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MULTIPLE POINTS OF A SIMPLICIAL MAP AND IMAGE-COMPUTING SPECTRAL SEQUENCES

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Abstract. The Image-Computing Spectral Sequence computes the homology of the image of a finite map from the alternating homology of the multiple point spaces of the map. A related spectral sequence, obtained by Gabrielov, Vorobjov and Zell, computes the homology of the image of a closed map from the homology of k-fold fibred products of the map. We give new proofs of these results, in case the map can be triangulated. Thanks to work of Hardt, this holds for a very wide range of maps, and in particular for most of the finite maps of interest in singularity theory. The proof seems conceptually simpler and more revealing than earlier proofs.

1. Introduction

Spectral sequences are a powerful tool to compute (co)homology groups. One way to obtain a spectral sequence is from a double complex: it has associated a total complex which has two canonical filtrations, given respectively, by the columns and rows of the double complex; each of these filtrations gives rise to a spectral sequence, both of them converging to the homology of the total complex. One particular case which is very useful, is when one of the spectral sequences *collapses onto one of the axes*, that is, all the rows (or columns) of the first page of the spectral sequence are exact. In this case it is straightforward to see what is the homology of the total complex. Classical examples of this situation are the Čech-de Rham (double) complex, which is used to prove that Čech cohomology is isomorphic to de Rham cohomology [BT82, Example 14.16], and the Hochschild-Serre spectral sequence which relates the homology of a group to the homology of a normal subgroup and the corresponding quotient group [Bro94, §VII.6].

The Image-Computing Spectral Sequence (icss), computes the homology of the image of a finite map $f\colon X \to Y$ from the alternating homology of the multiple point spaces $D^k(f)$ of f (see Subsection 2.1 below for definitions of these terms). It was introduced by Victor Goryunov and the second author in [GM93], for rational cohomology, and further developed by Goryunov, in [Gor95], for integer homology, to study the topology of the image of a stable perturbation of a map germ $\mathbb{C}^n \to \mathbb{C}^p$ with $n < p$. Besides Singularity Theory [Hou97, Hou99, Hou07, Hou05], the icss has been used in Convex Geometry [KM08, CGG14]. In [GVZ04, BR17], a related spectral sequence, which we call the GVZ Spectral Sequence (gvzss), after its authors, Gabrielov, Vorobjov and Zell, was used to give bounds of Betti numbers of semialgebraic sets. It computes the image of a closed map $f\colon X \to Y$ from the homology of k-fold fibred products $W^k(f)$ of X.

The aim of this article is to construct the gvzss and the icss, respectively, for a finite map $f : X \to Y$, from the double complex of chains of k-fold fibred products of X and the double complex of alternating chains of multiple point spaces of f, by proving that their first spectral sequences collapse onto the p-axis. These proofs seem conceptually simpler and more revealing than earlier proofs.

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As usual, computing anything directly with singular chains is essentially impossible, and so for the ICSS, Goryunov in [Gor95] worked instead with alternating cellular homology. In this paper, the new approach is to work with (alternating) simplicial homology, assuming a triangulation of the map f – that is, simplicial complexes K and L and a simplicial map $F : |K| \to |L|$, together with homeomorphisms $|K| \simeq X, |L| \simeq Y$ giving a commutative diagram

$$\begin{tikzcd} {|K|} \arrow[r, "\simeq"] \arrow[d, "F"'] & X \arrow[d, "f"] \\ {|L|} \arrow[r, "\simeq"] & Y \end{tikzcd} .$$

We show in Section 4 that a triangulation of f gives rise to triangulations of $W^k(f)$ and $D^k(f)$ for each k, and that these triangulations fit together in a remarkably simple way. We will refer to them as the triangulations *associated to the simplicial structure of f*.

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2. Double complexes of multiple point spaces

Let $f \colon X \to Y$ be a continuous surjective map with finite fibres. For each $k \geq 0$, the *k-fold product of X fibred over f* is the subspace of X^k defined by

$$W^k(f) = \{(x_1, \dots, x_k) \in X^k \mid f(x_1) = \cdots = f(x_k)\}.$$

This space is simple to define but hard to work with. Besides the "strict multiple points", where $x_i \neq x_j$ for each $i \neq j$, which make up what we call $D^k(f)_S$, it contains the "small diagonal" where all the x_j are equal, and all k-tuples $(x_1, \dots, x_k)$ obtained, from an ℓ-tuple in $D^\ell_S(f)$ with $\ell < k$, by duplicating some of the members of the ℓ-tuple. This plethora of components, in general of different dimensions, makes it unwieldy. The space $D^k(f)$, defined as the closure of $D^k(f)_S$ in X^k, is simpler. For example, if f is a smooth or complex analytic map and is stable in the sense of Thom and Mather, and at all of its non-immersive points, df has kernel rank 1, then $D^k(f)$ is non-singular, by the results of [MM89] (see also [MNnB20, Theorem 9.4]). Thus, if f_t is a stable perturbation of a multi-germ f with isolated instability, and if at all of its non-immersive points df_t has kernel rank ≤ 1, then the multiple point spaces of f_t are smoothings of the (possibly singular) multiple point spaces of f. The condition on the kernel rank always holds for stable maps, if $\dim M < \dim N \leq 6$, by the Thom Transversality Theorem: if $f : M \to N$ is a stable map then its jet extension map $j^k f : M \to J^k(M, N)$ is transverse to all $\mathscr{A}$-invariant strata in $J^k(M, N)$ (see e.g. [MNnB20, page 346]), and when $\dim M < \dim N$ then the set of 1-jets with differential of corank > 1 has codimension greater than 5, so that $j^k f$ cannot meet it.

Note that at points of kernel rank ≥ 2, the multiple point spaces may not be smooth even when f is stable. The properties of $D^k(f)$ are discussed further in Section 6.

Let $\pi_\ell \colon W^k(f) \to X$ with $\ell = 1, \dots, k$ be the restriction of the ℓ'th standard projection. Consider the map $f^{(k)} \colon W^k(f) \to Y$ defined by $f^{(k)}(\mathbf{x}) = f\big(\pi_\ell(\mathbf{x})\big)$ for some ℓ with $1 \leq \ell \leq k$. Evidently this is independent of the choice of ℓ, so for convenience we take $\ell = 1$. There are also projections $\varepsilon^{i,k} \colon W^k(f) \to W^{k-1}(f)$, $i = 1, \dots, k$, forgetting the i'th component.

Let $C_n(W^k(f))$ be the usual free abelian group of singular n-chains in $W^k(f)$ and consider the singular chain complex $C_\bullet(W^k(f), \partial^k_\bullet)$ with the usual boundary map

$$\partial^k_n \colon C_n(W^k(f)) \to C_{n-1}(W^k(f)).$$

The map $\varepsilon^{i,k}\colon W^k(f)\to W^{k-1}(f)$ induces a morphism of chain complexes,

$$\varepsilon^{i,k}_{\#}\colon C_\bullet(W^k(f))\to C_\bullet(W^{k-1}(f)).$$

We denote by $\varepsilon^{i,k}_{\#,n}\colon C_n(W^k(f))\to C_n(W^{k-1}(f))$ the corresponding homomorphism on n-chains. Define the morphism $\rho^k_n\colon C_n(W^k(f))\to C_n(W^{k-1}(f))$ by

$$\rho^k_n=\sum_{i=1}^{k}(-1)^{i-1}\varepsilon^{i,k}_{\#,n}. \tag{1}$$

In Section 4 we will prove that $\rho^{k-1}_n\circ\rho^k_n=0$ and $f_{\#,n}\circ\rho^2_n=0$ (Lemmata 4.12 and 4.14), hence for each n, the sequence of n-chains

$$\dots\to C_n(W^k(f))\xrightarrow{\rho^k_n}\cdots\xrightarrow{\rho^3_n}C_n(W^2(f))\xrightarrow{\rho^2_n}C_n(X)\xrightarrow{f_{\#,n}}C_n(Y)\to 0, \tag{2}$$

is a chain complex. Our main technical result, leading to the collapse of the spectral sequences described in the Introduction, is Theorem 2.5 below: the chain complex (2), and the analogous chain complex (6) of alternating chains on the spaces $D^k(f)$, are exact.

2.1. **Alternating homology of multiple point spaces.** We define the k'th *multiple point space* $D^k(f)$ as the closure, in X^k, of the set of strict multiple points[1]

$$D^k_{\mathrm{S}}(f)=\{(x_1,\dots,x_k)\in X^k : x_i\neq x_j \text{ if } i\neq j, f(x_i)=f(x_j) \text{ for all } i,j\}.$$

Notice that $D^k(f)$ is a subspace of $W^k(f)$, and the action of S_k restricts to an action on $D^k(f)$.

We also consider the restrictions to $D^k(f)$ of the maps π_ℓ, $f^{(k)}$ and $\varepsilon^{i,k}$, $i=1,\dots,k$. Since all the maps $\varepsilon^{i,k}$ are left-right equivalent to one another thanks to the symmetric group actions, when restricted to $D^k(f)$, we will use only $\varepsilon^{k,k}$, which we abbreviate to ε^k. Thus, we obtain a *tower of multiple point spaces*

$$\dots\to D^{k+1}(f)\xrightarrow{\varepsilon^{k+1}}D^k(f)\xrightarrow{\varepsilon^k}\cdots\xrightarrow{\varepsilon^3}D^2(f)\xrightarrow{\varepsilon^2}X\xrightarrow{f}Y. \tag{3}$$

Let V^k denote $W^k(f)$ or $D^k(f)$. Let $C_n(V^k)$ be the usual free abelian group of singular n-chains in V^k. We define the group of *alternating n-chains* $C^{\mathrm{Alt}}_n(V^k)$ by

$$C^{\mathrm{Alt}}_n(V^k)=\{c\in C_n(V^k) : \sigma_\#(c)=\operatorname{sign}(\sigma)c \text{ for all } \sigma\in S_k\}.$$

Since $\partial^k_n\circ\sigma_\#=\sigma_\#\circ\partial^k_n$ for each $\sigma\in S_k$, the usual boundary operator

$$\partial^k_n\colon C_n(V^k)\to C_{n-1}(V^k)$$

maps alternating chains to alternating chains, and hence $C^{\mathrm{Alt}}_\bullet(V^k)$ is a subcomplex of the usual singular chain complex. We call its homology the *alternating homology of V^k* and denote it by $AH_*(V^k)$. The notation $\mathrm{Alt}H_*(V^k)$ is used in [Gor95]; we prefer ours, since $\mathrm{Alt}H_*(V^k)$ can be misunderstood as the image of an operator $\mathrm{Alt} : H_*(V^k)\to H_*(V^k)$.

Remark 2.1. The ICSS was introduced in [GM93] in the context of rational homology, where $AH_*(V^k)$ coincides with the alternating isotype in $H_*(V^k)$, which we denote by $H^{\mathrm{Alt}}_*(V_k)$. ([MNnB20, p. 370]). In general the two are different – [MNnB20, p.371] gives the example of the quotient map from the unit disc in $\mathbb{R}^2$ to $\mathbb{RP}^2$, which identifies antipodal points on the boundary.

[1]A slight variant on this definition is used in e.g. [Gaf83], [NnBPnS17] when one wishes to describe how the multiple points change when one deforms a map. In this paper we will not need to do so, and so we use the simpler definition.

Motivation. By means of some elementary calculations with alternating cycles, one can easily appreciate how the alternating homology of the multiple-point spaces $D^k(f)$ contributes to the homology of the image of $f-$ see e.g. [MNnB20, 10.1.1].

Remark 2.2. Kevin Houston proved in [Hou99, Theorem 3.4] that the alternating homologies of $W^k(f)$ and $D^k(f)$ are canonically isomorphic:

$$AH_n(W^k(f)) = AH_n(D^k(f)),$$

– essentially, on the diagonals, i.e., on $W^k(f) \setminus D^k(f)$, there are no alternating chains [Hou99, Theorem 2.7] – see Lemma 4.18 below.

Let $d_j \colon \{1, \ldots, k-1\} \to \{1, \ldots, k\}$ be the monotone non-decreasing inclusion which does not have j in its image, i. e., given by

$$d_j(i) = \begin{cases} i & \text{if } i < j, \\ i+1 & \text{if } i \geq j. \end{cases}$$

Given a permutation $\sigma \in S_{k-1}$ we define the permutation $\bar{\sigma}^j \in S_k$ by

$$\bar{\sigma}^j(i) = \begin{cases} d_j(\sigma(i)) & \text{if } i < j, \\ j & \text{if } i = j, \\ d_j(\sigma(i-1)) & \text{if } i > j. \end{cases}$$

We have that $\mathrm{sign}(\bar{\sigma}^j) = \mathrm{sign}(\sigma)$. By a straightforward computation we get

$$\sigma \circ \varepsilon^{j,k} = \varepsilon^{j,k} \circ \bar{\sigma}^j. \tag{4}$$

The chain morphisms ρ_n^k and $\varepsilon_\#^{i,k}$ restrict to a chain morphism of the subcomplex of alternating chains:

Lemma 2.3. *There are the following inclusions:*

1. $\rho_n^k(C_n^{\mathrm{Alt}}(W^k(f)) \subset C_n^{\mathrm{Alt}}(W^{k-1}(f))$,
2. $\varepsilon_{\#,n}^{j,k}(C_n^{\mathrm{Alt}}(D^k(f)) \subset C_n^{\mathrm{Alt}}(D^{k-1}(f))$ *for* $j = 1, \ldots, k$.

Proof. 1. By (4), for any n-chain c in $W^k(f)$ and any $\sigma \in S_{k-1}$ we have

$$\sigma_\#(\varepsilon_{\#,n}^{j,k}(c)) = \varepsilon_{\#,n}^{j,k}(\bar{\sigma}_\#^j(c)), \quad \text{for } j = 1, \ldots, k. \tag{5}$$

From (5) and the definition of ρ_n^k given in (1) it follows that

$$\sigma_\#(\rho_n^k(c)) = \rho_n^k(\bar{\sigma}_\#^j(c)).$$

If c is alternating

$$\sigma_\#(\rho_n^k(c)) = \rho_n^k(\bar{\sigma}_\#^j(c)) = \rho_n^k(\mathrm{sign}(\bar{\sigma}^j)(c)) = \mathrm{sign}(\sigma)\rho_n^k(c).$$

2. Taking in (5) c an alternating n-chain in $D^k(f)$ we have

$$\sigma_\#(\varepsilon_{\#,n}^{j,k}(c)) = \varepsilon_{\#,n}^{j,k}(\bar{\sigma}_\#^j(c)) = \varepsilon_{\#,n}^{j,k}(\mathrm{sign}(\bar{\sigma}^j)(c)) = \mathrm{sign}(\sigma)\varepsilon_{\#,n}^{j,k}(c).$$

$\square$

It is convenient to include X and Y in the tower of multiple points spaces, as $D^1(f)$ and $D^0(f)$, and in this context to regard $f : X \to Y$ as $\varepsilon^1 : D^1(f) \to D^0(f)$.

Lemma 2.4. $\varepsilon_\#^{k-1} \circ \varepsilon_\#^k = 0$ *on* $C_\bullet^{\mathrm{Alt}}(D^k(f))$.

Proof. Consider the action of the odd permutation $(k-1,k)$. We have

$$\varepsilon^{k-1}\circ\varepsilon^{k}\circ(k-1,k)=\varepsilon^{k-1}\circ\varepsilon^{k},$$

and so

$$\varepsilon^{k-1}_{\#,n}\circ\varepsilon^{k}_{\#,n}\circ(k-1,k)_{\#}=\varepsilon^{k-1}_{\#,n}\circ\varepsilon^{k}_{\#,n}.$$

But if c is an alternating n-chain,

$$\varepsilon^{k-1}_{\#,n}\circ\varepsilon^{k}_{\#,n}\circ(k-1,k)_{\#}(c)=\varepsilon^{k-1}_{\#,n}\circ\varepsilon^{k}_{\#,n}(-c)=-\varepsilon^{k-1}_{\#,n}\varepsilon^{k}_{\#,n}(c).$$

□

It follows from Lemma 2.3–2 with $j=k$ and Lemma 2.4 that for each n, the sequence induced on alternating n-chains by the tower of multiple point spaces (3),

$$\ldots\to C_n^{\mathrm{Alt}}(D^k(f))\xrightarrow{\varepsilon^{k}_{\#,n}}\cdots\xrightarrow{\varepsilon^{3}_{\#,n}}C_n^{\mathrm{Alt}}(D^2(f))\xrightarrow{\varepsilon^{2}_{\#,n}}C_n(X)\xrightarrow{f_{\#,n}}C_n(Y)\to 0, \tag{6}$$

is a chain complex.

In Section 3, using a standard sign trick, we will obtain two first quadrant double complexes $(C_n(W^k(f)),\partial_n^k,\varrho_n^k)$ and $(C_n^{\mathrm{Alt}}(D^k(f)),\partial_n^k,\epsilon^k_{\#,n})$ from the commutative diagrams of free abelian groups and homomorphisms given by (2) and (6). Given any first quadrant double complex, two standard spectral sequences compute the homology of the total complex. The GVZSS and ICSS are the spectral sequences resulting from these double complexes by first applying the differential ∂_q to get, respectively, $E^1_{p,q}=H_q(W^{p+1}(f))$ and $E^1_{p,q}=AH_q(D^{p+1}(f))$ with differentials $(\varrho_q^{p+1})_*$ and $(\epsilon_q^{p+1})_*$. The opposite spectral sequences, in which one first applies the differentials ϱ^q and ϵ^q, are hard to make sense of with (alternating) singular homology. Our main technical result here is the following.

Theorem 2.5. *Let $f\colon X\to Y$ be a finite surjective simplicial map. Let V^k be $W^k(f)$ or $D^k(f)$. Then each projection $\varepsilon^{i,k}\colon V^k\to V^{k-1}$ is a simplicial map with respect to the associated triangulations of the V^k, and for each $n\in\mathbb{N}$, the resulting complexes* (2) *and* (6)*, now with simplicial chains, are resolutions of $C_n(Y)$ (i.e., are exact).*

The proof is given in Section 4.

From the theorem, it follows immediately that the first spectral sequences collapse onto the p-axis and we have $E^1_{p,0}=C_q(Y)$ with differential the usual boundary operator, thus showing that the homology of the total complex is that of Y.

Here, and in what follows, X and Y are geometric simplicial complexes, and C_n and C_n^{Alt} denote groups of simplicial chains, and alternating simplicial chains.

Before proceeding with the proof, let us point out that the images of "generic" smooth or complex analytic maps $f\colon M^n\to N^p$ (where M and N are smooth or complex analytic manifolds with $n<p$) are in general very singular, whereas, as noted above, the multiple point spaces $D^k(f)$ are in general less so, and this makes their homology more accessible than that of the image itself.

3. The GVZSS and the ICSS from the double complexes

In this section, given a finite surjective simplicial map $f\colon X\to Y$ between geometric simplicial complexes X and Y, using Theorem 2.5 we obtain the GVZSS and the ICSS from the second spectral sequence associated to the double complexes given by the complexes (2) and (6) respectively.

3.1. **Spectral sequences arising from a double complex.** Here we summarise the definitions and main results on spectral sequences that we use. Good references for spectral sequences are [McC01, Rot09, Wei94]. For some intuition on the ICSS which can help navigate this material, see [Mon18, §1.2 and §1.6] or [MNnB20, pp 373ff]..

A *homology spectral sequence* consists of the following data:

(1) A family $\{E^r_{p,q}\}$ of modules for all integers p, q and $r \geq r_0$ for some $r_0 \in \mathbb{Z}$. The *total degree* of the term $E^r_{p,q}$ is $n = p + q$.
(2) Homomorphisms $d^r_{p,q} \colon E^r_{p,q} \to E^r_{p-r,q+r-1}$ such that $d^r \circ d^r = 0$.
(3) Isomorphisms between $E^{r+1}_{p,q}$ and the homology of $E^r_{*,*}$ at the term $E^r_{p,q}$:

$$E^{r+1}_{p,q} \cong \frac{\ker d^r_{p,q}}{\operatorname{im} d_{p+r,q-r+1}}.$$

The collection of modules $E^r_{p,q}$ for fixed r, together with the morphisms $d^r_{p,q}$, is often referred to as the r'th *page* of the spectral sequence. Thus, item 3 above says that the $r+1$'st page is obtained by taking the homology of the r'th page.

A *first quadrant spectral sequence* is one with $E^r_{p,q} = 0$ unless $p \geq 0$ and $q \geq 0$. If this condition holds for r_0, then it holds for all $r \geq r_0$. For fixed p and q, $E^r_{p,q} = E^{r+1}_{p,q}$ for all $r > \max\{p, q+1\}$, because then every arrow $d^r_{p,q}$ beginning at $E^r_{p,q}$ must end at 0, and every arrow ending at $E^r_{p,q}$ must begin at 0. We write $E^\infty_{p,q}$ for this stable value of $E^r_{p,q}$.

A homology spectral sequence is *bounded* if for each n there are only finitely many nonzero terms of total degree n in $E^1_{*,*}$. Thus a first quadrant spectral sequence is bounded. A bounded spectral sequence *converges* to a graded module H_*, denoted by $E^r_{p,q} \Longrightarrow H_{p+q}$, if for each n, H_n has a finite filtration

$$0 = F^s H_n \subseteq \cdots \subseteq F^{p-1} H_n \subseteq F^p H_n \subseteq F^{p+1} H_n \subseteq \cdots \subseteq F^t H_n = H_n,$$

and there are isomorphisms

$$E^\infty_{p,q} \cong F^p H_{p+q} / F^{p-1} H_{p+q}.$$

A *double complex* is an ordered triple (C, d', d'') where $C = (C_{p,q})$ is a bigraded module, $d'_{p,q} \colon C_{p,q} \to C_{p-1,q}$ and $d''_{p,q} \colon C_{p,q} \to C_{p,q-1}$ such that $d'_{p-1,q} \circ d'_{p,q} = 0$, $d''_{p,q-1} \circ d''_{p,q} = 0$ and

$$d'_{p,q-1} \circ d''_{p,q} + d''_{p-1,q} \circ d'_{p,q} = 0.$$

(Note that for some authors, this equation is replaced by $d'_{p,q-1} \circ d''_{p,q} = d''_{p-1,q} \circ d'_{p,q}$, so that the squares commute rather than anticommuting.) A double complex $(C_{p,q})$ is a *first quadrant double complex* if $C_{p,q} = 0$ whenever $p < 0$ or $q < 0$.

If C is a double complex, its *total complex*, denoted by $\operatorname{Tot}(C)$, is the chain complex [Rot09, Lemma 10.5] with n'th term

$$\operatorname{Tot}(C)_n = \bigoplus_{p+q=n} C_{p,q},$$

with differentials $D_n \colon \operatorname{Tot}(C)_n \to \operatorname{Tot}(C)_{n-1}$ given by

$$D_n = \sum_{p+q=n} (d'_{p,q} + d''_{p,q}).$$

We can define two filtrations of $\operatorname{Tot}(C)$ by

$${}^{I}F^p(\operatorname{Tot}(C)_n) = \bigoplus_{i \leq p} C_{i,n-i}, \qquad {}^{II}F^p(\operatorname{Tot}(C)_n) = \bigoplus_{j \leq p} C_{n-j,j}.$$

These two filtrations determine two spectral sequences [Rot09, Theorem 10.16]. If C is a first quadrant double complex, both filtrations are bounded and both spectral sequences converge to the homology of the total complex of C. The first filtration gives the first spectral sequence

$$
\begin{aligned}
{}^{I}E^0_{p,q} &= C_{p,q}, \quad \text{with differentials } d^0_{p,q} = d''_{p,q},\\
{}^{I}E^1_{p,q} &= H''_q(C_{p,*}), \quad \text{with differentials } d^1_{p,q} = (d'_{p,q})_*,\\
{}^{I}E^2_{p,q} &= H'_pH''_q(C) \Rightarrow H_{p+q}(\mathrm{Tot}(C)).
\end{aligned}
\tag{7}
$$

The second filtration gives the second spectral sequence

$$
\begin{aligned}
{}^{II}E^0_{p,q} &= C_{q,p}, \quad \text{with differentials } d^0_{p,q} = d'_{p,q},\\
{}^{II}E^1_{p,q} &= H'_q(C_{*,p}), \quad \text{with differentials } d^1_{p,q} = (d''_{p,q})_*,\\
{}^{II}E^2_{p,q} &= H''_pH'_q(C) \Rightarrow H_{p+q}(\mathrm{Tot}(C)).
\end{aligned}
\tag{8}
$$

Here $(d'_{p,q})_*$ and $(d''_{p,q})_*$ denote respectively, the homomorphisms induced in homology by the differentials $d'_{p,q}$ and $d''_{p,q}$ of the double complex. Both spectral sequences converge to the homology of the total complex of C. In both, the higher differentials involve both d' and d'', and can be constructed by diagram chasing. In each of these cases, the filtration on the homology of the total complex is determined by the spectral sequence itself, and is not an additional independent datum. If a first quadrant spectral sequence converges to H_n, then each H_n has a finite filtration of length $n+1$

$$
0 = F_{-1}H_n \subset F_0H_n \subset \cdots \subset F_{n-1}H_n \subset F_nH_n = H_n.
\tag{9}
$$

The bottom part $F_0H_n = E^\infty_{0,n}$ lies on the q-axis, the intermediate quotients are

$$
F_pH_n/F_{p-1}H_n \cong E^\infty_{p,n-p}
$$

and the top quotient $H_n/F_{n-1}H_n \cong E^\infty_{n,0}$ lies on the p-axis.

A spectral sequence (E^r, d^r) *collapses* onto the p-axis if $E^2_{p,q} = 0$ for all $q \neq 0$. In this case $E^\infty_{p,q} = E^2_{p,q}$ and $H_n(\mathrm{Tot}(C)) \cong E^2_{n,0}$ [Rot09, Proposition 10.21]. More generally, a spectral sequence *collapses at* E^r if all the differentials $d^s_{p,q}$ for $s \geq r$ and all p, q are zero; in this case $E^s_{p,q} = E^r_{p,q}$ for $s \geq r$, and so $E^\infty_{p,q} = E^r_{p,q}$.

3.2. **The** GVZSS **and the** ICSS**.** Let $f\colon X \to Y$ be a finite surjective simplicial map. Since ρ^k_n and $\varepsilon^{i,k}_{\#,n}$ are chain morphisms, the diagrams of free abelian groups and homomorphisms given by the arrays

$$
(C_n(W^k(f)), \partial^k_n, \rho^k_n), \qquad \text{and} \qquad (C^{\mathrm{Alt}}_n(D^k(f)), \partial^k_n, \varepsilon^k_{\#,n})
$$

commute, and every row and column is a chain complex. Applying the usual sign trick [Rot09, Example 10.4], we define

$$
\varrho^k_n = (-1)^n\rho^k_n = \sum_{i=1}^{k}(-1)^{n+i-1}\varepsilon^{i,k}_{\#,n}, \qquad \text{and} \qquad \epsilon^k_n = (-1)^n\varepsilon^k_{\#,n},
\tag{10}
$$

to get (first quadrant) double complexes $(C_n(W^{k+1}(f)), \partial_n^{k+1}, \varrho_n^{k+1})$ with $n, k \geq 0$:

(11)
$$\begin{tikzcd}
\vdots \arrow[d, "\varrho_0^{k+1}"'] & \vdots \arrow[d, "\varrho_1^{k+1}"'] & & \vdots \arrow[d, "\varrho_l^{k+1}"'] & \\
C_0(W^k(f)) \arrow[d, "\varrho_0^k"'] & C_1(W^k(f)) \arrow[l, "\partial_1^k"'] \arrow[d, "\varrho_1^k"'] & \cdots \arrow[l, "\partial_2^k"'] & C_l(W^k(f)) \arrow[l, "\partial_l^k"'] \arrow[d, "\varrho_l^k"'] & \cdots \arrow[l, "\partial_{l+1}^k"'] \\
\vdots \arrow[d, "\varrho_0^3"'] & \vdots \arrow[d, "\varrho_1^3"'] & & \vdots \arrow[d, "\varrho_l^3"'] & \\
C_0(W^2(f)) \arrow[d, "\varrho_0^2"'] & C_1(W^2(f)) \arrow[l, "\partial_1^2"'] \arrow[d, "\varrho_1^2"'] & \cdots \arrow[l, "\partial_2^2"'] & C_l(W^2(f)) \arrow[l, "\partial_l^2"'] \arrow[d, "\varrho_l^2"'] & \cdots \arrow[l, "\partial_{l+1}^2"'] \\
C_0(X) & C_1(X) \arrow[l, "\partial_1"'] & \cdots \arrow[l, "\partial_2"'] & C_l(X) \arrow[l, "\partial_l"'] & \cdots \arrow[l, "\partial_{l+1}"']
\end{tikzcd}$$

and $(C_n^{\text{Alt}}(D^{k+1}(f)), \partial_n^{k+1}, \epsilon_n^{k+1})$ with $n, k \geq 0$:

(12)
$$\begin{tikzcd}
\vdots \arrow[d, "\epsilon_0^{k+1}"'] & \vdots \arrow[d, "\epsilon_1^{k+1}"'] & & \vdots \arrow[d, "\epsilon_l^{k+1}"'] & \\
C_0^{\text{Alt}}(D^k(f)) \arrow[d, "\epsilon_0^k"'] & C_1^{\text{Alt}}(D^k(f)) \arrow[l, "\partial_1^k"'] \arrow[d, "\epsilon_1^k"'] & \cdots \arrow[l, "\partial_2^k"'] & C_l^{\text{Alt}}(D^k(f)) \arrow[l, "\partial_l^k"'] \arrow[d, "\epsilon_l^k"'] & \cdots \arrow[l, "\partial_{l+1}^k"'] \\
\vdots \arrow[d, "\epsilon_0^3"'] & \vdots \arrow[d, "\epsilon_1^3"'] & & \vdots \arrow[d, "\epsilon_l^3"'] & \\
C_0^{\text{Alt}}(D^2(f)) \arrow[d, "\epsilon_0^2"'] & C_1^{\text{Alt}}(D^2(f)) \arrow[l, "\partial_1^2"'] \arrow[d, "\epsilon_1^2"'] & \cdots \arrow[l, "\partial_2^2"'] & C_l^{\text{Alt}}(D^2(f)) \arrow[l, "\partial_l^2"'] \arrow[d, "\epsilon_l^2"'] & \cdots \arrow[l, "\partial_{l+1}^2"'] \\
C_0(X) & C_1(X) \arrow[l, "\partial_1"'] & \cdots \arrow[l, "\partial_2"'] & C_l(X) \arrow[l, "\partial_l"'] & \cdots \arrow[l, "\partial_{l+1}"']
\end{tikzcd}$$

Proposition 3.1. *The first spectral sequences* ${}^{I}E^r_{p,q}$ *of the double complexes* (11) *and* (12) *collapse onto the* p*-axis, and thus in both cases* $E^2_{n,0} \cong H_n(\text{Tot}(C)) \cong H_n(Y)$.

Proof. We give the proof for the double complex (12). The proof for the double complex (11) is completely analogous. By (7), the first page of the first spectral sequence is given by taking the homology with respect to the vertical differentials ϵ^q in diagram (12), i.e.,

$${}^{I}E^1_{p,q} \cong H_q((C_p^{\text{Alt}}(D^{\bullet+1}(f)), \epsilon_p^{\bullet})).$$

By Theorem 2.5 the chain complex (6) is exact, hence we have

$${}^{I}E^1_{p,q} = \begin{cases} C_p(X)/\text{im } \epsilon^2_{\#,p} \cong C_p(Y) & \text{if } q = 0, \\ 0 & \text{if } q \neq 0. \end{cases}$$

The differential $(\partial_q)_*\colon {}^{I}E^1_{p,0} = C_p(X)/\mathrm{im}\ \epsilon^2_{\#,p} \to C_{p-1}(X)/\mathrm{im}\ \epsilon^2_{\#,p-1} = {}^{I}E^1_{p-1,0}$ corresponds under the isomorphism to the boundary map $\partial_q\colon C_p(Y) \to C_{p-1}(Y)$, thus we have

$$ {}^{I}E^2_{p,q} = \begin{cases} H_p(Y) & \text{if } q = 0, \\ 0 & \text{if } q \neq 0. \end{cases} $$

Therefore, the spectral sequence ${}^{I}E^r_{p,q}$ collapses on the p-axis and $H_n(\mathrm{Tot}(C)) \cong H_n(Y)$. □

Corollary 3.2. *1. The second spectral sequence ${}^{II}E^1_{p,q}$ of the double complex (11) converges to $H_\bullet(Y)$, i.e.,*

$$ {}^{II}E^1_{p,q} \cong H_q(W^{p+1}(f)) \Longrightarrow H_{p+q}(Y), \tag{13} $$

with differentials given by $(\varrho^{p+1}_q)_*$.

2. The second spectral sequence ${}^{II}E^1_{p,q}$ of the double complex (12) converges to $H_\bullet(Y)$, i.e.,

$$ {}^{II}E^1_{p,q} \cong AH_q(D^{p+1}(f)) \Longrightarrow H_{p+q}(Y), \tag{14} $$

with differentials given by $(\epsilon^{p+1}_q)_*$.

Proof. By (8), the first page of the second spectral sequence is given by taking the homology with respect to the horizontal differentials ∂_q in diagram (11) (resp. in (12)), so we get $H_q(W^{p+1}(f))$ (resp. $AH_q(D^{p+1}(f))$). It converges to the homology of the total complex, which by Proposition 3.1 is isomorphic to $H_\bullet(Y)$. □

The second spectral sequence (13) of the double complex (11) is the *GVZ Spectral Sequence*. The second spectral sequence (14) of the double complex (12) is the *Image-Computing Spectral Sequence*. Note that in view of Remark 2.2, the spectral sequence with $E^1_{p,q} \cong AH_q(W^{p+1}(f))$ coincides with the ICSS, and thus also converges to $H_n(Y)$.

4. Finite simplicial maps

To complete the construction of the GVZSS and the ICSS it only remains to prove Theorem 2.5. This is the aim of this section.

We suppose that X and Y are geometric simplicial complexes embedded in $\mathbb{R}^N, \mathbb{R}^P$ respectively, and that $f : X \to Y$ is simplicial, surjective and finite-to-one. We begin by establishing some notation and proving some elementary lemmas. An *ordered simplex* is an affine homeomorphism

$$ (t_0, \ldots, t_n) \mapsto \sum_i t_i v_i $$

from the standard n-simplex into X or Y which forms part of its simplicial structure; the image of an ordered simplex is a *geometric simplex*. Thus, an ordered simplex of X is determined by a geometric k-simplex together with an ordering of its vertices. We will use the symbol Δ to denote an ordered simplex, and $|\Delta|$ to denote its image. When we need to speak of a geometric simplex without specifying an ordering of its vertices, we will use the symbol Γ. We write $C_k(X)$ for the free abelian group generated by the ordered k-simplices of X. If f is a simplicial map and Δ an ordered simplex of X, then $f(|\Delta|)$ is a geometric simplex of Y and $f_\#(\Delta)$ an ordered simplex. Thus $f(|\Delta|) = |f_\#(\Delta)|$. If $\Delta = [v_0, \ldots, v_n]$ then $|\mathring{\Delta}|$ denotes its *interior*:

$$ |\mathring{\Delta}| = \{\sum_i t_i v_i : \sum_i t_i = 1, t_i > 0, i = 0, \ldots, n\}. $$

The following properties of any simplicial complex K are well known:

(1) Each point $x \in K$ lies in the interior of a unique geometric simplex of K, which we denote by Γ_x.

(2) Write $|\Delta| \leq |\Delta'|$ when $|\Delta|$ is a face of $|\Delta'|$. Then if $|\Delta'| \cap |\mathring{\Delta}| \neq 0$ it follows that $|\Delta| \leq |\Delta'|$. For two simplices meet only along a common face of each, so $\Delta = \overline{\mathring{\Delta}}$ is a face of Δ'.

(3) For each point x,

$$\operatorname{star}(x) := |\mathring{\Gamma}_x| \cup \bigcup_{\Gamma_x \leq |\Delta|} |\mathring{\Delta}|$$

is an open neighbourhood of x. For suppose that $(x_n) \to x$. By local finiteness of K we can suppose that all x_n lie in the interior of one simplex Δ. Then $x \in \Delta \cap \mathring{\Gamma}_x$, so by the preceding statement, $\Gamma_x \leq \Delta$, and thus all the x_n lie in $\operatorname{star}(x)$.

Lemma 4.1. *f maps each geometric n-simplex of X isomorphically to an n-simplex of Y.*

Proof. By definition of simplicial map, on the simplex $\Delta := [v_0, \ldots, v_n]$, we have

$$f(\sum_i t_i v_i) = \sum_i t_i f(v_i).$$

If the vertices $f(v_0), \ldots, f(v_n)$ were affinely dependent then this map would not be finite to one. Thus $\Delta' := [f(v_0), \ldots, f(v_n)]$ is an n-simplex of Y and $f : |\Delta| \to |\Delta'|$ is an isomorphism. ☐

Lemma 4.2. *If $f(x_1) = f(x_2)$ then the barycentric coordinates of x_1 in Γ_{x_1}, with respect to a suitable ordering of the vertices, are the same as those of x_2 in Γ_{x_2}.*

Proof. Write $y := f(x_1) = f(x_2)$. If $\Gamma_y = |[w_1, \ldots, w_n]|$ then for $i = 1, 2$ it is possible to order the vertices $v_j^{(i)}$ of Γ_{x_i} so that $f(v_j^{(i)}) = w_j$. Since f is linear on Γ_{x_i}, the barycentric coordinates of x_i in Γ_{x_i} are the same as those of y in Γ_y. ☐

Lemma 4.3. *$f_\# : C_n(X) \to C_n(Y)$ is surjective.*

Proof. Let Γ' be a k-simplex in Y, and $y \in \mathring{\Gamma}'$. Pick $x \in X$ such that $f(x) = y$. By Lemma 4.1, $f : \Gamma_x \to \Gamma'$ is an isomorphism. It follows that for each Δ' such that $\Gamma' = |\Delta'|$, there is an ordered k-simplex Δ in X such that $f_\#(\Delta) = \Delta'$, so that $f_\#$ is surjective. ☐

Remark 4.4. The statement of Lemma 4.3 is false in singular homology when $n > 0$: for example, the figure 8 is the image of the circle $X = S^1$ under a generically 1-to-1 immersion with one double point. A 1-simplex in the figure 8 which turns a corner at the vertex of the 8 is not the image of any 1-simplex in X.

4.1. **Triangulating $W^k(f)$.** Recall first that since $X \subset \mathbb{R}^N$, we have $W^k(f) \subset (\mathbb{R}^N)^k$. Let $y \in Y$. Suppose that Γ_y is an n-simplex. Pick $\mathbf{x} \in W^k(f)$ such that $f^{(k)}(\mathbf{x}) = y$, i.e., $f(\pi_\ell(\mathbf{x})) = y$ for $\ell = 1, \ldots, k$. An order of the vertices of Γ_y, say $\Gamma_y = |[w_0, \ldots, w_n]|$, determines an order for the vertices of each simplex

$$\Gamma_{\pi_\ell(\mathbf{x})} = \left| \left[v_0^{(\ell)}, \ldots, v_n^{(\ell)} \right] \right|,$$

where $f(v_j^{(\ell)}) = w_j$ for $j = 0, \ldots, n$. Note that the simplices $\Gamma_{\pi_\ell(\mathbf{x})}$ need not be pairwise distinct, since x_ℓ may be equal to x_m for some $\ell \neq m$.

The simplex

$$\Gamma_{\mathbf{x}}^{(k)} := \left| \left[\left(v_0^{(1)}, \ldots, v_0^{(k)}\right), \ldots, \left(v_n^{(1)}, \ldots, v_n^{(k)}\right) \right] \right| \tag{15}$$

is contained in $W^k(f)$ since all its vertices are. We also denote the geometric simplex (15) by

$$\left(\Gamma_{\pi_1(\mathbf{x})} \times \cdots \times \Gamma_{\pi_k(\mathbf{x})}\right) / Y.$$

We claim that $\mathbf{x}$ is in the interior of $\Gamma_{\mathbf{x}}^{(k)}$. Write y in barycentric coordinates $y = \sum_j t_j w_j$; since y is in the interior of Γ_y, $t_j > 0$ for $j = 0, \ldots, n$. By Lemma 4.2 the barycentric coordinates of each point $\pi_\ell(\mathbf{x})$ in $\Gamma_{\pi_\ell(\mathbf{x})}$ with the vertices ordered as described, are the same as those of y in Γ_y. Thus

$$\mathbf{x} = \sum_j t_j(v_j^{(1)}, \ldots, v_j^{(k)})$$

lies in the interior of $\Gamma_{\mathbf{x}}^{(k)}$ since $t_j > 0$ for $j = 0, \ldots, n$.

Figure 1 illustrates the 2-simplex $\Gamma_y^{(2)}$ for a point $y \in Y$ and the 2-simplex $\Gamma_{\mathbf{x}}^{(2)}$ for a point $\mathbf{x} = (x_1, x_2) \in W^2(f)$ with $f(x_1) = f(x_2) = y$ for two cases:

(a) when the 2-simplices $\Gamma_{x_1}^{(2)}$ and $\Gamma_{x_2}^{(2)}$ have different vertices,
(b) when the 2-simplices $\Gamma_{x_1}^{(2)}$ and $\Gamma_{x_2}^{(2)}$ have some common vertices.

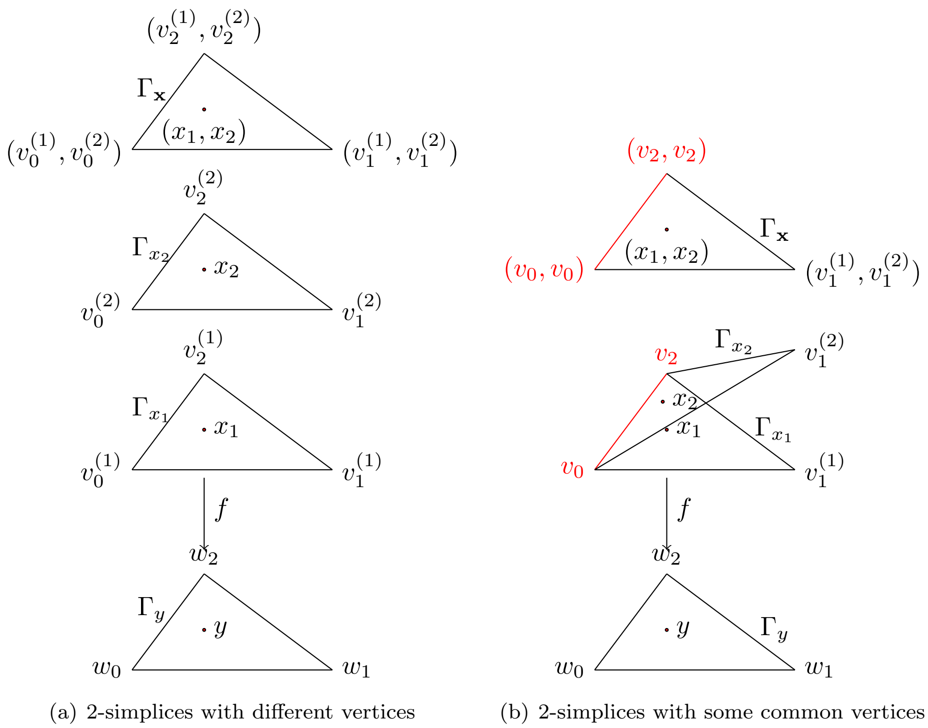

FIGURE 1. 2-simplices in $D^2(f)$

Lemma 4.5. *Let $\mathbf{x} \in W^k(f)$. Then*

(1) the geometric simplex $\Gamma_{\mathbf{x}}^{(k)}$ given in (15) is the unique geometric simplex of $W^k(f)$ containing $\mathbf{x}$ in its interior, and
(2) for each point $\mathbf{x} \in W^k(f)$,

$$\mathbf{x} \in \left|\left[\left(v_0^{(1)}, \ldots, v_0^{(k)}\right), \ldots, \left(v_n^{(1)}, \ldots, v_n^{(k)}\right)\right]\right| \quad \Leftrightarrow \quad \pi_\ell(\mathbf{x}) \in |[v_0^{(\ell)}, \ldots, v_n^{(\ell)}]| \text{ for } \ell = 1, \ldots, k. \tag{16}$$

Proof. Suppose $\mathbf{x}'$ lies in the interior of the n-simplex $\Gamma_{\mathbf{x}}^{(k)}$. Then, with the notation of the construction,

$$\mathbf{x}' = \sum_j t'_j \left(v_j^{(1)}, \ldots, v_j^{(k)}\right),$$

for some t'_j, with $t'_j > 0$ for $j = 0, \ldots, n$ and $\sum_j t'_j = 1$. Then $\pi_\ell(\mathbf{x}') = \sum_j t'_j v_j^{(\ell)}$, and $\pi_\ell(\mathbf{x}') \in \mathring{\Gamma}_{\pi_\ell(\mathbf{x})}$, so that $\Gamma_{\pi_\ell(\mathbf{x}')} = \Gamma_{\pi_\ell(\mathbf{x})}$. It follows that $\Gamma^{(k)}_{\mathbf{x}'} = \Gamma^{(k)}_{\mathbf{x}}$. Thus each point of $W^k(f)$ lies in the interior of a unique geometric simplex.

Let $\sigma^{(k)}$ and σ denote, for a moment, the simplices on the LHS and RHS of (16). By construction, the statement is evidently true with interior($\sigma^{(k)}$) and interior(σ) in place of $\sigma^{(k)}$ and σ. But then it follows that it must hold also for points in their closure. □

By Lemma 4.5 the notation $\Gamma^{(k)}_{\mathbf{x}}$ is consistent with the notation Γ_x for the unique geometric simplex of X containing x in its interior. Note that the "k" in $\Gamma^{(k)}_{\mathbf{x}}$ refers to the fact that we are speaking of a simplex in $W^k(f)$; it is not the dimension of the simplex.

Proposition 4.6. *The forgoing construction gives $W^k(f)$ the structure of a simplicial complex.*

Proof. Omitting a vertex of a simplex $\Gamma^{(k)}$ in $W^k(f)$ we obtain again a geometric simplex in $W^k(f)$, and by Lemma 4.5 for each point $\mathbf{x} \in W^k(f)$, $\Gamma^{(k)}_{\mathbf{x}}$ is the unique geometric simplex containing $\mathbf{x}$ in its interior.

Lemma 4.5 also shows that the intersection of two simplices of $W^k(f)$ is a face of each – this property is inherited from the triangulation of X. □

It is important to note that the action of the symmetric group S_k on X^k restricts to an action on $W^k(f)$, and moreover gives rise to an action on the triangulation we have just described. In particular, for $\sigma \in S_k$ we have

$$\sigma_\#(\Gamma^{(k)}_{\mathbf{x}}) = \Gamma^{(k)}_{\sigma(\mathbf{x})}. \tag{17}$$

Lemma 4.7. *Any geometric simplex in this triangulation which is mapped to itself by $\sigma \in S_k$ is in fact pointwise fixed.*

Proof. If the simplex Γ is mapped to itself by σ then its vertices are permuted. Suppose that the i'th vertex of Γ is equal to the j'th vertex of $\sigma_\#(\Gamma)$,

$$(v_i^{(1)}, ..., v_i^{(k)}) = (v_j^{(\sigma(1))}, ..., v_j^{(\sigma(k))}).$$

Then, since $v_i^{(1)} = v_j^{(\sigma(1))}$, we have $f(v_i^{(1)}) = f(v_j^{(\sigma(1))})$. But $f(v_i^{(\sigma(1))}) = f(v_i^{(1)})$ by the choice of the ordering of the vertices of the simplices Γ_{x_1} and $\Gamma_{x_{\sigma(1)}}$, as explained in the first paragraph of Subsection 4.1. So $f(v_i^{(\sigma(1))}) = f(v_j^{(\sigma(1))})$, and since, by Lemma 4.1, f is 1-to-1 on $\Gamma_{x_{\sigma(1)}}$, we must have $i = j$. Thus each vertex of the simplex Γ is mapped to itself by σ, and so Γ is pointwise fixed. □

Triangulating $D^k(f)$. Let $y \in Y$, suppose that Γ_y is an n-simplex, and suppose

$$\{x_1, \ldots, x_k\} \subset f^{-1}(y)$$

with $x_i \neq x_{i'}$ if $i \neq i'$. Thus, the point $\mathbf{x} = (x_1, \ldots, x_k)$ is a point in $D^k_S(f)$. By Lemma 4.1, the preimage of Γ_y contains the geometric n-simplices Γ_{x_i}, each of which is mapped isomorphically to Γ_y. Suppose that $\Gamma_y = |[w_0, \ldots, w_n]|$, and that $y = \sum_j t_j w_j$. Let $v_j^{(i)}$ be the unique vertex of Γ_{x_i} such that $f(v_j^{(i)}) = w_j$, for $i = 1, \ldots, k$ and $j = 0, \ldots, n$. Note that it may happen that $v_j^{(i)} = v_j^{(i')}$ for some $i \neq i'$. By Lemma 4.2, the barycentric coordinates of each point x_i in Γ_{x_i}, with the vertices ordered as described, are the same as those of y in Γ_y. Thus, the geometric n-simplex

$$\left|\left[\left(v_0^{(1)}, \ldots, v_0^{(k)}\right), \ldots, \left(v_n^{(1)}, \ldots, v_n^{(k)}\right)\right]\right| \tag{18}$$

is contained in $D^k(f)$, and

$$\mathbf{x} = (x_1, \ldots, x_r) = \sum_j t_j \left(v_j^{(1)}, \ldots, v_j^{(k)} \right)$$

lies in its interior: all of the t_{j_i} are strictly positive since $y \in \mathring{\Gamma}_y$.

As before, we denote the geometric simplex (18) by $\Gamma_{\mathbf{x}}^{(k)}$ or by

$$\left(\Gamma_{x_1} \times \cdots \times \Gamma_{x_k} \right) / Y.$$

Its interior is contained in $D_S^k(f)$, but some of its faces may not be. When $k = 2$, it is more commonly written as $\Gamma_{x_1} \times_Y \Gamma_{x_2}$.

Let us consider now more general points $\mathbf{x} \in D^k(f)$, where there may be repetitions in the k components. Such points appear in the closure of $D_S^k(f)$. We claim that the simplex (15) constructed for $W^k(f)$ above is contained in $D^k(f)$, and contains $\mathbf{x}$ in its interior.

To see this, observe that by definition of $D^k(f)$ as the closure of $D_S^k(f)$, there is a sequence $\left(x_n^{(1)}, \ldots, x_n^{(k)} \right)_{n \in \mathbb{N}}$ in $D_S^k(f)$ converging to $\mathbf{x}$. By local finiteness of the triangulation of X, by passing to a subsequence we may suppose each individual sequence $(x_n^{(\ell)})_{n \in \mathbb{N}}$ is contained in the interior of a single simplex, which we denote by $\Gamma^{(\ell)}$. Note that $\Gamma^{(\ell)} \neq \Gamma^{(\ell')}$ if $\ell \neq \ell'$ since, by Lemma 4.1, f is one-to-one on each simplex. Then by the argument of the first part of the construction,

$$\left(\Gamma^{(1)} \times \cdots \times \Gamma^{(k)} \right) / Y$$

is contained in $D^k(f)$. Since $\Gamma_{\pi_\ell(\mathbf{x})} \leq \Gamma^{(\ell)}$ for $\ell = 1, \ldots, k$, it follows that the simplex (15) is contained in $D^k(f)$. Again, $\mathbf{x}$ is contained in its interior, since the barycentric coordinates of each point $\pi_\ell(\mathbf{x})$ in $|[v_0^{(\ell)}, \ldots, v_n^{(\ell)}]|$ are equal to those of y in $[w_0, \ldots, w_n]$, which are all strictly positive.

By Lemma 4.5 $\Gamma_{\mathbf{x}}^{(k)}$ is the unique geometric simplex of $D^k(f)$ containing $\mathbf{x}$ in its interior.

Notation. As described in the construction, each of the simplices $\Delta_\ell := [v_0^{(\ell)}, \ldots, v_n^{(\ell)}]$ appearing in (15) satisfies $f_\#(\Delta_\ell) = [w_0, \ldots, w_n]$. It will be useful to refer to the ordered n simplex of $D^k(f)$ appearing in (15) as $(\Delta_1 \times \cdots \times \Delta_k)/Y$.

Corollary 4.8. *The forgoing construction gives $D^k(f)$ the structure of a simplicial complex. In fact, it is a simplicial subcomplex of $W^k(f)$ with the triangulation associated to the simplicial map f.*

Proof. It is obvious that the simplices constructed in $D^k(f)$ are also simplices of $W^k(f)$. To prove that $D^k(f)$ is a simplicial subcomplex of $W^k(f)$, it is necessary to show that if we omit any vertex of a simplex of $D^k(f)$ then the resulting simplex is still a simplex of $D^k(f)$. This is also obvious, since by omitting a vertex of a simplex $\Gamma^{(k)}$ we obtain a geometric simplex in the topological closure of $\Gamma^{(k)}$. □

We also refer to this triangulation of $D^k(f)$ as the *triangulation associated to the simplicial map f*.

Proposition 4.9. *Let V^k be $W^k(f)$ or $D^k(f)$. For each k and $1 \leq i \leq k$, the map*

$$\varepsilon^{i,k} \colon V^k \to V^{k-1}$$

is a simplicial map, with respect to the triangulations associated to the simplicial map f.

Proof. It is clear from (15) that $\varepsilon^{i,k}(\Gamma_{\mathbf{x}}^{(k)}) = \Gamma_{\varepsilon^{i,k}(\mathbf{x})}^{(k-1)}$. The two geometric simplices have the same dimension, and $\varepsilon^{i,k}$ preserves the barycentric coordinates of each point in the simplex $\Gamma_{\mathbf{x}}^{(k)}$. □

Remark 4.10. Let V^k be $W^k(f)$ or $D^k(f)$. Consider the map $f^{(k)}\colon V^k \to Y$ defined by $f^{(k)}(\mathbf{x}) = f\big(\pi_\ell(\mathbf{x})\big)$ for some ℓ with $1 \le \ell \le k$. Evidently this is independent of the choice of ℓ, so for convenience we take $\ell = 1$. We have

$$f^{(k)}\left(\sum_j t_j\Big(v_j^{(1)}, v_j^{(2)}, \dots, v_j^{(k)}\Big)\right) = f\Big(\sum_j t_j v_j^{(1)}\Big) = \sum_j t_j f\big(v_j^{(1)}\big),$$

the last equality because f is linear on each simplex of X. Thus $f^{(k)}$ defines an isomorphism $\Gamma_{\mathbf{x}}^{(k)} \to \Gamma_{f(\pi_\ell(\mathbf{x}))}$. Conversely, for any n-simplex Γ in Y, each component of any point $\mathbf{x}$ in V^k such that $f^{(k)}(\Gamma_{\mathbf{x}}^{(k)}) = \Gamma_y$, lies in the interior of a simplex which is mapped isomorphically to Γ_y.

Again, let V^k denote $W^k(f)$ or $D^k(f)$. Let $\Delta = [w_0, \dots, w_n]$ be an ordered n-simplex of Y. Denote by $C_n(V^k)|_\Delta$ and $C_n^{\mathrm{Alt}}(V^k)|_\Delta$ the collection of n-chains, and alternating n-chains, respectively, consisting of linear combinations of ordered n-simplices $\Delta^{(k)}$ such that $f_\#^{(k)}(\Delta^{(k)}) = \Delta$. Every such simplex is of the form $(\Delta_1 \times \cdots \times \Delta_k)/Y$, where each Δ_ℓ is a simplex of X such that $f_\#(\Delta_\ell) = \Delta$. Note that by V^1 we mean simply X, so that both $C_n(V^1)|_\Delta$ and $C_n^{\mathrm{Alt}}(V^1)|_\Delta$ are equal to the free abelian group generated by the n-simplices Δ' of X such that $f_\#(\Delta') = \Delta$. We have

$$\varepsilon_\#^{i,k}\big((\Delta_1 \times \cdots \times \Delta_k)/Y\big) = (\Delta_1 \times \cdots \times \widehat{\Delta_i} \times \cdots \times \Delta_k)/Y,$$

which implies the following simple but important lemma.

Lemma 4.11. *$C_n(V^\bullet)|_\Delta$ and $C_n^{\mathrm{Alt}}(V^\bullet)|_\Delta$ are subcomplexes of the complexes $C_n(V^\bullet)$ and $C_n^{\mathrm{Alt}}(V^\bullet)$.*

4.2. **The sequence $(C_n(W^\bullet(f)), \varrho^\bullet)$ is a resolution of $C_n(Y)$.** Now we shall prove that the sequence (2) is exact.

Lemma 4.12. $\rho_n^{k-1} \circ \rho_n^k = 0$.

Proof. This is the usual computation. Denoting by $\widehat{v_n^{(i)}}$ the entry that has to be removed we have

$$\begin{aligned}
&\rho_n^{k-1} \circ \rho_n^k([(v_0^{(1)}, \dots, v_0^{(k)}), \dots, (v_n^{(1)}, \dots, v_n^{(k)})])\\
&\qquad = \rho_n^{k-1}\Big(\sum_{i=1}^k (-1)^{i-1}[(v_0^{(1)}, \dots, \widehat{v_0^{(i)}}, \dots, v_0^{(k)}), \dots, (v_n^{(1)}, \dots, \widehat{v_n^{(i)}}, \dots, v_n^{(k)})]\Big)\\
&\quad = \sum_{j<i} (-1)^{i+j-2}[(v_0^{(1)}, \dots, \widehat{v_0^{(j)}}, \dots, \widehat{v_0^{(i)}}, \dots, v_0^{(k)}), \dots, (v_n^{(1)}, \dots, \widehat{v_n^{(j)}}, \dots, \widehat{v_n^{(i)}}, \dots, v_n^{(k)})]\\
&\qquad + \sum_{j>i} (-1)^{i+j-3}[(v_0^{(1)}, \dots, \widehat{v_0^{(i)}}, \dots, \widehat{v_0^{(j)}}, \dots, v_0^{(k)}), \dots, (v_n^{(1)}, \dots, \widehat{v_n^{(i)}}, \dots, \widehat{v_n^{(j)}}, \dots, v_n^{(k)})].
\end{aligned}$$

The latter two summations cancel since after switching i and j in the second sum, it becomes the negative of the first. □

Remark 4.13. It is immediate from Lemma 4.12 and the definition of ϱ_n^k given in (10) that we also have $\varrho_n^{k-1} \circ \varrho_n^k = 0$. Hence all the columns of the double complex (11) are chain complexes.

Lemma 4.14. $f_{\#,n} \circ \rho_n^2 = 0$.

Proof.

$$\begin{aligned}f_{\#,n}\circ\rho_n^2\big([(v_0^{(1)},v_0^{(2)}),\dots,(v_n^{(1)},v_n^{(2)})]\big)&=f_{\#,n}\big([(v_0^{(2)}),\dots,(v_n^{(2)})]-[(v_0^{(1)}),\dots,(v_n^{(1)})]\big)\\&=[(f(v_0^{(2)})),\dots,(f(v_n^{(2)}))]-[(f(v_0^{(1)})),\dots,(f(v_n^{(1)}))]=0\end{aligned}$$

since $f(v_i^{(1)})=f(v_i^{(2)})$ for $i=1,\dots,n$. □

By Lemma 4.12 and Lemma 4.14 the sequence (2) is a chain complex. It is also a chain complex if we replace ρ_n^k by ϱ_n^k.

Lemma 4.15. *For each n, the complex $(C_n(W^\bullet(f)),\varrho_n^\bullet)$ splits as a direct sum of subcomplexes:*

$$C_n(W^\bullet(f))=\bigoplus C_n(W^\bullet(f))|_\Delta. \tag{19}$$

Here the direct sum is over ordered n-simplices Δ in Y.

Proof. We have already seen, in Lemma 4.11, that each of the $(C_n(W^\bullet(f))|_\Delta,\varrho_n^\bullet)$ *is* a subcomplex. From the definition of $C_n(W^\bullet(f))|_\Delta$ we have that if $\Delta\neq\Delta'$, then

$$C_n(W^k(f))|_\Delta\cap C_n(W^k(f))|_{\Delta'}=0.$$

□

Proposition 4.16. *For each $n\geq 0$, the complex*

$$\dots\to C_n(W^k(f))\xrightarrow{\varrho_n^k}\cdots\xrightarrow{\varrho_n^3}C_n(W^2(f))\xrightarrow{\varrho_n^2}C_n(X)\xrightarrow{f_{\#,n}}C_n(Y)\to 0$$

is exact.

Proof. We have $C_n(Y)=\bigoplus\mathbb{Z}\cdot\Delta$, where the direct sum is over ordered n-simplices of Y. We claim that for each ordered n-simplex Δ of Y, the complex

$$\dots\to C_n(W^k(f))|_\Delta\xrightarrow{\varrho_n^k}\cdots\xrightarrow{\varrho_n^3}C_n(W^2(f))|_\Delta\xrightarrow{\varrho_n^2}C_n(X)|_\Delta\xrightarrow{f_{\#,n}}\mathbb{Z}\cdot\Delta\to 0$$

is exact. Because of the splitting of $(C_n(W^\bullet(f)),\varrho_n^\bullet)$ as a direct sum of subcomplexes $(C_n(W^\bullet(f))|_\Delta,\varrho_n^\bullet)$ given by Lemma 4.15 this will prove the proposition.

Suppose $\Delta=[w_0,\dots,w_n]$. Let $\Delta'=[v_0,\dots,v_n]$ be an ordered n-simplex in X such that $f_\#(\Delta')=\Delta$ with $f(v_j)=w_j$ for $j=1,\dots,n$. Let

$$\Delta^{(k)}=\Big[(v_0^{(1)},\dots,v_0^{(k)}),\dots,(v_n^{(1)},\dots,v_n^{(k)})\Big]$$

be an ordered n-simplex in $C_n(W^k(f))|_\Delta$, that is, an ordered n-simplex in $W^k(f)$ such that $f^{(k)}(\Delta^{(k)})=\Delta$ and $f(v_j^{(i)})=w_j$ for $i=1,\dots,k$ and $j=1,\dots,n$. Define homomorphisms

$$s_k^{\Delta'}\colon C_n(W^k(f))|_\Delta\to C_n(W^{k+1}(f))|_\Delta$$

by

$$s_k^{\Delta'}([(v_0^{(1)},\dots,v_0^{(k)}),\dots,(v_n^{(1)},\dots,v_n^{(k)})])=[(v_0,v_0^{(1)},\dots,v_0^{(k)}),\dots,(v_n,v_n^{(1)},\dots,v_n^{(k)})]$$

for $k\geq 1$ and

$$\begin{gathered}s_0^{\Delta'}\colon\mathbb{Z}\cdot\Delta\to C_n(X)|_\Delta\\ \Delta=[w_0,\dots,w_n]\mapsto\Delta'=[v_0,\dots,v_n].\end{gathered}$$

Hence we have the following diagram

$$\begin{tikzcd} {} \arrow[r, "\varrho_n^{k+2}"] & C_n(W^{k+1}(f))|_\Delta \arrow[r, "\varrho_n^{k+1}"] \arrow[d, "Id"'] & C_n(W^k(f))|_\Delta \arrow[r, "\varrho_n^k"] \arrow[d, "Id"'] \arrow[dl, "s_k^{\Delta'}"'] & C_n(W^{k-1}(f))|_\Delta \arrow[r, "\varrho_n^{k-1}"] \arrow[d, "Id"'] \arrow[dl, "s_{k-1}^{\Delta'}"'] & {} \\ {} \arrow[r, "\varrho_n^{k+2}"] & C_n(W^{k+1}(f))|_\Delta \arrow[r, "\varrho_n^{k+1}"] & C_n(W^k(f))|_\Delta \arrow[r, "\varrho_n^k"] & C_n(W^{k-1}(f))|_\Delta \arrow[r, "\varrho_n^{k-1}"] & {} \end{tikzcd}$$

We have the following identities:

- $\varepsilon_{\#,n}^{1,k} \circ s_k^{\Delta'} = Id$,

$$\begin{aligned}
&\varepsilon_{\#,n}^{1,k} \circ s_k^{\Delta'}([(v_0^{(1)}, \dots, v_0^{(k)}), \dots, (v_n^{(1)}, \dots, v_n^{(k)})]) \\
&\qquad = \varepsilon_{\#,n}^{1,k}([(v_0, v_0^{(1)}, \dots, v_0^{(k)}), \dots, (v_n, v_n^{(1)}, \dots, v_n^{(k)})]) \\
&\qquad\qquad = [(v_0^{(1)}, \dots, v_0^{(k)}), \dots, (v_n^{(1)}, \dots, v_n^{(k)})].
\end{aligned}$$

- $\varepsilon_{\#,n}^{i+1,k+1} \circ s_k^{\Delta'} = s_{k-1}^{\Delta'} \circ \varepsilon_{\#,n}^{i,k}$ for $i \geq 1$,

$$\begin{aligned}
&\varepsilon_{\#,n}^{i+1,k+1} \circ s_k^{\Delta'}([(v_0^{(1)}, \dots, v_0^{(k)}), \dots, (v_n^{(1)}, \dots, v_n^{(k)})]) \\
&\qquad = \varepsilon_{\#,n}^{i+1,k+1}([(v_0, v_0^{(1)}, \dots, v_0^{(k)}), \dots, (v_n, v_n^{(1)}, \dots, v_n^{(k)})]) \\
&\qquad = [(v_0, v_0^{(1)}, \dots, \widehat{v_0^{(i)}}, \dots, v_0^{(k)}), \dots, (v_n, v_n^{(1)}, \dots, \widehat{v_n^{(i)}}, \dots, v_n^{(k)})],
\end{aligned}$$

$$\begin{aligned}
&s_{k-1}^{\Delta'} \circ \varepsilon_{\#,n}^{i,k}([(v_0^{(1)}, \dots, v_0^{(k)}), \dots, (v_n^{(1)}, \dots, v_n^{(k)})]) \\
&\qquad s_{k-1}^{\Delta'}([(v_0^{(1)}, \dots, \widehat{v_0^{(i)}}, \dots, v_0^{(k)}), \dots, (v_n^{(1)}, \dots, \widehat{v_n^{(i)}}, \dots, v_n^{(k)})]) \\
&\qquad\qquad = [(v_0, v_0^{(1)}, \dots, \widehat{v_0^{(i)}}, \dots, v_0^{(k)}), \dots, (v_n, v_n^{(1)}, \dots, \widehat{v_n^{(i)}}, \dots, v_n^{(k)})].
\end{aligned}$$

- $f_\# \circ s_0^{\Delta'} = Id$

From this, by the definition of ϱ_n^k given in (10), it follows that

$$\begin{aligned}
\varrho_n^{k+1} \circ s_k^{\Delta'} + s_{k-1}^{\Delta'} \circ \varrho_n^k &= \sum_{i=1}^{k+1} (-1)^{n+i-1} \varepsilon_{\#,n}^{i,k+1} \circ s_k^{\Delta'} + s_{k-1}^{\Delta'} \circ \sum_{i=1}^{k} (-1)^{n+i-1} \varepsilon_{\#,n}^{i,k} \\
&= \varepsilon_{\#,n}^{1,k} \circ s_k^{\Delta'} + \sum_{i=1}^{k} (-1)^{n+i} \varepsilon_{\#,n}^{i+1,k+1} \circ s_k^{\Delta'} + \sum_{i=1}^{k} (-1)^{n+i-1} s_{k-1}^{\Delta'} \circ \varepsilon_{\#,n}^{i,k} \\
&= \varepsilon_{\#,n}^{1,k} \circ s_k^{\Delta'} + \sum_{i=1}^{k} (-1)^{n+i} s_{k-1}^{\Delta'} \circ \varepsilon_{\#,n}^{i,k} + \sum_{i=1}^{k} (-1)^{n+i-1} s_{k-1}^{\Delta'} \circ \varepsilon_{\#,n}^{i,k} \\
&= Id.
\end{aligned}$$

Therefore, $s_k^{\Delta'}$ with $k \geq 0$ defines a contracting homotopy and the complex is exact. □

4.3. **The sequence $\left(C_n^{\mathrm{Alt}}(D^\bullet(f)), \epsilon^\bullet\right)$ is a resolution of $C_n(Y)$.** Now it is the turn to prove that the sequence (6) is exact.

Lemma 4.17. $\varepsilon_{\#,n}^2 \left(C_n^{\mathrm{Alt}}(D^2(f))\right) = \ker\left(f_{\#,n} \colon C_n(X) \to C_n(Y)\right)$.

Proof. We have already seen in Lemma 2.4 that

$$\varepsilon_{\#,n}^2 \left(C_n^{\mathrm{Alt}}(D^2(f))\right) \subseteq \ker\left(f_{\#,n} : C_n(X) \to C_n(Y)\right).$$

Now we prove the opposite inclusion.

Suppose that $c := \sum_i m_i\Delta_i$ is a chain in $\ker f_{\#,n}\colon C_n(X) \to C_n(Y)$:

$$\sum_i m_i f_{\#,n}(\Delta_i) = 0.$$

Write $\|c\| := \sum_i |m_i|$. We can assume that for each $i \neq j$ with $m_i \neq 0 \neq m_j$, we have $\Delta_i \neq \Delta_j$. For each simplex Δ_{i_1} with $m_{i_1} \neq 0$, let $\Delta_{i_2}, \ldots, \Delta_{i_r}$ be the simplices with non-zero coefficients in c such that $f_{\#,n}(\Delta_{i_j}) = f_{\#,n}(\Delta_{i_1})$. We can assume that $m_{i_1} > 0$. Thus $\sum_{j=1}^r m_{i_j} = 0$. Without loss of generality we can assume that $m_{i_2} < 0$. Let $\Delta_{i_1} = [v_0, \ldots, v_n]$ and $\Delta_{i_2} = [v'_0, \ldots, v'_n]$; our assumptions means that $f(v_j) = f(v'_j)$ for $j = 0, \ldots, n$. Thus, as explained in the construction of the triangulation of the $D^k(f)$ preceding Corollary 4.8, $\Delta_{i_1} \times_Y \Delta_{i_2} = [(v_0, v'_0), \ldots, (v_n, v'_n)]$ is an n-simplex in the triangulation of $D^2(f)$. Then

$$c' := \Delta_{i_1} \times_Y \Delta_{i_2} - \Delta_{i_2} \times_Y \Delta_{i_1} = [(v_0, v'_0), \ldots, (v_n, v'_n)] - [(v'_0, v_0), \ldots, (v'_n, v_n)]$$

is an alternating n-chain on $D^2(f)$ and

$$\varepsilon^2_{\#,n}(c') = \Delta_{i_1} - \Delta_{i_2}.$$

The chain $c - \varepsilon^2_{\#}(c')$ lies in $\ker f_{\#,n}$, and moreover $\|c - \varepsilon^2_{\#,n}(c')\| < \|c\|$. By induction on $\|c\|$, we conclude that

$$\ker f_{\#,n} \subset \varepsilon^2_{\#,n}\left(C_n^{\mathrm{Alt}}(D^2(f))\right).$$

Since the opposite inclusion always holds, this is an equality. □

For any simplex $\Delta^{(k)}$ in the triangulation of $D^k(f)$, let

$$\mathrm{Alt}_{\mathbb{Z}}(\Delta^{(k)}) = \sum_{\sigma \in S_k} \mathrm{sign}(\sigma)\sigma_{\#}(\Delta^{(k)}). \tag{20}$$

Clearly $\mathrm{Alt}_{\mathbb{Z}}(\Delta^{(k)})$ is an alternating chain. We use the subindex $\mathbb{Z}$ to distinguish this operator from the more familiar projection operator $\mathrm{Alt}_{\mathbb{Q}} = \frac{1}{k!}\sum_{\sigma \in S_k} \mathrm{sign}(\sigma)\sigma_{\#}$.

Lemma 4.18. *1. If the simplex $\Delta^{(k)}$ in $D^k(f)$ is invariant under some non-trivial permutation $\sigma \in S_k$, then $\Delta^{(k)}$ does not appear in any irredundant expression of any alternating chain c, and moreover $\mathrm{Alt}_{\mathbb{Z}}(\Delta^{(k)}) = 0$.*

2. Let Δ_y be an n-simplex in Y, and let $\Delta_1, \ldots, \Delta_N$ be simplices in X for which $f_{\#}(\Delta_i) = \Delta_y$, with $\Delta_i \neq \Delta_j$ for $i \neq j$. For $1 \leq i_1 < \cdots < i_k \leq N$, denote the sequence $i_1, \ldots, i_k$ by I, and let $\Delta_I^{(k)}$ be the n-simplex $(\Delta_{i_1} \times \cdots \times \Delta_{i_k})/Y$ in $D^k(f)$. Then $C_n^{\mathrm{Alt}}(D^k(f))|_{\Delta_y}$ is freely generated over $\mathbb{Z}$ by the alternating chains $\mathrm{Alt}_{\mathbb{Z}}(\Delta_I^{(k)})$ for such I.

Proof. 1. If $\Delta^{(k)}$ is invariant under some non-trivial permutation $\sigma \in S_k$, pick $i \neq j$ such that $\sigma(i) = j$. The fact that $\Delta^{(k)}$ is invariant under σ means that $|\Delta^{(k)}|$ is contained in the set $\{x_i = x_j\} \subset X^k$, and hence $\Delta^{(k)}$ is invariant under (i,j). Suppose that the coefficient of $\Delta^{(k)}$ in some alternating chain c on $D^k(f)$ is m. Because (i,j) leaves $\Delta^{(k)}$ fixed, its coefficient in $(i,j)_{\#}(c)$ is also m. But because c is alternating and $\mathrm{sign}(i,j) = -1$, we must have $m = -m$. Thus $m = 0$.

If $\Delta^{(k)}$ is invariant under some non-trivial permutation, so is $\sigma_{\#}(\Delta^{(k)})$ for each $\sigma \in S_k$. It follows that $\mathrm{Alt}(\Delta_i^{(k)}) = 0$.

2. By part 1 of the lemma, every simplex $\Delta^{(k)}$ appearing non-trivially in an alternating chain is in the S_k orbit of some simplex $\Delta_I^{(k)}$ with I as described. The argument just given shows that for any such simplex $\Delta_I^{(k)}$, the simplices $\sigma_{\#}(\Delta_I^{(k)}), \sigma \in S_k$, are pairwise distinct, and thus that the coefficient of $\Delta_I^{(k)}$ in $\mathrm{Alt}_{\mathbb{Z}}(\Delta_I^{(k)})$ is equal to 1. Now let c be an alternating simplicial n-chain on $D^k(f)$. We claim that c is a linear combination of chains $\mathrm{Alt}(\Delta_I^{(k)})$. To see this, suppose that $m \neq 0$ is the coefficient of $\Delta_I^{(k)}$ in an irredundant expression of c as linear combination of

ordered simplices. Let $\|c\|$ be the sum of the absolute values of the coefficients of the simplices in such an expression. Because c is alternating, for each $\sigma \in S_k$, the coefficient of $\sigma_\#(\Delta_I^{(k)})$ is $\mathrm{sign}(\sigma)m$. It follows that

$$\|c - m\mathrm{Alt}_{\mathbb{Z}}(\Delta_I^{(k)})\| < \|c\|.$$

Hence, by induction on $\|c\|$, c is a linear combination of alternating chains $\mathrm{Alt}_{\mathbb{Z}}(\Delta_I^{(k)})$.

That there are no relations among distinct chains $\mathrm{Alt}_{\mathbb{Z}}\Delta(k)_I$ is obvious: if $I = \{i_1, \dots, i_k\}$ and $I' \neq I$ then the simplex $(\Delta_{i_1} \times \cdots \times \Delta_{i_k})/Y$ does not appear in $\mathrm{Alt}_{\mathbb{Z}}\Delta_{I'}^{(k)}$. □

Remark 4.19. The lemma shows that $C_n^{\mathrm{Alt}}(D^k(f))$ is equal to the image of the "Alternation Operator" $\mathrm{Alt}_{\mathbb{Z}} : C_n(D^k(f)) \to C_n^{\mathrm{Alt}}(D^k(f))$. We learned this from Kevin Houston's paper [Hou97]. This is a special feature of this S_k-action. In contrast, consider the action of S_3 on $\mathbb{R}\times\mathbb{R}^3$ defined by $\sigma(t, x_1, x_2, x_3) = ((\mathrm{sign}\ \sigma)t, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$. The point $(1, x, x, x)$ is then invariant only under even permutations. If we apply $\mathrm{Alt}_{\mathbb{Z}}$ to $(1, x, x, x)$, we get $3((1, x, x, x)-(-1, x, x, x))$. The alternating 0-chain $(1, x, x, x) - (-1, x, x, x)$ is not in the image of $\mathrm{Alt}_{\mathbb{Z}}$.

Lemma 4.20. *For each n, the complex $(C_n^{\mathrm{Alt}}(D^\bullet(f)), \epsilon_n^\bullet)$ splits as a direct sum of complexes:*

$$C_n^{\mathrm{Alt}}(D^\bullet(f)) = \bigoplus C_n^{\mathrm{Alt}}(D^\bullet(f))|_\Delta. \tag{21}$$

Here the direct sum is over ordered n-simplices Δ in Y.

Proof. We have already seen, in Lemma 4.11, that each of the $(C_n^{\mathrm{Alt}}(D^\bullet(f))|_\Delta, \epsilon_n^\bullet)$ *is* a subcomplex. If $\Delta \neq \Delta'$, then

$$C_n^{\mathrm{Alt}}(D^k(f))|_\Delta \cap C_n^{\mathrm{Alt}}(D^k(f))|_{\Delta'} = 0.$$

The splitting (21) now follows from Lemma 4.18. □

Now we prove the main theorem.

Proposition 4.21. *For each $n \geq 0$, the complex of alternating simplicial chains*

$$\dots \to C_n^{\mathrm{Alt}}(D^k(f)) \xrightarrow{\epsilon_n^k} \cdots \xrightarrow{\epsilon_n^3} C_n^{\mathrm{Alt}}(D^2(f)) \xrightarrow{\epsilon_n^2} C_n(X) \xrightarrow{f_{\#,n}} C_n(Y) \to 0,$$

is exact.

Proof. We have $C_n(Y) = \bigoplus \mathbb{Z}\cdot\Delta$, where the direct sum is over n-simplices of Y. We claim that for each n-simplex Δ of Y, the complex

$$\dots \to C_n^{\mathrm{Alt}}(D^k(f))|_\Delta \xrightarrow{\epsilon_n^k} \cdots \xrightarrow{\epsilon_n^3} C_n^{\mathrm{Alt}}(D^2(f))|_\Delta \xrightarrow{\epsilon_n^2} C_n(X)|_\Delta \xrightarrow{f_{\#,n}} \mathbb{Z}\cdot\Delta \to 0 \tag{22}$$

is exact. Because of the splitting of $(C_n^{\mathrm{Alt}}(D^\bullet(f)), \epsilon^\bullet)$ as a direct sum of subcomplexes $(C_n^{\mathrm{Alt}}(D^\bullet(f))|_\Delta, \epsilon_n^\bullet)$, this will prove the proposition. Let $\Delta_1, \dots, \Delta_N$ be the distinct simplices Δ' of X such that $f_\#(\Delta') = \Delta$. To prove exactness of (22), we show it is chain-isomorphic to a well-known exact complex, namely the augmented oriented simplicial chain complex of the standard $N-1$-simplex $\mathbf{\Delta^{N-1}} := [e_1, \dots, e_N]$, with degree shifted by 1, $C_\bullet^{\mathrm{or}}(\mathbf{\Delta^{N-1}})[1]$,

$$\cdots \xrightarrow{\partial} C_{k-1}^{\mathrm{or}}(\mathbf{\Delta^{N-1}}) \cdots \xrightarrow{\partial} C_1^{\mathrm{or}}(\mathbf{\Delta^{N-1}}) \xrightarrow{\partial} C_0^{\mathrm{or}}(\mathbf{\Delta^{N-1}}) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0 \tag{23}$$

with ∂ the usual simplicial boundary map: for a $k-1$ face $[e_{i_1}, \dots, e_{i_k}]$ of $\mathbf{\Delta^{N-1}}$,

$$\partial([e_{i_1}, \dots, e_{i_k}]) = \sum_{j=1}^{k} (-1)^{j-1} [e_{i_1}, \dots, \widehat{e_{l_j}}, \dots, e_{i_k}],$$

and $\varepsilon(\sum_i m_i e_i) = \sum_i m_i$. In fact to obtain a *bona fide* chain isomorphism, we would have to modify the sign of the boundary operators in (23). Since our aim is simply to prove exactness of (22), it is enough instead to construct an isomorphism $\varphi_k : C_n^{\mathrm{Alt}}(D^k(f))|_\Delta \to C_{k-1}^{\mathrm{or}}(\mathbf{\Delta^{N-1}})$ for each k such that the following diagram

$$\begin{tikzcd} C_n^{\mathrm{Alt}}(D^k(f)) \arrow[r, "\epsilon_n^k"] \arrow[d, "\varphi_k"'] & C_n^{\mathrm{Alt}}(D^k(f)) \arrow[d, "\varphi_{k-1}"'] \\ C_{k-1}^{\mathrm{or}}(\mathbf{\Delta^{N-1}}) \arrow[r, "\partial"] & C_{k-2}^{\mathrm{or}}(\mathbf{\Delta^{N-1}}) \end{tikzcd}$$

is almost commutative, that is

$$\varphi_{k-1} \circ \epsilon_n^k = (-1)^{k+n-1} \partial \circ \varphi_k. \tag{24}$$

Recall from Lemma 4.18 that $C_n^{\mathrm{Alt}}(D^k(f))|_\Delta$ is freely generated by the n-chains

$$\mathrm{Alt}_{\mathbb{Z}}(\Delta_{i_1} \times \cdots \times \Delta_{i_k})/Y,$$

where $1 \le i_1 < \cdots < i_k \le N$. When $k = 1$, this of course just reduces to the single simplex Δ_{i_1}. We map each such generator to the oriented $k-1$ face $[e_{i_1}, \ldots, e_{i_k}]$ of $\mathbf{\Delta^{N-1}}$, obtaining in this way an isomorphism

$$\varphi_k : C_n^{\mathrm{Alt}}(D^k(f))|_\Delta \to C_{k-1}^{\mathrm{or}}(\mathbf{\Delta^{N-1}}).$$

And we define $\varphi_0(m\Delta) = m$. To prove (24), recall first that $\epsilon_n^k = (-1)^n \varepsilon_{\#,n}^k$. Since $\varepsilon_{\#,n}^k$ maps alternating chains to alternating chains, so does ϵ_n^k; moreover since

$$\varepsilon_{\#,n}^k((\Delta_{i_1} \times \cdots \times \Delta_{i_k})/Y) = (\Delta_{i_1} \times \cdots \times \Delta_{i_{k-1}})/Y,$$

it follows that $\epsilon_n^k(\mathrm{Alt}_{\mathbb{Z}}(\Delta_{i_1} \times \cdots \times \Delta_{i_k})/Y)$ must be a linear combination of chains

$$\mathrm{Alt}_{\mathbb{Z}}((\Delta_{i_1} \times \cdots \times \widehat{\Delta_{i_\ell}} \times \cdots \times \Delta_{i_k})/Y)$$

with $1 \le \ell \le k$, and it is necessary only to determine the coefficient of each. The simplex

$$(\Delta_{i_1} \times \cdots \times \widehat{\Delta_{i_\ell}} \times \cdots \times \Delta_{i_k})/Y$$

itself appears only once in the expression of

$$\varepsilon_{\#,n}^k(\mathrm{Alt}_{\mathbb{Z}}((\Delta_{i_1} \times \cdots \times \Delta_{i_k})/Y)$$

as a linear combination of simplices, namely as

$$\varepsilon_{\#,n}^k \mathrm{sign}(\sigma_{k,\ell})(\sigma_{k,\ell})_\#((\Delta_{i_1} \times \cdots \times \Delta_{i_k})/Y),$$

where $\sigma_{k,\ell}$ is the permutation taking the ordered set $(0, \ldots, k)$ to $(0, \ldots, \hat{\ell}, \ldots, k, \ell)$. This permutation has sign $(-1)^{k-\ell}$. It follows that this, multiplied by $(-1)^n$, is the coefficient of $\mathrm{Alt}_{\mathbb{Z}}((\Delta_{i_1} \times \cdots \times \widehat{\Delta_{i_\ell}} \times \cdots \times \Delta_{i_k})/Y)$ in $\epsilon^k(\mathrm{Alt}_{\mathbb{Z}}(\Delta_{i_1} \times \cdots \times \Delta_{i_k})/Y)$. The coefficient of $[e_{i_1}, \ldots, \widehat{e_{i_\ell}}, \ldots, e_{i_k}]$ in $\partial\varphi_k(\mathrm{Alt}_{\mathbb{Z}}((\Delta_{i_1} \times \cdots \times \Delta_{i_k})/Y)$ is $(-1)^{\ell-1}$, thus showing that (24) holds, and completing the proof. □

Putting together Proposition 4.9, Proposition 4.16 and Proposition 4.21 we obtain the proof of Theorem 2.5, completing the construction of the GVZSS and the ICSS.

4.4. **Comparison with other proofs.** The first construction of the (cohomological) ICSS, for rational cohomology only, in [GM93], was based on the exactness of the sequence of locally constant sheaves

$$0 \longrightarrow \mathbb{Q}_Y \longrightarrow f_*(\mathbb{Q}_X) \longrightarrow f_*^{(2)}(\mathrm{Alt}\mathbb{Q}_{D^2(f)}) \longrightarrow f_*^{(3)}(\mathrm{Alt}\mathbb{Q}_{D^3(f)}) \longrightarrow \cdots.$$

Exactness was proved by the same argument as used here in Proposition 4.21. Goryunov in [Gor95] used a geometric realisation Y_G of the semi-simple object $D^\bullet(f)$ due to Vassiliev and described in [Vas01]: it is formed by beginning with X embedded in some high-dimensional

Euclidean space, and then adding, for each pair of points x_1, x_2 with $f(x_1) = f(x_2)$, the 1-simplex $[x_1, x_2]$, for each triple x_1, x_2, x_3 with $f(x_1) = f(x_2) = f(x_3)$ the 2-simplex $[x_1, x_2, x_3]$, etc. The embedding of X in $\mathbb{R}^N$ is chosen so general that none of these added simplices meet one another except by the standard face inclusions. There is then a natural surjection $Y_G \to Y$, and for each $y \in Y$ the preimage in Y_G is a simplex. From this it follows that $Y_G \to Y$ is a homotopy equivalence, so that the homology of Y may be computed as the homology of Y_G. Goryunov showed in [Gor95] that the alternating homology $AH_q(D^k(f))$ was isomorphic to the relative homology $H_{q+k-1}(Y_G^k, Y_G^{k-1})$, where Y_G^k is X together with the added simplices of dimension $\leq k$, and the ICSS (14) is thus reduced to the spectral sequence for the homology of the filtered space Y_G.

Our argument here is clearly closely related to Goryunov's proof. It seems likely that the total complex of the double complex $(C_\bullet^{\mathrm{Alt}}(D^\bullet(f)), \partial, \epsilon^\bullet)$ is isomorphic to the simplicial chain complex of some triangulation of Y_G. It would be interesting to prove this.

In [Hou07] Houston generalises the ICSS and in particular also takes into account an action of another group.

The construction of the GVZSS in [GVZ04] for a closed map $f\colon X \to Y$ uses the fibred join $X *_Y X$ to define the *join space* $J^f(X)$ as the quotient space of the disjoint union of spaces

$$J_p^f(X) = \underbrace{X *_Y \cdots *_Y X}_{p+1 \text{ times}}, \quad p = 0, 1, \ldots$$

identifying $J_{p-1}^f(X)$ with each of its images $\phi_i(J_{p-1}^f(X))$ in $J_p^f(X)$ for $i = 0, \ldots, p$, where ϕ_i is the natural embedding which does not have the i'th copy of X in its image. Hence, there is a natural filtration of the join space $J^f(X)$ given by $\tilde{J}_0^f(X) \subset \tilde{J}_1^f(X) \subset \cdots \subset \tilde{J}_p^f(X) \subset \cdots \subset \tilde{J}^f(X)$ where $\tilde{J}_p^f(X)$ is the image of $J_p^f(X)$ in $J^f(X)$ under the quotient. The authors proved that the quotient space $\tilde{J}_p^f(X)/\tilde{J}_{p-1}^f(X)$ is homotopically equivalent to the p'th suspension $S^p(W^{p+1}(f))$ of $W^{p+1}(f)$. There is a natural map $F\colon J^f(X) \to Y$ induced by f. It is proved that the fibres of F are homologically trivial and by the Vietoris-Begle Theorem $H_*(J^f(X)) \cong H_*(Y)$. The GVZSS (13) follows from the spectral sequence associated to the filtered space $J^f(X)$. Since this construction uses the homotopy equivalence $\tilde{J}_p^f(X)/\tilde{J}_{p-1}^f(X) \simeq S^p(W^{p+1}(f))$ and the isomorphism $\tilde{H}_{p+q}(S^p(W^{p+1})) \cong \tilde{H}_q(W^{p+1})$ it is very difficult to give the differentials even of the first page of the spectral sequence. In contrast, our construction gives these differentials explicitly.

Remark 4.22 (Simplicial vs Singular)**.** Our theorem relates the alternating simplicial homology of the $D^k(f)$ to the simplicial homology of the image, Y. It is well known that singular and simplicial homology coincide. In fact a variant of the standard proof shows that the same goes for alternating simplicial homology and alternating singular homology.

5. Cohomology

The complex $\mathrm{Hom}_{\mathbb{Z}}(C_n^{\mathrm{Alt}}(D^{\bullet+1}(f)), \mathbb{Z})$ is exact, since $C_n^{\mathrm{Alt}}(D^{\bullet+1}(f))$ is a resolution of a free $\mathbb{Z}$-module, $C_n(Y)$. This shows that there is an ICSS also for the cohomology of the image, if we define alternating cohomology to be the homology of the complex $\mathrm{Hom}_{\mathbb{Z}}(C_\bullet^{\mathrm{Alt}}(D^k(f)), \mathbb{Z})$.

An apparently different approach is to define alternating cochains by analogy with the definition of alternating chains:

$$C^n_{\mathrm{Alt}}(D^k(f)) := \{\psi \in C^n(D^k(f)) : \sigma^\#(\psi) = \mathrm{sign}(\sigma)\psi \text{ for all } \sigma \in S_k\}.$$

In fact, thanks to Lemma 4.18, the two approaches are equivalent.

Given $\psi \in \operatorname{Hom}(C_n^{\mathrm{Alt}}(D^k(f)),\mathbb{Z})$, define a cochain $\mathrm{Alt}_{\mathbb{Z}}^*\psi \in C^n(D^k(f))$ by composing with $\mathrm{Alt}_{\mathbb{Z}} : C_n(D^k(f)) \to C_n^{\mathrm{Alt}}(D^k(f))$. If $\sigma \in S_k$ we check that for any chain $c \in C_n(D^k(f))$,

$$\begin{aligned}\mathrm{Alt}_{\mathbb{Z}}^*\psi(\sigma_\# c) = \psi(\mathrm{Alt}_{\mathbb{Z}}(\sigma_\#(c)) &= \psi\left(\sum_{\tau\in S_k}\operatorname{sign}(\tau)\tau_\#\sigma_\#(c)\right)\\ &= \operatorname{sign}(\sigma)\psi\left(\sum_{\tau\in S_k}\operatorname{sign}(\tau)\operatorname{sign}(\sigma)(\tau\circ\sigma)_\#(c)\right)\\ &= \operatorname{sign}(\sigma)(\psi)(\mathrm{Alt}_{\mathbb{Z}}(c)) = \operatorname{sign}(\sigma)\mathrm{Alt}_{\mathbb{Z}}^*\psi(c).\end{aligned}$$

Thus $\mathrm{Alt}_{\mathbb{Z}}^*\psi \in C^n_{\mathrm{Alt}}(D^k(f))$. In fact $\mathrm{Alt}_{\mathbb{Z}}^*$ gives a homomorphism of cochain complexes

$$\operatorname{Hom}_{\mathbb{Z}}(C_\bullet^{\mathrm{Alt}}(D^k(f)),\mathbb{Z}) \to C^\bullet_{\mathrm{Alt}}(D^k(f)),$$

and thus a homomorphism of cohomology groups.

It has an inverse $\theta : C^\bullet_{\mathrm{Alt}}(D^k(f)) \to \operatorname{Hom}_{\mathbb{Z}}(C_\bullet^{\mathrm{Alt}}(D^k(f)),\mathbb{Z})$, defined as follows. Suppose that $\varphi \in C^n_{\mathrm{Alt}}(D^k(f))$ and $c \in C_n^{\mathrm{Alt}}(D^k(f))$. We can write $c = \sum_i m_i\mathrm{Alt}_{\mathbb{Z}}(c_i)$, where each c_i is a simplex. This representation is not unique, because for any $\sigma_i \in S_k$, the alternating chain $\mathrm{Alt}_{\mathbb{Z}}(c_i)$ can also be written as $\operatorname{sign}(\sigma_i)\mathrm{Alt}_{\mathbb{Z}}(\sigma_{i\#}(c_i))$. However, because φ is alternating, we have

$$\sum_i m_i\varphi(c_i) = \sum_i \varphi\big(m_i\operatorname{sign}(\sigma_i)\sigma_{i\#}(c_i)\big),$$

so that $\theta(\varphi)$ is well-defined by the formula

$$\theta(\varphi)\Big(\sum_i m_i\mathrm{Alt}_{\mathbb{Z}}(c_i)\Big) = \sum_i m_i\varphi(c_i).$$

Once again, θ is a homomorphism of cochain complexes.

Since, for any simplex c,

$$\mathrm{Alt}_{\mathbb{Z}}^*(\theta(\varphi))(c) = \theta(\varphi)(\mathrm{Alt}_{\mathbb{Z}}(c)) = \varphi(c),$$

and for any alternating chain $c = \sum_i m_i\mathrm{Alt}_{\mathbb{Z}}(c_i)$,

$$\theta(\mathrm{Alt}_{\mathbb{Z}}^*(\psi))(c) = \sum_i m_i\mathrm{Alt}_{\mathbb{Z}}^*(\psi)(c_i) = \sum_i m_i\psi(\mathrm{Alt}_{\mathbb{Z}}(c_i)) = \psi(c),$$

θ and $\mathrm{Alt}_{\mathbb{Z}}^*$ are mutually inverse. We conclude that

$$H^*\left(\operatorname{Hom}_{\mathbb{Z}}(C_\bullet^{\mathrm{Alt}}(D^k(f)),\mathbb{Z})\right) \simeq H^*(C^\bullet_{\mathrm{Alt}}(D^k(f)).$$

6. Applications in Singularity Theory

In this section we briefly survey some applications of the ICSS in singularity theory. First, finite analytic and subanalytic maps can all be triangulated, thanks to the following theorem of Hardt (Theorem 3.1 in [Har77]).

Theorem 6.1. *1. Let $P \subset \mathbb{R}^n$ be a closed finite-dimensional sub-analytic set and $N \subset \mathbb{R}^m$ a real analytic space, and let $f\colon P \to N$ be a proper light subanalytic map. Then there exist simplicial complexes $\mathscr{G}$ and $\mathscr{H}$ and homeomorphisms $g\colon \mathscr{G} \to P$ and $h\colon \mathscr{H} \to f(P)$, and a simplicial map $p\colon \mathscr{G} \to \mathscr{H}$ such that $f = h\circ p\circ g^{-1}$.*

2. Suppose, in the same situation, that $\mathscr{Q}$ and $\mathscr{R}$ are locally finite families of closed subanalytic sets in P and $f(P)$, respectively. Then the simplicial complexes $\mathscr{G}$ and $\mathscr{H}$, and the homeomorphisms g and h, can all be chosen so that $g^{-1}(Q)$ is a subcomplex of $\mathscr{G}$ and $h^{-1}(R)$ is a subcomplex of $\mathscr{H}$ for all $Q \in \mathscr{Q}$ and $R \in \mathscr{R}$.

A map is *light* if the preimage of a discrete set is discrete. Every real analytic space (and hence every complex analytic space) is subanalytic, and the same goes for analytic maps. Thus, our construction of the ICSS applies, for example, to a stable perturbation f_t of an $\mathscr{A}$-finite map-germs $(\mathbb{F}^n, S) \to (\mathbb{F}^p, 0)$ (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) when $n \leq p$, and, when $n \geq p$, to the restriction to the critical space of a stable perturbation of an $\mathscr{A}$-finite germ. The ICSS calculates the homology of the image in the former case, and of the discriminant (set of critical values) in the latter. The ICSS was first developed in this context.

If f_0 is a map-germ $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ with $n < p$, of corank ≤ 1 (i.e., $\dim \ker df_0 = 1$) then $D^k(f_0)$ is an isolated complete intersection singularity (ICIS) provided its expected dimension, $n - (k-1)(p-n)$, is non-negative, see [MM89]. If f_t is a stable perturbation of f_0 then each $D^k(f_t)$ is smooth, and of expected dimension. It is thus a Milnor fibre of an ICIS, and hence, crucially, has reduced homology only in middle dimension (i.e., $n-(k-1)(p-n)$), see [Ham71] (or [Loo84] for an account in English). In [Gor95, §2], Goryunov shows that the homology $AH_*(X)$ of the alternating chain complex on the Milnor fibre X of an S_k-invariant ICIS coincides with the alternating part of its regular homology, $H_*^{\mathrm{Alt}}(X)$, and thus it too is concentrated in middle dimension. From this it follows that the ICSS collapses at E^1, and the homology of the image Y_t can be read off from it. To simplify the resulting formula, let $r := p - n$. Then if, first, $r > 1$, we have

$$H_q(Y_t) = \begin{cases} \mathbb{Z} & \text{if } q = 0, \\ H^{\mathrm{Alt}}_{q-k+1}(D^k(f_t)) & \text{if } q = n - (k-1)(r-1) \geq 0, \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

If $r = 1$, then all the multiple point spaces contribute to the n'th homology of the image, and the graded module arising from the resulting filtration on $H_n(Y_t)$ satisfies

$$\mathrm{Gr} H_n(Y_t) \simeq \bigoplus_{k=2}^{n+1} H^{\mathrm{Alt}}_{q-r(k-1)}(D^k(f_t)). \tag{26}$$

It is known, by a Morse-theoretic argument, that in this case the image has the homotopy type of a wedge of n-spheres.

In [Hou97, Theorem 4.6], Kevin Houston proved the remarkable theorem that even without the hypothesis on the corank of f, $AH_*(D^k(f_t))$ is concentrated in middle dimension, and is free abelian, so that the ICSS collapses at E^1 and the formulae (25) and (26), with AH_* replacing H_*^{Alt}, continue to hold. This is all the more remarkable because when f_0 has corank > 1, the multiple point spaces $D^k(f_0)$ are no longer complete intersections, and, indeed, examples (see e.g. [Mon18]) show that their homology is not necessarily concentrated in middle dimension. It is only the alternating homology that has this pleasant property. Houston has to replace $H_*^{\mathrm{Alt}}(D^k(f_0))$ by $AH_*(D^k(f_0))$ since it is not known whether the two coincide when $D^k(f_0)$ is not an ICIS. Note, however, that for rational homology, the two always coincide. An example (not from singularity theory) where they do not coincide is given by the quotient map $q : B^2 \to \mathbb{RP}^2$, where B^2 is the closed unit disc in $\mathbb{R}^2$ and the map q identifies antipodal points on the boundary S^1. Here $D^2(q)$ is isomorphic to S^1 itself, and its involution is identified with the antipodal map. Evidently $H_0^{\mathrm{Alt}}(D^2(q))$ is trivial, but one calculates that $AH_0(D^2(q)) \simeq \mathbb{Z}/2\mathbb{Z}$.

Section 3 of [GM93] gives an expression for the rank of the alternating cohomology of the spaces $D^k(f)$ in terms of the Milnor number of the fixed point sets of permutations $\sigma \in S_k$. These have recently been extended by Giménez Conejero in the preprint [GimCon22].

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