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HOMOTOPICAL CANCELLATION THEORY FOR GUTIERREZ-SOTOMAYOR SINGULAR FLOWS

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Abstract. In this article, we present a dynamical homotopical cancellation theory for Gutierrez-Sotomayor singular flows φ, GS-flows, on singular surfaces M. This theory generalizes the classical theory of Morse complexes of smooth dynamical systems together with the corresponding cancellation theory for non-degenerate singularities. This is accomplished by defining a GS-chain complex for (M,φ) and computing its spectral sequence (E^r,d^r). As r increases, algebraic cancellations occur, causing modules in E^r to become trivial. The main theorems herein relate these algebraic cancellations within the spectral sequence to a family $\{M_r,\varphi_r\}$ of GS-flows φ_r on singular surfaces M_r, all of which have the same homotopy type as M. The surprising element in these results is that the dynamical homotopical cancellation of GS-singularities of the flows φ_r are in consonance with the algebraic cancellation of the modules in E^r of its associated spectral sequence. Also, the convergence of the spectral sequence corresponds to a GS-flow $\varphi_{\bar{r}}$ on $M_{\bar{r}}$, for some $\bar{r}$, with the property that $\varphi_{\bar{r}}$ admits no further dynamical homotopical cancellation of GS-singularities.

1. Introduction

The qualitative study of vector fields on smooth manifolds M^n via algebraic and differential topological tools has its origin in the foundational work of Poincaré and subsequently major contributions were made by Morse, Peixoto and Smale. For a historical overview see [19]. For more references see [16–18].

If X is a Morse-Smale vector field on M^n with no periodic orbits, then any hyperbolic rest point p of index k can be written in standard form, on a neighborhood of p with local coordinates $(x_1,\dots,x_n)$, such that

$$X = -\sum_{i=1}^{k} x_i\frac{\partial}{\partial x_i} + \sum_{i=k+1}^{n} x_i\frac{\partial}{\partial x_i}.$$

A Morse Smale flow φ_X with hyperbolic singularities and no periodic orbits is generated by X.

Smale proves a local cancellation theorem for hyperbolic singularities of consecutive indices, p and q, on M^n, where $n>5$ or $n=2$. Roughly speaking, whenever the algebraic intersection number of p and q is ± 1, this pair of singularities is cancelled, in a neighborhood U, and produces a flow φ' which coincides with φ_X outside of U and φ' restricted to U is topologically equivalent to a tubular flow. See [12].

In [2,3], global cancellation theorems were obtained for Morse-Smale flows with no periodic orbits on smooth manifolds M^n, $n>5$ and $n=2$, by considering a spectral sequence analysis of

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the associated Morse chain complex. Our objective in this work is to explore these techniques in a new context, namely Gutierrez-Sotomayor singular flows.

In [9], Gutierrez and Sotomayor presented cone ($\mathcal{C}$), Whitney ($\mathcal{W}$), double crossing ($\mathcal{D}$) and triple crossing ($\mathcal{T}$) singularities, which they called *simple*, for C^k vector fields tangent to a 2-dimensional compact subset M of $\mathbb{R}^k$, as well as, their characterization and genericity theorems for C^1-structurally stable vector fields.

For the first time, in [15], the flows associated to these vector fields, with no periodic orbits or limit cycles, were studied using Conley index theory and named *Gutierrez-Sotomayor flows*, GS-flows for short. Also regular hyperbolic ($\mathcal{R}$), as well as, $\mathcal{C}$, $\mathcal{W}$, $\mathcal{D}$ and $\mathcal{T}$ singularities are referred to as *GS-singularities* in [15]. Furthermore, the Conley index of each GS-singularity was computed. The existence of Lyapunov functions for GS-flows was established and a GS-handle theory was introduced in order to construct isolating blocks for each GS-singularity.

In this work, we take this analysis a step further, by analyzing global GS-flows on singular closed surfaces. Our goal is to investigate the connections of flow lines of GS-flows under a spectral sequence analysis of a chain complex associated to it. This method was successful in [2, 3, 10] in order to obtain cancellation theorems in smooth settings, for gradient flows of Morse functions, as well as, for circle-valued Morse functions.

However, it is a great challenge to adapt the smooth theory to the singular setting, more specifically for GS-flows. One wishes to maintain the principles that undergird the former setting in the latter. In order for the theory to retain its basic structure and be a valid generalization, the definitions and postulates of the singular setting must encompass the definitions and postulates of the smooth setting. Hence, one must face the problem of defining intersection numbers in the absence of differentiability, as well as, defining a chain complex generated by GS-singularities.

Furthermore, a generalized notion of cancellation must be presented for GS-singularities. This will be captured by defining a dynamical homotopical cancellation which is a generalization of the classical notion of cancellation in the smooth case as Figure 1 suggests. In a classical cancellation, the manifold before and after the cancellation are always homeomorphic. In a homotopical cancellation, the singular manifold before and after the cancellation are of the same homotopy type and may not be homeomorphic. Roughly, the idea behind a dynamical homotopical cancellation is to consider a set of three singularities x, x' and y and the flow lines u, u' joining them in a neighborhood U which, through a homotopy will be taken to a neighborhood $\bar{U}$ containing a GS-singularity $\overline{x}'$. The regions U and $\bar{U}$ are of the same homotopy type and this homotopy respects the number of singular regions (droplets and folds) that exist in U.

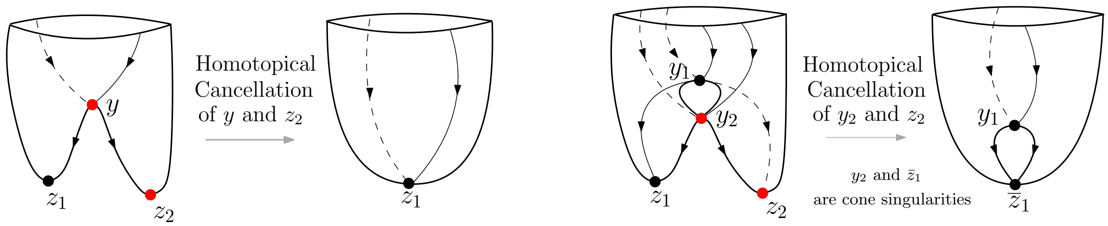


FIGURE 1. Dynamical homotopical cancellations on a smooth (left) and on a singular (right) manifold.

In order for these homotopies to be well defined, we will consider a larger class of GS-singularities which include n-sheet cone, Whitney, double and triple attractors and repellers. For simplicity, henceforth, we will continue to refer to these as GS-singularities for GS-flows.

Since these more general GS-flows have not been previously considered in the literature, these fundamental concepts have to be established herein in order to get the theory off the ground. This in itself is already quite a formidable endeavor, since the GS-singularities comprise a large class of different singularity types, which must be dealt with in a case by case analysis.

The main contribution of this work is that, with the introduction of a generalization of these concepts, several homotopical cancellation theorems for GS-flows are proven. In our opinion, what is most striking in these theorems, is that the dynamical homotopical cancellations within the flow occur in consonance with the algebraic cancellations of the unfolding, i.e., with the turning of the pages, of the associated spectral sequence. In order to appreciate the beauty of these results, we finalize this paper with three examples from the realms of flows with cone, Whitney and double crossing singularities. See Section 6.

This paper is organized as follows. Section 2 is an introduction to GS-vector fields and their associated flows. In order to define a Gutierrez-Sotomayor chain complex, we need to establish a regularization process of GS-singularities, referred to as its Morsification, which is presented in Section 3. In Section 4, we make use of the Morsification process to introduce GS-intersection numbers and hence obtain a differential for a chain complex generated by the GS-singularities, which we refer to as a GS-chain complex. Next, we prove local dynamical homotopical cancellation theorems for GS-singularities in Section 5. Moreover, in Section 6, we generalize the theory developed in [2, 3] to obtain global homotopical cancellation theorems for flows on singular surfaces with $\mathcal{R}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{W}$ or $\mathcal{T}$ singularities. This is accomplished by associating the algebraic cancellations that occur in a spectral sequence of a filtered GS-chain complex of a GS-flow with the dynamical homotopical cancellations that occur within the flow. The flow chart in Figure 2 provides an overview of the development of the results, and helps to understand the interrelationships among the sections.

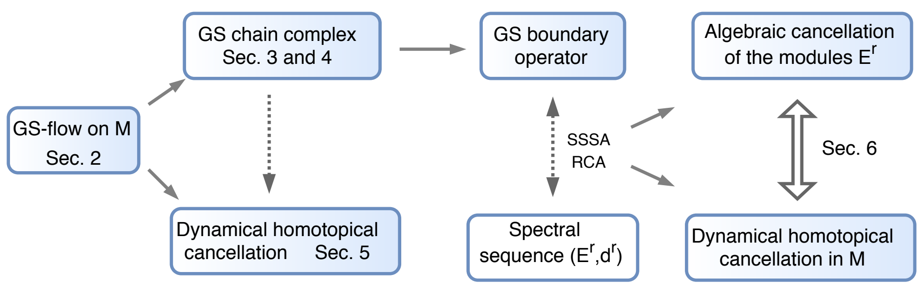


FIGURE 2. Flow chart: an overview of the development of the results herein.

2. Gutierrez-Sotomayor Flows

2.1. **Gutierrez-Sotomayor Vector Fields.** In [9], Gutierrez and Sotomayor presented a characterization for manifolds with singularities where the degeneracy is restricted in order to admit only those that appear in a stable manner. This means that the regularity conditions in the definition of the smooth surfaces in $\mathbb{R}^3$, given in terms of implicit functions and immersions, are broken stably, giving rise to cone, Whitney's umbrella, double crossing and triple crossing singularities. The authors refer to these singularities as *simple*.[1] Hence, in [9] cite the following definition of a two-dimensional manifold with simple singularities is given in terms of local charts.

Definition 2.1. *A subset $M \subset \mathbb{R}^l$ is called a two-dimensional* ***manifold with simple singularities*** *if for every point $p \in M$ there are a neighbourhood V_p of p in M and a local chart, a C^∞-diffeomorphism $\Psi : V_p \to \mathcal{G}$ such that $\Psi(p) = 0$, where $\mathcal{G}$ is one of the following subsets of $\mathbb{R}^3$:*
$\mathcal{R} = \{(x, y, z); z = 0\}$, *plane;*
$\mathcal{C} = \{(x, y, z); z^2 - y^2 - x^2 = 0\}$, *cone;*
$\mathcal{W} = \{(x, y, z); zx^2 - y^2 = 0\}$, *Whitney's umbrella*[2]*;*
$\mathcal{D} = \{(x, y, z); xy = 0\}$, *double crossing;*
$\mathcal{T} = \{(x, y, z); xyz = 0\}$, *triple crossing.*

We denote by $M(\mathcal{G})$ the set of points $p \in M$ such that $\Psi(p) = 0$ for a local chart $\Psi : V_p \to \mathcal{G}$, where $\mathcal{G} = \mathcal{R}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{W}$ or $\mathcal{T}$. Thus $M(\mathcal{R})$ is a smooth two-dimensional manifold called the ***regular part*** of M, $M(\mathcal{D})$ is a one-dimensional smooth manifold, while $M(\mathcal{C})$, $M(\mathcal{W})$ and $M(\mathcal{T})$ are discrete sets. Also, the set M endowed with the partition $\{M(\mathcal{G}), \mathcal{G}\}$ is a stratified set in the sense of Thom.

A vector field X of class C^r on $\mathbb{R}^l$ is said to be *tangent* to a manifold $M \subset \mathbb{R}^l$ with simple singularities if it is tangent to the smooth submanifolds $M(\mathcal{G})$, for all $\mathcal{G}$. The space of such vector fields is denoted by $\mathfrak{X}^r(M)$ and it is endowed with the C^r-compact open topology.

In [9], Gutierrez and Sotomayor characterized a set of structurally stable vector fields $\Sigma^r(M)$ contained in $\mathfrak{X}^r(M)$. This set contains vector fields with finitely many hyperbolic singularities (regular, cone, Whitney, double crossing and triple crossing singularities) and periodic orbits, as well as, singular limit cycles with no saddle connections in the regular part of M and the additional property that the α and ω-limit sets of a trajectory is either a singularity, a periodic orbit or a singular cycle.

In this paper, we consider vector fields in $\Sigma^r(M)$ with no periodic orbits nor limit cycles. Hence, we will consider vector fields in $\Sigma^r(M)$ having only hyperbolic simple singularities. Locally some of these singularities are depicted in Figure 3 and by considering the reverse flow, one obtains the complete set.

2.2. **Isolating blocks for GS-singularities.** Given a vector field $X \in \Sigma^r(M)$, we refer to the associated flow as a Gutierrez-Sotomayor flow φ_X on M, GS-flow for short. We define GS singularities as the hyperbolic singularities of X: regular, cone, Whitney, double crossing and triple crossing singularities.

An ***isolating block*** for a GS-singularities p of a GS-flow φ is an isolating neighborhood $N \subset M$ of p such that the exit set $N^- = \{x \in N; \varphi([0, T), x) \nsubseteq N, \forall T > 0\}$ is closed. The existence of isolating blocks for GS-singularities is a consequence of the existence of Lyapunov functions f in a neighborhood of p. Hence, if $f(p) = c$, let $\epsilon > 0$ be such that there are no critical values in $[c - \epsilon, c + \epsilon]$, then the connected component of $f^{-1}([c - \epsilon, c + \epsilon])$ which contains p, N, is an isolating block for p. Also, $N^- = f^{-1}(c - \epsilon) \cap N$. It is worth noting that an isolating block can also be defined for a maximal invariant set of a GS-flow. See [4, 14, 15] for more details.

[1]We draw attention to the fact that the term *simple singularities* is used in a different context in the classical theory of singularities, [21].

[2]Since the axis $z < 0$ is excluded this is referred to most commonly as a cross cap singularity; we will maintain the nomenclature in [9] with this understanding.

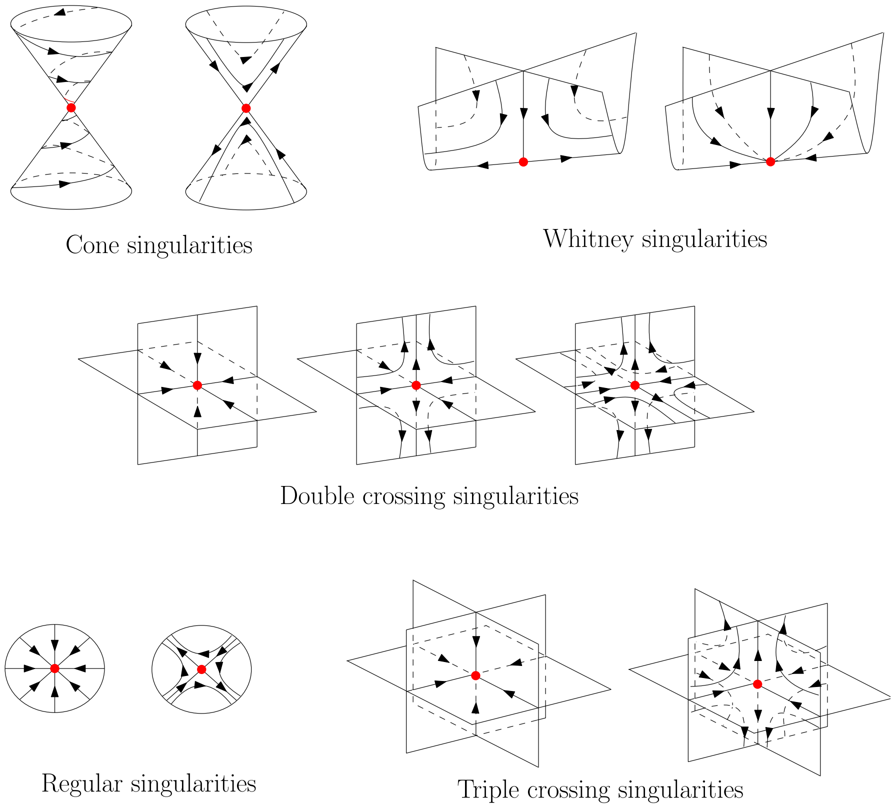


FIGURE 3. Local types of GS-singularities.

The next theorem characterizes the relation between the first Betti number of the boundary of an isolating block for the singularity p, with the number of boundary components and the ranks of the homology Conley index. The proof can be found in [14, 15].

Theorem 2.1 (Poincaré-Hopf equality)**.** *Let (N, N^-) be an index pair for a GS-singularity p and (h_0, h_1, h_2) be the ranks of the homology Conley index of p. Then*

$$(h_2 - h_1 + h_0) - (h_2 - h_1 + h_0)^* = e^+ - \mathcal{B}^+ - e^- + \mathcal{B}^-, \tag{1}$$

where * indicates the index of the time-reversed flow, e^+ (resp., e^-) is the number of entering (resp., exiting) boundary components of N and $\mathcal{B}^+ = \sum_{k=1}^{e^+} b_k^+$ (resp., $(\mathcal{B}^- = \sum_{k=1}^{e^-} b_k^-)$, where $b_k^+(b_k^-)$ is the first Betti number of the k-th entering (resp., exiting) boundary components of N.

For each type of GS-singularity, we now define its nature which corresponds to the local behavior of the flow on a chart around the singularity.

Definition 2.2. *Let p be a GS-singularity of*

(1) *regular or cone type, denote its **nature** by a (resp., r) if p is an attractor (resp., repeller); by s if p is a saddle.*
(2) *Whitney type, denote its **nature** by a (resp., r) if p is an attractor (resp., repeller); s_s (resp., s_u) if p is a saddle and its stable (resp., unstable) manifold is singular.*
(3) *double crossing typem denote its **nature** by a^2 (resp., r^2) if p is an attractor (resp., repeller); sa (resp., sr) if p is a saddle formed by a regular saddle and a regular attractor*

(resp., repeller); ss_s (resp., ss_u) if p is a saddle formed by two regular saddles which are identified along their stable (resp., unstable) manifolds.

(4) *triple crossing type, denote its **nature** by a^3 (resp., r^3) if p is an attractor (resp., repeller); ssa (resp., ssr) if p is a saddle formed by two regular saddles and a regular attractor (resp., repeller).*

In [14,15], the construction of isolating blocks was undertaken for each GS-singularity according to their type $\mathcal{C}$, $\mathcal{W}$, $\mathcal{D}$ or $\mathcal{T}$, and their nature according to the table below.

type	nature	e_v^-	e_v^+	weight
$\mathcal{D}$	a^2	0	1	$b_1^+ = 3$
	sa	1	1	$b_1^+ = b_1^- + 2$
	sa	2	1	$b_1^+ = b_1^- + b_2^- + 1$
	ss_s	1	1	$b_1^+ = b_1^- + 2$
	ss_s	1	2	$b_1^- = b_1^+ + b_2^+ - 3$
	ss_s	2	1	$b_1^+ = b_1^- + b_2^- + 1$
	ss_s	2	2	$b_1^+ + b_2^+ = b_1^- + b_2^- + 2$
	ss_s	3	1	$b_1^+ = b_1^- + b_2^- + b_3^-$
	ss_s	3	2	$b_1^+ + b_2^+ = b_1^- + b_2^- + b_3^- + 1$
	ss_s	4	1	$b_1^+ = b_1^- + b_2^- + b_3^- + b_4^- - 1$
	ss_s	4	2	$b_1^+ + b_2^+ = b_1^- + b_2^- + b_3^- + b_4^-$
				Reversed flow
	ss_u	2	4	$b_1^- + b_2^- = b_1^+ + b_2^+ + b_3^+ + b_4^+$
	ss_u	1	4	$b_1^- = b_1^+ + b_2^+ + b_3^+ + b_4^+ - 1$
	ss_u	2	3	$b_1^- + b_2^- = b_1^+ + b_2^+ + b_3^+ + 1$
	ss_u	1	3	$b_1^- = b_1^+ + b_2^+ + b_3^+$
	ss_u	2	2	$b_1^- + b_2^- = b_1^+ + b_2^+ + 2$
	ss_u	1	2	$b_1^- = b_1^+ + b_2^+ + 1$
	ss_u	2	1	$b_1^+ = b_1^- + b_2^- - 3$
	ss_u	1	1	$b_1^- = b_1^+ + 2$
	sr	1	2	$b_1^- = b_1^+ + b_2^+ + 1$
	sr	1	1	$b_1^- = b_1^+ + 2$
	r^2	1	0	$b_1^- = 3$

type	nature	e_v^-	e_v^+	weight
$\mathcal{R}$	a	0	1	$b_1^+ = 1$
	s	1	1	$b_1^- = b_1^+$
	s	1	2	$b_1^- = b_1^+ + b_2^+ - 1$
	s	2	1	$b_1^+ = b_1^- + b_2^- - 1$
	r	1	0	$b_1^- = 1$
$\mathcal{C}$	a	0	2	$b_1^+ = b_2^+ = 1$
	s	1	1	$b_1^- = b_1^+$
	s	2	2	$b_1^- + b_2^- = b_1^+ + b_2^+$
	r	2	0	$b_1^- = b_2^- = 1$
$\mathcal{W}$	a	0	1	$b_1^+ = 2$
	s_s	1	1	$b_1^+ = b_1^- + 1$
	s_s	2	1	$b_1^+ = b_1^- + b_2^-$
				Reversed flow
	s_u	1	2	$b_1^- = b_1^+ + b_2^+$
	s_u	1	1	$b_1^- = b_1^+ + 1$
	r	1	0	$b_1^- = 2$
$\mathcal{T}$	a^3	0	1	$b_1^+ = 7$
	ssa	1	1	$b_1^+ = b_1^- + 2$
	ssa	2	1	$b_1^+ = b_1^- + b_2^- + 1$
				Reversed flow
	ssr	1	2	$b_1^- = b_1^+ + b_2^+ + 1$
	ssr	1	1	$b_1^- = b_1^+ + 2$
	r^3	1	0	$b_1^- = 7$

2.3. **Super Attractors and Repellers.** In this work, we will study homotopical cancellations within an isolating block containing the maximal invariant set of three GS-singularities, one saddle and two attractors (resp. repellers), and their connecting orbits. This homotopy produces a super attractor (resp., super repeller) singularity. See Figure 4.

Let $D^A \subseteq \mathbb{R}^2$ ($D^R \subseteq \mathbb{R}^2$) be the unit disc of center $p = (0,0)$ and X the attracting radial (resp., repelling) vector field on D with attracting (resp., repelling) singularity p.

Definition 2.3. *A generalized GS-singularity p is:*

(1) *a **super attractor** (resp., **super repeller**) of type:*

  (a) *n-sheet cone of attracting (resp. repelling) nature when obtained by identifying the center points p_i of n discs $D_i, i = 2, ..., n$, where D_i has defined on it an attracting (resp., repelling) radial vector field.*

  (b) *n-sheet Whitney of attracting (resp., repelling) nature when obtained by identifying the center points p_i and some radii of the n discs D_i^A (resp., D_i^R), $i = 1, ..., n$, where D_i^A (resp., D_i^R) has defined on it an attracting (resp. repelling) radial vector field. Moreover, $n-2$ discs D_i^A (resp., D_i^R) have the property that exactly two radii are identified to raddi of two distinct discs. The remaining discs have the property that exactly one radius is identified to a radius of another disc. See Figure 5.*

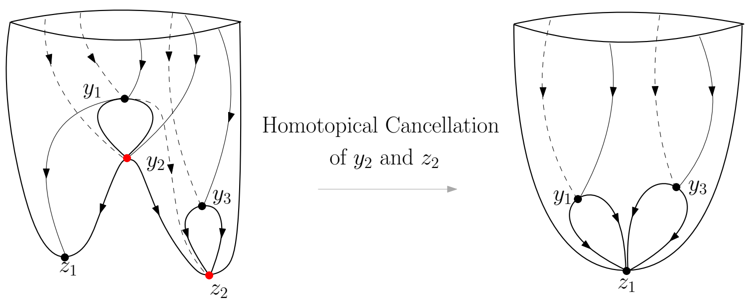


FIGURE 4. Homotopical cancellation of a saddle cone and an attracting cone singularities.

(c) *n-sheet double crossing of attracting (resp. repelling) nature, $n = 2, 3, \ldots$, when obtained by identifying the center points p_i of n discs $D_i^A, i = 0, ..., n-1$, where each D_i^A (resp., D_i^R) is defined as above. Moreover, we identify exactly one diameter of each disc $D_i, i = 1, ..., n-1$ to distinct diameters d_i of the disc D_0, i.e.,*

$$D_i \cap D_j \setminus \{p\} = \emptyset, \text{ and } D_i \cap D_0 = d_i, \; i \neq j, i, j = 1, \ldots n-1.$$

See Figure 5.

(d) *n-sheet triple crossing of attracting (resp., repelling) nature, $n = 2k+1$, when obtained by identifying the center points p_i of n discs $D_0, D_i^1, D_i^2, i = 1, ..., n$, where each disc is defined as above. Moreover, consider the sets of distinct diameters $\{d_{0,i}^1, d_{0,i}^2\}$ in D_0, $\{d_i^1, \partial_i^1\}$ in D_i^1 and $\{d_i^2, \partial_i^2\}$ in D_i^2, $i = 1, \ldots, n$. We identify the diameters ∂_i^1 and ∂_i^2, the diameters d_i^1 and $d_{0,i}^1$, and the diameters d_i^2 and $d_{0,i}^2$, so that all of discs D_i^1, D_i^2 are pairwise disjoint, i.e.,*

$$(D_i^1 \cup D_i^2) \cap (D_j^1 \cup D_j^2) = \emptyset, i \neq j, j = 1, \ldots n.$$

See Figure 5.

(2) *a **$\mathcal{C}$-type** (resp., **$\mathcal{W}$,$\mathcal{D}$, $\mathcal{T}$-type**) singularity of saddle nature if it is a $\mathcal{C}$-type (resp., $\mathcal{W}$,$\mathcal{D}$, $\mathcal{T}$-type) GS-singularity of saddle nature.*

Given an n-sheet generalized GS-singularity p, define the ***singularity type number*** $m(p)$ of p as $n-1$ if p is of $\mathcal{C}$-type or $\mathcal{D}$-type; n if p is of $\mathcal{W}$-type; k if p is of $\mathcal{T}$-type, where $n = 2k+1$. Note that a $\mathcal{C}$-type (resp., $\mathcal{W}$,$\mathcal{D}$, $\mathcal{T}$-type) singularity of saddle nature has type number equal to 1. Also, a regular singularity always has type number equal to zero.

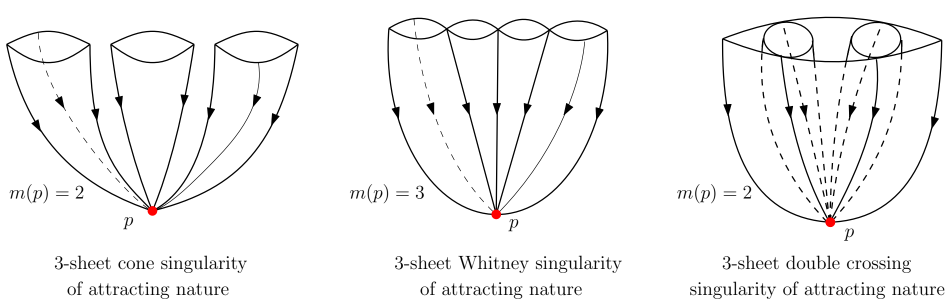


FIGURE 5. Examples of super attractor GS-singularities.

We now define the nature of super attractors and repellers:

Definition 2.4. *Let $p \in M$ be a super attractor or a super repeller singularity. Denote its* ***nature*** *by:*

- *a (resp., r) if p is an attracting (resp., repelling) n-sheet cone or Whitney;*
- *a^n (resp., r^n) if p is an attracting (resp., repelling) n-sheet double or triple crossing.*

Definition 2.5. *Denote by $\mathfrak{M}(\mathcal{GS})$ the set of two-dimensional stratified manifold with generalized GS-singularities. Given $M \in \mathfrak{M}(\mathcal{GS})$, define the set $\mathfrak{X}^r_{\mathcal{GS}}(M)$ of generalized GS-vector fields on M so that for each $X \in \mathfrak{X}^r_{\mathcal{GS}}(M)$ the following conditions are satisfied:*

(1) *X has finitely many generalized GS-singularities;*
(2) *X has no periodic orbits nor limit cycles;*
(3) *The α and ω- limit set of every trajectory of X is a generalized GS-singularity;*
(4) *There are no saddle connections in the regular part of M.*

The corresponding flow φ_X associated to a GS-vector field $X \in \mathfrak{X}^r_{\mathcal{GS}}(M)$ is called a ***generalized Gutierrez-Sotomayor flow*** on M, ***generalized GS-flow*** for short.

Since we are interested in working with vector fields that possess only one type of generalized GS-singularities in addition to regular singularities, we establish the following notation for subsets of $\mathfrak{M}(\mathcal{GS})$ and $\mathfrak{X}^r_{\mathcal{GS}}(M)$: $\mathfrak{M}(\mathcal{GC})$ (resp., $\mathfrak{M}(\mathcal{GW})$, $\mathfrak{M}(\mathcal{GD})$, $\mathfrak{M}(\mathcal{GT})$) denotes the set of stratified 2-manifolds with generalized GS-singularities of regular and cone (resp., Whitney, double crossing, triple crossing) types; $\mathfrak{X}_{\mathcal{GC}}(M)$ (resp., $\mathfrak{X}_{\mathcal{GC}}(W)$, $\mathfrak{X}_{\mathcal{GC}}(D)$, $\mathfrak{X}_{\mathcal{GC}}(T)$) denotes the set of all vector fields on $M \in \mathfrak{M}(\mathcal{GC})$ (resp., $\mathfrak{M}(\mathcal{GW})$, $\mathfrak{M}(\mathcal{GD})$, $\mathfrak{M}(\mathcal{GT})$) which only possess regular and generalized cone (resp., Whitney, double crossing, triple crossing) singularities.

Hereafter we will refer to generalized GS-flows as GS-flows omitting the term "generalized".

3. Morsification of Gutierrez-Sotomayor Flows on isolating blocks

Let $M \in \mathfrak{M}(\mathcal{GS})$ be a compact stratified 2-manifold and $X \in \mathfrak{X}_{\mathcal{GS}}(M)$ be a GS-vector field on M, where $\mathcal{S} = \mathcal{C}$, $\mathcal{W}$, $\mathcal{D}$ or $\mathcal{T}$. Consider the Gutierrez-Sotomayor flow φ_X on M associated to X. In this section, our goal is to establish a regularization process of the GS-singularities which will produce a smooth 2-manifold $\widetilde{M}$ together with a smooth flow with regular singularities. For this purpose we must define a set $\mathcal{SP}(M)$, the ***singular part*** of M, as the union of all non regular singularities and folds, i.e. the union $M(\mathcal{C}) \cup M(\mathcal{W}) \cup M(\mathcal{D}) \cup M(\mathcal{T})$. We refer to this regularization process as the Morsification of GS-singularities.

Definition 3.1. *Let $M \in \mathfrak{M}(\mathcal{GS})$ be a compact stratified 2-manifold, $X \in \mathfrak{X}_{\mathcal{GS}}(M)$ a GS-vector field on M and φ_X the GS-flow associated to X. An isolating block (N, φ_X) admits a* ***Morsification*** *if there exists a quadruple $(\widetilde{N}, \varphi_{\widetilde{X}}, \mathfrak{h}, \mathfrak{p})$ such that*

(1) *$\widetilde{N}$ is a smooth 2-manifold;*
(2) *$\widetilde{\varphi}$ is a smooth flow on $\widetilde{N}$ having only hyperbolic regular singularities and no saddle connections;*
(3) *$\mathfrak{h} : N \to \widetilde{N}$ is a multivalued map such that $\mathfrak{h}$ restricted to*

$$N \setminus \{\mathcal{SP}(N) \cup \{x \in N \mid \omega(x) = p \text{ or } \alpha(x) = p, \text{where } p \text{ is a saddle cone singularity}\}\}$$

is a homeomorphism;

(4) *$\mathfrak{p} : \widetilde{N} \to N$ is the projection map and $\mathfrak{h} \circ \mathfrak{p} = id|_{\widetilde{N}}$.*

In this case, one says that (N, φ_X) admits a Morsification to $(\widetilde{N}, \varphi_{\widetilde{X}})$, or that $(\widetilde{N}, \varphi_{\widetilde{X}})$ is a Morsification of (N, φ).

Theorem 3.1. *Let $M \in \mathfrak{M}(\mathcal{GS})$ be a singular 2-manifold, $X \in \mathfrak{X}_{\mathcal{GS}}(M)$ a GS-vector field on M and φ_X the GS-flow associated to X, where $\mathcal{S} = \mathcal{C}, \mathcal{W}, \mathcal{D}$ or $\mathcal{T}$. Given a GS-singularity p and an isolating block (N, φ_X) for p, there exists a Morsification $(\widetilde{N}, \varphi_{\widetilde{X}})$, where $\widetilde{N}$ is an isolating block w.r.t. the regularized flow $\varphi_{\widetilde{X}}$.*

Now we procedure to the proof of Theorem 3.1, which will be done in the following subsections for each type of singularities.

3.1. **Morsification of Cone Singularities.** Let p be a cone singularity in $M \in \mathfrak{M}(\mathcal{GC})$ and N be an isolating block for p with GS-flow φ_X where $X \in \mathfrak{X}_{\mathcal{GC}}(M)$. Consider the boundaries N^- and N^+ of N which constitute the exit and entering sets of φ_X, respectively. Next it is shown how to Morsify the GS-flow on N to obtain a regular flow on a smooth isolating block $\widetilde{N}$. Considering a Morsification of all isolating blocks for singularities of M, one can glue them together to form a flow on a smooth 2-manifold $\widetilde{M}$.

Proposition 3.1. *Let $M \in \mathfrak{M}(\mathcal{GC})$ be a singular 2-manifold, $X \in \mathfrak{X}_{\mathcal{GC}}(M)$ a GS-vector field on M and φ_X the GS-flow associated to X. Given a cone singularity p and an isolating block N for p, there exists a Morsification $(\widetilde{N}, \varphi_{\widetilde{X}})$ where $\partial\widetilde{N} = \partial N$.*

Proof. The proof is done by constructing a quadruple $(\widetilde{N}, \varphi_{\widetilde{X}}, \mathfrak{h}, \mathfrak{p})$ for each type of singularity.

1) Let p be a repelling (resp. attracting) n-sheet cone singularity.

Consider a 2-sphere with n-holes $\widetilde{N}$, with exit set $\widetilde{N}^- = \sqcup_{j=1}^n \widetilde{N}_j^-$ (resp., entering set $\widetilde{N}^+ = \sqcup_{j=1}^n \widetilde{N}_j^+$) homeomorphic to $N^- = \sqcup_{j=1}^n N_j^-$ (resp. $N^+ = \sqcup_{j=1}^n N_j^+$) and containing a regular repelling (resp., attracting) singularity $\tilde{p}$ and regular saddle singularities $\tilde{p}_i'$, $i = 1,\dots, n-1$. For each $j = 1,\dots, n$, the components of the exit set $N_j^-, \widetilde{N}_j^-$ (resp., entering sets $N_j^+, \widetilde{N}_j^+$) are homeomorphic to S^1. Denote the homeomorphisms which preserve counterclockwise orientation on the boundary by $h_j^- : N_j^- \to \widetilde{N}_j^-$ (resp., $h_j^+ : N_j^+ \to \widetilde{N}_j^+$). Let $\varphi_{\widetilde{X}}$ be a flow on $\widetilde{N}$ that satisfies the following conditions: for each i, there are two orbits $\tilde{u}_1(\tilde{p}, \tilde{p}_i')$ and $\tilde{u}_2(\tilde{p}, \tilde{p}_i')$ such that $\omega(\tilde{u}_1) = \tilde{p}_i{}' = \omega(\tilde{u}_2)$ and $\alpha(\tilde{u}_1) = \tilde{p} = \alpha(\tilde{u}_2)$ (resp., $\omega(\tilde{u}_1) = \tilde{p} = \omega(\tilde{u}_2)$ and $\alpha(\tilde{u}_1) = \tilde{p}_i{}' = \alpha(\tilde{u}_2)$). See Figure 6.

For each $i = 1,\dots, n-1$, chose points x_i, y_i where $x_i \in N_i^-$ and $y_i \in N_{i+1}^-$ (resp., $x_i \in N_i^+$ and $y_i \in N_{i+1}^+$). Denote by $A = \{\{x_i, y_i\} \mid i = 1,\dots, n-1\}$ the set of these points. Given $x \in N \setminus \{p\}$, there exists $x^- \in N_j^-$ (resp., $x^+ \in N_j^+$), for some $j = 1, \dots, n$, where x belongs to the orbit $u(p, x^-)$ (resp., $u(x^+, p)$). Define the multivalued map $\mathfrak{h} : N \to \widetilde{N}$ by:

$$\mathfrak{h}(u(p,x)) = \begin{cases} u(\tilde{p}, h_j^-(x)) \ \ (\text{resp., } u(h_j^+(x), \tilde{p})), & \text{if } x \notin A \\ \{u(\tilde{p}, \tilde{p}_i'), u(\tilde{p}_i', h_j^-(x))\} \ \ (\text{resp., } \{u(h_j^+(x), \tilde{p}_i'), u(\tilde{p}_i', \tilde{p})\}), & \text{if } x \in A \end{cases}.$$

Note that $\mathfrak{h}$ is a multivalued extension of the homeomorphisms h_j^- (resp., h_j^+), i.e., $\mathfrak{h}|_{N_j^-} = h_j^-$ (resp., $\mathfrak{h}|_{N_j^+} = h_j^+$).

Consider the closed region

$$D_i = \{u(\tilde{p}, \tilde{p}_i'), u(\tilde{p}_i', h_j^-(x)) \mid x \in A, i = 1, \dots, n-1, j = 1, \dots, n\}$$

$$(\text{resp., } D_i = \{u(h_j^+(x), \tilde{p}_i'), u(\tilde{p}_i', \tilde{p}) \mid x \in A, i = 1, \dots, n-1, j = 1, \dots, n\}).$$

Define the projection map $\mathfrak{p} : \widetilde{N} \to N$ by

$$\mathfrak{p}(u(\tilde{x}, \tilde{y})) = \begin{cases} \mathfrak{h}^{-1}(u(\tilde{x}, \tilde{y})), & \text{if } u(\tilde{x}, \tilde{y}) \notin D_i \\ u(p, (h_j^-)^{-1}(\tilde{y})) \ (\text{resp., } u((h_j^+)^{-1}(\tilde{x}), p)), & \text{if } u(\tilde{x}, \tilde{y}) \in D_i, \ y \neq \tilde{p}_i' \ (\text{resp., } x \neq \tilde{p}_i') \\ p, & \text{if } \tilde{x}, \tilde{y} \in \{\tilde{p}, \tilde{p}_i'\} \end{cases}$$

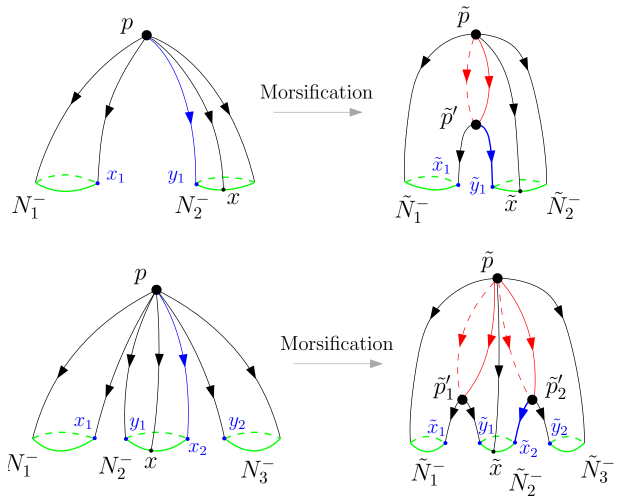


FIGURE 6. Isolating blocks for repelling cone singularities and their Morsifications.

2) Let p be a saddle cone singularity and N its isolating block.

There are two cases to consider, the first being the case where the boundary of the exit and entering sets of N are disconnected and the second where they are connected.

2.1) Consider the case where the boundaries N^- and N^+ of the singular block N are both disconnected, i.e. $N_i^- \simeq S^1$ and $N_i^+ \simeq S^1, i = 1, 2$ The Morsified block $\widetilde{N}$, is a sphere with 4 holes, corresponding to the boundaries $\widetilde{N}_i^- \simeq S^1$ and $\widetilde{N}_i^+ \simeq S^1, i = 1, 2$, corresponding to the connected component of the exit set $\widetilde{N}^-$ and entering set $\widetilde{N}^+$, respectively. See Figure 7.

For each $i = 1, 2$, note that $W^u(p) \cap N_i^-$ is a unique point. Denote this point by x_i^- and consider $u(p, x_i^-)$ the orbit that connects p and x_i^-. Similarly, consider $x_i^+ = W^s(p) \cap N_i^+$ and $u(x_i^+, p)$ the orbit that connects x_i^+ and p. See Figure 7. Let $A = \{x_i^-, x_i^+ \mid i = 1, 2\}$.

Consider a multivalued map $\mathfrak{h}_i^- : N_i^- \to \widetilde{N}_i^-$, such that $\mathfrak{h}_i^-(x_i^-) = \{a_i^-, b_i^- \mid a_i \neq b_i\}$, $\mathfrak{h}_i^-(N_i^- \setminus \{x_i^-\}) = \widetilde{N}_i^- \setminus [a_i^-, b_i^-]$, and $\mathfrak{h}_i^-$ restricted to $N_i^- \setminus \{x_i^-\}$ is a homeomorphism which preserves the counterclockwise orientation on the boundaries. Similarly, consider a multivalued map $\mathfrak{h}_i^+ : N_i^+ \to \widetilde{N}_i^+$, where $\mathfrak{h}_i^+(x_i^+) = \{a_i^+, b_i^+ \mid a_i^+ \neq b_i^+\}$ and $\mathfrak{h}_i^+(N_i^+ \setminus \{x_i^+\}) = \widetilde{N}_i^+ \setminus [a_i^+, b_i^+]$. Given $x \in N \setminus \{p\}$ such that $x \notin W^u(p) \cup W^s(p)$, there exist $x^+ \in N_i^+$ and $x^- \in N_i^-$ such that x belongs to the orbit $u(x^+, x^-)$. If $x \in W^u(p) \cup W^s(p)$ then x is on the orbit $u(x_i^+, p)$ or $u(p, x_i^-)$, for some $i = 1, 2$. Define the multivalued map $\mathfrak{h} : N \to \widetilde{N}$ by

$$\mathfrak{h}(u(x, y)) = \begin{cases} u(\mathfrak{h}_i^+(x), \mathfrak{h}_i^-(y)), & \text{if } x, y \notin A \cup \{p\} \\ u(\mathfrak{h}_i^+(x), \tilde{p}) \cup u(\mathfrak{h}_i^+(x), \tilde{p}'), & \text{if } x \in A \text{ and } x^- = p \\ u(\tilde{p}, \mathfrak{h_i}^-(y)) \cup u(\tilde{p}', \mathfrak{h}_i^-(y)), & \text{if } x = p \text{ and } y \in A \\ \{\tilde{p}, \tilde{p}'\}, & \text{if } x = p = y \end{cases}.$$

Consider $\varphi_{ij} : (a_i^+, b_i^+) \to (a_j^-, b_j^-)$ a homeomorphism which preserves the orientation, where $i, j = 1, 2$, with $i \neq j$. Given $\tilde{x}^+ \in (a_1^+, b_1^+)$, let $\varphi_{12}(\tilde{x}^+) = \tilde{x}^- \in (a_2^-, b_2^-)$. Consider $u(\tilde{x}^+, \tilde{x}^-)$ an orbit that connects $\tilde{x}^+$ and $\tilde{x}^-$. Analogously, given $\tilde{x}^+ \in (a_2^+, b_2^+)$, let $\varphi_{21}(\tilde{x}^+) = \tilde{x}^- \in (a_1^-, b_1^-)$. Consider $u(\tilde{x}^+, \tilde{x}^-)$ an orbit that connects $\tilde{x}^+$ and $\tilde{x}^-$.

Consider the closed region $D_{ij} = \varphi_{ij}(a_i^+, b_i^+) \cup \{\mathfrak{h}(u(x,y)) \mid x, y \in A \cup \{p\}\}$. Define the projection map $\mathfrak{p} : \widetilde{N} \to N$ by

$$\mathfrak{p}(u(\tilde{x},\tilde{y})) = \begin{cases} \mathfrak{h}^{-1}(u(\tilde{x},\tilde{y})), & \text{if } u(\tilde{x},\tilde{y}) \notin D_{ij} \\ \{u(x_i^+, p), u(p, x_j^-)\}, & \text{if } u(\tilde{x},\tilde{y}) \in D_{ij} \end{cases}.$$

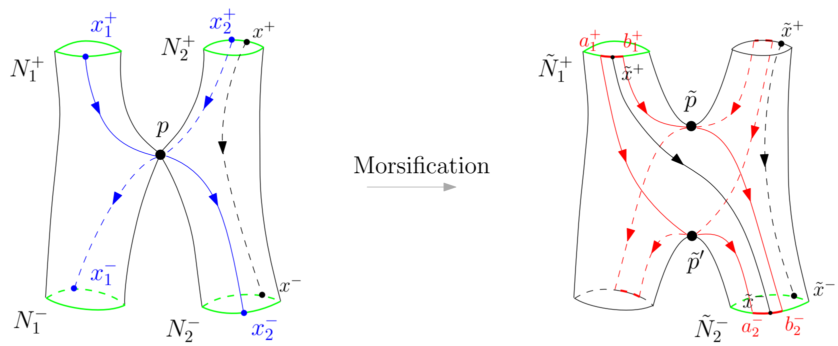


FIGURE 7. Isolating block for a saddle cone singularity and its Morsification.

2.2) Now, consider the case where the block N has connected boundaries $N^- \simeq S^1$ and $N^+ \simeq S^1$. The Morsified block $\widetilde{N}$ is a torus minus 2 disks, i.e, with boundaries $\widetilde{N}^- \simeq S^1$ and $\widetilde{N}^+ \simeq S^1$, corresponding to the exit set and entering set, respectively. See Figure 8.

Let $x_1^-, x_2^- \in N^-$ be the points in $W^u(p) \cap N^-$ and $u(p, x_i^-)$ be the orbit that connects p and $x_i^-, i = 1, 2$. Consider $x_1^+, x_2^+ \in N^+$ as points in $W^u(p) \cap N^+$ and $u(x_i^+, p)$ the orbit that connects $x_i^+, i = 1, 2$ and p. See Figure 8. Let $A = \{x_i^+, x_i^- \mid i = 1, 2\}$. Consider the arcs $C_1^- = (x_1^-, x_2^-)$ and $C_2^- = (x_2^-, x_1^-)$ in N^- as well as $C_1^+ = (x_1^+, x_2^+)$ and $C_2^+ = (x_2^+, x_1^+)$ in N^+ with counterclockwise orientation.

Consider a multivalued map $\mathfrak{h}^- : N^- \to \widetilde{N}^-$, where

$$\mathfrak{h}^-(x_i^-) = \{a_i^-, b_i^-\}, \quad \mathfrak{h}^-(C_1^-) = (b_1^-, a_2^-), \quad \mathfrak{h}^-(C_2^-) = (b_2^-, a_1^-),$$

and $\mathfrak{h}^-$ restricted to $N^- \setminus \{x_1^-, x_2^-\}$ is a homeomorphism which preserves the counterclockwise orientation on the boundaries. Similarly, consider a multivalued map $\mathfrak{h}^+ : N^+ \to \widetilde{N}^+$, where $\mathfrak{h}^+(x_i^+) = [a_i^+, b_i^+]$, $\mathfrak{h}^-(C_1^+) = (b_1^+, a_2^+)$, $\mathfrak{h}^-(C_2^+) = (b_2^+, a_1^+)$ and $\mathfrak{h}^+(x_i^+)$ restricted to $N^+ \setminus \{x_1^+, x_2^+\}$ is a homeomorphism that preserves the counterclockwise orientation on boundaries. Given $x \in N \setminus \{p\}$ and $x \notin W^u(p) \cup W^s(p)$, there exist $x^+ \in N_i^+$ and $x^- \in N_i^-$ such that x belongs to the orbit $u(x^+, x^-)$. If $x \in W^u(p) \cup W^s(p)$ then x is in the orbit $u(x_i^+, p)$ or $u(p, x_i^-)$, for some $i = 1, 2$. Define the multivalued map $\mathfrak{h} : N \to \widetilde{N}$ by

$$\mathfrak{h}(u(x,y)) = \begin{cases} u(\mathfrak{h}_i^+(x), \mathfrak{h}_i^-(y)), & \text{if } x, y \notin A \cup \{p\} \\ u(\mathfrak{h}_i^+(x'), \tilde{p}) \cup u(\mathfrak{h}_i^+(x), \tilde{p}'), & \text{if } x \in A \text{ and } x^- = p \\ u(\tilde{p}, \mathfrak{h}_{\mathfrak{i}}{}^-(y)) \cup u(\tilde{p}', \mathfrak{h}_i^-(y)), & \text{if } x = p \text{ and } y \in A \\ \{\tilde{p}, \tilde{p}'\}, & \text{if } x = p = y \end{cases}.$$

Let $\varphi_{ij} : (a_i^+, b_i^+) \to (a_j^-, b_j^-)$ be a homeomorphism which preserves orientation, where $i, j = 1, 2, i \neq j$. Given $\tilde{x}^+ \in (a_1^+, b_1^+)$, let $\varphi_{12}(\tilde{x}^+) = \tilde{x}^- \in (a_2^-, b_2^-)$. Let $u(\tilde{x}^+, \tilde{x}^-)$ be an orbit that connects $\tilde{x}^+$ and $\tilde{x}^-$.

Analogously, given $\tilde{x}^+ \in (a_2^+, b_2^+)$, let $\varphi_{21}(\tilde{x}^+) = \tilde{x}^- \in (a_1^-, b_1^-)$. Let $u(\tilde{x}^+, \tilde{x}^-)$ be an orbit that connects $\tilde{x}^+$ and $\tilde{x}^-$.

Consider the closed region $D_{ij} = \varphi_{ij}(a_i^+, b_i^+) \cup \{\mathfrak{h}(u(x,y)) \mid x, y \in A \cup \{p\}\}$. Define the projection map $\mathfrak{p} : \widetilde{N} \to N$ by

$$\mathfrak{p}(u(\tilde{x},\tilde{y})) = \begin{cases} \mathfrak{h}^{-1}(u(\tilde{x},\tilde{y})), & \text{if } u(\tilde{x},\tilde{y}) \notin D_{ij} \\ \{u(x_i^+, p), u(p, x_j^-)\}, & \text{if } u(\tilde{x},\tilde{y}) \in D_{ij} \end{cases} .$$

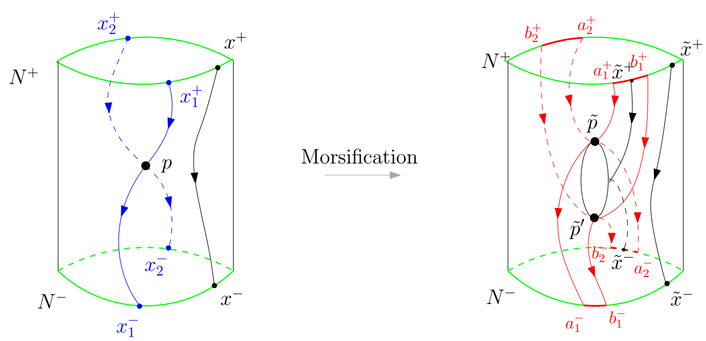


FIGURE 8. Isolating block for a saddle cone singularity and its Morsification.

□

Combinatorially the isolating blocks for cone singularities together with its Morsification can be seen as the Lyapunov (semi)graphs in Figure 9.

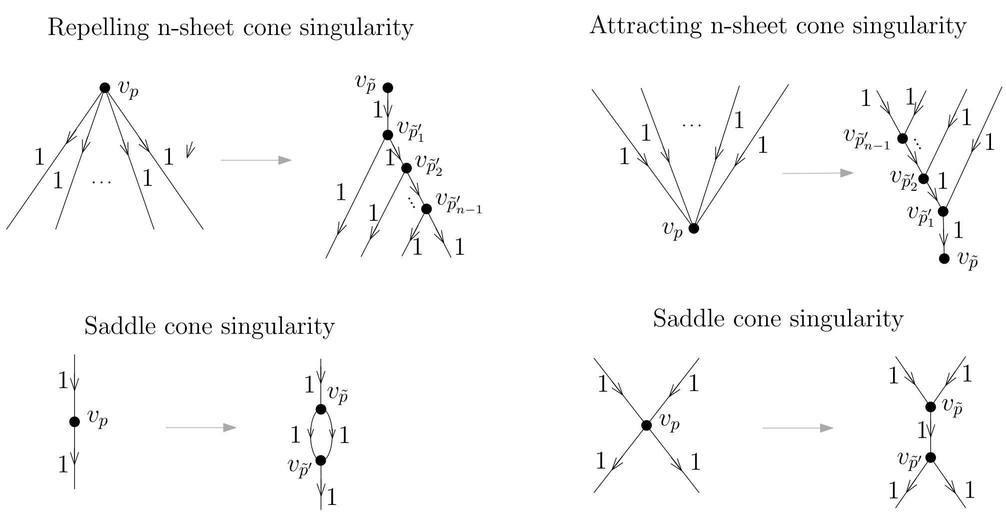


FIGURE 9. Morsification of a Lyapunov semigraph with a vertex associated to a cone singularity.

3.2. **Morsification of Whitney Singularities.** Let p be a Whitney singularity in $M \in \mathfrak{M}(\mathcal{GW})$ and N be an isolating block for p with GS-flow φ_X, where $X \in \mathfrak{X}_{\mathcal{GW}}(M)$. Consider the boundaries N^- and N^+ of the block N which constitute the exit and entering sets of φ_X, respectively. Next it is shown how to Morsify the GS-flow on N to obtain a regular flow on a smooth isolating block $\widetilde{N}$.

Proposition 3.2. *Let $M \in \mathfrak{M}(\mathcal{GW})$ be a singular 2-manifold, $X \in \mathfrak{X}_{\mathcal{GW}}(M)$ a GS-vector field on M and φ_X the GS-flow associated to X. Given a Whitney singularity p and an isolating block N for p, there exists a Morsification $(\widetilde{N}, \varphi_{\widetilde{X}})$ such that each orbit of φ_X in $\mathcal{SP}(M)$ admits a duplication of the orbits in N.*

Proof. The proof follows by considering each type of singularity and constructing a regular isolating block with a smooth flow defined on it.

1) Let p be a repelling (resp. attracting) n-sheet Whitney singularity and N its isolating block.

Consider a regular isolating block $\widetilde{N}$, homeomorphic to D^2, containing a regular repelling (resp. attracting) singularity $\tilde{p}$ with exit set $\widetilde{N}^-$ (resp. entering set $\widetilde{N}^+$) homeomorphic to S^1, as in Figure 10.

Let $x_i^- \in N^-$ (resp. $x_i^+ \in N^+$) be points associated to the orbit $u(p, x_i^-)$ (resp., $u(x_i^+, p)$) on $\mathcal{SP}(N)$ that connects p and x_i^-, (resp. p and x_i^+) for $i = 1, \ldots, n-1$. Define

$$A = \{x_i^-; i = 1, \ldots, n-1\} \text{ (resp., } A = \{x_i^+; i = 1, \ldots, n-1\}).$$

Consider the arcs $C_1 = (x_1, x_1)$, $C_2^1 = (x_1, x_2)$, $C_2^2 = (x_2, x_1)$, $\ldots$, $C_{n-1}^1 = (x_{n-2}, x_{n-1})$, $C_{n-1}^2 = (x_{n-1}, x_{n-2})$ and $C_n = (x_{n-1}, x_{n-1})$ in N^- (resp., N^+) oriented counterclockwise.

Consider a multivalued map $\mathfrak{h}^- : N^- \to \widetilde{N}^-$, where $\mathfrak{h}^-(x_i^-) = \{a_i^-, b_i^-\}$, $\mathfrak{h}^-(C_1) = (a_1^-, b_1^-)$, $\mathfrak{h}^-(C_2^1) = (a_1^-, a_2^-)$, $\mathfrak{h}^-(C_2^2) = (b_2^-, b_1^-), \ldots, \mathfrak{h}^-(C_{n-1}^1) = (a_{n-2}^-, a_{n-1}^-)$, $\mathfrak{h}^-(C_{n-1}^2) = (b_{n-1}^-, b_{n-2}^-)$, $\mathfrak{h}^-(C_n^1) = (a_{n-1}^-, b_{n-1}^-)$, and $\mathfrak{h}^-$ restricted to $N^- \setminus \bigcup_{i=1}^{n-1}\{x_i^-\}$ is a homeomorphism which preserves the counterclockwise orientation on the boundaries. Similarly, consider a multivalued map $\mathfrak{h}^+ : N^+ \to \widetilde{N}^+$, where $\mathfrak{h}^+(x_i^+) = \{a_i^+, b_i^+\}$, $\mathfrak{h}^+(C_1) = (a_1^+, b_1^+)$, $\mathfrak{h}^+(C_2^1) = (a_1^+, a_2^+)$, $\mathfrak{h}^+(C_2^2) = (b_2^+, b_1^+)$, $\ldots$, $\mathfrak{h}^+(C_{n-1}^1) = (a_{n-2}^+, a_{n-1}^+)$, $\mathfrak{h}^+(C_{n-1}^2) = (b_{n-1}^+, b_{n-2}^+)$, $\mathfrak{h}^+(C_n^1) = (a_{n-1}^+, b_{n-1}^+)$, and $\mathfrak{h}^+$ restricted to $N^+ \setminus \bigcup_{i=1}^{n-1}\{x_i^+\}$ is a homeomorphism which preserves the counterclockwise orientation on the boundaries. Given $x \in N \setminus \{p\}$, there exists $x^- \in N^-$ (resp., $x^+ \in N^+$), where x belongs to the orbit $u(p, x^-)$ (resp., $u(x^+, p)$). Define the multivalued map $\mathfrak{h} : N \to \widetilde{N}$ by:

$$\mathfrak{h}(u(p,x)) = \begin{cases} u(\tilde{p}, \mathfrak{h}^-(x)) \ \ \text{(resp., } u(\mathfrak{h}^+(x), \tilde{p})), & \text{if } x \notin A \\ \{u(\tilde{p}, a_i^-), u(\tilde{p}, b_i^-)\} \ \ \text{(resp., } \{u(a_i^+, \tilde{p}), u(b_i^+, \tilde{p})\}, & \text{if } x \in A \end{cases}$$

Consider the closed set $D = \{u(\tilde{p}, a_i^-), u(\tilde{p}, b_i^-); i = 1, \ldots, n-1\}$ (resp., $D = \{u(a_i^+, \tilde{p}), u(b_i^+, \tilde{p});$ $i = 1, \ldots, n-1\}$). Define the projection map $\mathfrak{p} : \widetilde{N} \to N$ as

$$\mathfrak{p}(u(\tilde{x}, \tilde{y})) = \begin{cases} \mathfrak{h}^{-1}(u(\tilde{x}, \tilde{y})), & \text{if } u(\tilde{x}, \tilde{y}) \notin D \\ u(p, x_i^-) \ \ \text{(resp. } u(x_i^+, p)), & \text{if } u(\tilde{x}, \tilde{y}) \in D \end{cases} .$$

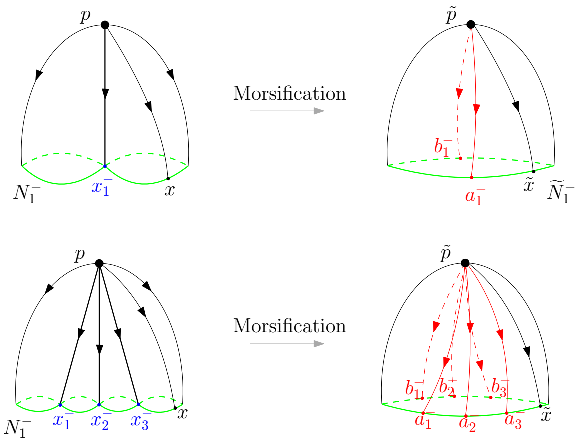


FIGURE 10. Isolating blocks for a repelling Whitney singularity and its Morsification.

Another possible Morsification for a repelling n-sheet Whitney singularity is a disjoint union of n repeller disks.

2) Let p be a saddle Whitney singularity of s_s-nature. (If p has s_u-nature the prove is completely analogous by using the reverse flow.)

Let N be an isolating block for p. One has two cases to consider, first when the exit set of N is disconnected and secondly when its is connected.

2.1) Consider the case where the exit set N^- is disconnected, i.e. N_i^- is homeomorphic to S^1 for $i = 1, 2$. The regular isolating block $\widetilde{N}$ is a sphere with 3 holes containing a regular singularity of saddle nature $\tilde{p}$ with entering set $\widetilde{N}^+$ homeomorphic to S^1 and exit set $\widetilde{N}^-$ with exactly two boundary components homeomorphic to S^1. See Figure 11. Let $x^+ \in N^+$ be the point belonging to a orbit $u(x^+, p)$ on $\mathcal{SP}(N)$. Define the multivalued map

$$\mathfrak{h}^+ : N^+ \to \widetilde{N}^+ \text{ by } \mathfrak{h}^+(x^+) = \{a^+, b^+\} \text{ and } \mathfrak{h}^+(N^+ \setminus \{x^+\}) = \widetilde{N}^+ \setminus \{a^+, b^+\},$$

such that $\mathfrak{h}^+$ restricted to $N^+ \setminus \{x^+\}$ is a homeomorphism which preserves the counter-clockwise orientation on the boundaries. Consider the trivial homeomorphisms $h_i : N_i^- \to \widetilde{N}_i^-$. Define the multivalued map $\mathfrak{h} : N \to \widetilde{N}$ as

$$\mathfrak{h}(u(x,y)) = \begin{cases} u(\mathfrak{h}^+(x), h_j(y)), & \text{if } x \neq x^+ \text{ and } x \neq p \\ u(\tilde{p}, h_j(y)), & \text{if } x \neq x^+ \text{ and } x = p \\ \{u(a^+, \tilde{p}), u(b^+, \tilde{p})\}, & \text{if } x = x^+ \end{cases}$$

Consider the closed set $D = \{u(a^+, \tilde{p}), u(b^+, \tilde{p})\}$. Define the projection map $\mathfrak{p} : \widetilde{N} \to N$ by

$$\mathfrak{p}(u(\tilde{x}, \tilde{y})) = \begin{cases} \mathfrak{h}^{-1}(u(\tilde{x}, \tilde{y})), & \text{if } u(\tilde{x}, \tilde{y}) \notin D \\ u(x^+, p), & \text{if } u(\tilde{x}, \tilde{y}) \in D \end{cases}'.$$

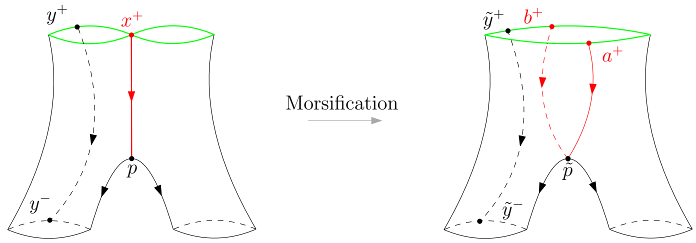


FIGURE 11. Isolating blocks for a saddle Whitney singularity and its Morsification.

2.2) Consider the case where the exit set N^- is connected, homeomorphic to S^1. The regular isolating block $\widetilde{N}$ is a sphere with 3 holes containing a regular singularity of saddle nature $\tilde{p}$ with exit set $\widetilde{N}^-$ homeomorphic to S^1 and entering set $\widetilde{N}^+$ with exactly two boundary components homeomorphic to S^1. See Figure 12. Let $x^+ \in N^+$ be the point in the orbit $u(x^+, p)$ on $\mathcal{SP}(N)$. Consider a multivalued map $\mathfrak{h}^+ : N^+ \to \widetilde{N}^+$, such that $\mathfrak{h}^+(x^+) = \{a^+, b^+\}$, $\mathfrak{h}^+(N^+ \setminus \{x^+\}) = \widetilde{N}^+ \setminus \{a^+, b^+\}$, and $\mathfrak{h}^+$ restricted to $N^+ \setminus \{x^+\}$ is a homeomorphism which preserves the counterclockwise orientation on the boundaries. Considering the trivial homeomorphism $h : N^- \to \widetilde{N}^-$, one defines the multivalued map $\mathfrak{h} : N \to \widetilde{N}$ as

$$\mathfrak{h}(u(x,y)) = \begin{cases} u(\mathfrak{h}(x), h_j(y)), & \text{if } x \neq x^+ \text{ and } x \neq p \\ u(\tilde{p}, h_j(y)), & \text{if } x \neq x^+ \text{ and } x = p \\ \{u(a^+, \tilde{p}), u(b^+, \tilde{p})\}, & \text{if } x = x^+ \end{cases}.$$

Consider the closed set $D = \{u(a^+, \tilde{p}), u(b^+, \tilde{p})\}$. Define the projection map $\mathfrak{p} : \widetilde{N} \to N$ by

$$\mathfrak{p}(u(\tilde{x}, \tilde{y})) = \begin{cases} \mathfrak{h}^{-1}(u(\tilde{x}, \tilde{y})), & \text{if } u(\tilde{x}, \tilde{y}) \notin D \\ u(x^+, p), & \text{if } u(\tilde{x}, \tilde{y}) \in D \end{cases}.$$

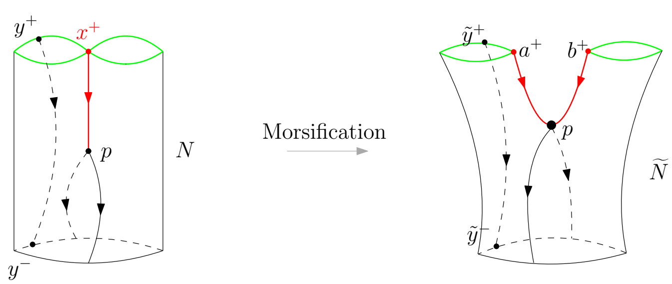


FIGURE 12. Isolating blocks for a saddle Whitney singularity and its Morsification.

□

Combinatorially the isolating blocks for Whitney singularities together with its Morsification can be seen as the Lyapunov (semi)graphs in Figure 13.

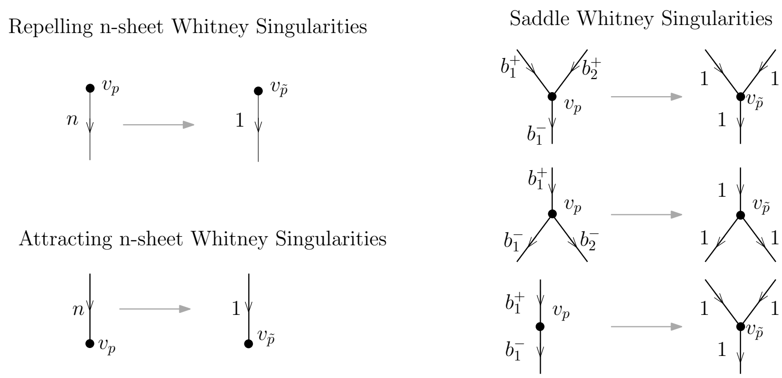


FIGURE 13. Morsification of a Lyapunov semigraph with vertex associated to a Whitney singularity.

3.3. **Morsification of Double and Triple Crossing Singularities.** Let p be a double crossing singularity in $M \in \mathfrak{M}(\mathcal{GD})$ and N be an isolating block for p with GS-flow φ_X where $X \in \mathfrak{X}_{\mathcal{GD}}(M)$. Consider the boundaries N^- and N^+ of the isolating block N which constitute the exit and entering sets of φ_X, respectively. Next it is shown how to Morsify the GS-flow on N to obtain a regular flow on a smooth isolating block $\widetilde{N}$. Considering a Morsification of all isolating blocks of singularities of M, one can glue them together to form a flow on a disconnected smooth surface $\widetilde{M}$.

Proposition 3.3. *Let $M \in \mathfrak{M}(\mathcal{GD})$ be a singular 2-manifold, $X \in \mathfrak{X}_{\mathcal{GD}}(M)$ a GS-vector field on M and φ_X the GS-flow associated to X. Given a double crossing singularity p and an isolating block N for p, there exists a Morsification $(\widetilde{N}, \varphi_{\widetilde{X}})$ such that each orbit of φ_X in $\mathcal{SP}(M)$ admits a duplication of orbits in N.*

Proof. Consider the different type of double crossing singularities.

1) Let p be a repelling (resp., attracting) n-sheet double crossing singularity. Consider a smooth block formed by n disjoint discs, $\widetilde{N} \simeq \sqcup_{i=1}^{n} D_i^2$, containing n repelling (resp., attracting) regular singularities $\tilde{p}_1, \ldots, \tilde{p}_n$ and having exit set $\widetilde{N}^-$ (resp. entering set $\widetilde{N}^+$) homeomorphic to a disjoint union of n circles, as in Figure 14.

   Let $A_i = \{x_i, y_i\}$, where $x_i, y_i \in N^-$ (resp., $x_i, y_i \in N^+$) be the points associated to the orbits $u(p, x_i)$ (resp., $u(x_i, p)$) on $\mathcal{SP}(N)$ that connects p and x_i, and $u(p, y_i)$ (resp., $u(y_i, p)$) that connects p and y_i, $i = 1, \ldots, n-1$.

   Consider the external arcs $C_1^1 = (y_1, x_1)$, $C_2^1 = (x_1, x_2)$, $C_2^2 = (y_2, y_1)$, $\ldots$,

$$C_{n-1}^1 = (x_{n-2}, x_{n-1}),\ C_{n-1}^2 = (y_{n-1}, y_{n-2}) \text{ and } C_n^1 = (x_{n-1}, y_{n-1})$$

   in N^- (resp., N^+) as well as $c_1^1 = (y_1, x_1)$, $c_1^2 = (x_1, y_1)$, $c_2^1 = (y_2, x_2)$, $c_2^2 = (x_2, y_2)$,..., $c_{n-1}^1 = (y_{n-1}, x_{n-1})$ and $c_{n-1}^2 = (x_{n-1}, y_{n-1})$ in N^- (resp., N^+) with counterclockwise orientations.

   Define the multivalued map $\mathfrak{h}^- : N^- \to \widetilde{N}^-$, by $\mathfrak{h}^-(x_i) = \{a_i, c_i\}$, $\mathfrak{h}^-(y_i) = \{b_i, d_i\}$, $\mathfrak{h}^-(C_1) = (b_1, a_1)$, $\mathfrak{h}^-(C_2^1) = (a_1, a_2)$, $\mathfrak{h}^-(C_2^2) = (b_2, b_1)$, $\ldots$, $\mathfrak{h}^-(C_{n-1}^1) = (a_{n-2}, a_{n-1})$, $\mathfrak{h}^-(C_{n-1}^2) = (b_{n-1}, b_{n-2})$, $\mathfrak{h}^-(C_n) = (a_{n-1}, b_{n-1})$, $\mathfrak{h}^-(c_1^1) = (d_1, c_1)$, $\mathfrak{h}^-(c_1^2) = (c_1, d_1)$, $\mathfrak{h}^-(c_2^1) = (d_2, c_2)$, $\mathfrak{h}^-(c_2^2) = (c_2, d_2)$, $\ldots$, $\mathfrak{h}^-(c_{n-1}^1) = (d_{n-1}, c_{n-1})$ and

$\mathfrak{h}^-(c_{n-1}^2) = (c_{n-1}, d_{n-1})$, such that $\mathfrak{h}^-$ restricted to $N^- \setminus \bigcup_{i=1}^{n-1}\{x_i, y_i\}$ is a homeomorphism which preserves orientation on the boundaries. Analogously, define the multivalued map $\mathfrak{h}^+ : N^+ \to \widetilde{N}^+$. Define the multivalued map $\mathfrak{h} : N \to \widetilde{N}$ by

$$\mathfrak{h}(u(p,x)) = \begin{cases} u(\tilde{p}_1, \mathfrak{h}^-(x)), \text{if } x \in C_i^* \\ u(\tilde{p}_i, \mathfrak{h}^-(x)), \text{if } x \in c_{i-1}^* \\ \{u(\tilde{p}_1, \mathfrak{h}^-(x)), u(\tilde{p}_{i+1}, \mathfrak{h}^-(x))\}, \text{if } x \in A_i \end{cases}$$

The case where p is an attracting double singularity, the prove follows analogously.

If $x \in A_i$, then $x = x_i$ ou $x = y_i$. Without loss of generality, suppose that $x = x_i$. Hence, the orbit $u(p, x_i)$ is mapped by $\mathfrak{h}$ to $u(\tilde{p}_1, a_i)$ and $u(\tilde{p}_{i+1}, c_i)$.

Finally, consider the closed set $D = \bigcup_{i=1}^n D_i$, where

$$D_i = \{u(\tilde{p}_1, \mathfrak{h}^-(x)), u(\tilde{p}_{i+1}, \mathfrak{h}^-(x)); x \in A_i\}.$$

Define the map $\mathfrak{p} : \widetilde{N} \to N$ by

$$\mathfrak{p}(u(\tilde{x}, \tilde{y})) = \begin{cases} \mathfrak{h}^{-1}(u(\tilde{x}, \tilde{y})), \text{if } u(\tilde{x}, \tilde{y}) \notin D \\ u(p, \mathfrak{h}^{-1}(\tilde{y})), \text{if } u(\tilde{x}, \tilde{y}) \in D_i \end{cases}.$$

See Figure 14.

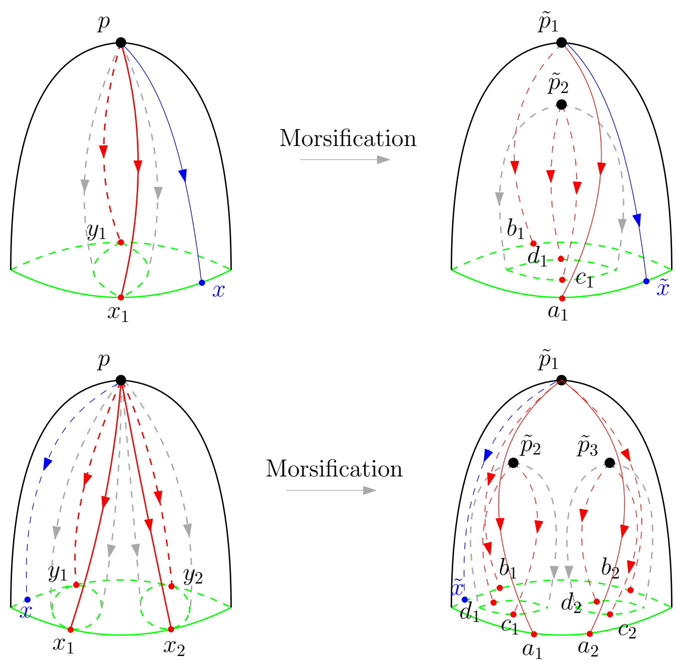


FIGURE 14. Isolating blocks for a repelling 2-sheet and 3-sheet double crossing singularities and their Morsification.

2) Let p be a saddle double crossing singularity of sa (resp., sr) nature. Let N be an isolating block for p, and consider the smooth disjoint block $\widetilde{N}$, containing two regular singularities $\tilde{p}_1$ and $\tilde{p}_2$, where $\tilde{p}_1$ is a saddle and $\tilde{p}_2$ is an attractor (resp., repeller).

The isolating block $\widetilde{N}$ has $\widetilde{N}^-$ and $\widetilde{N}^+$ as exit and entering sets, respectively, where each connected component is homeomorphic to S^1. There are two cases to be considered.

- $\widetilde{N}^-$ (resp., $\widetilde{N}^+$) is disconnected;
- $\widetilde{N}^-$ (resp., $\widetilde{N}^+$) connected.

2.1 Let $N_i^-, i = 1, 2$, be the connected components of the exit set.
Let $x^+, y^+ \in N^+$ be the points of the orbits $u(x^+, p)$ and $u(y^+, p)$ on $\mathcal{SP}(N)$. Consider the external arcs $C_1 = (y^+, x^+)$, $C_2 = (x^+, y^+)$, and the internal arcs $c_1 = (y^+, x^+)$, $c_2 = (x^+, y^+)$ in N^+ with counterclockwise orientations. Define the multivalued map $\mathfrak{h}^+ : N^+ \to \widetilde{N}^+$ by $\mathfrak{h}^+(x^+) = \{a^+, c^+\}$, $\mathfrak{h}^+(y^+) = \{b^+, d^+\}$ and $\mathfrak{h}^+(N^+ \setminus \{x^+, y^+\}) = \widehat{N}^+ \setminus \{a^+, b^+, c^+, d^+\}$, such that the map $\mathfrak{h}^+$ restricted to $N^+ \setminus \{x^+, y^+\}$ is a homeomorphism which preserves the orientation on the boundaries. Consider the trivial homeomorphisms $h_j : N_j^- \to \widetilde{N}_j^-$. Define the multivalued map $\mathfrak{h} : N \to \widetilde{N}$ by

$$\mathfrak{h}(u(x,y)) = \begin{cases} u(\mathfrak{h}^+(x), h_i(y)), \text{if } x \in C_i \\ u(\mathfrak{h}^+(x), \tilde{p}_2), \text{if } x \in c_i \\ \{u(\mathfrak{h}^+(x), \tilde{p}_1), u(\mathfrak{h}^+(x), \tilde{p}_2)\}, \text{if } x \in \{x^+, y^+\} \end{cases}.$$

If $x_1 \in C_i$, the orbit $u(x_1, y_1)$ is mapped by $\mathfrak{h}$ to $u(\tilde{x}_1, \tilde{y_1})$, where $\tilde{x}_1 = \mathfrak{h}^+(x_1)$ and $\tilde{y}_1 = h_i(y_1)$. If $x_2 \in c_i$, the orbit $u(x_2, p)$ is mapped by $\mathfrak{h}$ to $u(\tilde{x}_2, \tilde{p}_2)$, where $\tilde{x}_2 = \mathfrak{h}^+(x_2)$.
If $x = x^+$, the orbit $u(x^+, p)$ is mapped by $\mathfrak{h}$ to $u(a^+, \tilde{p}_1)$ and $u(c^+, \tilde{p}_2)$. Analogously for $x = y^+$.
Finally, consider the closed set $D = \{u(\mathfrak{h}^+(x), \tilde{p}_1), u(\mathfrak{h}^+(x), \tilde{p}_2)\}$. Define $\mathfrak{p} : \widetilde{N} \to N$ by

$$\mathfrak{p}(u(\tilde{x}, \tilde{y})) = \begin{cases} \mathfrak{h}^{-1}(u(\tilde{x}, \tilde{y})), \text{if } u(\tilde{x}, \tilde{y}) \notin D \\ u(\mathfrak{h}^{-1}(\tilde{x}), p), \text{if } u(\tilde{x}, \tilde{y}) \in D \end{cases}.$$

See Figure 15.

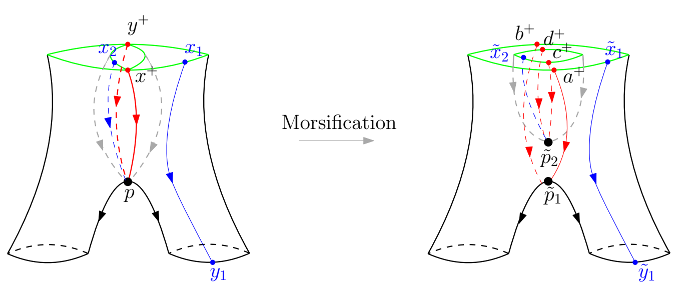


FIGURE 15. Isolating block for a saddle double crossing singularity of *sa*-nature and its Morsification.

2.2 Suppose that N^- and N^+ are both connected. Let $\widetilde{N}$ be the disjoint union of a 2-sphere minus three discs, $\widetilde{N}_1$, and an attracting disc, $\widetilde{N}_2$. The entering set of $\widetilde{N}$ is a disjoint union of three circles C_i, $i = 1, 2, 3$, where the entering set of $\widetilde{N}_1$ is C_1 and C_2 and the entering set of $\widetilde{N}_2$ is C_3.
Let $x^+, y^+ \in N^+$ be the points on the orbits $u(x^+, p)$ and $u(y^+, p)$ on $\mathcal{SP}(N)$. Consider the arcs $c_1 = (x^+, x^+)$, $c_2 = (y^+, y^+)$ $c_3 = (y^+, x^+)$, $c_4 = (x^+, y^+)$ in N^+ with counterclockwise orientation. Define the multivalued map $\mathfrak{h}^+ : N^+ \to \widetilde{N}^+$, by $\mathfrak{h}^+(x^+) = \{a^+, c^+\}$, $\mathfrak{h}^+(y^+) = \{b^+, d^+\}$, $\mathfrak{h}^+(c_1) = C_i$, for $i = 1, 2$, $\mathfrak{h}^+(c_3)$ is the arc (c^+, d^+) in $\widetilde{C_3}$, and $\mathfrak{h}^+(c_4)$ is the arc (d^+, c^+) in $\widetilde{C_3}$, such that the map $\mathfrak{h}^+$

restricted to $N^+ \setminus \{x^+, y^+\}$ is a homemomorphism which preserve orientation on the boundary. Consider the trivial homemomorphism $\mathfrak{h}^- : N^- \to \widetilde{N}^-$. Define the multivalued map $\mathfrak{h} : N \to \widetilde{N}$ by

$$\mathfrak{h}(u(x,y)) = \begin{cases} u(\mathfrak{h}^+(x), \mathfrak{h}^-(y)), \text{ if } x \in c_1 \cup c_2 \\ u(\mathfrak{h}^+(x), \tilde{p}_2), \text{ if } x \in c_3 \cup c_4 \\ \{u(\mathfrak{h}^+(x), \tilde{p}_1), u(\mathfrak{h}^+(x), \tilde{p}_2)\}, \text{if } x \in \{x^+, y^+\} \end{cases}.$$

If $x = x^+$, the orbit $u(x^+, p)$ is mapped by $\mathfrak{h}$ to $u(a^+, \tilde{p}_1)$ and $u(c^+, \tilde{p}_2)$. If $x = y^+$, the orbit $u(y^+, p)$ is mapped by $\mathfrak{h}$ to $u(b^+, \tilde{p}_1)$ and $u(d^+, \tilde{p}_2)$.
Finally consider the closed set $D = \{u(\mathfrak{h}^+(x), \tilde{p}_1), u(\mathfrak{h}^+(x), \tilde{p}_2)\}$. Define $\mathfrak{p} : \widetilde{N} \to N$ by

$$\mathfrak{p}(u(\tilde{x}, \tilde{y})) = \begin{cases} \mathfrak{h}^{-1}(u(\tilde{x}, \tilde{y})), \text{if } u(\tilde{x}, \tilde{y}) \notin D \\ u(\mathfrak{h}^{-1}(\tilde{x}), p), \text{if } u(\tilde{x}, \tilde{y}) \in D \end{cases}.$$

See Figure 16.

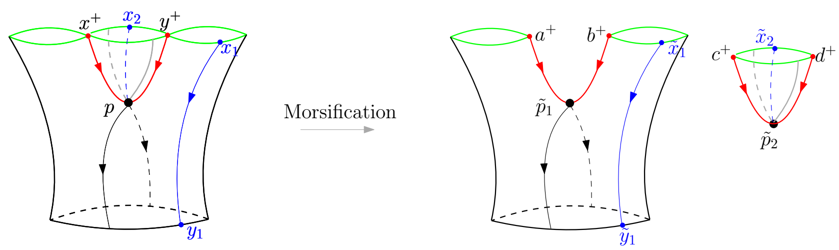


FIGURE 16. Isolating block for a saddle double crossing singularity of sa-nature and its Morsification.

3) Let p be a double crossing saddle singularity of ss_s-nature (ss_u-nature). Let N be an isolating block for p and consider the smooth isolating block $\widetilde{N}$, containing two regular saddle singularities $\tilde{p}_1$ and $\tilde{p}_2$.

The isolating block $\widetilde{N}$ has $\widetilde{N}^-$ and $\widetilde{N}^+$ as exit and entering sets, respectively. We will consider the following cases:

- $\widetilde{N}^+$ (resp., $\widetilde{N}^-$) is connected;
- $\widetilde{N}^+$ (resp., $\widetilde{N}^-$) is disconnected.

3.1) Consider the isolating block N in Figure 17 with exit set homeomorphic to four disjoint circles N^-_{ij}, where N^-_{1j} are the external boundaries and N^-_{2j} are the internal boundaries, $j = 1, 2$. Let $\widetilde{N}$ the a smooth isolating block formed by the disjoint union of two isolating blocks $\widetilde{N}_1$ and $\widetilde{N}_2$, for the regular saddle and attracting singularities $\tilde{p}_1$ and $\tilde{p}_2$, respectively. In $\widetilde{N}_1$ the exit set is $\widetilde{N}^-_{1j}$ and the entering set $\widetilde{N}^+_1$ and in $\widetilde{N}_2$ the exit set is $\widetilde{N}^-_{2j}$ and the entering set is $\widetilde{N}^+_2$, homeomorphic to S^1, $j = 1, 2$.
Let $x^+, y^+ \in N^+$ be points on the orbits $u(x^+, p)$ and $u(y^+, p)$ on $\mathcal{SP}(N)$. Consider the external arcs $C_1 = (y^+, x^+)$, $C_2 = (x^+, y^+)$ and the internal arcs $c_1 = (y^+, x^+)$, $c_2 = (x^+, y^+)$ in N^+ with counterclockwise orientation. Define the multivalued map $\mathfrak{h}^+ : N^+ \to \widetilde{N}^+$ by $\mathfrak{h}^+(x^+) = \{a^+, c^+\}$, $\mathfrak{h}^+(y^+) = \{b^+, d^+\}$ and $\mathfrak{h}^+(N^+ \setminus \{x^+, y^+\}) = \widetilde{N}^+ \setminus \{a^+, b^+, c^+, d^+\}$, such that $\mathfrak{h}^+$ restricted to $N^+ \setminus \{x^+, y^+\}$ is an orientation preserving homeomorphism. Consider the trivial homeomorphisms $h_{ij} : N^-_{ij} \to \widetilde{N}^-_{ij}$. Define the multivalued mpa $\mathfrak{h} : N \to \widetilde{N}$

by

$$\mathfrak{h}(u(x,y)) = \begin{cases} u(\mathfrak{h}^+(x), h_{ij}(y)), \text{if } x \in C_i \cup c_i \text{ and } x \neq p \\ u(\tilde{p}_1, h_{1j}(y)), \text{if } y \in N_{1j}^- \text{ and } x = p \\ u(\tilde{p}_2, h_{2j}(y)), \text{if } y \in N_{2j}^- \text{ and } x = p \\ \{u(\mathfrak{h}^+(x), \tilde{p}_1), u(\mathfrak{h}^+(x), \tilde{p}_2)\}, \text{se } x \in \{x^+, y^+\} \end{cases}$$

If $x_1 \in C_i$, the orbit $u(x_1, y_1)$ is mapped by $\mathfrak{h}$ to $u(\tilde{x}_1, \tilde{y_1})$,where $\tilde{x}_1 = \mathfrak{h}^+(x_1)$ and $\tilde{y}_1 = h_{1j}(y_1)$. If $x_2 \in c_i$, the orbit $u(x_2, y_2)$ is mapped by $\mathfrak{h}$ to $u(\tilde{x}_2, \tilde{y}_2)$, where $\tilde{x}_2 = \mathfrak{h}^+(x_2)$ and $\tilde{y_2} = h_{2j}(y_2)$.
If $x \in N_{1j}^-$, the orbit $u(p, x)$ is mapped by $\mathfrak{h}$ to $u(\tilde{p}_1, \tilde{x})$, where $\tilde{x} = h_{1j}(x)$. If $x \in N_{2j}^-$, the orbit $u(x, p)$ is mapped by $\mathfrak{h}$ to $u(\tilde{p}_2, \tilde{x})$, where $\tilde{x} = h_{2j}(x)$.
If $x = x^+$, the orbit $u(x^+, p)$ is mapped by $\mathfrak{h}$ to $u(a^+, \tilde{p}_1)$ and $u(c^+, \tilde{p}_2)$. Similarly, if $x = y^+$.
Finally, consider the closed set $D = \{u(\mathfrak{h}^+(x), \tilde{p}_1), u(\mathfrak{h}^+(x), \tilde{p}_2)\}$. Define $\mathfrak{p} : \widetilde{N} \to N$ by

$$\mathfrak{p}(u(\tilde{x}, \tilde{y})) = \begin{cases} \mathfrak{h}^{-1}(u(\tilde{x}, \tilde{y})), \text{se } u(\tilde{x}, \tilde{y}) \notin D \\ u(\mathfrak{h}^{-1}(\tilde{x}), p), \text{se } u(\tilde{x}, \tilde{y}) \in D \end{cases}$$

See Figure 17.

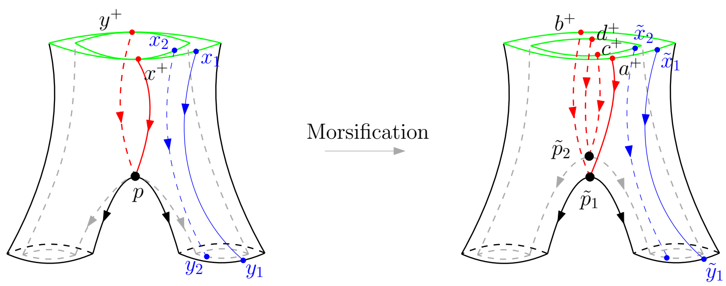


FIGURE 17. Isolating blocks for a saddle double crossing singularity of ss_s-nature and its Morsification.

3.2) This case follows with a similar proof. See Figure 18.

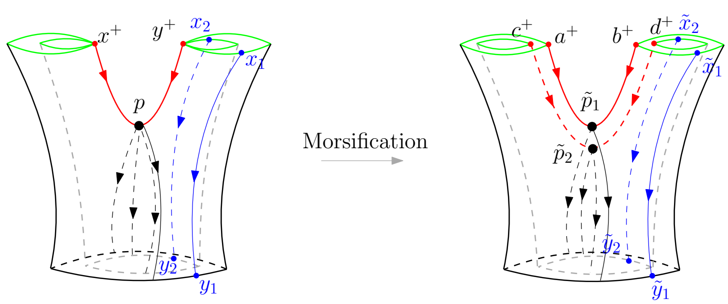


FIGURE 18. Isolating blocks for a saddle double crossing singularity of ss_s-nature and its Morsification.

□

Proposition 3.4. *Let $M \in \mathfrak{M}(\mathcal{GT})$ be a singular 2-manifold, $X \in \mathfrak{X}_{\mathcal{GT}}(M)$ a GS-vector field on M and φ_X the GS-flow associated to X. Given a triple crossing singularity p and an isolating block N for p, there exists a Morsification $(\widetilde{N}, \varphi_{\widetilde{X}})$ such that each orbit of φ_X in $\mathcal{SP}(M)$ admits a triplication of orbits in N.*

Proof. The proof follows the same steps as the previous one. □

Combinatorially the isolating blocks for double and triple singularities together with its Morsification can be seen as the Lyapunov (semi)graphs in Figures 19 and 20. By considering the opposite direction on the graphs in Figure 19, we obtain the graphs for attracting double crossing singularities.

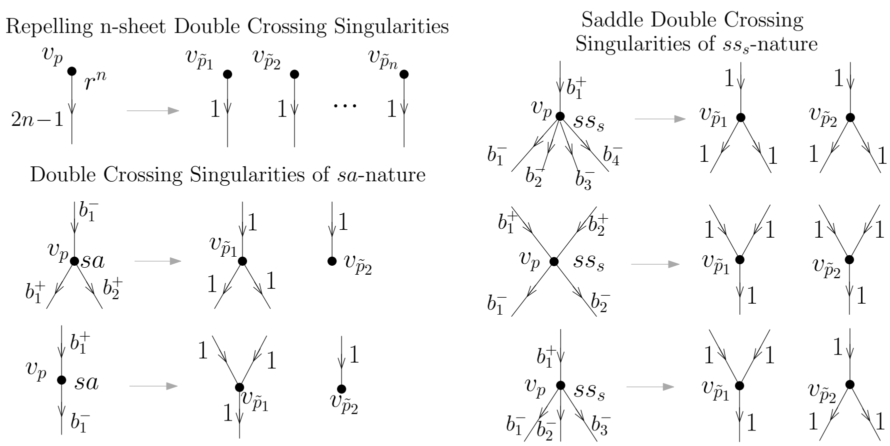


FIGURE 19. Morsification of a Lyapunov semigraph with vertex associated to a double crossing singularity.

There are other isolating blocks for saddle double crossing singularities which are not considered in this work.

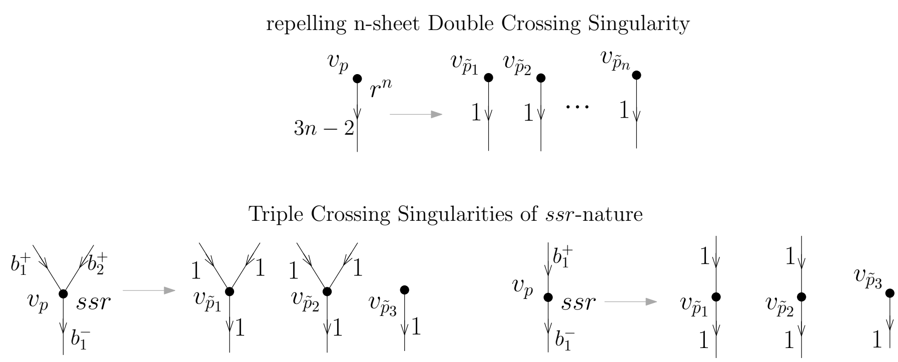


FIGURE 20. Morsification of a Lyapunov semigraph with vertex associated to a triple crossing singularity.

4. Gutierrez-Sotomayor Chain Complex

The goal of this section is to define a chain complex that describes the dynamics of a given GS-flow on a closed stratified 2-manifold $M \in \mathfrak{M}(\mathcal{GS})$, analogously to the Morse chain complex associated to a Morse-Smale flow. We make use of the Morsification process to introduce the notion of GS-intersection numbers which makes it possible to count the flow lines between consecutive GS singularities with sign, as in Subsection 4.2 below. In Subsection 4.1, we present a brief introduction on Morse chain complexes associated to Morse functions on closed manifolds.

4.1. **Morse Chain Complex.** Let M be a smooth closed n-manifold. A smooth function $f : M \to \mathbb{R}$ is called a *Morse function* if each critical point of f is nondegenerate, i.e. the Hessian matrix of f at p, H_p^f, is non-singular. The Morse index $ind_f(p)$ of a critical point p is the dimension of the maximal subspace where H_p^f is negative definite. Moreover, since M is a closed manifold, then the set of critical points of a Morse function is finite.

Fix a Riemannian metric g on M and let $f : M \to \mathbb{R}$ be a smooth Morse function. The identity $g(\nabla f, \cdot) = df(\cdot)$ uniquely determines a gradient vector field ∇f on M. Denote the flow associated to $-\nabla f$ by φ_f, which is called the *negative gradient flow*. The singularities of the vector field $-\nabla f$ correspond to the critical points of f.

A Morse function f is called a *Morse-Smale function* if, for each $x, y \in Crit(f)$, the unstable manifolds of φ_f at x, $W^u(x)$, and the stable manifold of φ_f at y, $W^s(y)$, intersect transversally. We define (f, g) as a Morse-Smale pair. Hereafter, in this subsection, assume that f is a Morse-Smale function, unless stated otherwise. In this case, the negative gradient flow φ_f is also called a *Morse flow*.

Given $x, y \in Crit(f)$, the *connecting manifold* of x and y is given by $\mathcal{M}_{xy} := W^u(x) \cap W^s(y)$. The connecting manifold $\mathcal{M}_{xy}$ is the set containing all points $p \in M$ such that $\omega(p) = y$ and $\alpha(p) = x$. The *moduli space* between x and y is defined by $\mathcal{M}_y^x(a) := \mathcal{M}_{xy} \cap f^{-1}(a)$, where a is a regular value between $f(x)$ and $f(y)$. The space $\mathcal{M}_y^x(a)$ is a set of points that are in 1-1 correspondence to the orbits running from x to y. For different choices of regular values a_1, a_2 there is a natural identification between $\mathcal{M}_y^x(a_1)$ and $\mathcal{M}_y^x(a_2)$ given by the flow. Hence, one uses the notation $\mathcal{M}_y^x$ for the moduli space. Whenever f is a Morse-Smale function, the connecting manifolds and the moduli spaces are orientable closed submanifolds of M of dimensions $\dim(\mathcal{M}_{xy}) = ind_f(x) - ind_f(y)$, and $\dim(\mathcal{M}_y^x) = ind_f(x) - ind_f(y) - 1$, respectively.

Once orientations are chosen for $W^u(x)$ and $W^u(y)$, these induce an orientation on $\mathcal{M}_{xy}$ denoted by $[\mathcal{M}_{xy}]_{ind}$, for $x, y \in Crit(f)$. The procedure given in [22] to obtain this orientation is:

(1) If $ind_f(y) > 0$, then
   (a) Let $\mathcal{V}_{\mathcal{M}_{xy}} W^s(y)$ be the normal bundle of $W^s(y)$ restricted to $\mathcal{M}_{xy}$. Consider the fiber $\mathcal{V}_y W^s(y)$ with an orientation given by the isomorphism

$$T_y W^u(y) \oplus T_y W^s(y) \simeq T_y M \simeq \mathcal{V}_y W^s(y) \oplus T_y W^s(y).$$

   The orientation on the fiber at y determines an orientation on the normal bundle $\mathcal{V}_{\mathcal{M}_{xy}} W^s(y)$ restricted to the submanifold $\mathcal{M}_{xy}$.

   (b) The orientation on $\mathcal{M}_{xy}$ is determined by the isomorphism

$$T_{\mathcal{M}_{xy}} W^u(x) \simeq T\mathcal{M}_{xy} \oplus \mathcal{V}_{\mathcal{M}_{xy}} W^s(y).$$

(2) If $ind_f(y) = 0$, then $\mathcal{V}_y W^s(y) = 0$. Hence, $T_{\mathcal{M}_{xy}} W^u(x) \simeq T\mathcal{M}_{xy}$.

Note that there are no restrictions on the orientability of the manifold M.

Given $x, y \in Crit(f)$ with $ind_f(x) - ind_f(y) = 1$, let $u \in \mathcal{M}_y^x$. The *characteristic sign* n_u of the orbit $\mathcal{O}(u)$ through u is defined via the identity $[\mathcal{O}(u)]_{ind} = n_u[\dot{u}]$, where $[\dot{u}]$ and $[\mathcal{O}(u)]_{ind}$

denote the orientations on $\mathcal{O}(u)$ induced by the flow and by $\mathcal{M}_{xy}$, respectively. The *intersection number* of x and y is defined by

$$n(x,y) = \sum_{u \in \mathcal{M}_y^x} n_u.$$

The intersection number between x and y counts, with sign, the flow lines from x to y. In the literature there are other ways to count such flow lines with orientations, for example, see [1], Chapter 7.

Fix an arbitrary orientation for the unstable manifolds $W^u(x)$, for each $x \in Crit(f)$, and denote by Or the set of these choices. The *Morse graded group* $C = \{C_k(f)\}$ is defined as the free abelian groups generated by the critical points of f and graded by their Morse index, i.e.,

$$C_k(f) := \bigoplus_{x \in Crit_k(f)} \mathbb{Z}\langle x \rangle,$$

where $\langle x \rangle$ denotes the pair consisting of the critical point x of f and the orientation chosen on $W^u(x)$. The *Morse boundary operator* $\partial_k(x) : C_k(f) \longrightarrow C_{k-1}(f)$ is given on a generator x of $C_k(f)$ by

$$\partial_k \langle x \rangle := \sum_{y \in Crit_{k-1}(f)} n(x,y) \langle y \rangle, \tag{2}$$

and it is extended by linearity to general chains.

The pair $(C_*(f), \partial_*)$ is a chain complex, that is, ∂ is of degree -1 and $\partial \circ \partial = 0$. This chain complex is called a *Morse chain complex*.

The proof that $\partial \circ \partial = 0$ follows by analyzing the 1-dimensional connected components of the moduli space $\mathcal{M}_z^x$, where $x \in Crit_k(f)$ and $z \in Crit_{k-2}(f)$, which can be either diffeomorphic to $(0,1)$ or to S^1, as in Figure 21. In [22], it is proved that if (u,v) and $(\tilde{u},\tilde{v})$ are two broken flow lines corresponding to the ends of a noncompact connected component of $\mathcal{M}_z^x$, then the *null cycle condition* is satisfied, i.e. $n_u n_v + n_{\tilde{u}} n_{\tilde{v}} = 0$.

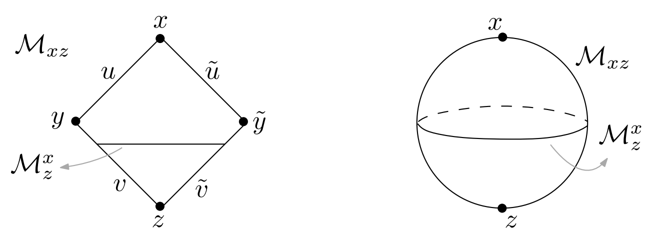


FIGURE 21. Possible connected components of $\mathcal{M}_{xz}$ for $x \in Crit_k(f)$ and $z \in Crit_{k-2}(f)$.

The *Morse homology groups* with integer coefficients are defined by

$$HM_k(M,f,g,Or;\mathbb{Z}) = \frac{\text{Ker } \partial_k}{\text{Im } \partial_{k+1}}, \quad \forall k \in \mathbb{Z}.$$

In [22], it was proved that, for two choices of Morse-Smale pairs (f^1,g^1) and (f^2,g^2) with orientations Or^1 and Or^2 on all unstable manifolds, the associated Morse homology groups $HM_k(M,f^1,g^1,Or^1;\mathbb{Z})$ and $HM_k(M,f^2,g^2,Or^2;\mathbb{Z})$ are naturally isomorphic, for all $k \in \mathbb{Z}$. Hence, this homology is simple denoted by $HM_*(M,\mathbb{Z})$. Moreover, one has that $HM_*(M;\mathbb{Z}) \cong H^{sing}(M;\mathbb{Z})$, i.e., the Morse homology of M is isomorphic to the singular homology of M.

4.2. **Gutierrez-Sotomayor Chain Complex.** Let $M \in \mathfrak{M}(\mathcal{GS})$ be a compact singular 2-manifold, $X \in \mathfrak{X}_{\mathcal{GS}}(M)$ a GS-vector field on M and φ_X the Gutierrez-Sotomayor flow on M associated to X. In this section, one defines a chain complex for a given GS-flow analogous to the Morse chain complex of a Morse-Smale flow. We will start by obtaining the characteristic signs of the flow lines on M from the characteristic signs of the flow lines on the smooth surface $\widetilde{M}$ obtained by a Morsification process. Subsequently, it is possible to define a GS-chain group and a GS-boundary map, as in the Definition 4.2.

Given $x, y \in Sing(X)$, define the ***connecting manifold*** of x and y by

$$\mathcal{M}_{xy} := \mathcal{M}_{xy}(X, M) := W^u(x) \cap W^s(y),$$

where W^s, W^u are the stable and unstable sets of the singularity, respectively. In other words, the connecting manifold $\mathcal{M}_{xy}$ is composed by the points $p \in M$ such that $\omega(p) = y$ and $\alpha(p) = x$. The ***moduli space*** between the singularities x and y is defined as the quotient of the connecting manifold $\mathcal{M}_{xy}$ by the natural action of $\mathbb{R}$ on the flow lines, i.e.,

$$\mathcal{M}_y^x := \mathcal{M}_{xy}/\mathbb{R}.$$

Define the nature numbers of a GS-singularity as follows:

Definition 4.1. *Denote by $Sing(X)$ the set of singularities of a vector field $X \in \mathfrak{X}_{\mathcal{GS}}(M)$. Given $p \in Sing(X)$, define $\eta_k(p)$ as the k-th **nature number** of p, where:*

- *$k = 2$ represents the repelling nature r;*
- *$k = 1$ represents the saddle nature s;*
- *$k = 0$ represents the attracting nature a.*

Two singularities x and y are said to be ***consecutive*** if $\eta_k(x)$ and $\eta_{k-1}(y)$ are both non zero, for some $k = 1, 2$.

For example, if p is a triple crossing singularity of ssa nature, we have that $\eta_2(p) = 0$, $\eta_1(p) = 2$ and $\eta_0(p) = 1$.

Note that in the Morse-Smale case, each singularity of index k has only one nature, implying that it contributes with only one generator for the k-th Morse chain group. The same holds for cone singularities and Whitney singularities. However, this is not the case for the double and triple crossing singularities, since they have at least two natures. Hence, these type of singularities will have more than one generator in the GS-chain groups associated with them. Moreover, for a double or triple crossing singularity x, the singularities associated to x by the Morsification process are in one-to-one correspondence with the collection of nature numbers of x.

In order to distinguish the generators provided by a GS-singularity x, we denote the generators of the nature of a singularity x by

$$\{h_k^i(x) \mid \; i = 1, \dots, \eta_k(x), k = 0, 1, 2\},$$

where $h_k^i(x)$ represents a generator of k-nature of the singularity x. The advantage of this notation is that this set $\{h_k^i(x) \mid x \in Sing(X),\; i = 1, \dots, \eta_k(x)\}$ will generate the k-chain group of the GS-chain complex that we define below.

Definition 4.2. *Given a GS-flow φ_X, the **Gutierrez-Sotomayor chain group** $C_k^{\mathcal{GS}}(M, X)$ with integer coefficients graded by the nature of the singularities is the free abelian group generated by the set of GS-singularities $Sing(X)$ of the vector field X, i.e.:*

$$C_k^{\mathcal{GS}}(M, X) := \bigoplus_{x \in Sing(X)} \left(\bigoplus_{i=1}^{\eta_k(x)} \mathbb{Z}\langle h_k^i(x) \rangle \right), \qquad k \in \mathbb{Z},$$

where $h_k^i(x)$ denotes a generator associated to the k-nature of the singularity x. The k-th ***Gutierrez-Sotomayor boundary map***, $\Delta_k^{\mathcal{GS}} : C_k^{\mathcal{GS}}(M,X) \to C_{k-1}^{\mathcal{GS}}(M,X)$, *is given on a generator* $h_k^i(x)$ *by*

$$\Delta_k^{\mathcal{GS}}\langle h_k^i(x)\rangle := \sum_{y\in Sing(X)} \left(\sum_{j=1}^{\eta_{k-1}(y)} n(h_k^i(x), h_{k-1}^j(y))\langle h_{k-1}^j(y)\rangle \right),$$

and it is extended by linearity to general chains.

The overarching idea is to make use of the Morsification of the GS-flow defined in Section 3, in order to define a GS-intersection number from the Morse counterpart. Intersection numbers depends heavily on the smooth structure of the manifold, i.e, the existence of tangent and normal bundles. The number $n(h_k^i(x), h_{k-1}^j(y))$ is called the GS-intersection number of the generators $h_k^i(x)$ and $h_{k-1}^j(y)$ and will be defined in the following subsections as the sum $\sum n_u$, over all flow lines $u \in \mathcal{M}_{h_{k-1}^j(y)}^{h_k^i(x)}$, where n_u is the GS-characteristic sign of the flow line u. This process relies on the Morsification process.

Furthermore, in the following subsections, we prove that, given $M \in \mathfrak{M}(\mathcal{GS})$ and $X \in \mathfrak{X}_{\mathcal{GS}}(M)$, the pair $(C_*^{\mathcal{GS}}(M,X), \Delta_*^{\mathcal{GS}})$ is a chain complex which we refer to as a ***Gutierrez-Sotomayor chain complex***.

Throughout this section, a GS-chain complex will be defined for flows associated to vector fields restricted to flows associated to vector fields X in $\mathfrak{X}_{\mathcal{GC}}(M)$, $\mathfrak{X}_{\mathcal{GW}}(M)$, $\mathfrak{X}_{\mathcal{GD}}(M)$ and $\mathfrak{X}_{\mathcal{GT}}(M)$.

4.3. **Gutierrez-Sotomayor complex for cone singularities.** In the previous section, we defined the Morsification process of a given GS-flow containing only regular and cone type singularities on a singular 2-manifold $M \in \mathfrak{M}(\mathcal{GC})$ in order to obtain a smooth manifold $\widetilde{M} \in \mathfrak{M}(\mathcal{R})$ with a smooth Morse flow $\widetilde{\varphi}_{\widetilde{X}}$ on it. Therefore, one can attach to each flow line of $\widetilde{\varphi}_{\widetilde{X}}$ a characteristic sign. Now, the idea is to transfer these signs to the corresponding flow lines of the singular flow φ_X.

We now define the transfer process of characteristic signs from the Morse setting to the GS-setting.

Definition 4.3 (Characteristic signs of flows lines of $\mathfrak{X}_{\mathcal{GC}}(M)$)**.** *Consider $x, y \in Sing(X)$ singularities of consecutive natures, where $X \in \mathfrak{X}_{\mathcal{GC}}(M)$. The* ***GS-characteristic sign*** *n_u of a flow line $u \in \mathcal{M}_{xy}$ is defined as follows:*

(1) *Let x be a singularity of repeller nature and y a cone singularity of saddle nature. Denote by $\tilde{y}, \tilde{y}'$ the singularities associated to y by the Morsification process, and by $\tilde{x}$ the singularity of repeller nature associated to x. It is easy to see that $\mathcal{M}_y^x \approx \widetilde{\mathcal{M}}_{\tilde{y}}^{\tilde{x}} \approx \widetilde{\mathcal{M}}_{\tilde{y}'}^{\tilde{x}}$. Hence, given $u \in \mathcal{M}_y^x$ there are corresponding flow lines $\tilde{u} \in \widetilde{\mathcal{M}}_{\tilde{y}}^{\tilde{x}}$ and $\tilde{u}' \in \widetilde{\mathcal{M}}_{\tilde{y}'}^{\tilde{x}}$ in the Morsified flow. Define*

$$n_u := \begin{cases} n_{\tilde{u}}, & \text{if } n_{\tilde{u}} = n_{\tilde{u}'} \\ 0, & \text{if } n_{\tilde{u}} \neq n_{\tilde{u}'} \end{cases}$$

(2) *Let x be a cone singularity of saddle nature and y a singularity of attractor nature. Denote by $\tilde{x}, \tilde{x}'$ the singularities associated to x by the Morsification process, and by $\tilde{y}$ the singularity of repeller nature associated to y. It is easy to see that $\mathcal{M}_y^x \approx \widetilde{\mathcal{M}}_{\tilde{y}}^{\tilde{x}} \approx \widetilde{\mathcal{M}}_{\tilde{y}}^{\tilde{x}'}$. Hence, given $u \in \mathcal{M}_y^x$ there are corresponding flow lines $\tilde{u} \in \widetilde{\mathcal{M}}_{\tilde{y}}^{\tilde{x}}$ and $\tilde{u}' \in \widetilde{\mathcal{M}}_{\tilde{y}}^{\tilde{x}'}$ in the*

Morsified flow. Define

$$n_u := \begin{cases} n_{\tilde{u}}, & \text{if } n_{\tilde{u}} = n_{\tilde{u}'} \\ 0, & \text{if } n_{\tilde{u}} \neq n_{\tilde{u}'} \end{cases}$$

(3) *For the other cases, one has that* $\mathcal{M}_y^x \approx \widetilde{\mathcal{M}}_{\tilde{y}}^{\tilde{x}}$. *For each* $u \in \mathcal{M}_y^x$, *define* $n_u := n_{\tilde{u}}$.

Once the characteristic signs are well defined for the flow lines of φ_X with $X \in \mathfrak{X}_{\mathcal{GC}}(M)$, the ***GS-intersection number*** between consecutive singularities x and y is defined as

$$n(x,y) := \sum_{u \in \mathcal{M}_y^x} n_u,$$

This sum is finite since the moduli space is compact. i.e., the sum of the characteristic signs of the flow lines connecting x to y. Thus, the GS-boundary map $\Delta_*^{\mathcal{GC}}$, described in Definition 4.2, is well defined for GS-flows on $M \in \mathfrak{M}(\mathcal{GC})$. Now, we will prove that $(C_*^{\mathcal{GC}}(M,X), \Delta_*^{\mathcal{GC}})$ is in fact a chain complex for $X \in \mathfrak{X}_{\mathcal{GC}}(M)$ and $M \in \mathfrak{M}(\mathcal{GC})$.

Lemma 4.1. *Let* $G_{\mathcal{GC}}$ *be the graph of the matrix associated to the GS-boundary map* $\Delta^{\mathcal{GC}}$ *and let* $y \in Sing(X)$ *be a cone singularity of saddle nature. Therefore, the incidence degree of the vertex* v_y *is null or* 2, *and in the latter case, the two edges are both positively or negatively incident in* v_y. *Consequently, there is no cycle in* $G(\Delta^{\mathcal{GC}})$ *containing* v_y.

Proof. Let $y \in Sing(X)$ be a cone singularity of saddle nature. Since, y has saddle nature, the positively and negatively incident edges appear in pairs and up to two positively incident edges and two negatively incident edges. Hence, the incident degree of v_y belongs to the set $\{0,2,4\}$.

Suppose that the incident degree of the vertex v_y is four. This is also the case for the vertices $v_{\tilde{y}}$ and $v_{\tilde{y}'}$ of the Morsified boundary map $\widetilde{\Delta}$, where $\tilde{y}, \tilde{y}'$ are the singularities associated to y by the Morsification process. Therefore, there are two vertices $v_{\tilde{x}_1}$, $v_{\tilde{x}_2}$ corresponding to repeller singularities and two vertices $v_{\tilde{z}_1}$, $v_{\tilde{z}_2}$ corresponding to attractor singularities which belong to distinct cycles in $G(\widetilde{\Delta})$ involving the vertices $v_{\tilde{y}}$ and $v_{\tilde{y}'}$. Moreover, these cycles can be chosen matching the ends of noncompact connected components of the moduli spaces $\widetilde{\mathcal{M}}_{\tilde{x}_1\tilde{z}_1}$ and $\widetilde{\mathcal{M}}_{\tilde{x}_2\tilde{z}_2}$. Let $(\tilde{u}_i, \tilde{v}_i) \in \widetilde{\mathcal{M}}_{\tilde{y}}^{\tilde{x}_i} \times \widetilde{\mathcal{M}}_{\tilde{z}_i}^{\tilde{y}}$ and $(\tilde{u}_i', \tilde{v}_i') \in \widetilde{\mathcal{M}}_{\tilde{y}'}^{\tilde{x}_i} \times \widetilde{\mathcal{M}}_{\tilde{z}_i}^{\tilde{y}'}$, be the broken orbits that correspond the ends of such component of $\widetilde{\mathcal{M}}_{\tilde{x}_i\tilde{z}_i}$, for $i = 1,2$, see Figure 22.

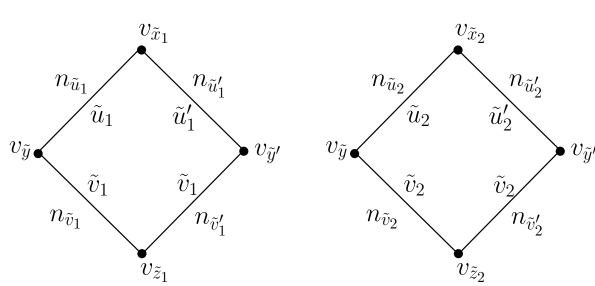


FIGURE 22. Connected components of $\widetilde{\mathcal{M}}_{\tilde{x}_1\tilde{z}_1}$ and $\widetilde{\mathcal{M}}_{\tilde{x}_2\tilde{z}_2}$.

Since these spaces correspond to moduli spaces of order two of a Morse flow,

$$n_{\tilde{u}_i}.n_{\tilde{v}_i} + n_{\tilde{u}_i'}.n_{\tilde{v}_i'} = 0, \quad \text{for} \quad i = 1,2,$$

as proved in [22]. Thus, for each $i = 1, 2$, there are exactly two possibilities for the characteristic signs:

$$\begin{cases} n_{\tilde{u}_i} = n_{\tilde{u}'_i} \\ n_{\tilde{v}_i} \neq n_{\tilde{v}'_i} \end{cases} \quad or \quad \begin{cases} n_{\tilde{u}_i} \neq n_{\tilde{u}'_i} \\ n_{\tilde{v}_i} = n_{\tilde{v}'_i} \end{cases}.$$

If the first (resp., second) possibility holds, then by the sign transfer process, one has that $n_{v_i} = 0$ (resp., $n_{u_i} = 0$). See Figure 23. In any case, it contradicts the assumption that there are four incident edges to the vertex v_y of $G(\Delta^{\mathcal{GC}})$. Hence, the incidence degree of v_y is either 0 or 2.

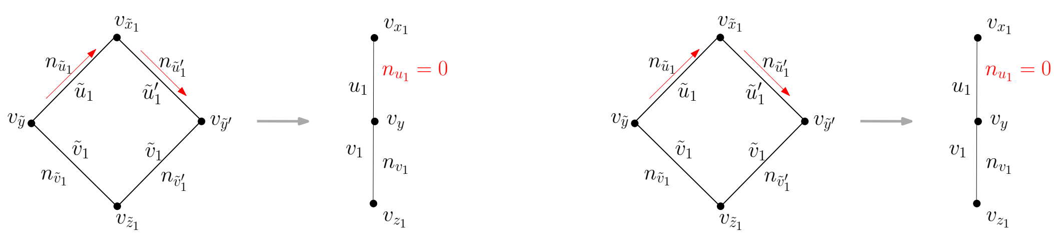


FIGURE 23. Transfer of characteristic signs of the ends of the noncompact connected components of $\widetilde{\mathcal{M}}_{\tilde{x}_1\tilde{z}_1}$ and $\widetilde{\mathcal{M}}_{\tilde{x}_2\tilde{z}_2}$

Now, we need to prove that if the incidence degree of v_y is 2 then both edges are positively incident or both are negatively incident to v_y. Since the characteristic signs of the two flow lines on the unstable manifold of a saddle are opposite, then $n_{\tilde{v}_2} = -n_{\tilde{v}_1}$ and $n_{\tilde{v}'_2} = -n_{\tilde{v}'_1}$. Thus:

$$\begin{cases} n_{\tilde{u}_1} = n_{\tilde{u}'_1} \\ n_{\tilde{v}_1} \neq n_{\tilde{v}'_1} \end{cases} \quad \Rightarrow \quad \begin{cases} n_{\tilde{u}_2} = n_{\tilde{u}'_2} \\ n_{\tilde{v}_2} \neq n_{\tilde{v}'_2} \end{cases}$$

and

$$\begin{cases} n_{\tilde{u}_1} \neq n_{\tilde{u}'_1} \\ n_{\tilde{v}_1} = n_{\tilde{v}'_1} \end{cases} \quad \Rightarrow \quad \begin{cases} n_{\tilde{u}_2} \neq n_{\tilde{u}'_2} \\ n_{\tilde{v}_2} = n_{\tilde{v}'_2} \end{cases}.$$

In other words, if the incidence degree of v_y is two then the two edges are positively or negatively incident to it. $\square$

The previous lemma is essential in the proof that $(C_*^{\mathcal{GC}}(M,X), \Delta_*^{\mathcal{GC}})$ is a chain complex.

Theorem 4.1. *Let $\Delta_*^{\mathcal{GC}}$ be the GS-boundary map associated to φ_X, where $X \in \mathfrak{X}_{\mathcal{GC}}(M)$. Then $\Delta_{k-1}^{\mathcal{GC}} \circ \Delta_k^{\mathcal{GC}} = 0$, for all $k \in \mathbb{Z}$.*

Proof. Given a singularity of repeller nature x and a singularity of attractor nature z, consider

$$\mathcal{B}^1_{xz} := \{(u,v) \mid u \in \mathcal{M}^x_y, v \in \mathcal{M}^y_z, \text{ for } y \in Sing(X) \text{ of saddle nature}\}.$$

With this notation, one can write the composition $\Delta^{\mathcal{GC}}_{k-1} \circ \Delta^{\mathcal{GC}}_{k}$ as follows:

$$
\begin{aligned}
\Delta^{\mathcal{GC}}_{k-1} \circ \Delta^{\mathcal{GC}}_{k}(x) &= \sum_{z \in Sing(X)} \left(\sum_{y \in Sing(X)} n(x,y)n(y,z) \right) z \\
&= \sum_{z \in Sing(X)} \left(\sum_{y \in Sing(X)} \sum_{u \in \mathcal{M}^x_y} \sum_{v \in \mathcal{M}^y_z} n_u n_v \right) z \\
&= \sum_{z \in Sing(X)} \left(\sum_{(u,v) \in \mathcal{B}^1_{xz}} n_u n_v \right) z \\
&= \sum_{z \in Sing(X)} \left(\sum (n_{u_i} n_{v_i} + n_{u_j} n_{v_j}) \right) z
\end{aligned}
$$

where the sum in the last equality is over the ends of the connected components of $\mathcal{M}^x_z$, for all $z \in Sing(X)$ of attractor nature.

In terms of the graph of the map $G(\Delta^{\mathcal{GC}})$, fixing x and z, each term $n_{u_i} n_{v_i} + n_{u_j} n_{v_j}$ of the last sum corresponds to a cycle in $G(\Delta^{\mathcal{GC}})$ connecting the vertices v_x and v_z. By Lemma 4.1, no cycle in $G(\Delta^{\mathcal{GC}})$ contains a vertex v_y, where y is a cone singularity of saddle nature. Therefore, the cycles in the graph $G(\Delta^{\mathcal{GC}})$ are also cycles in the graph $G(\widetilde{\Delta})$, where $\widetilde{\Delta}$ is the Morsified boundary map. Since the cycles we are considering correspond to the ends of noncompact connected components of the moduli space $\mathcal{M}^z_x$ of order 2, then $n_{u_i} n_{v_i} + n_{u_j} n_{v_j} = 0$. It follows that $\Delta^{\mathcal{GC}}_{k-1} \circ \Delta^{\mathcal{GC}}_{k} = 0$. □

We have shown that the pair $(C^{\mathcal{GC}}_*(M, X), \Delta^{\mathcal{GC}}_*)$ is in fact a chain complex whenever $X \in \mathfrak{X}_{\mathcal{GC}}(M)$ and $M \in \mathfrak{M}(\mathcal{GC})$.

Example 4.1. Consider a GS-flow φ_X defined on a singular manifold M and its Morsification $(\widetilde{M}, \varphi_{\widetilde{X}})$ as in Figure 24, where the characteristic sign transfer process is illustrated. Consider as well, the choice of orientations on the unstable manifolds of the critical points of $\widetilde{M}$. The GS-characteristic signs on the orbits of φ_X are obtained from this choice as shown in Figure 24.

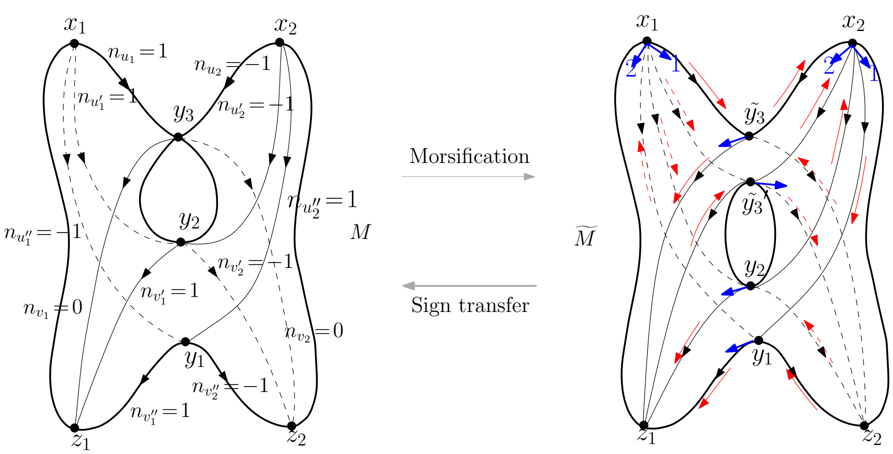


FIGURE 24. A GS-flow on a pinched torus with cone singularities and its Morsification.

Let us examine this example in more detail. For instance, in the sign transfer process, consider the connecting manifold $\widetilde{\mathcal{M}}_{x_1z_2}$ with its ends given by the broken flow lines $(\tilde{u_1}, \tilde{v_2})$ and $(\tilde{\tilde{u_1}}, \tilde{\tilde{v_2}})$. One has that $n_{\tilde{u_1}} = 1 = n_{\tilde{\tilde{u_1}}}$. On the other hand, since $n_{u_1} = 1$ and $n_{\tilde{v_2}} = -1 \neq 1 = n_{\tilde{\tilde{v_2}}}$, one has that $n_{v_2} = 0$. See Figure 25. Analogously, the same analysis holds for $\mathcal{M}_{x_2z_1}$, obtaining $n_{u_2} = -1$ and $n_{v_1} = 0$.

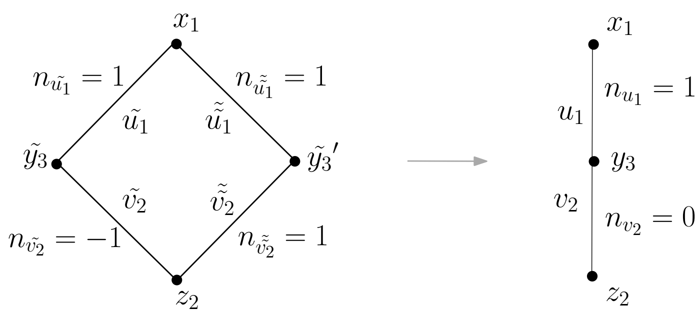


FIGURE 25. Characteristic sign tranfer.

The remaining orbits inherit the same characteristic signs as in $\widetilde{M}$, since all but y_3 are regular singularities.

The GS-chain groups are:

$C_2^{\mathcal{GC}}(M,X) = \mathbb{Z}\langle x_1\rangle \oplus \mathbb{Z}\langle x_2\rangle$, $C_1^{\mathcal{GC}}(M,X) = \mathbb{Z}\langle y_1\rangle \oplus \mathbb{Z}\langle y_2\rangle \oplus \mathbb{Z}\langle y_3\rangle$, $C_0^{\mathcal{GC}}(M,X) = \mathbb{Z}\langle z_1\rangle \oplus \mathbb{Z}\langle z_2\rangle$, and $C_k^{\mathcal{GC}}(M) = 0, k \neq 0,1,2$.

The GS-intersection numbers are:

$n(x_1, y_1) = n_{u_1''} = -1$, $n(x_2, y_1) = n_{u_2''} = 1$, $n(x_1, y_2) = n_{u_1'} = 1$, $n(x_2, y_2) = n_{u_2'} = -1$, $n(x_1, y_3) = n_{u_1} = 1$, $n(x_2, y_3) = n_{u_2} = -1$, $n(y_1, z_1) = n_{v_1''} = 1$, $n(y_1, z_2) = n_{v_2''} = -1$, $n(y_2, z_1) = n_{v_1'} = 1$, $n(y_2, z_2) = n_{v_2'} = -1$, $n(y_3, z_1) = n_{v_1} = 0$ and $n(y_3, z_2) = n_{v_2} = 0$. Therefore, the GS-boundary maps $\Delta_2^{\mathcal{GC}} : C_2 \to C_1$, $\Delta_1^{\mathcal{GC}} : C_1 \to C_0$ and $\Delta_0^{\mathcal{GC}} : C_0 \to 0$ are given by:

$$\Delta_2^{\mathcal{GSC}}(x_1) = -\langle y_1\rangle + \langle y_2\rangle + \langle y_3\rangle,\ \Delta_2^{\mathcal{GC}}(x_2) = \langle y_1\rangle - \langle y_2\rangle - \langle y_3\rangle$$

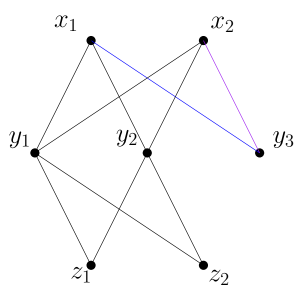
$$\Delta_1^{\mathcal{GC}}(y_1) = \langle z_1\rangle - \langle z_2\rangle,\ \Delta_1^{\mathcal{GC}}(y_2) = \langle z_1\rangle - \langle z_2\rangle,\ \Delta_1^{\mathcal{GC}}(y_3) = 0$$

$$\Delta_0^{\mathcal{GC}}(z_1) = 0 = \Delta_0(z_2),$$

respectively. Hence, the matrix associated to the GS-boundary map $\Delta^{\mathcal{GC}}$ and its graph $G_{\mathcal{GC}}$ are as in Figure 26.

$$\Delta^{\mathcal{GC}}(M) = \begin{array}{c|ccccccc} & z_1 & z_2 & y_1 & y_2 & y_3 & x_1 & x_2 \\ \hline z_1 & & & 1 & 1 & 0 & & \\ z_2 & & & -1 & -1 & 0 & & \\ y_1 & & & & & & -1 & 1 \\ y_2 & & & & & & 1 & -1 \\ y_3 & & & & & & 1 & -1 \\ x_1 & & & & & & & \\ x_2 & & & & & & & \end{array} \qquad \widetilde{\Delta}(\widetilde{M}) = \begin{array}{c|cccccccc} & \tilde{z}_1 & \tilde{z}_2 & \tilde{y}_1 & \tilde{y}_2 & \tilde{y}_3{}' & \tilde{y}_3 & \tilde{x}_1 & \tilde{x}_2 \\ \hline \tilde{z}_1 & & & 1 & 1 & -1 & 1 & & \\ \tilde{z}_2 & & & -1 & -1 & 1 & -1 & & \\ \tilde{y}_1 & & & & & & & -1 & 1 \\ \tilde{y}_2 & & & & & & & 1 & -1 \\ \tilde{y}_3{}' & & & & & & & 1 & -1 \\ \tilde{y}_3 & & & & & & & 1 & -1 \\ \tilde{x}_1 & & & & & & & & \\ \tilde{x}_2 & & & & & & & & \end{array}$$

$G(\Delta^{\mathcal{GC}})$



$G(\widetilde{\Delta})$

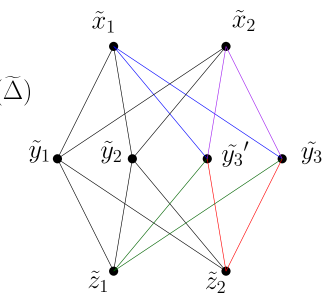


FIGURE 26. GS-boundary map and its matrix graph and its associated Morsification.

It is also interesting to verify that Lemma 4.1 holds in this example. Note that the orbits connecting $\tilde{y}_3$ and $\tilde{z}_1$, $\tilde{y}_3'$ and $\tilde{z}_1$ have opposite characteristic signs. Hence the characteristic sign of the orbit connecting y_3 and z_1 is zero. The same holds for the orbit that connects y_3 and z_2. Note that $\widetilde{G}$, the graph associated to $\widetilde{\Delta}$ has cycles containing the vertices $v_{\tilde{y}_3}$ and $v_{\tilde{y}_3{}'}$. However, the graph $G_{\mathcal{GC}}$ of the boundary map $\Delta^{\mathcal{GC}}$ has no cycle containing v_{y_3}. This is always the case, as it was proven in Lemma 4.1, which asserts that vertices associated to saddle cone singularities have only either two positively incident or two negatively incident edges. Hence, the cycles in $G_{\mathcal{GC}}$ are also in $\widetilde{G}$ and inherit the null cycle condition. △

4.4. **Gutierrez-Sotomayor complex for Whitney, double and triple crossing singularities.** Given a singular flow φ_X on a singular 2-manifold $M \in \mathfrak{M}(\mathcal{GS})$ associated to $X \in \mathfrak{X}_{\mathcal{GS}}(M)$, where $\mathcal{S} = \mathcal{W}, \mathcal{D}$ or $\mathcal{T}$, the Morsification process gives us a Morsified flow $\varphi_{\widetilde{X}}$ on a smooth manifold $\widetilde{M} \in \mathfrak{M}(\mathcal{GC})$ associated to $\widetilde{X} \in \mathfrak{X}(\mathcal{GC})$. In what follows, we define characteristic signs for the flow lines of φ_X by means of the transfer process of the characteristic signs of the orbits of the Morse flow $\varphi_{\widetilde{X}}$.

For the case of a vector field $X \in \mathfrak{X}_{\mathcal{GW}}(M)$, note that each Whitney singularity x of X has an associated regular singularity $\widetilde{x}$ in $\widetilde{X} \in \mathfrak{X}(\mathcal{GC})$, which makes the sign transfer process straightforward.

Definition 4.4 (Characteristic signs of flows lines of $\mathfrak{X}_{\mathcal{GW}}(M)$)**.** *Let $x, y \in Sing(X)$ be singularities of consecutive natures and $X \in \mathfrak{X}_{\mathcal{GW}}(M)$. The **GS-characteristic sign** n_u of a flow line $u \in \mathcal{M}_{xy}$ is defined as follows:*

(1) *If u does not belong to the singular part of M, define $n_u = n_{\tilde{u}}$;*
(2) *If u belongs to the singular part of M, define $n_u = 0$.*

The numbers n_u are well defined, since the orbits of φ_X in the regular part of M has exactly one corresponding orbit in the Morsified flow $\varphi_{\widetilde{X}}$.

In the context of flows φ_X where $X \in \mathfrak{X}_{\mathcal{GW}}(M)$, the GS-intersection number between consecutive singularities x and y is defined as $n(x,y) := \sum_{u \in \mathcal{M}_y^x} n_u$.

Now, consider the case of crossing singularities. If x is a double or a triple crossing singularity then it is associated to at least two singularities in the Morsified flow. In fact, if x is a n-sheet double crossing singularity of attractor (resp., repeller) nature, then there are n regular singularities $h_0^1(x), \dots, h_0^n(x) \in Sing(\widetilde{X})$ (resp., $h_2^1(x), \dots, h_2^n(x) \in Sing(\widetilde{X})$) of attractor (resp., repeller) nature associated to x via the Morsification process. Throughout this paper, assume that $h_0^1(x)$ (resp., $h_2^1(x)$) corresponds to the singularity of the external manifold and $h_0^2(x), \dots, h_0^n(x)$ (resp., $h_2^2(x), \dots, h_2^n(x)$) correspond to the singularities of the internal manifolds. If x is a double crossing singularity of sa-nature (resp., sr-nature), then there exist exactly two singularities associated to x via the Morsification process, namely, a regular saddle or saddle cone singularity $h_1^1(x)$ and a regular attracting singularity (resp., repelling) $h_0^1(x)$. If x is a double crossing singularity of ss_s- or ss_u-nature, then there are two saddle cone or regular saddle singularities $h_1^1(x)$ and $h_1^2(x)$ associated to x via the Morsification process.

Note that, given $u \in \mathcal{M}_{xy}$, if x and y are both double crossing singularities of consecutive natures, then u is a orbit on $\mathcal{SP}(N)$. Hence, there are exactly two flow lines in $\varphi_{\widetilde{X}}$ which project to u by the Morsification process, namely, $\tilde{u}^e$ and $\tilde{u}^i$, the first one is in the external manifold and the second one is in the internal manifold. Although x and y are considered to be consecutive singularities, one of the flow lines $\tilde{u}^e$ and $\tilde{u}^i$ maybe be connecting non consecutive singularities in the Morsified flow in $\varphi_{\widetilde{X}}$. In this sense, to simplify notation we will consider $n_{\tilde{u}} = 0$ whenever u is a flow line between non consecutive points. For example, this is the case of a double crossing singularity of sr-nature and a double crossing singularity of a^2-nature.

Definition 4.5 (Characteristic signs of flows lines of $\mathfrak{X}_{\mathcal{GD}}(M)$)**.** *Let $x, y \in Sing(X)$ be singularities of consecutive natures and $X \in \mathfrak{X}_{\mathcal{GD}}(M)$. The **GS-characteristic sign** n_u of a flow line $u \in \mathcal{M}_{xy}$ is defined as follows:*

(1) *If u does not belong to the singular part of M, define $n_u = n_{\tilde{u}}$;*
(2) *If u belongs to the singular part of M, define n_u to be the pair $n_u = (n_u^e, n_u^i) := (n_{\tilde{u}^e}, n_{\tilde{u}^i})$, where $n_{\tilde{u}^e}$ and $n_{\tilde{u}^i}$ are the characteristic signs of the flow lines $\tilde{u}^e$ and $\tilde{u}^i$, respectively.*

Note that, for vector fields having double crossing singularities, the characteristic sign of an orbit is defined in terms of the signs of the orbits of their corresponding singularities through the Morsified process. Hence, it is natural to define the intersection number between $h_k^j(x)$ and $h_{k-1}^\ell(y)$ by

$$n(h_k^j(x), h_{k-1}^\ell(y)) = \sum n_{\tilde{u}},$$

where the sum is over the flow lines $\tilde{u} \in \widetilde{\mathcal{M}}_{h_{k-1}^\ell(y)}^{h_k^j(x)}$, for $k = 1, 2$, $j = 1, \dots, \eta_k(x)$ and $\ell = 1, \dots, \eta_{k-1}(y)$.

Now, consider the case of triple crossing singularities. If x is an $(2n+1)$-sheet triple crossing singularity of a^{2n+1}-nature (resp., r^{2n+1}-nature), then there are $2n+1$ regular attracting (resp., repelling) singularities $h_0^1(x), \dots, h_0^{2n+1}(x) \in Sing(\widetilde{X})$ (resp., $h_2^1(x), \dots, h_2^{2n+1}(x) \in Sing(\widetilde{X})$) associated to x via the Morsification process. Assume that $h_0^1(x)$ (resp., $h_2^1(x)$) corresponds to the singularity of the external manifold and $h_0^2(x), \dots, h_0^{2n+1}(x)$ (resp., $h_2^2(x), \dots, h_2^{2n+1}(x)$) correspond to the singularities of the middle and internal manifolds. More specifically, the singularities $h_0^2(x), \dots, h_0^{n+1}(x)$ (resp., $h_2^2(x), \dots, h_2^{n+1}(x)$) belong to the middle manifolds and $h_0^{n+2}(x), \dots, h_0^{2n+1}(x)$ (resp., $h_2^{n+2}(x), \dots, h_2^{2n+1}(x)$) belong to the internal manifolds. If x is a triple crossing singularity of ssa-nature (resp., ssr-nature), then there are regular saddle or saddle cone singularities $h_1^1(x)$, $h_1^2(x)$ and a regular attracting (resp., repelling) singularity $h_0^1(x)$ (resp., $h_2^1(x)$) associated to x via the Morsification process.

Note that, given a orbit $u \in \mathcal{M}_{xy} \subset M$ where x and y are consecutive triple crossing singularities, then u is a orbit on $\mathcal{SP}(N)$. Hence, there are at most three orbits in $\varphi_{\widetilde{X}}$ which projects to u by the Morsfication process, namely,

$$\tilde{u}^e \in \widetilde{\mathcal{M}}_{h_k^1(x)h_{k-1}^1(y)}, \tilde{u}^m \in \widetilde{\mathcal{M}}_{h_k^{j_1}(x)h_{k-1}^{\ell_1}(y)} \text{ e } \tilde{u}^i \in \widetilde{\mathcal{M}}_{h_k^{j_2}(x)h_{k-1}^{\ell_2}(y)},$$

where $k = 1, 2,\ j_1, j_2 \in \{2, \ldots, \eta_k(x)\}$ and $\ell_1, \ell_2 \in \{2, \ldots, \eta_{k-1}(y)\}$. One has that $\tilde{u}^e$ belongs to the external manifold, $\tilde{u}^m$ belongs to the middle manifold and $\tilde{u}^i$ belongs to the internal manifold.

In other to simplify the notation, we consider $n_{\tilde{u}^i} = 0$ when $h_k^j(x)$ and $h_{k-2}^\ell(y)$ are not consecutive generators in the Morsified flow.

Definition 4.6 (Characteristic signs of flows lines of $\mathfrak{X}_{\mathcal{GT}}(M)$)**.** *Let $x, y \in Sing(X)$ be singularities of consecutive natures and $X \in \mathfrak{X}_{\mathcal{GT}}(M)$. The **characteristic sign** n_u of a flow line $u \in \mathcal{M}_{xy}$ is defined as follows:*

- *If u does not belong to the singular part of M, define $n_u = n_{\tilde{u}}$;*
- *If u belongs to the singular part of M, define n_u to be the triple*

$$n_u = (n_u^e, n_u^m, n_u^i) := (n_{\tilde{u}^e}, n_{\tilde{u}^m}, n_{\tilde{u}^i}),$$

where $n_{\tilde{u}^e}$, $n_{\tilde{u}^m}$ and $n_{\tilde{u}^i}$ are the characteristic signs of the flow lines $\tilde{u}^e$, $\tilde{u}^m$ and $\tilde{u}^i$, respectively.

Note that, for vector fields having triple crossing singularities, the characteristic sign of a flow line is defined in terms of the characteristic sign of the orbits of their corresponding singularities through the Morsification process. Hence, it is natural to define the intersection number between $h_k^j(x)$ and $h_{k-1}^\ell(y)$ as

$$n(h_k^j(x), h_{k-1}^\ell(y)) = \sum n_{\tilde{u}},$$

where the sum is over flow lines $\tilde{u} \in \widetilde{\mathcal{M}}^{h_k^j(x)}_{h_{k-1}^\ell(y)}$, for $k = 1, 2,\ j = 1, \ldots, \eta_k(x)$ and $\ell = 1, \ldots, \eta_{k-1}(y)$.

Now it remains to prove that the GS-boundary map $\Delta_*^{\mathcal{GS}}$, described in Definition 4.2, is well defined for GS-flows on $M \in \mathfrak{M}(\mathcal{GS})$, $\mathcal{S} = \mathcal{W}, \mathcal{D}, \mathcal{T}$.

Proposition 4.1. *Consider a flow φ_X in $M \in \mathfrak{M}(\mathcal{GS})$ associated to a vector field $X \in \mathfrak{X}_{\mathcal{GS}}(M)$, where $\mathcal{S} = \mathcal{W}, \mathcal{D}$ or $\mathcal{T}$. Let $\widetilde{\varphi}_{\widetilde{X}}$ be the flow in $\widetilde{M} \in \mathfrak{M}(\mathcal{GS})$ obtained by the Morsification process of $X \in \mathfrak{X}_{\mathcal{GS}}(M)$, then the boundary map $\Delta^{\mathcal{GS}}$ associated to φ_X is equal to the boundary map $\widetilde{\Delta}^{\mathcal{GC}}$ associated to $\widetilde{\varphi}_{\widetilde{X}}$.*

Proof. Given a flow line u of φ_X that does not belong to the singular part of M, the characteristic sign n_u of u is equal to the characteristic sign of its unique corresponding flow line $\tilde{u}$ in $\varphi_{\widetilde{X}}$. Therefore the intersection number between $x, y \in Sing(X)$ is equal to the intersection number $\tilde{x}, \tilde{y} \in Sing(\widetilde{X})$ whenever the connecting manifold $\mathcal{M}_{xy}$ is contained in the regular part of M.

Consider orbits in a connecting manifold $\mathcal{M}_{xy}$ which belong to the singular part of M. This means that both x and y are not regular singularities. The proof is done case by case according to the type of vector field $X \in \mathfrak{X}_{\mathcal{GS}}(M)$ where $\mathcal{S} = \mathcal{W}, \mathcal{D}, \mathcal{T}$.

- Let $X \in \mathfrak{X}_{\mathcal{GW}}(M)$ and $x, y \in Sing(X)$ be Whitney singularities then $u \in \mathcal{M}_{xy}$ belongs to the singular part of M. Denote by $\tilde{x}, \tilde{y} \in Sing(\widetilde{X})$ the Morsified singularities associated to x and y, respectively. The Morsified process matches u to two flow lines $\tilde{u}_1, \tilde{u}_2 \in \widetilde{M}_{\tilde{y}}^{\tilde{x}}$ with $n_{\tilde{u}_1} \neq n_{\tilde{u}_2}$. Hence, $n(\tilde{x}, \tilde{y}) = 0 = n(x, y)$.

- Let $X \in \mathfrak{X}_{\mathcal{GD}}(M)$ and $x, y \in Sing(X)$ be double crossing singularities then $u \in \mathcal{M}_{xy}$ belongs to the singular part of M. Consider the flow lines of $\widetilde{X}$,

$$\tilde{u}^e \in \widetilde{\mathcal{M}}_{h_k^1(x)h_{k-1}^1(y)} \text{ and } \tilde{u}^i \in \widetilde{\mathcal{M}}_{h_k^j(x)h_{k-1}^\ell(y)},$$

with $k = 1, 2$, $j \in \{2, \ldots, \eta_k(x)\}$ and $\ell \in \{2, \ldots, \eta_{k-1}(y)\}$. By the definition of transfer of signs, $n_u^e = n_{\tilde{u}^e}$ and $n_u^i = n_{\tilde{u}^i}$. Thus,

$$n(x, y) = \left(\sum_{u \in \mathcal{M}_{xy}} n_u^e \ , \sum_{u \in \mathcal{M}_{xy}} n_u^i \right) =$$

$$\left(\sum_{\tilde{u}^e} n_{\tilde{u}^e} \ , \sum_{\tilde{u}^i} n_{\tilde{u}^i} \right) = \Big(n(h_k^1(x)h_{k-1}^1(y)) \ , \ n(h_k^j(x)h_{k-1}^\ell(y) \Big).$$

- Let $X \in \mathfrak{X}_{\mathcal{GT}}(M)$ and $x, y \in Sing(X)$ be triple crossing singularities, then $u \in \mathcal{M}_{xy}$ belongs to the singular part of M. Consider the flow lines of $\widetilde{X}$,

$$\tilde{u}^e \in \widetilde{\mathcal{M}}_{h_k^1(x)h_{k-1}^1(y)}, \tilde{u}^m \in \widetilde{\mathcal{M}}_{h_k^{j_1}(x)h_{k-1}^{\ell_1}(y)} \text{ and } \tilde{u}^i \in \widetilde{\mathcal{M}}_{h_k^{j_2}(x)h_{k-1}^{\ell_2}(y)},$$

with $k = 1, 2$, $j_1, j_2 \in \{2, \ldots, \eta_k(x)\}$ and $\ell_1, \ell_2 \in \{2, \ldots, \eta_{k-1}(y)\}$. By the definition of sign transfer, one has that $n_u^e = n_{\tilde{u}^e}, n_u^m = n_{\tilde{u}^m}$ and $n_u^i = n_{\tilde{u}^i}$. Therefore,

$$n(x, y) = \left(\sum_{u \in \mathcal{M}_{xy}} n_u^e, \sum_{u \in \mathcal{M}_{xy}} n_u^m, \sum_{u \in \mathcal{M}_{xy}} n_u^i \right) = \left(\sum_{\tilde{u}^e} n_{\tilde{u}^e}, \sum_{\tilde{u}^m} n_{\tilde{u}^m}, \sum_{\tilde{u}^i} n_{\tilde{u}^i} \right)$$

$$= \Big(n(h_k^1(x)h_{k-1}^1(y)), n(h_k^{j_1}(x)h_{k-1}^{\ell_1}(y), n(h_k^{j_2}(x)h_{k-1}^{\ell_2}(y)) \Big).$$

In any case, the proposition follows. $\square$

Corollary 4.1. *Let $\Delta_*^{\mathcal{GS}}$ be the GS-boundary map associated to φ_X with $X \in \mathfrak{X}_{\mathcal{GS}}(M)$, where $\mathcal{S} = \mathcal{W}, \mathcal{D}$ or $\mathcal{T}$. Then $\Delta_{k-1}^{\mathcal{GS}} \circ \Delta_k^{\mathcal{GS}} = 0$, for all $k \in \mathbb{Z}$.*

Proof. It follows directly from Theorem 4.1 and Proposition 4.1. $\square$

Hence, the pair $(C_*^{\mathcal{GS}}(M, X), \Delta_*^{\mathcal{GS}})$ is indeed a chain complex, whenever $M \in \mathfrak{M}(\mathcal{GS})$, $X \in \mathfrak{X}_{\mathcal{GS}}(M)$ and $\mathcal{S} = \mathcal{W}, \mathcal{D}$ or $\mathcal{T}$.

4.4.1. *Examples.*

Example 4.2. Consider a GS-flow φ_X defined on a singular manifold $M \in \mathfrak{M}(\mathcal{GW})$ and its Morsification $(\widetilde{M}, \varphi_{\widetilde{X}})$, as in Figure 27. Note that z_2, y_1, x_1, x_3 are regular singularities and z_1, y_2, y_3, x_2 are Whitney singularities. Considering a choice of orientations on the unstable manifolds of the critical points of $\widetilde{M}$, the GS-characteristic signs on the orbits of φ_X are obtained, by the Definition 4.4.

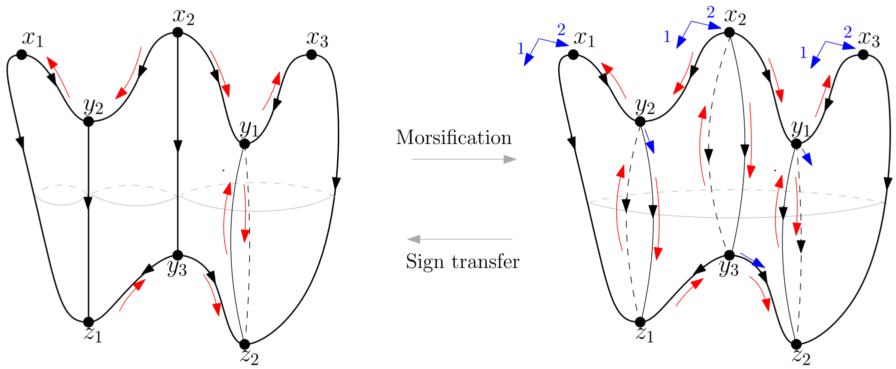


FIGURE 27. A GS-flow with Whitney singularities and its Morsification.

The GS-chain groups are:
$C_2^{\mathcal{GW}}(M) = \mathbb{Z}\langle x_1\rangle \oplus \mathbb{Z}\langle x_2\rangle \oplus \mathbb{Z}\langle x_3\rangle$, $C_1^{\mathcal{GW}}(M) = \mathbb{Z}\langle y_1\rangle \oplus \mathbb{Z}\langle y_2\rangle \oplus \mathbb{Z}\langle y_3\rangle$, $C_0^{\mathcal{GW}}(M) = \mathbb{Z}\langle z_1\rangle \oplus \mathbb{Z}\langle z_2\rangle$ and $C_k^{\mathcal{GW}}(M) = 0, k \neq 0, 1, 2$.

The GS- intersection numbers are:
$n(x_1, y_2) = -1$, $n(x_2, y_1) = 1$, $n(x_2, y_2) = 1$, $n(x_2, y_1) = 1$, $n(x_3, y_1) = 0$, $n(x_2, y_3) = 0$, $n(y_1, z_2) = 0$, $n(y_2, z_1) = 0$, $n(y_3, z_2) = 1$, $n(y_3, z_1) - 1$,

The GS-boundary maps $\Delta_2^{\mathcal{GW}} : C_2 \to C_1$, $\Delta_1^{\mathcal{GW}} : C_1 \to C_0$ and $\Delta_0^{\mathcal{GW}} : C_0 \to 0$ are given by:

$$\Delta_2^{\mathcal{GW}}(x_1) = -\langle y_2\rangle,\ \Delta_2^{\mathcal{GW}}(x_2) = \langle y_1\rangle + \langle y_2\rangle,\ \Delta_2^{\mathcal{GW}}(x_3) = -\langle y_1\rangle,$$

$$\Delta_1^{\mathcal{GW}}(y_1) = 0,\ \Delta_1^{\mathcal{GW}}(y_2) = 0,\ \Delta_1^{\mathcal{GW}}(y_3) = \langle z_2\rangle - \langle z_1\rangle.$$

Therefore, the matrix of the GS-boundary map $\Delta^{\mathcal{GW}}$ is as in Figure 28.

$\Delta^{\mathcal{GW}}(M) =$

	z_1	z_2	y_1	y_2	y_3	x_1	x_2	x_3
z_1			0	0	-1			
z_2			0	0	1			
y_1							1	-1
y_2						-1	1	
y_3								
x_1								
x_2								
x_3								

FIGURE 28. GS-boundary operator $\Delta^{\mathcal{GW}}(M)$.

$\triangle$

Example 4.3. Consider a GS-flow φ_X defined on a singular manifold M and its Morsification $(\widetilde{M}, \varphi_{\widetilde{X}})$ as in Figure 29. Considering a choice of orientations on the unstable manifolds of the critical points of $\widetilde{M}$, the GS-characteristic signs on the orbits of $(\widetilde{M}, \varphi_{\widetilde{X}})$ are obtained. By Definition 4.5, one gets the GS-characteristic signs of the orbits in φ_X.

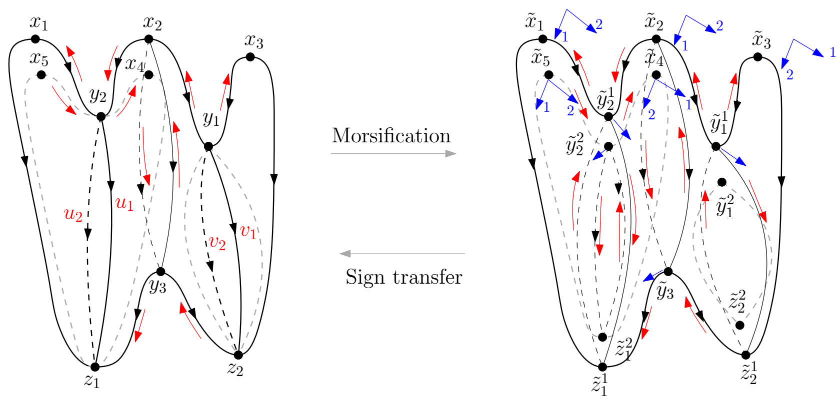


FIGURE 29. A GS-flow with double crossing singularities and its Morsification.

Since the orbits u_1, u_2, v_1 and v_2 are orbits on the singular part of M, hence the GS-characteristic sign is given by a pair: $n_{u_1} = (n_{u_1^1},\, n_{u_1^2}) = (1,-1)$, $n_{u_2} = (n_{u_2^1},\, n_{u_2^2}) = (-1,1)$, $n_{v_1} = (n_{v_1^1},\, n_{v_1^2}) = (1,0), n_{v_2} = (n_{v_2^1},\, n_{v_2^2}) = (-1,0)$. The groups of the GS-chain complex are given by

$$C_2^{\mathcal{GD}}(M) = \bigoplus_{i=1}^{5} \mathbb{Z}\langle x_i\rangle \oplus \mathbb{Z}\langle y_1^2\rangle$$

$$C_1^{\mathcal{GD}}(M) = \mathbb{Z}\langle y_1^1\rangle \oplus \mathbb{Z}\langle y_2^1\rangle \oplus \mathbb{Z}\langle y_2^2\rangle \oplus \mathbb{Z}\langle y_3\rangle, \qquad C_0^{\mathcal{GD}}(M) = \mathbb{Z}\langle z_1^1\rangle \oplus \mathbb{Z}\langle z_1^2\rangle \oplus \mathbb{Z}\langle z_2^1\rangle \oplus \mathbb{Z}\langle z_2^2\rangle$$

and $C_k^{\mathcal{GD}}(M) = 0, k \neq 0,1,2$. Note that y_1 is a double crossing singularity of saddle-repelling nature (sr), hence it is associated to a generator in $C_2^{\mathcal{GD}}(M)$ and a generator in $C_1^{\mathcal{GD}}(M)$. Analogously, the double crossing singularity y_3 has saddle-saddle nature (ss_s), hence it is associated to two generators in $C_1^{\mathcal{GD}}(M)$. Finally, each attracting double crossing singularity z_1 and z_2 is associated to two generators in $C_0^{\mathcal{GD}}(M)$. The GS-intersection numbers are: $n(x_1, y_2^1) = -1,\ n(x_2, y_2^1) = 1,\ n(x_2, y_1^1) = -1, n(x_2, y_3) = 0,\ n(x_3, y_1^1) = 1,\ n(x_4, y_2^2) = -1,$ $n(x_5, y_2^2) = 1, n(y_3, z_1^1) = 1,\ n(y_3, z_2^1) = -1, n(y_1, z_2) = n_{v_1} + n_{v_2} = (1,0) + (-1,0) = (0,0),$ $n(y_2, z_1) = n_{u_1} + n_{u_2} = (1,-1) + (-1,1) = (0,0)$. Therefore, the GS-boundary operator $\Delta_2^{\mathcal{GD}} : C_2 \to C_1$, $\Delta_1^{\mathcal{GD}} : C_1 \to C_0$ and $\Delta_0^{\mathcal{GD}} : C_0 \to 0$ are given by:

$$\Delta_2^{\mathcal{GD}}(x_1) = -\langle y_2^1\rangle,\ \Delta_2^{\mathcal{GD}}(x_2) = -\langle y_1^1\rangle + \langle y_2^1\rangle,\ \Delta_2^{\mathcal{GD}}(x_3) = -\langle y_1^1\rangle,$$

$$\Delta_2^{\mathcal{GD}}(x_4) = -\langle y_2^2\rangle,\ \Delta_2^{\mathcal{GD}}(x_5) = -\langle y_2^2\rangle,\ \Delta_1^{\mathcal{GD}}(y_3) = \langle z_1^1\rangle - \langle z_2^1\rangle,$$

$$\Delta_1^{\mathcal{GD}}(y_1^1) = 0,\ \Delta_1^{\mathcal{GD}}(y_2^1) = 0,\ \Delta_1^{\mathcal{GD}}(y_2^2) = 0,\ \Delta_0^{\mathcal{GD}}(z_1) = \Delta_0^{\mathcal{GD}}(z_2) = 0.$$

The matrix $\Delta^{\mathcal{GD}}$ of the GS-boundary operator is shown in Figure 30.

$$\Delta^{\mathcal{GD}}(M) = \begin{array}{c|cccccccccccccc}
 & z_1^1 & z_1^2 & z_2^1 & z_2^2 & y_1^1 & y_2^1 & y_2^2 & y_3 & x_1 & x_2 & x_3 & x_4 & x_5 & y_1^2 \\ \hline
z_1^1 & & & & & 0 & 0 & 0 & 1 & & & & & & \\
z_1^2 & & & & & 0 & 0 & 0 & 0 & & & & & & \\
z_2^1 & & & & & 0 & 0 & 0 & -1 & & & & & & \\
z_2^2 & & & & & 0 & 0 & 0 & 0 & & & & & & \\
y_1^1 & & & & & & & & & 0 & -1 & -1 & 0 & 0 & 0 \\
y_2^1 & & & & & & & & & -1 & 1 & 0 & 0 & 0 & 0 \\
y_2^2 & & & & & & & & & 0 & 0 & 0 & -1 & 1 & 0 \\
y_3 & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\
x_1 & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\
x_2 & & & & & & & & & & & & & & \\
x_3 & & & & & & & & & & & & & & \\
x_4 & & & & & & & & & & & & & & \\
x_5 & & & & & & & & & & & & & & \\
y_1^2 & & & & & & & & & & & & & &
\end{array}$$

FIGURE 30. GS-boundary operator $\Delta^{\mathcal{GD}}(M)$.

$\triangle$

5. Dynamical Homotopical Cancellation Theorem for GS-flows

In this section we prove a homotopical cancellation theorem for consecutive singularities of a Gutierrez-Sotomayor vector field.

In the homotopical cancellation process of singularities in $S = \mathcal{C}$, $\mathcal{W}$, $\mathcal{D}$ or $\mathcal{T}$, three singularities in S, one of saddle nature y and two of attracting (resp., repelling) nature z_1, z_2 (resp., x_1, x_2), give rise to a new singularity of attracting nature $\bar{z}$ (resp., repelling nature $\bar{x}$). Droplets or folds associated to these singularities are topological invariants and are registered in the singularity type number, see definition in Section 2. After the homotopical cancellation of y and z_i (resp., x_i), $i = 1$ or 2, $\bar{z}$ is the new singularity (resp., $\bar{x}$) and the type number of $\bar{z}$ is related to the types numbers $m(y)$, $m(z_1)$ and $m(z_2)$ (resp. $m(y)$, $m(x_1)$ and $m(x_2)$) of y, z_1 and z_2 (resp., y, x_1 and x_2). We say that $\bar{z}$ (resp., $\bar{x}$) ***inherits the type numbers*** $m(y)$, $m(z_1)$ and $m(z_2)$ as follows:

$$m(\bar{z}) = m(y) + m(z_1) + m(z_2) \quad \text{(resp., } m(\bar{x}) = m(y) + m(x_1) + m(x_2)\text{)}$$

for $S = \mathcal{C}$, $\mathcal{W}$, $\mathcal{D}$ or $\mathcal{T}$. Recall, that the type number of a regular singularity is always zero. Hence, as a consequence, whenever z_1 and z_2 are regular $\bar{z}$ will inherit a type number equal to zero.

For example, in Figure 4, the three singularities: z_1 a regular type singularity of attracting nature, y_2 a cone type singularity of saddle nature and z_2 a 2-sheet cone type singularity of attracting nature, are involved in the homotopical cancellation process. More specifically, they give rise to $\bar{z}_1$ which inherits the types of z_1, z_2 and y_2, i.e., z_1 contributes zero, y_2 contributes one and z_2 contributes 2, hence $\bar{z}_1$ is a 3-sheet cone type singularity of attracting nature. For more examples, see Subsection 6.3.

Definition 5.1. *Let $X \in \mathfrak{X}_{\mathcal{GS}}(M)$ be a Gutierrez-Sotomayor vector field on $M \in \mathfrak{M}(\mathcal{GS})$, for $\mathcal{S} = \mathcal{C}$ or $\mathcal{W}$. Let $p, q \in Sing(X)$ be consecutive singularities of k and $k-1$ nature numbers, respectively. One says that p and q are* ***dynamically homotopically cancelled*** *and that together with q' give rise to $\overline{q}'$ if there is a singular 2-manifold M' of the same homotopy type as M and there exists a vector field $X' \in \mathfrak{X}_{\mathcal{GS}}(M')$ which is topologically equivalent to X outside of a neighborhood V of the flow lines u_1 and u_2 which is given as follows:*

(1) *If $k=1$, then $\mathcal{M}_q^p=\{u_1\}$ and $\mathcal{M}_{q'}^p=\{u_2\}$, where q' is the other singularity of attracting nature connecting with p. Moreover, the vector field $X'\in\mathfrak{X}_{\mathcal{GS}}(M')$ on V contains only one n-sheet $\mathcal{S}$-singularity $\overline{q}'\in\mathfrak{X}_{\mathcal{GS}}(M')$ of attracting nature which inherits the type numbers of p, q and q'. In this case, p and q together with q' give rise to $\overline{q}'$.*

(2) *If $k=2$, then $\mathcal{M}_q^p=\{u_1\}$ and $\mathcal{M}_q^{p'}=\{u_2\}$, where p' is the other singularity of repelling nature connecting with q. Moreover, the vector field $X'\in\mathfrak{X}_{\mathcal{GS}}(M')$ on V contains only one singularity $\overline{p}'$ of repelling nature. In this case, p and q together with p' give rise to an n-sheet $\mathcal{S}$-singularity $\overline{p}'\in\mathfrak{X}_{\mathcal{GS}}(M')$ which inherits the type numbers of p, p' and q.*

Note that if $X\in\mathfrak{X}_{\mathcal{GS}}(M)$ is a GS-vector field on $M\in\mathfrak{M}(\mathcal{GS})$ with only regular singularities, then the GS homotopical cancellation of consecutive singularities coincides with the notion of cancellation of critical points established by Smale in the Morse setting. This follows, since Morse critical points have a unique generator and hence $q'=\overline{q}'$ and $p'=\overline{p}'$ in Definition 5.1.

Consider $M\in\mathfrak{M}(\mathcal{GS})$ a closed 2-manifold and a GS-flow φ_X on M associated to a GS-vector field $X\in\mathfrak{X}_{\mathcal{GS}}(M)$, where $\mathcal{S}=\mathcal{C},\mathcal{W},\mathcal{D}$ or $\mathcal{T}$. Given consecutive singularities x and y, suppose that:

- the GS-intersection number $n(x,y)$ is ±1, when $\mathcal{S}=\mathcal{C}$ or $\mathcal{W}$;
- one of the coordinates of the GS-intersection number $n(x,y)$ is equal to ±1, when $\mathcal{S}=\mathcal{D}$ or $\mathcal{T}$.

The theorems in this section guarantee that under these conditions, it is always possible to dynamically homotopically cancel the singularities with GS-intersection number equal to ±1.

Theorem 5.1 (Dynamical Homotopical Cancellation Theorem for GS-flows - Cases $\mathcal{C}$ and $\mathcal{W}$)**.** *Let $X\in\mathfrak{X}_{\mathcal{GS}}(M)$ be a Gutierrez-Sotomayor vector field on $M\in\mathfrak{M}(\mathcal{GS})$, for $\mathcal{S}=\mathcal{C}$ or $\mathcal{W}$. Let $p,q\in Sing(X)$ be consecutive singularities of k and $k-1$ nature numbers, respectively. If $n(p,q)$ is non zero, then p and q dynamically homotopically cancelled.*

Proof. Consider $k=1$, i.e. p is a saddle and q is an attracting singularity. The assumption that $n(p,q)$ is non zero guarantees that there exists a singularity q' of attracting nature, distinct from p and connecting with q. Let $C(p,q,q')$ be the set composed by the singularities p, q and q' and the flow lines $\mathcal{M}_q^p=\{u_1\}$ and $\mathcal{M}_{q'}^p=\{u_2\}$ connecting them. $C(p,q,q')$ is an isolated invariant set which is an attractor. Hence, choose an isolating block V of $C(p,q,q')$ such that the vector field X is transversal to the boundary of V, $\partial\overline{V}$.

Given a point $x\in M$, denote by $\gamma(x)$ the orbit through the point x. Clearly $\gamma(p)=p$ and $\gamma(q)=q$ and $\gamma(q')=q'$. Consider a path δ which has image coinciding with the juxtaposition of the paths

$$\gamma(q')*\gamma(u_2)*\gamma(p)*\gamma^{-1}(u_1)*\gamma(q),$$

i.e., δ is the path going from q to q' through the orbits of u_1 and u_2. Since δ is not a closed path, one can contract it to a point $\overline{q}'$, obtaining a new topological space $\overline{V}$ preserving the boundary $\partial\overline{V}$ of the same homotopy type as V, where the type number of $\overline{q}'$ is equal to

$$m(\overline{q}')=m(q)+m(q')+m(p).$$

Now consider an attracting singular vector field on $\overline{V}$ such that $\overline{q}'$ is a singularity of attracting nature and the vector field is transverse to the boundary of $\overline{V}$.

The result follows by gluing $M\setminus V$ and $\overline{V}$ together with their respective vector fields.

If $k=2$, the proof is analogous by considering the reverse flow. □

Definition 5.2. *Let $X\in\mathfrak{X}_{\mathcal{GS}}(M)$ be a Gutierrez-Sotomayor vector field on $M\in\mathfrak{M}(\mathcal{GS})$, for $\mathcal{S}=\mathcal{D}$ or $\mathcal{T}$. Let $p,q\in Sing(X)$ be consecutive singularities, $h_k^j(p)$ and $h_{k-1}^\ell(q)$ be the respective consecutive generators of their natures, where $j\in\{1,\dots,\eta_k(p)\}$ and $\ell\in\{1,\dots,\eta_{k-1}(q)\}$. One* *says that p and q are* ***dynamically homotopically cancelled*** *if there exists a vector field $X' \in \mathfrak{X}_{\mathcal{GS}}(M')$ which is topologically equivalent to X outside of a neighborhood V of the flow lines u_1 and u_2 which are given as follows:*

(1) *If $k = 1$, then $\widetilde{\mathcal{M}}^{h^j_k(p)}_{h^\ell_{k-1}(q)} = \{u_1\}$ and $\widetilde{\mathcal{M}}^{h^j_k(p)}_{h^i_{k-1}(q')} = \{u_2\}$, where q' is the other singularity of attracting nature connecting with p such that $h^i_{k-1}(q')$ and $h^\ell_{k-1}(q)$ belong to the same connected component of the Morsification $\widetilde{M}$. Moreover, the vector field $X' \in \mathfrak{X}_{\mathcal{GS}}(M')$ on V contains only one singularity $\overline{q}'$ whose generators match the union of all the generators of p, q and q' excluding the cancelled pair of generators $h^j_k(p)$ and $h^\ell_{k-1}(q)$. In this case, p and q together with q' give rise to an n-sheet $\mathcal{S}$-singularity $\overline{q}' \in \mathfrak{X}_{\mathcal{GS}}(M')$ which inherits the type numbers of p, q and q'.*

(2) *If $k = 2$, then $\widetilde{\mathcal{M}}^{h^j_k(p)}_{h^\ell_{k-1}(q)} = \{u_1\}$ and $\widetilde{\mathcal{M}}^{h^i_k(p')}_{h^\ell_{k-1}(q)} = \{u_2\}$, where p' is the other singularity of repelling nature connecting with q such that $h^i_k(p')$ and $h^j_k(p)$ belong to the same connected component of the Morsification $\widetilde{M}$. Moreover, the vector field $X' \in \mathfrak{X}_{\mathcal{GS}}(M')$ on V contains only one singularity $\overline{p}'$ whose generators match the union of all the generators of p, q and p' excluding the cancelled pair of generators $h^j_k(p)$ and $h^\ell_{k-1}(q)$. In this case, p and q together with p' give rise to an n-sheet $\mathcal{S}$-singularity $\overline{p}' \in \mathfrak{X}_{\mathcal{GS}}(M')$ which inherits the type numbers of p, q and p'.*

For example, in Figure 31, the three singularities: z_1 and z_2 are 2-sheet double crossing type singularities of attracting nature and y is a regular type singularity of saddle nature; are involved in the dynamical homotopical cancellation process. More specifically, they give rise to $\bar{z}_2$ which inherits the types of z_1, z_2 and y, i.e., z_1 contributes 1, y contributes zero and z_2 contributes 1, hence $\bar{z}_2$ is a 3-sheet double crossing type singularity of attracting nature. For more examples, see Subsection 6.3.

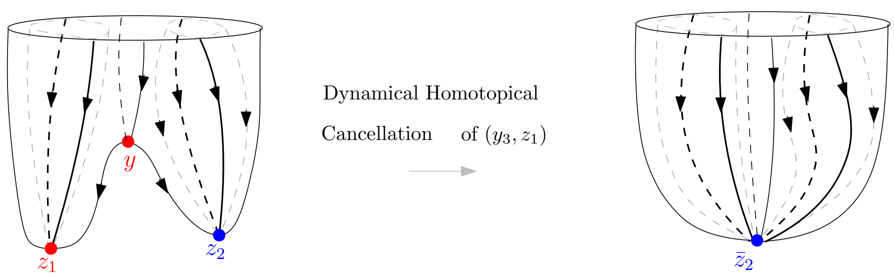


FIGURE 31. Dynamical Homotopical Cancellation of the singularities y and z_1.

Theorem 5.2 (Dynamical Homotopical Cancellation Theorem for GS-flows - Cases $\mathcal{D}$ and $\mathcal{T}$)**.** *Let $X \in \mathfrak{X}_{\mathcal{GS}}(M)$ be a Gutierrez-Sotomayor vector field on $M \in \mathfrak{M}(\mathcal{GS})$, for $\mathcal{S} = \mathcal{D}$ or $\mathcal{T}$. Let $p, q \in Sing(X)$ be consecutive singularities, $h^j_k(p)$ and $h^\ell_{k-1}(q)$ be the respective consecutive generators of their natures, where $j \in \{1, \dots, \eta_k(p)\}$ and $\ell \in \{1, \dots, \eta_{k-1}(q)\}$. If $n(h^j_k(p), h^\ell_{k-1}(q))$ is non zero for some j and ℓ, then p and q are dynamically homotopically cancelled.*

Proof. Firstly, one considers the Morsified manifold $\widetilde{M} \in \mathfrak{M}(\mathcal{GC})$ and the Morsified flow $\varphi_{\widetilde{X}}$ on $\widetilde{M}$ associated to $\widetilde{X} \in \mathfrak{X}(\mathcal{GC})$. One also needs to be aware that, in the Morsification process, the generators of double or triple crossing points will give rise to singularities in $\mathfrak{X}_{\mathcal{GS}}(\widetilde{M})$ which may be in distinct connected components of $\widetilde{M}$.

Consider $k = 1$, i. e. suppose that q is a singularity having at least one generator of attracting nature and p is a singularity having at least one generator of saddle nature. Since $n(h_k^j(p), h_{k-1}^\ell(q))$ is non zero for some j and ℓ, there is another singularity q' of attracting nature connecting with p. In other words, $n(h_k^j(p), h_{k-1}^{\ell'}(q'))$ is non zero, for some ℓ'. Moreover, the generator $h_k^j(p)$ corresponds to a saddle singularity and $h_{k-1}^\ell(q)$ and $h_{k-1}^{\ell'}(q')$ correspond to attracting singularities in the Morsified flow. Since they belong to the same connected component of $\widetilde{X}$, one can apply the Smale's cancellation theorem and cancel $h_k^j(p)$ and $h_{k-1}^\ell(q)$, as in the Morse case. After the cancellation, one obtains a Morsified flow $\varphi_{\widetilde{X}'}$ on $\widetilde{M}$ which coincides with $\varphi_{\widetilde{X}}$ outside a neighborhood V of the set $C(h_k^j(p), h_{k-1}^\ell(q), h_{k-1}^{\ell'}(q'))$ composed by the singularities $h_k^j(p)$, $h_{k-1}^\ell(q)$ and $h_{k-1}^{\ell'}(q')$ and the flow lines connecting them.

An orbit γ of the initial singular flow φ_X, which belongs to the singular part of M and has ω-limit set equal to p, q or q', is a fold. Moreover, γ admits a duplication γ^1 and γ^2 in $\varphi_{\widetilde{X}}$. Only one of these orbits has ω-limit set equal to $h_k^j(p)$, $h_{k-1}^\ell(q)$ or $h_{k-1}^{\ell'}(q')$, and after the cancellation, this orbit will have ω-limit set equal to the singularity $h_{k-1}^{\ell'}(q')$. Now considering all the connected components of $\widetilde{M}$ with the respective new Morse vector field $\widetilde{X}'$, one can identify all the orbits γ corresponding to orbits on $\mathcal{SP}(M)$ in the initial flow. As a result, one obtains a new singular manifold M' with a flow $\varphi_{X'}$ associated to a vector field $X' \in \mathfrak{X}_{\mathcal{GS}}(M')$ which coincides with X, up to topological equivalence, outside of a neighborhood V of the flow lines u_1 and u_2, as in the statement of the theorem.

If $k = 2$, the proof is analogous by considering the reverse flow. $\square$

6. Detecting Dynamical Homotopical Cancellations through Spectral Sequences

The use of algebraic tools to extract dynamical information has been explored in highly influential work, see [8, 12, 13]. Particularly, the use of spectral sequences has been used in dynamical systems (see [2,3,10]) as well as in computational topology (see [7,23]).

In this section, our main motivation is to establish how the algebraic cancellations in a spectral sequence of a filtered GS-chain complex affects dynamical homotopical cancellations within a GS-flow. In order to keep track of the changes of the differentials d^r and the generators of the modules E^r on each page (E^r, d^r) of the spectral sequence, the Spectral Sequence Sweeping Algorithm (SSSA) was developed in [5] and is presented in Section 6.1. It provides all of the algebraic cancellations that occur on each page, as well as, the new generators of the modules of the following page. With the complete information in hand of the algebraic cancellations in Section 6.2, we make use of the Row Cancellation Algorithm (RCA), which is the dynamical counterpart of the SSSA. We refer the reader to [5, 11] for more details on RCA and SSSA.

In Section 6.1, we give a brief overview of basic definitions for spectral sequences and state the SSSA. In Section 6.2, we present the main homotopical cancellation results, Theorem 6.2 and Theorem 6.3. In Section 6.3 examples of these theorems for flows on $M \in \mathfrak{M}(\mathcal{GS})$, where $\mathcal{S} = \mathcal{C}, \mathcal{W}$ or $\mathcal{D}$ are proved.

6.1. Spectral Sequence Sweeping Algorithm.

Let R be a principal ideal domain. A k-***spectral sequence*** E over R is a sequence $\{E^r, \partial^r\}_{r \geq k}$, such that

(1) E^r is a bigraded module over R, i.e., an indexed collection of R-modules $E^r_{p,q}$, for all $p, q \in \mathbb{Z}$;
(2) d^r is a differential of bidegree $(-r, r-1)$ on E^r, i.e., an indexed collection of homomorphisms $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$, for all $p, q \in \mathbb{Z}$, and $(d^r)^2 = 0$;

(3) for all $r \geq k$, there exists an isomorphism $H(E^r) \approx E^{r+1}$, where

$$H_{p,q}(E^r) = \frac{\text{Ker}d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}}{\text{Im}d^r : E^r_{p+r,q-r+1} \to E^r_{p,q}}.$$

Let $Z^k_{p,q} = Ker(d^k_{p,q} : E^k_{p,q} \to E^k_{p,q-1})$ and $B^k_{p,q} = Im(d^k_{p,q+1} : E^k_{p,q+1} \to E^k_{p,q})$, then $E^{k+1} = Z^k/B^k$ and

$$B^k \subseteq B^{k+1} \subseteq \ldots \subseteq B^r \subseteq \ldots \subseteq Z^r \subseteq \ldots \subseteq Z^{k+1} \subseteq Z^k.$$

Consider the bigraded modules $Z^\infty = \cap_r Z^r, B^\infty = \cup_r B^r$ and $E^\infty = Z^\infty/B^\infty$. The latter module is called the ***limit of the spectral sequence***. A spectral sequence $E = \{E^r, \partial^r\}$ is ***convergent*** if given p, q there is $r(p,q) \geq k$ such that for all $r \geq r(p,q)$, $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ is trivial. A spectral sequence $E = \{E^r, \partial^r\}$ is ***convergent in the strong sense*** if given $p, q \in \mathbb{Z}$ there is $r(p,q) \geq k$ such that $E^r_{p,q} \approx E^\infty_{p,q}$, for all $r \geq r(p,q)$.

Let (C, ∂) be a chain complex. An ***increasing filtration*** F on (C, ∂) is a sequence of submodules F_pC of C such that $F_pC \subset F_{p+1}C$, for all integer p, and the filtration is compatible with the gradation of C, i.e. F_pC is a chain subcomplex of C consisting of $\{F_pC_q\}$. A filtration F on C is called ***convergent*** if $\cap_p F_pC = 0$ and $\cup_p F_pC = C$. It is called *finite* if there are $p, p' \in \mathbb{Z}$ such that $F_pC = 0$ and $F_{p'}C = C$. Also, it is said to be ***bounded below*** if for any q there is $p(q)$ such that $F_{p(q)}C_q = 0$.

Given a filtration on C, the associated bigraded module $G(C)$ is defined as

$$G(C)_{p,q} = \frac{F_pC_{p+q}}{F_{p-1}C_{p+q}}.$$

A filtration F on C induces a filtration F on $H_*(C)$ defined by

$$F_pH_*(C) = \text{Im}\ [H_*(F_pC) \to H_*(C)].$$

If the filtration F on C is convergent and bounded below then the same holds for the induced filtration on $H_*(C)$.

The following theorem (see [20], Chapter 9) shows that one can associate a spectral sequence to a filtered chain complex whenever the filtration is convergent and bounded below.

Theorem 6.1 (Spanier, [20], Chapter 9)**.** *Let F be a convergent and bounded below filtration on a chain complex C. There is a convergent spectral sequence with*

$$E^0_{p,q} = \frac{F_pC_{p+q}}{F_{p-1}C_{p+q}} = G(C)_{p,q} \qquad \textit{and} \qquad E^1_{p,q} \approx H_{p+q}\left(\frac{F_pC_{p+q}}{F_{p-1}C_{p+q}}\right)$$

and E^∞ is isomorphic to the bigraded module $GH_*(C)$ associated to the induced filtration on $H_*(C)$.

This theorem is proved by expliciting algebraic formulas for the modules E^r, which are given by

$$E^r_{p,q} = \frac{Z^r_{p,q}}{Z^{r-1}_{p-1,q+1} + \partial Z^{r-1}_{p+r-1,q-r+2}},$$

where $Z^r_{p,q} = \{c \in F_pC_{p+q} \mid \partial c \in F_{p-r}C_{p+q-1}\}$.

Despite the fact that E^∞ does not determine $H_*(C)$ completely, it determines the bigraded module $GH_*(C)$, i.e. $E^\infty_{p,q} \approx GH_*(C)_{p,q}$. Moreover, it is a well known fact (see [6], Chapter 9) that, whenever $GH_*(C)_{p,q}$ is free and the filtration is bounded, then

$$\bigoplus_{p+q=k} GH_*(C)_{p,q} \approx H_{p+q}(C). \tag{3}$$

The Spectral Sequence Sweeping Algorithm (SSSA) was introduced in [5] in order to recover the modules and differentials of a spectral sequence associated to a finite chain complex over $\mathbb{Z}$ with a finest filtration. More specifically, let (C, ∂) be a finite chain complex such that each module C_k is finitely generated. Denote the generators of the C_k chain module by $h_k^1, \cdots, h_k^{c_k}$. One can reorder the set of the generators of C_* as

$$\{h_0^1, \cdots, h_0^{\ell_0}, h_1^{\ell_0+1}, \cdots, h_1^{\ell_1}, \cdots, h_k^{\ell_{k-1}+1}, \cdots, h_k^{\ell_k}, \cdots\},$$

where $\ell_k = c_0 + \cdots + c_k$ [3]. Let F be a finest filtration on C defined by

$$F_p C_k = \bigoplus_{h_k^\ell,\ \ell \leq p+1} \mathbb{Z}\langle h_k^\ell \rangle,$$

for $p \in \mathbb{N}$. The spectral sequence associated to (C_*, ∂_*) with this finest filtration has a special property: the only q for which $E_{p,q}^r$ is non-zero is $q = k - p$, where k is the index of the chain in $F_p C \setminus F_{p-1} C$. Hence, in this case, we omit reference to q. It is understood that E_p^r is in fact $E_{p,k-p}^r$. The SSSA presented below, provides an alternative way to obtain such modules as well as the differentials d^r's.

We can view the differential boundary map ∂ as the matrix Δ where the column j of Δ corresponds to the generator h_k^j of C_*, and the submatrix Δ_k corresponds to the k-th boundary map ∂_k. From now on, the boundary operator ∂ and the matrix Δ will be used interchangeably.

For completeness sake we give a summarized description of the Spectral Sequence Sweeping Algorithm below. For more details see [5, 11].

Spectral Sequence Sweeping Algorithm - SSSA

For a fixed diagonal r parallel and to the right of the main diagonal, the method described below must be applied simultaneously for all k.

Initialization Step:

(1) Let ξ_1 be the first diagonal of Δ that contains non-zero entries $\Delta_{i,j}$ in Δ_k, which will be called index k primary pivots. Define Δ^{ξ_1} to be Δ with the k- index *primary pivots* marked on the ξ_1-th diagonal.

(2) Consider the matrix Δ^{ξ_1}. Let ξ_2 be the first diagonal greater than ξ_1 which contains non-zero entries $\Delta_{i,j}^{\xi_1}$. The construction of Δ^{ξ_2} follows the procedure below. Given a non-zero entry $\Delta_{i,j}^{\xi_1}$ on the ξ_2-th diagonal of Δ^{ξ_1}:

If $\Delta_{s,j}^{\xi_1}$ contains an index k primary pivot for $s > i$, **then** the numerical value of the given entry remains the same, $\Delta_{i,j}^{\xi_2} = \Delta_{i,j}^{\xi_1}$, and the entry is left unmarked.

If $\Delta_{s,j}^{\xi_1}$ does not contain a primary pivot for $s > i$:

then if $\Delta_{i,t}^{\xi_1}$ contains a primary pivot, for $t < j$,

then define $\Delta_{i,j}^{\xi_2} = \Delta_{i,j}^{\xi_1}$ and mark the entry $\Delta_{i,j}^{\xi_2}$ as a *change-of-basis pivot.*

Else, define $\Delta_{i,j}^{\xi_2} = \Delta_{i,j}^{\xi_1}$ and permanently mark $\Delta_{i,j}^{\xi_2}$ as an index k *primary pivot.*

Iterative Step: Suppose by induction that Δ^ξ is defined for all $\xi \leq r$ with the primary and change-of-basis pivots marked on the diagonals smaller or equal to ξ. Without loss of generality, one can assume that there is at least one change-of-basis pivot on the r-th diagonal of Δ^r. Otherwise, define $\Delta^{r+1} = \Delta^r$ with primary pivots and change-of-basis pivots marked as in step (2) below.

[3]In order to simplify notation, we use the index f_k to denote the first column of Δ associated to a k-chain. Hence $f_k = \ell_{k-1} + 1$. Moreover, ℓ_k denotes the last column of Δ associated to a k-chain.

(1) **Change of basis.** Let $\Delta^r_{i,j}$ be a change-of-basis pivot in Δ^r_k. Denote by $\Delta^r_{i,t}$ the primary pivot in the i-th row, with $t < j$. Perform a change of basis on Δ^r by adding or subtracting the column t to the column j of Δ^r, in order to zero out the entry $\Delta^r_{i,j}$ without introducing non-zero entries in $\Delta^r_{s,j}$ for $s > i$.

Define T^r as the matrix which performs all the change of basis on all of the r-th diagonal. Define $\Delta^{r+1} = (T^r)^{-1}\Delta^r T^r$.

(2) **Markup.** Given a non-zero entry $\Delta^{r+1}_{i,j}$ on the $(r+1)$-th diagonal of Δ^{r+1}_k:
If $\Delta^{r+1}_{s,j}$ contains a primary pivot for $s > i$, **then** leave the entry $\Delta^{r+1}_{i,j}$ unmarked.
If $\Delta^{r+1}_{s,j}$ does not contain a primary pivot for $s > i$:
then if $\Delta^r_{i,t}$ contains a primary pivot, for $t < j$,
then mark $\Delta^r_{i,j}$ as a change-of-basis pivot.
Else permanently mark $\Delta^r_{i,j}$ as a primary pivot.

Final Step:
Repeat the above procedure until all diagonals have been considered.

According to the algorithm, if $\Delta^r_{i,j}$ is a change-of-basis pivot on the r-th diagonal of Δ^r_k, then once the corresponding change of basis has been performed, one obtains a new k-chain associated to column j of Δ^{r+1}, which will be denoted by $\sigma_k^{j,r+1}$. Observe that $\sigma_k^{j,r+1}$ is a linear combination over $\mathbb{Z}$ of columns t and j of Δ^r, i.e., $\sigma_k^{j,r+1}$ is a linear combination over $\mathbb{Z}$ of $\sigma_k^{t,r}$and $\sigma_k^{j,r}$. Hence,

$$\sigma_k^{j,r+1} \quad = \quad \underbrace{\sum_{\ell=f_k}^{j} c_\ell^{j,r} h_k^\ell}_{\sigma_k^{j,r}} \quad \pm \quad \underbrace{\sum_{\ell=f_k}^{j-1} c_\ell^{j-1,r} h_k^\ell}_{\sigma_k^{j-1,r}} \quad = \quad \mathsf{c}_{\mathsf{j}}^{\mathsf{j,r+1}} h_k^j + \mathsf{c}_{\mathsf{j-1}}^{\mathsf{j-1,r+1}} h_k^{j-1} + \cdots + \mathsf{c}_{\mathsf{f_k}}^{\mathsf{f_k,r+1}} h_k^{f_k}$$

where $\mathsf{c}_{\mathsf{k}}^{\ell,\mathsf{r+1}} \in \mathbb{Z}$, for $\ell = f_k, \cdots, j$. If Δ^r contains an index k primary pivot in the entry $\Delta^r_{s,\bar{\ell}}$ with $s > i$ and $\bar{\ell} < j$, then $q_{\bar{\ell}} = 0$. Of course, the first column of any Δ_k cannot undergo changes of basis, since there is no column to its left associated to a k-chain.

The family of matrices $\{\Delta^r\}$ produced by the SSSA has several properties, such as: there is at most one primary pivot in a fixed row or column; if the entry $\Delta^r_{j-r,j}$ is a primary pivot or a change-of-basis pivot, then $\Delta^r_{s,j} = 0$ for all $s > j - r$; Δ^r is a strictly upper triangular boundary map, for each r.

In [5] it is proved that the SSSA provides a system that spans the modules E^r in terms of the original basis of C_* and identifies all differentials $d^r_p : E^r_p \to E^r_{p-r}$ with primary and change-of -basis pivots on the r-th diagonal. A formula for the module $Z^r_{p,k-p}$ in terms of the chains σ_k's is

$$Z^r_{p,k-p} = \mathbb{Z}\Big[\mu^{p+1,r}\sigma_k^{p+1,r}, \mu^{p,r-1}\sigma_k^{p,r-1}, \cdots, \mu^{f_k,r-p-1+f_k}\sigma_k^{f_k,r-p-1+f_k}\Big], \tag{4}$$

where f_k is the first column of Δ associated to a k-chain, and $\mu^{j,\xi} = 0$ whenever there is a primary pivot on column j below row $(p - r + 1)$ and $\mu^{j,\xi} = 1$ otherwise. Moreover, if E^r_p and E^r_{p-r} are both non-zero, then the differential $d^r : E^r_p \to E^r_{p-r}$ is induced by multiplication by $\Delta^r_{p-r+1,p+1}$, whenever this entry is either a primary pivot, change-of-basis pivot or a zero with a column of zero entries below it.

6.2. **Algebraic Cancellation and Dynamical Homotopical Cancellation.** Our goal in this subsection is to establish a global homotopical cancellation result for GS-flows which follows closely the unfolding of its spectral sequence. In order to achieve this, we make use of the Row

Cancellation Algorithm (RCA), which reflects more closely the effect of dynamical homotopical cancellations on the modules E^r, while at the same time retaining the relevant information given by the primary and change of bases pivots of the SSSA.

Theorem 6.2 (Algebraic Cancellation and Dynamical Homotopical Cancellation via Spectral Sequence)**.** *Let $(C^{\mathcal{GS}}(M,X),\Delta^{\mathcal{GS}})$ be the GS-chain complex associated to a GS-flow φ_X on a singular 2-manifold $M\in\mathfrak{M}(\mathcal{GS})$, where $X\in\mathfrak{X}_{\mathcal{GS}}(M)$ and $\mathcal{S}=\mathcal{C},\mathcal{W},\mathcal{D}$ or $\mathcal{T}$. Let (E^r,d^r) be the associated spectral sequence for a finest filtration $F=\{F_pC^{\mathcal{GS}}\}$ on the chain complex.*

(1) *If $X\in\mathfrak{X}_{\mathcal{GC}}(M)$ or $X\in\mathfrak{X}_{\mathcal{GW}}(M)$, then the algebraic cancellations of the modules E^r of the spectral sequence are in one-to-one correspondence with dynamical homotopical cancellations of the singularities of X.*
(2) *If $X\in\mathfrak{X}_{\mathcal{GD}}(M)$ or $X\in\mathfrak{X}_{\mathcal{GT}}(M)$, then the algebraic cancellations of the modules E^r of the spectral sequence are in one-to-one correspondence with dynamical homotopical cancellations of the natures of the singularities of X.*

Moreover, the order of homotopical cancellation occurs as the gap r increases with respect to the filtration F.

We want to associate the data of the spectral sequence (E^r,d^r) with a dynamical continuation of the initial flow, by means of homotopical cancellations of the singularities and using as guide the family of matrices $\{\Delta^r\}_{\mathcal{GS}}$ produced by the SSSA, which codifies all data related to the modules and differentials of (E^r,d^r). However, it is easy to see that the matrices $\{\Delta^r\}_{\mathcal{GS}}$ are not in general realized as the GS-boundary operator associated to a GS-flow. Moreover, the changes of basis caused by pivots in row $j-r$ reflect all the changes in connecting orbits caused by the cancellation of the consecutive generators h_k^j and h_{k-1}^{j-r}. When we cancel the pair of generators h_k^j and h_{k-1}^{j-r}, then

- all the flow lines between the corresponding singularities associated to generators of k-nature and h_{k-1}^{j-r} are immediately removed and new connections are born;
- also all the flow lines between h_k^j and singularities associated to generators of $(k-1)$-nature are immediately removed.

In this sense, in order to interpret the SSSA as a dynamical homotopical cancellation, we have to perform the changes of basis that occur therein in a different order to reflect the death and birth of connections. More specifically, if $\Delta^r_{j-r,j}=\pm1$ is a primary pivot marked in step r of the SSSA all changes of basis caused by $\Delta^r_{j-r,j}$ must be performed in step $r+1$. The algorithm which reflects it is called the *Row cancellation Algorithm* (RCA) and it was first introduced in [2,3]. One emphasizes that whenever a primary pivot is marked, all the changes of basis caused by this pivot are performed in the next step.

Row Cancellation Algorithm - RCA

Initialization Step::

$$\left[\begin{array}{l} r=0\\ \tilde{\Delta}^r=\Delta\\ \tilde{T}^r=I\ (m\times m\text{ identity matrix})\end{array}\right.$$

Iterative Step:: (Repeated until all diagonals parallel and to the right of the main diagonal have been swept)

$$\left[\begin{array}{l} \textbf{Matrix } \tilde{\Delta} \textbf{ update}\\ \quad r\leftarrow r+1\\ \quad \tilde{\Delta}^r=(\tilde{T}^{r-1})^{-1}\tilde{\Delta}^{r-1}\tilde{T}^{r-1}\end{array}\right.$$

Markup
Sweep entries of $\tilde{\Delta}^r$ in the r-th diagonal:
If $\tilde{\Delta}^r_{j-r,j} \neq 0$ **and** $\tilde{\Delta}^r_{\bullet,j}$ does not contain a primary pivot
Then permanently mark $\tilde{\Delta}^r_{j-r,j}$ as a primary pivot

Matrix $\tilde{T}^r$ construction
$\tilde{T}^r \leftarrow I$
For each primary pivot $\tilde{\Delta}^r_{p-r,p}$ such that $j < m$, change the p-th row of $\tilde{T}^r$ as follows
$\tilde{T}^r_{p,\ell} \leftarrow -(1/\tilde{\Delta}^r_{p-r,p})\tilde{\Delta}^r_{p-r,\ell}$, for $\ell = p+1, \ldots, m$

Final Step::
Matrix $\tilde{\Delta}$ update
$r \leftarrow r+1$
$\tilde{\Delta}^r = (T^{r-1})^{-1}\tilde{\Delta}^{r-1}T^{r-1}$

In [3], it was proved the Primary Pivots Equality Theorem which states that the primary pivots on the r-th diagonal of $\widetilde{\Delta}^r$ marked in the r-th step of the RCA coincide with the ones on the r-th diagonal of Δ^r marked in the r-th step of the SSSA. More details on this algorithm can be found in [3].

Theorem 6.3 (Family of GS-Flows via Spectral Sequences)**.** *Let $(C^{\mathcal{GS}}(M,X), \Delta^{\mathcal{GS}})$ be the GS-chain complex associated to a GS-flow φ_X on a singular 2-manifold $M \in \mathfrak{M}(\mathcal{GS})$, where $X \in \mathfrak{X}_{\mathcal{GS}}(M)$ and $\mathcal{S} = \mathcal{C}, \mathcal{W}, \mathcal{D}$ or $\mathcal{T}$. The RCA for the GS-boundary map $\Delta^{\mathcal{GS}}$ produces a family of GS-flows $\{\varphi^1 = \varphi_X, \varphi^2, \ldots, \varphi^\omega\}$ where φ^r continues to φ^{r+1} by cancelling pairs of singularities of φ^r having gap r with respect to the filtration F. Moreover, the flow φ^ω is a minimal GS-flow in the sense that there is no more possible homotopical cancellations to be done.*

Proof. In order to prove the theorem, firstly one analyzes the local and global effects a homotopical cancellation of a pair of consecutive singularities has on the GS-boundary map $\Delta^{\mathcal{GS}}(M, \varphi_{X'})$ of the new flow $\varphi_{X'}$. Secondly, one constructs a family of GS-flows $\{\varphi^1 = \varphi_X, \varphi^2, \ldots, \varphi^\omega\}$ via the RCA in such way that the connections of the flow φ^r are codified in the r-th matrix produced by the RCA.

To simplify the exposition, one considers first the cases where $X \in \mathfrak{X}_{\mathcal{GS}}(M)$ for $\mathcal{S} = \mathcal{C}$ or $\mathcal{W}$, since in these cases each singularity posseses only one nature, hence there is only one generator corresponding to the nature of the singularity.

Without loss of generality, the set of orientations of the unstable manifolds for the Morsified flow $\varphi_{\widetilde{X}}$ will be considered, within this proof, as the one where all orientations of the unstable manifolds of repeller singularities are the same. This assumption guarantees that, whenever h_1 is a saddle singularity, the flow lines in $W^s(h_1)\backslash\{h_1\}$ either have opposite characteristic signs or they are null. On the other hand, by definition, the flow lines in $W^u(h_1)\backslash\{h_1\}$ always have opposite characteristic signs, if they are not null.

Throughout the proof, denote by $n(h_k, h_{k-1}, \varphi)$ the intersection number of h_k and h_{k-1} with respect to the flow φ.

Let h^j_k and h^{j-r}_{k-1} be consecutive singularities of the vector field X. By the Dynamical Homotopical Cancellation Theorem for GS-Flows (Theorem 5.1), if $n(h^j_k, h^{j-r}_{k-1}, \varphi_X) = \pm 1$ then these singularities can be cancelled, i.e. there is a GS-flow $\varphi_{X'}$ which coincides with φ_X outside a neighborhood of $\{h^j_k, h^i_{k-1}, h^{j-r}_{k-1}\} \cup \mathcal{O}(u_1) \cup \mathcal{O}(u_2)$, up to homotopy, where $\mathcal{M}^{h^j_k}_{h^{j-r}_{k-1}} = \{u_1\}$ and

$\mathcal{M}_{h^i_{k-1}}^{h^j_k} = \{u_2\}$. For $k=1$ (resp., $k=2$), let h_1^j (resp., h_1^{j-r}) be a saddle singularity that connects with the attracting (resp., repelling) singularities h_0^{j-r} and h_0^i (resp., h_2^j and h_2^p). If h_1^j cancels with h_0^{j-r} (resp., h_2^j cancels with h_1^{j-r}), then each saddle h_1^p (resp., h_1^i) which connects with h_0^{j-r} (resp., h_2^j) in φ_X will connect with h_0^i (resp., h_2^p) in $\varphi_{X'}$. Since the old and new connections have the same characteristic signs, then

$$n(h_1^p, h_0^i, \varphi_{X'}) = n(h_1^p, h_0^{j-r}, \varphi_X) + n(h_1^p, h_0^i, \varphi_X)$$

(resp., $n(h_2^p, h_1^i, \varphi_{X'}) = n(h_2^p, h_1^{j-r}, \varphi_X) + n(h_2^p, h_1^i, \varphi_X)$).

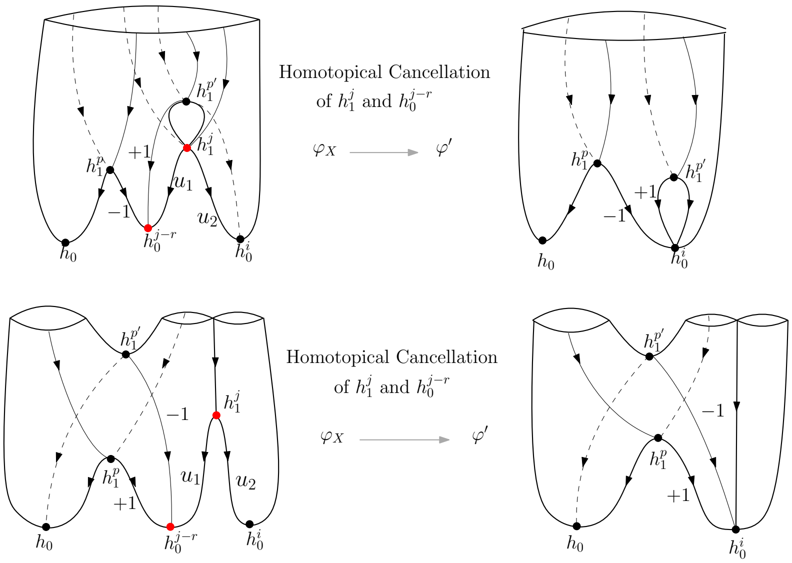


FIGURE 32. Birth and death of connections - characteristic signs.

Since the flow $\varphi_{X'}$ coincides with the flow φ_X outside a neighborhood U of

$$\{h_k^j, h_{k-1}^i, h_{k-1}^{j-r}\} \cup \mathcal{O}(u_1) \cup \mathcal{O}(u_2),$$

up to homotopy, the only intersection numbers that are modified after a homotopical cancellation are those $n(h_1^p, h_0^i)$, where h_1^p is such that $\mathcal{M}_{h_0^{j-r}}^{h_1^p} \neq \emptyset$ in the case of saddle-sink homotopical cancellation, and those $n(h_2^p, h_1^i)$, where h_2^p is such that $\mathcal{M}_{h_1^{j-r}}^{h_2^p} \neq \emptyset$, in the case of source-saddle homotopical cancellation.

The GS-boundary map $\Delta^{\mathcal{GS}}(M, X')$ can be obtained from $\Delta^{\mathcal{GS}}(M, X)$ in the following way:

- If a saddle singularity h_1^j is cancelled with an attracting singularity h_0^{j-r}, then define the matrix $\widetilde{\Delta}$ to be the matrix obtained from $\Delta^{\mathcal{GS}}(M, X)$ by replacing row i by the sum of row $(j-r)$ to row i. Then $\Delta^{\mathcal{GS}}(M, X')$ is the submatrix of $\widetilde{\Delta}$ which does not contain rows $j-r$, j and neither columns $j-r$, j.
- If a repelling singularity h_2^j is cancelled with a saddle singularity h_1^{j-r}, then define the matrix $\widetilde{\Delta}$ to be the matrix obtained from $\Delta^{\mathcal{GS}}(M, X)$ by replacing column p by the sum of column j to column p. Then $\Delta^{\mathcal{GS}}(M, X')$ is the submatrix of $\widetilde{\Delta}$ which does not contain rows $j-r$, j rows nor columns $j-r$, j.

It is straightforward to see that this corresponds to the row operations performed by the RCA.

Consider the matrices $\{\widetilde{\Delta}^r\}$ produced by the RCA when applied to $\Delta^{\mathcal{GS}}(M,X)$. Define $\varphi^1 = \varphi_X$ and φ^{r+1} to be a flow obtained from φ^r by cancelling all pairs of consecutive singularities corresponding to primary pivots on the r-th diagonal of $\widetilde{\Delta}^r$. In order to show that these flows are well defined, we have to prove that whenever a primary pivot $\Delta^r_{j-r,j}$ on the r-th diagonal of $\widetilde{\Delta}^r$ is marked, it is actually an intersection number between two consecutive singularities h^j_k and h^{j-r}_{k-1} of the flow φ^r and hence they can be cancelled by the Dynamical Homotopical Cancellation Theorem (Theorem 5.1).

Since $\varphi^1 = \varphi_X$, the GS-boundary map $\Delta^{\mathcal{GS}}(M,\varphi^1)$ is $\widetilde{\Delta}^1$. Let $\Delta^1_{j-1,j} = \pm 1$ be a primary pivot on the first diagonal of $\widetilde{\Delta}^1$. By definition, this primary pivot represents the intersection number between two singularities of the flow φ^1, namely h^j_k and h^{j-1}_{k-1}, which are consecutive since the gap between them is one. Using the Dynamical Homotopical Cancellation Theorem (Theorem 5.1), we can define a flow φ^2 by cancelling all pairs of consecutive singularities corresponding to primary pivots on the first diagonal of $\widetilde{\Delta}^1$. Moreover, the GS-boundary map $\Delta^{\mathcal{GS}}(M,\varphi^2)$ is the submatrix obtained from $\widetilde{\Delta}^2$ which does not contain the columns and rows corresponding to the cancelled singularities. Because of this and the fact that all non-zero entries of $\widetilde{\Delta}^2$ belong to $\Delta^{\mathcal{GS}}(M,\varphi^2)$, each non-zero entry of $\widetilde{\Delta}^2$ represents an intersection number between two singularities of φ^2. Observe that two singularities h^j_k and h^{j-2}_{k-1} of φ^2 with gap two in the filtration F are consecutive in the flow φ^2 since all the gap 1 singularities have been cancelled in the previous stage.

Suppose that φ^r is well defined, that is, each primary pivot $\Delta^{r-1}_{j-(r-1),j}$ on the diagonal $(r-1)$ of $\widetilde{\Delta}^{r-1}$ corresponds to the intersection number of consecutive singularities h^j_k and $h^{j-(r-1)}_{k-1}$ of φ^{r-1} and the GS-boundary map $\Delta^{\mathcal{GS}}(M,\varphi^r)$ is a submatrix of $\widetilde{\Delta}^r$ which does not contain columns and rows of $\widetilde{\Delta}^r$ corresponding to all primary pivots marked until the diagonal $r-1$. These correspond to all singularities of φ of gap less than or equal to $r-1$. Under these hypothesis singularities h^i_k and h^{i-r}_{k-1} of φ^r with gap r with respect to the filtration F are consecutive in the flow φ^r. Hence two singularities corresponding to a primary pivot on the diagonal r of $\widetilde{\Delta}^r$ can be cancelled, by the Dynamical Homotopical Cancellation Theorem (Theorem 5.1). Therefore, φ^{r+1} is a well defined flow obtained from φ^r by cancelling all pairs of critical points corresponding to primary pivots on the diagonal r of $\widetilde{\Delta}^r$. Moreover, the GS-boundary map $\Delta^{\mathcal{GS}}(M,\varphi^{r+1})$ is a submatrix of $\widetilde{\Delta}^{r+1}$ which does not contain columns and rows of $\widetilde{\Delta}^{r+1}$ corresponding to all primary pivots marked until step r. The flow φ_X continues to φ^r for all r.

Assume that $X \in \mathfrak{X}_{\mathcal{GS}}(M)$ for $\mathcal{S} = \mathcal{D}$ or $\mathcal{T}$. The proof follows the same steps as above by considering the homotopical cancellation of consecutive generators of the natures of the singularities. ☐

Proof of Theorem 6.2. By the Primary Pivots for Orientable Surfaces Theorem, see [2, 3], the primary pivots are always equal to ± 1 when working on orientable surfaces. Thus the differentials $d^r_p : E^r_p \to E^r_{p-r}$ induced by the primary pivots are isomorphisms and the ones associated to change of basis pivots always correspond to zero maps. Consequently, if d^r_p is non-zero differential, then, at the next stage of the spectral sequence, the algebraic cancellation $E^{r+1}_p = E^{r+1}_{p-r} = 0$ occurs.

An algebraic cancellation $E^{r+1}_p = E^{r+1}_{p-r} = 0$ is associated to a primary pivot $\Delta^r_{p-r+1,p+1} = \pm 1$ on the r-th diagonal of Δ^r produced by the r-th step of the SSSA. The row $p-r+1$ is associated to $h^{p-r+1}_{k-1} \in F_{p-r}C^{\mathcal{GS}}_{k-1} \backslash F_{p-r-1}C^{\mathcal{GS}}_{k-1}$ and the column $p+1$ is associated to $h^{p+1}_k \in F_pC^{\mathcal{GS}}_k \backslash F_{p-1}C^{\mathcal{GS}}_k$ in a gradient flow φ associated to f. By the Primary Pivots Equality Theorem in [3], the primary pivot $\Delta^r_{p-r+1,p+1} = \pm 1$ is also a primary pivot $\widetilde{\Delta}^r_{p-r+1,p+1} = \pm 1$ of the RCA. As it was shown in the proof of Theorem 6.3, the primary pivot $\Delta^r_{p-r+1,p+1}$ corresponds to the intersection number of two consecutive generators of natures of the singularities h^{p+1}_k and h^{p-r+1}_{k-1} of the flow φ^r. This pair can be homotopical cancelled by the Dynamical Homotopical Cancellation Theorem (Theorem 5.1).

Moreover, E^r_p and E^r_{p-r} correspond to generators of saddle and attractor (or repeller and saddle) natures, respectively, with gap r with respect to the filtration F. Therefore, the dynamical and algebraic cancellations occur with increasing gap. □

6.3. **Examples.** In this subsection we present some examples where we explore the algebraic cancellations of the modules of the spectral sequence and their corresponding dynamical homotopical cancellations.

Throughout this section, the primary pivots are the entries indicated by darker edge and the change of basis pivots are indicated by dashed edges, null entries are left blank and the diagonal being swept is indicated with a gray line.

Example 6.1. Consider the singular manifold $M \in \mathfrak{M}(\mathcal{GC})$ and a GS-flow φ_X associated to a vector field $X \in \mathfrak{X}_{\mathcal{GC}}(M)$ as in Figure 33. Consider as well a choice of orientations on the unstable manifolds of the critical points of a Morsification $\widetilde{M}$. Then we are able to determine the GS-characteristic signs of the orbits of φ_X , as it is shown in Figure 33.

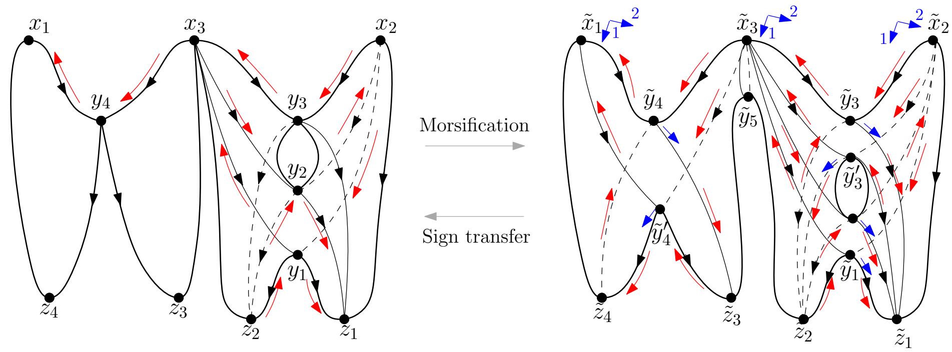


FIGURE 33. A GS-flow with cone singularities, its Morsification and sign tranfers.

The GS-chain groups are:

$$C_0(M,X)^{\mathcal{GC}} = \mathbb{Z}[\langle z_1\rangle,\langle z_2\rangle,\langle z_3\rangle,\langle z_4\rangle],\ C_1(M,X)^{\mathcal{GC}} = \mathbb{Z}[\langle y_1\rangle,\langle y_2\rangle,\langle y_3\rangle,\langle y_4\rangle],$$

$C_2(M,X)^{\mathcal{GC}} = \mathbb{Z}[\langle x_1\rangle,\langle x_2\rangle,\langle x_3\rangle]$, and $C_k(M) = 0, k \neq 0,1,2$. The GS-boundary operator $\Delta^{\mathcal{GS}}_*$ is given by the matrix in Figure 35a.

Consider a finest filtration on the GS-chain complex $(C^{\mathcal{GC}}_*(M,X), \Delta^{\mathcal{GC}}_*)$, namely,

$$F_0C^{\mathcal{GC}} = \mathbb{Z}[z_1],\ F_1C^{\mathcal{GC}} = \mathbb{Z}[z_1,z_2],\ F_2C^{\mathcal{GS}} = \mathbb{Z}[z_1,z_2,z_3],$$

$$F_3C^{\mathcal{GS}} = \mathbb{Z}[z_1,z_2,z_3,z_4],\ F_4C^{\mathcal{GC}} = \mathbb{Z}[z_1,z_2,z_3,z_4,y_1],\ F_5C^{\mathcal{GC}} = \mathbb{Z}[z_1,z_2,z_3,z_4,y_1,y_2],$$

$$F_6C^{\mathcal{GC}} = \mathbb{Z}[z_1,z_2,z_3,z_4,y_1,y_2,y_3],\ F_7C^{\mathcal{GC}} = \mathbb{Z}[z_1,z_2,z_3,z_4,y_1,y_2,y_3,y_4],$$

$$F_8C^{\mathcal{GC}} = \mathbb{Z}[z_1,z_2,z_3,z_4,y_1,y_2,y_3,y_4,x_1],\ F_9C^{\mathcal{GC}} = \mathbb{Z}[z_1,z_2,z_3,z_4,y_1,y_2,y_3,y_4,x_1,x_2]$$

and

$$F_{10}C^{\mathcal{GC}} = \mathbb{Z}[z_1,z_2,z_3,z_4,y_1,y_2,y_3,y_4,x_1,x_2,x_3].$$

The spectral sequence associated to $(C_*^{\mathcal{GC}}(M,X), \Delta_*^{\mathcal{GC}})$ enriched with the filtration F is shown in Figure 34.

$$
\begin{array}{llllllllllll}
E^0: & \mathbb{Z}[z_1] & \mathbb{Z}[z_2] & \mathbb{Z}[z_3] & \mathbb{Z}[z_4] & \mathbb{Z}[y_1] & \mathbb{Z}[y_2] & \mathbb{Z}[y_3] & \mathbb{Z}[y_4] & \mathbb{Z}[x_1] & \mathbb{Z}[x_2] & \mathbb{Z}[x_3] \\
E^1: & \mathbb{Z}[z_1] & \mathbb{Z}[z_2] & \mathbb{Z}[z_3] & \mathbb{Z}[z_4] & \mathbb{Z}[y_1] & \mathbb{Z}[y_2] & \mathbb{Z}[y_3] & \mathbb{Z}[y_4] & \xleftarrow{d_8^1} \mathbb{Z}[x_1] & \mathbb{Z}[x_2] & \mathbb{Z}[x_3] \\
E^2: & \mathbb{Z}[z_1] & \mathbb{Z}[z_2] & \mathbb{Z}[z_3] & \mathbb{Z}[z_4] & \mathbb{Z}[y_1] & \mathbb{Z}[y_2] & \mathbb{Z}[y_3] & 0 & 0 & \mathbb{Z}[x_2] & \mathbb{Z}[x_3] \\
E^3: & \mathbb{Z}[z_1] & \mathbb{Z}[z_2] & \mathbb{Z}[z_3] & \mathbb{Z}[z_4] & \xleftarrow{d_4^3} \mathbb{Z}[y_1] & \mathbb{Z}[y_2] & \mathbb{Z}[y_3] & 0 & 0 & \xleftarrow{d_9^3} \mathbb{Z}[x_2] & \mathbb{Z}[x_3] \\
E^4: & \mathbb{Z}[z_1] & 0 & \mathbb{Z}[z_3] & \mathbb{Z}[z_4] & 0 & \mathbb{Z}[y_2] & 0 & 0 & 0 & 0 & \mathbb{Z}[x_3]
\end{array}
$$

FIGURE 34. The spectral sequence for $(C_*^{\mathcal{GC}}(M,X), \Delta_*^{\mathcal{GC}})$ with filtration F.

Applying the SSSA to the GS-boundary differential $\Delta^{\mathcal{GC}}$, one obtains the sequence of matrices $\Delta^1, \cdots, \Delta^5$ as in Figures 35b,$\cdots$, 35f, respectively, where the singularities are identified by $h_0^i = z_i$, $h_1^{i+4} = y_i$, for $i = 1 \dots 4$ and $h_2^{i+8} = z_i$, for $i = 1, \dots, 3$.

	h_0^1	h_0^2	h_0^3	h_0^4	h_1^5	h_1^6	h_1^7	h_1^8	h_2^9	h_2^{10}	h_2^{11}
h_0^1					1	1					
h_0^2					−1	−1					
h_0^3											
h_0^4											
h_1^5										1	−1
h_1^6										−1	1
h_1^7										1	−1
h_1^8									−1		1
h_2^9											
h_2^{10}											
h_2^{11}											

(A) Δ^0, the GS-boundary operator.

	$\sigma_0^{1,1}$	$\sigma_0^{2,1}$	$\sigma_0^{3,1}$	$\sigma_0^{4,1}$	$\sigma_1^{5,1}$	$\sigma_1^{6,1}$	$\sigma_1^{7,1}$	$\sigma_1^{8,1}$	$\sigma_2^{9,1}$	$\sigma_2^{10,1}$	$\sigma_2^{11,1}$
$\sigma_0^{1,1} = h_0^1$					1	1					
$\sigma_0^{2,1} = h_0^2$					−1	−1					
$\sigma_0^{3,1} = h_0^3$											
$\sigma_0^{4,1} = h_0^4$											
$\sigma_1^{5,1} = h_1^5$										1	−1
$\sigma_1^{6,1} = h_1^6$										−1	1
$\sigma_1^{7,1} = h_1^7$										1	−1
$\sigma_1^{8,1} = h_1^8$									$\boxed{-1}$		1
$\sigma_2^{9,1} = h_2^9$											
$\sigma_2^{10,1} = h_2^{10}$											
$\sigma_2^{11,1} = h_2^{11}$											

(B) Δ^1, sweeping 1-st diagonal.

	$\sigma_0^{1,2}$	$\sigma_0^{2,2}$	$\sigma_0^{3,2}$	$\sigma_0^{4,2}$	$\sigma_1^{5,2}$	$\sigma_1^{6,2}$	$\sigma_1^{7,2}$	$\sigma_1^{8,2}$	$\sigma_2^{9,2}$	$\sigma_2^{10,2}$	$\sigma_2^{11,2}$
$\sigma_0^{1,2} = h_0^1$					1	1					
$\sigma_0^{2,2} = h_0^2$					−1	−1					
$\sigma_0^{3,2} = h_0^3$											
$\sigma_0^{4,2} = h_0^4$											
$\sigma_1^{5,2} = h_1^5$										1	−1
$\sigma_1^{6,2} = h_1^6$										−1	1
$\sigma_1^{7,2} = h_1^7$										1	−1
$\sigma_1^{8,2} = h_1^8$									−1		1
$\sigma_2^{9,2} = h_2^9$											
$\sigma_2^{10,2} = h_2^{10}$											
$\sigma_2^{11,2} = h_2^{11}$											

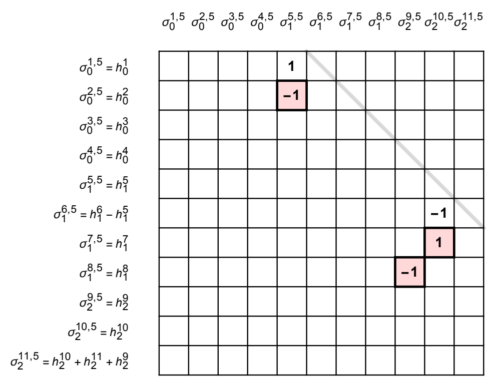
(c) Δ^2, sweeping 2-nd diagonal.

	$\sigma_0^{1,3}$	$\sigma_0^{2,3}$	$\sigma_0^{3,3}$	$\sigma_0^{4,3}$	$\sigma_1^{5,3}$	$\sigma_1^{6,3}$	$\sigma_1^{7,3}$	$\sigma_1^{8,3}$	$\sigma_2^{9,3}$	$\sigma_2^{10,3}$	$\sigma_2^{11,3}$
$\sigma_0^{1,3} = h_0^1$					1	1					
$\sigma_0^{2,3} = h_0^2$					−1	−1					
$\sigma_0^{3,3} = h_0^3$											
$\sigma_0^{4,3} = h_0^4$											
$\sigma_1^{5,3} = h_1^5$										1	−1
$\sigma_1^{6,3} = h_1^6$										−1	1
$\sigma_1^{7,3} = h_1^7$										1	−1
$\sigma_1^{8,3} = h_1^8$									−1		1
$\sigma_2^{9,3} = h_2^9$											
$\sigma_2^{10,3} = h_2^{10}$											
$\sigma_2^{11,3} = h_2^{11}$											

(d) Δ^3, sweeping 3-rd diagonal.

	$\sigma_0^{1,4}$	$\sigma_0^{2,4}$	$\sigma_0^{3,4}$	$\sigma_0^{4,4}$	$\sigma_1^{5,4}$	$\sigma_1^{6,4}$	$\sigma_1^{7,4}$	$\sigma_1^{8,4}$	$\sigma_2^{9,4}$	$\sigma_2^{10,4}$	$\sigma_2^{11,4}$
$\sigma_0^{1,4} = h_0^1$					1	1					
$\sigma_0^{2,4} = h_0^2$					−1	−1					
$\sigma_0^{3,4} = h_0^3$											
$\sigma_0^{4,4} = h_0^4$											
$\sigma_1^{5,4} = h_1^5$										1	−1
$\sigma_1^{6,4} = h_1^6$										−1	1
$\sigma_1^{7,4} = h_1^7$										1	−1
$\sigma_1^{8,4} = h_1^8$									−1		
$\sigma_2^{9,4} = h_2^9$											
$\sigma_2^{10,4} = h_2^{10}$											
$\sigma_2^{11,4} = h_2^{11} + h_2^9$											

(e) Δ^4, sweeping 4-th diagonal.



(F) Δ^5, sweeping 5-th diagonal.

FIGURE 35. Sequence of matrices produced by the SSSA.

As proven in Theorem 6.2, the primary pivots detect algebraic cancellations of the modules of the spectral sequence. More specifically,

- the primary pivot $\Delta^1_{8,9}$ detects the algebraic cancellation of the modules E^1_8 and E^1_7;
- the primary pivot $\Delta^2_{7,10}$ detects the algebraic cancellation of the modules E^2_9 and E^2_1;
- the primary pivot $\Delta^2_{2,5}$ detects the algebraic cancellation of the modules E^2_4 and E^2_1.

On the other hand, these algebraic cancellations are associated to dynamical cancellations by Theorem 6.2, namely:

- the algebraic cancellation of E^1_8 and E^1_7 determines the dynamical homotopical cancellation of the singularities (x_1, y_4).
- the algebraic cancellation of E^2_9 and E^2_1 determines the dynamical homotopical cancellation of the singularities (x_2, y_3).
- the algebraic cancellation of E^2_4 and E^2_1 determines the dynamical homotopical cancellation of the singularities (y_1, z_2).

Figure 36 shows the dynamical homotopical cancellations of the pair of singularities (x_1, y_4), (x_2, y_3) and (y_1, z_2), respectively.

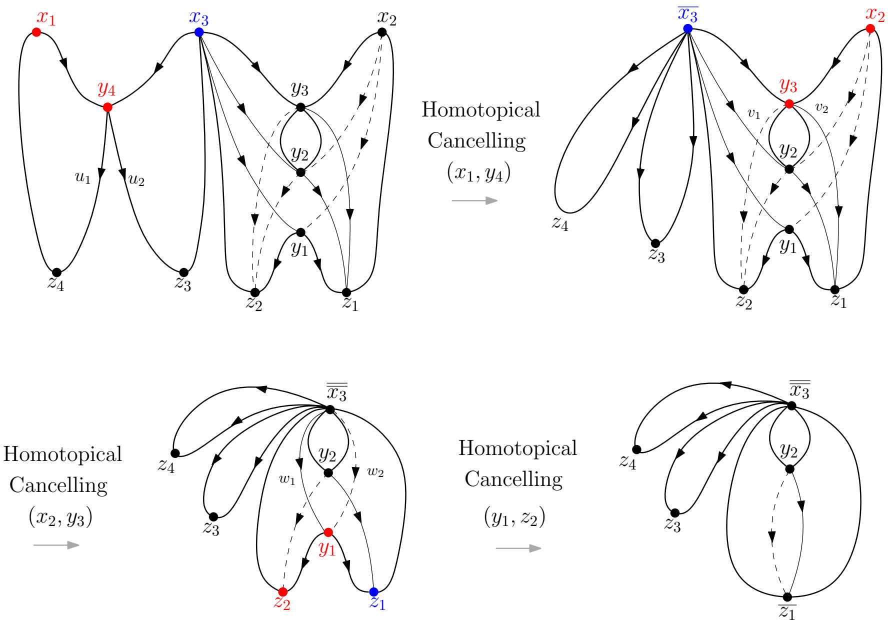


FIGURE 36. Homotopical Cancellation the pair of singularities (x_1, y_4), (x_2, y_3) and (y_1, z_2), sucessively.

Example 6.2. Consider the singular manifold $M \in \mathfrak{M}(\mathcal{GW})$ and the GS-flow φ_X associated to a vector field $X \in \mathfrak{X}_{\mathcal{GW}}(M)$ as in Figure 27. The GS-chain complex associated to this flow is presented in Example 4.2. Consider a finest filtration on $(C_*^{\mathcal{GW}}(M,X), \Delta_*^{\mathcal{GW}})$, namely,

$$F_0C^{\mathcal{GW}} = \mathbb{Z}[z_1],\ F_1C^{\mathcal{GW}} = \mathbb{Z}[z_1, z_2],\ F_2C^{\mathcal{GW}} = \mathbb{Z}[z_1, z_2, y_1],\ F_3C^{\mathcal{GW}} = \mathbb{Z}[z_1, z_2, y_1, y_2],$$
$$F_4C^{\mathcal{GW}} = \mathbb{Z}[z_1, z_2, y_1, y_2, y_3],\ F_5C^{\mathcal{GW}} = \mathbb{Z}[z_1, z_2, y_1, y_2, y_3, x_1],$$
$F_6C^{\mathcal{GW}} = \mathbb{Z}[z_1, z_2, y_1, y_2, y_3, x_1, x_2]$ and $F_7C^{\mathcal{GW}} = \mathbb{Z}[z_1, z_2, y_1, y_2, y_3, x_1, x_2, x_3]$.

The spectral sequence associated to $(C_*^{\mathcal{GW}}(M,X), \Delta_*^{\mathcal{GW}})$ enriched with the filtration F is shown in Figure 37.

E^0 :	E^0_0	E^0_1	E^0_2	E^0_3	E^0_4	E^0_5	E^0_6	E^0_7
E^1 :	$\mathbb{Z}[z_1]$	$\mathbb{Z}[z_2]$	$\mathbb{Z}[y_1]$	$\mathbb{Z}[y_2]$	$\mathbb{Z}[y_3]$	$\mathbb{Z}[x_1]$	$\mathbb{Z}[x_2]$	$\mathbb{Z}[x_3]$
E^2 :	$\mathbb{Z}[z_1]$	$\mathbb{Z}[z_2]$	$\mathbb{Z}[y_1]$	$\mathbb{Z}[y_2]$	$\mathbb{Z}[y_3]$	$\mathbb{Z}[x_1]$	$\mathbb{Z}[x_2]$	$\mathbb{Z}[x_3]$
E^3 :	$\mathbb{Z}[z_1]$	$\mathbb{Z}[z_2]$	$\mathbb{Z}[y_1]$	0	$\mathbb{Z}[y_3]$	0	$\mathbb{Z}[x_2]$	$\mathbb{Z}[x_3]$
E^4 :	$\mathbb{Z}[z_1]$	0	$\mathbb{Z}[y_1]$	0	0	0	$\mathbb{Z}[x_2]$	$\mathbb{Z}[x_3]$
E^5 :	$\mathbb{Z}[z_1]$	0	0	0	0	0	0	$\mathbb{Z}[x_3]$

d^2_5, d^3_4, d^4_6

FIGURE 37. The spectral sequence for $(C_*^{\mathcal{GW}}(M,X), \Delta_*^{\mathcal{GW}})$ with filtration F.

Applying the SSSA to the GS-boundary differential $\Delta^{\mathcal{GW}}$, one obtains the sequence of matrices $\Delta^1, \cdots, \Delta^6$ as in Figures 38a,$\cdots$, 38f, respectively, where the singularities are identified by $h^i_0 = z_i$, for $i = 1, 2$, $h^{i+2}_1 = y_i$, for $i = 1 \ldots 3$ and $h^{i+5}_2 = x_i$, for $i = 1 \ldots 3$.

	$\sigma_0^{1,1}$	$\sigma_0^{2,1}$	$\sigma_1^{3,1}$	$\sigma_1^{4,1}$	$\sigma_1^{5,1}$	$\sigma_2^{6,1}$	$\sigma_2^{7,1}$	$\sigma_2^{8,1}$
$\sigma_0^{1,1}=h_0^1$					-1			
$\sigma_0^{2,1}=h_0^2$					1			
$\sigma_1^{3,1}=h_1^3$							1	-1
$\sigma_1^{4,1}=h_1^4$						-1	1	
$\sigma_1^{5,1}=h_1^5$								
$\sigma_2^{6,1}=h_2^6$								
$\sigma_2^{7,1}=h_2^7$								
$\sigma_2^{8,1}=h_2^8$								

(A) Δ^1, sweeping 1-st diagonal.

	$\sigma_0^{1,2}$	$\sigma_0^{2,2}$	$\sigma_1^{3,2}$	$\sigma_1^{4,2}$	$\sigma_1^{5,2}$	$\sigma_2^{6,2}$	$\sigma_2^{7,2}$	$\sigma_2^{8,2}$
$\sigma_0^{1,2}=h_0^1$					-1			
$\sigma_0^{2,2}=h_0^2$					1			
$\sigma_1^{3,2}=h_1^3$							1	-1
$\sigma_1^{4,2}=h_1^4$						-1	1	
$\sigma_1^{5,2}=h_1^5$								
$\sigma_2^{6,2}=h_2^6$								
$\sigma_2^{7,2}=h_2^7$								
$\sigma_2^{8,2}=h_2^8$								

(B) Δ^2, sweeping 2-nd diagonal.

	$\sigma_0^{1,3}$	$\sigma_0^{2,3}$	$\sigma_1^{3,3}$	$\sigma_1^{4,3}$	$\sigma_1^{5,3}$	$\sigma_2^{6,3}$	$\sigma_2^{7,3}$	$\sigma_2^{8,3}$
$\sigma_0^{1,3}=h_0^1$					-1			
$\sigma_0^{2,3}=h_0^2$					1			
$\sigma_1^{3,3}=h_1^3$							1	-1
$\sigma_1^{4,3}=h_1^4$						-1	1	
$\sigma_1^{5,3}=h_1^5$								
$\sigma_2^{6,3}=h_2^6$								
$\sigma_2^{7,3}=h_2^7$								
$\sigma_2^{8,3}=h_2^8$								

(C) Δ^3, sweeping 3-rd diagonal.

	$\sigma_0^{1,4}$	$\sigma_0^{2,4}$	$\sigma_1^{3,4}$	$\sigma_1^{4,4}$	$\sigma_1^{5,4}$	$\sigma_2^{6,4}$	$\sigma_2^{7,4}$	$\sigma_2^{8,4}$
$\sigma_0^{1,4} = h_0^1$					-1			
$\sigma_0^{2,4} = h_0^2$					$\boxed{1}$			
$\sigma_1^{3,4} = h_1^3$							$\boxed{1}$	-1
$\sigma_1^{4,4} = h_1^4$						$\boxed{-1}$		
$\sigma_1^{5,4} = h_1^5$								
$\sigma_2^{6,4} = h_2^6$								
$\sigma_2^{7,4} = h_2^6 + h_2^7$								
$\sigma_2^{8,4} = h_2^8$								

(D) Δ^4, sweeping 4-th diagonal.

	$\sigma_0^{1,5}$	$\sigma_0^{2,5}$	$\sigma_1^{3,5}$	$\sigma_1^{4,5}$	$\sigma_1^{5,5}$	$\sigma_2^{6,5}$	$\sigma_2^{7,5}$	$\sigma_2^{8,5}$
$\sigma_0^{1,5} = h_0^1$					-1			
$\sigma_0^{2,5} = h_0^2$					$\boxed{1}$			
$\sigma_1^{3,5} = h_1^3$							$\boxed{1}$	$\boxed{-1}$
$\sigma_1^{4,5} = h_1^4$						$\boxed{-1}$		
$\sigma_1^{5,5} = h_1^5$								
$\sigma_2^{6,5} = h_2^6$								
$\sigma_2^{7,5} = h_2^6 + h_2^7$								
$\sigma_2^{8,5} = h_2^8$								

(E) Δ^5, sweeping 5-th diagonal.

	$\sigma_0^{1,6}$	$\sigma_0^{2,6}$	$\sigma_1^{3,6}$	$\sigma_1^{4,6}$	$\sigma_1^{5,6}$	$\sigma_2^{6,6}$	$\sigma_2^{7,6}$	$\sigma_2^{8,6}$
$\sigma_0^{1,6} = h_0^1$					-1			
$\sigma_0^{2,6} = h_0^2$					$\boxed{1}$			
$\sigma_1^{3,6} = h_1^3$							$\boxed{1}$	
$\sigma_1^{4,6} = h_1^4$						$\boxed{-1}$		
$\sigma_1^{5,6} = h_1^5$								
$\sigma_2^{6,6} = h_2^6$								
$\sigma_2^{7,6} = h_2^6 + h_2^7$								
$\sigma_2^{8,6} = h_2^6 + h_2^7 + h_2^8$								

(F) Δ^6, sweeping 6-th diagonal.

FIGURE 38. Sequence of matrices produced by the SSSA.

As proven in Theorem 6.2, the primary pivots detect algebraic cancellations of the modules of the spectral sequence. More specifically,

- the primary pivot $\Delta^2_{4,6}$ detects the algebraic cancellation of the modules E^2_5 and E^2_3;
- the primary pivot $\Delta^3_{2,5}$ detects the algebraic cancellation of the modules E^3_4 and E^3_1;
- the primary pivot $\Delta^4_{3,7}$ detects the algebraic cancellation of the modules E^4_6 and E^4_2.

On the other hand, these algebraic cancellations are associated to dynamical homotopical cancellations by Theorem 6.2, namely:

- the algebraic cancellation of E^1_5 and E^1_3 determines the dynamical homotopical cancellation of the singularities (x_1, y_2).
- the algebraic cancellation of E^3_4 and E^3_1 determines the dynamical homotopical cancellation of the singularities (y_3, z_2).
- the algebraic cancellation of E^4_6 and E^4_2 determines the dynamical homotopical cancellation of the singularities (x_2, y_1).

Figure 39 shows the dynamical cancellations of the pair of singularities (x_1, y_2), (y_3, z_2) and (x_2, y_1), respectively.

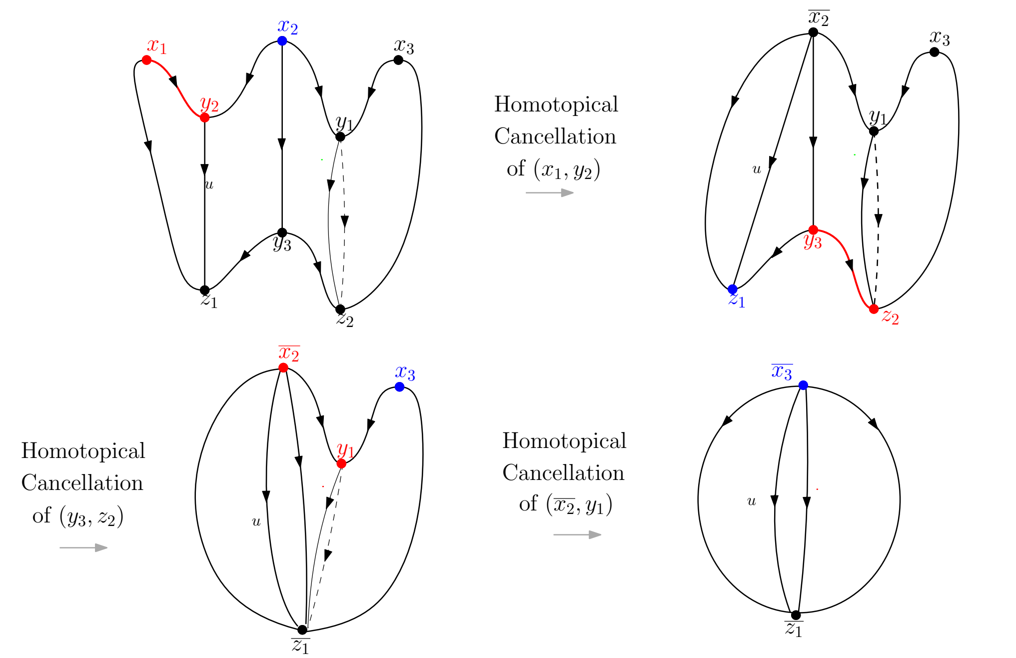


FIGURE 39. Homotopical cancellation the pair of singularities (x_1, y_2), (y_3, z_2) and (x_2, y_1), sucessively.

Example 6.3. Consider the singular manifold $M \in \mathfrak{M}(\mathcal{GD})$ and the GS-flow φ_X associated to a vector field $X \in \mathfrak{X}_{\mathcal{GD}}(M)$ as in Figure 29. The GS-chain complex associated to (M, X) is presented in Example 4.3. The GS-boundary operator $\Delta^{\mathcal{GS}}_*$ is given by the matrix in Figure 30.

Consider a finest filtration on $(C^{\mathcal{GD}}_*(M, X), \Delta^{\mathcal{GD}}_*)$, namely,

$$F_0C^{\mathcal{GD}} = \mathbb{Z}[z^e_1],\ F_1C^{\mathcal{GS}} \setminus F_0C^{\mathcal{GD}} = \mathbb{Z}[z^i_1],\ F_2C^{\mathcal{GD}} \setminus F_1C^{\mathcal{GD}} = \mathbb{Z}[z^e_2],\ F_3C^{\mathcal{GD}} \setminus F_2C^{\mathcal{GD}} = \mathbb{Z}[z^i_2],$$
$$F_4C^{\mathcal{GD}} \setminus F_3C^{\mathcal{GD}} = \mathbb{Z}[y^e_1],\ F_5C^{\mathcal{GD}} \setminus F_4C^{\mathcal{GD}} = \mathbb{Z}[y^i_1],\ F_6C^{\mathcal{GD}} \setminus F_5C^{\mathcal{GD}} = \mathbb{Z}[y^e_2],$$

$$F_7C^{\mathcal{GD}} \setminus F_6C^{\mathcal{GD}} = \mathbb{Z}[y_2^i],\ F_8C^{\mathcal{GD}} \setminus F_7C^{\mathcal{GD}} = \mathbb{Z}[y_3],\ F_9C^{\mathcal{GD}} \setminus F_8C^{\mathcal{GD}} = \mathbb{Z}[x_1],$$

$$F_{10}C^{\mathcal{GD}} \setminus F_9C^{\mathcal{GD}} = \mathbb{Z}[x_2],\ F_{11}C^{\mathcal{GD}} \setminus F_10C^{\mathcal{GD}} = \mathbb{Z}[x_3],\ F_{12}C^{\mathcal{GD}} \setminus F_11C^{\mathcal{GD}} = \mathbb{Z}[x_4]$$

and $F_{13}C^{\mathcal{GD}} \setminus F_12C^{\mathcal{GD}} = \mathbb{Z}[x_5]$. The spectral sequence associated to $(C_*^{\mathcal{GD}}(M,X), \Delta_*^{\mathcal{GD}})$ enriched with the filtration F is shown in Figure 34.

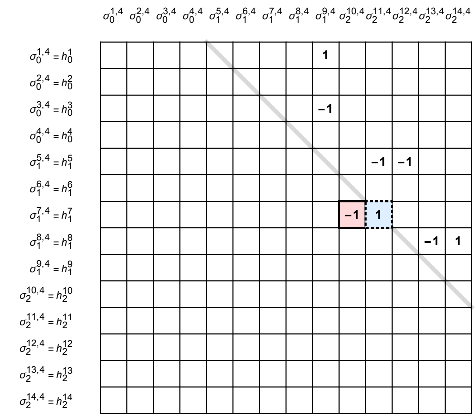
$$\begin{array}{rcccccccccccccc}
E^0,E^1,E^2: & \mathbb{Z}[z_1^e] & \mathbb{Z}[z_1^i] & \mathbb{Z}[z_2^e] & \mathbb{Z}[z_e^i] & \mathbb{Z}[y_1^e] & \mathbb{Z}[y_1^i] & \mathbb{Z}[y_2^e] & \mathbb{Z}[y_2^i] & \mathbb{Z}[y_3] & \mathbb{Z}[x_1] & \mathbb{Z}[x_2] & \mathbb{Z}[x_3] & \mathbb{Z}[x_4] & \mathbb{Z}[x_5] \\
 & & & & & & & & & d_9^3 & & & & & \\
E^3: & \mathbb{Z}[z_1^e] & \mathbb{Z}[z_1^i] & \mathbb{Z}[z_2^e] & \mathbb{Z}[z_e^i] & \mathbb{Z}[y_1^e] & \mathbb{Z}[y_1^i] & \mathbb{Z}[y_2^e] & \mathbb{Z}[y_2^i] & \mathbb{Z}[y_3] & \mathbb{Z}[x_1] & \mathbb{Z}[x_2] & \mathbb{Z}[x_3] & \mathbb{Z}[x_4] & \mathbb{Z}[x_5] \\
E^4: & \mathbb{Z}[z_1^e] & \mathbb{Z}[z_1^i] & \mathbb{Z}[z_2^e] & \mathbb{Z}[z_e^i] & \mathbb{Z}[y_1^e] & \mathbb{Z}[y_1^i] & 0 & \mathbb{Z}[y_2^i] & \mathbb{Z}[y_3] & 0 & \mathbb{Z}[x_2] & \mathbb{Z}[x_3] & \mathbb{Z}[x_4] & \mathbb{Z}[x_5] \\
E^5: & \mathbb{Z}[z_1^e] & \mathbb{Z}[z_1^i] & \mathbb{Z}[z_2^e] & \mathbb{Z}[z_e^i] & \mathbb{Z}[y_1^e] & \mathbb{Z}[y_1^i] & 0 & \mathbb{Z}[y_2^i] & \mathbb{Z}[y_3] & 0\ d_{12}^5 & \mathbb{Z}[x_2] & \mathbb{Z}[x_3] & \mathbb{Z}[x_4] & \mathbb{Z}[x_5] \\
 & & & & d_8^6 & & & & & & d_9^6 & & & & \\
E^6: & \mathbb{Z}[z_1^e] & \mathbb{Z}[z_1^i] & \mathbb{Z}[z_2^e] & \mathbb{Z}[z_e^i] & \mathbb{Z}[y_1^e] & \mathbb{Z}[y_1^i] & \mathbb{Z}[y_2^e] & 0 & \mathbb{Z}[y_3] & \mathbb{Z}[x_1] & \mathbb{Z}[x_2] & \mathbb{Z}[x_3] & 0 & \mathbb{Z}[x_5] \\
E^6: & \mathbb{Z}[z_1^e] & \mathbb{Z}[z_1^i] & 0 & \mathbb{Z}[z_e^i] & 0 & \mathbb{Z}[y_1^i] & \mathbb{Z}[y_2^e] & 0 & 0 & \mathbb{Z}[x_1] & 0 & \mathbb{Z}[x_3] & 0 & \mathbb{Z}[x_5]
\end{array}$$

FIGURE 40. The spectral sequence for $(C_*^{\mathcal{GD}}(M,X), \Delta_*^{\mathcal{GD}})$ with filtration F.

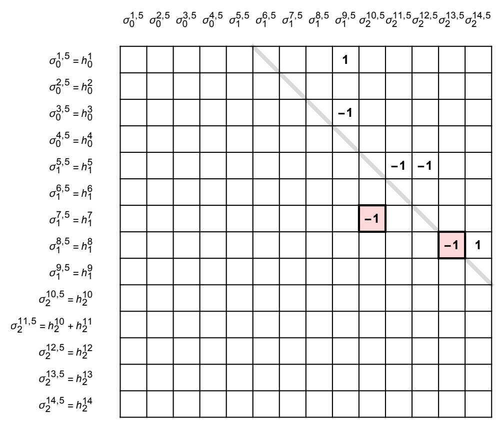
Applying the SSSA to the GS-boundary differential $\Delta^{\mathcal{GD}}$, one obtains the sequence of matrices $\Delta^1, \cdots, \Delta^8$ as in Figures 41a,$\cdots$, 41f, respectively, where the singularities are identified by $h_0^1 = z_1^e$, $h_0^2 = z_1^i$, $h_0^3 = z_2^e$, $h_0^4 = z_2^i$, $h_1^5 = y_1^e$, $h_1^6 = y_1^i$, $h_1^7 = y_2^e$, $h_1^8 = y_2^i$, $h_1^9 = y_3$ and $h_2^{j+9} = x_j$, for $j = 1 \ldots 5$.

$$\begin{array}{r|cccccccccccccc|}
 & \sigma_0^{1,3} & \sigma_0^{2,3} & \sigma_0^{3,3} & \sigma_0^{4,3} & \sigma_1^{5,3} & \sigma_1^{6,3} & \sigma_1^{7,3} & \sigma_1^{8,3} & \sigma_1^{9,3} & \sigma_2^{10,3} & \sigma_2^{11,3} & \sigma_2^{12,3} & \sigma_2^{13,3} & \sigma_2^{14,3} \\
\sigma_0^{1,3}=h_0^1 & & & & & & & & & 1 & & & & & \\
\sigma_0^{2,3}=h_0^2 & & & & & & & & & & & & & & \\
\sigma_0^{3,3}=h_0^3 & & & & & & & & & -1 & & & & & \\
\sigma_0^{4,3}=h_0^4 & & & & & & & & & & & & & & \\
\sigma_1^{5,3}=h_1^5 & & & & & & & & & & & -1 & -1 & & \\
\sigma_1^{6,3}=h_1^6 & & & & & & & & & & & & & & \\
\sigma_1^{7,3}=h_1^7 & & & & & & & & & & \boxed{-1} & 1 & & & \\
\sigma_1^{8,3}=h_1^8 & & & & & & & & & & & & & -1 & 1 \\
\sigma_1^{9,3}=h_1^9 & & & & & & & & & & & & & & \\
\sigma_2^{10,3}=h_2^{10} & & & & & & & & & & & & & & \\
\sigma_2^{11,3}=h_2^{11} & & & & & & & & & & & & & & \\
\sigma_2^{12,3}=h_2^{12} & & & & & & & & & & & & & & \\
\sigma_2^{13,3}=h_2^{13} & & & & & & & & & & & & & & \\
\sigma_2^{14,3}=h_2^{14} & & & & & & & & & & & & & &
\end{array}$$

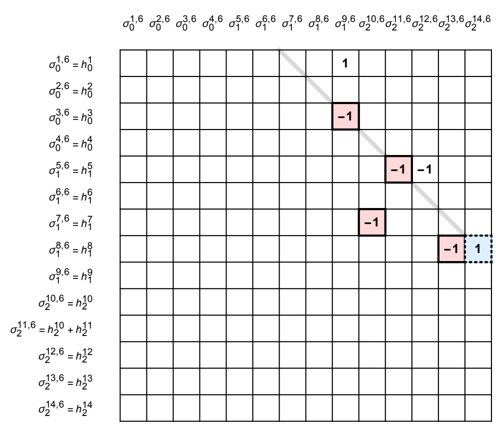
(A) Δ^3, sweeping 3-rd diagonal.



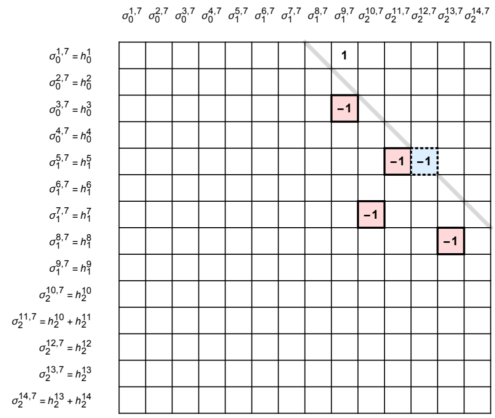
(B) Δ^4, sweeping 4-th diagonal.



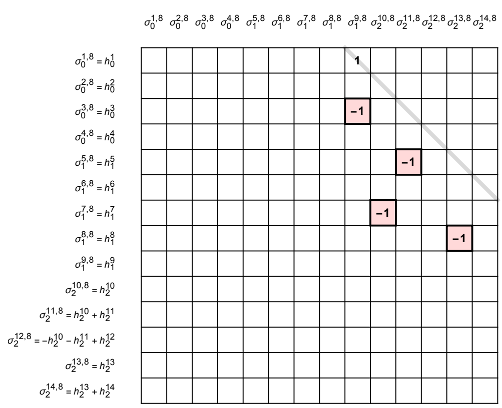
(C) Δ^5, sweeping 5-th diagonal..



(D) Δ^6, sweeping 6-th diagonal.



(E) Δ^7, sweeping 7-th diagonal.



(F) Δ^8, sweeping 8-th diagonal.

FIGURE 41. Sequence of matrices produced by the SSSA.

As proven in Theorem 6.2, the primary pivots detect algebraic cancellations of the modules of the spectral sequence. More specifically,

- the primary pivot $\Delta^3_{7,10}$ detects the algebraic cancellation of the modules E^3_9 and E^3_6;
- the primary pivot $\Delta^5_{8,13}$ detects the algebraic cancellation of the modules E^5_{12} and E^5_7;
- the primary pivot $\Delta^6_{3,9}$ detects the algebraic cancellation of the modules E^6_8 and E^6_2;
- the primary pivot $\Delta^6_{5,11}$ detects the algebraic cancellation of the modules E^6_{10} and E^6_4.

On the other hand, these algebraic cancellations are associated to dynamical homotopical cancellations, namely:

- the algebraic cancellation of E^3_9 and E^3_6 determines the dynamical cancellation of the singularities (y^e_2, x_1).
- the algebraic cancellation of E^5_{12} and E^5_7 determines the dynamical homotopical cancellation of the singularities (y^i_2, x_4).

- the algebraic cancellation of E^6_8 and E^6_2 determines the dynamical homotopical cancellation of the singularities (z^e_2, y_3).
- the algebraic cancellation of E^6_{10} and E^6_4 determines the dynamical homotopical cancellation of the singularities $(y^e_1, \bar{x}_3)$.

Figure 42 shows the dynamical cancellation of the pair of singularities (y^e_2, x_1), (y^i_2, x_4), (z^e_2, y_3) and $(y^e_1, \bar{x}_3)$, respectively.

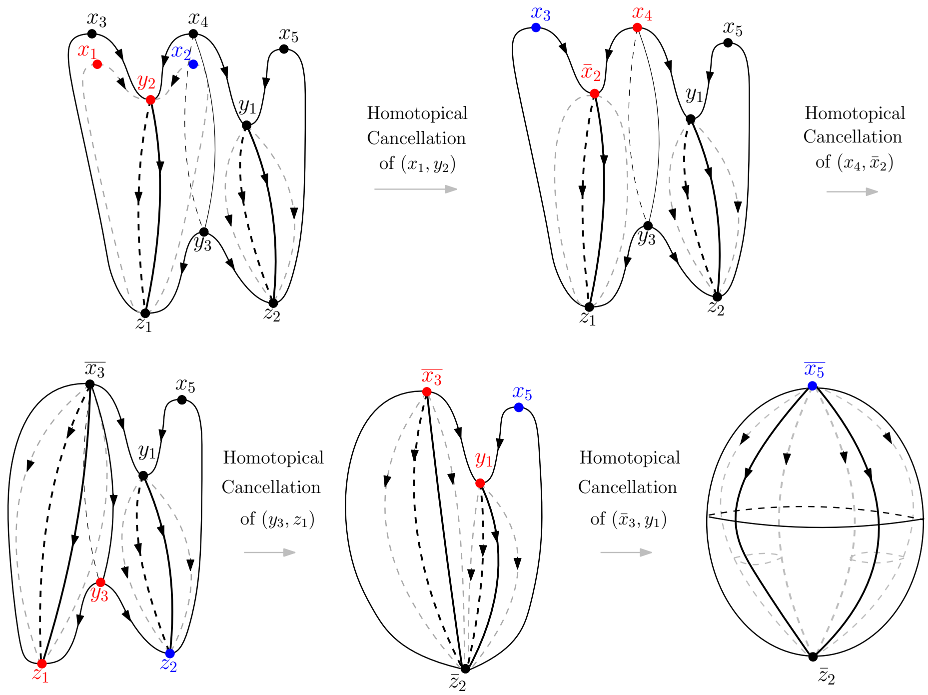


FIGURE 42. Homotopical cancellation of the pair of generators (y^e_2, x_1), (y^i_2, x_4), (z^e_2, y_3) and $(y^e_1, \bar{x}_3)$, sucessively.

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