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LINEAR ISOPERIMETRIC INEQUALITY FOR NORMAL AND INTEGRAL CURRENTS IN COMPACT SUBANALYTIC SETS

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Abstract. The isoperimetric inequality for a smooth compact Riemannian manifold A provides a positive constant c depending only on A, so that whenever a k-dimensional integral current T in A bounds some integral current S in A, S can be chosen to have mass at most c times the $(k+1)/k$ power of the mass of T. Although such an inequality still holds for any compact Lipschitz neighborhood retract A, it may fail in case A contains a single polynomial singularity. Here, replacing this power by one, we verify the linear inequality, the mass of S being bounded by c times the mass of T, is valid for any compact algebraic, semialgebraic, or even subanalytic set A. In such a set, this linear inequality holds not only for integral currents, which have integer coefficients, but also for normal currents having real coefficients and generally for normal flat chains having coefficients in any complete normed Abelian group. A relative version for a subanalytic pair (A, B) is also true, and there are applications to variational and metric properties of subanalytic sets.

1. Introduction

Assume that A is a smooth compact Riemannian manifold and that k is a positive integer. The following hold.

(A) The singular homology group $H_k(A;\mathbb{Z})$ of A with integer coefficients and the homology group $\mathbf{H}_k(A;\mathbb{Z})$ of A defined by means of integral currents are isomorphic.

(B) If an integral current $T \in \mathbf{I}_k(A)$ equals ∂S_0 for some $S_0 \in \mathbf{I}_{k+1}(A)$, then there exists an $S \in \mathbf{I}_{k+1}(A)$ such that $\partial S = T$ and $\mathbf{M}(S) \leqslant \mathbf{c}(A)\mathbf{M}(T)^{(k+1)/k}$.

(C) Each homology class in $\mathbf{H}_k(A;\mathbb{Z})$ admits a mass minimizing representative, i.e. given $T_0 \in \mathbf{I}_k(A)$ with $\partial T_0 = 0$, the following variational problem admits a minimizer:

$$(\mathscr{P}) \begin{cases} \text{minimize } \mathbf{M}(T) \\ \text{among } T \in \mathbf{I}_k(A) \text{ with } T - T_0 = \partial S \text{ for some } S \in \mathbf{I}_{k+1}(A). \end{cases}$$

These have been established by H. Federer and W.H. Fleming, [11, 6.3]. Note that a smooth isometric embedding $A \subseteq \mathbb{R}^n$ of the manifold into some Euclidean space exhibits A as a Lipschitz neighborhood retract. This means that there exists an open neighborhood U of A in $\mathbb{R}^n$ and a Lipschitzian retraction $f : U \to A$ onto A (one can choose f to have the same class of smoothness as A). It ensues that $\mathbf{H}_k(A;\mathbb{Z})$ and $\mathbf{H}_k(U;\mathbb{Z})$ are isomorphic, and conclusion (A) now follows from the the deformation theorem, [10, 4.4.2] which shows that each integral cycle $T \in \mathbf{I}_k(U)$ is homologous to some polyhedral cycle $P \in \mathbf{I}_k(U)$, of comparable mass, supported in a fixed complex. Conclusion (B) also follows from a careful application of this deformation theorem. In order to establish (C), one considers a mass minimizing sequence $\langle T_j \rangle_j$ for $(\mathscr{P})$. According to (B), there are $S_j \in \mathbf{I}_{k+1}(A)$ such that $\partial S_j = T_j - T_0$ and

$$\mathbf{M}(S_j) \leqslant \mathbf{c}(n)\mathbf{M}(T_j - T_0)^{(k+1)/k} \leqslant \mathbf{c}\left(n, \mathbf{M}(T_0)\right)$$

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if j is large enough. Referring to the compactness theorem of integral currents, corresponding subsequences of $\langle T_j\rangle_j$ and $\langle S_j\rangle_j$ converge in flat norm to, respectively, $T \in \mathbf{I}_k(A)$ and $S \in \mathbf{I}_{k+1}(A)$ such that $\partial S = T - T_0$, thus T and T_0 are homologous in A. As $\mathbf{M}$ is lower semicontinuous with respect to convergence in the flat norm, T minimizes mass in its homology class.

In this paper we study these questions with A being a compact subanalytic subset of $\mathbb{R}^n$, and $\mathbf{I}_k(A) = \mathbf{I}_k(\mathbb{R}^n) \cap \{T : \operatorname{spt} T \subseteq A\}$. Whereas **semialgebraic sets** [2] are defined by finitely many polynomials, the larger class of **subanalytic sets** [1] includes sets defined locally by real analytic functions as well as their images under proper real analytic maps. Since such sets may fail to be Lipschitz neighborhood retracts, the methods of [11] do not apply, and the isoperimetric inequality of (B) may in fact fail, as the following example illustrates.

For $N = 2, 3, \ldots$, we define the semialgebraic set

$$A_N = \mathbb{R}^3 \cap \left\{(x, y, z) : z^{2N} = x^2 + y^2 \text{ and } 0 \leqslant z \leqslant 1\right\},$$

i.e. A_N is obtained from the rotation around the z axis of the graph of $z = x^{1/N}$, $0 \leqslant x \leqslant 1$; it has a cusp at the origin. Given $0 < h < 1$, we consider $T_h \in \mathbf{I}_1(A_N)$ an oriented circle of multiplicity 1 on A_N, at height h. It is not hard to show that there exists a unique $S_h \in \mathbf{I}_2(A_N)$ such that $\partial S_h = T_h$, and that

$$\mathbf{M}(S_h) = 2\pi \int_0^h z^N \sqrt{1 + (N z^{N-1})^2}\, dz\,.$$

Since $\mathbf{M}(T_h) = 2\pi r = 2\pi h^N$, where r is the radius of the circle $A_N \cap \{z = h\}$, we infer the following. For each $q > 1$, choosing N so large that $q = \frac{N+1}{N} + \varepsilon$ for some $\varepsilon > 0$,

$$\lim_{h\to 0^+} \frac{\mathbf{M}(S_h)}{\mathbf{M}(T_h)^q} = \lim_{h\to 0^+} \frac{2\pi\left(\frac{h^{N+1}}{N+1}\right)}{(2\pi h^N)^q} = \lim_{h\to 0^+} \frac{(2\pi)^{1-q}}{N+1}\left(\frac{1}{h}\right)^{N\varepsilon} = \infty\,.$$

This shows that, in conclusion (B) above, we cannot hope for an inequality $\mathbf{M}(S) \leqslant \mathbf{c}(A)\mathbf{M}(T)^q$ with an exponent $q > 1$ depending only on the dimension of T and the constant $\mathbf{c}(A)$ depending only on the semialgebraic set A. In fact, if q is allowed to depend only on the dimension of T, then $q = 1$ is the only possible choice, as illustrated by these simple calculations. Incidentally, the computations also show that A_N is not a Lipschitz neighborhood retract: If there were a Lipschitzian retraction $f : U \to A_N$, considering $D_h \in \mathbf{I}_2(\mathbb{R}^3)$ the unique flat disk in $\mathbb{R}^3$ with $\partial D_h = T_h$, and h small enough for $\operatorname{spt} D_h \subseteq U$, we would have $f_\# D_h = S_h$ and in turn

$$\frac{2\pi h^{N+1}}{N+1} \leqslant \mathbf{M}(S_h) = \mathbf{M}(f_\# D_h) \leqslant \pi(\operatorname{Lip} f)^2 h^{2N},$$

a contradiction as $h \to 0^+$.

Our main result is as follows.

Theorem. — *Let $A \subseteq \mathbb{R}^n$ be a compact subanalytic set. There exists $\mathbf{c}(A) > 0$ with the following property. For every $k = 0, 1, 2, \ldots$ and every $S_0 \in \mathbf{I}_{k+1}(A)$, there exists $S \in \mathbf{I}_{k+1}(A)$ such that $\partial S = \partial S_0$ and $\mathbf{M}(S) \leqslant \mathbf{c}(A)\mathbf{M}(\partial S)$.*

This is known as a *linear* isoperimetric inequality because of the absence of an exponent in the righthand term. Moreover, the linear inequality holds, via the same proof, with $\mathbb{Z}$ replaced by any normed, complete Abelian group G of coefficients. When $G = \mathbb{R}$, with the absolute value norm, the fact that both $\mathbf{M}(\lambda S) = \lambda \mathbf{M}(S)$ and $\mathbf{M}(\lambda \partial S) = \lambda \mathbf{M}(\partial S)$ for arbitrarily small positive λ shows that both sides of a valid isoperimetric inequality must be homogeneous of the same degree, which can therefore be set equal to 1.

Our proof is based on two facts of metric nature, regarding a compact subanalytic set A. A basic topological property is that A is triangulable [14], [15], i.e. there exists a simplicial complex $\mathscr{K}$ and a subanalytic homeomorphism $\phi : |\mathscr{K}| \to A$. For $x \in A$, there exists $r(x) > 0$ such that the intersections with open balls $|\mathscr{K}| \cap \mathbf{U}(\phi^{-1}(x), r)$ are contractible for $0 < r \leqslant r(x)$, and it ensues that A is locally contractible. Here we will use a strengthening due to G. Valette, [21] stating that the local homotopy $h : [0,1] \times (A \cap U) \to A \cap U$ from the identity to a constant can be chosen to be Lipschitz. This already implies A is "locally acyclic" with respect to the chain complex of integral currents and in turn, that $H_k(A;\mathbb{Z})$ and $\mathbf{H}_k(A;\mathbb{Z})$ are isomorphic, see [5], which is analogous to (A) above. See also [12]. A second consequence of triangulability is that if $x \in A$ and $0 < s \leqslant s(x)$ is small enough, then, for $s/2 < t < t' < s$, the spherical links $A \cap \operatorname{Bdry} \mathbf{U}(x,t)$ and $A \cap \operatorname{Bdry} \mathbf{U}(x,t')$ are subanalytically homeomorphic. Here we use the work of A. Parusiński, [19] on Lipschitz stratification of subanalytic sets to find, away from the origin, intervals of radii for links which are uniformly bilipschitz equivalent.

After our Main Theorem and the analog of (B) above are established, the analog of (C) follows along the same lines sketched at the beginning of this introduction. It suffices to observe that for proving (B), the particular power $(k+1)/k$ plays no significant role. An existence theorem for the Plateau problem in semialgebraic sets has been obtained recently by Q. Funk using different methods, [12].

As discussed in §5 , the linear isoperimetric inequality for compact subanalytic sets is also valid for normal currents with $\mathbb{R}$ coefficients or even normal flat chains with coefficients in any complete normed Abelian group. It plays a role in relating not only various homology theories but also a duality between a homology based on normal currents and a cohomology based on normal cochains called charges [7].

A large number of geometric variational problems involving support constraints, boundary constraints, or free boundaries can be formulated with various groups of chains, cycles, boundaries, or homologies. Generalizing (C), we give in §5.9, the existence theory for one problem related to minimizing mass in a relative homology class of a pair $B \subseteq A$ of compact subanalytic sets. This uses a relative isoperimetric inequality which bounds the the part of the mass in $A \setminus B$ of a suitable chain in terms of the part of the mass in $A \setminus B$ of its boundary. There remain many interesting regularity questions concerning solutions of this and other variational problems in semialgebraic and subanalytic sets.

The referee advised addressing the generalization from (globally) subanalytic sets to sets definable in some o-minimal structure. For a polynomially bounded o-minimal structure, the editor has suggested that both of the properties of A remain valid and has kindly provided the references. Specifically, the local Lipschitz contraction result, Theorem 3.3, holds in polynomially bounded structures as per [20] and [22]. Furthermore, the Lipschitz stratification in the sense of Mostowski, Theorem 3.5, holds in such structures according to, for instance, [18] and [13]. Whether our main result holds in the general realm of (non necessarily polynomially bounded) o-minimal structures is an open question. The naive 3-dimensional algebraic cusp given above cannot be modified to an o-minimal one that fails the linear isoperimetric inequality. More specifically, let $f : [0,1] \to \mathbb{R}^+$ be nonnegative and consider the set

$$A_f = \mathbb{R}^3 \cap \{(x,y,z) : f(z) = \sqrt{x^2+y^2} \text{ an } 0 \leqslant z \leqslant 1\}.$$

We also assume that $f(0) = 0$, $f(z) > 0$ whenever $z > 0$, and that f is differentiable at 0 and $f'(0) = 0$ (so that the origin of $\mathbb{R}^3$ is a cusp of A_f). Letting S_h and T_h be defined similarly to the example above, we note that

$$\limsup_{h \to 0^+} \frac{M(S_h)}{M(T_h)} = \limsup_{h \to 0^+} \frac{1}{f(h)} \int_0^h f(z)\sqrt{1+(f'(z))^2}dz.$$

In order for this limit superior to be ∞, it is easy to see that f should oscillate in a neighborhood of 0. In fact, if f is definable in some o-minimal structure, then there exists a neighborhood of 0 where f is differentiable and f' is positive and bounded (see e.g. [23, Ch. 3 §1 and Ch.7 §3]). In that case, the above limit superior vanishes.

Interesting inequalities concerning functions defined on singular algebraic varieties or subanalytic domains are found in works of L. Bos and P. Milman [4], [3], B. Hua and F.H. Lin [personal communication, 2013], and A.Valette and G. Valette [20] .

Our notation regarding integral and normal currents and flat chains is consistent with that of [10].

2. Bilipschitz Equivalence

2.1 (Linear isoperimetric inequality). — Let $k = 0, 1, 2, \ldots$ and $A \subseteq \mathbb{R}^n$. We say that A **satisfies the linear isoperimetric inequality of dimension** k whenever the following holds. There exists $\mathbf{c}(A, k) > 0$ such that for each $S_0 \in \mathbf{I}_{k+1}(A)$ there exists $S \in \mathbf{I}_{k+1}(A)$ with $\partial S = \partial S_0$ and $\mathbf{M}(S) \leqslant \mathbf{c}(A, k)\mathbf{M}(\partial S)$.

2.2 (Bilipschitz equivalence). — Let $X, Y \subseteq \mathbb{R}^n$ and $\mathbb{L} > 0$. We say that a bijective map $\phi : X \to Y$ is $\mathbb{L}$**-bilipschitz** if ϕ is Lipschitz, ϕ^{-1} is Lipschitz and $\max\{\operatorname{Lip}\phi, \operatorname{Lip}\phi^{-1}\} \leqslant \mathbb{L}$. In case such map ϕ exists, we say that X and Y are **bilipschitz equivalent**.

Proposition 2.3. — *Let $k = 0, 1, 2, \ldots$, and $X, Y \subseteq \mathbb{R}^n$. Assume that X and Y are bilipschitz equivalent. It follows that X satisfies the linear isoperimetric inequality of dimension k if and only if Y does.*

Proof. Let $\phi : X \to Y$ be an $\mathbb{L}$-bilipschitz homeomorphism. One infers from Kirszbraun's Theorem [10, 2.10.43] that ϕ, ϕ^{-1} admit extensions $f, g : \mathbb{R}^n \to \mathbb{R}^n$ with $\max\{\operatorname{Lip} f, \operatorname{Lip} g\} \leqslant \mathbb{L}$. Assume that X satisfies the linear isoperimetric inequality of dimension k with constant $\mathbf{c} > 0$. Let $S_0 \in \mathbf{I}_{k+1}(Y)$. Define $S_0' = g_\# S_0 \in \mathbf{I}_{k+1}(\mathbb{R}^n)$. Notice that $\operatorname{spt}(S_0') \subseteq X$, [10, 4.1.14 p. 371]. Thus, there exists $S' \in \mathbf{I}_{k+1}(X)$ such that $\partial S' = \partial S_0'$ and $\mathbf{M}(S') \leqslant \mathbf{c}\mathbf{M}(\partial S')$. Define $S = f_\# S'$ and notice, as before, that $S \in \mathbf{I}_{k+1}(Y)$. Further, note that $f_\# g_\# S = (f \circ g)_\# S = S$ where the last equality follows from [10, 4.1.15 p. 372] and the fact that $f \circ g = \mathrm{id}_{\mathbb{R}^n}$ on $\operatorname{spt} S$. Finally,

$$\begin{aligned}\mathbf{M}(S) = \mathbf{M}(f_\# g_\# S) &\leqslant (\operatorname{Lip} f)^{k+1}\mathbf{M}(S') \leqslant (\operatorname{Lip} f)^{k+1}\mathbf{c}\mathbf{M}(\partial S') \\ &= (\operatorname{Lip} f)^{k+1}\mathbf{c}\mathbf{M}(g_\# \partial S) \leqslant (\operatorname{Lip} f)^{k+1}(\operatorname{Lip} g)^k\mathbf{c}\mathbf{M}(\partial S) \leqslant \mathbf{c}\mathbb{L}^{2k+1}\mathbf{M}(\partial S).\end{aligned}$$

□

3. Two Properties of Subanalytic Sets

This section contains a very brief introduction to subanalytic geometry with emphasis on the two main technical properties of subanalytic sets that are critical to our argument, with some sketches as well as full references to proofs.

3.1 (Semialgebraic sets). — A set $A \subseteq \mathbb{R}^n$ is **semialgebraic** if it is the finite union $A = \cup_i A_i$ of sets A_i defined by finitely many polynomial equalities or inequalities, i.e. $A_i = \cap_j A_{i,j}$ where each $A_{i,j}$ is of the form $\mathbb{R}^n \cap \{\operatorname{sign} P = \varepsilon\}$ where $P \in \mathbb{R}[X_1, \ldots, X_n]$ and $\varepsilon \in \{-1, 0, 1\}$. Letting $\mathscr{S}_n$, $n = 1, 2, \ldots$, denote the collection of semialgebraic subsets of $\mathbb{R}^n$, it is obvious that $\mathscr{S}_n$ is a Boolean algebra with respect to set theoretic operations. Furthermore, $\pi(A) \in \mathscr{S}_n$ for any $A \in \mathscr{S}_{n+1}$, where π denotes the canonical projection of $\mathbb{R}^{n+1}$ on the first n coordinates (the Tarski-Seidenberg Theorem [2, §1.4 and Theorem 2.2.1]). Finally, each $A \in \mathscr{S}_1$ is a finite union of (possibly degenerate) intervals. These properties make $(\mathscr{S}_n)_n$ an example of an **o-minimal structure**, which enjoys various useful geometric properties, including triangulability (see [23, Chapter 1 §3] or [2, §2.3]).

3.2 (Subanalytic sets). — Replacing polynomials by families of real anatytic functions defined locally leads to the notion of **semianalytic** subsets of $\mathbb{R}^n$. These again form a Boolean algebra. However, the projected image $\pi(A)$ of a semianalytic set may fail to be semianalytic [1, Example 2.14], motivating the following definition. A subset A of $\mathbb{R}^n$ is **subanalytic** if every $x \in \mathbb{R}^n$

has an open neighborhood U such that $U \cap A$ is the projected image of a relatively compact semianalytic subset (of some higher dimensional Euclidean space, [1, Definition 3.2]). Using the important Theorem of the Complement [1, Theorem 3.10], one verifies that subanalytic subsets of a Euclidean space form a Boolean algebra. While projections are okay for bounded subanalytic sets, behavior at infinity must be contolled for general ones. A **globally subanalytic** subset of $\mathbb{R}^n$ is a one whose image in the standard compactifying projective space $\mathbb{RP}^n \supseteq \mathbb{R}^n$ is a subanalytic subset of $\mathbb{RP}^n$. Then the subclass of globally subanalytic sets in $\mathbb{R}^n$ contains all bounded subanalytic sets and does constitute an o-minimal structure, like the class of semialgebraic sets does.

A *function* is called semialgebraic (respectively globally subanalytic) if its graph is semialgebraic (respectively globally subanalytic). For example the homeomorphism defining the triangulation of a compact semialgebraic (respectively subanalytic) set may be chosen to be semialgebraic (respectively globally subanalytic) [14], [23, Chapter 8]. In terms of the standard ambient metric, a globally subanalytic mapping is automatically Hölder continuous, although not all o-minimal structures in $\mathbb{R}^n$ have this property [16].

A compact subanalytic set A, being triangulable, is locally contractible at each point, and by the above discussions, such local contractions can be chosen to be Hölder continuous. The following stronger result of G. Valette, [21, Theorem 2.3.1] is that these local contractions may be chosen to be *Lipschitz*, even though A itself may fail to be a Lipschitz neighborhood retract (as seen in the cusp example of the introduction).

Theorem 3.3. — *Any point a in a closed subanalytic subset A of $\mathbb{R}^n$ has a compact subanalytic neighborhood $K \subseteq A$ and a Lipschitz deformation contraction $h : [0,1] \times K \to K$ so that $h(0,x) = x$ and $h(1,x) = a$ for $x \in K$ and $h(t,a) = a$ for $t \in [0,1]$.*

From this we obtain at once a local version of the desired linear isoperimetric inequality.

Corollary 3.4. — *Suppose that $J \in \mathbf{I}_k(K)$ and that $\partial J = 0$ in case $k > 0$. Then the chain $H := -h_\# ([\![0,1]\!] \times J)$ belongs to $\mathbf{I}_{k+1}(K)$ and satisfies*

$$\partial H \;=\; \begin{cases} J & \text{if} \quad k > 0 \\ J - J(1)[\![a]\!] & \text{if} \quad k = 0 \end{cases} \qquad \text{and} \qquad \mathbf{M}(H) \leqslant (\operatorname{Lip} h)^{k+1} \mathbf{M}(J) \ .$$

Proof. Clearly $H \in \mathcal{R}_{k+1}(K)$ because h is Lipschitz, and the homotopy formula [10, 4.1.9] shows that, for $k > 0$,

$$\begin{aligned} \partial H \;=\; h_\# \partial([\![0,1]\!] \times J) \;&=\; h_\#([\![1]\!] \times J) - h_\#([\![0]\!] \times J) - h_\#([\![0,1]\!] \times \partial J) \\ &=\; \quad 0 \quad - \quad J \quad - \quad 0 \quad \in \mathcal{R}_k(K) \ , \end{aligned}$$

Here $h(0, \,\cdot\,) = \mathrm{id}$ and $h_\#([\![1]\!] \times J) = 0$, by [10, 4.1.20], because $\operatorname{spt} h_\#([\![1]\!] \times J) \subseteq \{a\}$ and $h_\#([\![1]\!] \times J) \in \mathbf{I}_k(\mathbb{R}^n)$ with $k \geqslant 1$. In case $k = 0$, [10, 4.1.9] shows that

$$\partial H \;=\; h_\#([\![1]\!] \times J) - h_\#([\![0]\!] \times J) \;=\; J(1)[\![a]\!] \;-\; J$$

because. $h_\#([\![1]\!] \times J)(f) = J(f(a)) = f(a)J(1)$ for $f \in \mathscr{C}_c^\infty(\mathbb{R}^n)$. In either case, we easily estimate

$$\mathbf{M}(H) \;\leqslant (\operatorname{Lip} h)^{k+1} \mathbf{M}\left([\![0,1]\!] \times J\right) \;=\; (\operatorname{Lip} h)^{k+1} \mathbf{M}(J) \ .$$

□

To use this Corollary in the proof of the Main Theorem via local modifications of the given current S, one needs to partition S into finitely many small pieces each contained in a Lipschitz contractible piece of A. The choice of the partition necessarily depends on the current S. Since one still needs the mass bounds in all constructions to be independent of this choice, we first verify some bilipschitz equivalences. We will verify these by using the following result of A. Parusiński [19, Theorem 1.6(1)] :

Theorem 3.5 (Bilipschitz Triviality). — *Suppose X is a compact subanalytic subset of $\mathbb{R}^\ell \times \mathbb{R}^n$ and $\pi : X \to \mathbb{R}^\ell$ is the projection. Then there is a compact nowhere dense subanalytic subset Z of $\pi(X)$ such that X is locally bilipschitz trivial over $\pi(X) \setminus Z$; that is, for each $y \in \pi(X) \setminus Z$, there is a neighborhood U_y of y in $\pi(X)$ and a bilipschitz homeomorphism*

$$\Phi_y : \pi^{-1}(U_y) \to U_y \times \pi^{-1}\{y\}$$

such that $\pi \circ \Phi_y = \pi$ and $\Phi_y|\pi^{-1}\{y\} = (y, \mathbf{id})$.

Such a *bilipschitz trivializing homeomorphism over* π gives uniformly bilipschitz maps between the fibers of π. Here $\Phi_y = (\pi, \Psi_y)$ for some retraction Ψ_y of $\pi^{-1}(U_y)$ onto $\pi^{-1}\{y\}$, and Φ_y induces, for any pair z, w in U_y, the bilipschitz homeomorphism

$$(\Phi_y)^{-1}\left(w, \Psi_y \,\middle|\, \pi^{-1}\{z\}\right) \;:\; \pi^{-1}\{z\} \;\to\; \pi^{-1}\{w\}$$

whose bilipschitz constants are bounded independently of z and w.

We will need an extra property for the trivialization. Given any finite family $\mathcal{E}$ of compact subanalytic subsets of X, such local bilipschitz trivializations can be found to apply simultaneously to $\pi|E$ for every $E \in \mathcal{E}$. Specifically, we verify below how the proof in [19] may be slightly modified to show that:

(#) *There exist Z, U_y, Φ_y so that $\Phi_y\big(E \cap \pi^{-1}(U_y)\big) = U_y \times (E \cap \pi^{-1}\{y\})$ for all $E \in \mathcal{E}$.*

Proof. To describe how to obtain (#), we briefly outline the proof of [19, Theorem 1.6(1)]. The main step [19, Theorem 1.4], involves showing the existence of a *Lipschitz stratification* [19, Definition 1.1] $\mathcal{X}$ of X. This key notion, defined by Mostowski in [17], is a condition on the angles of the tangent planes of every stratum on approach to the frontier of the stratum. As already stated and shown in [19, Theorem 1.4], one may insist that the Lipschitz stratification $\mathcal{X}$ be *compatible with* the given family $\mathcal{E}$. This means that any stratum $S \in \mathcal{X}$ which intersects some $E \in \mathcal{E}$ must lie entirely in E. This extra compatibility will eventually imply (#). Letting $X_i = \bigcup\{S \in \mathcal{X} \;:\; \dim \mathrm{S} \leqslant \mathrm{i}\}$ gives the associated filtration $\emptyset \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{\ell+n} = X$ by closed subanalytic sets. Note how the stratification $\mathcal{X}$ may be recovered from the filtration by taking the components of $X_i \setminus X_{i-1}$ for $i = 1, \dots, \dim \mathrm{X}$. Next, [19, Proposition 1.3] establishes the important extendability from X_{i-1} to X_i of Lipschitz vectorfields *compatible with* $\mathcal{X}$, i.e. tangent to the strata in $\mathcal{X}$. For the proof of [19, Theorem 1.6(1)] one also insists that the stratification have the rank of $\pi|S$ constant for all $S \in \mathcal{X}$. Constant rank refinements are basic to many stratification results. For use with subanalytic sets, see e.g. [9]. To choose a suitable exceptional set Z, one may take a subanalytic stratification $\mathcal{T}$ of $\pi(X)$, compatible with $\{\pi(S) \;: S \in \mathcal{X}\}$, and let

$$Z \;=\; \bigcup\{\mathrm{Fron}\,\mathrm{T} \;:\mathrm{T} \in \mathcal{T}\}.$$

Clearly Z is compact, subanalytic, and nowhere dense in $\pi(X)$. Moreover, each $T \in \mathcal{T}$ with $T \cap Z = \emptyset$, is locally of maximal dimension in $\pi(X)$. For each $S \in \mathcal{X}$, and each component C of $S \cap \pi^{-1}(T)$ with $S \in \mathcal{X}$, π maps C submersively onto T. Any $y \in \pi(X) \setminus Z$ is contained in one such T, and one may take U_y to be a topological ball in T. As in the end of the proof of [19, Theorem 1.6(1)] (see also [17, §2]), the trivilalizing map is derived from linear combinations of lifted Lipschitz $\mathcal{X}$-compatible vectorfields. The lifts are obtained by repeatedly extending, using [19, Proposition 1.3], from X_{i-1} to X_i, starting with $i - 1 = \dim \mathrm{T}$. The Lipschitz property of the lifted vectorfields leads to the bilipschitz nature of the trivilalizing map while the $\mathcal{X}$-compatibility of the lifted vectorfields implies that the trivializing map Φ_y preserves the X_i, i.e. satisfies (#) with E replaced by each X_i. It follows that it preserves the components of $X_i \setminus X_{i-1}$, namely every stratum in $\mathcal{X}$. Finally, by the constructed compatibility of $\mathcal{X}$ with $\mathcal{E}$, each $E \in \mathcal{E}$ partitions into some of the strata from $\mathcal{X}$, and we conclude (#) holds for every $E \in \mathcal{E}$. □

We will now apply Theorem 3.5 and **(#)** to finite unions of spherical links in a subanalytic set as the radii of the spheres vary.

3.6 (Finite unions of links). — Any $a \in A$ and $r > 0$ determine a **spherical link**

$$L^a_r := A \cap \operatorname{Bdry} \mathbf{U}(a, r).$$

Also for ℓ-tuples $\vec{a} = (a_1, \dots, a_\ell) \in A^\ell$ and $\vec{r} = (r_1, \dots, r_\ell) \in (\mathbb{R}^*_+)^\ell$ we may denote the corresponding **union of links** $L^{\vec{a}}_{\vec{r}} := \cup_{i=1}^{\ell} L^{a_i}_{r_i}$.

To indicate individual rescalings of the r_i, we use, for $\vec{\lambda} = (\lambda_1, \dots, \lambda_\ell)$ and $\vec{r} = (r_1, \dots, r_\ell)$, the (useful, but nonstandard) notation $\vec{\lambda}\vec{r} = (\lambda_1 r_1, \dots, \lambda_\ell r_\ell)$.

Theorem 3.7 (Link Bilipschitz Equivalence). — *Suppose $\vec{a}$, $\vec{r}$ are such ℓ-tuples with $L^{a_i}_{\lambda r_i} \neq \emptyset$ for all $\lambda \in [0, 1]$ and $i \in \{1, \dots, \ell\}$. There are numbers $\frac{1}{2} < \lambda^{\vec{a}}_{\vec{r},i} < \mu^{\vec{a}}_{\vec{r},i} < 1$ so that, for any $\lambda_i, \mu_i \in [\lambda^{\vec{a}}_{\vec{r},i}, \mu^{\vec{a}}_{\vec{r},i}]$, the two corresponding link unions $L^{\vec{a}}_{\vec{\lambda}\vec{r}}$ and $L^{\vec{a}}_{\vec{\mu}\vec{r}}$ are uniformly bilipschitz equivalent by a map sending $L^{a_i}_{\lambda_i r_i}$ to $L^{a_i}_{\mu_i r_i}$ for $i = 1, \dots, \ell$.*

Proof. For $i = 1, \dots, \ell$,

$$L_i = \left\{ (\vec{\lambda}, x) \ : \ \vec{\lambda} \in [\frac{1}{2}, 1]^\ell, \ x \in A, \ |x - a_i|^2 = \lambda_i^2 r_i^2 \right\} = \bigcup_{\vec{\lambda} \in [\frac{1}{2}, 1]^\ell} \{\vec{\lambda}\} \times L^{a_i}_{\lambda_i r_i} .$$

is compact and subanalytic. We will apply Theorem 3.5 and **(#)** with $\mathcal{E} = \{L_1, \dots, L_\ell\}$ and

$$X = \cup_{i=1}^{\ell} L_i = \left\{ (\vec{\lambda}, x) \ : \ \vec{\lambda} \in [\frac{1}{2}, 1]^\ell, \ x \in A, \ |x - a_i|^2 = \lambda_i^2 r_i^2 \text{ for some i} \right\} = \bigcup_{\vec{\lambda} \in [\frac{1}{2}, 1]^\ell} \{\vec{\lambda}\} \times L^{\vec{a}}_{\vec{\lambda}\vec{r}} ,$$

Note that, by the hypothesis, the projection π of X to $Y \equiv [\frac{1}{2}, 1]^\ell$ is surjective. Applying Theorem 3.5 and **(#)**, we find that the resulting exceptional subanalytic subset Z of Y has dimension at most $\ell - 1$, and we can choose any noncollapsed rectangle

$$\Pi_{l=1}^{\ell} [\lambda^{\vec{a}}_{\vec{r},i}, \ \mu^{\vec{a}}_{\vec{r},i}] \ \subseteq \ (\mathrm{Int\,Y}) \setminus \mathrm{Z}$$

to see that, there are uniform bilipschitz equivalences of the link unions

$$L^{\vec{a}}_{\vec{\lambda}\vec{r}} \ \approx \ \{\vec{\lambda}\} \times L^{\vec{a}}_{\vec{\lambda}\vec{r}} \ = \ \pi^{-1}\{\vec{\lambda}\} \ \approx \ \pi^{-1}\{\vec{\mu}\} \ = \ \{\vec{\mu}\} \times L^{\vec{a}}_{\vec{\mu}\vec{r}} \ \approx \ L^{\vec{a}}_{\vec{\mu}\vec{r}} .$$

Moreover, applying **(#)** to each E_i shows that this bilipschitz equivalence of link unions maps the individual link $L^{a_i}_{\lambda_i r_i}$ to $L^{a_i}_{\mu_i r_i}$ and gives the uniform bilipschitz equivalence of the individual links. ☐

4. Proof of the Theorem

In this section we will use $\mathbf{c}$ (rather than $\mathbf{c}(A)$) to denote a constant depending only on A. Its value may increase in the course of the proof, even in a single chain of inequalities. Since dividing A into its finitely many path components decomposes both S_0 and ∂S_0, we may assume that A itself is path-connected. We of course ignore the case A is a singleton, which does not support any nonzero boundary. Since A is triangulable and has no isolated points, every point $a \in A$ has nonempty links $L^a_r = A \cap \operatorname{Bdry} \mathbf{U}(a, r) \neq \emptyset$ for all sufficiently small r.

4.1. **Partition into contractible regions.** By compactness of A and Theorem 3.3, we may now choose a finite family of open balls

$$\{ \ \mathbf{U}(a_i \, , \, r_i/2) \ : \ i = 1, 2, \dots, \ell \ \}$$

covering A where the a_i are distinct points of A and each $\mathbf{U}(a_i, r_i) \cap A$ lies in a compact neighborhood $K_i \subseteq A$ of a_i having a Lipschitz deformation contraction h_i to $\{a_i\}$, Also we may assume that all the links $L^{a_i}_{\lambda r_i} \neq \emptyset$, $0 < \lambda \leqslant 1$.

We would like to make, for some fixed $\vec{\lambda} \in [1/2, 1]^\ell$, separate adjustments of S_0 in each of the open balls $U_i^{\lambda_i} := \mathbf{U}(a_i, \lambda_i r_i)$. Since the $U_i^{\lambda_i}$ will likely overlap, we will work with the corresponding disjoint open sets

$$W_1^{\vec{\lambda}} := U_1^{\lambda_1}, \quad W_2^{\vec{\lambda}} := U_2^{\lambda_2} \setminus \operatorname{Clos} U_1^{\lambda_1}, \quad \ldots , \quad W_\ell^{\vec{\lambda}} := U_\ell^{\lambda_\ell} \setminus \bigcup_{h=1}^{\ell-1} \operatorname{Clos} U_i^{\lambda_h} ,$$

noting that $A \subseteq \cup_{i=1}^{\ell} \operatorname{Clos} W_i^{\vec{\lambda}}$ and $\operatorname{Bdry} W_i^{\vec{\lambda}} \subseteq \cup_{h=1}^{i} \operatorname{Bdry} U_h^{\lambda_h}$, a finite union of $n-1$ spheres.

We will eventually choose the radii $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ in order which determine the sets $W_1^{\vec{\lambda}}$, $W_2^{\vec{\lambda}}, \ldots, W_\ell^{\vec{\lambda}}$ (because $W_i^{\vec{\lambda}}$ depends only on $\lambda_1, \ldots, \lambda_i$).

In choosing the radii λ_i, there are some Lebesgue null sets of λ that one must avoid. To describe one, first, note that, for any finite Radon measure α on $\mathbb{R}^n$, the set

$$\Lambda_\alpha := \{\lambda > 0 \ : \ \alpha(L^{a_i}_{\lambda r_i}) > 0 \text{ for some } i \ \}$$

is at most countable because $\{\lambda : \ \alpha(L^{a_i}_{\lambda r_i}) > 1/j\}$ is finite for each $j \in \mathbb{N}$. One useful consequence, for any positive $\lambda_1, \ldots, \lambda_i \notin \Lambda_{\|S_0\|+\|\partial S_0\|}$, is that

$$(\|S_0\| + \|\partial S_0\|) \left(\operatorname{Bdry}(W_i^{\vec{\lambda}})\right) = 0 \ , \quad S_0 \llcorner \operatorname{Clos}\left(W_i^{\vec{\lambda}}\right) = S_0 \llcorner W_i^{\vec{\lambda}} \ ,$$
$$(\partial S_0) \llcorner \operatorname{Clos}\left(W_i^{\vec{\lambda}}\right) = (\partial S_0) \llcorner W_i^{\vec{\lambda}},$$

for $i = 1, \ldots, \ell$. The disjoint open sets $W_i^{\vec{\lambda}}$ now give the two decompositions

$$S_0 \ = \ \sum_{i=1}^{\ell} S_0 \llcorner W_i^{\vec{\lambda}} \quad \text{and} \quad (\partial S_0) \ = \ \sum_{i=1}^{\ell} (\partial S_0) \llcorner W_i^{\vec{\lambda}} \ , \tag{1}$$

obtained by ignoring the boundaries of the $W_i^{\vec{\lambda}}$.

4.2. **Proof of the case $k = 0$.**

Proof. We will make essentially two different applications of Corollary 3.4 with $k = 0$. First, for $\vec{\lambda} = (\lambda_1, \ldots, \lambda_\ell)$ as above and $i \in \{1, 2, \ldots, \ell\}$, we apply Corollary 3.4 with $J = (\partial S_0) \llcorner W_i^{\vec{\lambda}}$ to find that

$$E_i \ := \ -h_{i\#} \left([\![0,1]\!] \times [(\partial S_0) \llcorner W_i^{\vec{\lambda}}] \right) \in \mathbf{I}_1(A)$$

has

$$\partial E_i = (\partial S_0) \llcorner W_i^{\vec{\lambda}} \ - \ g_i [\![a_i]\!] \quad \text{and} \quad \mathbf{M}(E_i) \leqslant \operatorname{Lip}(h_i) \mathbf{M}[(\partial S_0) \llcorner W_i^{\vec{\lambda}}] \ ,$$

where $g_i = (\partial S_0) \left(\mathbb{1}_{W_i^{\vec{\lambda}}} \right) \in \mathbb{Z}$ is the total multiplicity of ∂S_0 in $W_i^{\vec{\lambda}}$. Observe that by (1),

$$\sum_{i=1}^{\ell} g_i \ = \ \sum_{i=1}^{\ell} [(\partial S_0) \llcorner W_i^{\vec{\lambda}})](1) \ = \left(\sum_{i=1}^{\ell} (\partial S_0) \llcorner W_i^{\vec{\lambda}} \right) (1) \ = \ (\partial S_0)(1) \ = \ 0 \ . \tag{2}$$

Second, we can use Corollary 3.4 to bound the intrinsic diameter of A. Consider the simple situation in Corollary 3.4 when J is a single point with multiplicity, say $J = g[\![b]\!]$ where $g \in \mathbb{Z}$ and $b \in K$. The resulting $H = -h_{\#}\left([\![0,1]\!] \times g[\![b]\!]\right)$ is $-g$ times the path $t \mapsto h(t, b)$ from b to a and

$$\partial H \ = \ g[\![b]\!] - g[\![a]\!] \quad \text{and} \quad \mathbf{M}(H) \ = \ |g| \text{length}\,(h(\cdot, b)) \ \leqslant \ |g| \operatorname{Lip} h \ .$$

Thus the intrinsic diameter of K is $\leqslant 2 \times \text{length}\,(h(\cdot, b)) \leqslant 2 \operatorname{Lip} h$. Inasmuch as the intrinsic diameter of the union of two intersecting sets is at most the sum of their intrinsic diameters, we readily deduce that the intrinsic diameter of A is at most $D(A) := 2 \sum_{i=1}^{\ell} \operatorname{Lip}(h_i)$.

To complete the proof of the $k = 0$ case, we choose, for each $i = 1, \ldots, \ell$, a curve $\gamma_i : [0,1] \to A$ from a_1 to a_i with $\mathrm{length}(\gamma_i) \leqslant D(A)$. We now define

$$S \ := \ \sum_{i=1}^{\ell} E_i + g_i \gamma_{i\#}[\![0,1]\!]$$

and verify, by (2), that

$$\begin{aligned} \partial S \ &= \ \sum_{i=1}^{\ell} \Big((\partial S_0) \llcorner W_i^{\vec\lambda} \ - \ g_i [\![a_i]\!] \Big) \ + \ \big(g_i[\![a_i]\!] - g_i[\![a_1]\!]\big) \\ &= \ \partial S_0 \ - \ \left(\sum_{i=1}^{\ell} g_i \right) \cdot [\![a_1]\!] \ = \ \partial S_0 \ - \ 0 \ , \end{aligned}$$

and

$$\begin{aligned} \mathbf{M}(S) \ &\leqslant \ \sum_{i=1}^{\ell} \mathbf{M}(E_i) + \sum_{i=1}^{\ell} |g_i| D(A) \\ &\leqslant \ \sum_{i=1}^{\ell} \mathrm{Lip}(h_i) \mathbf{M}[(\partial S_0) \llcorner W_i^{\vec\lambda}] \ + \ D(A) \mathbf{M}[(\partial S_0) \llcorner W_i^{\vec\lambda}] \ \leqslant \mathbf{cM}(\partial S_0) \ . \end{aligned}$$

□

4.3. **Proof of the case $k \geqslant 1$.**

Proof. We use induction on $\dim A$. For $\dim A = 0$, the current $S_0 \in \mathbf{I}_{k+1}(A)$ necessarily vanishes, and the theorem is trivially true. Also for $\dim A = 1$, $S_0 \in \mathbf{I}_{k+1}(A)$ is nonvanishing only for $k = 0$, in which case the theorem was established in the previous section §4.2. So we now assume that $k \geqslant 1$, that $\dim A \geqslant 2$ and inductively, that the theorem is true for any compact subanalytic set of dimension less than $\dim A$.

For this inductive step, note that the links $L^{a_i}_{\lambda_i r_i} = A \cap \mathrm{Bdry}\, U_i^{\lambda_i}$, as well as all the sets $A \cap \mathrm{Bdry}\, W_i^{\vec\lambda}$, are compact and subanalytic of dimension $< \dim A$. We need some more discussion about the choice of the radii λ_i , the corresponding balls $U_i^{\lambda_i}$, disjoint open sets $W_i^{\vec\lambda}$ and their boundaries. First, the distinctness of the centers of the $U_i^{\lambda_i}$ already guarantees that their spherical boundaries intersect in sets of dimension less than or equal to $n-2$.

We describe a particular subdivision of $\bigcup_{i=1}^{\ell} \mathrm{Bdry}\, W_i^{\vec\lambda}$ by using the sequence of closed spherical regions

$$\Gamma_i^{\vec\lambda} := (\mathrm{Bdry}\, U_i^{\lambda_i}) \cap (\mathrm{Bdry}\, W_i^{\vec\lambda}) \quad = \quad (\mathrm{Bdry}\, U_i^{\lambda_i}) \setminus \bigcup_{h=1}^{i-1} U_h^{\lambda_h} \ ,$$

whose relative interiors $\Gamma_i^{\vec\lambda\circ}$ are disjoint. While each $\Gamma_i^{\vec\lambda}$ clearly does not overlap any $W_h^{\vec\lambda}$ for $h \leqslant i$, we will also be interested in the cover of $\Gamma_i^{\vec\lambda}$ by the closed $n-1$ dimensional spherical regions

$$\Gamma_{i,j}^{\vec\lambda} := \Gamma_i^{\vec\lambda} \cap \mathrm{Clos}\, W_j^{\vec\lambda} \quad \text{for} \quad j = i+1, \ldots, \ell \ ,$$

whose relative interiors $\Gamma_{i,j}^{\vec\lambda\circ}$ are disjoint and whose relative boundaries are contained in $n-2$ dimensional spheres. Note that

$$A \cap \mathrm{Bdry}\, W_i^{\vec\lambda} \quad = \quad A \cap \left(\Gamma_i^{\vec\lambda} \ \cup \ \bigcup_{h=1}^{i-1} \Gamma_{h,i}^{\vec\lambda} \right) \quad = \quad A \cap \left(\bigcup_{h=1}^{i-1} \Gamma_{h,i}^{\vec\lambda} \ \cup \ \bigcup_{j=i+1}^{\ell} \Gamma_{i,j}^{\vec\lambda} \right) \ .$$

Again all the corresponding interior spherical regions, $\Gamma_{h,i}^{\vec{\lambda}\circ}$ or $\Gamma_{i,j}^{\vec{\lambda}\circ}$, are disjoint for each fixed i, with h ranging in $\{1,\ldots,i-1\}$ and j ranging in $\{i+1,\ldots,\ell\}$.

Many of the partitioning domains $W_i^{\vec{\lambda}}$ or many of their boundary regions $\Gamma_{i,j}^{\vec{\lambda}\circ}$ may be empty. In any case, for each nonempty $W_i^{\vec{\lambda}}$, the boundary $\operatorname{Bdry} W_i^{\vec{\lambda}}$ has a single nonempty "outward-pointing" (or "concave") region $\Gamma_i^{\vec{\lambda}} \subseteq \operatorname{Bdry} U_i^{\lambda_i}$, which may decompose into various $\Gamma_{i,j}^{\vec{\lambda}}$ for some later $j > i$. And any remainder $\operatorname{Bdry} W_i^{\vec{\lambda}} \setminus \Gamma_i^{\vec{\lambda}}$ then consists of "inward-pointing" (or "convex") regions $\Gamma_{h,i}^{\vec{\lambda}}$ coming from distinct spheres for some earlier $h < i$.

Each set $A\cap\Gamma_i^{\vec{\lambda}}$ is contained in the link union $L_{\vec{\lambda}\vec{r}}^{\vec{a}}$. Moreover, $A\cap\Gamma_i^{\vec{\lambda}}$ is mapped to $A\cap\Gamma_i^{\vec{\mu}}$ under the bilipschitz equivalence that maps $L_{\vec{\lambda}\vec{r}}^{\vec{a}}$ to $L_{\vec{\mu}\vec{r}}^{\vec{a}}$, obtained in Theorem 3.7. In fact, since each individual link $L_{\lambda_i r_i}^{a_i} = A\cap \operatorname{Bdry} U_i^{\lambda_i}$ is mapped to the corresponding link $L_{\mu_i r_i}^{a_i} = A\cap \operatorname{Bdry} U_i^{\mu_i}$, a similar $\vec{\lambda} \to \vec{\mu}$ transfer property holds for corresponding finite intersections, set differences, connected components, or disjoint unions of such links. We easily see that each region $\Gamma_i^{\vec{\lambda}}$ partitions into finitely many pieces, each given as a connected component of a difference of finite intersections of spheres $\operatorname{Bdry} U_j^{\lambda_j}$. Thus sets obtained by intersecting these with A all inherit the desired $\vec{\lambda} \to \vec{\mu}$ transfer property.

Since each set $A\cap\Gamma_i^{\vec{\lambda}}$ is compact and subanalytic of dimension $< \dim A$, it has, by our induction on $\dim A$, its own linear isoperimetric inequality. Moreover, by the bilipschitz equivalence from Theorem 3.7 and Proposition 2.3, we may now assume

(*) *The linear isoperimetric inequality is true with the* **same constant c** *for*

$$A\cap\Gamma_1^{\vec{\lambda}}\ ,\ \ldots\ ,\ A\cap\Gamma_\ell^{\vec{\lambda}} \quad \textbf{for all} \quad \vec{\lambda}\in R_{\vec{r}}^{\vec{a}}\ .$$

The uniformity of this estimate will allow us to choose and fix an ℓ tuple of scaling factors $\vec{\lambda} = (\lambda_1,\ldots,\lambda_\ell) \in R_{\vec{r}}^{\vec{a}}a$ depending on the given chain $S_0 \in \mathbf{I}_{k+1}(A)$, but still have mass estimates independent of S_0.

As we did in §4.1, we again consider some Lebesgue null sets of λ_i to avoid to guarantee some desired properties relative to the given chain.

For any fixed $S \in \mathbf{I}_{k+1}(\mathbb{R}^n)$ and positive $\lambda_1,\ldots,\lambda_i \notin \Lambda_{\|S\|+\|\partial S\|}$, we see as before that

$$(\|S\|+\|\partial S\|)\left(\operatorname{Bdry}(W_h^{\vec{\lambda}})\right) = 0\ , \quad S\llcorner \operatorname{Clos}\left(W_h^{\vec{\lambda}}\right) = S\llcorner W_h^{\vec{\lambda}}\ ,$$
$$(\partial S)\llcorner \operatorname{Clos}\left(W_h^{\vec{\lambda}}\right) = (\partial S)\llcorner W_h^{\vec{\lambda}},$$

for $h = 1,\ldots,i$. Now we find, in $\cup_{h=1}^i U_h^{\lambda_h}$, the two decompositions

$$S\llcorner\left(\cup_{h=1}^i U_h^{\lambda_h}\right) \;=\; \sum_{h=1}^i S\llcorner W_h^{\vec{\lambda}} \quad \text{and} \quad (\partial S)\llcorner\left(\cup_{h=1}^i U_h^{\lambda_h}\right) \;=\; \sum_{h=1}^i (\partial S)\llcorner W_h^{\vec{\lambda}}\ , \tag{3}$$

obtained by ignoring the boundaries of the $W_h^{\vec{\lambda}}$.

Using the Lipschitz map $u_i(x) = r_i^{-1}|x-a_i|$, we have that $U_i^\lambda \;=\; \{x : u_i(x) < \lambda\}$, and there is another exceptional null set $\Lambda_i^S \subseteq [0,1]$, on the complement of which one obtains integral chain slices satisfying the formulas

$$\begin{aligned} \langle S,u_i,\lambda\rangle &= \partial(S\llcorner U_i^\lambda) - (\partial S)\llcorner U_i^\lambda \quad \in \mathbf{I}_k(A\cap \operatorname{Bdry} U_i^\lambda)\ , \\ \partial\langle S,u_i,\lambda\rangle \;&=\; -\langle \partial S,u_i,\lambda\rangle = -\partial[(\partial S)\llcorner U_i^\lambda] \quad \in \mathbf{I}_{k-1}(A\cap \operatorname{Bdry} U_i^\lambda)\ . \end{aligned} \tag{4}$$

and the corresponding measures satisfy

$$\|\partial(S\llcorner U_i^\lambda)\| \;=\; \|\partial S\|\llcorner U_i^\lambda \;+\; \|\langle S,u_i,\lambda\rangle\|\ , \quad \|\partial[(\partial S)\llcorner U_i^\lambda]\,\| \;=\; \|\langle \partial S,u_i,\lambda\rangle\|\ ,$$

with $\|\partial S\| \llcorner U_i^\lambda$ and $\|\langle S, u_i, \lambda\rangle\|$ being mutually singular.

Moreover, since $\operatorname{Lip} u_i = r_i^{-1}$, we have the integral slice mass inequality

$$\int_{\lambda_{\vec{r},i}^{\vec{a}}}^{\mu_{\vec{r},i}^{\vec{a}}} \mathbf{M}\langle \partial S, u_i, \lambda\rangle \, d\lambda \leqslant r_i^{-1}\mathbf{M}(\partial S) \ . \tag{5}$$

We will also need a similar discussion for a fixed $Q \in \mathbf{I}_k(A \cap \Gamma_i^{\vec{\lambda}})$. If $j \in \{i+1, \dots, \ell\}$ and $\lambda_j \notin \Lambda_{\|Q\|}$, then $\|Q\| \Big(\operatorname{Bdry}(W_j^{\vec{\lambda}})\Big) = 0$. In case this is true for every such $j \in \{i+1, \dots, \kappa\}$, one gets the decomposition

$$Q \llcorner (U_1^{\lambda_1} \cup \dots \cup U_\kappa^{\lambda_\kappa}) \ = \ \sum_{j=i+1}^{\kappa} Q \llcorner \Gamma_{i,j}^{\vec{\lambda}} \ . \tag{6}$$

There is also another exceptional null set $\Lambda_i^Q \subseteq [0,1]$ so that each $\lambda \in [0,1] \setminus \Lambda_i^Q$ gives an integral chain slices $\langle Q, u_i, \lambda\rangle$ satisfying

$$\langle Q, u_i, \lambda\rangle = \partial(Q \llcorner U_i^\lambda) - (\partial Q) \llcorner U_i^\lambda \quad \in \mathbf{I}_{k-1}(A \cap \Gamma_i^{\vec{\lambda}}) \ , \tag{7}$$

with the orthogonal decomposition of measures

$$\|\partial(Q \llcorner U_i^\lambda)\| \ = \ \|\partial Q\| \llcorner U_i^\lambda \ + \ \|\langle Q, u_i, \lambda\rangle\| \ .$$

We shall now choose below $\lambda_1, \lambda_2, \dots, \lambda_\ell$ in order, and use these radii to construct chains $H_1, \dots H_\ell \in \mathbf{I}_{k+1}(A)$ so that

$$\partial(H_1 + \dots + H_\ell) \ = \ \partial S_0 \quad \text{and} \quad \mathbf{M}(H_1) \ , \dots , \ \mathbf{M}(H_\ell) \ \leqslant \mathbf{c}\mathbf{M}(\partial S_0) \ ,$$

with $\mathbf{c}$ depending only on A, and independent of S_0. This will complete the proof by letting $S = H_1 + \dots + H_\ell$.

For our first choice, we apply (4) and the second inequality of (5) with $i = 1$, $S = S_0$, to find a "good" radius

$$\lambda_1 \in [\lambda_{\vec{r},1}^{\vec{a}}, \mu_{\vec{r},1}^{\vec{a}}] \ \setminus \ \Big(\Lambda_{\|S_0\| + \|\partial S_0\|} \cup \Lambda_1^{S_0}\Big)$$

so that the chain

$$R_1 \ := \ \partial\langle S_0, u_1, \lambda_1\rangle \ = \ -\langle \partial S_0, u_1, \lambda_1\rangle \ = -\partial[(\partial S_0) \llcorner U_1^{\lambda_1}] \ \in \mathbf{I}_{k-1}(A \cap \Gamma_1^{\vec{\lambda}})$$

has mass satisfying

$$\mathbf{M}(R_1) \ \leqslant \ [(\mu_{\vec{r},1}^{\vec{a}} - \lambda_{\vec{r},1}^{\vec{a}}) r_i]^{-1} \mathbf{M}(\partial S_0) \ = \ \mathbf{c}\mathbf{M}(\partial S_0) \ .$$

Since the chain $P_1 := \langle S_0, u_1, \lambda_1\rangle \in \mathbf{I}_k(A \cap \Gamma_1^{\vec{\lambda}})$ has $\partial P_1 = R_1$, we may apply our dimension induction (*), with A replaced by the lower dimensional set $A \cap \Gamma_1^{\vec{\lambda}}$ and S_0 replaced by P_1, to obtain a chain $Q_1 \in \mathbf{I}_k(A \cap \Gamma_1^{\vec{\lambda}})$ with $\partial Q_1 = \partial P_1 = R_1$ and

$$\mathbf{M}(Q_1) \ \leqslant \mathbf{c}\mathbf{M}(\partial Q_1) \ = \ \mathbf{c}\mathbf{M}(R_1) \ \leqslant \ \mathbf{c}\mathbf{M}(\partial S_0) \ .$$

Note that $J_1 \ := \ (\partial S_0) \llcorner W_1^{\vec{\lambda}} \ + \ Q_1 \quad \in \mathbf{I}_k(A \cap \operatorname{Clos} U_1^{\lambda_1})$ has

$$\partial J_1 \ = \ \partial[(\partial S_0) \llcorner W_1^{\vec{\lambda}}] + \ \partial P_1 \ = \ -R_1 + R_1 \ = \ 0 \ ,$$

and

$$\mathbf{M}(J_1) \ \leqslant \ \mathbf{M}(\partial S_0) + \mathbf{M}(Q_1) \ \leqslant \ \mathbf{c}\mathbf{M}(\partial S_0) \ .$$

Inasmuch as the cycle J_1 has support in $A \cap \mathbf{U}(a_1, r_1)$, we infer from Corollary 3.4 that the contraction

$$H_1 \ := \ -h_{1\#}\,(\llbracket 0,1 \rrbracket \times J_1) \ \in \mathbf{I}_{k+1}(A)$$

has $\partial H_1 \ = \ J_1$ and

$$\mathbf{M}(H_1) + \mathbf{M}(\partial H_1) \ \leqslant \ \big((\operatorname{Lip} h_1)^{k+1} + 1\big)\, \mathbf{M}(J_1) \ \leqslant \ \mathbf{c}\mathbf{M}(\partial S_0) \ .$$

Letting $S_1 := S_0 - H_1$, we have that

$$\mathbf{M}(\partial S_1) \leqslant \mathbf{M}(\partial S_0) + \mathbf{M}(\partial H_1) \leqslant \mathbf{cM}(\partial S_0) ,$$

and we easily see that one may replace S_0 by S_1 to prove the theorem. The advantage in passing from S_0 to S_1 is that subtractng H_1 essentially "moves the boundary" out of the ball $U_1^{\lambda_1}$. In fact, the chain

$$\partial S_1 = \partial S_0 - \partial H_1 = \partial S_0 - (\partial S_0) \llcorner W_1^{\vec{\lambda}} - Q_1 = (\partial S_0) \llcorner (A \setminus W_1^{\vec{\lambda}}) - Q_1$$

has

$$\operatorname{spt}(\partial S_1) \subseteq (A \setminus W_1^{\vec{\lambda}}) \cup \Gamma_1^{\vec{\lambda}} = A \setminus U_1^{\lambda_1} .$$

We wish to continue moving the boundary in steps out of the remaining balls $U_i^{\lambda_i}$. However, to find H_2 for the next modification $S_2 = S_1 - H_2$, or generally H_i for $S_i = S_{i-1} - H_i$, is somewhat more involved. For example, to get the boundary of S_2 out of both balls

$$U_1^{\lambda_1} \cup U_2^{\lambda_2} = W_1^{\vec{\lambda}} \cup \Gamma_{1,2}^{\vec{\lambda}\,o} \cup W_2^{\vec{\lambda}} ,$$

we will below need to choose H_2 to attach to H_1 along the interface $\Gamma_{1,2}^{\vec{\lambda}\,o}$. For the reader's convenience, we will first go through this second step, involving the choice of H_2, carefully before describing the general ith step.

For the second step, we again start with the choice of radius λ_2. By applying (4), (5), (6), (7) with $i = 2$, $S = S_1$, and $Q = Q_1$. We obtain

$$\lambda_2 \in [\lambda_{\vec{r},2}^{\vec{a}}, \mu_{\vec{r},2}^{\vec{a}}] \setminus \left(\Lambda_{\|S_0\|+\|\partial S_0\|+\|S_1\|+\|\partial S_1\|+\|Q_1\|} \cup \Lambda_1^{S_0} \cup \Lambda_1^{S_1} \cup \Lambda_1^{Q_1}\right)$$

so that the chain

$$R_2 := \partial\langle S_1, u_2, \lambda_2\rangle = -\langle \partial S_1, u_2, \lambda_2\rangle = -\partial[(\partial S_1) \llcorner U_2^{\lambda_2}]$$

has mass satisfying

$$\mathbf{M}(R_2) \leqslant [(\mu_{\vec{r},2}^{\vec{a}} - \lambda_{\vec{r},2}^{\vec{a}}) r_2]^{-1} \mathbf{M}(\partial S_1) \leqslant \mathbf{cM}(\partial S_1) \leqslant \mathbf{cM}(\partial S_0) .$$

Since the chain $P_2 := \langle S_1, u_2, \lambda_2\rangle \in \mathbf{I}_k(A \cap \Gamma_2^{\vec{\lambda}})$ has $\partial P_2 = R_2$, we may again apply our dimension induction (*), this time with A replaced by the lower dimensional set $A \cap \Gamma_2^{\vec{\lambda}}$ and S_0 replaced by P_2, to obtain a chain $Q_2 \in \mathbf{I}_k(A \cap \Gamma_2^{\vec{\lambda}})$ with $\partial Q_2 = \partial P_2 = R_2$ and

$$\mathbf{M}(Q_2) \leqslant \mathbf{cM}(\partial Q_2) = \mathbf{cM}(R_2) \leqslant \mathbf{cM}(\partial S_0) .$$

The chain $Q_{1,2} := Q_1 \llcorner \Gamma_{1,2}^{\vec{\lambda}} = Q_1 \llcorner \operatorname{Clos}\left(W_2^{\vec{\lambda}}\right)$ shows up in the boundary calculation

$$\begin{aligned}
\partial(S_1 \llcorner U_2^{\lambda_2}) &= \langle S_1, u_2, \lambda_2\rangle + (\partial S_1) \llcorner U_2^{\lambda_2} \\
&= P_2 + (\partial S_1) \llcorner \operatorname{Clos}\left(W_2^{\vec{\lambda}}\right) \\
&= P_2 + (\partial S_0) \llcorner \operatorname{Clos}\left(W_2^{\vec{\lambda}}\right) - (\partial H_1) \llcorner \operatorname{Clos}\left(W_2^{\vec{\lambda}}\right) \\
&= P_2 + (\partial S_0) \llcorner \operatorname{Clos}\left(W_2^{\vec{\lambda}}\right) - (\partial S_0) \llcorner \left(W_1^{\vec{\lambda}} \cap \operatorname{Clos}\left(W_2^{\vec{\lambda}}\right)\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad - Q_1 \llcorner \operatorname{Clos}\left(W_2^{\vec{\lambda}}\right) \\
&= P_2 + (\partial S_0) \llcorner W_2^{\vec{\lambda}} + 0 - Q_{1,2} .
\end{aligned}$$

Thus the chain $J_2 := (\partial S_0) \llcorner W_2^{\vec{\lambda}} + Q_2 - Q_{1,2}$ has

$$\partial J_2 = \partial[(\partial S_0) \llcorner W_2^{\vec{\lambda}} + P_2 - Q_{1,2}] = \partial^2(S_1 \llcorner U_2^{\vec{\lambda}}) = 0 ,$$

and

$$\mathbf{M}(J_2) \leqslant \mathbf{M}(\partial S_0) + \mathbf{M}(Q_2) + \mathbf{M}(Q_1) \leqslant \mathbf{c}\mathbf{M}(\partial S_0) .$$

Since J_2 also has support in $A \cap \mathbf{U}(a_2, r_2)$, we find from Corollary 3.4 that the contraction

$$H_2 := -h_{2\#}(\llbracket 0,1 \rrbracket \times J_2) \in \mathbf{I}_{k+1}(A)$$

has $\partial H_2 = J_2$ and

$$\mathbf{M}(H_2) + \mathbf{M}(\partial H_2) \leqslant \big((\operatorname{Lip} h_2)^{k+1} + 1\big)\mathbf{M}(J_2) \leqslant \mathbf{c}\mathbf{M}(\partial S_0) .$$

Letting $S_2 := S_1 - H_2 = S_0 - H_1 - H_2$, we have

$$\mathbf{M}(\partial S_2) \leqslant \mathbf{c}[\mathbf{M}(\partial S_1) + \mathbf{M}(\partial H_2)] \leqslant \mathbf{c}\mathbf{M}(\partial S_0) ,$$

and

$$\begin{aligned}\partial S_2 &= \partial S_1 - \partial H_2 = (\partial S_0) \llcorner (A \setminus W_1^{\vec\lambda}) - Q_1 - Q_2 + Q_{1,2} - (\partial S_0) \llcorner W_2^{\vec\lambda} \\ &= (\partial S_0) \llcorner [A \setminus (W_1^{\vec\lambda} \cup W_2^{\vec\lambda})] - Q_1 + Q_{1,2} - Q_2 \\ &= (\partial S_0) \llcorner [A \setminus (U_1^{\lambda_1} \cup U_2^{\lambda_2})] - Q_1 + Q_{1,2} - Q_2 .\end{aligned}$$

because $\|\partial S_0)\|(\operatorname{Bdry} U_1^{\lambda_1}) = 0$. Inasmuch as

$$Q_1 \llcorner U_1^{\lambda_1} = 0 , \quad Q_1 \llcorner U_2^{\lambda_2} = Q_{1,2} , \quad Q_{1,2} \llcorner U_1^{\lambda_1} = 0 , \quad Q_2 \llcorner U_1^{\lambda_1} = 0 , \quad Q_2 \llcorner U_2^{\lambda_2} = 0 ,$$

we see that

$$\begin{aligned}(\partial S_2) \llcorner (U_1^{\lambda_1} \cup U_2^{\lambda_2}) &= 0 - (Q_1 - Q_{1,2}) \llcorner (U_1^{\lambda_1} \cup U_2^{\lambda_2}) - Q_2 \llcorner (U_1^{\lambda_1} \cup U_2^{\lambda_2}) \\ &= 0 - (Q_{1,2} - Q_{1,2}) - 0 = 0 ,\end{aligned}$$

and conclude that

$$\operatorname{spt}(\partial S_2) \subseteq A \setminus (U_1^{\lambda_1} \cup U_2^{\lambda_2}) ,$$

which completes the second step.

Modifying the above, we now describe how to obtain the ith step from the $(i-1)$st. We assume that $i \leqslant \ell - 1$ and that we have already chosen, for each $h \in \{1, \ldots, i-1\}$,

$$\lambda_h > 0 , \quad Q_h \in \mathbf{I}_k(A \cap \Gamma_h^{\vec\lambda}) , \quad H_h \in \mathbf{I}_{k+1}(A) , \quad S_h := S_0 - \sum_{\eta=1}^{i-1} H_\eta \in \mathbf{I}_{k+1}(A) ,$$

to satisfy the three conditions:

(I)$_h$ $\quad \partial H_h = (\partial S_0) \llcorner W_h^{\vec\lambda} + Q_h - \sum_{\eta=1}^{h-1} Q_{\eta,h}$ where $Q_{\eta,h} := Q_\eta \llcorner \Gamma_{\eta,h}^{\vec\lambda}$,

(II)$_h$ $\quad \mathbf{M}(Q_h) + \mathbf{M}(H_h) + \mathbf{M}(\partial H_h) + \mathbf{M}(\partial S_h) \leqslant \mathbf{c}\mathbf{M}(\partial S_0)$.

(III)$_h$ $\quad \operatorname{spt}(\partial S_h) \subseteq A \setminus (U_1^{\lambda_1} \cup \cdots \cup U_h^{\lambda_h})$,

(Here the formula for ∂H_1 is correct with the convention $\sum_{\eta=1}^{0} Q_{\eta,h} = 0$.)

We first choose λ_i by applying (4), (5), (6), and (7), this time with $i = i$, $S = S_{i-1}$, and $Q = Q_1, \ldots, Q_{i-1}$. We obtain

$$\lambda_i \in [\lambda_{\vec r,i}^{\vec a}, \mu_{\vec r,i}^{\vec a}] \setminus \Lambda_{\sum_{h=0}^{i-1} \|S_h\| + \|\partial S_h\| + \|Q_h\|} \setminus \cup_{h=0}^{i-1} \left(\Lambda_1^{S_h} \cup \Lambda_1^{Q_h}\right)$$

so that the chain

$$R_i := \partial\langle S_{i-1}, u_i, \lambda_i\rangle = -\langle \partial S_{i-1}, u_i, \lambda_i\rangle = -\partial[(\partial S_{i-1}) \llcorner U_i^{\lambda_i}] \in \mathbf{I}_{k-1}(A \cap \Gamma_i^{\vec\lambda})$$

has mass satisfying

$$\mathbf{M}(R_i) \leqslant [(\mu_{\vec r,i}^{\vec a} - \lambda_{\vec r,i}^{\vec a}) r_i]^{-1} \mathbf{M}(\partial S_{i-1}) \leqslant \mathbf{c}\mathbf{M}(\partial S_{i-1}) \leqslant \mathbf{c}\mathbf{M}(\partial S_0) .$$

Since the chain $P_i := \langle S_{i-1}, u_i, \lambda_i \rangle \in \mathbf{I}_{k-1}(A \cap \Gamma_i^{\vec{\lambda}})$ has $\partial P_i = R_i$, we may again apply our dimension induction (*), this time with A replaced by the lower dimensional set $A \cap \Gamma_i^{\vec{\lambda}}$ and with S_0 replaced by P_i, to obtain a chain $Q_i \in \mathbf{I}_k(A \cap \Gamma_i^{\vec{\lambda}})$ such that $\partial Q_i = \partial P_i = R_i$ and

$$\mathbf{M}(Q_i) \leqslant \mathbf{cM}(\partial Q_i) = \mathbf{cM}(R_i) \leqslant \mathbf{cM}(\partial S_0) .$$

The chains $Q_{h,i} := Q_h \llcorner \Gamma_{h,i}^{\vec{\lambda}} = Q_h \llcorner \overline{W}_i^{\vec{\lambda}}$ for $h = 1, \ldots, i-1$, show up in the following boundary calculation using (4), $(\mathrm{III})_{i-1}$, and $(\mathrm{II})_h$,

$$\begin{aligned}
\partial(S_{i-1} \llcorner U_i^{\lambda_i}) &= \langle S_{i-1}, u_i, \lambda_i \rangle + (\partial S_{i-1}) \llcorner U_i^{\lambda_i} \\
&= \langle S_{i-1}, u_i, \lambda_i \rangle + (\partial S_{i-1}) \llcorner \overline{W}_i^{\vec{\lambda}} \\
&= P_i + (\partial S_0) \llcorner \overline{W}_i^{\vec{\lambda}} - \sum_{h=1}^{i-1} (\partial H_h) \llcorner \overline{W}_i^{\vec{\lambda}} \\
&= P_i + (\partial S_0) \llcorner \overline{W}_i^{\vec{\lambda}} - \sum_{h=1}^{i-1} \left((\partial S_0) \llcorner (W_h^{\vec{\lambda}} \cap \overline{W}_i^{\vec{\lambda}}) + Q_h \llcorner \overline{W}_i^{\vec{\lambda}} - \sum_{\eta=1}^{h} Q_{\eta,h} \llcorner \overline{W}_i^{\vec{\lambda}} \right) \\
&= P_i + (\partial S_0) \llcorner W_i^{\vec{\lambda}} - \sum_{h=1}^{i-1} (0 + Q_{h,i} - 0) .
\end{aligned}$$

Then the chain $J_i = (\partial S_0) \llcorner W_i^{\vec{\lambda}} + Q_i - \sum_{h=1}^{i-1} Q_{h,i} \in \mathbf{I}_k(A \cap \operatorname{Clos} U_i^{\lambda_i})$ satisfies

$$\partial J_i = \partial \left((\partial S_0) \llcorner W_i^{\vec{\lambda}} + P_i - \sum_{h=1}^{i-1} Q_{h,i} \right) = \partial^2 (S_{i-1} \llcorner U_i^{\lambda_i}) = 0$$

and, by $(\mathrm{II})_h$,

$$\mathbf{M}(J_i) \leqslant \mathbf{M}(\partial S_0) + \mathbf{M}(Q_i) + \sum_{h=1}^{i-1} \mathbf{M}(Q_h) \leqslant \mathbf{cM}(\partial S_0) .$$

Since this cycle has support in $A \cap \mathbf{U}(a_i, r_i)$, we see from Corollary 3.4 that the contraction

$$H_i := -h_{i\#} [\llbracket 0,1 \rrbracket \times J_i] \in \mathbf{I}_{k+1}(A)$$

has $\partial H_i = J_i$ and that

$$\mathbf{M}(H_i) + \mathbf{M}(\partial H_i) \leqslant \big((\operatorname{Lip} h_i)^{k+1} + 1 \big) \mathbf{M}(J_i) \leqslant \mathbf{cM}(\partial S_0) .$$

Letting $S_i := S_{i-1} - H_i = S_0 - \sum_{h=1}^{i} H_h$, we have from $(\mathrm{II})_{i-1}$ that

$$\mathbf{M}(\partial S_i) \leqslant \mathbf{c}[\mathbf{M}(\partial S_{i-1}) + \mathbf{M}(\partial H_i)] \leqslant \mathbf{cM}(\partial S_0) ,$$

and from $(\mathrm{I})_h$ that

$$\begin{aligned}
\partial S_i &= \partial S_0 - \sum_{h=1}^{i} \partial H_h \\
&= \partial S_0 - \sum_{h=1}^{i} \left((\partial S_0) \llcorner W_h^{\vec{\lambda}}) + Q_h - \sum_{\eta=1}^{h-1} Q_{\eta,h} \right) \\
&= (\partial S_0) \llcorner (A \setminus \cup_{h=1}^{i} W_h^{\vec{\lambda}}) - \sum_{h=1}^{i} \left(Q_h - \sum_{\eta=1}^{h-1} Q_{\eta,h} \right) \\
&= (\partial S_0) \llcorner (A \setminus \cup_{h=1}^{i} U_h^{\lambda_h}) - \sum_{h=1}^{i} Q_h + \sum_{h=2}^{i} \sum_{\eta=1}^{h-1} Q_{\eta,h}
\end{aligned}$$

because $\|\partial S_0)\|(\operatorname{Bdry} U_h^{\lambda_h}) = 0$ for $h = 1,\dots,i$. Abbreviating $\tilde U_i = \cup_{h=1}^{i} U_h^{\lambda_h}$, we see that $Q_i \llcorner \tilde U_i = 0$ because $\Gamma_i^{\vec\lambda} \cap \tilde U_i = \emptyset$, and that we may decompose $Q_h \llcorner \tilde U_i$ by applying (6) with Q, i, κ replaced by Q_h, h, i. We deduce that

$$\begin{aligned}(\partial S_i)\llcorner \tilde U_i \;&=\; (\partial S_0)\llcorner (A\setminus \tilde U_i)\llcorner \tilde U_i \;-\; \sum_{h=1}^{i-1} Q_h \llcorner \tilde U_i \;+\; \sum_{h=2}^{i}\sum_{\eta=1}^{h-1} Q_{\eta,h}\llcorner \tilde U_i \\ &=\; 0 \;-\; \sum_{h=1}^{i-1}\sum_{j=h+1}^{i} Q_{h,j} \;+\; \sum_{h=2}^{i}\sum_{\eta=1}^{h-1} Q_{\eta,h} \quad=\quad 0\,.\end{aligned}$$

Thus

$$\operatorname{spt}(\partial S_i) \;\subseteq\; A\setminus\, \tilde U_i \;=\; A\setminus\,(U_1^{\lambda_1}\cup\;\cdots\;\cup U_h^{\lambda_h}\;)\,,$$

and we have now verified $(\mathrm{I})_i$, $(\mathrm{II})_i$, and $(\mathrm{III})_i$ and completed the ith step for all $i \leqslant \ell-1$. In particular,

$$\operatorname{spt}(\partial S_{\ell-1}) \;\subseteq\; A\setminus \cup_{i=1}^{\ell-1} U_i^{\lambda_i} \;\subseteq\; A\setminus \cup_{i=1}^{\ell-1}\mathbf{U}(a_i, r_i/2) \;\subseteq A\cap \mathbf{U}(a_\ell, r_\ell/2)\,.$$

Finally by defining $H_\ell \;:=\; -h_{\ell\#}\,([\![0,1]\!]\times \partial S_{\ell-1})$ and $S = H_1 + \cdots + H_\ell$, we find that

$$\partial H_\ell \;=\; \partial S_{\ell-1} \;=\; \partial S_0 \;-\; \sum_{i=1}^{\ell-1}\partial H_i\;;\quad \text{hence,}\quad \partial S \;= \partial S_0\,,$$

and, by using $(\mathrm{II})_1,\dots,(\mathrm{II})_{\ell-1}$, that

$$\mathbf{M}(S) \;\leqslant\; \mathbf{c}\sum_{i=1}^{\ell-1}\mathbf{M}(\partial H_i) + \mathbf{M}(H_\ell) \;\leqslant\; \mathbf{c}\mathbf{M}(\partial S_0) + (\operatorname{Lip} h_\ell)^k \mathbf{M}(\partial S_{\ell-1}) \;\leqslant\; \mathbf{c}\mathbf{M}(\partial S_0)\,,$$

which completes the proof. □

5. Applications

5.1 (Normal Currents). — The exact same proof shows that *if $A \subseteq \mathbb{R}^n$ is compact and subanalytic, $k = 0,1,2,\dots$ and $S_0 \in \mathbf{N}_{k+1}A)$ is a normal current of dimension $k+1$ supported in A, then there exists $S \in \mathbf{N}_{k+1}(A)$ such that $\partial S = \partial S_0$ and $\mathbf{M}(S) \leqslant \mathbf{c}(A)\mathbf{M}(\partial S)$.* This seems to be new even in the case when A is a compact real analytic submanifold of $\mathbb{R}^n$. In fact, it also holds in case A is a compact submanifold of $\mathbb{R}^n$ of class C^∞, by 2.3 and the fact that such A is C^∞ diffeomorphic to an algebraic manifold, according to the Nash-Tognoli Theorem [2, §14.1].

5.2 (Other coefficients groups). — We observe that the proof further generalizes to the case of a general normed, complete, Abelian group G of coefficients, [25], [6]. Here $\mathcal{R}_k(A;G)$ and $\mathscr{F}_k(A;G)$ denote the groups consisting of those k dimensional, respectively rectifiable and flat chains with coefficients in G, supported in A, and

$$\begin{aligned}\mathcal{I}_{k+1}(A;G) &= \mathcal{R}_{k+1}(A;G)\cap\{S : \partial S \in \mathcal{R}_k(A;G)\}\\ \mathscr{N}_k(A;G) &= \mathscr{F}_k(A;G)\cap\{S : \mathbf{M}(S)+\mathbf{M}(\partial S) < \infty\}\end{aligned}$$

where $\mathbf{M}$ denotes the usual Euclidean Hausdorff mass of a rectifiable chain in $\mathbb{R}^n$, relaxed to the class of flat chains. *If $k = 1,2,\dots$ and $S_0 \in \mathcal{I}_{k+1}(A;G)$ (resp. $S_0 \in \mathscr{N}_{k+1}(A;G)$), then there exists $S \in \mathcal{I}_{k+1}(A;G)$ (resp. $S \in \mathscr{N}_{k+1}(A;G)$) such that*

$$\partial S = \partial S_0 \qquad\text{and}\qquad \mathbf{M}(S) \leqslant \mathbf{c}(A)\mathbf{M}(\partial S).$$

This indeed encompasses the previous cases since

$$\mathcal{I}_{k+1}(A;\mathbb{Z}) \cong \mathbf{I}_{k+1}(A) \qquad\text{and}\qquad \mathscr{N}_{k+1}(A;\mathbb{R}) \cong \mathbf{N}_{k+1}(A).$$

We ought to say a word about the case $k=0$ of the proof. Here one replaces the expression $(\partial S_0)\left(\mathbb{1}_{W_i^{\vec\lambda}}\right)$ using the total multiplicity morphism $\chi : \mathscr{F}_0(A;G) \to G$, see [6, 4.3.3] to find that $g_i = \chi[(\partial S_0) \llcorner W_i^{\vec\lambda}]$, while recalling that χ is finitely additive and that $|\chi(T)| \leqslant \mathscr{F}(T) \leqslant \mathbf{M}(T)$.

5.3 (Comparing homology groups). — We use the same notations as in 5.2, and we define the groups of cycles and boundaries

$$\mathbf{Z}_k^{\mathcal{I}}(A;G) = \mathcal{I}_k(A;G) \cap \{T : \partial T = 0\}$$
$$\mathbf{B}_k^{\mathcal{I}}(A;G) = \mathcal{I}_k(A;G) \cap \{T : T = \partial S \text{ for some } S \in \mathcal{I}_{k+1}(A;G)\}$$

as well as the corresponding homology group $\mathbf{H}_k^{\mathcal{I}}(A;G) = \mathbf{Z}_k^{\mathcal{I}}(A;G)/\mathbf{B}_k^{\mathcal{I}}(A;G)$. One checks that $\mathbf{H}_0^{\mathcal{I}}(\{0\};G) = G$ and, as in [5, 3.7, 3.9, 3.10, 3.14] one shows that the functors $\mathbf{H}_k^{\mathcal{I}}(\cdot;G)$ and $\check{H}_k(\cdot;G)$ (Čech homology with coefficients in G) are naturally equivalent on the category of $(\mathbf{H}^{\mathcal{I}},k)$ locally connected subsets of Euclidean space and their Lipschitz maps. According to Theorem 3.3 each compact subanalytic set $A \subseteq \mathbb{R}^n$ is $(\mathbf{H}^{\mathcal{I}},k)$ locally connected, recall [5, 3.11]. Thus in that case, $\mathbf{H}_k^{\mathcal{I}}(A;G) \cong \check{H}_k(A;G) \cong H_k(A;G)$ where $H_k(A;G)$ denotes singular homology and the last equivalence holds because A is triangulable.

5.4 (Homology of Normal Currents). — We can repeat the argument made in 5.3 with the usual normal currents. Letting

$$\mathbf{Z}_k(A) = \mathbf{N}_k(A) \cap \{T : \partial T = 0\}$$
$$\mathbf{B}_k(A) = \mathbf{N}_k(A) \cap \{T : T = \partial S \text{ for some } S \in \mathbf{N}_{k+1}(A)\}$$

and $\mathbf{H}_k(A) = \mathbf{Z}_k(A)/\mathbf{B}_k(A)$ we note that $\mathbf{H}_0(\{0\}) = \mathbb{R}$ and, referring to [5, 3.14] again that $\mathbf{H}_k(A) \cong \check{H}_k(A;\mathbb{R}) \cong H_k(A;\mathbb{R})$ in case $A \subseteq \mathbb{R}^n$ is compact and subanalytic.

5.5 (Cohomology of Charges). — A complex of cochains on a compact subset $A \subseteq \mathbb{R}^n$ is defined and studied in [7]. A **charge of degree k on A** is a linear functional $\alpha : \mathbf{N}_k(A) \to \mathbb{R}$ with the following property. For every $\varepsilon > 0$ there exists $\theta > 0$ such that $|\alpha(T)| \leqslant \theta\mathbf{F}(T) + \varepsilon\mathbf{N}(T)$ where $\mathbf{F}(T)$ is the flat norm of T and $\mathbf{N}(T) = \mathbf{M}(T) + \mathbf{M}(\partial T)$. According to the main result of [8], *a linear functional $\alpha : \mathbf{N}_k \to \mathbb{R}$ is a charge of degree k if and only if there exist continuous forms $\omega \in C(\mathbb{R}^n, \bigwedge_k \mathbb{R}^n)$ and $\zeta \in C(\mathbb{R}^n, \bigwedge_{k-1} \mathbb{R}^n)$ such that*

$$\alpha(S) = \int_{\mathbb{R}^n} \langle \omega, \vec{S} \rangle d\|S\| + \int_{\mathbb{R}^n} \langle \zeta, \vec{\partial S} \rangle d\|\partial S\|$$

whenever $S \in \mathbf{N}_k(A)$, in other words $\alpha = \omega + d\zeta$. Let $\mathbf{CH}^k(A)$ denote the vector space of charges of degree k in A. The notions of cocycle and coboundary for charges in A are readily defined in terms of their exterior derivatives $d = \partial^*$:

$$\mathbf{Z}^k(A) = \mathbf{CH}^k(A) \cap \{\alpha : d\alpha = 0\}$$
$$\mathbf{B}^k(A) = \mathbf{CH}^k(A) \cap \left\{ d\beta : \beta \in \mathbf{CH}^{k-1}(A) \right\} .$$

This in turn yields a cohomology space $\mathbf{H}^k(A) = \mathbf{Z}^k(A)/\mathbf{B}^k(A)$. Furthermore $\mathbf{CH}^k(A)$ is given a structure of Banach space with the norm

$$\|\alpha\| = \sup\{\alpha(S) : S \in \mathbf{N}_k(A) \text{ and } \mathbf{N}(S) \leqslant 1\} .$$

The relevance of the linear isoperimetric inequality 5.1 in this context is as follows. We recall [7, Chapter 14] that the compact set $A \subseteq \mathbb{R}^n$ is called **q-bounded**, $q = 0,1,2,\ldots$, whenever the following holds: There exists $\mathbf{c}(A,q) > 0$ such that for every $T \in \mathbf{B}_q(A)$ there exists $S \in \mathbf{N}_{q+1}(A)$ with $\partial S = T$ and $\mathbf{M}(S) \leqslant \mathbf{c}(A,q)\mathbf{M}(T)$. It follows from [7, 14.4 and 13.10] that A is q-bounded if and only $\mathbf{B}^q(A)$ is closed in $\mathbf{CH}^q(A)$. In that case $\mathbf{H}^q(A)$ is a Banach space. Furthermore, according to [7, 14.9] $\mathbf{H}^q(A)$ is the strong dual of $\mathbf{H}_q(A)$ equipped with an appropriate locally convex vector topology, [7, Chapter 12]. From §5.1 it readily follows that:

If $A \subseteq \mathbb{R}^n$ is compact and subanalytic, then A is q-bounded for all $q = 0, 1, 2, \ldots$ and

$$\mathbf{H}^q(A) \cong \mathbf{H}_q(A)^* \cong H^q(A;\mathbb{R}),$$

the singular cohomology with real coefficients.

The latter may be interpreted as a de Rham Theorem in this context.

5.6 (Plateau problem in a homology classes). — Let G be a complete normed Abelian group with norm $|\cdot|$ and assume it satisfies the following extra two conditions.

(A) $G \cap \{g : |g| \leqslant \kappa\}$ is compact for every $\kappa > 0$;
(B) G is a White group, i.e. G contains no nonconstant curve of finite length.

We also let $A \subseteq \mathbb{R}^n$ be a compact subanalytic set, and $k = 1, 2, \ldots$. Recall the notations of 5.2. Given $T_0 \in \mathcal{R}_k(A;G)$ with $\partial T_0 = 0$, the following variational problem admits a minimizer:

$$(\mathscr{P}) \begin{cases} \text{minimize } \mathbf{M}(T) \\ \text{among } T \in \mathcal{R}_k(A;G) \text{ with } T - T_0 = \partial S \text{ for some } S \in \mathcal{R}_{k+1}(A;G). \end{cases}$$

As we show below this is a consequence of our linear isoperimetric inequality and of work of B. White.

Existence of solution. Let $\langle T_j \rangle_j$ be a minimizing sequence of $(\mathscr{P})$. For each j there exists $S_j \in \mathcal{R}_{k+1}(A;G)$ such that $T_j - T_0 = \partial S_j$. According to the linear isoperimetric inequality 5.2, we may choose $\hat{S}_j \in \mathcal{I}_{k+1}(A;G)$ so that $\partial \hat{S}_j = \partial S_j$ and $\mathbf{M}(\hat{S}_j) \leqslant \mathbf{c}(A)\mathbf{M}(\partial \hat{S}_j)$. Thus $T_j - T_0 = \partial \hat{S}_j$ and

$$\mathbf{M}(\hat{S}_j) \leqslant \mathbf{c}(A)\mathbf{M}(\partial \hat{S}_j) = \mathbf{c}(A)\mathbf{M}(T_j - T_0) \leqslant \mathbf{c}(A)[1 + \inf(\mathscr{P}) + \mathbf{M}(T_0)] < \infty .$$

for j sufficiently large. Since A is compact, the deformation theorem [24] together with condition (A) above imply that both sets $\{T_j : j = 1, 2, \ldots\}$ and $\{\hat{S}_j : j = 1, 2, \ldots\}$ are totally bounded in the flat norm $\mathscr{F}$. Consequently there are integers $j_1 < j_2 < \ldots$ and flat G chains $T \in \mathscr{F}_k(A;G)$ and $\hat{S} \in \mathscr{F}_{k+1}(A;G)$ such that $\lim_i \mathscr{F}(T - T_{j_i}) = 0 = \lim_i \mathscr{F}(\hat{S} - \hat{S}_{j_i})$. Note that

$$T - T_0 = \lim_{i \to \infty} (T_{j_i} - T_0) = \lim_{i \to \infty} \partial \hat{S}_{j_i} = \partial \hat{S} .$$

Inasmuch as $\mathbf{M}$ is lower semicontinuous with respect to $\mathscr{F}$ convergence, one infers that $\mathbf{M}(T) < \infty$ and $\mathbf{M}(\hat{S}) < \infty$. It therefore follows from [25] and condition (B) above that $T \in \mathcal{R}_k(A;G)$ and $\hat{S} \in \mathcal{R}_{k+1}(A;G)$. □

5.7 (A linear relative isoperimetric inequality). — Here we work with a pair $B \subseteq A$ of compact subanalytic subsets of $\mathbb{R}^n$ and verify an isoperimetric inequality generalizing our Main Theorem.

Assuming that $U := \mathbb{R}^n \setminus B$ and that S is a chain of finite mass with support in A, we will be interested in the **mass of S in U**,

$$\mathbf{M}(S \llcorner U) = \|S\|(U) = \|S\|(A \setminus B) = \mathbf{M}[S \llcorner (A \setminus B)] ,$$

rather than the total mass $\mathbf{M}(S)$. For integral currents, our relative version is:

Theorem. — *There is a constant $\mathbf{c}(A, B) > 0$ so that, for every $k \in \{1, 2, \ldots\}$ and every $S_0 \in \mathbf{I}_k(A)$, there exists an $S \in \mathbf{I}_k(A)$ satisfying*

$$(\partial S) \llcorner U = (\partial S_0) \llcorner U \quad \text{and} \quad \mathbf{M}(S \llcorner U) \leqslant \mathbf{c}(A, B)\mathbf{M}[(\partial S) \llcorner U] ,$$

Proof. Note that this does not follow by simply applying the statement of the Main Theorem to $S \llcorner U$ because the righthand side of the inequality is missing the term

$$\mathbf{M}[\partial(S \llcorner U)] - \mathbf{M}[(\partial S) \llcorner U].$$

Nevertheless, we can obtain the desired S by slightly modifying the arguments in §3 and §4. Of course, the various constructions and choices, as well as the final chain S, will now depend on B (and $U = \mathbb{R}^n \setminus B$) as well as on A and S_0.

For the new relative version of Theorem 3.3, we simply require the extra statement that

if $a \in B$*, then* **the Lipschitz contraction** h **preserves** $B \cap K$*, that is,*

$$h\left([0,1] \times (B \cap K)\right) \ = \ B \cap K \ .$$

This extra property has already been essentially treated in $(A_n)(1)$ of the proof of [21, Th.2.3.1]. There, the inductive argument, in G. Valette's notation, gives, for any finite collection X_1, X_2, …, X_s of compact subanalytic subsets of $\mathbb{R}^n$, a single Lipschitz neighborhood contraction r of a neighborhood U_ε of x_0 whose restrictions simultaneously contract the $U_\varepsilon \cap X_j$. So here we are simply using the two sets $X_1 = A$, $X_2 = B$ to get the desired contractions $h = r$ of $K = X_1 \cap \mathrm{Clos}(U_\varepsilon)$ to $a = x_0$ that preserves $B \cap K$.

In the new relative version of Corollary 3.4, the fact that h preserves both K and $B \cap K$ implies that h preserves $U \cap K = K \setminus B$, that is, $h([0,1) \times (U \cap K)) \ = \ U \cap K$. Thus we obtain, by applying $\llcorner U$ in the proof of Corollary 3.4, that

$$H \llcorner U \ = \ -h_\# \left([0,1] \times J\right) \llcorner U \ = \ -h_\# \left([0,1] \times (J \llcorner U)\right) \ ,$$

which gives the additional conclusion

$$\mathbf{M}(H \llcorner U) \leqslant (\mathrm{Lip}\, h)^{k+1} \mathbf{M}(J \llcorner U) \ , \tag{8}$$

which is essentially the local case of our linear relative isoperimetric inequality.

If the point $a \notin B$ and K is small enough so that $K \cap B = \emptyset$, we can still use the contraction h as before and inequality (8) remains true because $J \llcorner U = J$ and $H \llcorner U = H$.

For the new version of Theorem 3.7, we need the extra property that the resulting bilipschitz equivalence of A links also preserves the B links, ie. for $\vec{\lambda}, \vec{\mu}$ in $R^{\vec{a}}_{\vec{r}}$,

$$B \cap \mathrm{Bdry}\, \mathbf{U}(a_i, \lambda r_i) \ = \ B \cap L^{a_i}_{\lambda r_i} \ \to \ B \cap \mathrm{Bdry}\, \mathbf{U}(a_i, \mu r_i) \ = \ B \cap L^{a_i}_{\mu r_i} \ .$$

Our present proof will give this property if we simply add the requirement that our stratification $\mathscr{S}$ also be compatible with the set $\mathbb{R} \times B$.

We now repeat all the constructions of §4, look at the restrictions to U, and estimate the masses in U in terms of $\mathbf{M}[(\partial S_0) \llcorner U]$ rather than $\mathbf{M}(\partial S_0)$. In these estimates we will use the symbol $\mathbf{c}$ to abbreviate a constant $\mathbf{c}(A, B)$, depending only on A and B.

When we first employ compactness to find the finite collection of balls $\mathbf{U}(a_i, r_i/2)$ covering A, we may choose just from those balls $\mathbf{U}(a, r/2)$ where the center $a \in A$ and **either** $a \in B$ **or** $\mathbf{B}(a, r) \cap B = \emptyset$. Thus, when we repeat the constructions with the resulting $U_i^{\lambda_i} = \mathbf{U}(a_i, \lambda_i r_i)$, we can use either the new or old version of Theorem 3.7, and Corollary 3.4 depending on whether $a_i \in B$ or $a_i \notin B$.

We again argue by induction on $\dim A$. As before, the bilipschitz equivalence discussion gives us the assumption (*) on the inductive validity of the new relative theorem with the same constant when A, B is replaced by every pair $A \cap \Gamma_i^{\vec{\lambda}}$, $B \cap \Gamma_i^{\vec{\lambda}}$.

For $\lambda \in \mathbb{R} \setminus \Lambda_i^S$, the slices of S and ∂S by u_i are given by integration, and so they may be restricted to the open set U. That is, these slices all commute with the operation $\llcorner U$. In particular, simply applying $\llcorner U$ to every term in (4), (5), and (7) results in

$$\langle S \llcorner U, u_i, \lambda\rangle \;=\; \langle S, u_i, \lambda\rangle \llcorner U \;=\; [\partial(S \llcorner U_i^\lambda)] \llcorner U \;-(\partial S)\llcorner (U_i^\lambda \cap U)\ ,$$

$$\langle (\partial S)\llcorner U, u_i, \lambda\rangle \;=\; \langle \partial S, u_i, \lambda\rangle \llcorner U \;=\; \partial[(\partial S)\llcorner U_i^\lambda]\llcorner U\ ,$$

$$\int_{\lambda^{\vec a}_{\vec r,i}}^{\mu^{\vec a}_{\vec r,i}} \mathbf{M}(\langle S, u_i, \lambda\rangle \llcorner U)\, d\lambda \;\leqslant\; r_i^{-1}\mathbf{M}(S\llcorner U)$$

$$\int_{\lambda^{\vec a}_{\vec r,i}}^{\mu^{\vec a}_{\vec r,i}} \mathbf{M}(\langle \partial S, u_i, \lambda\rangle \llcorner U)\, d\lambda \;\leqslant\; r_i^{-1}\mathbf{M}((\partial S)\llcorner U)\ .$$

$$\langle Q \llcorner U, u_i, \lambda\rangle \;=\; \langle Q, u_i, \lambda\rangle \llcorner U \;=\; [\partial(Q \llcorner U_i^\lambda)] \llcorner U \;-(\partial Q)\llcorner (U_i^\lambda \cap U)\ ,$$

In the ith step of the new proof, we find λ_i so that the slice at $R_i \llcorner U \;:=\; \partial\langle S_{i-1}, u_i, \lambda_i\rangle \llcorner U$, has mass

$$\mathbf{M}(R_i \llcorner U) \;\leqslant\; \mathbf{cM}[(\partial S_{i-1})\llcorner U] \;\leqslant\; \mathbf{cM}[(\partial S_0)\llcorner U]\ .$$

Now the induction on dimension allows us to to replace $P_i = \langle S_{i-1}, u_i, \lambda_i\rangle$ by a chain $Q_i \in \mathbf{I}_{k-1}(A\cap \Gamma_i^{\vec\lambda})$ such that $\partial Q_i = \partial P_i = R_i$ and

$$\mathbf{M}(Q_i \llcorner U) \;\leqslant\; \mathbf{cM}[(\partial Q_i)\llcorner U] \;=\; \mathbf{cM}(R_i\llcorner U) \;\leqslant\; \mathbf{cM}[(\partial S_0)\llcorner U]\ .$$

By defining $Q_{\eta,h}$ and H_i exactly as before, based on $Q_1, \ldots, Q_i$, S_0, and contractions h_i, we now find mass-in-U estimates

$$\mathbf{M}(H_i\llcorner U) + \mathbf{M}[(\partial H_i)\llcorner U] \;\leqslant\; \mathbf{c}\left(\mathbf{M}[(\partial S_0)\llcorner U] \;+\; \sum_{h=1}^{i}\mathbf{M}(Q_h\llcorner U)\right) \;\leqslant\; \mathbf{cM}[(\partial S_0)\llcorner U],$$

since the h_i preserves $U\cap K_i$. With the last modification H_ℓ also defined exactly as before, we similarly verify $\mathbf{M}(H_i\llcorner U) \leqslant \mathbf{cM}[(\partial S_0)\llcorner U]$. It follows, as before, that $S = H_1 + \cdots + H_\ell$ satisfies the relative theorem. □

5.8 (Remark). — Again both the statement and proof of Theorem 5.7 carry over to chains with coefficients in a complete normed abelian group G. The application below in §5.9 generalizes §5.6 and so involves the rectifiability and the group assumptions of §5.6(A)(B). But we will no longer be assuming that all chains and their boundaries have finite mass or are rectifiable everywhere. Noting that the statement and proof of Theorem 5.7 involve the behavior of the chains and their masses only in U, we see that we can further generalize Theorem 5.7 to flat chains T where both T and ∂T are rectifiable with finite **mass in** U. Here a chain $T \in \mathscr{F}_k(A;G)$ is **rectifiable with finite mass in** U provided that

$$T\llcorner U_i \in \mathcal{R}_k(A;G) \quad\text{and}\quad \sup_i \mathbf{M}(T\llcorner U_i) < \infty\ ,$$

for some open sets $U_1 \subseteq U_2 \subseteq \cdots$ with $\cup_{i=1}^\infty U_i = U$. In this case $\langle T\llcorner U_i\rangle_i$ is $\mathbf{M}$ Cauchy, and we let $T\llcorner U$ denote the $\mathbf{M}$ limit. This limit is well-defined independently of the choice of open sets, is rectifiable, and coincides with the usual definition of $T\llcorner U$ in case $\mathbf{M}(T) < \infty$. Also one checks that $\operatorname{spt}(T - T\llcorner U) \subseteq B$ [6, §5.5].

5.9 (A relative homology Plateau problem). — First recall that, for rectifiable chains, the groups of relative cycles, relative boundaries, and relative homology can be defined:

$$\mathbf{Z}_k^{\mathcal{R}}(A,B;G) = \mathcal{R}_k(A;G)\cap\{T : \operatorname{spt}(\partial T)\subseteq B\}$$
$$\mathbf{B}_k^{\mathcal{R}}(A,B;G) = \mathcal{R}_k(A;G)\cap\{T : \operatorname{spt}(T-\partial S)\subseteq B \text{ for some } S\in\mathcal{R}_{k+1}(A;G)\}$$
$$\mathbf{H}_k^{\mathcal{R}}(A,B;G) = \mathbf{Z}_k^{\mathcal{R}}(A,B;G)/\mathbf{B}_k^{\mathcal{R}}(A,B;G)\ .$$

Here we discuss how the relative isoperimetric inequality of §5.7 is useful for the following relative homology Plateau problem that generalizes §5.6.

Given $T_0 \in \mathcal{R}_k(A;G)$ where $B \subseteq A$ are compact subanalytic subsets of $\mathbb{R}^n$ and G satisfies §5.6(A)(B), consider the problem:

$$(\mathscr{P}_B) \begin{cases} \text{minimize } \mathbf{M}(T) \\ \text{among } T \in \mathcal{R}_k(A;G) \text{ with } \mathrm{spt(T - T_0 - \partial S) \subseteq B \text{ for some } S \in \mathcal{R}_{k+1}(A;G)}. \end{cases}$$

Note that in case $\mathrm{spt}(\partial T_0) \subseteq B$, i.e. $T_0 \in \mathbf{Z}_k^{\mathcal{R}}(A,B;G)$, one is minimizing mass in the relative homology class

$$[T_0] \ := \ \mathbf{Z}_k^{\mathcal{R}}(A,B;G) \cap \big\{ T \ : \ T - T_0 \in \mathbf{B}_k^{\mathcal{R}}(A,B;G) \big\} \ \in \ \mathbf{H}_k^{\mathcal{R}}(A,B;G) \ .$$

Existence of solution. Since T_0 is admissible, one easily obtains a mass minimizing sequence $\langle T_j \rangle_j$ in $\mathcal{R}_k(A;G)$ and $\langle S_j \rangle_j$ in $\mathcal{R}_{k+1}(A;G)$ with $\mathrm{spt}(T_j - T_0 - \partial S_j) \subseteq B$. The sequence $\mathbf{M}(T_j)$ has a finite upper bound which we may assume is $\mathbf{M}(T_0)$. The chain $T \llcorner U$ is admissible whenever T is because $\mathrm{spt}(T - T \llcorner U) \subseteq B$. Also we may assume $T_0 = T_0 \llcorner U$ because $T_0 \llcorner U$ gives the same admissible class as T_0 does .

Even though the sequence $\mathbf{M}(\partial T_j)$ is not obviously bounded above, the equation

$$(\partial T_j) \llcorner U = (\partial T_0) \llcorner U$$

gives a bound on the mass in U of ∂T_j. While there is no bound for the mass in U of S_j, the equation $(\partial S_j) \llcorner U = T_j \llcorner U \ - \ T_0 \llcorner U$ shows that ∂S_j has bounded mass in U and is rectifiable in U. Just like in §5.6, we can now use Theorem 5.7, with $S_0 = S_j$, and Remark 5.8 to replace S_j by another chain $\hat{S}_j$, with the same boundary in U, to assure that the sequence $\hat{S}_j$ has bounded mass in U. Specifically,

$$\mathbf{M}(\hat{S}_j \llcorner U) \leqslant \mathbf{cM}((\partial S_j) \llcorner U) = \mathbf{cM}[(T_j - T_0) \llcorner U] \leqslant 2\mathbf{cM}(T_0) \ .$$

To construct the desired rectifiable chains $S \in \mathcal{R}_{k+1}(A;G)$ and $T \in \mathcal{R}_k(A;G)$ so that

$$\mathrm{spt(T - T_0 - \partial S) \subseteq B}$$

and T is a mass minimizer for $(\mathscr{P}_B)$, we will take limits inside of U away from B and then use a diagonal argument. Accordingly, we define, for $\delta > 0$, $U_\delta := \ \{x \in U \ : \ u(x) > \delta\}$ where $u(x) = \mathrm{dist(x, B)}$. Inasmuch as

$$\begin{aligned} \int_0^\infty \liminf_{j\to\infty} [\mathbf{M}\langle \hat{S}_j, u, t\rangle + \mathbf{M}\langle T_j, u, t\rangle]\, dt \ &\leqslant \ \liminf_{j\to\infty} \int_0^\infty [\mathbf{M}\langle \hat{S}_j, u, t\rangle + \mathbf{M}\langle T_j, u, t\rangle]\, dt \\ &\leqslant \ \sup_j \mathbf{M}(\hat{S}_j \llcorner U) + \sup_j \mathbf{M}(T_j \llcorner U) \ \leqslant \ (2\mathbf{c} + 1)\mathbf{M}(T_0) \ < \ \infty \ , \end{aligned}$$

we can choose a sequence $t_i \downarrow 0$ so that, for all i,

$$\liminf_{j\to\infty} [\mathbf{M}\langle \hat{S}_j, u, t_i\rangle + \mathbf{M}\langle T_j, u, t_i\rangle] \ < \ \infty \ , \quad \mathbf{M}\langle T_0, u, t_i\rangle \ < \ \infty, \quad \mathbf{M}\langle \partial T_0, u, t_i\rangle \ < \ \infty \ .$$

We can also insist that, for all i and j,

$$\begin{gathered} \langle \hat{S}_j, u, t_i\rangle \in \mathcal{R}_k(A;G) \ , \quad \langle T_j, u, t_i\rangle \in \mathcal{R}_{k-1}(A;G) \ , \\ \partial(\hat{S}_j \llcorner U_{t_i}) = (\partial \hat{S}_j) \llcorner U_{t_i} + \langle \hat{S}_j, u, t_i\rangle \ , \quad \partial(T_j \llcorner U_{t_i}) = (\partial T_j) \llcorner U_{t_i} + \langle T_j, u, t_i\rangle \ . \end{gathered}$$

Inasmuch as $(\partial\hat{S}_j)\llcorner U = (T_j - T_0)\llcorner U$ and $(\partial T_j)\llcorner U = (\partial T_0)\llcorner U$, we also have, for every $i = 1, 2, \ldots,$ that

$$\begin{aligned}
&\liminf_{j\to\infty}\Big[\mathbf{M}(\hat{S}_j\llcorner U_{t_i}) + \mathbf{M}\partial(\hat{S}_j\llcorner U_{t_i}) + \mathbf{M}(T_j\llcorner U_{t_i}) + \mathbf{M}\partial(T_j\llcorner U_{t_i})\Big]\\
&\leqslant \sup_j \Big(\mathbf{M}(\hat{S}_j\llcorner U) + \mathbf{M}[(\partial\hat{S}_j)\llcorner U] + \mathbf{M}(T_j\llcorner U) + \mathbf{M}[(\partial T_j)\llcorner U]\Big)\\
&\qquad\qquad\qquad + \liminf_{j\to\infty}\Big[\mathbf{M}\langle\hat{S}_j, u, t_i\rangle + \mathbf{M}\langle T_j, u, t_i\rangle\Big]\\
&\leqslant (2\mathbf{c}+2+1)\mathbf{M}(T_0) + \mathbf{M}[(\partial T_0)\llcorner U] + \liminf_{j\to\infty}\Big[\mathbf{M}\langle\hat{S}_j, u, t_i\rangle + \mathbf{M}\langle T_j, u, t_i\rangle\Big] \ < \ \infty\ .
\end{aligned}$$

We find a subsequence $\langle i_j^{(1)}\rangle_j$ of $\langle j\rangle_j$ giving, as $j\to\infty$, flat convergences

$$\hat{S}_{i_j^{(1)}}\llcorner U_{t_1} \to \hat{S}^{(1)} \in \mathcal{R}_{k+1}(A;G) \quad\text{and}\quad T_{i_j^{(1)}}\llcorner U_{t_1} \to T^{(1)} \in \mathcal{R}_k(A;G)\ .$$

For $m = 2, 3, \ldots,$ we similarly inductively find subsequences $\langle i_j^{(m)}\rangle_j$ of $\langle i_j^{(m-1)}\rangle_j$ and, as $j\to\infty$, flat convergences

$$\hat{S}_{i_j^{(m)}}\llcorner U_{t_m} \to \hat{S}^{(m)} \in \mathcal{R}_{k+1}(A;G) \quad\text{and}\quad T_{i_j^{(m)}}\llcorner U_{t_m} \to T^{(m)} \in \mathcal{R}_k(U;G)\ .$$

The lower semicontinuity of $\mathbf{M}$ implies that $\mathbf{M}(\hat{S}^{(m)}) \leqslant 2\mathbf{c}\mathbf{M}(T_0)$ and $\mathbf{M}(\hat{T}^{(m)}) \leqslant \mathbf{M}(T_0)$.

For $\ell < m$, $U_{t_\ell} \subseteq U_{t_m}$, and one has $\hat{S}^{(\ell)} = \hat{S}^{(m)}\llcorner U_{t_\ell}$ and $\hat{T}^{(\ell)} = \hat{T}^{(m)}\llcorner U_{t_\ell}$. It follows that the sequences $\hat{S}^{(m)}$ and $\hat{T}^{(m)}$ are $\mathbf{M}$-Cauchy and $\mathbf{M}$-convergent to chains $S \in \mathcal{R}_{k+1}(A;G)$ and $T \in \mathcal{R}_k(A;G)$, respectively, characterized by having $S\llcorner U_{t_m} = \hat{S}^{(m)}$ and $T\llcorner U_{t_m} = T^{(m)}$ for every $m \in \{1, 2, \ldots\}$. Taking the diagonal subsequence $\langle j'\rangle_j = \langle i_j^{(j)}\rangle_j$, one now has, for all m, the flat convergences

$$\lim_{j\to\infty}\hat{S}_{j'}\llcorner U_{t_m} = \hat{S}^{(m)} = S\llcorner U_{t_m} \quad\text{and}\quad \lim_{j\to\infty} T_{j'}\llcorner U_{t_m} = \hat{T}^{(m)} = T\llcorner U_{t_m}\ .$$

To verify the boundary relation that $R := T - T_0 - \partial S$ has support in B, it suffices to show that $R\llcorner U_t = 0$ for a.e. $t > 0$. For each $m \in \{1, 2, \ldots\}$, a.e. $t \in [t_{m+1}, t_m]$, and $j' > m$, we have that

$$R\llcorner U_t \ = \ (T - T_0 - \partial S)\llcorner U_t \ = \ (T_{j'} - T_0 - \partial\hat{S}_{j'})\llcorner U_t + (T - T_{j'})\llcorner U_t - (\partial S - \partial\hat{S}_{j'})\llcorner U_t\ .$$

Taking the flat norm, integrating, and applying [6][Thm.5.2.3(2)] gives

$$\begin{aligned}
\int_{t_{m+1}}^{t_m}\mathscr{F}(R\llcorner U_t)\,dt \ &= \ \int_{t_{m+1}}^{t_m}\mathscr{F}[(T - T_0 - \partial S)\llcorner U_t]\,dt\\
&\leqslant \ \int_{t_{m+1}}^{t_m}\Big(0 + \mathscr{F}[(T - T_{j'})\llcorner U_t] + \mathscr{F}[(\partial S - \partial\hat{S}_{j'})\llcorner U_t)]\Big)\,dt\\
&\leqslant \ (t_m - t_{m+1} + 1)\Big[\mathscr{F}(T - T_{j'}) + \mathscr{F}\partial(S - \hat{S}_{j'})\Big]\\
&\leqslant \ (t_m - t_{m+1} + 1)\Big[\mathscr{F}(T - T_{j'}) + \mathscr{F}(S - \hat{S}_{j'})\Big] \ \to \ 0 \text{ as } j\to\infty\ .
\end{aligned}$$

We conclude that $(T - T_0 - \partial S)\llcorner U_t = 0$ for a.a. positive t, and $\operatorname{spt}(T - T_0 - \partial S) \subseteq B$.

Since T is thus admissible for $(\mathscr{P}_B)$ and $\langle T_j\rangle_j$ is a mass minimizing sequence,

$$\mathbf{M}(T) \geqslant \mathscr{M} := \lim_{j\to\infty}\mathbf{M}(T_j).$$

To get the reverse inequality, we may for any $\varepsilon > 0$, choose an m sufficiently large so that $\mathbf{M}(T^{(m)}) > \mathbf{M}(T) - \varepsilon$. We may combine this with the mass lower semicontinuity, under the flat convergence of $T_{i_j^{(m)}} \llcorner U_{t_m}$ to $T^{(m)}$, to deduce that

$$\mathbf{M}(T) - \varepsilon \;<\; \mathbf{M}(T^{(m)}) \;\leqslant\; \liminf_{j\to\infty} \mathbf{M}(T_{i_j^{(m)}} \llcorner U_{t_m}) \;\leqslant\; \liminf_{j\to\infty} \mathbf{M}(T_{i_j^{(m)}}) \;=\; \mathscr{M}.$$

Letting $\varepsilon \downarrow 0$ gives $\mathbf{M}(T) \leqslant \mathscr{M}$, showing that T is the desired mass minimizer for $(\mathscr{P}_B)$. $\square$

5.10 (A Poincaré inequality). — *Let A be a k dimensional compact subanalytic subset of $\mathbb{R}^n$ and let M denote the manifold of regular points of A . If M is connected and orientable, then there is a finite constant $\mathbf{c}(A)$ so that for any $f \in \mathbf{BV}(M)$ and any $m \in \mathbb{R}$ satisfying $\mathscr{H}^k\{x : f(x) < m\} = \mathscr{H}^k\{x : f(x) > m\}$, (i.e. m is a median of f), one has the inequalities*

$$(1) \quad \int_M |f - m| \, d\mathscr{H}^k \;\leqslant\; \mathbf{c}(A) \int_M \|Df\| \quad \text{and}$$

$$(2) \quad \int_M |f - \overline{f}| \, d\mathscr{H}^k \;\leqslant\; 2\mathbf{c}(A) \int_M \|Df\| \quad \text{where} \quad \overline{f} \;=\; \mathscr{H}^k(A)^{-1} \int f \, d\mathscr{H}^k \;.$$

Proof. Since $A \setminus M$ is subanalytic of dimension $< k$ [1, §7], we may assume $\overline{M} = A$. For (1) we may subtract the constant function m to assume that $m = 0$. Thus the two sets

$$M_- = \{x \in M \,:\, f(x) < 0\} \qquad \text{and} \qquad M_+ = \{x \in M \,:\, f(x) > 0\}$$

have the same $\mathscr{H}^k$ measure, which is finite because $\mathscr{H}^k(M) = \mathscr{H}^k(A) < \infty$ [10, 3.4.8(13)]. Fix an orientation for M, and let $[\![M]\!]$ denote the corresponding k dimensional rectifiable current. Here, the flat chain $\partial[\![M]\!]$ is also rectifiable because $\operatorname{spt}\partial[\![M]\!] \subseteq B := \overline{M} \setminus M$ and the constancy theorem [10, 4.1.31] may be applied to the $k-1$ dimensional strata. From [10, 4.5.9(12)], we see that for almost all $s > 0$, the chain $[\![M]\!]_s := [\![M]\!] \llcorner \{x \in M \,:\, f(x) > s\}$ is rectifiable and of finite mass and finite boundary mass in $U : \mathbb{R}^n \setminus B$.

Applying the linear relative isoperimetric inequality, Theorem 5.7, with $S_0 = [\![M]\!]_s$ we find an $S_s \in \mathbf{I}_k(A)$ satisfying

$$(\partial S_s) \llcorner M \;=\; (\partial[\![M]\!]_s) \llcorner M \quad \text{and} \quad \mathbf{M}(S_s \llcorner M) \;\leqslant\; \mathbf{c}(A)\mathbf{M}[(\partial[\![M]\!]_s \llcorner M] \;,$$

Inasmuch as $\partial(S_s - [\![M]\!]_s) \llcorner M = 0$, the constancy theorem gives an integer j so that

$$(S_s - [\![M]\!]_s)] \llcorner M \;= j[\![M]\!] \quad \text{and} \quad S_s \llcorner M = (j-1)\,([\![M]\!] \; - [\![M]\!]_s) + j[\![M]\!]_s \;.$$

Since $\mathbf{M}([\![M]\!]_s) \leqslant \mathscr{H}^k(M_+) = \frac{1}{2}\mathscr{H}^k(M) = \frac{1}{2}\mathbf{M}([\![M]\!])$, we deduce that

$$\mathbf{M}([\![M]\!]_s) \;\leqslant\; \mathbf{M}(S_s \llcorner M) \;\leqslant\; \mathbf{c}(A)\mathbf{M}\,(\partial[\![M]\!]_s \llcorner M) \;.$$

We now use the BV coarea formula on M, whose proof follows from the Euclidean case [10, 4.5.9(13)], to see that

$$\int_{M_+} |f| \, d\mathscr{H}^k \;=\; \int_0^\infty \mathscr{H}^k\{f > s\} \, ds \;=\; \int_0^\infty \mathbf{M}([\![M]\!]_s) \, ds$$
$$\leqslant\; \mathbf{c}(A) \int_0^\infty \mathbf{M}\,(\partial[\![M]\!]_s) \llcorner M) \;\; ds \;=\; \mathbf{c}(A) \int_{M_+} \|Df\| \;.$$

The same argument applied to $-f$ gives

$$\int_{M_-} |f| \, d\mathscr{H}^k \;\leqslant\; \mathbf{c}(A) \int_{M_-} \|Df\| \;,$$

and the conclusion of (1)

$$\begin{aligned}\int_M |f|\, d\mathscr{H}^k &= \int_{M_-} |f|\, d\mathscr{H}^k + \int_{M_+} |f|\, d\mathscr{H}^k \\ &\leqslant \mathbf{c}(A) \int_{M_-} \|Df\| + \mathbf{c}(A) \int_{M_+} \|Df\| \leqslant \mathbf{c}(A) \int_M \|Df\| \,.\end{aligned}$$

Conclusion (2) easily follows from (1) because

$$\int_M |f - \overline{f}|\, d\mathscr{H}^k \leqslant \int_M |f - m|\, d\mathscr{H}^k + \int_M |m - \overline{f}|\, d\mathscr{H}^k \,,$$

and

$$|m - \overline{f}| = \mathscr{H}^k(A)^{-1} \Big| \int_M (m - f)\, d\mathscr{H}^k \Big| \leqslant \mathscr{H}^k(A)^{-1} \int_M |f - m|\, d\mathscr{H}^k \,,$$

□

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