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NEW SINGULARITY INVARIANTS: THE SHEAF $\beta_X^\bullet$.

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ABSTRACT. The graded coherent sheaf $\alpha_X^\bullet$ constructed in [3] for any reduced complex space X is stable by exterior product but not by the de Rham differential. We construct here a new graded coherent sheaf $\beta_X^\bullet$ containing $\alpha_X^\bullet$ and stable both by exterior product and by the de Rham differential. It is the unique minimal such sheaf inside the sheaf $L_X^\bullet$ of meromorphic forms on X which become holomorphic on any desingularization of X, which has these properties and this sub-sheaf is coherent.

We show that it has again the "pull-back property" for holomorphic maps $f : X \to Y$ between reduced complex spaces such that $f^{-1}(S_Y)$ has empty interior in X. Moreover, this graded coherent sheaf $\beta_X^\bullet$ comes with a natural coherent exhaustive filtration which is also compatible with the pull-back by such holomorphic maps. We show on some simple examples that these sheaves and their natural filtrations are new invariants on singular complex spaces.

1. Introduction

1.1. **New singularity invariants.** In the article [3] (see also the erratum [4]) we introduce on a reduced complex space X the coherent graded sheaf $\alpha_X^\bullet$ of locally bounded meromorphic forms on X. They are characterized by the fact that they satisfy locally an integral dependence relation over the symmetric algebra of the sheaf of usual holomorphic forms modulo torsion. Then the normalized Nash blow-up makes them holomorphic. But the story is not as simple as in the case of functions (the standard normalization of a reduced complex space). First because the normalized Nash blow-up $\nu : \tilde{X} \to X$ is not a finite map in general and the sheaf $\alpha_{\tilde{X}}^\bullet$ is in general bigger than the sheaf of holomorphic forms on $\tilde{X}$ modulo torsion. To show that the iteration of this process stops, and when this is the case that it gives a desingularization of X, is still an open problem.

But there is another fact which comes into the picture: the new holomorphic forms on $\tilde{X}$ coming from sections of the sheaf $\alpha_X^\bullet$ have holomorphic differentials. But on X it is not true that the differential of a locally bounded meromorphic form is still locally bounded. This remark is the initial point of the present paper.

We construct on any reduced complex space X a graded sheaf $\beta_X^\bullet$ containing the graded sheaf $\alpha_X^\bullet$, stable by exterior product (as the sheaf $\alpha_X^\bullet$ is), stable by the de Rham differential, and as small as possible.

We show that there is an unique minimal such sheaf inside the sheaf $L_X^\bullet$ of meromorphic forms on X which become holomorphic on any desingularization of X and that this sub-sheaf is coherent. Then we show that the pull-back map $\hat{f} : f^*(\alpha_Y^\bullet) \to \alpha_X^\bullet$ defined in [3] (see also the erratum [4]) for any holomorphic map $f : X \to Y$ between reduced complex spaces such that $f^{-1}(S_Y)$ has empty interior in X (where S_Y is the singular set in Y) extends to the sheaf $\beta^\bullet$.

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The construction of this sheaf $\beta_X^\bullet$ highlights the existence of a natural finite filtration inside the sheaf $\beta_X^\bullet$ by coherent sub-sheaves, the first one being $\alpha_X^\bullet$ and the last one being $\beta_X^\bullet$. Then we prove that this (graded) filtration is coherent and also compatible with the pull-back by any holomorphic map $f : Y \to X$ as above. Then the graded sheaf $\beta_X^\bullet$ with its graded filtration defines new singularity invariants, the first one being the graded sheaf $\alpha_X^\bullet$.

We conclude the article with some simple examples of computation of these new singularity invariants.

1.2. Reminder of the sheaf $\alpha_X^\bullet$.

NOTATIONS. For a reduced complex space X we denote by $\Omega_X^\bullet$ the graded sheaf of holomorphic (Kähler) differential forms on X, $L_X^\bullet$ the graded sheaf of meromorphic forms on X which are holomorphic on any desingularization of X and $\omega_X^\bullet$ the sheaf of $\bar\partial-$closed currents on X of type $(\bullet, 0)$ modulo its torsion sub-sheaf. Of course these sheaves coincide on the smooth part of X and satisfy on X the graded inclusions

$$\Omega_X^\bullet\big/torsion \subset L_X^\bullet \subset \omega_X^\bullet.$$

Recall that the graded sheaf $\alpha_X^\bullet$ constructed in [3] is the integral closure in the sheaf $\omega_X^\bullet$ of the sheaf $\Omega_X^\bullet\big/torsion$. This means that the germ of a section σ of the sheaf ω_X^p at a point x of the reduced complex space X is in $\alpha_{X,x}^p$ if and only if there exists a monic homogeneous polynomial P with coefficients in the symmetric algebra $S^\bullet(\Omega_{X,x}^p)$ of the $\mathcal{O}_{X,x}-$module $\Omega_{X,x}^p$ such that the germ $P(\sigma)$ vanishes on the smooth part of X near x.

The following properties of these graded sheaves $\alpha_X^\bullet$ are proved in [3] and see [4] for ii).

i) The sheaf $\alpha_X^\bullet$ is a graded coherent sub-sheaf of $L_X^\bullet$. So it has no torsion.

ii) For any holomorphic map $f : X \to Y$ between reduced complex spaces such that $f^{-1}(S_Y)$ has empty interior in X (where S_Y is the singular set in Y) there exists a natural graded pull-back map
$$\hat{f}^* : f^*(\alpha_Y^\bullet) \to \alpha_X^\bullet$$
which is compatible with the usual pull-back of holomorphic differential forms. Moreover if $g : Y \to Z$ is a holomorphic map between reduced complex spaces such that $g^{-1}(S_Z)$ has empty interior in Y and such that $f^{-1}(g^{-1}(S_Z))$ has empty interior in X we have $\hat{f}^* \circ \hat{g}^* = \widehat{f \circ g}^*$.

iii) For any germ $\sigma \in \alpha_{X,x}^p$ we may find finitely may germs $\omega_j \in \Omega_{X,x}^p, j \in [1, N]$ and finitely many germs at x of $\mathscr{C}^\infty$ functions on $X \setminus S_X$ which are bounded near x such that we have $\sigma = \sum_{j=1}^N \rho_j.\omega_j$ on $X \setminus S_X$ near x.

2. DEFINITION OF $\beta_X^\bullet$ AND THE PULL-BACK PROPERTY

2.1. Construction of the sheaf $\beta_X^\bullet$. Let X be a reduced complex space and denote by $\alpha_X^\bullet$ the graded sheaf on X introduced in [3] (see the reminder in the previous paragraph).

Lemma 2.1.1. *The sheaf $\alpha_X^\bullet$ is stable by exterior product.*

PROOF. Recall that, by definition, the sheaf $\alpha_X^\bullet$ is a sub-sheaf of the sheaf $\omega_X^\bullet$ and the following characterization is proved in [3] Th. 3.0.2 (see property iii) in the previous paragraph) : a section σ on the open set $U \subset X$ of the sheaf $\omega_X^\bullet$ is a section on U of $\alpha_X^\bullet$ if it may be written locally on U as $\sigma = \sum_{j\in J} \rho_j.\omega_j$ where ω_j are holomorphic forms on U and ρ_j are $\mathscr{C}^\infty$ functions on the complement of the singular set S in U which are bounded near S. Is it clear that the exterior product of two such sections on U of $\alpha_X^\bullet$ can be written in the same way locally on U and then define a current on U which is $\bar\partial-$closed on $U \setminus S$. So to conclude the lemma, it is enough to prove that this current admits a $\bar\partial-$closed extension to U. In fact, as the sheaf $\alpha_X^\bullet$

is a sub-sheaf of the sheaf $L_X^\bullet$ obtained by the direct image of the sheaf $\Omega_{\tilde{X}}^\bullet$ where $\tau : \tilde{X} \to X$ is any desingularisation of X and as this sheaf $L_X^\bullet$ is stable by exterior product, the conclusion follows from the inclusion $L_X^\bullet \subset \omega_X^\bullet$. ■

We remark that the sheaf $\alpha_X^\bullet$ is a graded $\Omega_X^\bullet-$module but is not stable in general by the de Rham differential. For instance in

$$X := \{(x, y, z) \in \mathbb{C}^3 \ / \ x.y = z^2\},$$

the differential form $dx \wedge dy/z = -d\big(z.dx/x - z.dy/y\big)$ is not in α_X^2 but the form $z.dx/x - z.dy/y$ is a section of α_X^1 (see [3] or Paragraph 3.2 below).

A construction. Define $\alpha_X^\bullet[0] := \alpha_X^\bullet$ and for any integer $p \geq 0$ and any integer $q \geq 0$ define

$$\alpha_X^q[p+1] := \sum_{r=0}^{q} \Big(\alpha_X^r[p] \wedge \alpha_X^{q-r}[p]\Big) + \sum_{r=0}^{q-1} \Big(\alpha_X^r[p] \wedge d\big(\alpha_X^{q-r-1}[p]\big)\Big) \subset L_X^q. \tag{1}$$

Recall that the sheaf $L_X^\bullet$ is stable by exterior products and by the de Rham differential.

Proposition 2.1.2. *We have the following properties:*

1. *For each integer p the sheaf $\alpha_X^\bullet[p]$ is stable by exterior product with $\Omega_X^\bullet/torsion$. Moreover for each pair of integers p, q we have $\alpha_X^0.\alpha_X^q[p] = \alpha_X^q[p]$ (the equality comes from the fact that $1 \in \alpha_X^0$).*
2. *For each pair of integers p, q the sheaf $\alpha_X^q[p]$ is $\mathcal{O}_X-$coherent sub-sheaf of L_X^q. So the sheaf $\alpha_X^q[p]$ is torsion free.*
3. *For each pair of integers p, q the sub-sheaf $\alpha_X^q[p]$ is contained in $\alpha_X^q[p+1]$.*
4. *For each pair of integers p, q and q' we have $\alpha_X^q[p] \wedge \alpha_X^{q'}[p] \subset \alpha_X^{q+q'}[p+1]$.*
5. *For each pair of integers p, q and r we have $\alpha_X^r[p] \wedge d\big(\alpha_X^q[p]\big) \subset \alpha_X^{q+r+1}[p+1]$. In particular $d\big(\alpha_X^q[p]\big) \subset \alpha_X^{q+1}[p+1]$.*

Proof. Property (1) is an obvious consequence of the definition of these sheaves by an induction on p.

As L_X^q is a coherent sheaf on X which is torsion free, to prove (2) it is enough to prove that $\alpha_X^q[p+1]$ is a finite type $\mathcal{O}_X-$module. We shall prove this by an induction on $p \geq 0$.

So assume the coherence of the sheaf $\alpha_X^q[p]$ for each q. Then we want to prove that $\alpha_X^q[p+1]$ is finitely generated. Let $(g_{j,r})$ be a finite set of generators of the sheaf $\alpha_X^r[p]$. Then we shall show that the elements $g_{i,r} \wedge g_{j,q-r}$ and $g_{i,r} \wedge dg_{j,q-r-1}$ for all choices of i, j and $r \leq q$, generate $\alpha_X^q[p+1]$. The only point which is not obvious is the fact that for any sections $u \in \alpha_X^r[p]$ and $v \in \alpha_X^{q-r-1}[p]$ the wedge product $u \wedge dv$ is in the sheaf generated by our "candidates" generators. But then write

$$u = \sum_i a_i.g_{i,r} \quad \text{and} \quad v = \sum_j b_j.g_{j,q-r-1}$$

where a_i and b_j are holomorphic functions. Then

$$dv = \sum_j db_j \wedge g_{j,q-r-1} + \sum_j b_j.dg_{j,q-r-1}.$$

So in the wedge products $u \wedge dv$ the terms are linear combinations of our candidates generators excepted those like $a_i.g_{i,r} \wedge db_j \wedge g_{j,q-r-1}$. This point is solved by Condition (1) which is already proved. Points (3), (4), and (5) are obvious. ■

Now we remark that the sequence of coherent sub-sheaves $\alpha_X^\bullet[p]$ of the coherent sheaf $L_X^\bullet$ is increasing. So it is locally stationary on X.

Definition 2.1.3. *Define the coherent sub-sheaf $\beta_X^\bullet$ as the union of the increasing sequence of coherent sub-sheaves $\alpha_X^\bullet[p], p \geq 0$ of the coherent sheaf $L_X^\bullet$.*

Corollary 2.1.4. *The graded sub-sheaf $\beta_X^\bullet$ of the graded coherent differential sheaf $L_X^\bullet$ is coherent, stable by exterior product and by the de Rham differential.*

Proof. The assertion is local, so we may assume that $\beta_X^\bullet = \alpha_X^\bullet[p] \quad \forall p \geq p_0$. Then the corollary is a consequence of Properties (4) and (5) above. ■

Theorem 2.1.5. *For any holomorphic map $f : X \to Y$ between reduced complex spaces such that $f^{-1}(S(Y))$ has empty interior in X, there exists an unique pull-back*

$$\hat{f}^* : f^*(\beta_Y^\bullet) \to \beta_X^\bullet$$

which is compatible with the pull-back of the $L^\bullet-$sheaves (and so with the pull-back of the $\alpha^\bullet-$sheaves ; see [4]) and which is graded of degree 0 and compatible with the exterior product and the de Rham differential.

For any holomorphic maps $f : X \to Y$ and $g : Y \to Z$ between reduced complex spaces, such that $f^{-1}\big(S(Y) \cup g^{-1}(S(Z))\big)$ has no interior point in X and $g^{-1}(S(Z))$ has no interior point in Y we have

$$\widehat{f \circ g}^*(\sigma) = \hat{f}^*(\hat{g}^*(\sigma)) \qquad \forall \sigma \in \beta_Z^\bullet.$$

Moreover, for each integer $p \geq 0$ the pull-back $\hat{f}^*$ induces a pull-back

$$\hat{f}^*[p] : f^*(\alpha_Y^\bullet[p]) \to \alpha_X^\bullet[p]$$

and in the previous situation $\widehat{f \circ g}^*[p] = \hat{g}^*[p] \circ \hat{f}^*[p]$ for each $p \geq 0$.

So we shall construct in fact a (graded) naturally filtered sheaf $(\beta_X^\bullet, (\alpha_X^\bullet[p])_{p\in\mathbb{N}})$ such that the pull-back constructed in the previous theorem is compatible with these filtrations and with the composition of suitable holomorphic maps.

In the following we make the convention that $\alpha_X^\bullet[-1] := \alpha_X^\bullet[0] := \alpha_X^\bullet$.

Proof. Assume that X is an irreducible complex space and that $f(X)$ is not contained in the singular locus $S(Y)$ of Y. Note that our hypothesis implies that there exists a dense Zariski open set X'' in $X \setminus S(X)$ such that the restriction of f to X'' takes values in $Y \setminus S(Y)$.

Assume also that for some integer $p \geq 0$ we have constructed for any $q \leq p$ a pull-back morphism

$$\hat{f}^*[q] : f^*(\alpha_Y^\bullet[q]) \to \alpha_X^\bullet[q]$$

with the following properties

- 1_p It induces the usual pull-back of the sheaves of holomorphic forms when it is restricted to the smooth parts of X and Y. Note that this implies that the restriction of $\hat{f}^*[p]$ to $f^*(\alpha_Y^\bullet[q])$ is equal to $\hat{f}^*[q]$ because, by definition, the sections of the sheaves under consideration are determined by their restrictions to an open dense subset.
- 2_p For s, t in $\alpha_Y^\bullet[p-1]$ we have $\hat{f}^*[p](s \wedge t) = \hat{f}^*[p-1](s) \wedge \hat{f}^*[p-1](t)$.
- 3_p For any u in $\alpha_Y^\bullet[p-1]$ such that du is in $\alpha_Y^{\bullet+1}[p]$[1] we have $d(\hat{f}^*[p-1](u)) = \hat{f}^*[p](du)$.

[1]For $p \geq 1$ $u \in \alpha_X^\bullet[p-1]$ implies $du \in \alpha_X^{\bullet+1}[p]$ is automatic; but not for $p = 0$ with our convention.

Then we want to construct $\hat{f}^*[p+1] : f^*(\alpha_N^\bullet[p+1]) \to \alpha_M^\bullet[p+1]$ satisfying again the properties above for $p+1$.

It is clear that that our inductive hypothesis given by Conditions $1_p, 2_p$ and 3_p is true for $p = 0$ (but 2_0 is obtained by looking at points in X'' and using the absence of torsion).

Now we shall show that if it is satisfied for some $p \geq 0$ then it is also satisfied for $p+1$.

CONSTRUCTION OF $\hat{f}^*[p+1]$. Let ξ be a section in $\alpha_Y^\bullet[p+1]$. We may write

$$\xi = \sum_{j=0}^{J} \beta_j \wedge \gamma_j + \sum_{j=0}^{J} u_j \wedge dv_j$$

where $\beta_j, \gamma_j, u_j, v_j$ are sections of the sheaf $\alpha_Y^\bullet[p]$. It is clear that our Conditions $1_{p+1}, 2_{p+1}, 3_{p+1}$ imply that we must put

$$\hat{f}^*[p+1](\xi) = \sum_{j=0}^{J} \hat{f}^*[p](\beta_j) \wedge \hat{f}^*[p](\gamma_j) + \sum_{j=0}^{J} \hat{f}^*[p](u_j) \wedge d(\hat{f}^*[p](v_j)).$$

Now the main point is to prove that if we change the choice of writing ξ in such a way, the value of $\hat{f}^*[p+1](\xi)$ stays the same. In other words, we have to prove that if $\xi = 0$ is written as above then we find $\hat{f}^*[p+1](\xi) = 0$.

To prove this is quite simple because it is enough to look on X''. On this open dense subset we have simply taken the usual pull-back of the holomorphic form ξ restricted to the smooth part of Y by the holomorphic map $f' : X'' \to Y \setminus S(Y)$ induced by f. As this pull-back commutes with exterior product and de Rham differential, its result is independent of the way in which we have written ξ above. This implies our claim because the sheaf $\alpha_X^\bullet[p+1]$ has no torsion.

To verify Properties $1_{p+1}, 2_{p+1}$ and 3_{p+1} is then obvious because it is enough to check them on X''.

This completes the proof of the existence of pull-back morphisms $\hat{f}^*[p]$ for each $p \geq 0$ and then for the sheaves $\beta^\bullet$. And it also gives the compatibility of these pull-backs with the exterior product and the de Rham differential.

The only point which we have to make precise to complete the proof of Theorem 2.1.5 is the "functorial" aspect of these pull-backs. But this is again an easy consequence of the non-existence of torsion for the sheaves we consider. ■

Proposition 2.1.6. *Let X be a reduced complex space. Then for each $q \geq 0$ we have $\beta_X^q = \alpha_X^q[q]$. If X is normal, for $q \geq 1$ we have $\beta_X^q = \alpha_X^q[q-1]$.*

PROOF. First we remark that, by definition α_X^0 is the sheaf of locally bounded meromorphic functions on X (so it is equal to $\mathcal{O}_X$ if and only if X is normal), and that $\beta_X^0 = \alpha_X^0$ by definition.

We remark also that for each p and each q we have $\alpha_X^0[p] = \alpha_X^0$ and $\alpha_X^q[p] \wedge \alpha_X^0 = \alpha_X^q[p]$ (see prop. 2.1.2 above).

Fix an integer $q_0 \geq 1$ and assume that for any integer $q < q_0$ we have $\alpha_X^q[p-1] = \beta_X^q$ for some integer $p \geq 1$. This means that $\alpha_X^q[p-1] = \alpha_X^q[p]$ for these q. By definition we have

$$\alpha_X^{q_0}[p+1] = \sum_{h=0}^{q_0} \alpha_X^h[p] \wedge \alpha_X^{q_0-h}[p] + \sum_{h=0}^{q_0-1} d(\alpha_X^h[p]) \wedge \alpha_X^{q_0-h-1}[p].$$

But our assumption allows one to replace p by $p-1$ in the right-hand side of the equality above except for the terms $h = q_0$ in the first sum. So we find

$$\alpha_X^{q_0}[p+1] \subset \alpha_X^{q_0}[p] \wedge \alpha_X^0[p] + \sum_{h=0}^{q_0-1} \alpha_X^h[p-1] \wedge \alpha_X^{q_0-h}[p-1] + \sum_{h=0}^{q_0-1} d(\alpha_X^h[p-1]) \wedge \alpha_X^{q_0-h-1}[p-1].$$

But $\alpha_X^{q_0}[p] \wedge \alpha_X^0[p] = \alpha_X^{q_0}[p]$ and then all sheaves in the right-hand side are contained in $\alpha_X^{q_0}[p]$; this gives the equality $\alpha_X^{q_0}[p+1] = \alpha_X^{q_0}[p]$.

Now using that, for each $q \geq 0$, α_X^q is stable by multiplication by elements in α_X^0, we obtain

$$\alpha_X^1[1] = \alpha_X^1[0] + \sum_{i,j=1}^{I} \mathcal{O}_X.g_j.dg_i,$$

where $g_1, \dots, g_I$ generate the coherent $\mathcal{O}_X-$module α_X^0. This implies that $\alpha_X^1[1]$ is stable by multiplication by α_X^0 and this implies the equality

$$\alpha_X^1[2] = \alpha_X^1[1] + \sum_{i,j=1}^{I} \mathcal{O}_X.g_j.dg_i = \alpha_X^1[1].$$

So we have $\alpha_X^1[1] = \beta_X^1$. This allows us to begin our induction on q_0 for $q_0 = 1$ with $p = 2$.

Then by induction on $q_0 \geq 1$ we conclude that for each $q \geq 1$ we have $\beta_X^q = \alpha_X^q[q]$.

In the case where X is normal, we may take $I = \{1\}$ and $g_1 = 1$ and this shows that $\alpha_X^1[0] = \beta_X^1$ and the induction gives now, if we begin with $q_0 = 1$ and $p = 1$, the equality $\alpha_X^q[q-1] = \beta_X^q$ for each $q \geq 1$. ■

Remark. This shows that for a normal complex space we always have the equality $\beta_X^1 = \alpha_X^1$, so the sheaf $\beta_X^\bullet$ is "new" only in degrees at least equal to 2 when X is normal.

2.2. **A finer filtration.**

We shall show that there exists another natural filtration for the sheaf $\beta_X^\bullet$ which is finer than the filtration $(\alpha_X^\bullet[p])_{p\geq 0}$ and which is also compatible with the pull-back by any holomorphic map $f : Y \to X$ such that $f^{-1}(S(X))$ has empty interior in Y. This filtration gives finer invariants for singular complex spaces.

Definition 2.2.1. *Let X be a reduced complex space. We define the increasing filtration $\alpha_X^\bullet\langle p\rangle$ of the graded sheaf $\beta_X^\bullet$ by induction on $p \in \mathbb{N}$ with the following conditions*

- $\alpha_X^\bullet\langle 0\rangle := \alpha_X^\bullet$
- $\alpha_X^\bullet\langle p+1\rangle := \alpha_X^\bullet\langle p\rangle + \alpha_X^\bullet\langle p\rangle \wedge d\alpha_X^\bullet$.

in the graded sense. This means explicitly that for each $q \geq 0$ we have

$$\alpha_X^q\langle p+1\rangle := \alpha_X^q\langle p\rangle + \sum_{r=0}^{q-1} \alpha_X^r\langle p\rangle \wedge d\alpha_X^{q-r-1}.$$

Proposition 2.2.2. *For each pair of integers p and q the sheaf $\alpha_X^q\langle p\rangle$ is a coherent sub-sheaf of the sheaf β_X^q and we have the following properties for each pair of integers p, p':*

(1) $\alpha_X^\bullet\langle p\rangle \wedge \alpha_X^\bullet\langle p'\rangle \subset \alpha_X^\bullet\langle p+p'\rangle$.
(2) $d\alpha_X^\bullet\langle p\rangle \subset \alpha_X^\bullet\langle p+1\rangle$.
(3) $\alpha_X^p\langle p\rangle = \beta_X^p$.

Proof. The coherence of the sheaves $\alpha_X^q\langle p\rangle$ is obtained by an induction in a similar way as for the sheaves $\alpha_X^q[p]$.

Note that for each p we have $\alpha_X^\bullet\langle p\rangle \subset \beta_X^\bullet$ by an easy induction on $p \geq 0$.

We shall prove (1) by induction on $p \geq 0$. For $p = 0$ the inclusion is clear as $\alpha_X^\bullet\langle p'\rangle$ is stable by wedge product on $\alpha_X^\bullet$ by an obvious induction on $p' \geq 0$. Assume that (1) is true for p (for each given $p' \geq 0$). Then

$$\alpha_X^\bullet\langle p+1\rangle \wedge \alpha_X^\bullet\langle p'\rangle = \alpha_X^\bullet\langle p\rangle \wedge \alpha_X^\bullet\langle p'\rangle + \alpha_X^\bullet\langle p\rangle \wedge d\alpha_X^\bullet \wedge \alpha_X^\bullet\langle p'\rangle$$

is a subset of

$$\alpha_X^\bullet\langle p+p'\rangle + \alpha_X^\bullet\langle p+p'\rangle \wedge d\alpha_X^\bullet = \alpha_X^\bullet\langle p+1+p'\rangle$$

thanks to our induction hypothesis and the anti-commutativity and associativity of the wedge-product.

We shall also prove (2) by induction on $p \geq 0$. The case $p = 0$ is clear by definition of $\alpha_X^\bullet\langle 1\rangle$ as $1 \in \alpha_X^0$. So assume that (2) is true for p. Then

$$d\alpha_X^\bullet\langle p+1\rangle = d\alpha_X^\bullet\langle p\rangle + d\alpha_X^\bullet\langle p\rangle \wedge d\alpha_X^\bullet$$

$$d\alpha_X^\bullet\langle p+1\rangle \subset \alpha_X^\bullet\langle p+1\rangle + \alpha_X^\bullet\langle p+1\rangle \wedge d\alpha_X^\bullet = \alpha_X^\bullet\langle p+2\rangle$$

concluding our induction.

From (1) and (2) we see that $\gamma_X^\bullet = \cup_{p=0}^\infty \alpha_X^\bullet\langle p\rangle$ is stable by $\wedge$ and d. So, as $\beta_X^\bullet$ is the smallest graded sub-sheaf of $L_X^\bullet$ which is stable under $\wedge$ and d and contains $\alpha_X^\bullet$, we conclude that $\gamma_X^\bullet = \beta_X^\bullet$. The proof of (3) is then obtained by an easy induction (analogous to the induction in Lemma 2.2.4 below). ■

Remark. It is an easy exercise to show that for each $p \geq 0$ we have

$$\alpha_X^\bullet\langle p\rangle \subset \alpha_X^\bullet[p] \subset \alpha_X^\bullet\langle 2^p - 1\rangle.$$

So we have $\beta_X^q = \cup_{n\in\mathbb{N}}\, \alpha_X^q\langle n\rangle$.

The following corollary of Theorem 2.1.5 shows that this finer filtration is also compatible with the pull-back for the sheaf $\beta^\bullet$.

Corollary 2.2.3. *For any holomorphic map $f : X \to Y$ between reduced complex spaces such that $f^{-1}(S(Y))$ has empty interior in X, the (graded) pull-back*

$$\hat{f}^* : f^*(\beta_Y^\bullet) \to \beta_X^\bullet$$

is compatible with the graded filtrations $\alpha^\bullet\langle p\rangle$ of the graded sheaves $\beta^\bullet$.

proof. It is enough to check this compatibility property when X and Y are complex manifolds, and this case is clear. ■

In the case where X is normal we may improve Property (3) in Proposition 2.2.2 above.

Lemma 2.2.4. *Assume that X is a normal complex space. Then $\beta_X^q = \alpha_X^q\langle q-1\rangle$ for each $q \geq 1$.*

Proof. Let us prove this lemma by induction on $q \geq 1$. For $q = 1$ we have

$$\alpha_X^1\langle 1\rangle = \alpha_X^1 + \alpha_X^0.d\alpha_X^0.$$

But the normality of X gives $\alpha_X^0 = \mathcal{O}_X$, and as α_X^1 contains $\Omega_X^1/torsion$ we obtain

$$\alpha_X^1\langle 1\rangle = \alpha_X^1\langle 0\rangle = \alpha_X^1$$

and Property (3) in Proposition 2.2.2 above implies $\beta_X^1 = \alpha_X^1$.

Assume that the lemma is proved for $q \geq 1$. Then

$$\alpha_X^{q+1}\langle q+1\rangle = \alpha_X^{q+1}\langle q\rangle + \sum_{r=0}^{q} \alpha_X^r\langle q\rangle \wedge d\alpha_X^{q-r}$$

and for $r \in [1,q]$ we have $\alpha_X^r\langle q\rangle = \alpha_X^r\langle r-1\rangle$ by our induction hypothesis. So the term $r = 0$ in the sum above is $\mathcal{O}_X.d\alpha_X^q \subset \alpha_X^{q+1}\langle q\rangle$ and the term for $r \geq 1$ is contained in $\alpha_X^r\langle q-1\rangle \wedge d\alpha_X^{q-r} \subset \alpha_X^{q+1}\langle q\rangle$; so the sum above is contained in $\alpha_X^{q+1}\langle q\rangle$ and this implies the equality $\alpha_X^{q+1}\langle q+1\rangle = \alpha_X^{q+1}\langle q\rangle$. Then Lemma 2.2.5 below allows us to conclude. ■

Our next simple lemma may help for computing the sheaf β_X^q.

Lemma 2.2.5. *Assume that for an integer q_0 we have $\alpha_X^q\langle p+1\rangle = \alpha_X^q\langle p\rangle$ for a given integer p and for each $q \in [0,q_0]$. Then $\beta_X^q = \alpha_X^q\langle p\rangle$ for each $q \in [0,q_0]$.*

Of course this lemma is useful only when q_0 is smaller that p. For instance, if

$$\alpha_X^p = \Omega_X^p\big/torsion \quad \text{for} \quad p \in [0, q-1],$$

then this implies $\alpha_X^q\langle 1\rangle = \alpha_X^q$ and the lemma implies $\beta_X^q = \alpha_X^q$.

Proof. By induction on q_0 it is clear that we may assume that $\beta_X^q = \alpha_X^q\langle p\rangle$ for $q \in [0, q_0-1]$, and it is enough to show that $\beta_X^{q_0} = \alpha_X^{q_0}\langle p\rangle$. We have

$$\alpha_X^{q_0}\langle p+2\rangle = \alpha_X^{q_0}\langle p+1\rangle + \sum_{r=0}^{q_0-1} \alpha_X^r\langle p+1\rangle \wedge d\alpha_X^{q_0-r-1}$$

$$= \alpha_X^{q_0}\langle p\rangle + \sum_{r=0}^{q_0-1} \alpha_X^r\langle p\rangle \wedge d\alpha_X^{q_0-r-1} = \alpha_X^{q_0}\langle p+1\rangle,$$

because $\alpha_X^r\langle p+1\rangle = \alpha_X^r\langle p\rangle$ for $r \in [0, q_0-1]$ as $\beta_X^r = \alpha_X^r\langle p\rangle$. So we obtain that $\alpha_X^{q_0}\langle n\rangle = \alpha_X^{q_0}\langle p\rangle$ for each $n \geq p$ and then $\beta_X^{q_0} = \alpha_X^{q_0}\langle p\rangle$. ■

Remark. It is not difficult to see that for each pair of integers p and q the sub-$\mathcal{O}_X-$module $\alpha_X^q\langle p\rangle$ of L_X^q is generated by sections of the type

$$\alpha_X^{q_0} \wedge d\alpha_X^{q_1} \wedge \cdots \wedge d\alpha_X^{q_j} \quad \text{for } j \in [0,p],$$

where $q_1, \ldots, q_j \in [0, q-1]$ and $q_0 + q_1 + \cdots + q_j + j = q$. When X is normal this gives the equalities

$$\beta_X^1 = \alpha_X^1, \quad \beta_X^2 = \alpha_X^2 + \mathcal{O}_X.d\alpha_X^1, \quad \beta_X^3 = \alpha_X^3 + \alpha_X^1 \wedge d\alpha_X^1 + \mathcal{O}_X.d\alpha_X^2 \ldots$$

2.3. **The product theorem.**

Notation. Let X and Y be reduced complex spaces. Then if

$$p_1 : X \times Y \to X \quad \text{and} \quad p_2 : X \times Y \to Y$$

are the projections, for a sheaf $\mathcal{F}$ of $\mathcal{O}_X-$modules and a sheaf $\mathcal{G}$ of $\mathcal{O}_Y-$modules, we define

$$\mathcal{F} \boxtimes \mathcal{G} := p_1^*(\mathcal{F}) \otimes_{\mathcal{O}_{X\times Y}} p_2^*(\mathcal{G}).$$

Theorem 2.3.1. *Let X and Y be two reduced complex spaces. Then the sheaves $\alpha_{X\times Y}^\bullet$ and $\beta_{X\times Y}^\bullet$ are given by the following formulas:*

$$\alpha_{X\times Y}^\bullet = \alpha_X^\bullet \boxtimes \alpha_Y^\bullet$$
$$\beta_{X\times Y}^\bullet = \beta_X^\bullet \boxtimes \beta_Y^\bullet$$

as graded sheaves. Moreover we have

$$\text{(F)} \qquad \alpha_{X\times Y}^\bullet\langle p\rangle = \sum_{h=0}^{p} \alpha_X^\bullet\langle h\rangle \boxtimes \alpha_Y^\bullet\langle p-h\rangle$$

as graded filtrations of the graded sheaf $\beta_{X\times Y}^\bullet$

Proof. The formula for the $\alpha-$sheaves is an easy exercice using two desingularisations $\sigma : \tilde{X} \to X$ and $\tau : \tilde{Y} \to Y$ which are normalizing respectively for the sheaves $\Omega_X^\bullet\big/torsion$ and $\Omega_Y^\bullet\big/torsion$, as the product map $\sigma\times\tau : \tilde{X}\times\tilde{Y} \to X\times Y$ is a desingularisation of $X\times Y$ which normalizes the sheaf $\Omega_{X\times Y}^\bullet$.

We shall prove now Formula (F) which implies the formula for the $\beta-$sheaves, thanks to the remark following Proposition 2.2.2.

We make an induction on $p \geq 0$. The case $p = 0$ is already proved, so assume that Formula (F) is proved for p. Then using the equality

$$\mathcal{O}_{X\times Y}.d\alpha_{X\times Y}^\bullet = \mathcal{O}_X.d\alpha_X^\bullet \boxtimes \alpha_Y^\bullet + \alpha_X^\bullet \boxtimes \mathcal{O}_Y.d\alpha_Y^\bullet$$

we obtain:

$$\begin{aligned}
\alpha_{X\times Y}^\bullet\langle p+1\rangle &= \alpha_{X\times Y}^\bullet\langle p\rangle + \alpha_{X\times Y}^\bullet\langle p\rangle \wedge d\alpha_{X\times Y}^\bullet \\
&= \sum_{h=0}^{p} \alpha_X^\bullet\langle h\rangle \boxtimes \alpha_Y^\bullet\langle p-h\rangle + \big(\sum_{h=0}^{p} \alpha_X^\bullet\langle h\rangle \boxtimes \alpha_Y^\bullet\langle p-h\rangle\big) \wedge d\big(\alpha_X^\bullet \boxtimes \alpha_Y^\bullet\big) \\
&= \sum_{h=0}^{p} \big(\alpha_X^\bullet\langle h\rangle + \alpha_X^\bullet\langle h\rangle \wedge d\alpha_X^\bullet\big) \boxtimes \big(\alpha_Y^\bullet\langle p-h\rangle\big) \\
&+ \sum_{h=0}^{p} \big(\alpha_X^\bullet\langle h\rangle\big) \boxtimes \big(\alpha_Y^\bullet\langle p-h\rangle + \alpha_Y^\bullet\langle p-h\rangle \wedge d\alpha_Y^\bullet\big) \\
&= \sum_{h=0}^{p} \alpha_X^\bullet\langle h+1\rangle \boxtimes \alpha_Y^\bullet\langle p-h\rangle + \sum_{h=0}^{p} \alpha_X^\bullet\langle h\rangle \boxtimes \alpha_Y^\bullet\langle p-h+1\rangle
\end{aligned}$$

which gives Formula (F) for $p+1$. ∎

The following trivial corollary will be used in an example below.

Corollary 2.3.2. *Let X be a reduced complex space. Consider on $X\times D$ a L^{p+1} form $\omega\wedge f(z).dz$ where D is a disc in $\mathbb{C}$ with coordinate z, $f : D \to \mathbb{C}$ a holomorphic function on D which is not identically zero, and where ω is a L^p-form on X. Then ω is a section of α_X^p if and only if $\omega\wedge f(z).dz$ is a section of $\alpha_{X\times D}^{p+1}$.*

3. Examples

3.1. The curve $X := \{x^3 = y^5\} \subset \mathbb{C}^2$.

Lemma 3.1.1. *On the curve $X := \{x^3 = y^5\} \subset \mathbb{C}^2$ we have*

$$\alpha^0_X = \beta^0_X = L^0_X = \mathcal{O}_X \oplus \mathbb{C}\,.y^2/x \oplus \mathbb{C}\,.y^4/x^2 \oplus \mathbb{C}\,.y^3/x \oplus \mathbb{C}\,.y^4/x$$
$$\omega^0_X = L^0_X + \mathcal{O}_X.y/x^2$$
$$\alpha^1_X = \Omega^1_X \oplus \mathbb{C}\,.y^2.dy/x \oplus \mathbb{C}\,.y^3.dy/x.$$
$$\beta^1_X = \alpha^1_X\langle 1\rangle = L^1_X.$$
$$L^1_X = \Omega^1_X \oplus \mathbb{C}\,.y.dy/x \oplus \mathbb{C}\,.y^3.dy/x^2 \oplus \mathbb{C}\,.y^2.dy/x \oplus \mathbb{C}\,.y^3.dy/x.$$
$$\omega^1_X = \Omega^1_X/torsion + \mathcal{O}_X.dy/x^2.$$

Proof. Let $\nu : \mathbb{C} \to X$ be the normalization given by

$$t \in \mathbb{C} \mapsto \nu(t) := (t^5, t^3).$$

Then $L^0_{X,0} = \nu_*(\mathbb{C}\{t\})$ and we have

$$L^0_X = \mathcal{O}_X \oplus \mathbb{C}\,.\nu_*(t) \oplus \mathbb{C}\,.\nu_*(t^2) \oplus \mathbb{C}\,.\nu_*(t^4) \oplus \mathbb{C}\,.\nu_*(t^7)$$

and the equalities

$$\nu_*(t) = y^2/x,\ \nu_*(t^2) = y^4/x^2,\ \nu_*(t^4) = y^3/x,\ \text{and } \nu_*(t^7) = y^4/x.$$

The equality $\alpha^0_X = L^0_X$ is a consequence of the fact that the algebra $\mathbb{C}\{t\}$ is integral over the sub-algebra $\mathbb{C}\{t^5, t^3\}$.

The sheaf ω^0_X is contained in $(1/x^2).\mathcal{O}_X$ thanks to Lemma 6.1.1 in [3] using the projection $\mathbb{C}^2 \to \mathbb{C}, (x,y) \mapsto y$ which induced a proper finite map $\pi : X \to \mathbb{C}$ surjective of degree 3. Then using the characterization of the sheaf ω^p_X given in [1] Proposition 1, we see that $1/x \not\in \omega^0_X$ (and also $1/x^2$) because $Trace_\pi[dx/x] = 5dy/y \not\in \Omega^1_{\mathbb{C}}$. But y/x^2 belongs to ω^0_X because for any $f \in \mathcal{O}_X$ we have $Trace_\pi[f.y/x^2] \in \mathcal{O}_{\mathbb{C}}$ and $Trace_\pi[f.y.dx/x^2]$ belongs to $\Omega^1_{\mathbb{C}}$.

So, as for any reduced complex space X, $L^\bullet_X$ is a subsheaf of $\omega^\bullet_X$, we obtain that $\omega^0_X = L^0_X + \mathcal{O}_X.y/x^2$.

The sheaf L^1_X is, by definition, given by $L^1_X = \nu_*(\Omega^1_{\mathbb{C}})$.

As Ω^1_X contains $\nu_*(t^2.dt) = dy/3, \nu_*(t^4.dt) = dx/5, \nu_*(t^5.dt) = y.dy/3$, and $\nu_*(t^n.dt)$ for each integer $n \geq 7$ because

$$\nu_*(t^7.dt) = y.dx/5,\ \nu_*(t^8.dt) = y^2.dy/3,\ \nu_*(t^9.dt) = x.dx/5$$

and for $n \geq 10$ we have $\nu_*(t^n.dt) = y.\nu_*(t^{n-3}.dt)$, we conclude, as Ω^1_X has no torsion, that

$$L^1_X = \Omega^1_X \oplus \mathbb{C}\,.y.dy/x \oplus \mathbb{C}\,.y^3.dy/x^2 \oplus \mathbb{C}\,.y^2.dy/x \oplus \mathbb{C}\,.y^3.dy/x.$$

As it is clear that $\nu_*(dt), \nu_*(t.dt)$ are not integral over Ω^1_X and that

$$\big(y^2.dy/x\big)^2 = 3dx.dy/5 \qquad \big(y^3.dy/x\big)^2 = 3y^2.dx.dy/5$$

$$\alpha^1_X = \Omega^1_X \oplus \mathbb{C}\,.y^2.dy/x \oplus \mathbb{C}\,.y^3.dy/x.$$

We know that $\beta^0_X = \alpha^0_X$ and that $\beta^1_X = \alpha^1_X\langle 1\rangle$ by Proposition 2.1.6. The only "new" contribution to $\alpha^1_X\langle 1\rangle$ comes from the sheaf $\alpha^0_X.d\alpha^0_X$. As $\nu_*(t)$ is in $\alpha^0_X = \nu_*(\mathcal{O}_{\mathbb{C}})$, $d\alpha^0_X$ contains $\nu_*(dt)$ and so $\alpha^1_X\langle 1\rangle$ contains L^1_X. Then $\beta^1_X = L^1_X = \nu_*(\Omega^1_{\mathbb{C}})$.

The equality $\omega^1_X = \Omega^1_X + \mathcal{O}_X.dy/x^2$ is already given by Lemma 6.1.1 in [3]. ■

3.2. **The surfaces S_k.** Consider the surfaces $S_k := \{(x,y,z) \in \mathbb{C}^3 \ / \ x.y = z^k\}$ for k an integer at least equal to 2.

In the following lemma, we determine the sheaves $\alpha_{S_k}^\bullet$ and $\beta_{S_k}^\bullet$. We also correct Lemma 6.2.2 of [3] which is wrong for $k \geq 4$.

Lemma 3.2.1. *Let $m := [k/2]$ be the integral part of $k/2$. Then we have*

$$\beta_{S_k}^1 = \alpha_{S_k}^1 = \Omega_{S_k}^1/torsion + \mathcal{O}_{S_k}.x.dy/z^m$$
$$\alpha_{S_k}^2 = \Omega_{S_k}^2/torsion + \mathcal{O}_{S_k}.\frac{dx \wedge dy}{z^{m-1}}$$
$$\beta_{S_k}^2 = \alpha_{S_k}^2\langle 1\rangle = \Omega_{S_k}^2/torsion + \mathcal{O}_{S_k}.\frac{dx \wedge dy}{z^m}.$$

Proof. The first assertion is a consequence of the equality $\alpha_M^1 = \beta_M^1$ for any normal complex space which is proved in Proposition 2.1.6. The computation of $\alpha_{S_k}^1$ is an obvious consequence of Lemma 6.2.3 in [3]. Note that the equalities

$$x.dy/z^m + y.dx/z^m = k.z^{k-m-1}.dz \quad \text{and} \quad (x.dy/z^m).(y.dx/z^m) = z^{k-2m}.(dx).(dy)$$

give the integral dependance relation of $x.dy/z^m$ on $S^\bullet(\Omega_{S_k}^1/torsion)$.

Let us now prove the second assertion.

We remark first that we have on S_k the relations

$$x.dx \wedge dy = k.z^{k-1}.dx \wedge dz \quad y.dx \wedge dy = k.z^{k-1}.dz \wedge dy$$

and using the equality $x.y = z^k$ this implies

$$dx \wedge dy = k.y.dx \wedge dz/z = -k.x.dy \wedge dz/z.$$

Dividing by z^{m-1} this gives

$$\left(\frac{dx \wedge dy}{z^{m-1}}\right)^2 = -k^2.z^{k-2m}.(dx \wedge dz).(dy \wedge dz) \quad \text{in } S^2(\Omega_{S_k}^2/torsion).$$

This proves that $dx \wedge dy/z^{m-1}$ is a section of the sheaf $\alpha_{S_k}^2$.

We want to prove now that the meromorphic form

$$\frac{dx \wedge dy}{z^m} = k.y.\frac{dx \wedge dz}{z^{m+1}} = -k.x.\frac{dy \wedge dz}{z^{m+1}}$$

which corresponds to $k^2.(a.b)^{k-m}.da \wedge db$ via the quotient map

$$q_k : \mathbb{C}^2 \to S_k \quad (a,b) \mapsto (x = a^k, y = b^k, z = a.b)$$

is not in $\alpha_{S_k}^2$.

As the fiber F_0 of the sheaf $F := q_k^{-1}(\Omega_{S_k}^2/torsion)$ at 0 is the $\mathbb{C}\{a^k, b^k, a.b\}-$submodule of $\mathbb{C}\{a^k, b^k, a.b\}.da \wedge db$ generated by $a^k.da \wedge db, b^k.da \wedge db, (a.b)^{k-1}.da \wedge db$, we have to show that $(a.b)^{k-m-1}.da \wedge db$ is not integral on F_0. This an easy consequence of the fact that for $q < k/2$ there is no positive constant C such that for $a > 0$ and $b > 0$ small enough we have the inequality $(a.b)^q \leq C.(a^k + b^k)$.

Note that Lemma 6.1.1 in [3] gives us that the sheaf $L_{S_k}^2$ is equal to

$$\Omega_{S_k}^2/torsion + \mathcal{O}_{S_k}.dx \wedge dy/z^{k-1},$$

so the second assertion is proved.

To prove the last assertion we remark first that the form $d(x.dy/z^m)$ is in $\alpha^2_{S_k}\langle 1\rangle = \beta^2_{S_k}$ (this last equality is proved in Lemma 2.2.4 as S_k is normal). But we have on S_k, using the equality $[(k-1)/2] + [k/2] = k-1$:

$$y.dx + x.dy = k.z^{k-1}.dz \qquad \text{so}$$
$$x.dy \wedge dx = k.z^{k-1}.dz \wedge dx \quad \text{and then}$$
$$\frac{dy \wedge dx}{z^m} = k.y.\frac{dz \wedge dx}{z^{m+1}}$$

This gives $d(x.dy/z^m) = (1 - m/k).dx \wedge dy/z^m$.

So the inclusion of $\Omega^2_{S_k}\big/torsion + \mathcal{O}_{S_k}.dx\wedge dy/z^m$ in $\beta^2_{S_k}$ is proved. The equality $\alpha^2_{S_k}\langle 1\rangle = \beta^2_{S_k}$ easily implies the equality in the previous inclusion, as we have the inclusion $\alpha^1_{S_k} \wedge \alpha^1_{S_k} \subset \alpha^2_{S_k}$ and $\alpha^0_{S_k}.d(\alpha^1_{S_k}) \subset \Omega^2_{S_k}\big/torsion + \mathcal{O}_{S_k}.\frac{dx\wedge dy}{z^m}$ thanks to the computation above. ∎

So for $k \geq 4$ we have strict inclusions between $\Omega^2_{S_k}\big/torsion, \alpha^2_{S_k}, \beta^2_{S_k}$ and $L^2_{S_k} = \omega^2_{S_k}$.

3.3. $M_k := \{x.y = u^k.v\}$.

Let $m := [k/2]$ be the integral part of the integer $k \geq 1$.

Lemma 3.3.1. *The meromorphic $1-$form $\omega_m := x.dy/u^m$ belongs to $\alpha^1_{M_k}$ but for $k \geq 2$ the differential $d\omega_m$ is not in $\alpha^2_{M_k}$.*

Proof. We have

$$x.dy/u^m + y.dx/u^m = d(xy)/u^m = d(u^k v)/u^m = k.u^{k-1-m}v.du + u^{k-m}.dv \quad \text{and}$$
$$\big(x.dy/u^m\big).\big(y.dx/u^m\big) = x.y.(dx).(dy)/u^{2m} = u^{k-2m}.v.(dx).(dy)$$

so ω_m satisfies the following integral dependance relation on $\Omega^1_M\big/torsion$

$$\omega_m^2 - (k.u^{k-m-1}.v.du + u^{k-m}.dv).\omega_m + u^{k-2m}.v.(dx).(dy) = 0. \tag{2}$$

Now we have

$$d\omega_m = \frac{dx \wedge dy}{u^m} - m.\frac{x.du \wedge dy}{u^{m+1}}.$$

But now we restrict this $2-$form to the surface $S_k := \{v = 1\} \cap M_k$ which cuts the $1-$dimensional singular set $\{x = y = u = 0\}$ of M_k only at the point $x = y = u = 0$ and $v = 1$, and we find, as we have on this surface $x.dy + y.dx = k.u^{k-1}.du$ which implies $y.dx \wedge dy = k.u^{k-1}.du \wedge dy$ and then $u.dx \wedge dy = k.x.du \wedge dy$,

$$(d\omega_m)_{|\{v=1\}} = (1 - m/k).dx \wedge dy/u^m$$

which is not in $\alpha^2_{S_k}$ for $k \geq 2$ (see Lemma 3.2.1). So $d\omega_m$ is not in $\alpha^2_{M_k}$ thanks to Theorem 2.1.5. ∎

Lemma 3.3.2. *The $2-$form $w := \omega_m \wedge dv$ belongs to $\alpha^2_{M_k}$ but dw is not in $\alpha^3_{M_k}$ for $k \geq 2$.*

Proof. The first assertion is obvious using the previous lemma as $\alpha^\bullet_{M_k}$ is stable by wedge products and contains $\Omega^\bullet_{M_k}\big/torsion$.

To prove the second assertion consider the following holomorphic map

$$\pi : S_k \times \mathbb{C} \to M_k, \quad ((x, y, u)), v) \mapsto (x.v, y, u, v).$$

Then $\pi^*(dw) = dx\wedge dy\wedge dv/u^m - m.x.du\wedge dy\wedge dv/u^{m+1}$. Using Corollary 2.3.2 of Theorem 2.3.1 and the fact that we have on $S_k \times \mathbb{C}$

$$\pi^*(dw) = v.dv \wedge \big((k-m).x.du \wedge dy/u^{m+1}\big)$$

we conclude that $\pi^*(dw)$ is not a section of $\alpha^3_{S_k\times\mathbb{C}}$, concluding the proof. ∎

Corollary 3.3.3. *For $k \geq 4$ we have on M_k*

$$\Omega^1_{M_k}/torsion \subset \alpha^1_{M_k} = \beta^1_{M_k} \subset L^1_{M_k}$$
$$\Omega^2_{M_k}/torsion \subset \alpha^2_{M_k} \subset \beta^2_{M_k} \subset L^2_{M_k}$$
$$\Omega^3_{M_k}/torsion \subset \alpha^3_{M_k} \subset \beta^3_{M_k} \subset L^3_{M_k}$$

where all inclusions are strict.

We leave to the reader the easy proof using the previous computations. ■

3.4. **Fermat surfaces.** In the previous example we use, for instance, the fact that a holomorphic map $f : S_k \to X$ allows us to show that a section of the sheaf $L^\bullet_X$ is not a section of $\alpha^\bullet_X$ or $\beta^\bullet_X$. We shall illustrate now on Fermat surfaces

$$F_n := \{(a, b, z) \in \mathbb{C}^3 \ / \ a^n - b^n = z^n\}$$

for $n \geq 3$, the fact that holomorphic maps $f : F_n \to S_n$ help to give some non-trivial[2] sections in $\alpha^\bullet_{F_n}$ and $\beta^\bullet_{F_n}$.

Let $\zeta := exp(2i\pi/n)$ and define for $q \in [0, n-1]$ the holomorphic map

$$f_q : F_n \to S_n \quad \text{defined by} \quad f_q(a, b, z) := (a - \zeta^q.b, \eta_q, z) \quad \text{where} \quad \eta_q := (a^n - b^n)/(a - \zeta^q.b).$$

Then, thanks to Lemma 3.2.1 and the pull-back theorem for $\alpha^\bullet$ and $\beta^\bullet$ sheaves, we obtain that for each $q \in [0, n-1]$ with $m := [n/2]$:

$$\hat{f_q}^*(y.dx/z^m) = \eta_q.d(a - \zeta^q.b)/z^m = z^{n-m}.\frac{d(a - \zeta^q.b)}{(a - \zeta^q.b)} \quad \text{belongs to} \quad \alpha^1_{F_n}$$

$$\hat{f_q}^*(dy \wedge dx/z^{m-1}) = d\eta_q \wedge d(a - \zeta^q.b)/z^{m-1}) = n.z^{n-m}.dz \wedge \frac{d(a - \zeta^q.b)}{(a - \zeta^q.b)} \text{ belongs to } \alpha^2_{F_n}$$

$$\hat{f_q}^*(dy \wedge dx/z^m) = d\eta_q \wedge d(a - \zeta^q.b)/z^m) = n.z^{n-m-1}.dz \wedge \frac{d(a - \zeta^q.b)}{(a - \zeta^q.b)} \text{ belongs to } \beta^2_{F_n}.$$

We remark that the degrees of homogeneity in (a, b, z) of the forms above are equal to $n-m+1$ or $n-m$; this will also be the case if $n = 2m$ and if we use the map $g : F_n \to S_n$ given by $g(a, b, z) = \big((a^2 - b^2), (a^n - b^n)/(a^2 - b^2), z\big)$ to pull-back forms.

The following lemma shows that we cannot find all sections in $\alpha^2_{F_n}$ by this method.

Lemma 3.4.1. *For $n = 2m \geq 4$ the form $(a.b)^m.da \wedge db/z^{2m-1}$ is a section of $\alpha^2_{F_{2m}}$.*

For $n = 2m + 1 \geq 3$ the form $(a.b)^m.da \wedge db/z^{2m}$ is a section of $\alpha^2_{F_{2m+1}}$.

Moreover, for $n \geq 8$, these forms are not in the $\mathcal{O}_{F_n}-$submodule generated by holomorphic forms and by the pull-backs of sections of $\alpha^2_{S_n}$ by the holomorphic maps f_q, for $q \in [0, n-1]$, described above.

Proof. On F_n we have the equalities

$$a^{n-1}.da \wedge db = z^{n-1}.dz \wedge db \quad \text{and}$$
$$b^{n-1}.da \wedge db = z^{n-1}.dz \wedge da$$

so we have

$$(dz \wedge da).(dz \wedge db) = \frac{(a.b)^{n-1}.(da \wedge db)^2}{z^{2n-2}}$$

[2]For instance, if $n = 4$, we shall obtain that $d(a^2 - b^2) \wedge d(a^2 + b^2)/z^2 = 8a.b.da \wedge db/z^2$ is in $\beta^2_{F_4}$. This form is homogeneous of degree 2 and even in a, b and z and there is no section in $\Omega^2_{F_4}$ with these two properties.

which implies
$$\left(\frac{(a.b)^m.da\wedge db}{z^{2m-1}}\right)^2 = a.b.(dz\wedge da).(dz\wedge db) \quad \text{for } n = 2m$$
and
$$\left(\frac{(a.b)^m.da\wedge db}{z^{2m}}\right)^2 = (dz\wedge da).(dz\wedge db) \quad \text{for } n = 2m+1.$$
Note that the forms $\frac{(a.b)^m.da\wedge db}{z^{2m-1}}$ and $\frac{(a.b)^m.da\wedge db}{z^{2m}}$ have degrees of homogeneity 3 and 2 in (a,b,z) respectively which are strictly smaller than m for $n\geq 8$.

To see that these forms are not holomorphic is then a simple exercise using the homogeneity on F_n (the list of holomorphic $2-$forms homogeneous of degree 2 or 3 is quite short!); we leave it to the reader. ■

Remark. For $n = 2m$ the form $(a.b)^{m-1}.da\wedge db/z^m$ is also a section of $\beta^2_{F_{2m}}$. To see that $(a.b)^{m-1}.da\wedge db/z^m$ is a section of $\beta^2_{F_{2m}}$ consider the map
$$f : F_{2m}\to S_{2m} \quad \text{given by} \quad f(a,b,z) = (a^m - b^m, a^m + b^m, z)$$
and compute the pull back of the form $dx\wedge dy/z^m$ which is a section of $\beta^2_{S_{2m}}$.

This shows that we may use more holomorphic maps from F_n to S_n. But as long as they respect homogeneity, they will not produce the sections given in the previous lemma.

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