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ON THE DEFORMATION OF THE EXCEPTIONAL UNIMODAL SINGULARITIES

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Abstract. Ebeling and Takahashi considered the deformation of an isolated surface singularity $f(x, y, z) - txyz$ ($t \in \mathbb{C}$) for any invertible polynomial f in three variables. In particular, they deformed each of the 14 exceptional unimodal singularities into a cusp singularity. However, their proof is purely algebraic and requires a detailed knowledge of normal forms. In this article, instead of algebraic treatment of the singularity, we observe the critical points of the squared distance function restricted to the singular complex surface in $\mathbb{C}^3$. We show that only one additional critical point emerges via the deformation if and only if f is one of the 14 exceptional unimodal singularities. Moreover, we can determine the approximate location of the critical point when the parameter t is a small positive number. This would be helpful to describe the change of the topology of the complex surface by means of the Morse theory.

1. Introduction

V. I. Arnol'd defined the notion of modality of a function-germ and classified all the hypersurface singularities of modality equal or smaller than 2 (see [1]). The singularities of modality 1 are called unimodal singularities, which are listed below. Throughout this paper, (x, y, z) denotes the coordinates on $\mathbb{C}^3$ unless otherwise stated.

(1) Simple elliptic singularities (parabolic singularities)

$$\begin{aligned}
\tilde{E}_6 &: \ x^3 + y^3 + z^3 + axyz, \ a^3 + 27 \neq 0, \\
\tilde{E}_7 &: \ x^2 + y^4 + z^4 + axyz, \ a^4 - 64 \neq 0, \\
\tilde{E}_8 &: \ x^2 + y^3 + z^6 + axyz, \ a^6 - 432 \neq 0.
\end{aligned}$$

(2) Cusp singularities (hyperbolic singularities)

$$T_{pqr} \colon x^p + y^q + z^r + axyz, \ a \neq 0, \ \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$

(3) 14 exceptional singularities

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$$
\begin{aligned}
&E_{12}\colon x^2+y^3+z^7+ayz^5, \quad E_{13}\colon x^2+y^3+yz^5+az^8,\\
&E_{14}\colon x^3+y^2+yz^4+ayz^6,\\
&Z_{11}\colon x^2+y^5+yz^3+ay^4z, \quad Z_{12}\colon x^2+zy^3+yz^4+ay^2z^3,\\
&Z_{13}\colon x^2+xy^3+yz^3+ay^5z,\\
&W_{12}\colon x^5+y^2+yz^2+ax^3z^2, \quad W_{13}\colon x^2+xy^2+yz^4+az^6,\\
&Q_{10}\colon x^3+y^4+yz^2+axy^3, \quad Q_{11}\colon x^3+xy^3+yz^2+ay^5,\\
&Q_{12}\colon x^3+zy^2+yz^3+axz^4,\\
&S_{11}\colon x^4+xy^2+yz^2+ax^3y, \quad S_{12}\colon x^2y+y^2z+z^3x+az^5,\\
&U_{12}\colon x^4+zy^2+yz^2+ax^2(y^2+yz+z^2).
\end{aligned}
$$

On the other hand, there is an important class of polynomials, called quasihomogeneous polynomials. A complex polynomial $f(z_1,\dots,z_n)$ is called quasihomogeneous with weight system $(w_1,\dots,w_n,d)\in\mathbb{Z}_{>0}^{n+1}$ if

$$f(\lambda^{w_1}z_1,\dots,\lambda^{w_n}z_n)=\lambda^d f(z_1,\dots,z_n)$$

for $\lambda\in\mathbb{C}^*$. We note that simple singularities (modality 0) and simple elliptic singularities are quasihomogeneous, while cusp singularities are not. The exceptional unimodal singularities are quasihomogeneous only when $a=0$. Now, we focus on a special class of quasihomogeneous polynomials.

Definition 1.1 (Invertible polynomials)**.** A quasihomogeneous polynomial $f(z_1,\dots,z_n)$ is said to be invertible if the following conditions are satisfied:

(1) the number of monomials in the polynomial $f(z_1,\dots,z_n)$ is n, namely,

$$f(z_1,\dots,z_n)=\sum_{i=1}^n a_i\prod_{j=1}^n {z_j}^{E_{ij}}$$

for some $a_i\in\mathbb{C}^*$ and nonnegative integers E_{ij} for $i,j=1,\dots,n$;

(2) the matrix $E=(E_{ij})$ is invertible over $\mathbb{Q}$;

(3) $f(z_1,\dots,z_n)$ and $f^t(z_1,\dots,z_n)$ have singularities only at the origin of $\mathbb{C}^n$ which are isolated, where $f^t(z_1,\dots,z_n)$ is defined by

$$f^t(z_1,\dots,z_n)=\sum_{i=1}^n a_i\prod_{j=1}^n {z_j}^{E_{ji}}.$$

The terminology "invertible polynomial" was introduced by Kreuzer [3].

In this paper, we only consider invertible polynomials in three variables. They are classified as follows up to rescaling the variables.

Proposition 1.2 ([1], see also [2])**.** *An invertible polynomial $f(x,y,z)$ in three variables is reduced to one of the following five types by a complex rescaling of the variables;*

$$
\begin{aligned}
&f_{1,(p,q,r)}(x,y,z)=x^p+y^q+z^r \ (2\le p\le q\le r),\\
&f_{2,(p,q,r)}(x,y,z)=x^p+y^q+yz^r \ (2\le p,q,r),\\
&f_{3,(p,q,r)}(x,y,z)=x^p+zy^q+yz^r \ (2\le p, 2\le q\le r),\\
&f_{4,(p,q,r)}(x,y,z)=x^p+xy^q+yz^r \ (2\le p,q,r),\\
&f_{5,(p,q,r)}(x,y,z)=x^py+y^qz+z^rx \ (1\le p\le q\le r).
\end{aligned}
$$

The 14 exceptional singularities are represented by some invertible polynomials. Indeed, when $a = 0$, they are the germs of the surface singularities

$$\begin{array}{lll} E_{12} : f_{1,(2,3,7)} = 0, & E_{13} : f_{2,(2,3,5)} = 0, & E_{14} : f_{2,(3,2,4)} = 0, \\ Z_{11} : f_{2,(2,5,3)} = 0, & Z_{12} : f_{3,(2,3,4)} = 0, & Z_{13} : f_{4,(2,3,3)} = 0, \\ W_{12} : f_{2,(5,2,2)} = 0, & W_{13} : f_{4,(2,2,4)} = 0, & \\ Q_{10} : f_{2,(3,4,2)} = 0, & Q_{11} : f_{4,(3,3,2)} = 0, & Q_{12} : f_{3,(3,2,3)} = 0, \\ S_{11} : f_{4,(4,2,2)} = 0, & S_{12} : f_{5,(2,2,3)} = 0, & U_{12} : f_{3,(4,2,2)} = 0, \end{array}$$

at the origin of $\mathbb{C}^3$.

Using this description, Ebeling and Takahashi [2] constructed a deformation of each exceptional singularity into a cusp singularity. Concretely, they showed the following theorem.

Theorem 1.3 ([2])**.** *Let $f(x, y, z)$ be an invertible polynomial. Through the normal form of Proposition 1.2, we associate to f the tuple of integers $\Gamma_f = (\gamma_1, \gamma_2, \gamma_3)$ as follows:*

$$\begin{aligned} \Gamma_{1,(p,q,r)} &= (p,\ q,\ r), \\ \Gamma_{2,(p,q,r)} &= (p,\ q,\ p(r-1)), \\ \Gamma_{3,(p,q,r)} &= (p,\ p(q-1),\ p(r-1)), \\ \Gamma_{4,(p,q,r)} &= (p,\ p(r-1),\ qr - r + 1), \\ \Gamma_{5,(p,q,r)} &= (qr - q + 1,\ rp - r + 1,\ pq - p + 1). \end{aligned}$$

(1) *If $\frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \frac{1}{\gamma_3} > 1$, then, by a suitable polynomial change of coordinates, the polynomial $f(x, y, z) - xyz$ can be transformed to a polynomial of the following form:*

$$x^{\gamma_1} + y^{\gamma_2} + z^{\gamma_3} - xyz + \sum_{i=1}^{\gamma_1 - 1} a_i x^i + \sum_{j=1}^{\gamma_2 - 1} b_j y^j + \sum_{k=1}^{\gamma_3 - 1} c_k z^k + c.$$

(2) *If $\frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \frac{1}{\gamma_3} = 1$, then the polynomial $f(x, y, z) - xyz$ can be written as*

$$x^{\gamma_1} + y^{\gamma_2} + z^{\gamma_3} + axyz$$

for some $a \neq 0$ after a suitable holomorphic transformation of coordinates.

(3) *If $\frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \frac{1}{\gamma_3} < 1$, then the polynomial $f(x, y, z) - xyz$ can be written as*

$$x^{\gamma_1} + y^{\gamma_2} + z^{\gamma_3} + axyz$$

for some $a \neq 0$ after a suitable holomorphic transformation of coordinates.

Definition 1.4 (Gabrielov numbers)**.** The numbers $(\gamma_1, \gamma_2, \gamma_3)$ in Theorem 1.3 are called the Gabrielov numbers of f.

In the third case of Theorem 1.3, the 1-parameter family of polynomials

$$\{f(x, y, z) - txyz\}_{t \in \mathbb{C}}$$

is a deformation of the singularity of f into a cusp singularity. In particular, it gives a deformation of each exceptional singularity into a cusp singularity, since the Gabrielov numbers

$$\begin{array}{lll}
\Gamma_{1,(2,3,7)} = (2,3,7), & \Gamma_{2,(2,3,5)} = (2,3,8), & \Gamma_{2,(3,2,4)} = (3,2,9), \\
\Gamma_{2,(2,5,3)} = (2,5,4), & \Gamma_{3,(2,3,4)} = (2,4,6), & \Gamma_{4,(2,3,3)} = (2,4,7), \\
\Gamma_{2,(5,2,2)} = (5,2,5), & \Gamma_{4,(2,2,4)} = (2,6,5), & \\
\Gamma_{2,(3,4,2)} = (3,4,3), & \Gamma_{4,(3,3,2)} = (3,3,5), & \Gamma_{3,(3,2,3)} = (3,3,6), \\
\Gamma_{4,(4,2,2)} = (4,4,3), & \Gamma_{5,(2,2,3)} = (5,4,3), & \Gamma_{3,(4,2,2)} = (4,4,4)
\end{array}$$

satisfy the inequality

$$\frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \frac{1}{\gamma_3} < 1.$$

The aim of this paper is to understand these deformations from the topological view point. However, it is hard to analyze the topological change of the complex surface from their arguments since they are purely algebraic. Instead, we consider the following critical point problem and understand the topological change by using the Morse theory.

We set $\rho(x,y,z) = \sqrt{|x|^2+|y|^2+|z|^2}$ and

$$\Sigma_t = \{(x,y,z) \in \mathbb{C}^3 \mid f(x,y,z) - txyz = 0\}$$

for a complex number t, and take the restriction $\rho^2|_{\Sigma_t}$ of the squared distance function ρ^2 to the singular complex surface Σ_t. We consider the critical point problem of the function $\rho^2|_{\Sigma_t}$. When $t = 0$, the function has no critical point except the origin since f is an invertible polynomial. On the other hand, when $t \neq 0$, the function must have additional critical points since f and $f - txyz$ define singularities of different types with respect to Arnold's classification. Namely, the deformation by Ebeling and Takahashi must be understood as a phenomenon that a critical point splits into plural ones.

As the main theorem of this article (Theorem 2.2), we show that for any nonzero complex number t, the function $\rho^2|_{\Sigma_t}$ has only one critical point except the origin if and only if f is any of the 14 exceptional unimodal singularities. Moreover, we can determine the approximate location of the critical point when the parameter t is a small positive number (Remark 2.4). This would be helpful to describe the change of the topology of the complex surface by means of the Morse theory.

2. The critical point problem

In this section, we prove Theorem 2.2, the main theorem of this article. In order to do so, we need the following proposition. Let $(z_1,\ldots,z_n)$ be the coordinates on $\mathbb{C}^n$ and $\rho = \sqrt{|z_1|^2+\cdots+|z_n|^2}$.

Proposition 2.1. *Let g be a polynomial in n complex variables $z_1,\ldots,z_n$ and Σ be the complex hypersurface defined by $g(z_1,\ldots,z_n) = 0$. A point $(z_1,\ldots,z_n)$ on Σ is a critical point of the restricted squared distance function $\rho^2|_\Sigma$ if and only if the gradient vector of g is linearly dependent with the vector $(\bar{z}_1,\ldots,\bar{z}_n)$.*

We apply this to the case where $g = f(x,y,z) - txyz$. Then, a point $(x,y,z) \in \Sigma_t$ satisfying $xyz \neq 0$ is a critical point of $\rho^2|_{\Sigma_t}$ if and only if

$$\frac{x\partial_x f - txyz}{|x|^2} = \frac{y\partial_y f - txyz}{|y|^2} = \frac{z\partial_z f - txyz}{|z|^2}.$$

Using these equations, we can show the following main theorem.

Theorem 2.2. *Let f be an invertible polynomial in three variables. Then, for any nonzero complex number t, the function $\rho^2|_{\Sigma_t}$ has only one critical point except the origin if and only if f is any of the fourteen exceptional unimodal singularities.*

Proof. We set $\theta_1 = \arg x$, $\theta_2 = \arg y$, $\theta_3 = \arg z$, and

$$\begin{aligned}
\Delta_1(p,q,r) &= pqr - pq - qr - rp,\\
\Delta_2(p,q,r) &= (p-1)(q-1)(r-1) - q - r + 1,\\
\Delta_3(p,q,r) &= (p-1)(q-1)(r-1) - q - r + 2,\\
\Delta_4(p,q,r) &= (p-1)(q-1)(r-1) - q,\\
\Delta_5(p,q,r) &= (p-1)(q-1)(r-1) - 1.
\end{aligned}$$

By a complex rescaling of the variables, the invertible polynomial f can be written in the form of Proposition 1.2. Then, in the 1-parameter family

$$f(x,y,z) - txyz,$$

the parameter t absorbs the rescaling constant. Thus, we may assume that f is originally written as $f = f_{k,(p,q,r)}$ $(k = 1,2,3,4,5)$. Moreover, we assume that the parameter t is a positive real number. It is easily checked in the following argument that this assumption does not change the number of critical points. Now, we prove the five cases each by each.

(I). $f = x^p + y^q + z^r$ $(p \le q \le r)$.

In this case, a critical point (x,y,z) must satisfy

$$\frac{px^p - txyz}{|x|^2} = \frac{qy^q - txyz}{|y|^2} = \frac{rz^r - txyz}{|z|^2},$$

and we denote the value of these fractions by l. Then, we obtain

$$x^p = \frac{1}{p}(l|x|^2 + txyz),\ y^q = \frac{1}{q}(l|y|^2 + txyz),\ z^r = \frac{1}{r}(l|z|^2 + txyz).$$

These three equations and the defining equation of Σ_t imply

$$\begin{aligned}
&\left(\frac{|x|^2}{p} + \frac{|y|^2}{q} + \frac{|z|^2}{r}\right) l - \left(1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}\right) txyz = 0\\
&\iff l = \frac{pqr - pq - qr - rp}{qr|x|^2 + rp|y|^2 + pq|z|^2} txyz.
\end{aligned}$$

Substituting this to the above three equations, we have

$$\begin{cases}
x^p = \dfrac{(qr-q-r)|x|^2 + r|y|^2 + q|z|^2}{qr|x|^2 + rp|y|^2 + pq|z|^2} txyz,\\
y^q = \dfrac{r|x|^2 + (rp-r-p)|y|^2 + p|z|^2}{qr|x|^2 + rp|y|^2 + pq|z|^2} txyz,\\
z^r = \dfrac{q|x|^2 + p|y|^2 + (pq-p-q)|z|^2}{qr|x|^2 + rp|y|^2 + pq|z|^2} txyz.
\end{cases} \tag{1}$$

Comparing the arguments of both sides of each equation, it follows

$$\begin{aligned}
&(p-1)\theta_1 - \theta_2 - \theta_3 \equiv -\theta_1 + (q-1)\theta_2 - \theta_3\\
&\quad \equiv -\theta_1 - \theta_2 + (r-1)\theta_3 \equiv 0 \pmod{2\pi}.
\end{aligned}$$

In other words, the vector

$$\begin{pmatrix} p-1 & -1 & -1 \\ -1 & q-1 & -1 \\ -1 & -1 & r-1 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}$$

is an element of the lattice $2\pi\mathbb{Z}^3$. The determinant of the coefficient matrix is

$$\Delta_1(p,q,r) = pqr - pq - qr - rp = \left(1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}\right) pqr.$$

This implies that if the simultaneous equations (1) have m positive real solutions, then they have $m\Delta_1$ complex solutions. In particular, there are only positive real solutions if and only if $(p,q,r) = (2,3,7)$. Now we put

$$\alpha(x,y,z) = \frac{(qr-q-r)|x|^2 + r|y|^2 + q|z|^2}{qr|x|^2 + rp|y|^2 + pq|z|^2},$$
$$\beta(x,y,z) = \frac{r|x|^2 + (rp-r-p)|y|^2 + p|z|^2}{qr|x|^2 + rp|y|^2 + pq|z|^2},$$
$$\gamma(x,y,z) = \frac{q|x|^2 + p|y|^2 + (pq-p-q)|z|^2}{qr|x|^2 + rp|y|^2 + pq|z|^2}.$$

Then the simultaneous equations (1) are equivalent to

$$x^{\Delta_1} = \alpha^{qr-q-r}\beta^r\gamma^q t^{qr},$$
$$y^{\Delta_1} = \alpha^r\beta^{rp-r-p}\gamma^p t^{rp},$$
$$z^{\Delta_1} = \alpha^q\beta^p\gamma^{pq-p-q} t^{pq}.$$

Moreover, the functions α, β and γ satisfy

$$\alpha + \beta + \gamma = 1,$$
$$\frac{1}{p} < \alpha < 1 - \frac{1}{q} - \frac{1}{r},$$
$$\frac{1}{q} < \beta < 1 - \frac{1}{p} - \frac{1}{r},$$
$$\frac{1}{r} < \gamma < 1 - \frac{1}{p} - \frac{1}{q}.$$

In the following, we only focus on the case where $(p,q,r) = (2,3,7)$ and prove that $m = 1$. In this case, the equations

$$x = \alpha^{11}\beta^7\gamma^3 t^{21},$$
$$y = \alpha^7\beta^5\gamma^2 t^{14},$$
$$z = \alpha^3\beta^2\gamma t^6$$

give a biholomorphism between the complex surfaces $\{\alpha+\beta+\gamma = 1\} \cap (\mathbb{C}^*)^3$ and $\Sigma_t \cap (\mathbb{C}^*)^3$. Hence, the critical point problem can be written by α, β, γ as

$$\frac{2\alpha - 1}{\alpha^{22}\beta^{14}\gamma^6 t^{42}} = \frac{3\beta - 1}{\alpha^{14}\beta^{10}\gamma^4 t^{28}} = \frac{7\gamma - 1}{\alpha^6\beta^4\gamma^2 t^{12}},\ \alpha + \beta + \gamma = 1.$$

Thus, a critical point is nothing but a fixed point of the map

$$T\colon \{\alpha+\beta+\gamma = 1\} \cap (\mathbb{R}_{>0})^3 \to \{\alpha+\beta+\gamma = 1\} \cap (\mathbb{R}_{>0})^3$$

given by

$$T(\alpha,\beta,\gamma)=\begin{pmatrix}\dfrac{1}{2}+\dfrac{1}{42}\cdot\dfrac{21x^2}{21x^2+14y^2+6z^2}\\ \dfrac{1}{3}+\dfrac{1}{42}\cdot\dfrac{14y^2}{21x^2+14y^2+6z^2}\\ \dfrac{1}{7}+\dfrac{1}{42}\cdot\dfrac{6z^2}{21x^2+14y^2+6z^2}\end{pmatrix},$$

where $x=\alpha^{11}\beta^7\gamma^3t^{21}$, $y=\alpha^7\beta^5\gamma^2t^{14}$, and $z=\alpha^3\beta^2\gamma t^6$. As is proved later in Proposition 2.3, T is a contraction on the closed set

$$\left\{\alpha+\beta+\gamma=1,\ \ \frac{1}{2}\le\alpha\le\frac{11}{21},\ \ \frac{1}{3}\le\beta\le\frac{5}{14},\ \ \frac{1}{7}\le\gamma\le\frac{1}{6}\right\},$$

and hence, it has only one fixed point by the Banach fixed point theorem. Therefore, the simultaneous equations have only one solution except the origin.

(II). $f=x^p+y^q+yz^r$.
A critical point (x,y,z) must satisfy

$$\frac{px^p-txyz}{|x|^2}=\frac{qy^q+yz^r-txyz}{|y|^2}=\frac{ryz^r-txyz}{|z|^2},$$

and we denote the value of these fractions by l. Then,

$$l=\frac{pqr-p(q-1)-qr-rp}{qr|x|^2+rp|y|^2+p(q-1)|z|^2}txyz,$$

and we obtain the simultaneous equations

$$\begin{cases} x^p=\dfrac{(q-1)(r-1)|x|^2+r|y|^2+(q-1)|z|^2}{qr|x|^2+rp|y|^2+p(q-1)|z|^2}txyz,\\ y^q=\dfrac{(r-1)|x|^2+(rp-r-p)|y|^2+|z|^2}{qr|x|^2+rp|y|^2+p(q-1)|z|^2}txyz,\\ yz^r=\dfrac{q|x|^2+p|y|^2+(pq-p-q)|z|^2}{qr|x|^2+rp|y|^2+p(q-1)|z|^2}txyz.\end{cases}\tag{2}$$

Thus, $\dfrac{x^p}{txyz}$, $\dfrac{y^q}{txyz}$, and $\dfrac{yz^r}{txyz}$ are real positive numbers. Hence, we have

$$\begin{aligned}(p-1)\theta_1-\theta_2-\theta_3&\equiv-\theta_1+(q-1)\theta_2-\theta_3\\ &\equiv-\theta_1+(r-1)\theta_3\equiv 0\pmod{2\pi},\end{aligned}$$

namely,

$$\begin{pmatrix}p-1&-1&-1\\-1&q-1&-1\\-1&0&r-1\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\end{pmatrix}\in 2\pi\mathbb{Z}^3.$$

The determinant of the coefficient matrix is

$$\Delta_2(p,q,r)=(p-1)(q-1)(r-1)-q-r+1,$$

and it is equal to 1 if and only if

$$(p,q,r)=(2,3,5),(3,2,4),(2,5,3),(5,2,2),(3,4,2).$$

In these cases, x, y and z are all real positive numbers. We put

$$\alpha(x,y,z) = \frac{(q-1)(r-1)|x|^2 + r|y|^2 + (q-1)|z|^2}{qr|x|^2 + rp|y|^2 + p(q-1)|z|^2},$$
$$\beta(x,y,z) = \frac{(r-1)|x|^2 + (rp-r-p)|y|^2 + |z|^2}{qr|x|^2 + rp|y|^2 + p(q-1)|z|^2},$$
$$\gamma(x,y,z) = \frac{q|x|^2 + p|y|^2 + (pq-p-q)|z|^2}{qr|x|^2 + rp|y|^2 + p(q-1)|z|^2}.$$

Then, the simultaneous equations (2) are equivalent to

$$\begin{aligned} x^{\Delta_2} &= \alpha^{(q-1)(r-1)}\beta^{r-1}\gamma^q t^{qr}, \\ y^{\Delta_2} &= \alpha^r\beta^{pr-p-r}\gamma^p t^{pr}, \\ z^{\Delta_2} &= \alpha^{q-1}\beta\gamma^{pq-p-q}t^{p(q-1)}. \end{aligned}$$

Now, we focus only on the five cases

$$(p,q,r) = (2,3,5), (3,2,4), (2,5,3), (5,2,2), (3,4,2).$$

Then, just as in the case of (I), a critical point corresponds to a fixed point of some map, which is proved to be a contraction in Proposition 2.3. Therefore, the simultaneous equations have only one solution except the origin.

(III). $f = x^p + zy^q + yz^r$ $(q \le r)$.
A critical point (x,y,z) must satisfy

$$\frac{px^p - txyz}{|x|^2} = \frac{qzy^q + yz^r - txyz}{|y|^2} = \frac{zy^q + ryz^r - txyz}{|z|^2},$$

and we denote the value of these fractions by l. Then,

$$l = \frac{pqr - pq - qr - rp + p + 1}{(qr-1)|x|^2 + p(r-1)|y|^2 + p(q-1)|z|^2}txyz,$$

and we obtain the simultaneous equations

$$\begin{cases} x^p = \dfrac{(q-1)(r-1)|x|^2 + (r-1)|y|^2 + (q-1)|z|^2}{(qr-1)|x|^2 + p(r-1)|y|^2 + p(q-1)|z|^2}txyz, \\ zy^q = \dfrac{(r-1)|x|^2 + (rp-r-p)|y|^2 + |z|^2}{(qr-1)|x|^2 + p(r-1)|y|^2 + p(q-1)|z|^2}txyz, \\ yz^r = \dfrac{(q-1)|x|^2 + |y|^2 + (pq-p-q)|z|^2}{(qr-1)|x|^2 + p(r-1)|y|^2 + p(q-1)|z|^2}txyz. \end{cases} \tag{3}$$

Thus, $\dfrac{x^p}{txyz}$, $\dfrac{zy^q}{txyz}$, and $\dfrac{yz^r}{txyz}$ are real positive numbers. Hence, we have

$$\begin{pmatrix} p-1 & -1 & -1 \\ -1 & q-1 & 0 \\ -1 & 0 & r-1 \end{pmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} \in 2\pi\mathbb{Z}^3.$$

The determinant of the coefficient matrix is

$$\Delta_3(p,q,r) = (p-1)(q-1)(r-1) - q - r + 2,$$

and it is equal to 1 if and only if

$$(p,q,r) = (2,3,4), (3,2,3), (4,2,2).$$

In these cases, x, y and z are all real positive numbers. We put

$$\alpha(x,y,z) = \frac{(q-1)(r-1)|x|^2 + (r-1)|y|^2 + (q-1)|z|^2}{(qr-1)|x|^2 + p(r-1)|y|^2 + p(q-1)|z|^2},$$
$$\beta(x,y,z) = \frac{(r-1)|x|^2 + (pr-p-r)|y|^2 + |z|^2}{(qr-1)|x|^2 + p(r-1)|y|^2 + p(q-1)|z|^2},$$
$$\gamma(x,y,z) = \frac{(q-1)|x|^2 + |y|^2 + (pq-p-q)|z|^2}{(qr-1)|x|^2 + p(r-1)|y|^2 + p(q-1)|z|^2}.$$

Then, the simultaneous equations (3) are equivalent to

$$x^{\Delta_3} = \alpha^{(q-1)(r-1)}\beta^{r-1}\gamma^{q-1}t^{qr-1},$$
$$y^{\Delta_3} = \alpha^{r-1}\beta^{pr-p-r}\gamma t^{p(r-1)},$$
$$z^{\Delta_3} = \alpha^{q-1}\beta\gamma^{pq-p-q}t^{p(q-1)}.$$

Now, we consider only the three cases

$$(p,q,r) = (2,3,4), (3,2,3), (4,2,2).$$

Then, just as in the case of (I), a critical point corresponds to a fixed point of some map, which is proved to be a contraction in Proposition 2.3. Therefore, the simultaneous equations have only one solution except the origin.

(IV). $f = x^p + xy^q + yz^r$.
A critical point (x,y,z) must satisfy

$$\frac{px^p + xy^q - txyz}{|x|^2} = \frac{qxy^q + yz^r - txyz}{|y|^2} = \frac{ryz^r - txyz}{|z|^2},$$

and we denote the value of these fractions by l. Then,

$$l = \frac{(p-1)(q-1)(r-1) - q}{qr|x|^2 + (p-1)r|y|^2 + (pq-p+1)|z|^2} txyz,$$

and we obtain the simultaneous equations

$$\begin{cases} x^p = \dfrac{(q-1)(r-1)|x|^2 + |y|^2 + (q-1)|z|^2}{qr|x|^2 + (p-1)r|y|^2 + (pq-p+1)|z|^2} txyz, \\ xy^q = \dfrac{(r-1)|x|^2 + (pr-p-r)|y|^2 + |z|^2}{qr|x|^2 + (p-1)r|y|^2 + (pq-p+1)|z|^2} txyz, \\ yz^r = \dfrac{q|x|^2 + (p-1)|y|^2 + (p-1)(q-1)|z|^2}{qr|x|^2 + (p-1)r|y|^2 + (pq-p+1)|z|^2} txyz. \end{cases} \tag{4}$$

Thus, $\dfrac{x^p}{txyz}$, $\dfrac{xy^q}{txyz}$, and $\dfrac{yz^r}{txyz}$ are real positive numbers. Hence, we have

$$\begin{pmatrix} p-1 & -1 & -1 \\ 0 & q-1 & -1 \\ -1 & 0 & r-1 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} \in 2\pi\mathbb{Z}^3.$$

The determinant of the coefficient matrix is

$$\Delta_4(p,q,r) = (p-1)(q-1)(r-1) - q,$$

and it is equal to 1 if and only if

$$(p,q,r) = (2,3,3), (2,2,4), (3,3,2), (4,2,2).$$

In these cases, x, y and z are all real positive numbers. We put

$$\alpha(x,y,z) = \frac{(q-1)(r-1)|x|^2 + |y|^2 + (q-1)|z|^2}{qr|x|^2 + (p-1)r|y|^2 + (pq-p+1)|z|^2},$$
$$\beta(x,y,z) = \frac{(r-1)|x|^2 + (pr-p-r)|y|^2 + |z|^2}{qr|x|^2 + (p-1)r|y|^2 + (pq-p+1)|z|^2},$$
$$\gamma(x,y,z) = \frac{q|x|^2 + (p-1)|y|^2 + (p-1)(q-1)|z|^2}{qr|x|^2 + (p-1)r|y|^2 + (pq-p+1)|z|^2}.$$

Then, the simultaneous equations (4) are equivalent to

$$x^{\Delta_4} = \alpha^{(q-1)(r-1)}\beta^{r-1}\gamma^{q}t^{qr},$$
$$y^{\Delta_4} = \alpha\beta^{pr-p-r}\gamma^{p-1}t^{(p-1)r},$$
$$z^{\Delta_4} = \alpha^{q-1}\beta\gamma^{(p-1)(q-1)}t^{pq-p+1}.$$

Now, we focus only on the four cases

$$(p,q,r) = (2,3,3),(2,2,4),(3,3,2),(4,2,2).$$

Then, just as in the case of (I), a critical point corresponds to a fixed point of some map, which is proved to be a contraction in Proposition 2.3. Therefore, the simultaneous equations have only one solution except the origin.

(V). $f = x^p y + y^q z + z^r x$ $(p \le q \le r)$.
A critical point (x,y,z) must satisfy

$$\frac{px^p y + z^r x - txyz}{|x|^2} = \frac{x^p y + qy^q z - txyz}{|y|^2} = \frac{y^q z + rz^r x - txyz}{|z|^2},$$

and we denote the value of these fractions by l. Then,

$$l = \frac{(p-1)(q-1)(r-1)-1}{(qr-r+1)|x|^2 + (pr-p+1)|y|^2 + (pq-q+1)|z|^2}txyz,$$

and we obtain the simultaneous equations

$$\tag{5} \begin{cases} x^p y = \dfrac{(q-1)(r-1)|x|^2 + (r-1)|y|^2 + |z|^2}{(qr-r+1)|x|^2 + (pr-p+1)|y|^2 + (pq-q+1)|z|^2}txyz, \\ y^q z = \dfrac{|x|^2 + (p-1)(r-1)|y|^2 + (p-1)|z|^2}{(qr-r+1)|x|^2 + (pr-p+1)|y|^2 + (pq-q+1)|z|^2}txyz, \\ z^r x = \dfrac{(q-1)|x|^2 + |y|^2 + (p-1)(q-1)|z|^2}{(qr-r+1)|x|^2 + (pr-p+1)|y|^2 + (pq-q+1)|z|^2}txyz. \end{cases}$$

Thus, $\dfrac{x^p y}{txyz}$, $\dfrac{y^q z}{txyz}$, and $\dfrac{z^r x}{txyz}$ are real positive numbers. Hence, we have

$$\begin{pmatrix} p-1 & 0 & -1 \\ -1 & q-1 & 0 \\ 0 & -1 & r-1 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} \in 2\pi\mathbb{Z}^3.$$

The determinant of the coefficient matrix is

$$\Delta_5(p,q,r) = (p-1)(q-1)(r-1) - 1,$$

and it is equal to 1 if and only if $(p,q,r)=(2,2,3)$. In this case, x, y and z are all real positive numbers. We put

$$\begin{aligned}
\alpha(x,y,z) &= \frac{(q-1)(r-1)|x|^2+(r-1)|y|^2+|z|^2}{(qr-r+1)|x|^2+(pr-p+1)|y|^2+(pq-q+1)|z|^2},\\
\beta(x,y,z) &= \frac{|x|^2+(p-1)(r-1)|y|^2+(p-1)|z|^2}{(qr-r+1)|x|^2+(pr-p+1)|y|^2+(pq-q+1)|z|^2},\\
\gamma(x,y,z) &= \frac{(q-1)|x|^2+|y|^2+(p-1)(q-1)|z|^2}{(qr-r+1)|x|^2+(pr-p+1)|y|^2+(pq-q+1)|z|^2}.
\end{aligned}$$

Then, the simultaneous equations (5) are equivalent to

$$\begin{aligned}
x^{\Delta_5} &= \alpha^{(q-1)(r-1)}\beta\gamma^{q-1}t^{qr-r+1},\\
y^{\Delta_5} &= \alpha^{r-1}\beta^{(p-1)(r-1)}\gamma t^{pr-p+1},\\
z^{\Delta_5} &= \alpha\beta^{p-1}\gamma^{(p-1)(q-1)}t^{pq-q+1}.
\end{aligned}$$

Now, we assume $(p,q,r)=(2,2,3)$. Then, just as in the case of (I), a critical point corresponds to a fixed point of some map, which is proved to be a contraction in Proposition 2.3. Therefore, the simultaneous equations have only one solution except the origin.

Therefore, the function $\rho^2|_{\Sigma_t}$ has only one critical point except the origin if and only if f is an exceptional unimodal singularity. $\square$

Now, we show Proposition 2.3 to complete the proof of Main Theorem.

Proposition 2.3. *In the following fourteen cases, each map $T\colon M\to M$ is a contraction.*

(1) $(p,q,r)=(2,3,7)$.

$$T\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \begin{pmatrix}\dfrac{1}{2}+\dfrac{1}{42}\cdot\dfrac{21x^2}{21x^2+14y^2+6z^2}\\ \dfrac{1}{3}+\dfrac{1}{42}\cdot\dfrac{14y^2}{21x^2+14y^2+6z^2}\\ \dfrac{1}{7}+\dfrac{1}{42}\cdot\dfrac{6z^2}{21x^2+14y^2+6z^2}\end{pmatrix},\quad \begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}\alpha^{11}\beta^7\gamma^3t^{21}\\ \alpha^7\beta^5\gamma^2t^{14}\\ \alpha^3\beta^2\gamma t^6\end{pmatrix},$$
$$M=\left\{\alpha+\beta+\gamma=1,\ \ \frac{1}{2}\le\alpha\le\frac{11}{21},\ \ \frac{1}{3}\le\beta\le\frac{5}{14},\ \ \frac{1}{7}\le\gamma\le\frac{1}{6}\right\}.$$

(2) $(p,q,r) = (2,3,5), (3,2,4), (2,5,3), (5,2,2), (3,4,2)$.

$$T(\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}) = \begin{pmatrix} \dfrac{1}{p} + \dfrac{1}{pqr}\cdot\dfrac{qrx^2}{qrx^2+rpy^2+p(q-1)z^2} \\ \dfrac{1}{q} + \dfrac{1}{pqr}\cdot\dfrac{rpy^2}{qrx^2+rpy^2+p(q-1)z^2} \\ \dfrac{q-1}{qr} + \dfrac{1}{pqr}\cdot\dfrac{p(q-1)z^2}{qrx^2+rpy^2+p(q-1)z^2} \end{pmatrix},$$

$$\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}\alpha^{(q-1)(r-1)}\beta^{r-1}\gamma^{q}t^{qr}\\ \alpha^{r}\beta^{pr-p-r}\gamma^{p}t^{pr}\\ \alpha^{q-1}\beta\gamma^{pq-p-q}t^{p(q-1)}\end{pmatrix},$$

$$\begin{aligned} M =&\{\alpha+\beta+\gamma=1,\ \frac{1}{p}\le\alpha\le\frac{(q-1)(r-1)}{qr},\\ &\frac{1}{p(q-1)}\le\beta\le\frac{pr-p-r}{pr},\ \frac{1}{r}\le\gamma\le\frac{pq-p-q}{p(q-1)}\}.\end{aligned}$$

(3) $(p,q,r) = (2,3,4), (3,2,3), (4,2,2)$.

$$T(\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}) = \begin{pmatrix} \dfrac{1}{p} + \dfrac{1}{p(qr-1)}\cdot\dfrac{(qr-1)x^2}{(qr-1)x^2+p(r-1)y^2+p(q-1)z^2} \\ \dfrac{r-1}{qr-1} + \dfrac{1}{p(qr-1)}\cdot\dfrac{p(r-1)y^2}{(qr-1)x^2+p(r-1)y^2+p(q-1)z^2} \\ \dfrac{q-1}{qr-1} + \dfrac{1}{p(qr-1)}\cdot\dfrac{p(q-1)z^2}{(qr-1)x^2+p(r-1)y^2+p(q-1)z^2} \end{pmatrix},$$

$$\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}\alpha^{(q-1)(r-1)}\beta^{r-1}\gamma^{q-1}t^{qr-1}\\ \alpha^{r-1}\beta^{pr-p-r}\gamma t^{p(r-1)}\\ \alpha^{q-1}\beta\gamma^{pq-p-q}t^{p(q-1)}\end{pmatrix},$$

$$\begin{aligned} M =&\{\alpha+\beta+\gamma=1,\ \frac{1}{p}\le\alpha\le\frac{(q-1)(r-1)}{qr-1},\\ &\frac{1}{p(q-1)}\le\beta\le\frac{pr-p-r}{p(r-1)},\ \frac{1}{p(r-1)}\le\gamma\le\frac{pq-p-q}{p(q-1)}\}.\end{aligned}$$

(4) $(p,q,r) = (2,3,3), (2,2,4), (3,3,2), (4,2,2)$.

$$T(\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}) = \begin{pmatrix} \dfrac{1}{p} + \dfrac{1}{pqr}\cdot\dfrac{qrx^2}{qrx^2+(p-1)ry^2+(pq-p+1)z^2} \\ \dfrac{p-1}{pq} + \dfrac{1}{pqr}\cdot\dfrac{(p-1)ry^2}{qrx^2+(p-1)ry^2+(pq-p+1)z^2} \\ \dfrac{pq-p+1}{pqr} + \dfrac{1}{pqr}\cdot\dfrac{(pq-p+1)z^2}{qrx^2+(p-1)ry^2+(pq-p+1)z^2} \end{pmatrix},$$

$$\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}\alpha^{(q-1)(r-1)}\beta^{r-1}\gamma^{q}t^{qr}\\ \alpha\beta^{pr-p-r}\gamma^{p-1}t^{(p-1)r}\\ \alpha^{q-1}\beta\gamma^{(p-1)(q-1)}t^{pq-p+1}\end{pmatrix},$$

$$\begin{aligned} M =&\{\alpha+\beta+\gamma=1, \frac{1}{(p-1)r}\le\alpha\le\frac{(q-1)(r-1)}{qr},\\ &\frac{1}{pq-p+1}\le\beta\le\frac{pr-p-r}{(p-1)r},\ \frac{1}{r}\le\gamma\le\frac{(p-1)(q-1)}{pq-p+1}\}.\end{aligned}$$

(5) $(p,q,r)=(2,2,3)$.

$$T\left(\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}\right)=\begin{pmatrix}\dfrac{4}{13}+\dfrac{1}{13}\cdot\dfrac{4x^2}{4x^2+5y^2+3z^2}\\ \dfrac{5}{13}+\dfrac{1}{13}\cdot\dfrac{5y^2}{4x^2+5y^2+3z^2}\\ \dfrac{3}{13}+\dfrac{1}{13}\cdot\dfrac{3z^2}{4x^2+5y^2+3z^2}\end{pmatrix},\quad \begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}\alpha^2\beta\gamma t^4\\ \alpha^2\beta^2\gamma t^5\\ \alpha\beta\gamma t^3\end{pmatrix},$$

$$M=\left\{\alpha+\beta+\gamma=1,\ \frac{1}{3}\le\alpha\le\frac{1}{2},\ \frac{1}{4}\le\beta\le\frac{1}{3},\ \frac{1}{5}\le\gamma\le\frac{1}{3}\right\}.$$

Proof. We only show the case of (1), since the same argument works in the other cases. We prove that the Lipschitz constant of the map $T\colon M\to M$ is less than 1. In order to do so, we see T as the composition of the three maps F, G, H given by

$$F(\alpha,\beta,\gamma)=(21\alpha^{22}\beta^{14}\gamma^{6}t^{42},14\alpha^{14}\beta^{10}\gamma^{4}t^{28},6\alpha^{6}\beta^{4}\gamma^{2}t^{12}),$$
$$G(x_1,x_2,x_3)=\left(\frac{x_1}{x_1+x_2+x_3},\frac{x_2}{x_1+x_2+x_3},\frac{x_3}{x_1+x_2+x_3}\right),$$
$$H(y_1,y_2,y_3)=\left(\frac{y_1+21}{42},\ \frac{y_2+14}{42},\ \frac{y_3+6}{42}\right).$$

We put

$$X=21\alpha^{22}\beta^{14}\gamma^{6}t^{42},\ Y=14\alpha^{14}\beta^{10}\gamma^{4}t^{28},\ Z=6\alpha^{6}\beta^{4}\gamma^{2}t^{12}.$$

Then, the Jacobian J_F of F is described as

$$J_F=\begin{pmatrix}X&0&0\\0&Y&0\\0&0&Z\end{pmatrix}\begin{pmatrix}22&14&6\\14&10&4\\6&4&2\end{pmatrix}\begin{pmatrix}\alpha^{-1}&0&0\\0&\beta^{-1}&0\\0&0&\gamma^{-1}\end{pmatrix}.$$

Hence, the differential map dF maps the tangent vectors $\begin{pmatrix}1\\-1\\0\end{pmatrix}$, $\begin{pmatrix}1\\0\\-1\end{pmatrix}$ to the vectors

$$\begin{pmatrix}X(22\alpha^{-1}-14\beta^{-1})\\ Y(14\alpha^{-1}-10\beta^{-1})\\ Z(6\alpha^{-1}-4\beta^{-1})\end{pmatrix},\quad\begin{pmatrix}X(22\alpha^{-1}-6\gamma^{-1})\\ Y(14\alpha^{-1}-4\gamma^{-1})\\ Z(6\alpha^{-1}-2\gamma^{-1})\end{pmatrix},$$

respectively. Since $\frac{1}{2}\le\alpha\le\frac{11}{21}$, $\frac{1}{3}\le\beta\le\frac{5}{14}$, $\frac{1}{7}\le\gamma\le\frac{1}{6}$, the norms of these vectors are both less than $8\max\{X,Y,Z\}$. Moreover, the operator norms of the linear maps $dG_{(X,Y,Z)}$ and dH_* are $\dfrac{1}{X+Y+Z}$ and $\dfrac{1}{42}$, respectively, and the singular values of the matrix $\begin{pmatrix}1&1\\-1&0\\0&-1\end{pmatrix}$ are 1 and $\sqrt{3}$. Since

$$\frac{1}{42}\cdot\frac{8\max\{X,Y,Z\}}{X+Y+Z}<\frac{4}{21},$$

the operator norm of the differential map dT_p at any point $p\in M$ is less than $\frac{4}{21}$. Therefore, the Lipschitz constant of T is less than 1. □

Remark 2.4. Suppose that the invertible polynomial f is written in the form of Proposition 1.2 and represents one of the fourteen exceptional singularities. In this case, it is remarkable that the coordinates of the critical point of the function $\rho^2|_{\Sigma_t}$ are positive real when the parameter t is any positive real number. Moreover, Proposition 2.3 shows the approximate location of the critical point when the positive parameter t is small. We explain the case where $(p, q, r) = (2, 3, 7)$ as an example. In this case, if $(\alpha_0, \beta_0, \gamma_0)$ is the fixed point of the map T, then the coordinates of the critical point are $(\alpha_0^{11}\beta_0^7\gamma_0^3 t^{21}, \alpha_0^7\beta_0^5\gamma_0^2 t^{14}, \alpha_0^3\beta_0^2\gamma_0 t^6)$. Since $(\alpha_0, \beta_0, \gamma_0)$ is close to $(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$ when t is a small positive number, the critical point is approximated by

$$\Big(\frac{t^{21}}{2^{14}3^{10}}, \frac{t^{14}}{2^9 3^7}, \frac{t^6}{2^4 3^3}\Big).$$

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