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ON THE CHARACTERISTIC CURVES ON A SURFACE IN $\mathbb{R}^4$

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Abstract. We study some robust features of characteristic curves on smooth surfaces in $\mathbb{R}^4$. These curves are analogous to the asymptotic curves in the elliptic region. A $P_3(c)$-point is an isolated special point at which the unique characteristic (or asymptotic) direction is tangent to the parabolic curve. At this point, by considering the cross-ratio invariant, we show that the 2-jet of the curve formed by the inflections of the characteristic curves is projectively invariant. In addition, we exhibit the possible configurations of the characteristic curves at a $P_3(c)$-point.

1. Introduction

For surfaces in $\mathbb{R}^3$, an asymptotic direction is a self-conjugate tangent direction, and a characteristic direction is a tangent direction such that the angle it forms with its conjugate direction is extremal. At a hyperbolic (resp. parabolic or elliptic) point there are two (resp. one or 0) asymptotic directions and at an elliptic (resp. parabolic or hyperbolic) point there are two (resp. one or 0) characteristic directions. The asymptotic and characteristic curves are the integral curves of asymptotic and characteristic directions, respectively. It is well known that the characteristic curves are, in many ways, analogous to the asymptotic curves in the elliptic region (see [4, 5, 20]) and both curves are given, in a local chart, by a binary differential equation (BDE)

$$A(x,y)dx^2 + 2B(x,y)dxdy + C(x,y)dy^2 = 0, \tag{1}$$

where the coefficients A, B, and C are smooth functions defined in an open subset U of $\mathbb{R}^2$. The discriminant curve of equation (1) of the asymptotic and characteristic curves coincides with the parabolic curve. At cusps of Gauss the unique asymptotic and characteristic direction is tangent to the parabolic curve (see for example [1]). Although asymptotic curves can be also defined using the contact of the surface with lines, the characteristic curves do not satisfy this property.

In [20], Oliver used Uribe-Vargas's cr-invariant defined in [24], to show that the topological type of the singularity of the characteristic curves at a cusp of Gauss is invariant under projective transformations. Furthermore, the locus of inflection points of the characteristic curves (characteristic inflection curve) has some geometrical meaning. In particular, he classified a cusp of Gauss in terms of the relative position of the parabolic curve, the characteristic inflection curve and conodal curve. In this paper, we extend the results in [20] on characteristic curves for surfaces in $\mathbb{R}^4$.

The study of the differential geometry of immersed surfaces in 4-space was carried out by several authors, for example [2, 3, 10, 11, 16, 17, 19, 21, 23]. The study of characteristic curves did not receive the same treatment in the current literature. The definition of characteristic

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curves for surfaces in $\mathbb{R}^4$ is inspired from (and is analogous to) that for surfaces in $\mathbb{R}^3$ in the following way: for surfaces in $\mathbb{R}^3$, there is a relation between the BDEs of the asymptotic curves, of the characteristic curves and of the lines of principal curvature. By considering the BDE (1) as a point in the projective plane, the BDEs of the asymptotic curves and of the lines of principal curvature determine the BDE of the characteristic curves, such that the three BDEs define (at each point on the surface) a self polar triangle in the projective plane. In fact, the BDE of the asymptotic curves determines the other two BDEs ([4, 23]). Asymptotic directions are also defined on surfaces in $\mathbb{R}^4$ and are given by a BDE (see §3). Its equation is used to define in a unique way, two other BDEs such that the three equations form a self-polar triangle in the projective plane. One of them is what is called the BDE of the characteristic curves (called a *characteristic* BDE, for short) (see [23]). In this sense, the asymptotic and characteristic directions on surface in $\mathbb{R}^4$ behave as solutions of BDEs in the same way as its analogue on surfaces in $\mathbb{R}^3$.

For a surface in $\mathbb{R}^4$, the asymptotic directions are also captured by the contact of the surface with lines. This contact reveals aspects of the differential geometry of the surface in the closure of its hyperbolic region and is described by the $\mathcal{A}$-singularities of the family of orthogonal projections to 3-spaces. The projection along an asymptotic directions at a point on the parabolic set may have a $P_3(c)$-point. Away from inflection points, the characteristic and asymptotic curves are generically a family of cusps at ordinary parabolic points and have a folded singularity at a $P_3(c)$-point.

This point has similar behavior to the cusps of Gauss on surfaces in $\mathbb{R}^3$ (see [3, 10, 19, 24]). In [9, 10], we defined the cr-invariant at $P_3(c)$-points and showed that the S_2-curve, flecnodal curve and multi-local singularities curves are robust features of the surface in 4-space (Euclidean, affine or projective). Although the characteristic curves are not projective invariant of the surface, our goal is to produce results on the characteristic curves at $P_3(c)$-points similar to those results of Oliver [20]. At a $P_3(c)$-point, we show that the 2-jet of the curve formed by the inflection points of the characteristic curves (*characteristic inflection curve*) and the topological type of singularity of the characteristic curves in the elliptic domain are invariants under projective transformations. In addition, we list the possible configurations of the parabolic, S_2 and characteristic inflection curves using the cross-ratio invariant of this set of curves.

2. Binary differential equation

To study the configurations of characteristic curves, we need some results on BDEs which are studied extensively (see for example [22] for a survey article). We recall some results concerning the configurations of the solution curves of a BDE. A BDE defines two directions in the region where $\delta = B^2 - AC > 0$, a double (repeated) direction on the set $\Delta = \{\delta = 0\}$ and no direction where $\delta < 0$. The set Δ is the discriminant of the BDE. For generic BDEs and at generic points on Δ, the integral curves of (1) is a family of cusps, and the discriminant curve is a smooth curve traced by these cusps, except at isolated points called folded singularity (see below).

Consider the manifold of contact elements to the plane, that is, $PT^*\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}P^1$, and take the affine chart $q = dx/dy$, then $PT^*\mathbb{R}^2$ is endowed with the canonical contact structure determined by the 1-form $dx - qdy$. The projection associated to the contact structure is $\pi : PT^*\mathbb{R}^2 \to \mathbb{R}^2$ and given by $\pi(x, y, q) = (x, y)$. When the coefficients of a BDE do not vanish simultaneously, we may assume that $A \neq 0$ and take

$$\Omega(x, y, q) = A(x, y)q^2 + 2B(x, y)q + C(x, y). \tag{2}$$

The set $\Omega = 0$ is a surface $\mathcal{M}$. The directions defined by (1) lift to a single valued field

$$\xi = \Omega_q \partial y + q\Omega_q \partial x - (\Omega_y + q\Omega_x)\partial q \tag{3}$$

on $\mathcal{M}$ obtained by intersecting the contact planes with the tangent planes to $\mathcal{M}$. (See, for example, [8] for a suitable lifted field). The regions where $\delta > 0$, the image of $\pi|_{\mathcal{M}}$ is a two-fold covering. The critical set of $\pi|_{\mathcal{M}}$ given by $\Omega = \Omega_q = 0$ is called the criminant (its projection is the discriminant curve).

Stable topological models of (1) arise when the discriminant is a regular curve (or is empty). At almost all points of the discriminant, the field ξ is regular i.e., the unique direction at a point of the discriminant is transverse to it, then the BDE is smoothly equivalent to $dx^2 + ydy^2 = 0$ ([6], [7]). When ξ has an elementary singularity, the unique direction is tangent to the discriminant at that point, then equation (1) is smoothly equivalent to $dx^2 + (-x + \lambda y^2)dy^2 = 0$ with $\lambda \neq 0, \frac{1}{16}$ ([8]); the corresponding point in the plane is called a *folded singularity* of the BDE. There are three topological models: a *folded saddle* if $\lambda < 0$, a *folded node* if $0 < \lambda < \frac{1}{16}$ and a *folded focus* if $\frac{1}{16} < \lambda$. These occur when the lifted field ξ has a saddle, node or focus, respectively (see Figure 1 and [8]).

A solution curve of (1) has an *inflection point* at the projection of a point on $\mathcal{M}$ where

$$\Omega = \Omega_y + q\Omega_x = 0. \tag{4}$$

There is a smooth curve of such points which is tangent to the discriminant curve at folded singularities of equation (1) ([5]).

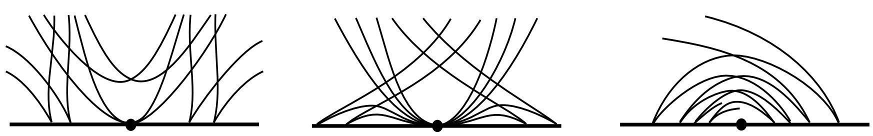

FIGURE 1. A folded saddle (left), node (center) and focus (right).

3. Characteristic curves on surfaces in $\mathbb{R}^4$

Let M be a regular surface in $\mathbb{R}^4$. For a given point $p \in M$, consider the unit circle in T_pM parametrized by $\theta \in [0, 2\pi]$. The curvature vectors $\eta(\theta)$ of the normal sections of M by the hyperplane $\langle\theta\rangle \oplus N_pM$ form an ellipse in the normal plane N_pM called the *curvature ellipse* and is the image this unit circle by a pair of quadratic forms

$$(Q_1, Q_2) = (ax^2 + 2bxy + cy^2, lx^2 + 2mxy + ny^2),$$

where a, b, c, l, m, n are the coefficients of the second fundamental form of M at p ([16]). Points on the surface are classified according to the position of the point p with respect to the ellipse (N_pM is viewed as an affine plane through p). The point p is called *elliptic/parabolic/hyperbolic* if it is inside/on/outside the ellipse at p, respectively.

Following the approach in [2], a binary form $Ax^2 + 2Bxy + Cy^2$ is represented by its coefficients $(A, B, C) \in \mathbb{R}^3$, there is a cone Γ given by $B^2 - AC = 0$ representing the perfect squares. If the forms Q_1 and Q_2 are independent, they determine a line in the projective plane $\mathbb{R}P^2$ and the cone a conic that we still denoted by Γ. This line meets the conic in $0/1/2$ points according as $\delta(p) < 0/ = 0/ > 0$, where

$$\delta(p) = (an - cl)^2 - 4(am - bl)(bn - cm).$$

A point p is elliptic/parabolic/hyperbolic if $\delta < 0/ = 0/ > 0$. The parabolic set is denoted by Δ-set. If Q_1 and Q_2 are dependent, the rank of the matrix $\begin{pmatrix} a & b & c \\ l & m & n \end{pmatrix}$ is 1 (provided either of the forms is non-zero); the corresponding points on the surface are referred to as inflection points. There is an action of $\mathbf{GL}(2,\mathbb{R}) \times \mathbf{GL}(2,\mathbb{R})$ on pairs of binary forms. The orbits of this action are as follows (see for example [13]):

(x^2, y^2)	hyperbolic point
$(xy, x^2 - y^2)$	elliptic point
(x^2, xy)	parabolic point
$(x^2 \pm y^2, 0)$	inflection point
$(x^2, 0)$	degenerate inflection point
$(0, 0)$	degenerate inflection point.

The *asymptotic directions* (labelled by conjugate directions in [16]) are defined as the directions along θ such that the curvature vector $\eta(\theta)$ is tangent to the curvature ellipse (see also [17]). A curve on M whose tangent direction at each point is an asymptotic direction is called an *asymptotic curve*. The asymptotic curves of M are solution curves of the BDE

$$\Psi(x, y, q) = (am - bl)q^2 + (an - cl)q + (bn - cm) = 0, \tag{5}$$

([17, 16]). We call this equation the asymptotic BDE. The discriminant of the BDE (5) is the Δ-set and is a generic smooth curve on surface. Away from inflection points, at a hyperbolic (resp. parabolic or elliptic) point there are 2 (resp. 1 or 0) asymptotic directions at that point.

Since we do not distinguish between a BDE and its non-zero multiples, at each point (x, y), we can view a BDE (1) as a quadratic form in dx, dy and represent it by the point $(A : 2B : C)$ in $\mathbb{R}P^2$. To a point $(A : 2B : C)$ is associated a polar line with respect to the conic Γ. Three points in $\mathbb{R}P^2$ form a self-polar triangle if the polar of any of the three points is the line through the remaining two points. In our case the point $(A : 2B : C)$ is parametrized by $(x, y) \in U$ (for more details, see [15] chapter 7). The metric on M is given by $ds^2 = X_1dx^2 + 2X_2dxdy + X_3dy^2$ and determines a point $(X_1 : 2X_2 : X_3)$ in the projective plane. It turns out that the polar line of $(X_1 : 2X_2 : X_3)$ consists of BDEs whose solutions are orthogonal curves on M ([4, 23]). This polar line intersects the polar line of the asymptotic BDE (5) at a unique point (P) which represent a BDE, called the BDE of the lines of principal curvature ([23]). The BDEs (A) of the asymptotic curves and the BDE (P) determine a unique BDE (C), the characteristic BDE, such the three of them form a self-polar triangle in the projective plane. In fact, (C) is the Jacobian of (A) and (P) ([23]), and if the surface M is parametrized by $\phi(x, y)$, the characteristic BDE is given by

$$\begin{aligned} \Phi(x, y, q) =& (L(GL - EN) - 2M(FL - EM))q^2 + 2((M(EN + GL) - 2LNF))q \\ &+ 2M(GM - FN) - N(GL - EN) = 0, \end{aligned} \tag{6}$$

where $E = \langle \phi_x, \phi_x \rangle$, $F = \langle \phi_x, \phi_y \rangle$, $G = \langle \phi_y, \phi_y \rangle$, $L = (am - bl)$, $2M = (an - cl)$ and $N = (bn - cm)$. A *characteristic curve* is the a curve on M whose tangent direction at each point is a characteristic direction. The discriminant curve of the BDE (6) coincides with the parabolic set. At elliptic point there are two characteristic directions and at each parabolic point there is one.

The asymptotic directions can be described via the singularities of the projections of M to 3-spaces (see [2]). Consider the family of orthogonal projections given by

$$\begin{array}{rcl} P : M \times S^3 & \to & TS^3 \\ (p, \mathbf{u}) & \mapsto & (\mathbf{u}, p - \langle p, \mathbf{u} \rangle \mathbf{u}). \end{array}$$

For $\mathbf{u}$ fixed, the projection can be viewed, locally at a point p, as a map germ

$$P_{\mathbf{u}} : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0).$$

(Two germs f and g are said to be $\mathcal{A}$-equivalent and write $f \sim_{\mathcal{A}} g$, if $g = k \circ f \circ h^{-1}$ for some germs of diffeomorphisms h and k of, respectively, the source and target.) The generic $\mathcal{A}$-singularities of $P_{\mathbf{u}}$ are those that have $\mathcal{A}_e$-codimension ≤ 3 (which is the dimension of S^3), see Table 1 and Table 2.

Table 1. The generic local singularities of orthogonal projections of M to 3-spaces ([18]).

Name	Normal form	$\mathcal{A}_e$-codimension
Immersion	$(x, y, 0)$	0
Cross-cap	(x, y^2, xy)	0
$B_k^{\pm}$	$(x, y^2, x^2y \pm y^{2k+1}), k = 2, 3$	k
$S_k^{\pm}$	$(x, y^2, y^3 \pm x^{k+1}y), k = 1, 2, 3$	k
$C_k^{\pm}$	$(x, y^2, xy^3 \pm x^k y), k = 3$	k
H_k	$(x, xy + y^{2k+2}, y^3), k = 2, 3$	k
$P_3(c)$	$(x, xy + y^3, xy^2 + cy^4),\ c \neq 0, \frac{1}{2}, 1, \frac{3}{2}$	3^*

* The codimension of $P_3(c)$ is that of its stratum.

Table 2. Bi-germs of $\mathcal{A}_e$-codimension 2 of orthogonal projections of M to 3-spaces ([14]).

Name	Normal Form	$\mathcal{A}_e$-codimension
$[A_2]$	$(x, y, 0; X, Y, X^2 + Y^3)$	2
$(A_0S_0)_2$	$(x, y, 0; Y^2, XY + Y^5, X)$	2
$A_0S_1^{\pm}$	$(x, y, 0; Y^3 \pm X^2Y, Y^2, X)$	2
$A_0S_0\|A_1^{\pm}$	$(x, y, 0, X, XY, Y^2 \pm X^2)$	2

For a complete table see [14].

The projection $P_{\mathbf{u}}$ is singular at p if and only if $\mathbf{u} \in T_pM$. The singularity is a cross-cap unless $\mathbf{u}$ is an asymptotic direction at p. The $\mathcal{A}_e$-codimension 2 singularities occur on curves on a generic surface and the $\mathcal{A}_e$-codimension 3 ones occur at special points on these curves. When projecting the surface along an asymptotic direction at a parabolic point, the projection may have a $P_3(c)$-singularity ([3, 10]). If we call S_2-curve (resp. B_2, $(A_0S_0)_2$, $A_0S_1^{\pm}$, $A_0S_0|A_1^{\pm}$-curve) the closure of the set of points p on M for which there exists a projection $P_{\mathbf{u}}$ having an S_2 (resp. B_2, $(A_0S_0)_2$, $A_0S_1^{\pm}$, $A_0S_0|A_1^{\pm}$)-singularity at p, then these curves meet the parabolic set tangentially at a $P_3(c)$-singularity (see Proposition 3.1 and for a complete proof [9, 10]). At a $P_3(c)$-singularity the unique asymptotic (or characteristic) direction is tangent to the parabolic set. This point is called a $P_3(c)$-point and is also a point where the asymptotic (or characteristic) curves have a folded singularity (see §2).

Throughout this paper, we consider the family of orthogonal projections P where the map $P_{\mathbf{u}}$ has $P_3(c)$-point. We can take $\mathbf{u} = (0, 1, 0, 0)$ as an asymptotic direction. We choose local coordinates at p such that the surface is given in Monge form

$$\phi(x, y) = (x, y, f^1(x, y), f^2(x, y))$$

where $(j^1f^1(0,0), j^1f^2(0,0)) = (0,0)$ and with 2-jet of $(f^1, f^2) = (Q_1, Q_2)$. We denote by (X, Y, Z, W) the coordinates in $\mathbb{R}^4$ and we parametrize the directions near $\mathbf{u}$ by $(u, 1, v, w)$. Instead of the orthogonal projection to the plane $(u, 1, v, w)^\perp$, we project to the fixed plane (X, Z, W). The modified family of projections is given by

$$\begin{array}{rccl} P: & (\mathbb{R}^2 \times \mathbb{R}^3, 0) & \to & (\mathbb{R}^3, 0) \\ & ((x, y), (u, v, w)) & \mapsto & P_{\mathbf{u}} = (x - uy, f^1(x, y) - vy, f^2(x, y) - wy), \end{array}$$

with $P_0(x, y) = (x, f^1(x, y), f^2(x, y))$. As the $P_3(c)$-point belongs to Δ-set and if we denote by $o(k)$ the terms of order greater than k in $x_1, \ldots, x_r$, then we can take $(Q_1, Q_2) = (x^2, xy)$ and write

$$\begin{aligned} f^1(x,y) &= x^2 + \textstyle\sum_{i=0}^3 a_{3i}x^{3-i}y^i + \sum_{i=0}^4 a_{4i}x^{4-i}y^i + \sum_{i=0}^5 a_{5i}x^{5-i}y^i + o(5), \\ f^2(x,y) &= xy + \textstyle\sum_{i=0}^3 b_{3i}x^{3-i}y^i + \sum_{i=0}^4 b_{4i}x^{4-i}y^i + \sum_{i=0}^5 b_{5i}x^{5-i}y^i + o(5). \end{aligned} \tag{7}$$

The 2-jet of the coefficients of a, b, c, l, m, and n of (Q_1, Q_2) are given as follows

$$\begin{array}{rcl} a & = & \frac{1}{2}f^1_{xx} = 1 + 3a_{30}x + a_{31}y + 6a_{40}x^2 + 3a_{41}xy + a_{42}y^2, \\ b & = & \frac{1}{2}f^1_{xy} = a_{31}x + a_{32}y + \frac{3}{2}a_{41}x^2 + 2a_{42}xy + \frac{3}{2}a_{43}y^2, \\ c & = & \frac{1}{2}f^1_{yy} = a_{32}x + 3a_{33}y + a_{42}x^2 + 3a_{43}xy + 6a_{44}y^2, \\ l & = & \frac{1}{2}f^2_{xx} = 3b_{30}x + b_{31}y + 6b_{40}x^2 + 3b_{41}xy + b_{42}y^2, \\ m & = & \frac{1}{2}f^2_{xy} = \frac{1}{2} + b_{31}x + b_{32}y + \frac{3}{2}b_{41}x^2 + 2b_{42}xy + \frac{3}{2}b_{43}y^2, \\ n & = & \frac{1}{2}f^2_{yy} = b_{32}x + 3b_{33}y + b_{42}x^2 + 3b_{43}xy + 6b_{44}y^2. \end{array}$$

The curve formed by the locus of geodesic inflection points of the characteristic (resp. asymptotic) curves we call *characteristic inflection curve* (resp. *flecnodal curve* (see [9, 10])) and denoted by C_h-curve (resp. F_l-curve). We have the following result.

Proposition 3.1. *Let M be a surface in $\mathbb{R}^4$ given in Monge form as in (7), and suppose that the origin is a $P_3(c)$-point. Then we have the following initial terms of the following curves:*

a) *the parabolic curve (Δ-curve):*

$$x = \frac{6a_{32}b_{33} - 9b_{33}^2 - 6a_{44}}{a_{32}}y^2 + o(2).$$

b) *the B_2-curve:*

$$x = \frac{2(3a_{32}^3b_{33} - 4a_{32}^2b_{33}^2 - 3a_{44}a_{32}^2 - 8a_{44}a_{32}b_{33} + 12a_{44}b_{33}^2 + 8a_{44}^2)}{a_{32}(a_{32} - 2b_{33})^2}y^2 + o(2).$$

c) *the S_2-curve:*

$$x = \frac{6(a_{32}^3b_{33} + 48a_{32}^2b_{33}^2 - 72a_{32}b_{33}^3 - a_{44}a_{32}^2 - 72a_{44}a_{32}b_{33} + 36a_{44}b_{33}^2 + 24a_{44}^2)}{a_{32}(a_{32} + 6b_{33})^2}y^2 + o(2).$$

d) *the $A_0S_1^\pm$-curve:*

$$x = \frac{3a_{32}^2b_{33}^2 - 4a_{32}a_{44}b_{33} + 3a_{44}b_{33}^2 + 2a_{44}^2}{a_{32}(4a_{32}b_{33} - 4b_{33}^2 - 3a_{44})}y^2 + o(2).$$

e) *the $(A_0S_0)_2$-curve:*

$$x = \frac{12a_{32}b_{33} - 9b_{33}^2 - 6a_{44}}{a_{32}}y^2 + o(2).$$

f) *the $A_0S_0|A_1^\pm$-curve:*

$$x = \frac{3a_{32}^2b_{33}^2 - 16a_{32}a_{44}b_{33} + 12a_{44}b_{33}^2 + 8a_{44}^2}{4(a_{32}b_{33} - b_{33}^2 - a_{44})a_{32}}y^2 + o(2).$$

g) *the F_l-curve:*

$$x = \frac{6(a_{32}b_{33} - a_{44})(24a_{32}b_{33} - 36b_{33}^2 + a_{32}^2 - 24a_{44})}{a_{32}(6b_{33} - a_{32})^2}y^2 + o(2).$$

h) *the C_h-curve:*

$$x = \frac{6(a_{32}b_{33} - a_{44} - 3b_{33}^2)(36b_{33}^2 - 24a_{32}b_{33} + a_{32}^2 + 24a_{44})}{a_{32}(a_{32} + 6b_{33})^2}y^2 + o(2).$$

All the above curves are tangent to the parabolic curve at the $P_3(c)$-point and any two have contact of order 2 at the origin.

Proof. The singularity of the projection P_0 is $\mathcal{A}$-equivalent to a $P_3(c)$-singularity when $a_{33} = 0$, $a_{32}, a_{44}, b_{33} \neq 0$, $a_{44}/(a_{32}b_{33}) \neq 0, 1/2, 1, 3/2$, and $5a_{32}b_{33} - 6b_{33}^2 - 4a_{44} \neq 0$ ([9, 21]). All the curves Δ, B_2, S_2, $A_0S_1^{\pm}$, $(A_0S_0)_2$, $A_0S_0|A_1^{\pm}$ are determined in [9, 10] using adjacencies of the $P_3(c)$-singularity.

The curves in g) and h) are obtained using the asymptotic and characteristic BDEs. In fact, the 2-jet of the characteristic BDE (6) is written as

$$\begin{aligned} j^2\Phi =& q^2 + (2b_{32}x + 6b_{33}y)q + (2a_{32}b_{32} - 6a_{31}b_{33} + a_{31}a_{32} + 12b_{32}b_{33} + 3a_{43})xy + a_{32}x \\ & (a_{42} + a_{32}^2 + 2b_{32}^2 - 2a_{31}b_{32} + 3a_{30}a_{32} + 4b_{31}a_{32})x^2 + (18b_{33}^2 - 6a_{32}b_{33} + 6a_{44})y^2. \end{aligned} \tag{8}$$

Thus, we can write by the implicit function theorem

$$x = \frac{6(-a_{44} + a_{32}b_{33} - 3b_{33}^2)}{a_{32}}y^2 - (6b_{33}a_{32})yq - \frac{1}{a_{32}}q^2 + o(2).$$

Substituting the expression of x into $\Phi_y + q\Phi_x = 0$ we obtain

$$\Big(18b_{33}^2 - 6a_{44} + 6a_{32}b_{33}\Big)y + \Big(3b_{33} + \frac{1}{2}a_{32}\Big)q + o(1) = 0.$$

Again, solving implicitly the last equality, we get

$$q = \frac{12(a_{32}b_{33} - a_{44} - 3b_{33}^2)}{(6b_{33} + a_{32})}y + o(1).$$

Substituting q in the expression of x gives the 2-jet of the characteristic inflection curve. The 2-jet of the flecnodal curve is also determined in [9, 10] using the same approach above for the asymptotic BDE. □

We denote the tangent lines to the Legendrian lifts of the parabolic, B_2, S_2, flecnodal, characteristic inflection, $(A_0S_0)_2$, A_0S_1, and $A_0S_0|A_1^{\pm}$ curves in PT^*M at a $P_3(c)$-point by l_P, l_B, l_S, l_F, l_C, $l_{s_{02}}$, l_{s_1}, and $l_{s_{01}}$, respectively. We denote by l_g the contact element at the $P_3(c)$-point (i.e., the vertical line in the contact plane at that point).

Remark 3.1. By Proposition 3.1, l_P, l_S , l_B , l_F, and l_C are distinct unless

$$(5a_{32}b_{33} - 6b_{33}^2 - 4a_{44}) = 0.$$

This condition is precisely that for the family of the orthogonal projections P to fail to be a versal unfolding of a $P_3(c)$-singularity ([10]). In a generic one-parameter family of surfaces case (see [3, Proposition 4.3]) there are double $P_3(c)$-points.

Theorem 1. *At a generic $P_3(c)$-point, the 2-jet of the C_h-curve is projectively invariant.*

Proof. The cross-ratio of lines l_P, l_g, l_S, l_C is given by

$$(l_P, l_g : l_S, l_C) = \frac{c_S - c_P}{c_C - c_P} = \frac{\frac{9(5a_{32}b_{33}-6b_{33}^2-4a_{44})^2}{a_{32}(a_{32}+6b_{33})^2}}{-\frac{9(5a_{32}b_{33}-6b_{33}^2-4a_{44})^2}{a_{32}(a_{32}+6b_{33})^2}} = -1,$$

where c_P, c_S, and c_C are the coefficients of order 2 of the parabolic curve, S_2-curve and C_h-curve, respectively. The result follows from the fact that the 2-jet of the C_h-curve depends on the S_2-curve and parabolic curve which are projective invariants ([9, 10]). □

Proposition 3.2. *The topological type of the singularity of the characteristic BDE at a $P_3(c)$-point is invariant under projective transformations.*

Proof. The singularity type is determined by equation (8). It is given by the type of the singularity of the lifted field ξ: a saddle, node or focus. Since a $P_3(c)$-point is a folded singularity, the characteristic BDE is locally smoothy equivalent to $dx^2 + (-x + \lambda y^2)dy^2 = 0$, where

$$\lambda = -\frac{3}{2}\frac{(5a_{32}b_{33} - 6b_{33}^2 - 4a_{44})}{a_{32}^2}$$

determines the topological type of singularity if and only if $\lambda \neq 0, \frac{1}{16}$ (see [5]). Observe that the coefficients a_{44} and b_{33} of λ depend on a combination of the cross-ratios $\rho_1 = (l_P, l_B : l_S, l_F)$, $\rho_2 = (l_P, l_g : l_{s_{01}}, l_{s_{02}})$, $\rho_3 = (l_P, l_g : l_{s_1}, l_{s_{02}})$, and a_{32}. In fact,

$$\begin{aligned}
\rho_1 &= \frac{a_{32} - 3b_{33}}{a_{32} - 6b_{33}},\\
\rho_2 &= -\frac{21a_{32}^2b_{33}^2 - 60a_{32}b_{33}^3 + 36b_{33}^4 - 32a_{32}a_{44}b_{33} + 48a_{44}b_{33}^2 + 16a_{44}^2}{24a_{32}(a_{32}b_{33} - b_{33}^2 - a_{44})b_{33}},\\
\rho_3 &= -\frac{21a_{32}^2b_{33}^2 - 60a_{32}b_{33}^3 + 36b_{33}^4 - 32a_{32}a_{44}b_{33} + 48a_{44}b_{33}^2 + 16a_{44}^2}{6(4a_{32}b_{33} - 4b_{33}^2 - 3a_{44})a_{32}b_{33}}.
\end{aligned}$$

Using ρ_1 we get $b_{33} = \dfrac{1}{3}\dfrac{(\rho_1 - 1)a_{32}}{2\rho_1 - 1}$. From ρ_2 and ρ_3 it follows that

$$6a_{32}b_{33}((3\rho_3 - 4\rho_2 + 1)a_{44} + 4b_{33}(6\rho_2 - \rho_3)(a_{32} - b_{33})) = 0.$$

Replacing b_{33} in the above equation, we obtain

$$a_{44} = \frac{4}{9}\frac{a_{32}^2(\rho_1 - 1)(\rho_2 - \rho_3)(5\rho_1 - 2)}{(2\rho_1 - 1)^2(4\rho_2 - 3\rho_3 - 1)}.$$

Since $a_{32} \neq 0$, substituting b_{33} and a_{44} into λ, shows that the type of singularity of the characteristic BDE depends only on the values of the cross-ratios ρ_1, ρ_2 and ρ_3, all of which are projective invariants. □

At a $P_3(c)$-point, the 4-jet of the parametrization $\phi(x, y) = (x, y, f^1(x, y), f^2(x, y))$ of the surface M is equivalent, by projective transformations, to the normal form

$$(x, y, x^2 + xy^2 + \alpha y^4, xy + \beta y^3 + \psi), \tag{9}$$

where $6\beta^2 + 4\alpha - 15\beta + 5 \neq 0$, $\alpha \neq 0, 1/2, 1, 3/2$, and ψ is a polynomial of degree 4 (see [11]).

According to Proposition 3.2, we can use the normal form (9) to present the topological type of the singularity of the characteristic BDE at a $P_3(c)$-point. In [9, 10] we showed that α and β in (9) are also projective invariants described as functions of ρ_1, ρ_2 and ρ_3. This allows us to recalculate the expressions of the curves in Proposition 3.1 in terms of α and β. In fact, consider representing M locally as a surface $\bar{M}$ in $\mathbb{P}^4$, given in the affine chart $\{[x : y : z : w : 1]\}$ in Monge form $[x : y : f^1(x, y) : f^2(x, y) : 1]$. We can take (f^1, f^2) with 4-jet as in (9) and use the equations of the curves in Proposition 3.1 with $a_{32} = 1$, $a_{44} = \alpha$, and $b_{33} = \beta$.

Theorem 2. *At a $P_3(c)$-point, the characteristic BDE has a folded singularity if and only if $\gamma = -(5\beta - 6\beta^2 - 4\alpha) \neq 0, \frac{1}{24}$. The singularity is a folded saddle if $\gamma < 0$, a folded node if $0 < \gamma < \frac{1}{24}$, and folded focus if $\gamma > \frac{1}{24}$.*

Proof. The proof follows from Proposition 3.2. Note that $\lambda = -\frac{3}{2}(5\beta - 6\beta^2 - 4\alpha) \neq 0, \frac{1}{16}$. Thus the singularity of the characteristic BDE is determined by values of γ. $\square$

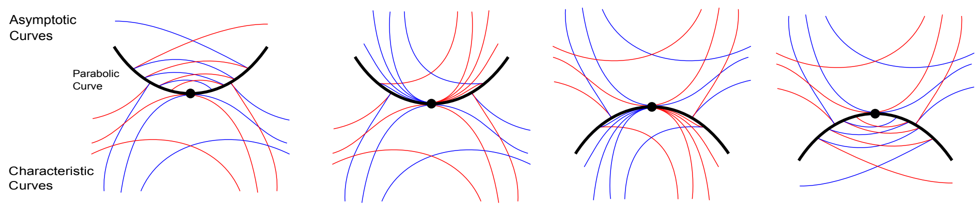

FIGURE 2. The asymptotic and characteristic curves at a $P_3(c)$-point. $\gamma < -1/24$ (first); $-1/24 < \gamma < 0$ (second); $0 < \gamma < 1/24$ (third) and $\gamma > 1/24$ (fourth).

Remark 3.2. The types of the singularities of the asymptotic and characteristic BDE are not related ([23]). However, for surfaces in $\mathbb{R}^4$, thanks to Theorem 2, the types of these singularities have opposite indices at a $P_3(c)$-point, that is, on one side of the parabolic curve we have a folded saddle and on the other a folded node or focus or vice-versa. This also happens for surfaces in $\mathbb{R}^3$ at cusps of Gauss [4]. Figure 2 shows the generic configurations of asymptotic and characteristic curves at a folded singularity.

Following the approach in [20], we denoted by ρ_c the cross-ratio $(l_P, l_g : l_C, l_B)$ and call it the *characteristic cross-ratio*. It can be written in terms of the coefficients of normal form (9) as follows

$$\rho_c = -\frac{9(2\beta - 1)^2}{(1 + 6\beta)^2}.$$

As the generic relative positions of the relevant curves at a $P_3(c)$-point are determined by their 2-jets, we can give the their relative positions in terms of the values of ρ_c. In what follows, we present the relative positions of the curves Δ, B_2, S_2, F_l, and C_h.

Theorem 3. *Let c_P, c_B, c_S, c_F, and c_C be the coefficients of order 2 associated to curves Δ, B_2, S_2, F_l, and C_h, respectively, at a $P_3(c)$-point of a smooth surface in $\mathbb{R}^4$. Then there are 4 possible relative positions of these curves depending on the values of ρ_c:*

(i) *If $\rho_c < -9$, then $c_C < c_P < c_B < c_F < c_S$*
(ii) *If $-9 < \rho_c < -1$, then $c_C < c_P < c_B < c_S < c_F$*
(iii) *If $-1 < \rho_c < -1/9$, then $c_C < c_P < c_S < c_B < c_F$*
(iv) *If $-1/9 < \rho_c < 0$, then $c_C < c_P < c_S < c_F < c_B$.*

Proof. The proof follows from Proposition 3.1 with $a_{32} = 1$, $a_{44} = \alpha$ and $b_{33} = \beta$. It is easy to check that the coefficients c_P, c_B, c_S, c_F, and c_C satisfy $c_C < c_P < c_B, c_S, c_F$ for all value of

α, β. Furthermore,

$$c_B - c_S = \frac{8(6\beta-1)(4\alpha+6\beta^2-5\beta)^2}{(2\beta-1)^2(1+6\beta)^2},$$
$$c_B - c_F = \frac{8(3\beta-1)(4\alpha+6\beta^2-5\beta)^2}{(2\beta-1)^2(1+6\beta)^2},$$
$$c_S - c_F = -\frac{216\beta(4\alpha+6\beta^2-5\beta)^2}{(6\beta-1)^2(1+6\beta)^2}.$$

Since $4\alpha+6\beta^2-5\beta \neq 0$ (see Remark 3.1), we have $c_B > c_S$ if and only if $\beta > 1/6$; $c_B > c_F$ if and only if $\beta > 1/3$; and $c_S > c_F$ if and only if $\beta < 0$. This and the fact that $\rho_c = -\frac{9(2\beta-1)^2}{(1+6\beta)^2}$, for each value of β we obtain the desired result. □

Theorem 4. *With notation in Theorem 3, consider the 2-jets of curves Δ, S_2, and C_h represented by the parabolas $x = c_P \cdot y^2$, $x = c_S \cdot y^2$, and $x = c_C \cdot y^2$, respectively. There are four possible configurations for Δ, S_2, and C_h and these are determined by α and β. They are described by Figure 3.*

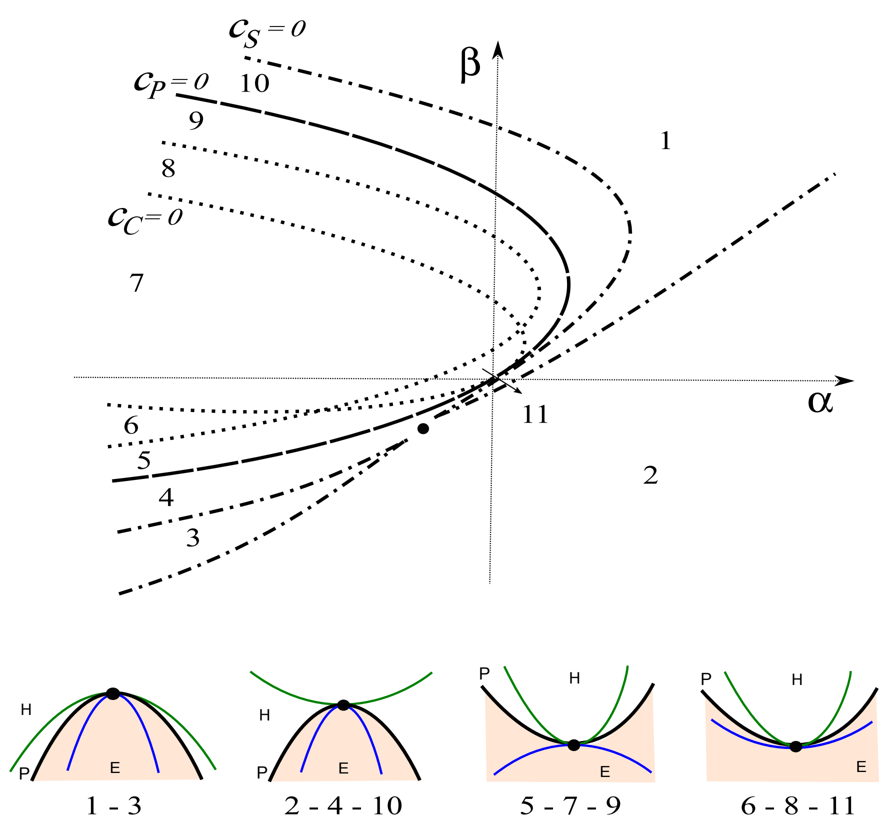

FIGURE 3. Partition of (α, β)-plane. The bottom pictures are the configurations of Δ-curve (black), S_2-curve (green), and C_h-curve (blue) at a $P_3(c)$-point. H, P, and E mean hyperbolic, parabolic, and elliptic region, respectively.

Proof. Consider the 2-jets of the parametrisation of the Δ-curve, S_2-curve, C_h-curve with the second order coefficients given by

$$\begin{aligned} c_P &= 3(2\beta - 3\beta^2 - 2\alpha), \\ c_S &= \frac{6(36\alpha\beta^2 - 72\beta^3 + 24\alpha^2 - 72\alpha\beta + 48\beta^2 - \alpha + \beta)}{(1+6\beta)^2}, \\ c_C &= \frac{6(-3\beta^2 - \alpha + \beta)(36\beta^2 + 24\alpha - 24\beta + 1)}{(1+6\beta)^2}. \end{aligned}$$

The generic configurations of these curves occur when α and β avoid the set

$$\{c_P = 0\} \cup \{c_S = 0\} \cup \{c_C = 0\}.$$

The conditions $c_P = 0$, $c_S = 0$, and $c_C = 0$ determine curves in (α, β)-plane represented by dashed curve, dot-dashed curve, and doted curve in Figure 3, respectively. Then the (α, β)-plane is partitioned into 11 open regions. There are four different configurations of the Δ-curve, S_2-curve, and C_h-curve that are given at the bottom of Figure 3. For instance, in regions 1 and 3, the configurations of the Δ-curve, S_2-curve, C_h-curve are described in the first bottom picture; in regions 2, 4 and 10, the configurations are described in the second bottom picture and so on. □

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References

[1] T. Banchoff, T. Gaffney and C. McCrory, Cusps of Gauss mappings. *Research Notes in Mathematics* 55 (1982), Pitman (Advanced Publishing Program).

[2] J. W. Bruce and A. C. Nogueira, Surfaces in $\mathbb{R}^4$ and duality. *Quart. J. Math. Oxford Ser.* Ser. (2) 49 (1998), 433–443.

[3] J. W. Bruce and F. Tari, Families of surfaces in $\mathbb{R}^4$. *Proc. Edinb. Math. Soc.* 45 (2002), 181–203.

[4] J. W. Bruce and F. Tari, Dupin indicatrices and families of curve congruence. *Trans. Amer. Math. Soc.* 357 (2004), 267–285. DOI: 10.1090/s0002-9947-04-03497-x

[5] J. W. Bruce and F. Tari, Duality and implicit differential equations. *Nonlinearity.* 13 (2000), 791–811. DOI: 10.1088/0951-7715/13/3/315

[6] M. Cibrario, Sulla reduzione a forma delle equatione linearialle derviate parziale di secondo ordine di tipo misto, *Accamdemia di Scienze e Lettrere, Instituto Lomardo Redicconti.* 65 (1932), 889–906. DOI: 10.1007/bf03017728

[7] L. Dara, Singularites generiques des equations differentielles multi-forms, *Bol. Soc. Bras. Mat.* No. 6 (1975) 95–129. DOI: 10.1007/bf02584779

[8] A. A. Davydov, *Qualitative control theory.* Translations of Mathematical Monographs 142, AMS, Providence, RI, 1994.

[9] J. L. Deolindo-Silva, Cr-invariantes para surperfícies em $\mathbb{R}^4$. Doctoral Thesis, Instituto de Ciências Matemáticas e de Computação, University of São Paulo, São Carlos, 2016. DOI: 10.11606/T.55.2016.tde-03102016-104837

[10] J. L. Deolindo-Silva, Cross-ratio invariants for surfaces in 4-space (2018). arχiv: 1807.11133

[11] J. L. Deolindo-Silva and Y. Kabata, Projective classification of jets of surfaces in 4-space. *Hiroshima Mathematical Journal*, 49 (2019) 35–46. DOI: 10.32917/hmj/1554516037

[12] R. A. Garcia and J. Sotomayor, Harmonic mean curvature lines on surfaces immersed in $\mathbb{R}^3$. *Bull. Braz. Math. Soc.*, 34 (2) (2003), 303–331.

[13] C. G. Gibson, *Singular points of smooth mappings.* Pitman Research Notes in Mathematics, vol. 25 (1979).

[14] C. A. Hobbs and N. P. Kirk, On the classification and bifurcation of multi-germs of maps from surfaces to 3-space, Math. Scand. 89 (2001), 57–96

[15] S. Izumyia, M. C. Romero-Fuster, M. A. S. Ruas and F. Tari, *Differential geometry from a singularity theory viewpoint.* World Scientific, 2016. DOI: 10.1142/9108

[16] J. A. Little, On the singularities of submanifolds of heigher dimensional Euclidean space. *Annli Mat. Pura et Appl.* (4A) 83 (1969), 261–336.

[17] D. K. H. Mochida, M. C. Romero-Fuster and M. A. S. Ruas, The geometry of surfaces in 4-space from a contact viewpoint. *Geometriae Dedicata* 54 (1995), 323–332. DOI: 10.1007/bf01265348

[18] D. M. Q. Mond, On the classification of germs of maps from $\mathbb{R}^2$ to $\mathbb{R}^3$. *Proc. London Math. Soc.* 50 (1985), 333–369.

[19] J. J. Nuño-Ballesteros and F. Tari, Surfaces in $\mathbb{R}^4$ and their projections to 3-spaces. *Roy. Proc. Edinburgh Math. Soc.* 137A (2007), 1313–1328.

[20] J. Oliver, On the characteristic curves on a smooth surface. *J. London Math. Soc.* (2) 83 (2011), 755–767.

[21] R. Oset-Sinha and F. Tari, Projections of surfaces in $\mathbb{R}^4$ to $\mathbb{R}^3$ and the geometry of their singular images. *Rev. Mat. Iberoam.* 31, 1 (2015), 33–50. DOI: 10.4171/rmi/825

[22] F. Tari, Pairs of foliations on surfaces. *Real and Complex Singularities*, Edited by M. Manoel, M. C. Romero Fuster, C. T. C. Wall. London Mathematical Society Lecture Notes Series 380 (2010), 305–337. DOI: 10.1017/cbo9780511731983.023

[23] F. Tari, Self-adjoint operators on surfaces in $\mathbb{R}^n$. *Differential Geom. Appl.* 27 (2009), 296–306.

[24] R. Uribe-Vargas, A projective invariant for swallowtails and godrons, and global theorems on the flecnodal curve. *Mosc. Math. J.* 6 (2006), 731–768. DOI: 10.17323/1609-4514-2006-6-4-731-768

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