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BRUNELLA–KHANEDANI–SUWA VARIATIONAL RESIDUES FOR INVARIANT CURRENTS

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To Israel Vainsencher, on the occasion of his 70th birthday

ABSTRACT. In this work we prove a Brunella–Khanedani–Suwa variational type residue theorem for currents invariant by holomorphic foliations. As a consequence, we provide conditions for the accumulation of the leaves to the intersection of the singular set of a holomorphic foliation with the support of an invariant current.

1. INTRODUCTION

In [20] B. Khanedani and T. Suwa introduced an index for singular holomorphic foliations on complex compact surfaces called the *Variational index*. In [22] D. Lehmann and T. Suwa generalized the variational index for higher dimensional holomorphic foliations. In particular, they showed that if V is an m-dimensional complex subvariety invariant by a holomorphic foliation $\mathscr{F}$ of dimension $k \geq 1$ on an n-dimensional complex compact manifold X, then

$$c_1^{m-k+1}(\det(N\mathscr{F}^*)) \cdot [V] = (-1)^{m-k+1} \sum_{\lambda} \mathrm{Res}_{c_1^{m-k+1}}(\mathscr{F}; S_\lambda),$$

where S_λ is a connected component of $S(\mathscr{F}, V) := (\mathrm{Sing}(\mathscr{F}) \cap V) \cup \mathrm{Sing}(V)$ (here $\mathrm{Sing}(\mathscr{F})$ and $\mathrm{Sing}(V)$ denotes the singular set of $\mathscr{F}$ and V respectively), $[V]$ is the integration current of V and $N\mathscr{F}^*$ is the conormal sheaf of $\mathscr{F}$. In the case such that X is a complex surface and $S(\mathscr{F}, V)$ is an isolated set, then for each $p \in S(\mathscr{F}, V)$

$$-\mathrm{Res}_{c_1}(\mathscr{F}; p) = \mathrm{Var}(\mathscr{F}, V, p),$$

where $\mathrm{Var}(\mathscr{F}, V, p)$ denotes the Variational index of $\mathscr{F}$ along V at p as defined by Khanedani and Suwa in [20].

M. Brunella in [1] studied the Khanedani–Suwa variational index and its relations with GSV and Camacho–Sad indices. See also [25, II, Proposition 1.2.1].

In [25] M. McQuillan, in his proof of the Green-Griffiths conjecture (for a projective surface X with $c_1^2(X) > c_2(X)$), showed that if X is a complex surface of general type and $\mathscr{F}$ is a holomorphic foliation on X, then $\mathscr{F}$ has no entire leaf which is Zariski dense. See [14, 26, 18, 15] for more details about the Green-Griffiths conjecture and generalizations. M. Brunella in [2] provided an alternative proof of McQuillan's result by showing that the following non-positivity result holds: if $[T_f]$ is the Ahlfors current associated to a Zariski dense entire curve $f : \mathbb{C} \to X$ which is tangent to $\mathscr{F}$, then

$$c_1(N\mathscr{F}^*) \cdot [T_f] = \sum_{p \in \mathrm{Sing}(\mathscr{F}) \cap \mathrm{Supp}(T_f)} \frac{1}{2\pi i} [T_f](\chi_{U_p} d(\phi_p \beta_p)) \leq 0,$$

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where χ_{U_p} denotes the characteristc function of a neighborhood U_p of $p \in \mathrm{Sing}(\mathscr{F}) \cap \mathrm{Supp}(T_f)$, see section 3 for more details.

To continue we consider a singular holomorphic foliation $\mathscr{F}$, of dimension $k \geq 1$, on a compact complex manifold X of dimension at least two. We recall that a positive closed current T in X is *invariant* by $\mathscr{F}$ if $T_{|\mathscr{F}} \equiv 0$, that is, $T(\eta) = 0$ for every test form η vanishing along the leaves of $\mathscr{F}$, so that $T(\eta)$ depends only on the restriction of η to the leaves.

In [3] M. Brunella proved a more general variational index type Theorem for positive closed currents of bidimension $(1,1)$ which are invariant by one-dimensional holomorphic foliations, with isolated singularities, on compact complex manifolds. More precisely, he showed that if T is an invariant positive closed current of bidimension $(1,1)$, then

$$c_1(\det(N\mathscr{F}^*)) \cdot [T] = \sum_{p \in \mathrm{Sing}(\mathscr{F}) \cap \mathrm{Supp}(T)} \frac{1}{2\pi i}[T](\chi_{U_p} d(\phi_p \beta_p)).$$

Compare this formula with the so called *asymptotic Chern class* of a foliation on complex surfaces introduced in [7]. Moreover, Brunella showed in the same work that a generic one-dimensional holomorphic foliation on complex projective spaces has no invariant measure. In [19, Corollary 1.2] L. Kaufmann showed that there is no diffuse foliated cycle directed by embedded Lipschitz laminations of dimension $k \geq n/2$ on $\mathbb{P}^n$.

We denote the class of a closed current T of bidimension (p,p) in the cohomology group $H^{n-p,n-p}(X)$ by $[T]$. In order to provide a generalization of the above results, we define the residue of $\mathscr{F}$ relative to T along a connected component of the singular set of $\mathscr{F}$, (see Def. 3.1 in Sect. 3.). In this work we prove the following result.

Theorem 1.1. *Let $\mathscr{F}$ be a holomorphic foliation of dimension $k \geq 1$, on a compact complex manifold X, of dimension n, with $\dim(\mathrm{Sing}(\mathscr{F})) \leq k-1$. Write $\bigcup_\lambda Z_\lambda \subset \mathrm{Sing}(\mathscr{F})$, a decomposition of the components of dimension $k-1$ into connected components and let U_λ be a regular neighborhood of Z_λ. For $p \geq k$, if T is a positive closed current of bidimension (p,p) invariant by $\mathscr{F}$, then*

$$c_1^{p-k+1}(\det(N\mathscr{F}^*)) \cdot [T] = \sum_{Z_\lambda \subset \mathrm{Supp}(T) \cap \mathrm{Sing}(\mathscr{F})} \mathrm{Res}(\mathscr{F}, T, Z_\lambda).$$

A compact non-empty subset $\mathcal{M} \subset X$ is said to be a *minimal set* for $\mathscr{F}$ if the following properties are satisfied

(i) $\mathcal{M}$ is invariant by $\mathscr{F}$;
(ii) $\mathcal{M} \cap \mathrm{Sing}(\mathscr{F}) = \emptyset$;
(iii) $\mathcal{M}$ is minimal with respect to these properties.

The problem of existence of minimal sets for codimension one holomorphic foliations on $\mathbb{P}^n$ was considered by Camacho–Lins Neto–Sad in [7]. To our knowledge, this problem remains open for $n = 2$. If $\mathscr{F}$ is a codimension one holomorphic foliation on $\mathbb{P}^n$, with $n \geq 3$, Lins Neto [23] proved that $\mathscr{F}$ has no minimal sets.

M. Brunella stated in [4] the following conjecture:

Conjecture. *Let X be a compact connected complex manifold of dimension $n \geq 3$, and let $\mathscr{F}$ be a codimension one holomorphic foliation on X such that $N\mathscr{F}$ is ample. Then every leaf of $\mathscr{F}$ accumulates to $\mathrm{Sing}(\mathscr{F})$.*

In [5], M. Brunella and C. Perrone proved the above Conjecture for codimension-one holomorphic foliations on projective manifolds with cyclic Picard group. In [9] the natural conjecture has been stated:

Conjecture (Generalized Brunella's conjecture)**.** *Let X be a compact connected complex manifold of dimension $n \geq 3$, and let $\mathscr{F}$ be a holomorphic foliation of codimension $r < n$ on X such that $\det(N\mathscr{F})$ is ample. Then every leaf of $\mathscr{F}$ accumulates to $\operatorname{Sing}(\mathscr{F})$, provided $n \geq 2r+1$.*

The main result in [9] suggests that the property of accumulation of the leaves of a foliation $\mathscr{F}$ to its singular set (*or nonexistence of minimal sets of $\mathscr{F}$*) depends on the existence of strongly q-convex spaces which contains the singularities of $\mathscr{F}$. In [7] was proved that there is no invariant measure with support on a nontrivial minimal set of a foliation on $\mathbb{P}^2$. We observe that in $\mathbb{P}^n$ we have that $\det(N_{\mathscr{F}})$ is ample for every foliation $\mathscr{F}$. The following Corollary 1.2 generalize the result in [7, Theorem 2].

Corollary 1.2. *Let $\mathscr{F}$ be a holomorphic foliation, of dimension $k \geq 1$, on a projective manifold X such that $\dim(\operatorname{Sing}(\mathscr{F})) \leq k-1$ and $\det(N\mathscr{F})$ is ample. Suppose that $h^{n-p,n-p}(X) = 1$, for some $p \geq k$. If T is a positive closed current of bidimension (p,p) invariant by $\mathscr{F}$, then $\operatorname{Supp}(T) \cap \operatorname{Sing}(\mathscr{F}) \neq \emptyset$. In particular, there is no invariant positive closed current of bidimension (p,p) with support on a nontrivial minimal set of $\mathscr{F}$.*

Compare Corollary 1.2 with [19, Corollary 5.5]. Since $h^{n-p,n-p}(\mathbb{P}^n) = 1$, this result holds for foliations on $\mathbb{P}^n$, in particular if $V \subset \mathbb{P}^n$ is an $\mathscr{F}$-invariant complex subvariety, then $V \cap \operatorname{Sing}(\mathscr{F}) \neq \emptyset$. This is the Esteves–Kleiman result [17, Proposition 3.4, pp. 12].

We can also apply Theorem 1.1 to Ahlfors' currents associated to $f : \mathbb{C}^k \to X$, a holomorphic map of generic maximal rank, which is a leaf of the foliation $\mathscr{F}$. To see this fix a Kähler form ω on X. On $\mathbb{C}^k$ we take the homogeneous metric form

$$\omega_0 := dd^c \ln |z|^2,$$

and denote by

$$\sigma = d^c \ln |z|^2 \wedge \omega_0^{k-1}$$

the Poincaré form. Consider $\eta \in A^{1,1}(X)$ and for any $r > 0$ define

$$T_{f,r}(\eta) = \int_0^r \frac{dt}{t} \int_{B_t} f^*\eta \wedge \omega_0^{k-1},$$

where $B_t \subset \mathbb{C}^k$ is the ball of radius t. Then we consider the positive currents $\Phi_r \in A^{1,1}(X)'$ defined by

$$\Phi_r(\eta) := \frac{T_{f,r}(\eta)}{T_{f,r}(\omega)}.$$

This gives a family of positive currents of bounded mass from which we can extract a subsequence Φ_{r_n} which converges to a current $[T_f] \in A^{1,1}(X)'$ called an Ahlfors' current of f, see [18, Claim 2.1].

This construction has been generalized in [6] by Burns–Sibony and [16] by De Thélin. In order to associate to $f : \mathbb{C}^k \to X$ positive closed currents of any bidimension (s,s), $1 \leq s \leq k$ (also called Ahlfors' currents) certain extra technical conditions are necessary, which we will not consider in this paper.

We obtain another consequence of Theorem 1.1 as follows:

Corollary 1.3. *Let $\mathscr{F}$ be a holomorphic foliation, of dimension $k \geq 1$, on a projective manifold X such that $\dim(\operatorname{Sing}(\mathscr{F})) \leq k-1$ and $\det(N\mathscr{F})$ is ample. Let $f : \mathbb{C}^k \to X$ be a holomorphic map of generic* *maximal rank which is a leaf of the foliation. Suppose that $h^{n-p,n-p}(X) = 1$, for some $p \geq k$, and that there exist an Ahlfors' current of bidimension (p,p) associated to f. Then $\overline{f(\mathbb{C}^k)} \cap \mathrm{Sing}(\mathscr{F}) \neq \emptyset$.*

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2. Singular holomorphic foliations

Let X be a connected compact complex manifold of dimension n. A holomorphic *distribution* $\mathscr{F}$ of dimension k on X is a nonzero coherent subsheaf $T\mathscr{F} \subsetneq TX$ of generic rank k such that $TX/T\mathscr{F} := N\mathscr{F}$ is torsion free. We have an exact sequence of sheaves

$$0 \longrightarrow T\mathscr{F} \longrightarrow TX \longrightarrow N\mathscr{F} \longrightarrow 0. \tag{2.1}$$

The sheaves $T\mathscr{F}$ and $N\mathscr{F}$ are called the *tangent* and the *normal* sheaves of $\mathscr{F}$, respectively. The codimension of $\mathscr{F}$ is the generic rank of $N\mathscr{F}$ which is equal to $n-k$. The singular locus of $\mathscr{F}$ is

$$\mathrm{Sing}(\mathscr{F}) = \{p \in X : (N\mathscr{F})_p \text{ , is not a free } \mathscr{O}_p - \text{module}\}. \tag{2.2}$$

Condition $N\mathscr{F}$ to be torsion free implies codim$(\mathrm{Sing}(\mathscr{F})) \geq 2$. The sheaf $N\mathscr{F}^*$ is called the conormal sheaf of the the distribution $\mathscr{F}$.

Now, by taking the double dual of the $(n-k)$-th wedge product of the inclusion

$$N\mathscr{F}^* \longrightarrow \Omega^1_X$$

we get a map

$$(\wedge^{n-k} N\mathscr{F}^*)^{**} \longrightarrow \Omega^{n-k}_X.$$

Since $N\mathscr{F}$ and $N\mathscr{F}^*$ are torsion-free, it follows from [21, Proposition 5.6.10] and [21, Proposition 5.6.12] that $(\wedge^{n-k} N\mathscr{F}^*)^{**} \simeq \det(N\mathscr{F}^*) \simeq \det(N\mathscr{F})^*$. This gives rise to a nonzero twisted holomorphic $(n-k)$-form $\omega \in H^0(X, \Omega^{n-k}_X \otimes \det(N\mathscr{F})^{**}) \simeq H^0(X, \Omega^{n-k}_X \otimes \det(N\mathscr{F}))$, which is locally decomposable outside $\mathrm{Sing}(\mathscr{F})$. To say that $\omega \in H^0(X, \Omega^{n-k}_X \otimes \det(N\mathscr{F}))$ is locally decomposable outside $\mathrm{Sing}(\mathscr{F})$ means that, in a neighborhood U of all point $p \in X \setminus \mathrm{Sing}(\mathscr{F})$, ω decomposes as the wedge product of $n-k$ local 1-forms $\omega|_U = \omega_1 \wedge \cdots \wedge \omega_{n-k}$.

We say that a codimension $n-k$ distribution $\mathscr{F}$ is a *foliation* if the induced twisted holomorphic $(n-k)$-form $\omega \in H^0(X, \Omega^{n-k}_X \otimes \det(N\mathscr{F}))$ is integrable. To say that it is integrable means that for all local decomposition ω on $p \in X \setminus \mathrm{Sing}(\mathscr{F})$ one has $d\omega_j \wedge \omega = 0$ for $1 \leq j \leq n-k$. In terms of sheaves, the integrability condition is equivalent to $dN\mathscr{F}^*|_U \subset N\mathscr{F}^*|_U \wedge \Omega^1_U$, where $U := X \setminus \mathrm{Sing}(\mathscr{F})$. By the exact sequence (2.1) and from [21, Proposition 5.6.9] we have the following adjunction formula

$$KX = K\mathscr{F} \otimes \det(N\mathscr{F})^*,$$

where $K\mathscr{F} = \det(T\mathscr{F})^*$ denotes the canonical bundle of $\mathscr{F}$. For more details on singular holomorphic distributions and foliations see [10, 13, 17, 27].

2.1. Holomorphic foliations on complex projective spaces.

Let $\omega \in \mathrm{H}^0(\mathbb{P}^n, \Omega^{n-k}_{\mathbb{P}^n}(m))$ be the twisted $(n-k)$-form induced by a holomorphic foliation $\mathscr{F}$ of dimension k on $\mathbb{P}^n$.

Take a generic non-invariant linearly embedded subspace $i : L \simeq \mathbb{P}^{n-k} \hookrightarrow \mathbb{P}^n$. We have an induced non-trivial section

$$i^*\omega \in \mathrm{H}^0(L, \Omega_L^{n-k}(m)) \simeq \mathrm{H}^0(\mathbb{P}^{n-k}, \mathcal{O}_{\mathbb{P}^{n-k}}(k-n-1+m)),$$

since $\Omega_{\mathbb{P}^{n-k}}^{n-k} = \mathcal{O}_{\mathbb{P}^{n-k}}(k-n-1)$. The *degree* of $\mathscr{F}$ is defined by

$$\deg(\mathscr{F}) := \deg(Z(i^*\omega)) = k-n-1+m.$$

In particular, $\omega \in \mathrm{H}^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^k(\deg(\mathscr{F})+n-k+1))$. That is, $\det(N\mathscr{F}) = \mathcal{O}_{\mathbb{P}^n}(\deg(\mathscr{F})+n-k+1)$ is ample.

A holomorphic foliation, of degree d, can be induced by a polynomial $(n-k)$-form on $\mathbb{C}^{n+1}$ with homogeneous coefficients of degree $d+1$, see for instance [12, 13].

3. The variational residue and proof of Theorem 1.1

From section 2, a holomorphic foliation of dimension k is given by a twisted integrable holomorphic $(n-k)$-form $\omega \in H^0(X, \Omega_X^{n-k} \otimes \det(N\mathscr{F}))$ which is equivalent to giving a family $(\{V_\mu\}, \{\omega_\mu\})_{\mu\in\Lambda}$, where $\mathcal{V} = \{V_\mu\}_{\mu\in\Lambda}$ is an open cover of X by Stein open sets, ω_μ is an integrable holomorphic $(n-k)$-form defined in V_μ and locally decomposable in $V_\mu \setminus \mathrm{Sing}(\mathscr{F})$. That is, for each $p \in V_\mu$, there is an open neighborhood $V_p \subset V_\mu$ of p such that

$$\omega_{\mu|V_p} = \omega_1^\mu \wedge \cdots \wedge \omega_{n-k}^\mu,$$

where ω_j^μ is a holomorphic 1-form and $d\omega_j^\mu \wedge \omega_\mu = 0$ for $1 \le j \le n-k$.

The integrability condition tells us that, in $V_\mu \setminus \mathrm{Sing}(\mathscr{F})$, there is a C^∞ 1-form α_μ satisfying:

(i) $d\omega_\mu = \alpha_\mu \wedge \omega_\mu$, for all $\mu \in \Lambda$. α_μ is not unique, but its restriction to the leaves of $\mathscr{F}$ is, provided ω_μ is fixed.

(ii) α_μ is of type $(1,0)$ since ω_μ is holomorphic and $\alpha_{\mu|\mathscr{F}}$ is holomorphic. This last fact follows from: if we assume that around a regular point the foliation $\mathscr{F}$ is generated by $\partial/\partial z_i$, $i = 1, \ldots, k$, then $\iota_{\partial/\partial z_i}(d\omega_\mu) = \left(\iota_{\partial/\partial z_i}\alpha_\mu\right)\omega_\mu$. In particular, if $k=1$ then $\alpha_{\mu|\mathscr{F}}$ is closed and $d\alpha_{\mu|\mathscr{F}} = 0$.

In the overlapping $V_{\mu\nu}$ we have $\omega_\mu = f_{\mu\nu}\omega_\nu$, with $f_{\mu\nu} \in \mathscr{O}^*(V_{\mu\nu})$ and the cocycle $\{f_{\mu\nu}\}_{\mu,\nu\in\Lambda}$ determines the line bundle $\det(N\mathscr{F})$. Hence

$$\left(\alpha_\mu - \alpha_\nu - \frac{df_{\mu\nu}}{f_{\mu\nu}}\right) \wedge \omega_\mu = 0. \tag{3.1}$$

This shows that $\alpha_\mu - \alpha_\nu - \dfrac{df_{\mu\nu}}{f_{\mu\nu}}$ is a C^∞ local section of the conormal bundle $N\mathscr{F}^*$ of the regular foliation $\mathscr{F}_{|X\setminus\mathrm{Sing}(\mathscr{F})}$.

By fixing a small neighbourhood U of $\mathrm{Sing}(\mathscr{F})$ and we can regularize each α_ν on U , i.e. we choose a smooth $(1,0)$-form $\tilde{\alpha_\nu}$ on V_ν coinciding with α_ν outside of $V_\nu \cap U$. More precisely, we can define $\tilde{\alpha_\nu} = \varphi_\nu\alpha_\nu$, where $\varphi_\nu : U \longrightarrow \mathbb{R}$ is a C^∞ function satisfying $0 < \varphi_\nu \le 1$ in $U \setminus \mathrm{Sing}(\mathscr{F})$ and $\varphi_\nu = 1$ in $U \setminus (V_\nu \cap U)$. Then the smooth $(1,0)$-forms

$$\gamma_{\mu\nu} = \frac{df_{\mu\nu}}{f_{\mu\nu}} - \tilde{\alpha_\nu} + \tilde{\alpha_\mu}$$

vanish on $\mathscr{F}$ outside of U. The cocycle $\gamma_{\mu\nu}$ can be trivialized, i.e, $\gamma_{\mu\nu} = \tilde{\gamma_\mu} - \tilde{\gamma_\nu}$, where $\tilde{\gamma_\mu}$ is a smooth $(1,0)$-form on V_μ vanishing on $\mathscr{F}$ outside of $V_\mu \cap U$. Hence, by setting $\beta_\mu = \tilde{\alpha_\nu} + \tilde{\gamma_\nu}$ we get

(3.2) $$\beta_\mu = \beta_\nu + \frac{df_{\mu\nu}}{f_{\mu\nu}},\ d\beta_\mu = d\beta_\nu \text{ in } V_{\mu\nu},\ d\omega_\mu = \beta_\mu \wedge \omega_\mu \text{ and } d\beta_\mu \wedge \omega_\mu = 0 \text{ outside of } V_\nu \cap U.$$

By the second equality in 3.2, the 2-forms $\{d\beta_\mu\}$ piece together and we have a global C^∞ 2-form on X which we denote by $d\beta$ and from

$$\frac{df_{\mu\nu}}{f_{\mu\nu}} = \beta_\nu - \beta_\mu$$

we conclude that 2-form $\frac{1}{2\pi i} d(\beta)$ represents $c_1(\det N(\mathscr{F}))$, since $\{f_{\mu\nu}\}_{\mu,\nu\in\Lambda}$ is a cocycle of $\det(N\mathscr{F})$. Therefore, the 2-form $\frac{1}{2\pi i} d(-\beta)$ represents $c_1(\det N\mathscr{F}^*)$.

We shall briefly digress on the geometric meaning of this smooth 2-form $\frac{1}{2\pi i} d(\beta)$ (see [8] 6.2.4): the first equality in 3.2 tells us that the 1-forms $\{\beta_\mu\}$ behave as connection matrices of $\det(N\mathscr{F})$, in V_μ, for some connection. In this case it is natural to consider the basic connections (in the sense of Bott, see [11]).

Fix a C^∞ decomposition

$$TX_{|X\setminus \mathrm{Sing}(\mathscr{F})} = (N\mathscr{F} \oplus T\mathscr{F})_{|X\setminus \mathrm{Sing}(\mathscr{F})},$$

where $N_{\mathscr{F}}$ and $T_{\mathscr{F}}$ are the normal and tangent bundles, respectively, of the regular foliation $\mathscr{F}_{|X\setminus \mathrm{Sing}(\mathscr{F})}$.

Let V_μ be the domain of a local trivialization of $N\mathscr{F}$ and $\{v_1^\mu, \ldots, v_{n-k}^\mu\}$ be a local frame for $N\mathscr{F}_{|V_\mu}$ such that $\omega_\mu(v_1^\mu, \ldots, v_{n-k}^\mu) \equiv 1$. For a suitable basic connection ∇ and ζ any section of $T\mathscr{F}_{|V_\mu}$, we have that

$$\beta_\mu(\zeta) = \mathrm{tr}(\theta^\mu)(\zeta)$$

if, and only if, $d\omega_\mu = \beta_\mu \wedge \omega_\mu$, where θ^μ is the connection matrix in V_μ of ∇ relative to the frame $\{v_1^\mu, \ldots, v_{n-k}^\mu\}$. In particular, the 1-forms $\{\beta_\mu\}$ piece together to give a well defined global form β on $X \setminus \mathrm{Sing}(\mathscr{F})$. It follows that $\frac{1}{2\pi i} d\beta = \mathrm{tr}(K_\nabla) = c_1(K_\nabla)$ where $K_\nabla = \{K_\nabla^\mu\}_{\mu\in\Lambda}$ is the curvature form of ∇ and the class $\frac{1}{2\pi i} d\beta = c_1(N\mathscr{F}) = -c_1(\det N\mathscr{F}^*)$.

Before defining the residue let's recall the concept of tubular neighborhood of an analytic set in our context (see [24]).

Let $\mathscr{F}$ be a singular foliation of dimension $k \geq 1$ on X, as above, and consider

$$\mathrm{Sing}(\mathscr{F}) = \bigcup_\lambda Z_\lambda$$

a decomposition of its singular locus into connected components. Take a Whitney stratification $\mathcal{S}_\lambda$ of Z_λ and let W_λ be any open set containing Z_λ. By the proof of Proposition 7.1 of [24], we can construct a family of tubular neighborhoods $\{T_{S_\lambda,\rho_{S_\lambda}}\}$, with $|T_{S_\lambda,\rho_{S_\lambda}}| \subset W_\lambda$, $\pi_{S_\lambda} : |T_{S_\lambda,\rho_{S_\lambda}}| \longrightarrow S_\lambda$ the projection and ρ_{S_λ} the tubular (or distance) function, for each stratum S_λ of $\mathcal{S}_\lambda$, satisfying the commutation relations which give *control data* for $\mathcal{S}_\lambda$: if S_λ and S'_λ are strata with $S_\lambda < S'_\lambda$ then

$$\begin{cases} \pi_{S_\lambda} \circ \pi_{S'_\lambda}(p) = \pi_{S_\lambda}(p) \\ \rho_{S_\lambda} \circ \pi_{S'_\lambda}(p) = \rho_{S_\lambda}(p). \end{cases}$$

This allows for the construction of an open set U_λ such that $Z_\lambda \subset U_\lambda \subset W_\lambda \subset X$, $\overline{U_\lambda}$ is a (real) C^0 manifold of dimension $2n$ with boundary ∂U_λ, which we call a *regular neighborhood* of Z_λ. By shrinking W_λ, we may assume $U_\lambda \cap U_{\tilde{\lambda}} = \emptyset$ for $\lambda \neq \tilde{\lambda}$. We call $\{U_\lambda\}_{\lambda\in L}$ a *system of regular neighborhoods* of $\mathrm{Sing}(\mathscr{F})$. Also, each $Z_\lambda = \coprod_{i=1,\ldots,m} S_\lambda^i$ (disjoint union), where the S_λ^i are the strata of $\mathcal{S}_\lambda$. Each S_λ^i

is a complex manifold and consider the S_λ^i which have maximum dimension. The union of these strata is precisely the regular part Z_λ^* of Z_λ. A *volume element* υ_{Z_λ} of Z_λ is a volume element of Z_λ^*.

Definition 3.1. *Let $\mathscr{F}$ be a singular foliation of dimension $k \geq 1$, as above, and consider*

$$\bigcup_\lambda Z_\lambda \subset \mathrm{Sing}(\mathscr{F})$$

a decomposition of the components of dimension $k-1$ into connected components. For $p \geq k$, suppose T is a positive closed current of bidimension (p,p) which is invariant by $\mathscr{F}$. The residue of $\mathscr{F}$ relative to T along Z_λ is

$$\mathrm{Res}(\mathscr{F}, T, Z_\lambda) = \left(\frac{1}{2\pi i}\right)^{p-k+1} \frac{T\left(\chi_{Z_\lambda} d(-\beta)^{p-k+1} \wedge \upsilon_{Z_\lambda}\right)}{\mathrm{vol}(Z_\lambda)} \cdot [Z_\lambda],$$

where χ_{Z_λ} denotes the characteristic function, υ_{Z_λ} is a volume element of Z_λ.

Now we are able to prove the

Theorem 1.1. *Let $\mathscr{F}$ be a holomorphic foliation of dimension k on a complex compact manifold X with $\dim(\mathrm{Sing}(\mathscr{F})) \leq k-1$. Write $\bigcup_\lambda Z_\lambda \subset \mathrm{Sing}(\mathscr{F})$, a decomposition of the components of dimension $k-1$ into connected components and let U_λ be a regular neighborhood of Z_λ. For $p \geq k$, if T is a positive closed current of bidimension (p,p) invariant by $\mathscr{F}$ then,*

$$c_1^{p-k+1}(\det(N\mathscr{F}^*)) \cdot [T] = \sum_{Z_\lambda \subset \mathrm{Supp}(T) \cap \mathrm{Sing}(\mathscr{F})} \mathrm{Res}(\mathscr{F}, T, Z_\lambda).$$

Proof. In order to show geometrically that

$$c_1^{p-k+1}(\det(N\mathscr{F}^*)) \cdot [T]$$

localizes at $\mathrm{Supp}(T) \cap \mathrm{Sing}(\mathscr{F})$ we will use the concept of regular neighborhood.

Let $\{U_\lambda\}_{\lambda \in L}$ be a system of regular neighborhoods of $\mathrm{Sing}(\mathscr{F})$. Since outside U_λ we have that $d\beta_{|\mathscr{F}} = 0$ in $X \setminus \mathrm{Sing}(\mathscr{F})$ and T is $\mathscr{F}$-invariant, we get

$$T\left(\chi_{Z_\lambda} d(-\beta)^{p-k+1}\right) = 0$$

in $X \setminus U_\lambda$. By squeezing U_λ via the tubular functions used to construct it, we conclude that

$$\mathrm{Supp}\, T\left(\chi_{Z_\lambda} d(-\beta)^{p-k+1}\right) \subseteq Z_\lambda$$

which gives

$$\left(\frac{1}{2\pi i}\right)^{p-k+1} T\left(\chi_{Z_\lambda} d(-\beta)^{p-k+1}\right) = \mu_{Z_\lambda}\,[Z_\lambda]$$

for some $\mu_{Z_\lambda} \in \mathbb{C}$. Now, since

$$\left(\frac{1}{2\pi i}\right)^{p-k+1} T\left(\chi_{Z_\lambda} d(-\beta)^{p-k+1} \wedge \upsilon_{Z_\lambda}\right) = \mu_{Z_\lambda}\,[Z_\lambda]\,(\upsilon_{Z_\lambda}),$$

$[Z_\lambda](v_{Z_\lambda}) = \mathrm{vol}(Z_\lambda)$ and $\frac{1}{2\pi i}d(-\beta)$ represents $c_1(\det N\mathscr{F}^*)$ we have that

$$\begin{aligned} c_1^{p-k+1}(\det(N\mathscr{F}^*))\cdot[T] &= T\left(\left(\frac{1}{2\pi i}\right)^{p-k+1} d(-\beta)^{p-k+1}\right) \\ &= \sum_{Z_\lambda\subset \mathrm{Supp}(T)\cap \mathrm{Sing}(\mathscr{F})} \left(\frac{1}{2\pi i}\right)^{p-k+1} T\left(\chi_{Z_\lambda} d(-\beta)^{p-k+1}\right) \\ &= \sum_{Z_\lambda\subset \mathrm{Supp}(T)\cap \mathrm{Sing}(\mathscr{F})} \mathrm{Res}(\mathscr{F}, T, Z_\lambda). \end{aligned}$$

□

Remark. The reason for taking the $(p-k+1)$-th power of $c_1(\det(N\mathscr{F}^*))$ is because the current $c_1^{p-k+1}(\det(N\mathscr{F}^*))\cdot[T]$ has compact support in the components of dimension $k-1$ of the singular set of the foliation $\mathscr{F}$, i.e, it is a current of bidimension $(k-1, k-1)$.

3.1. **Proof of Corollaries 1.2 and 1.3.** It is enough to prove the Corollary 1.2. The result is a straightforward consequence of Theorem 1.1. In fact, suppose by contradiction that T is a closed positive current of bidimension (p,p) invariant by $\mathscr{F}$ and that $\mathrm{Supp}(T)\cap\mathrm{Sing}(\mathscr{F}) = \emptyset$. Then, it follows from Theorem 1.1 that

$$c_1^{p-k+1}(\det(N\mathscr{F}^*))\cdot[T] = 0.$$

Since $h^{n-p,n-p}(X) = 1$ and $\det(N\mathscr{F}^*)$ is ample, then $[T] = b\cdot c_1^{n-p}(\det(N\mathscr{F}))\in H^{n-p,n-p}(X)$, for some $b>0$. Therefore, we have

$$c_1^{p-k+1}(\det(N\mathscr{F}^*))\cdot[T] = (-1)^{p-k+1} b\cdot c_1^{n-k+1}(\det(N\mathscr{F}))\neq 0.$$

This is a contradiction.

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