \cdots + x_{m+1}^k \partial_{x_{m+1}^{k-1}} \,. \end{aligned}$$

Hence it is the variable x_1^{k-1} that well-parametrizes the integral lines of Z_1'. Therefore, one differentiates the one-step-old coordinates x_j^{k-1}, $j = 2, 3, \dots, m+1$, with respect to x_1^{k-1}, and gets

$$\frac{d\,x_j^{k-1}}{d\,x_1^{k-1}} = x_j^k\,, \quad j = 2, 3, \dots, m+1 \tag{4}$$

(Think about the family of Pfaffian equations describing dually the field Z_1'. That family comprises the equations $d\,x_j^{k-1} - x_j^k\,d\,x_1^{k-1} = 0$,
$j = 2, 3, \dots, m+1$.)

Case (b). $j_k > 1$. Now the identification and handling of a coordinate which well-parametrizes the integral curves of Z_1' goes slightly differently. However, Observation 1 is being used one more time:

$$\begin{aligned} Z_1' &= x_2^k Z_1 + x_3^k Z_2 + \cdots + x_{j_k}^k Z_{j_k - 1} + Z_{j_k} + x_{j_k+1}^k Z_{j_k+1} + \cdots + x_{m+1}^k Z_{m+1} \\ &= x_2^k \underbrace{\left(\cdots + \partial_{x_1^{k-1}} + \cdots\right)}_{Z_1} + x_3^k \partial_{x_2^{k-1}} + \cdots + x_{j_k}^k \partial_{x_{j_k-1}^{k-1}} + \partial_{x_{j_k}^{k-1}} \\ &\quad + x_{j_k+1}^k \partial_{x_{j_k+1}^{k-1}} + \cdots + x_{m+1}^k \partial_{x_{m+1}^{k-1}} \,. \end{aligned}$$

Therefore, this time it is the variable $x_{j_k}^{k-1}$ that well-parametrizes the integral lines of Z_1'. And, moreover, one differentiates now with respect to that variable **groupwise**:

$$\frac{d\,x_1^{k-1}}{d\,x_{j_k}^{k-1}} = x_2^k\,, \tag{5}$$

$$\frac{d\,x_{j-1}^{k-1}}{d\,x_{j_k}^{k-1}} = x_j^k\,, \quad j = 3, \dots, j_k\,, \tag{6}$$

$$\frac{d\,x_j^{k-1}}{d\,x_{j_k}^{k-1}} = x_j^k\,, \quad j = j_k + 1, \dots, m+1\,. \tag{7}$$

At this moment the packs of coordinates building up the EKR systems $\mathbf{j}_1.\mathbf{j}_2 \dots \mathbf{j}_r$ on the Differential Side, and the $C(p_1 p_2 \dots p_r)$ systems on the Algebraic Side have gotten much closer to each other. How to establish a possible correspondence among *all* EKR's and *all* $C(p_1p_2\dots p_r)$'s? This is the objective of the following Section. Certain precisely defined reorderings of those rows with superfluous coordinates from Section 3.1 will do.

4. A one-to-one correspondence

In order to quickly demonstrate the correspondence, some preparatory steps are necessary. The actual extended $(r+1)\times(m+1)$ matrix of EKR variables

$$\left(x_i^j\right)_{\substack{i=0,1,\dots,r\\ j=1,2,\dots,m+1}} \tag{8}$$

will instantly be reordered row-wise, that is – within its rows. In fact, let σ_k mean the permutation *cycle* $\begin{pmatrix} 1 & 2 & \cdots & j_k-1 & j_k \\ 2 & 3 & \cdots & j_k & 1 \end{pmatrix}$ in the permutation group S_{m+1}. Let, moreover, $\langle \mathbf{j}_k\rangle$ be the mapping sending the *ordered* $(m+1)$-tuple of variables $\left(x_1^{k-1},\, x_2^{k-1},\dots,\, x_{m+1}^{k-1}\right)$ to the *ordered* $(m+1)$-tuple $\left(x^k_{\sigma_k(1)},\, x^k_{\sigma_k(2)},\dots,\, x^k_{\sigma_k(m+1)}\right)$. With these notations, we reorder the entries of (8) row-wise, replacing its k-th row $(1\le k\le r)$ by the row

$$\langle \mathbf{j}_k\rangle\circ\langle \mathbf{j}_{k-1}\rangle\circ\cdots\circ\langle \mathbf{j}_1\rangle\left(x_1^0,\, x_2^0,\dots,\, x_{m+1}^0\right). \tag{9}$$

(Naturally enough, the 0-th row is kept unchanged.)

Definition 2. For $k=1,\,2,\dots,\,r$, a natural number p_k is the position, in the $(k-1)$-st row of the form (9), of the variable $x^{k-1}_{j_k}$.[3]

In this way there is obtained a length r sequence $(p_1,\,p_2,\dots,\,p_r)$ of integers from

$$\{1,\,2,\dots,\,m,\,m+1\}.$$

(p_1 is, of course, 1.)

Here is an example of the translation procedure $\mathbf{j}\longrightarrow p$, when $m=3$ and $r=10$, $\mathbf{j}_1.\mathbf{j}_2\dots\mathbf{j}_{10}\in\Upsilon$ (cf. Definition 1 in Section 1):

k	$\mathbf{j}_k$	$\sigma_k\circ\sigma_{k-1}\circ\cdots\circ\sigma_1(1\ 2\ 3\ 4)$				p_k
		1	2	3	4	
1	**1**	1	2	3	4	1
2	**2**	2	1	3	4	2
3	**1**	2	1	3	4	2
4	**3**	3	2	1	4	3
5	**2**	3	1	2	4	2
6	**1**	3	1	2	4	2
7	**4**	4	2	3	1	4
8	**4**	1	3	4	2	1
9	**3**	2	1	4	3	2
10	**4**	3	2	1	4	3

Theorem 1. *Let $\mathbf{j}_1.\mathbf{j}_2\dots\mathbf{j}_{r-1}.\mathbf{j}_r$ be any EKR satisfying the least upward jumps rule.* • *The reordered matrix with the rows (9) is precisely the $(r+1)\times(m+1)$ matrix of variables $C(p_1p_2\dots p_r)$ started with $\left(x_1,\dots,x_{m+1}\right)=\left(x_1^0,\dots,\,x_{m+1}^0\right)$.* •• *Moreover, the sequence $(p_1,\,p_2,\dots,\,p_r)$ also satisfies the least upward jumps rule.*

[3] Because $\sigma_k(j_k)=1$, p_k is also — handier, if posterior to the definition itself — the position of the variable x_1^k in the k-th row, *cf.* $(*)$ in the proof of Theorem 1 below.

Proof. We will show by induction on $k = 1, 2, \ldots, r$ that

$$\langle \mathbf{j}_k\rangle \circ \langle \mathbf{j}_{k-1}\rangle \circ \cdots \circ \langle \mathbf{j}_1\rangle \big(x_1^0,\, x_2^0, \ldots,\, x_{m+1}^0\big) \\ = \big(x_1(p_1 \ldots p_k),\, x_2(p_1 \ldots p_k), \ldots,\, x_{m+1}(p_1 \ldots p_k)\big)\,.$$

[$k = 1$]:
$\mathbf{j}_1 = 1$, the variable $x_{j_1}^{1-1} = x_1^0$ is at the 1st position in $\big(x_1^0,\, x_2^0, \ldots,\, x_{m+1}^0\big)$, hence $p_1 = 1$. Next, $x_1(1) = x_1 = x_1^0 = x_1^1$ and also $x_j(1) = \frac{dx_j}{dx_1} = \frac{dx_j^0}{dx_1^0} = x_j^1$ for $j \neq 1$ so that the case $k = 1$ is checked.

[$k-1 \Rightarrow k$]:
from the inductive assumption we know that

$$\langle \mathbf{j}_{k-1}\rangle \circ \cdots \circ \langle \mathbf{j}_1\rangle \big(x_1^0,\, x_2^0, \ldots,\, x_{m+1}^0\big) \\ = \big(x_1(p_1 \ldots p_{k-1}),\, x_2(p_1 \ldots p_{k-1}), \ldots,\, x_{m+1}(p_1 \ldots p_{k-1})\big)\,.$$

We know from Definition 2 that the p_k-th entry on the LHS above is $x_{j_k}^{k-1}$, hence also

$$x_{p_k}(p_1 \ldots p_{k-1}) = x_{j_k}^{k-1}.$$

Let us write now, directly below the row of variables on the LHS, the derivatives of x_j^{k-1}, $j \neq j_k$, with respect to $x_{j_k}^{k-1}$ and **in the same order as** on the LHS before differentiation. And
($*$) write $x_1^k = x_{j_k}^{k-1}$ below the entry $x_{j_k}^{k-1}$.
In view of (4), (5), (6) and (7), the outcome of these actions is precisely

$$\langle \mathbf{j}_k\rangle \circ \langle \mathbf{j}_{k-1}\rangle \circ \cdots \circ \langle \mathbf{j}_1\rangle \big(x_1^0,\, x_2^0, \ldots,\, x_{m+1}^0\big).$$

Now comes the decisive moment in the proof. One step down on the RHS we write
(†) below $x_j(p_1 \ldots p_{k-1})$: $x_j(p_1 \ldots p_{k-1}\, p_k) = \frac{d\, x_j(p_1 \ldots p_{k-1})}{d\, x_{p_k}(p_1 \ldots p_{k-1})}$ for $j \neq p_k$
and
(‡) below $x_{p_k}(p_1 \ldots p_{k-1})$: $x_{p_k}(p_1 \ldots p_{k-1}\, p_k) = x_{p_k}(p_1 \ldots p_{k-1})$.
In differentiating the same as on the LHS functions with respect to the same as on the LHS variable, we have simply mimicked our previous actions below the row of functions standing on the LHS. Therefore, the two newly obtained ordered rows of functions are identical. The induction step is done.

[The 'Moreover' part]:
In order to justify this, we will show two things. Namely
(i) Suppose $n \in \{1, 2, \ldots, m\}$, $1 \leq s \leq r$, $j_1, j_2, \ldots, j_s \leq n$. Then $p_1, p_2, \ldots, p_s \leq n$.
(ii) If the assumptions in (i) are slightly changed in that $j_1, j_2, \ldots, j_{s-1} \leq n$, but $j_s = n+1$ (recalling, the EKR code under consideration satisfies the least upward jumps rule), then $p_s = n+1$.
This will do, because from (i) and (ii) taken together there follows already that the sequence $p_1, p_2, \ldots, p_r$ satisfies the least upward jumps rule.

Item (i). Because

$$j_1,\, j_2, \ldots,\, j_{s-1} \leq n\,, \tag{10}$$

the permutations $\sigma_1, \sigma_2, \ldots, \sigma_{s-1}$ only mix numbers within the set $\{1, 2, \ldots, n\}$, and leave untouched numbers bigger than n through $m+1$. Since $j_1, j_2, \ldots, j_s$ are in this distinguished set, the positions of the variables $x^{l-1}_{j_l}$, $l = 1, 2, \dots, s$, in the *reordered* rows with running numbers $l-1$ are not bigger than n.

Item (ii). The premise (10), put to work in the previous argument, still holds, hence the number $n+1$ stays still under all the permutations $\sigma_1, \sigma_2, \dots, \sigma_{s-1}$. By consequence, the variable $x^{s-1}_{j_s} = x^{s-1}_{n+1}$ stands in the $(n+1)$-st position in the *reordered* row No $s-1$. This means that $p_s = n+1$.
Now Theorem 1 is proved in its entirety. □

4.1. **The injectivity of the mapping $\mathbf{j} \to p$.** Any given EKR system of coordinates $\mathbf{j}_1.\mathbf{j}_2 \dots \mathbf{j}_r$ has been accordingly reordered to just become a certain $C(p_1p_2 \dots p_r)$ system of coordinates. This EKR system is, in certain precise sense, the *only* EKR system associated to the singularity class $j_1.j_2 \dots j_r$.
(In fact, Theorem 1 and Remark 5 in [M3] say that the germ at the origin of *only* $\mathbf{j}_1.\mathbf{j}_2 \dots \mathbf{j}_r$ sits in $j_1.j_2 \dots j_r$.[4] As a matter of recollection, the construction of the singularity classes in general width $m \geq 2$ – the notion recalled in section 1.1 – was given in [M1].)
Consequently, that system $C(p_1p_2 \dots p_r)$ is the only one associated to the class $j_1.j_2 \dots j_r$. In view of this, the injectivity of the mapping $\mathbf{j} \to p$ is, in principle, clear. However, in order that this paper be as self-contained as possible, we note formally the following.

Observation 2. *The mapping* $(\mathbf{j}_1, \mathbf{j}_2, \dots, \mathbf{j}_r) \mapsto (p_1, p_2, \dots, p_r)$ *sending the sequences satisfying the least upward jumps rule to sequences satisfying the least upward jumps rule as well is injective.*

Proof. Let $\big(\mathbf{j}_1, \mathbf{j}_2, \dots, \mathbf{j}_r\big) \neq \big(\mathbf{j}'_1, \mathbf{j}'_2, \dots, \mathbf{j}'_r\big)$ and

$$\big(\mathbf{j}_1, \mathbf{j}_2, \dots, \mathbf{j}_{k-1}\big) = \big(\mathbf{j}'_1, \mathbf{j}'_2, \dots, \mathbf{j}'_{k-1}\big) \tag{11}$$

while $\mathbf{j}_k \neq \mathbf{j}'_k$ (the first occurrence of the deviation between the two $\mathbf{j}$-sequences under consideration). The coincidence (11) clearly implies

$$\langle \mathbf{j}_{k-1} \rangle \circ \cdots \circ \langle \mathbf{j}_1 \rangle = \langle \mathbf{j}'_{k-1} \rangle \circ \cdots \circ \langle \mathbf{j}'_1 \rangle \,. \tag{12}$$

Now observe that p_k is the position number of the variable $x^{k-1}_{j_k}$ in the ordered row of functions

$$\langle \mathbf{j}_{k-1} \rangle \circ \cdots \circ \langle \mathbf{j}_1 \rangle \big(x^0_1,\, x^0_2, \dots,\, x^0_{m+1}\big)\,,$$

while p'_k is the position number of the variable $x^{k-1}_{j'_k}$ in the same (see (12)) ordered row of functions. The two variables brought out are different, hence their positions in that row are different: $p_k \neq p'_k$. □

Corollary 1. *For every $r \geq 1$ the mapping $\mathbf{j} \to p$ is a bijection in the set Υ.*

(An injective mapping $\Upsilon \to \Upsilon$, with the set Υ finite.)

4.2. **On the cardinality of the sets Υ.** The lujr limitation imposed on the sequences with values in $\{1, 2, \dots, m+1\}$ does not seem to have been in use before the appearance of the texts [M1, M2]. It is of interest to know, how many such sequences exist, as a function of the width m and length r. That is to say, how many singularity classes exist for given m and r. Here is a sample of experimental data.

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[4] Adjacencies occurring among the singularity classes are closely related to that uniqueness. When a singularity class $k_1.k_2 \dots k_r$ is adjacent to $k'_1.k'_2 \dots k'_r$, then it is *not* visible in the chart $\mathbf{k}'_1.\mathbf{k}'_2 \dots \mathbf{k}'_r$. Whereas the thicker $k'_1.k'_2 \dots k'_r$ *is* visible in $\mathbf{k}_1.\mathbf{k}_2 \dots \mathbf{k}_r$.

the width m	#(words of length r satisfying the lujr)
2	$\frac{1}{3!}3^r + \frac{1}{2}\,, \qquad r \geq 3$
3	$\frac{1}{4!}4^r + \frac{1}{4}2^r + \frac{1}{3}\,, \qquad r \geq 4$
4	$\frac{1}{5!}5^r + \frac{1}{12}3^r + \frac{1}{6}2^r + \frac{3}{8}\,, \qquad r \geq 5$
5	$\frac{1}{6!}6^r + \sum(\mathit{lesser\ bases})^r\,, \qquad r \geq 6$

The restrictions $r \geq m+1$ in the above table are, formally speaking, redundant. Yet the special m-flags – precisely due to the lujr principle – start to fully manifest their properties only from the length $r = m+1$ onwards! In these optics, the following is important.

Question 1. Keeping $m \geq 2$ [arbitrary] fixed, is the leading term in the power expansion of $\#(\Upsilon) = \#$(words of length r satisfying the lujr) always $\frac{1}{(m+1)!}(m+1)^r$?
Or, relaxing the question a bit, does the following equality hold?

$$\lim_{r\to\infty} \frac{\#(\Upsilon)}{(m+1)^r} \;=\; \frac{1}{(m+1)!}\,. \tag{13}$$

Attention. The case $m = 1$ (Goursat flags) falls off this pattern. The number of Kumpera-Ruiz classes of length r is $\frac{1}{4}\cdot 2^r$. This deviation from a general rule is due to the *Engel theorem* for Goursat distributions of corank 2. Engel forces, in the Kumpera-Ruiz systems of coordinates, on top of $\mathbf{j}_1 = 1$, also $\mathbf{j}_2 = 1$. For special m-flags, $m \geq 2$, it is well known that there is no analogous theorem – see Proposition 1, (iii) in [M2].

It turns out that a proof of equality (13) is just round the corner.
Let, for $p = 1, 2, \ldots, m, m+1$, $N(r,p)$ denote the number of lujr words of length r having the *maximal* letter p. (For instance $N(r,1) = 1$, $N(r,2) = 2^{r-1} - 1$.) Let also, more generally, $T(r,p)$ be the number of *all* words whatsoever of length r over the alphabet $\{1, 2, \ldots, p\}$ which use *all* letters from this alphabet. Then $T(r, m+1) < (m+1)^r =$

$$T(r,m+1) + \binom{m+1}{m}T(r,m) + \binom{m+1}{m-1}T(r,m-1) + \cdots + \binom{m+1}{1}T(r,1)$$
$$< T(r,m+1) + 2^{m+1}T(r,m) < T(r,m+1) + 2^{m+1}m^r\,.$$

This implies

$$\frac{T(r,m+1)}{(m+1)^r} < 1 < \frac{T(r,m+1)}{(m+1)^r} + 2^{m+1}\left(\frac{m}{m+1}\right)^r$$

and further

$$\frac{T(r,m+1)}{(m+1)^r} \longrightarrow 1$$

when r tends to infinity. Now the lujr enters the situation with $T(r,m+1) = (m+1)!N(r,m+1)$, so that the above limit gets transformed into

$$\frac{N(r,m+1)}{(m+1)^r} \underset{r\to\infty}{\longrightarrow} \frac{1}{(m+1)!}\,. \tag{14}$$

Naturally enough, (14) generalizes to

$$\frac{N(r,p)}{p^r} \underset{r\to\infty}{\longrightarrow} \frac{1}{p\,!}$$

for $p = m,\ m-1, \dots, 1$. This in turn implies that, for the same p's,

$$\frac{N(r,p)}{(m+1)^r} = \frac{N(r,p)}{p^r}\left(\frac{p}{m+1}\right)^r \underset{r\to\infty}{\longrightarrow} \frac{1}{p\,!}\cdot 0 = 0\,. \tag{15}$$

Putting together (14) and (15),

$$\frac{N(r,m+1)+N(r,m)+\cdots+N(r,1)}{(m+1)^r} \underset{r\to\infty}{\longrightarrow} \frac{1}{(m+1)!}\,,$$

that is

$$\frac{\#(\Upsilon)}{(m+1)^r} \underset{r\to\infty}{\longrightarrow} \frac{1}{(m+1)!}\,. \qquad \square$$

4.3. **Algorithmic translation $\mathbf{j} \longleftarrow p$.** The plain *existence* of the mapping inverse to $\mathbf{j} \longrightarrow p$ is already established. However, how to describe that inverse mapping in algorithmic terms? To answer this, we ask what has been (in Definition 2) the gist of the defining the p-sequence out of any given $\mathbf{j}$-sequence? Simply as it has been,

$$p_k = \Big(\sigma_k \circ \sigma_{k-1} \circ \cdots \circ \sigma_1\Big)^{-1}(1)\,.$$

Or else, writing this equivalently,

$$1 = \sigma_k \circ \sigma_{k-1} \circ \cdots \circ \sigma_1(p_k) = \begin{pmatrix} 1 & 2 & \cdots & j_k-1 & j_k \\ 2 & 3 & \cdots & j_k & 1 \end{pmatrix} \Big(\sigma_{k-1} \circ \cdots \circ \sigma_1(p_k)\Big)\,.$$

This implies

$$j_k = \sigma_{k-1} \circ \cdots \circ \sigma_1(p_k)\,. \tag{16}$$

It is the sought after recurrence: $\mathbf{j}_k$ gets retrieved from the data consisting of p_k and the previous length $k-1$ permutation $\sigma_{k-1} \circ \cdots \circ \sigma_1$. Putting the same formally, the starting data are $\mathbf{j}_1 = \mathbf{1}$ and $\sigma_1 = \mathrm{id}$. Then, for $2 \le k \le r$, supposing $\mathbf{j}_1, \dots, \mathbf{j}_{k-1}$ already ascertained, hence $\sigma_1, \dots, \sigma_{k-1}$ known, the value of $\mathbf{j}_k$ is gotten via the formula (16). That is, $\mathbf{j}_2$ (and $\sigma_2\,!$) is gotten from p_2 and σ_1, then $\mathbf{j}_3$ is gotten from p_3, σ_1 and σ_2 (known from the previous step), and so on. Below given is an example illustrating this recursive procedure.

One starts from a given sequence $p_1\, p_2 \dots p_r \in \Upsilon$ (invariably $m = 3$ and $r = 10$). Reiterating, one knows that $\mathbf{j}_1 = \mathbf{1}$, so that the permutation in the first row is but 1 2 3 4:

k	p_k	$\mathbf{j}_k$	$\sigma_k \circ \sigma_{k-1} \circ \cdots \circ \sigma_1$			
			1	2	3	4
1	1	**1**	1	2	3	4
2	2	**2**	2	1	3	4
3	1	**2**	1	2	3	4
4	2	**2**	2	1	3	4
5	3	**3**	3	2	1	4
6	1	**3**	1	3	2	4
7	2	**3**	2	1	3	4
8	3	**3**	3	2	1	4
9	4	**4**	4	3	2	1
10	4	**1**				

5. The $\mathbf{j} \longleftarrow p$ procedure in action

Example 1 which terminates Section 2 deserves to be analyzed in depth. And all the more so that the then displayed C-chart $C(32123)$ (excerpted, recalling, from [CCKS]) fell off the framework of the lujr.

5.1. **Reading off the local geometry hidden in Example 1.** What is that C-chart in tower's stage $M(5)$ recalled in Example 1? It is a certain EKR chart visualising one concrete singularity class on the DG side ([M1, M3]). Which one? To answer, one firstly needs to bring that chart within the framework of sequences of indices satisfying the lujr — and that is easy. To get it done, one just transposes the first and third columns in Example 1, together with the corresponding transposition of all 1's and all 3's in the indices featuring in Example 1. So, up to the order of coordinates used, in the original Example 5.1 in [CCKS] there is displayed the chart $C(12321)$:

$$\begin{array}{rcl}
x_1 & x_2 & x_3 \\
\underline{x_1(1)} & x_2(1) & x_3(1) \\
x_1(12) & \underline{x_2(12)} & x_3(12) \\
x_1(123) & x_2(123) & \underline{x_3(123)} \\
x_1(1232) & \underline{x_2(1232)} & x_3(1232) \\
\underline{x_1(12321)} & x_2(12321) & x_3(12321)
\end{array}$$

Having observed this, the translation – according to the algorithm of section 4.3 – into an EKR chart is immediate:

k	p_k	$\mathbf{j}_k$	$\sigma_k \circ \sigma_{k-1} \circ \cdots \circ \sigma_1$		
			1	2	3
1	1	**1**	1	2	3
2	2	**2**	2	1	3
3	3	**3**	3	2	1
4	2	**2**	3	1	2
5	1	**3**			

Only at this moment it becomes visible that the chart in Example 1 (so, ultimately, in Example 5.1 in [CCKS]) serves the purpose of watching the singularity class 1.2.3.2.3 living in the fifth stage of the Special 2-Flags Monster Tower. It is a deeply singular codimension-6 class (cf. Proposition 4 in [M3]), sitting inside a codimension-4 *sandwich class* $1.\underline{2}.\underline{2}.\underline{2}.\underline{2}$. (The definition of the sandwich classes is given in section 3.2 in [M3].)

It so happens that this latter class coincides with the *intersection locus* I_W, $W = RV_2V_3V_4V_5$ in [CCKS] – compare the pages from 860 onwards there. Such a coincidence is not random in that each sandwich class is certain intersection locus, if of a very basic type. The matching recipe is as follows.

$$\big(\mathbf{1} \text{ in the } j\text{-th sandwich code position}\big) \Leftrightarrow \big(R \text{ in the } j\text{-th locus code position}\big)$$

and

$$\big(\underline{\mathbf{2}} \text{ in the } j\text{-th sandwich code position}\big) \Leftrightarrow \big(V_j \text{ in the } j\text{-th locus code position}\big)\,.$$

For instance, the locus I_U, $U = RRV_3RV_5$, is precisely the sandwich class $1.1.\underline{2}.1.\underline{2}$. However, such a perfect matching does not extend to further refinements of the sandwich classes vis à vis finer intersection loci.

On the whole, the intersection loci I_W with all admissible words W of length k do not form a stratification of the k-th stage $M(k)$ in the Semple tower. Instead, there is a natural stratification of $M(k)$ built by the pairwise disjoint sets

$$S_W : = I_W \setminus \bigcup_{I_{W'} \subsetneq I_W} I_{W'} \tag{17}$$

(W, W' – admissible RV words of length k).

Attention. The symbol S_W appears explicitly neither in [CCKS] nor in [CK], but the construction (17) is implicit in [CCKS] in the last paragraph on p. 860 and in the first paragraph on p. 864. It is more than natural to ask how the strata (17) are positioned with respect to the singularity classes $j_1.j_2 \dots j_k$ living – as the reader knows from Sections 2 and 3 – on the same space $M(k)$. The numbers of the RV strata grow quickly with k; much quicker than the numbers of the singularity classes (see section 4.2 and the table on p. 195 in [MP2]). Yet, in general, the former are *not* a refinement of the latter (cf. in this respect a badly misfired question on p. 195 in [MP2]). Some examples that first come to mind are given in the following section.

5.2. **Singularity classes quickly come across with RV strata.** The comparison of these natural (if both coarse) stratifications constructed on the DG and AG sides of a one and the same theory starts deceivingly simply in length 2: $1.1 = S_{RR}$ and $1.2 = S_{RV_2}$ (remember the defining formula (17)). Also length 3 is clear, with five singularity classes and six RV strata: $1.1.1 = S_{RRR}$, $1.1.2 = S_{RRV_3}$,

$$1.2.1 = S_{RV_2R} \cup S_{RV_2V_2}\,, \tag{18}$$

$1.2.2 = S_{RV_2V_3}$, $1.2.3 = S_{RV_2V_{23}}$ (cf., for instance, Figure 13 in [CK]). It is known (Theorem 4 in [MP1]) that the most involved class of length 3, 1.2.1 is the union of *three* orbits of the local equivalence by the underlying diffeomorphisms of the base manifold. Their relation to the splitting (18) is as follows. The only codimension one orbit is the stratum S_{RV_2R}. The two remaining orbits (of codimensions two and three) build up the stratum $S_{RV_2V_2}$ — it is also the first instance of an RV stratum not being an orbit of the local equivalence. (The classes 1.1.1, 1.1.2, 1.2.2 and 1.2.3 are just orbits, and this independent of the base manifold dimension $m + 1 \geq 3$.)

For the present discussion in length 4 it is important to have the orbits building up the class 1.2.1 described in the EKR coordinates **1.2.1**, that is – Theorem 1 – in the coordinates $C(122)$. With [MP1], pages 13–15 at hand,

$$S_{RV_2R} = \{x_1(12) = 0\,,\;\; x_1(122) \neq 0\}\,,$$

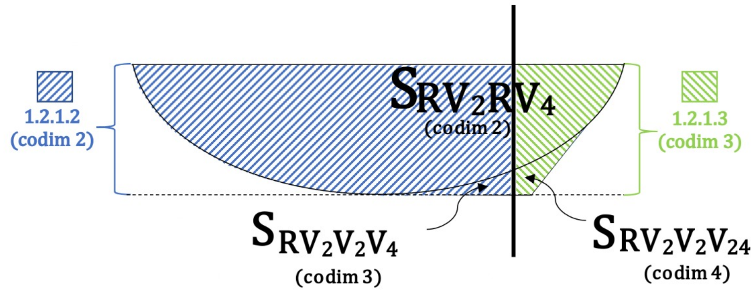
while the codimension two and three orbits are

$$S_{RV_2V_2} = \{x_1(12) = x_1(122) = 0 \neq x_3(122)\} \cup \{x_1(12) = x_1(122) = x_3(122) = 0\}\,.$$

Problems already start in length 4. We will illustrate them on just one sandwich class $1.\underline{2}.1.\underline{2}$ being the union of two singularity classes 1.2.1.2 (of codimension 2) and 1.2.1.3 (of codimension 3). Passing from an intersection locus to its parts – RV loci,

$$1.\underline{2}.1.\underline{2}\;\; (= I_{RV_2RV_4}) = S_{RV_2RV_4} \cup S_{RV_2V_2V_4} \cup S_{RV_2V_2V_{24}}\,. \tag{19}$$

The first and thickest term in the union on the right-hand side comes across with both singularity classes 1.2.1.2 and 1.2.1.3. In turn, the second term in the union sits entirely in 1.2.1.2, while the third one sits in 1.2.1.3. Graphically this situation looks as follows.



Formula (19) and the figure above show that the RV strata quickly deviate from the singularity classes. Needless to say, the orbits of the local classification are a refinement of either stratification. We conclude this paper by explicitly describing these orbits.

We start with the class 1.2.1.2 (blue on the figure). The EKR system **1.2.1.2** which is pertinent for the class 1.2.1.2 is, on the AG side, the $C(1221)$ system. The description of the *four* orbits dissecting 1.2.1.2 (pages 25 − 28 in [MP1]) goes as follows in the $C(1221)$ terms.

(i) The [lion's] part of the stratum $S_{RV_2RV_4}$ inside 1.2.1.2 is the union of a codimension-2 orbit

$$\{x_1(12) = 0\,, \quad x_1(122) \neq 0\,, \quad x_2(1221) = 0\,, \quad x_3(1221) \neq 0\}$$

and codimension-3 orbit

$$\{x_1(12) = 0\,, \quad x_1(122) \neq 0\,, \quad x_2(1221) = 0\,, \quad x_3(1221) = 0\}\,.$$

(ii) The stratum $S_{RV_2V_2V_4}$ entirely contained in 1.2.1.2 is the union of
a codimension-3 orbit

$$\{x_1(12) = 0\,, \quad x_1(122) = 0\,, \quad x_3(122) \neq 0\,, \quad x_2(1221) = 0\}$$

and codimension-4 orbit

$$\{x_1(12) = 0\,, \quad x_1(122) = 0\,, \quad x_3(122) = 0\,, \quad x_2(1221) = 0\}\,.$$

As regards the class 1.2.1.3 (green on the figure), the pertinent 'viewing' system of coordinates is **1.2.1.3**, that is, $C(1223)$. Section 4.2 in [MP1] says that the *three* orbits dissecting 1.2.1.3 are as follows:

(iii) the remaining part of the stratum $S_{RV_2RV_4}$ inside 1.2.1.3 is a single codimension-3 orbit

$$\{x_1(12) = 0\,, \quad x_1(122) \neq 0\,, \quad x_1(1223) = 0\,, \quad x_2(1223) = 0\}\,.$$

(iv) The stratum $S_{RV_2V_2V_{24}}$ entirely contained in 1.2.1.3 is the union of
a codimension-4 orbit

$$\{x_1(12) = 0\,, \quad x_1(122) = 0\,, \quad x_3(122) \neq 0\,, \quad x_1(1223) = 0\,, \quad x_2(1223) = 0\}$$

and codimension-5 orbit

$$\{x_1(12) = 0\,, \quad x_1(122) = 0\,, \quad x_3(122) = 0\,, \quad x_1(1223) = 0\,, \quad x_2(1223) = 0\}\,.$$

5.3. **Reading off the [intriguing] geometry hidden in an example of Lejeune-Jalabert.** We are going to conclude with yet another question. To that end we recall an example of a one-parameter family of space (3D) analytic curves, originally due to A. Campillo and later extensively discussed in [L-J]. That family $\{\gamma_\alpha\}$ is defined in the space $\mathbb{R}^3$ with coordinates x_1, x_2, x_3 by the equations

$$\gamma_\alpha: \quad x_1 = t^8\,, \qquad x_2 = t^{10} + t^{13}\,, \qquad x_3 = t^{12} + \alpha\, t^{15}\,, \tag{20}$$

where α is a real parameter. In order to better understand the geometric character of the family $\{\gamma_\alpha\}$, a series of five Nash blow-ups[5] is being performed over each one curve γ_α. The eventual curve is horizontal with respect to the bundle of focal 3-spaces [at points of the stage $M(5)$ of the Semple tower with the base $\mathbb{R}^3$].

The proper C-chart in which one views the result of these five blow-ups is $C(12223)$ *provided* $\alpha \neq \frac{52}{25}$. (It is also the chart Z^{12223} in [L-J], p. 1310. Curiously, for the special value $\alpha = \frac{52}{25}$ the proper glasses are different: $C(1222221)$.) Wishing to record this – rare in the singularity theory *sensu largo* – phenomenon, we supply the intermediate steps of computations. In use is the C-language (instead of the original Z-language of [L-J]). The departing point are the coordinate functions (20) of γ_α. And then

$$x_2(1) = \frac{5}{4}t^2 + \frac{13}{8}t^5\,, \qquad x_3(1) = \frac{3}{2}t^4 + \frac{15}{8}\alpha\, t^7\,,$$

$$x_1(12) = \frac{16}{5}t^6 + \cdots\,, \qquad x_3(12) = \frac{12}{5}t^2 + \frac{105\,\alpha - 156}{20}t^5 + \cdots\,,$$

$$x_1(122) = \frac{192}{25}t^4 + \cdots\,, \qquad x_3(122) = \frac{48}{25} + \frac{21}{50}\big(25\,\alpha - 52\big)t^3 + \cdots\,,$$

$$x_1(1222) = \frac{1536}{125}t^2 + \cdots\,, \qquad x_3(1222) = \frac{63}{125}\big(25\,\alpha - 52\big)t + \cdots\,,$$

$$x_1(12223) = \frac{3072}{63\big(25\,\alpha - 52\big)}t + \cdots\,, \qquad x_2(12223) = \frac{625}{126\big(25\,\alpha - 52\big)}t + \cdots\,.$$

Upon doing one more Cartan prolongation (or, the same thing, Nash blow-up) the two brand new active coordinates are already non-zero at the reference point:

$$x_1(122233)|_0 = \frac{128\,000}{1323\big(25\,\alpha - 52\big)^2}\,, \qquad x_2(122233)|_0 = \frac{78\,125}{7938\big(25\,\alpha - 52\big)^2}\,.$$

The indices in the underlying chart $C(122233)$ satisfy the lujr; the EKR name for this chart is **1.2.1.1.3.1**. This latter chart, as we know, properly describes its associated singularity class $\mathcal{C} = 1.2.1.1.3.1$, one among the two-step prolongations of the most involved in length 4 class 1.2.1.1 (see [MP1] and section 8.2 in [MP2]).

Saying the same differently, the sixth prolongation of γ_α hits at $t = 0$ the class $\mathcal{C}$ at a point P_α having the pair of last EKR coordinates $x_6 = \frac{78\,125}{7938\left(25\,\alpha - 52\right)^2}$ and $y_6 = \frac{128\,000}{1323\left(25\,\alpha - 52\right)^2}$.

Question 2. Are arbitrary two points P_α and P_β, $\alpha \neq \frac{52}{25} \neq \beta$, equivalent in $\mathcal{C}$? (Their projections two levels down, $\pi^6_4\big(P_\alpha\big)$ and $\pi^6_4\big(P_\beta\big)$, are – cf. Section 6 in [MP1] – identical in the class 1.2.1.1.)

When $\alpha = \frac{52}{25}$, the local geometry becoming visible in the prolongations of $\gamma_{\frac{52}{25}}$ is cardinally different. Two more Nash blow-ups are required to regularize this particular curve, and the chart needed to observe it (already mentioned) is $C(1222221)$.

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