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LOOPS IN GENERALIZED REEB GRAPHS ASSOCIATED TO STABLE CIRCLE-VALUED FUNCTIONS

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Abstract. Let N be a smooth compact, connected and orientable 2-manifold with or without boundary. Given a stable circle-valued function $\gamma : N \to S^1$, we introduced a topological invariant associated to γ, called generalized Reeb graph. It is a generalized version of the classical and well known Reeb graph. The purpose of this paper is to investigate the number of loops in generalized Reeb graphs associated to stable circle-valued functions $\gamma : N \to S^1$. We show that the number of loops depends on the genus of N, the number of boundary components of N, and the number of open saddles of γ. In particular, we show a class of functions whose generalized Reeb graphs have the maximal number of loops.

1. Introduction

The Reeb graph was introduced by Reeb in [13] and it is well known that it is a complete topological invariant for Morse functions from S^2 to $\mathbb{R}$, where S^2 is the standard sphere in $\mathbb{R}^3$ (see [1, 14]).

Although originally introduced as a tool in Morse theory, the Reeb graphs have several applications in Computational Geometry, Computer Graphics, Engineering, Applied Mathematics, etc. A more extensive discussion of Reeb graphs and their variations in geometric modeling and visualization applications can be found in [4, 7].

An interesting problem related to Reeb graphs in the context of computational geometry is to investigate the number of loops of such graphs. The number of loops in a Reeb graph of a Morse function over a 2-manifold (orientable or non-orientable) with and without boundary was investigated in [5]. Later, some of these results were generalized in [8].

In this paper we study a similar problem. We investigate the number of loops in a graph associated to a stable circle-valued function $\gamma : N \to S^1$, where N is a smooth compact, connected and orientable 2-manifold with or without boundary and S^1 is the standard sphere in $\mathbb{R}^2$. The study of stable circle-valued functions was initiated by S.P. Novikov in the early 1980's related with a hydrodynamic problem [11, 12]. Today we can find applications and connections to many geometrical problems. Recently, an interesting connection with Singularity theory was obtained by the authors related to the topological classification of finitely determined map germs from $(\mathbb{R}^3, 0)$ to $(\mathbb{R}^2, 0)$ (see [2, 3]).

A stable circle-valued function is defined as follows:

Definition 1.1. Let N be a smooth compact, connected and orientable 2-manifold with boundary ∂N (including the case when $\partial N = \emptyset$), and let P be a smooth 1-manifold. We say that $\gamma : N \to P$ is *stable* if:

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(1) γ is Morse with distinct critical values;
(2) γ does not have critical points in ∂N;
(3) $\gamma|_{\partial N}$ is regular.

If $P = \mathbb{R}$ and $\gamma : N \to \mathbb{R}$ is stable, we can consider the following equivalence relation in N: given x, $y \in N$, $x \sim y$ if and only if $\gamma(x) = \gamma(y)$ and furthermore, x and y are in the same connected component of $\gamma^{-1}(\gamma(x))$. Reeb [13] showed that the quotient set $N/\sim$ admits a graph structure which is called *Reeb graph* associated to γ.

Intuitively, the Reeb graph associated to γ is obtained by contracting each connected component of the level curves of γ to points, where the vertices correspond to connected components of level curves containing critical points. Consider the following example, where $\gamma : N \to \mathbb{R}$ is the height function and N is a closed 2-manifold:

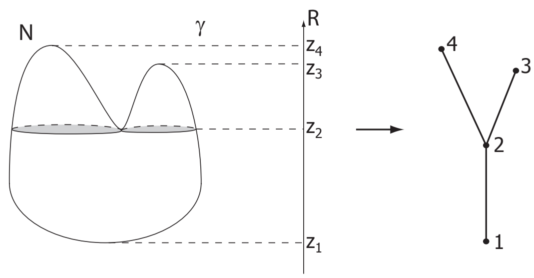

FIGURE 1. Reeb graph associated to the height function

When N is diffeomorphic to the sphere S^2, the Reeb graph is a tree (see [13]).

Since the Reeb graph gives the topological information about N, it is interesting to investigate the relation of its structure with topological elements such as Euler characteristic, Betti numbers, genus, etc. For instance, as motivation for this work, we can cite the following results:

Proposition 1.2. *([5, 8]) The Reeb graph of a Morse function over a connected orientable 2-manifold of genus g without boundary has g loops.*

Proposition 1.3. *([5, 8]) The Reeb graph of a Morse function over a connected orientable 2-manifold of genus g with $h \geqslant 1$ boundary components has between g and $2g + h - 1$ loops.*

Notice that the number of loops in the Reeb graph is given by the first Betti number of the graph, which is the rank of the first homology group. Also, it follows that the first Betti number of the 2-manifold N bounds from above the first Betti number of the graph, i.e.,

$$\text{number of loops} \leq \beta_1(\text{N}).$$

Figure 2 provides an example of a Reeb graph associated to $\gamma : N \to S^1$, where N is a 2-manifold with $h = 4$ boundary components and genus $g = 1$. The Reeb graph in this case has 3 loops, with $3 \leq 2g + h - 1 = 5 = \beta_1(N)$.

Remark 1.4. In the Reeb graph given in Figure 2, the slim traces indicate circle fibers and the bold traces arc fibers of γ, respectively. In Section 2, these different kind of traces in a Reeb graph are defined with more details.

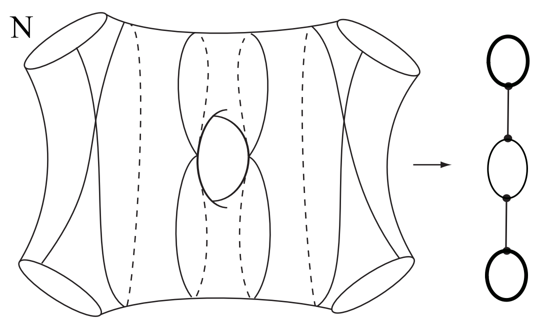

FIGURE 2. Reeb graph of a circle-valued Morse function γ.

In this work we obtain a similar relation to the number of loops, but now in a more general context, using stable circle-valued functions $\gamma : N \to S^1$ and the notion of generalized Reeb graphs.

2. The generalized Reeb graph

The generalized Reeb graph was introduced by the authors in [2, 3]. It is a generalized version of the classical Reeb graph, and it was inspired in Maksymenko's work [10].

Let $\gamma : N \to S^1$ be a stable circle-valued function, where N is a smooth connected, compact and orientable 2-manifold with or without boundary. Consider the following equivalence relation in N, analogous to the one given in the previous Section: given $x,\, y \in N$, $x \sim y$ if and only if $\gamma(x) = \gamma(y)$, where x and y are in the same connected component of $\gamma^{-1}(\gamma(x))$. The following result shows the structure of $N/\sim$:

Proposition 2.1. *Let N be a smooth connected, compact and orientable 2-manifold with or without boundary. Let $\gamma : N \to S^1$ be a stable circle-valued function. Then, the quotient space $N/\sim$ admits a graph structure as follows:*

1. *The vertices are the connected components of level curves $\gamma^{-1}(v)$, where $v \in S^1$ is a critical value;*
2. *Each edge is formed by points that correspond to connected components of level curves $\gamma^{-1}(v)$, where $v \in S^1$ is a regular value.*

Proof. Since γ is stable its critical points are isolated and N being compact, γ has a finite number of critical points. Moreover, N connected implies $N/\sim$ connected.

Let $v_1, \ldots, v_r$ be the critical values of γ. Then,

$$\gamma|N - \gamma^{-1}(\{v_1, \ldots, v_r\}) : N - \gamma^{-1}(\{v_1, \ldots, v_r\}) \to S^1 - \{v_1, \ldots, v_r\}$$

is regular, and the induced map

$$\tilde{\gamma} : (N - \gamma^{-1}(\{v_1, \ldots, v_r\}))/\sim \to S^1 - \{v_1, \ldots, v_r\}$$

is a local homeomorphism. Each connected component of $S^1 - \{v_1, \ldots, v_r\}$ is homeomorphic to an open interval, so each connected component of $(N - \gamma^{-1}(\{v_1, \ldots, v_r\}))/\sim$ is also homeomorphic to an open interval.

□

Remark 2.2. (1) *Let C_i be the connected components of ∂N, with $i = 1, \dots, n$. Then $\gamma|C_i : C_i \to S^1$ is a diffeomorphism.*
(2) *The level curves of γ intersect ∂N transversely.*

The possible topological types of the level curves of $\gamma : N \to S^1$ are:

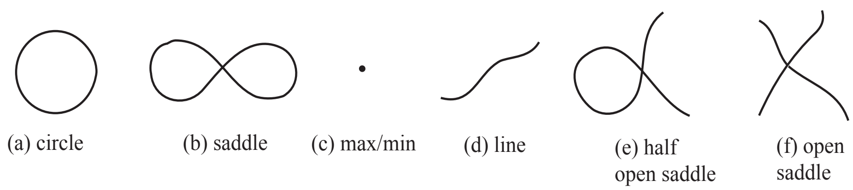

FIGURE 3. Topological types of level curves

By Remark 2.2 item (2), the level curves of γ that can intersect ∂N are only the types (d), (e) and (f). Furthermore, by item (1), each level curve of γ can intersects at most once a connected component C_i of ∂N, and these intersections happen in regular points.

The graph structure of $N/\sim$ given in Proposition 2.1 associated to a stable function $\gamma : N \to S^1$ will be denoted by Γ_γ. Each edge of Γ_γ can be of two types: one corresponds to connected components of circle type and will be denoted by a slim trace; another corresponds to connected components of interval type and will be denoted by a bold trace. We denote by Γ the subgraph of Γ_γ given by the slim edges with their respective vertices, and by Γ' the subgraph of Γ_γ given by the bold edges with their respective vertices (i.e., $\Gamma_\gamma = \Gamma \cup \Gamma'$).

Each vertex of the graph can be of six types, depending if the connected component has a maximum/minimum critical point, a saddle point, a half open saddle point, an open saddle point or a regular point. Then, the possible incidence rules of edges and vertices when $\gamma : N \to S^1$ is stable are given in Figure 4.

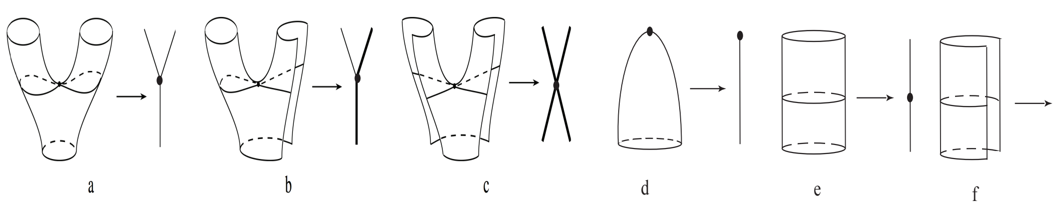

FIGURE 4. Incidence rules

We denote by S, S', S'', M, C and I the number of vertices of type (a) through (f), respectively.

Figure 5 represents some possible structures of the graph $N/\sim$ for stable maps from N to S^1. Notice that Γ and Γ' are not necessarily connected graphs.

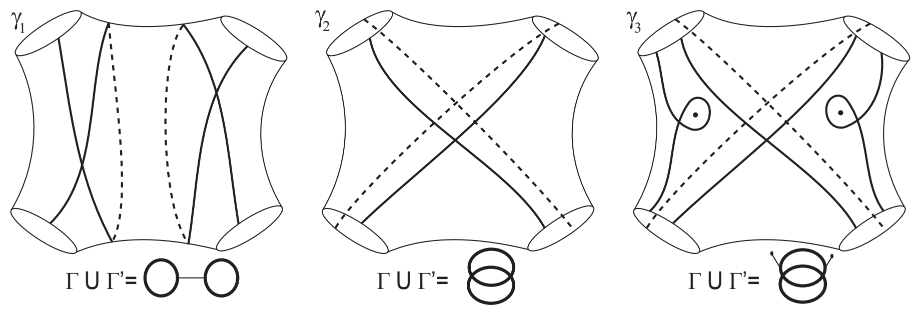

FIGURE 5. Graphs $N/\sim$ for stable maps $\gamma_i : N \to S^1$, $i = 1, 2, 3$

Let $v_1, \ldots, v_k \in S^1$ be the critical values of $\gamma : N \to S^1$. We choose a base point $v_0 \in S^1$ and an orientation. We can reorder the critical values such that $v_0 < v_1 < \ldots < v_k$ and we label each vertex with values $i \in \{1, \ldots, k\}$, if it corresponds to critical values v_i.

Definition 2.3. Let $\gamma : N \to S^1$ be a stable circle-valued function. The graph given by $N/\sim$ together with the types of edges and the labels of the vertices, as previously defined is called the *generalized Reeb graph* associated to γ.

Example 2.4. Consider the stable circle-valued functions $\gamma_1 : S^2 \to S^1$, $\gamma_2 : N \to S^1$, where N is a 2-manifold with boundary, as appear in Figure 5. The respective generalized Reeb graphs, Γ_{γ_1} and Γ_{γ_2}, are exhibited in Figure 6.

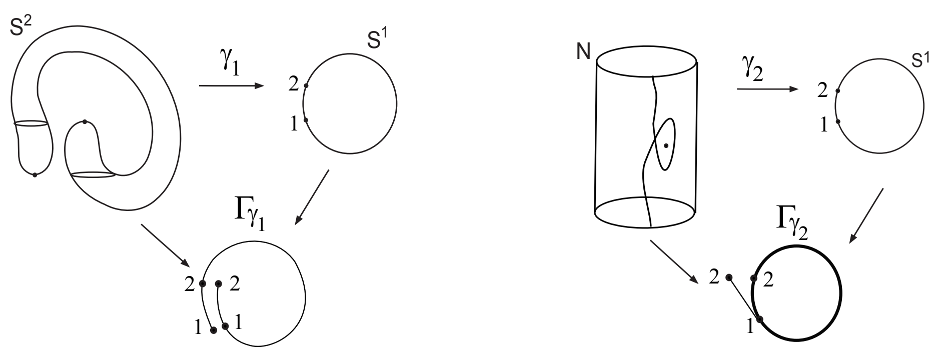

FIGURE 6. Generalized Reeb graphs

As previously stated, the main goal of this work is to investigate the number of loops in generalized Reeb graphs. This number is defined as follows:

Definition 2.5. Let Γ_γ be the generalized Reeb graph associated to the stable function $\gamma : N \to S^1$. The first Betti number of Γ_γ, denoted by $\beta_1(\Gamma_\gamma)$, is called the *number of loops* of Γ_γ.

In what follows, the notation β_i will indicate the ith Betti number.

3. Number of loops and other properties of Γ_γ

In this section we investigate the number of loops in generalized Reeb graphs and present some other properties of these graphs.

From now on, N will be a smooth connected, orientable and closed 2-manifold or N will be a 2-manifold with boundary obtained by taking a closed 2-manifold and removing h-disks. In the last case, by simplicity, we will simply say that N is a 2-manifold with boundary.

Theorem 3.1. *Let N be a closed 2-manifold of genus g and let $\gamma : N \to S^1$ be a non regular stable circle-valued function. Then the generalized Reeb graph Γ_γ of γ has g loops.*

Proof. First notice that Γ_γ is connected and $\chi(\Gamma_\gamma) = V - E$, where V, E denote the number of vertices and edges of Γ_γ, respectively.

On one hand, $V = M + S + I$ where M, S, I are the numbers of vertices of type: max/min, saddle or regular, respectively. Since γ is non regular, $V \neq 0$.

On the other hand, by Euler's formula $E = \frac{1}{2}\sum_{i=1}^{V} \deg(v_i)$ where $v_i \in V$ and $\deg(v_i)$ (the degree of v_i) is the number of edges incident to v_i. As γ is stable, the degree of each vertex of max/min type is 1, while of regular type is 2 and of saddle type is 3. Hence,

$$\chi(\Gamma_\gamma) = V - E = M + S + I - \frac{1}{2}(M + 2I + 3S) = \frac{M - S}{2} = \frac{2-2g}{2} = 1 - g.$$

Since Γ_γ is connected, it follows that $\beta_1(\Gamma_\gamma) = g$, i.e., Γ_γ has g loops.

□

Remark 3.2. If $\gamma : N \to S^1$ is a stable circle-valued function, where N is a closed 2-manifold with $\chi(N) \neq 0$, then γ is always non regular. In fact, suppose γ is regular. Then, γ should be surjective and from Ehresmann's fibration theorem [6], γ should be a locally trivial fibration. In particular, since F is a fiber of this fibration, it should happen that $0 \neq \chi(N) = \chi(S^1)\chi(F) = 0$, which is an absurd.

Corollary 3.3. *(Proposition 3.4 [2]) Let $\gamma : S^2 \to S^1$ be a stable circle-valued function. Then the generalized Reeb graph of γ is a tree.*

Remark 3.4. (1) Notice that the definition of generalized Reeb graph differs from the classical Reeb graph with respect to the vertices. In the classical case, the vertices are related just with the connected components of level curves $\gamma^{-1}(v)$ which contain a critical point. Hence, our generalized Reeb graph contains some extra vertices corresponding to the regular connected components of $\gamma^{-1}(v)$, where v is a critical value. Of course the classical Reeb graph can be obtained from the generalized one just by eliminating the extra vertices and joining the two adjacent edges. But in general, the generalized Reeb graph provides more information.

(2) The Figure 7 shows two stable functions $\gamma_1, \gamma_2 : S^2 \to S^1$ with their respective generalized Reeb graphs. Both functions share the same classical Reeb graph, but the generalized Reeb graphs are different. The stable function γ_1 is non surjective while γ_2 is surjective. Then γ_1 and γ_2 could not be topologically equivalent, i.e., there are no homeomorphisms $\phi : S^2 \to S^2$ and $\psi : S^1 \to S^1$ such that $\gamma_1 = \psi \circ \gamma_2 \circ \phi^{-1}$. This shows that the classical Reeb graph is not sufficient to distinguish between these two examples.

(3) If $\gamma : S^2 \to S^1$ is not surjective, then γ may be regarded as a Morse function from S^2 to $\mathbb{R}$ (via stereographic projection). In this case, the generalized Reeb graph can be obtained from the classical one just by adding the extra vertices each time that one passes through a critical value.

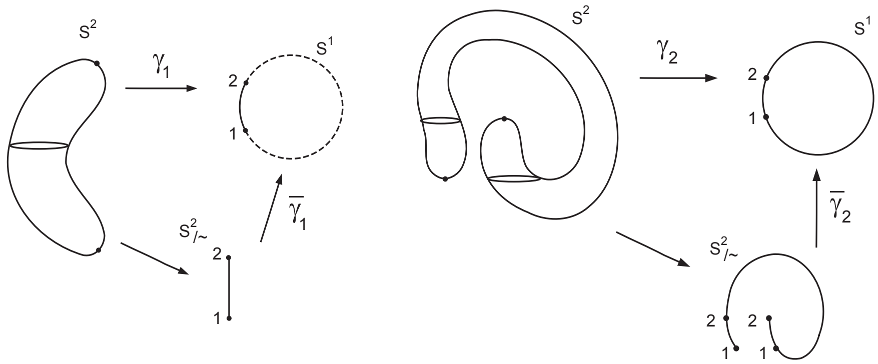

FIGURE 7. Stable functions and their generalized Reeb graphs

It is obvious that the labeling of vertices of the generalized Reeb graph is not uniquely determined, since it depends on the chosen orientations and the base points on each S^1. Different choices will produce either a cyclic permutation or a reversal of the labeling in the generalized Reeb graph.

The following result shows that the number of open saddles together with the genus and the number of boundary components of N, determine the number of loops in the generalized Reeb graph associated to $\gamma : N \to S^1$:

Theorem 3.5. *Let N be a 2-manifold with boundary and let $\gamma : N \to S^1$ be a stable circle-valued function. Then, the number of loops in Γ_γ is given by $g + \dfrac{h+S''}{2}$, where g is the genus of N, h is the number of connected components of ∂N and S'' is the number of vertices of open saddle type.*

Proof. Since Γ_γ is connected we have $\beta_0(\Gamma_\gamma) = 1$. The Euler characteristic of Γ_γ is given by $\chi(\Gamma_\gamma) = \beta_0(\Gamma_\gamma) - \beta_1(\Gamma_\gamma) = 1 - \beta_1(\Gamma_\gamma)$, where $\beta_1(\Gamma_\gamma)$ represents the number of loops in Γ_γ.

We also have that $\chi(\Gamma_\gamma) = V - E$, where V, E denote the number of vertices and edges of Γ_γ, respectively. Moreover, $V = M + S + S' + S'' + C + I$ where M, S, S', S'', C, I denote the numbers of vertices of each type listed in Section 2. On the other hand, by Euler's formula

$$E = \frac{1}{2}\sum_{i=1}^{V} deg(v_i)$$

where $v_i \in V$.

Since γ is stable, the degree of each vertex of max/min type is 1, while of regular type is 2 and saddle type is 3. Hence,

$$\chi(\Gamma_\gamma) = V - E = M + S + S' + S'' + C + I - \frac{1}{2}(M + 2C + 2I + 3S + 3S' + 4S'')$$

$$\Rightarrow \chi(\Gamma_\gamma) = \frac{M - S - S' - 2S''}{2} = \frac{\chi(N) - S''}{2} = 1 - g - \frac{(S''+h)}{2}.$$

Therefore, the number of loops is given by $\beta_1(\Gamma_\gamma) = g + \dfrac{(h+S'')}{2}$.

□

The next proposition shows that the first Betti number of N bounds the number of loops in a generalized Reeb graph, similar to what happens with the classical Reeb graph (see Section 1):

Proposition 3.6. *Let N be a 2-manifold with boundary and let $\gamma : N \to S^1$ be a stable circle-valued function. Then, the number of loops $=\ \beta_1(\Gamma_\gamma) \leq \beta_1(N)$.*

Proof. In the proof of Theorem 3.5 we showed that $2\chi(\Gamma_\gamma) = \chi(N) - S''$. Then,

$$\beta_1(N) = 2\beta_1(\Gamma_\gamma) - 1 - S''.$$

Note that

$$\chi(\Gamma_\gamma) = \chi(\Gamma \cup \Gamma') = \chi(\Gamma) + \chi(\Gamma') - \chi(\Gamma \cap \Gamma') = \chi(\Gamma) - S'' - S',$$

because

$$\chi(\Gamma') = V - E = S' + S'' + I - \frac{1}{2}(2S' + 4S'' + 2I) = -S''$$

and $\chi(\Gamma \cap \Gamma') = S'$.

However, since N is a 2-manifold with boundary, the number of connected components of Γ is at most S', which means that $\chi(\Gamma) \leq S' - \beta_1(\Gamma)$.

Then,

$$\chi(\Gamma_\gamma) = \chi(\Gamma) - S'' - S' \leq -\beta_1(\Gamma) - S'' \leq -S''.$$

Therefore,

$$\beta_0(\Gamma_\gamma) - \beta_1(\Gamma_\gamma) = \chi(\Gamma_\gamma) \leq -S'' \iff \beta_1(\Gamma_\gamma) \geq 1 + S''.$$

Consequently,

$$\beta_1(N) = 2\beta_1(\Gamma_\gamma) - (1 + S'') \geq \beta_1(\Gamma_\gamma) \Rightarrow \beta_1(\Gamma_\gamma) \leq \beta_1(N).$$

□

A consequence of Theorem 3.5 and Proposition 3.6 is the following relation

$$g + \frac{(h + S'')}{2} \leq 2g + h - 1 \Rightarrow S'' \leq 2g + h - 2.$$

The next result shows a class of functions whose generalized Reeb graphs have the maximal number of loops:

Theorem 3.7. *Let N be a 2-manifold with boundary and let $\gamma : N \to S^1$ be a stable circle-valued function. If $\beta_0(\Gamma) = S'$ then Γ_γ has the maximal number of loops, i.e., $\beta_1(\Gamma_\gamma) = 2g + h - 1$.*

Proof. Since γ is stable and $h \neq 0$, then $\Gamma' \neq \emptyset$. We divide the proof in two cases:

Case 1: $S' = 0$.

Since $\Gamma_\gamma = \Gamma \cup \Gamma'$ is connected, $\Gamma \cap \Gamma'$ is the set of vertices that correspond to the half open saddles type and $\Gamma' \neq \emptyset$, we have that $\Gamma = \emptyset$.

Consequently, $M = 0$ and $S = 0$. By the Poincaré-Hopf Theorem it follows that

$$2 - 2g - h = M - S - S' - S'' = -S'' \Rightarrow S'' = 2g + h - 2.$$

As

$$1 - \beta_1(\Gamma_\gamma) = \chi(\Gamma_\gamma) = \chi(\Gamma') = -S'' = -(2g + h - 2),$$

then $\beta_1(\Gamma_\gamma) = 2g + h - 1$.

Case 2: $S' \neq 0$.

Notice that the level curves of half open saddle type divide N in two connected components. Consider $\alpha_1, \dots, \alpha_{S'}$ the level curves of half open saddle type, and let v_i be the vertex corresponding to α_i in $\Gamma_\gamma = \Gamma \cup \Gamma'$, with $i = 1, \dots, S'$. Then, for each vertex v_i there are 3 incident edges, 2 bold traced edges and 1 slim traced edge.

Let B_i be the connected component of N determined by α_i that contains the level curves corresponding to the slim traced edges arriving at v_i. Since $\Gamma \cap \Gamma' = \{v_i, i = 1, \dots, S'\}$, $\Gamma_\gamma = \Gamma \cup \Gamma'$ is connected and $\beta_0(\Gamma) = S'$, then each connected component of Γ contains exactly one vertex v_i, $i = 1, \dots, S'$.

Assume that $B_i \cap \partial N \neq \emptyset$ for some $i = 1, \dots, S'$. Then, B_i contains the level curves of interval type. Consequently, it contains a level curve of half open saddle type. Hence, there is a connected component of Γ which contains two vertices corresponding to half open saddles. But this is a contradiction, therefore $B_i \cap \partial N = \emptyset$.

Since $\gamma|B_i$ is Morse for every $i = 1, \dots, S'$, it follows that B_i contains only level curves of saddle type, circle type and max/min type. Also, the subgraph $\Gamma_{\gamma|B_i}$ satisfies $1 - \beta_1(\Gamma_{\gamma|B_i}) = M_i - S_i$, where M_i is the number of vertices of max/min type and S_i is the number of vertices of saddle type of $\Gamma_{\gamma|B_i}$, respectively. It follows that

$$\sum_{i=1}^{S'} \left(1 - \beta_1(\Gamma_{\gamma|B_i})\right) = \sum_{i=1}^{S'} (M_i - S_i) \Rightarrow S' - \beta_1(\Gamma) = M - S \Rightarrow \beta_1(\Gamma) = -M + S + S'.$$

Also, notice that $\beta_0(\Gamma) = S'$ implies $\beta_0(\Gamma') = 1$, then

$$\chi(\Gamma') = -S'' \Rightarrow \beta_1(\Gamma') = 1 + S''.$$

Consequently,

$$\begin{aligned} \chi(\Gamma_\gamma) &= \chi(\Gamma) + \chi(\Gamma') - \chi(\Gamma \cap \Gamma') = \beta_0(\Gamma) - \beta_1(\Gamma) + \beta_0(\Gamma') - \beta_1(\Gamma') - S' \\ &= S' - (-M + S + S') + 1 - (1 + S'') - S' = M - S - S' - S'' = \chi(N). \end{aligned}$$

Therefore, $\beta_1(\Gamma_\gamma) = 2g + h - 1$.

□

The next picture illustrates a stable circle-valued function under the conditions of Theorem 3.7.

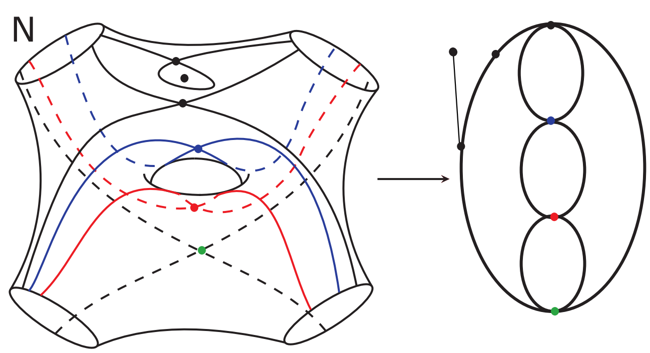

FIGURE 8. Stable circle-valued function with maximal number of loops in the generalized Reeb graph

Remark 3.8. Consider $\gamma : N \to S^1$ a stable circle-valued function, where N is a 2-manifold with boundary and genus zero. Notice that since $\beta_0(\Gamma) \leq S'$, if $\beta_0(\Gamma') = 1$ then $\beta_0(\Gamma) = S'$. Consequently, the number of loops of Γ_γ is maximal.

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