Journal of Singularities

Volume 21

Proceedings of the international conference
"Geometric and Algebraic Singularity Theory"
in honor of Goo Ishikawa on the
occasion of his 60th Birthday,
10-16 September 2017, Będlewo, Poland

Editors:
Osamu Saeki
Toru Ohmoto
Wojciech Domitrz

www.journalofsing.org

Journal *of* Singularities

Volume 21
2020

Proceedings of the international conference "Geometric and Algebraic Singularity Theory" in honor of Goo Ishikawa on the occasion of his 60th Birthday

Editors:
Osamu Saeki, Toru Ohmoto, and Wojciech Domitrz

ISSN: # 1949-2006

Table of Contents

Proceedings of the international conference
"Geometric and Algebraic Singularity Theory"

In honor of Goo Ishikawa on the occasion
of his 60th Birthday

In 2017, from September 10^{th} to September 16^{th}, the conference "Geometric and Algebraic Singularity Theory" was held at the Banach Center, Będlewo, Poland. This conference was organized especially to celebrate the 60^{th} Birthday of Professor Goo Ishikawa. There we brought together specialists of various domains, who represent algebraic, geometric and analytic approaches to numerous topics related to Singularity Theory; including, e.g.,

- Singularities of smooth maps and differential forms
- Lagrangian and Legendrian singularities
- Differential geometry of frontals and wavefronts
- Symplectic and contact geometry
- Subanalytic and semialgebraic sets
- Algebraic curves, moduli and resolutions
- Applications to physical systems, dynamics and control theory
- Global topology of smooth maps and their singularities.

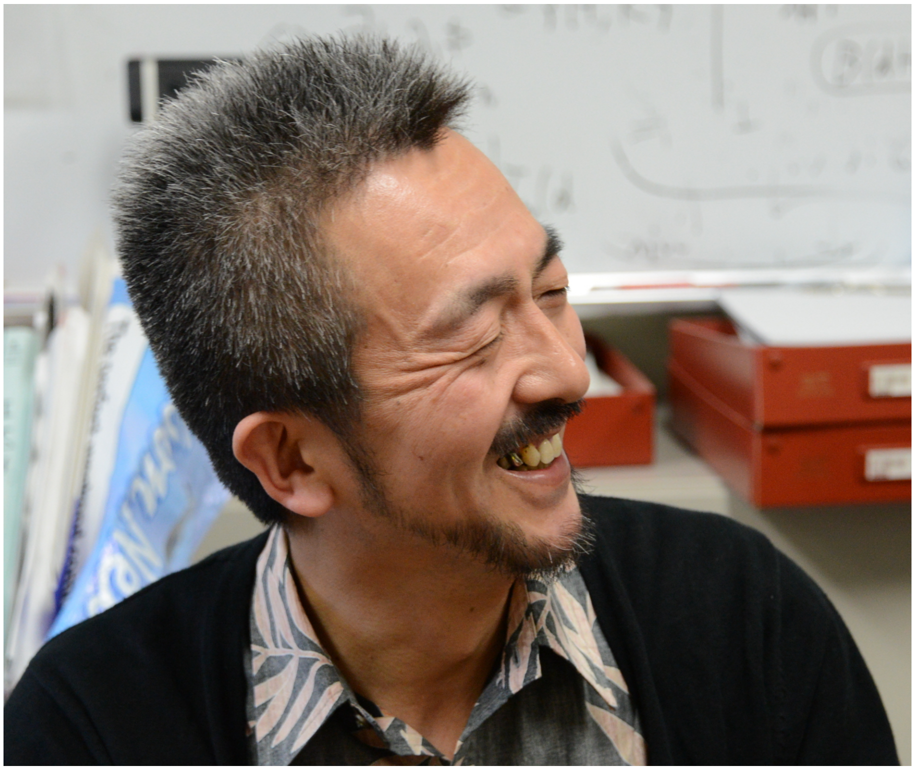
We had 38 talks and 7 poster presentations, and nearly 60 participants did exchange fruitful discussions during the conference.

In these proceedings we have collected several research papers mainly from the participants of the conference. All the papers have been refereed and are presented in the final form. We hope that this volume will give a wide range of readers, including graduate students and researchers in different fields, an opportunity to encounter deep and attractive aspects of the marvelous field of Singularities.

Finally, we would like to express our sincere gratitude to the Banach Center, Polish Academy of Science and Japan Society for the Promotion of Science for their support, to all contributors for the proceedings, and to all the participants and local organizers of the conference. We also thank the referees for helping us with the review process and the editors of the Journal of Singularities for making this special issue possible.

Osamu Saeki, Fukuoka
Toru Ohmoto, Sapporo
Wojciech Domitrz, Warsaw

Group Photo at the Banach Center, Będlewo, Sept. 2017.



Goo Ishikawa

T. Ohmoto

A note on Goo Ishikawa

Goo (Go-o) Ishikawa is a well known Japanese member of the international community of Singularity Theory. Over three decades, he has been running on the top front in his research fields, especially with focusing on geometry of singular mappings equipped with certain differential systems. It creates a new bridge between Singularity Theory and Differential Geometry (and its various applications).

He was born on November 2, 1957 in Fukushima, Japan, and grew up there. Afterwards, he entered Kyoto University and there he was fascinated by the beauty of mathematics. Around 1980, he began to study Singularity Theory of Mappings, which was one of the most hottest topics in that time – first he met J. Mather's fundamental papers and V. I. Arnol'd's attractive works related to symplectic/contact geometry, and perhaps those must have been merged into a 'kernel' of Goo's mathematics later. His advisor was Masahisa Adachi, who regularly organized "Differential Topology Seminar" at Kyoto University, and Goo was a main contributor. Many people gathered for this seminar, e.g., Shyuichi Izumiya, Masahiro Shiota, Shuzo Izumi, Satoshi Koike and Isao Nakai. In 1985, he got PhD at Kyoto University and began his first career at Nara Women's University. Three years after, he moved to Hokkaido University. Since then, he has been working surrounded by the beautiful nature of the northern earth.

When he was a PhD student, his handwriting seminar note on Hilbert's 16^{th} problem was widely circulated in topology community in Japan, and actually this became the theme of his PhD thesis, "*The number of singular points in a pencil of real plane algebraic curves*" (1985). On the other hand, he also worked on sheaves of C^∞-rings, influenced by works of Malgrange, Tougeron and others – his first original paper, *Families of functions dominated by distributions of $\mathcal{C}$-classes of mappings*, has been published in Ann. Inst. Fourier (1983), in which he introduced the notion of *ramification modules*. This notion took an important role at Goo's long-term project. He then started to explore singularities of tangent developables of curves in $\mathbb{R}^n$ in relation with the theory of singular Lagrange and Legendre singularities; here a typical singularity is of type *open swallowtail*. Also he studied, with his own techniques, singular Lagrange immersions having typical singularities named *open Whitney umbrellas*. The theory of *opening of map-germs*, introduced later by Goo himself, provides a unified method for characterizing those new important classes of singularities arising in various geometric applications. Indeed, Goo's attempt was to establish a Mather-type framework for a new classification theory of map-germs having integrability on certain differential systems. That is truly his own original theory and it has been quite successful – for example, its application has matured into the theory of *frontals* and *tangential mappings*. As for such kinds of classification problems, Goo produced several joint works especially with S. Janeczko, and also with I. Bogaevsky, A. Davydov, L. Wilson, H. Brodersen, etc. and with Japanese co-workers. For instance, Goo and Janeczko established a symplectic classification of plane curves, and Goo together with Y. Machida and M. Takahashi studied tangent surfaces in detail from the viewpoint of special geometry, e.g., D_4-geometry, and so on. Besides, in an earlier period (1987), Goo and Takuo Fukuda published a joint paper which provides a new algebraic formula for counting the number of cusps appearing in a generic perturbation of a given finite real and complex plane-to-plane map-germ. That was influential in two-folded ways; their formula in complex case was soon generalized by several authors into the case of higher dimension for Thom-Boardman singularities, and real enumerations using the Eeisenbud-Levine theorem attracted several younger people to find a new research direction.

self-portrait ©Goo

As known, Goo and his elder colleague and old friend, Shyuichi Izumiya, created "Sapporo School" in Singularity Theory – they have organized many conferences, raised many students, and especially, in 1998, they published a graduate course textbook entitled with *Applied Singularity Theory* (Ohyo-Tokuiten-ron), which was the first comprehensive book written in Japanese on Lagrange and Legendre singularity theory and applications. In 1994-1995, Goo visited the University of Liverpool as his sabbatical hosted by C.T.C. Wall. This experience has led to a deep and widespread development of his own research, resulting in many international collaborative researches and warmest friendships with foreign researchers. Since then, he has organized several international symposiums together with Shyuichi, including the 12^{th} International Research Institute of the Mathematical Society of Japan "Singularity Theory and Its Application" at Sapporo (2003), "Japanese-Polish working days" with S. Janeczko, Japanese-Russia bilateral project with A. Davidov and I. Bogaevsky, and so on. Also he has frequently been invited to Scientific Committees and to give keynote/plenary talks at many international conferences around the world.

On a broad range of topics, Goo Ishikawa has supervised more than five PhD students, e.g., T. Yamamoto, T. Fukunaga, W. Yukuno, A. Tsuchida, T. Yamashita, and has had 73 publications together with 26 co-authors (according to MathSciNet). He has written totally 13 books so far – there is one lecture note in English, *Singularities of Curves and Surfaces in Various Geometric Problems*, CAS Lecture Notes 10, Exact Sciences (2015), and three advanced textbooks were written in Japanese with several co-authors, e.g., *Applied Singularity Theory* mentioned above. There are five textbooks for undergraduate courses on linear algebra, calculus, sets and logic, topology, and four enlightenment booklets for general public readers, one of which is a lovely collection of his witty answers to students' funny questions on mathematics and life (this booklet has received positive ratings in reviews on amazon !).

Goo is still quite active on researches in mathematics. We wish you a happy birthday Goo, sincerely from all your friends and colleagues, and look forward to working with you for many years to come !

Journal *of* Singularities
Volume 21 (2020), 1-14

Proc. of Geometric and Algebraic Singularity Theory, Będlewo, 2017
DOI: 10.5427/jsing.2020.21a

THE FLAT GEOMETRY OF THE I_1 SINGULARITY: $(x,y) \mapsto (x, xy, y^2, y^3)$

P. BENEDINI RIUL, R. OSET SINHA

Abstract. We study the flat geometry of the least degenerate singularity of a singular surface in $\mathbb{R}^4$, the I_1 singularity parametrised by $(x,y) \mapsto (x, xy, y^2, y^3)$. This singularity appears generically when projecting a regular surface in $\mathbb{R}^5$ orthogonally to $\mathbb{R}^4$ along a tangent direction. We obtain a generic normal form for I_1 invariant under diffeomorphisms in the source and isometries in the target. We then consider the contact with hyperplanes by classifying submersions which preserve the image of I_1. The main tool is the study of the singularities of the height function.

1. Introduction

Singularity theory has played an important role on recent results on the differential geometry of singular surfaces. The geometry of the cross-cap (or Whitney umbrella), for instance, has been studied in depth: [5, 7, 8, 10, 11, 22, 24]. Also, the cuspidal edge, the most simple type of wave front, appears in many papers: [11, 14, 17, 18, 21, 25, 28].

In [16] the authors investigate the second order geometry of corank 1 surfaces in $\mathbb{R}^3$. Also, singular surfaces in $\mathbb{R}^4$ have been taken into account in [1], where corank 1 surfaces are the main object of study. In that paper, the curvature parabola is defined, inspired by the curvature parabola for corank 1 surfaces in $\mathbb{R}^3$ ([16]) and the curvature ellipse for regular surfaces in $\mathbb{R}^4$ ([15]). This curve is a plane curve that may degenerate into a half-line, a line or even a point and whose trace lies in the normal hyperplane of the surface. This special curve carries all the second order information of the surface at the singular point. Singular surfaces in $\mathbb{R}^4$ appear naturally as projections of regular surfaces in $\mathbb{R}^5$ along tangent directions. In this context, the authors associate to a regular surface $N \subset \mathbb{R}^5$ a corank 1 surface $M \subset \mathbb{R}^4$ and a regular surface $S \subset \mathbb{R}^4$. Furthermore, they compare the geometry of both surfaces M and S. An invariant called umbilic curvature (invariant under the action of $\mathcal{R}^2 \times \mathcal{O}(4)$, the subgroup of 2-jets of diffeomorphisms in the source and linear isometries in the target) is defined as well and used to study the singularities of the height function of corank 1 surfaces in $\mathbb{R}^4$.

In [13], the authors give a classification of all $\mathcal{A}$-simple map germs $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^4, 0)$. The singularity I_k given by $(x,y) \mapsto (x, xy, y^2, y^{2k+1})$, $k \geqslant 1$ is the first singular germ to appear in this classification. In [1], it is shown that this singularity is the only one whose curvature parabola is a non degenerate parabola. Also, when we consider $k = 1$, the singularity I_1 has an interesting geometric property. In [27], the authors show that given a regular surface $N \subset \mathbb{R}^5$, a tangent direction $\mathbf{u}$, in a point whose second fundamental form has maximal rank, is asymptotic if and only if the projection of N along $\mathbf{u}$ to a transverse 4-space has a $\mathcal{A}$-singularity worse than I_1. In a way, I_1 is to singular surfaces in $\mathbb{R}^4$ what the cross-cap is to singular surfaces in $\mathbb{R}^3$.

In this paper, we investigate the flat geometry of the singularity I_1, using its height function and providing geometric conditions for each possible singularity. Sections 2 and 3 are an overview

2000 *Mathematics Subject Classification.* Primary 57R45; Secondary 53A05, 58K05.
Key words and phrases. singular surface in 4-space, flat geometry, height function.
Work of P. Benedini Riul supported by CAPES - PVE 88887.122685/2016-00.
Work of R. Oset Sinha partially supported by DGICYT Grant MTM2015–64013–P.

of the differential geometry of regular surfaces in $\mathbb{R}^4$ and of the geometry of corank 1 surfaces in $\mathbb{R}^4$, respectively. We bring all the definitions and results from [1] that are going to be used throughout the paper.

The last section presents our results regarding the flat geometry of a surface whose local parametrisation is $\mathcal{A}$-equivalent to the singularity I_1. We classify submersions $(\mathbb{R}^4, 0) \to (\mathbb{R}, 0)$ up to changes of coordinates in the source that preserve the model surface $\mathtt{X}$ parametrised by I_1 (Theorem 4.7). Such changes of coordinates form a geometric subgroup $\mathcal{R}(\mathtt{X})$ of the Mather group $\mathcal{R}$ (see [3, 6]). Moreover, we study the singularities of the height function of a singular surface whose parametrisation is given by a generic normal form obtained by changes of coordinates in the source and isometries in the target (Theorem 4.8). These singularities are modeled by the ones of the submersions obtained before. Finally, we provide geometrical characterisations for each type of singularity of the height function.

Aknowledgements: the authors would like to thank M. A. S. Ruas and the referee for a careful reading of the paper and valuable suggestions.

2. The geometry of regular surfaces in $\mathbb{R}^4$

In this section we present some aspects of regular surfaces in $\mathbb{R}^4$. For more details, see [12]. Little, in [15], studied the second order geometry of submanifolds immersed in Euclidean spaces, in particular of immersed surfaces in $\mathbb{R}^4$. This paper has inspired a lot of research on the subject (see [2, 4, 9, 19, 20, 22, 24, 26], amongst others). Given a smooth surface $S \subset \mathbb{R}^4$ and $f : U \to \mathbb{R}^4$ a local parametrisation of S with $U \subset \mathbb{R}^2$ an open subset, let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ be an orthonormal frame of $\mathbb{R}^4$ such that at any $u \in U$, $\{\mathbf{e}_1(u), \mathbf{e}_2(u)\}$ is a basis for T_pS and $\{\mathbf{e}_3(u), \mathbf{e}_4(u)\}$ is a basis for N_pS at $p = f(u)$. The second fundamental form of S at p is the vector valued quadratic form $II_p : T_pS \to N_pS$ given by

$$II_p(\mathbf{w}) = (l_1 w_1^2 + 2m_1 w_1 w_2 + n_1 w_2^2)\mathbf{e}_3 + (l_2 w_1^2 + 2m_2 w_1 w_2 + n_2 w_2^2)\mathbf{e}_4,$$

where $l_i = \langle f_{xx}, \mathbf{e}_{i+2}\rangle$, $m_i = \langle f_{xy}, \mathbf{e}_{i+2}\rangle$ and $n_i = \langle f_{yy}, \mathbf{e}_{i+2}\rangle$ for $i = 1, 2$ are called the coefficients of the second fundamental form with respect to the frame above and $\mathbf{w} = w_1\mathbf{e}_1 + w_2\mathbf{e}_2 \in T_pS$. The matrix of the second fundamental form with respect to the orthonormal frame above is given by

$$\alpha = \begin{pmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{pmatrix}.$$

The *resultant* of the quadratic forms is a scalar invariant of the surface defined by Little in [15], given by

$$\delta = \frac{1}{4}(4(l_1 m_2 - m_1 n_2)(m_1 n_2 - n_1 m_2) - (l_1 n_2 - n_1 l_2)^2).$$

A point $p \in S$ is hyperbolic or elliptic according to whether $\delta(p)$ is negative or positive, respectively. If $\delta(p)$ is equal to zero, the point is parabolic or an inflection, according to the rank of α: p is parabolic if the rank is 2 and an inflection if it is less than 2.

A non zero tangent direction $\mathbf{u} \in T_pS$ is an *asymptotic direction* if there is a non zero vector $v \in N_pS$ such that

$$\langle II(\mathbf{u}, \mathbf{w}), v\rangle = 0, \quad \forall\ \mathbf{w} \in T_pS.$$

Furthermore, $v \in N_pS$ is a *binormal direction.*

One can obtain a lot of geometrical information of a regular surface $S \subset \mathbb{R}^4$, by studying the generic contact of the surface with hyperplanes. Such contact is measured by the singularities of the height function of S. Let $f : U \to \mathbb{R}^4$ be a local parametrisation of S. The family of height functions is given by

$$H : U \times \mathbb{S}^3 \to \mathbb{R}, \quad H(u, v) = \langle f(u), v\rangle.$$

Fixing $v \in \mathbb{S}^3$, the height function h_v of S is given by $h_v(u) = H(u,v)$ and has the following property: a normal direction v at $p = f(u) \in S$ is a binormal direction if and only if any tangent direction lying in the kernel of the Hessian of h_v at u is an asymptotic direction of S at p.

Definition 2.1. The *canal hypersurface* of the surface $S \subset \mathbb{R}^4$ is the 3-manifold

$$CS(\varepsilon) = \{p + \varepsilon v \in \mathbb{R}^4 |\ p \in S \text{ and } v \in (N_pS)_1\}$$

where $(N_pS)_1$ denotes the unit sphere in N_pS and ε is a small positive real number.

It is possible to consider $(N_pS)_1$ as a subset of $\mathbb{S}^3$ and as a consequence, identify (p,v) and $p + \varepsilon v$.

We shall denote the family of height functions on $CS(\varepsilon)$ by $\bar{H} : CS(\varepsilon) \times \mathbb{S}^3 \to \mathbb{R}$. So, given $w \in \mathbb{S}^3$, the height function of $CS(\varepsilon)$ along w is given by $\bar{h}_w : CS(\varepsilon) \to \mathbb{R}$, where $\bar{h}_w(p,v) = \bar{H}((p,v),w)$. Given a point $p \in M$, it is a singular point of h_v if and only if $(p,v) \in CS(\varepsilon)$ is a singular point of $\bar{h}_v$.

The *Gauss map* of the canal hypersurface $CS(\varepsilon)$, $G : CS(\varepsilon) \to \mathbb{S}^3$, is given by $G(p,v) = v$. Let $K_c : CS(\varepsilon) \to \mathbb{R}$ be the Gauss-Kronecker curvature function of $CS(\varepsilon)$. Then, the singular set of G is the parabolic set

$$K_c^{-1}(0) = \{p + \varepsilon v \in CS(\varepsilon) |\ h_v \text{ has a degenerate singularity at p}\}$$

of $CS(\varepsilon)$, which is a regular surface except at a finite number of singular points corresponding to the $D_4^{\pm}$-singularities oh $\bar{h}_v$. The regular part has regular curves corresponding to the cuspidal edge points of G and those curves may have special isolated points which are the swallowtail points of G.

One can characterise geometrically the degenerate singularities of generic height functions. Denote by γ the normal section of the surface S tangent to the asymptotic direction θ at p associated to the binormal direction v.

Theorem 2.2. [12] *Let p be a hyperbolic point on a height function generic surface $M \subset \mathbb{R}^4$. Then,*

(i) *p is an A_2 singularity of h_v if and only if γ has non vanishing torsion at p.*
(ii) *p is an A_3 singularity of h_v if and only if γ has a vanishing torsion at p and the direction θ is transversal to the curve of cuspidal edge points of the Gauss map G at p.*

A characterisation of the singularities of the height functions at a parabolic point can also be done.

Theorem 2.3. [12] *Let M be a height function generic surface in $\mathbb{R}^4$ and $p \in M$. Suppose p is a parabolic point, but not an inflection point. Then,*

(i) *p is an A_2-singularity of h_v if and only if θ is transversal to the parabolic curve.*
(ii) *p is an A_3-singularity of h_v if and only if θ is tangent to the parabolic curve with first order contact at p.*

3. Corank 1 surfaces in $\mathbb{R}^4$

3.1. **The curvature parabola.** Here we present a brief study of the differential geometry of corank 1 surfaces in $\mathbb{R}^4$ which can be found in [1]. Let M be a corank 1 surface in $\mathbb{R}^4$ at p. We take M as the image of a smooth map $g : \tilde{M} \to \mathbb{R}^4$, where $\tilde{M}$ is a smooth regular surface and $q \in \tilde{M}$ is a corank 1 point of g such that $g(q) = p$. Also, we consider $\phi : U \to \mathbb{R}^2$ a local coordinate system defined in an open neighbourhood U of q at $\tilde{M}$, and by doing this we may consider a local parametrisation $f = g \circ \phi^{-1}$ of M at p (see the diagram below).

$$\begin{tikzcd} \mathbb{R}^2 & U \subset \tilde{M} \arrow[l, "\phi"'] \arrow[r, "g"] & M \subset \mathbb{R}^4 \arrow[from=1-1, to=1-3, bend right, "f"'] \end{tikzcd}$$

The *tangent line* of M at p, T_pM, is given by Im dg_q, where $dg_q : T_q\tilde{M} \to T_p\mathbb{R}^4$ is the differential map of g at q. Hence, the *normal hyperplane* of M at p, N_pM, is the subspace satisfying $T_pM \oplus N_pM = T_p\mathbb{R}^4$.

Consider the orthogonal projection $\perp: T_p\mathbb{R}^4 \to N_pM$, $w \mapsto w^\perp$. The *first fundamental form* of M at p, $I : T_q\tilde{M} \times T_q\tilde{M} \to \mathbb{R}$ is given by

$$I(\mathbf{u}, \mathbf{v}) = \langle dg_q(\mathbf{u}), dg_q(\mathbf{v})\rangle, \qquad \forall\ \mathbf{u}, \mathbf{v} \in T_q\tilde{M}.$$

Since the map g has corank 1 at $q \in T_q\tilde{M}$, the first fundamental form is not a Riemannian metric on $T_q\tilde{M}$, but a pseudometric. Considering the local parametrisation of M at p, $f = g \circ \phi^{-1}$ and the basis $\{\partial_x, \partial_y\}$ of $T_q\tilde{M}$, the coefficients of the first fundamental form with respect to ϕ are:

$$E(q) = I(\partial_x, \partial_x) = \langle f_x, f_x\rangle(\phi(q)),\ F(q) = I(\partial_x, \partial_y) = \langle f_x, f_y\rangle(\phi(q)),$$
$$G(q) = I(\partial_y, \partial_y) = \langle f_y, f_y\rangle(\phi(q)).$$

Taking $\mathbf{u} = \alpha\partial_x + \beta\partial_y = (\alpha, \beta) \in T_q\tilde{M}$, we write $I(\mathbf{u}, \mathbf{u}) = \alpha^2E(q) + 2\alpha\beta F(q) + \beta^2G(q)$.

With the same conditions as above, the *second fundamental form* of M at p,

$$II : T_q\tilde{M} \times T_q\tilde{M} \to N_pM$$

in the basis $\{\partial_x, \partial_y\}$ of $T_q\tilde{M}$ is given by

$$II(\partial_x, \partial_x) = f_{xx}^\perp(\phi(q)),\ II(\partial_x, \partial_y) = f_{xy}^\perp(\phi(q)),\ II(\partial_y, \partial_y) = f_{yy}^\perp(\phi(q))$$

and we extend it to the whole space in a unique way as a symmetric bilinear map. It is possible to show that the second fundamental form does not depend on the choice of local coordinates on $\tilde{M}$.

For each normal vector $\nu \in N_pM$, the *second fundamental form along* ν, $II_\nu : T_q\tilde{M} \times T_q\tilde{M} \to \mathbb{R}$ is given by $II_\nu(\mathbf{u}, \mathbf{v}) = \langle II(\mathbf{u}, \mathbf{v}), \nu\rangle$, for all $\mathbf{u}, \mathbf{v} \in T_q\tilde{M}$. The coefficients of II_ν with respect to the basis $\{\partial_x, \partial_y\}$ of $T_q\tilde{M}$ are

$$l_\nu(q) = \langle f_{xx}^\perp, \nu\rangle(\phi(q)),\ m_\nu(q) = \langle f_{xy}^\perp, \nu\rangle(\phi(q)),$$
$$n_\nu(q) = \langle f_{yy}^\perp, \nu\rangle(\phi(q)).$$

Fixing an orthonormal frame $\{\nu_1, \nu_2, \nu_3\}$ of N_pM,

$$\begin{aligned} II(\mathbf{u}, \mathbf{u}) &= II_{\nu_1}(\mathbf{u}, \mathbf{u})\nu_1 + II_{\nu_2}(\mathbf{u}, \mathbf{u})\nu_2 + II_{\nu_3}(\mathbf{u}, \mathbf{u})\nu_3 \\ &= \sum_{i=1}^{3}(\alpha^2 l_{\nu_i}(q) + 2\alpha\beta m_{\nu_i}(q) + \beta^2 n_{\nu_i}(q))\nu_i, \end{aligned}$$

Moreover, the second fundamental form is represented by the matrix of coefficients

$$\begin{pmatrix} l_{\nu_1} & m_{\nu_1} & n_{\nu_1} \\ l_{\nu_2} & m_{\nu_2} & n_{\nu_2} \\ l_{\nu_3} & m_{\nu_3} & n_{\nu_3} \end{pmatrix}.$$

Definition 3.1. [1] Let $C_q \subset T_q\tilde{M}$ be the subset of unit tangent vectors and let $\eta_q : C_q \to N_pM$ be the map given by $\eta_q(\mathbf{u}) = II(\mathbf{u}, \mathbf{u})$. The *curvature parabola* of M at p, denoted by Δ_p, is the image of η_q, that is, $\eta_q(C_q)$.

The curvature parabola is a plane curve whose trace lies in the normal hyperplane of the surface. Also, this curve may degenerate into a half-line, a line or even a point.

Example 3.2. Consider $\tilde{M} = \mathbb{R}^2$ and the singular surface M locally parametrised by the I_1-singularity $f(x,y) = (x, xy, y^2, y^3)$. Taking coordinates (X,Y,Z,W) in $\mathbb{R}^4$, $q = (0,0)$ and $p = (0,0,0,0)$, the tangent line T_pM is the X-axis and N_pM is the YZW-hyperplane. The coefficients of the first fundamental form are given by $E(q) = 1$ and $F(q) = G(q) = 0$. Hence, if $\mathbf{u} = (\alpha, \beta) \in T_q\mathbb{R}^2$, $I(\mathbf{u},\mathbf{u}) = \alpha^2$ and $C_q = \{(\pm 1, y) : y \in \mathbb{R}\}$. The matrix of coefficients of the second fundamental form is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

when we consider the orthonormal frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$. Therefore, for $\mathbf{u} = (\alpha, \beta)$,

$$II(\mathbf{u},\mathbf{u}) = (0, 2\alpha\beta, 2\beta^2, 0)$$

and the curvature parabola Δ_p is a non-degenerate parabola which can be parametrised by $\eta(y) = (0, 2y, 2y^2, 0)$.

3.2. **Second order properties.** Given a regular surface $N \subset \mathbb{R}^5$, we consider the corank 1 surface M at p obtained by the projection of N in a tangent direction, via the map

$$\xi : N \subset \mathbb{R}^5 \to M.$$

The regular surface $N \subset \mathbb{R}^5$ can be taken, locally, as the image of an immersion $i : \tilde{M} \to N \subset \mathbb{R}^5$, where $\tilde{M}$ is the regular surface from the construction done before.

The points of N can be characterized according to the rank of its fundamental form at that point. Inspired by this classification, we have the following:

Definition 3.3. Given a corank 1 surface $M \subset \mathbb{R}^4$, we define the subset

$$M_i = \{p \in M : p \text{ is singular and } rank(II_p) = i\},\ i = 0,1,2,3.$$

Definition 3.4. The minimal affine space which contains the curvature parabola is denoted by $\mathcal{A}ff_p$. The plane denoted by E_p is the vector space: parallel to $\mathcal{A}ff_p$ when Δ_p is a non degenerate parabola, the plane through p that contains $\mathcal{A}ff_p$ when Δ_p is a non radial half-line or a non radial line and any plane through p that contains $\mathcal{A}ff_p$ when Δ_p is a radial half-line, a radial line or a point.

Let $S \subset \mathbb{R}^4$ be the regular surface locally obtained by projecting $N \subset \mathbb{R}^5$ via the map π into the four space given by $T_{\xi^{-1}(p)}N \oplus \xi^{-1}(E_p)$ (see the following diagram).

$$\begin{tikzcd}
 & & N \subset \mathbb{R}^5 \arrow[d, "\xi"] \arrow[dr, "\pi"] & \\
\mathbb{R}^2 \arrow[rr, bend right, "f"'] & \tilde{M} \arrow[l, "\phi"'] \arrow[r, "g"] \arrow[ur, "i"] & M \subset \mathbb{R}^4 & S \subset \mathbb{R}^4
\end{tikzcd}$$

Using the previous construction, one can relate the corank 1 singular surface $M \subset \mathbb{R}^4$ and the regular surface $S \subset \mathbb{R}^4$.

Definition 3.5. A non zero direction $\mathbf{u} \in T_q\tilde{M}$ is called *asymptotic* if there is a non zero vector $\nu \in E_p$ such that

$$II_\nu(\mathbf{u},\mathbf{v}) = \langle II(\mathbf{u},\mathbf{v}), \nu\rangle = 0 \ \ \forall\ \mathbf{v} \in T_q\tilde{M}.$$

Moreover, in such case, we say that ν is a *binormal direction.*

The normal vectors $\nu \in N_pM$ satisfying the condition $II_\nu(\mathbf{u}, \mathbf{v}) = 0$ are called *degenerate directions*, but only those in E_p are binormal directions. When $p \in M_1 \cup M_0$, the choice of E_p does not change the number of binormal directions. Furthermore, all directions $\mathbf{u} \in T_q\tilde{M}$ are asymptotic.

Definition 3.6. Given a binormal direction $\nu \in E_p$, the hyperplane through p and orthogonal to ν is called an *osculating hyperplane* to M at p.

Definition 3.7. Given a surface $M \subset \mathbb{R}^4$ with corank 1 singularity at $p \in M$. The point p is called:

(i) *elliptic* if there are no asymptotic directions at p;
(ii) *hyperbolic* if there are two asymptotic directions at p;
(iii) *parabolic* if there is one asymptotic direction at p;
(iv) *inflection* if there are an infinite number of asymptotic directions at p.

The next result compares the geometry of a corank 1 surface in $\mathbb{R}^4$ with the geometry of the associated regular surface $S \subset \mathbb{R}^4$ obtained.

Theorem 3.8. [1] *Let $M \subset \mathbb{R}^4$ be a surface with corank 1 singularity at $p \in M$ and $S \subset \mathbb{R}^4$ the regular surface associated to M.*

(i) *A direction $\boldsymbol{u} \in T_q\tilde{M}$ is an asymptotic direction of M if and only if it is also an asymptotic direction of the associated regular surface $S \subset \mathbb{R}^4$;*
(ii) *A direction $\nu \in N_pM$ is a binormal direction of M if and only if $\pi \circ \xi^{-1}(\nu) \in N_{\pi\circ\xi^{-1}(p)}S$ is a binormal direction of S.*
(iii) *The point p is an elliptic/hyperbolic/parabolic/inflection point if and only if $\pi\circ\xi^{-1}(p) \in S$ is an elliptic/hyperbolic/parabolic/inflection point, respectively.*

The singularity I_k, $k \geqslant 1$, given by the $\mathcal{A}$-normal form $(x, y) \mapsto (x, xy, y^2, y^{2k+1})$ has an interesting property: every map germ $\mathcal{A}$-equivalent to it prarametrises a corank 1 surface in $\mathbb{R}^4$ whose curvature parabola is a non degenerate parabola. Moreover, I_k are the only singularities having this property. Hence, every map germ $\mathcal{A}$-equivalent to I_k is $\mathcal{R}^2 \times \mathcal{O}(4)$-equivalent to the normal form $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^4, 0)$ where

$$f(x, y) = (x, xy + p(x, y), b_{20}x^2 + b_{11}xy + b_{02}y^2 + q(x, y), c_{20}x^2 + r(x, y))$$

with $b_{02} > 0$ and $p, q, r \in \mathcal{M}_2^3$. The proof of this assertion can be found in [1].

Proposition 3.9. [1] *Consider the $\mathcal{R}^2 \times \mathcal{O}(4)$ normal form of the singularity I_k given above. Then, the singularity I_k is hyperbolic, parabolic or elliptic if and only if b_{20} is positive, zero or negative, respectively.*

For corank 1 surfaces in $\mathbb{R}^4$ we have the following:

Definition 3.10. The non-negative number

$$\kappa_u(p) = d(p, \mathcal{A}ff_p)$$

is called the *umbilic curvature* of M at p.

The authors in [1] present explicit formulas of this invariant as well as geometric interpretations of it. Here, however, we shall restrict our study to the case where Δ_p is a non degenerate parabola.

Proposition 3.11. [1] *Let $\{\nu_1, \nu_2, \nu_3\}$ be an othonormal frame of N_pM such that $E_p = \{\nu_1, \nu_2\}$ and $E_p^\perp = \{\nu_3\}$. Then the following holds:*

$$\kappa_u(p) = \frac{|II_{\nu_3}(\boldsymbol{u}, \boldsymbol{u})|}{I(\boldsymbol{u}, \boldsymbol{u})} = |proj_{\nu_3}\eta(y)| = |\langle \eta(y), \nu_3\rangle|,$$

for any $\boldsymbol{u} \in T_q\tilde{M}$*, where* η *is a parametrisation of* Δ_p*.*

4. Flat geometry

In this section we study the contact of a singular surface $M \subset \mathbb{R}^4$ locally given by the $\mathcal{A}$-normal form $(x, y) \mapsto (x, xy, y^2, y^3)$ with hyperplanes. One can summarize the *modus operandi* in the following way: we fix a model of the singularity I_1 and study the contact with the zero fibres of submersions. We then associate the singularities of the height functions with the geometry studied in the previous section.

4.1. **Functions on** I_1**.** In this section, we classify germs of functions on $\mathtt{X} \subset \mathbb{R}^4$, where $\mathtt{X}$ is the germ of the model surface locally parametrised by the I_1 singularity. This technique was introduced in [5], where the authors study the contact between the Whitney umbrella (or crosscap) with planes. More recently, the same was done in [25] and [23] but this time the surfaces were the cuspidal edge and the cuspidal S_k singularities, respectively.

We denote by $\mathcal{E}_n$ the local ring of germs of functions $f : (\mathbb{R}^n, 0) \to \mathbb{R}$ and by $\mathcal{M}_n$ its maximal ideal. Let $(\mathtt{X}, 0) \subset (\mathbb{R}^n, 0)$ be a germ of a reduced analytic subvariety of $\mathbb{R}^n$ at 0 defined by an ideal I of $\mathcal{E}_n$. A diffeomorphism $k : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is said to preserve $\mathtt{X}$ if $(k(\mathtt{X}, 0)) = (\mathtt{X}, 0)$. The group of such diffeomorphisms is a subgroup of the group $\mathcal{R}$ and is denoted by $\mathcal{R}(\mathtt{X})$. This is one of Damon's "geometrical subgroups" of $\mathcal{A}$ (see [3, 6]).

Consider the $\mathcal{A}$-normal form of the I_1 singularity: $f(x, y) = (x, xy, y^2, y^3)$. Our aim is to classify germs of submersions $g : (\mathbb{R}^4, 0) \to (\mathbb{R}, 0)$ using the $\mathcal{R}(\mathtt{X})$ equivalence, where $\mathtt{X} = f(\mathbb{R}^2, 0)$ is our model surface. The ideal $I \lhd \mathcal{E}_4$ of irreducible polynomials defining $\mathtt{X}$ is given by

$$I = \langle Y^2 - X^2 Z,\ W^2 - Z^3,\ XW - YZ,\ YW - XZ^2 \rangle.$$

We shall denote by $\Theta(\mathtt{X})$ the $\mathcal{E}_4$-module of vector fields tangent to $\mathtt{X}$ (Derlog($\mathtt{X}$) in other texts). Hence, we have

$$\xi \in \Theta(\mathtt{X}) \Leftrightarrow \xi h(x) = dh_x(\xi(x)) \in I,\ \forall h \in I.$$

Proposition 4.1. $\Theta(\mathtt{X})$ *is generated by:*

$$\begin{aligned}
&\xi_1 = X\tfrac{\partial}{\partial X} + Y\tfrac{\partial}{\partial Y}, && \xi_2 = X^2\tfrac{\partial}{\partial X} + 2Y\tfrac{\partial}{\partial Z} + 3XZ\tfrac{\partial}{\partial W},\\
&\xi_3 = Y\tfrac{\partial}{\partial Y} + 2Z\tfrac{\partial}{\partial Z} + 3W\tfrac{\partial}{\partial W}, && \xi_4 = Y\tfrac{\partial}{\partial X} + XZ\tfrac{\partial}{\partial Y},\\
&\xi_5 = Z\tfrac{\partial}{\partial X} + W\tfrac{\partial}{\partial Y}, && \xi_6 = XZ\tfrac{\partial}{\partial Y} + 2W\tfrac{\partial}{\partial Z} + 3Z^2\tfrac{\partial}{\partial W},\\
&\xi_7 = W\tfrac{\partial}{\partial X} + Z^2\tfrac{\partial}{\partial Y}, && \xi_8 = (Y^2 - X^2 Z)\tfrac{\partial}{\partial W},\\
&\xi_9 = (YZ - XW)\tfrac{\partial}{\partial W}, && \xi_{10} = XW\tfrac{\partial}{\partial Y} + 2Z^2\tfrac{\partial}{\partial Z} + 3ZW\tfrac{\partial}{\partial W},\\
&\xi_{11} = (YW - XZ^2)\tfrac{\partial}{\partial W}, && \xi_{12} = (W^2 - Z^3)\tfrac{\partial}{\partial W},\\
&\xi_{13} = (W^2 - Z^3)\tfrac{\partial}{\partial Y}.
\end{aligned}$$

Proof. For notation purposes we write $(X, Y, Z, W) = (X_1, X_2, X_3, X_4)$. Let h_i, $i = 1, \ldots, 4$ be the functions which generate the ideal I in the order in which they appear in the definition of I. We are looking for vector fields $\xi = \sum_{i=1}^4 \xi_i \frac{\partial}{\partial X_i}$ on $\mathbb{R}^4$ such that for each $j = 1, \ldots, 4$ there exist functions $\alpha_i(X_1, \ldots, X_4)$ such that

$$\sum_{i=1}^4 \xi_i \frac{\partial h_j}{\partial X_i} = \sum_{i=1}^4 \alpha_i h_i.$$

Consider, for $j = 1, \dots, 4$, the map $\Phi_j : \mathcal{E}_4^8 \to \mathbb{R}$ given by

$$\Phi_j(\xi, \alpha) = \sum_{i=1}^{4} \xi_i \frac{\partial h_j}{\partial X_i} - \sum_{i=1}^{4} \alpha_i h_i,$$

where $\xi = (\xi_1, \dots, \xi_4) \in \mathcal{E}_4^4$ and $\alpha = (\alpha_1, \dots, \alpha_4) \in \mathcal{E}_4^4$. Let $A_j = \ker \Phi_j$. Let $\pi : \mathcal{E}_4^8 \to \mathcal{E}_4^4$ be the canonical projection given by $\pi(\xi, \alpha) = \xi$. Let $B_j = \pi(A_j)$. Then

$$\Theta(\mathtt{X}) = \bigcap_{j=1}^{4} B_j.$$

In order to obtain the A_j we use syzygies in the computer package Singular. It can be checked that all the vector fields obtained by this method are, in fact, liftable, i.e. there exists a vector field η on $\mathbb{R}^2$ such that $dh(\eta) = \xi \circ h$, and are therefore tangent to $\mathtt{X}$. □

The idea for classifying analytic function germs $g : (\mathbb{R}^4, 0) \to (\mathbb{R}, 0)$ up to $\mathcal{R}(\mathtt{X})$-equivalence is to use generalisations of the standard results for the group $\mathcal{R}$, that is, when $\mathtt{X} = \emptyset$. Since $\mathcal{R}(\mathtt{X})$ is one of the Damon's "geometrical subgroups" of $\mathcal{A}$, there are versions of the unfolding and determinacy theorems. In this classification, the orbits are obtained inductively on the jet level and the complete transversal method is also adapted for our action.

We define $\Theta_1(\mathtt{X}) = \{\xi \in \Theta(\mathtt{X}) : j^1\xi = 0\}$. Hence, from Proposition 4.1,

$$\Theta_1(\mathtt{X}) = \mathcal{M}_4.\{\xi_1, \dots, \xi_7\} + \mathcal{E}_4.\{\xi_8, \dots, \xi_{13}\}.$$

For each $f \in \mathcal{E}_4$, $\Theta(\mathtt{X}) \cdot f = \{\xi(f) : \xi \in \Theta(\mathtt{X})\}$. A similar definition is made for $\Theta_1(\mathtt{X}) \cdot f$. Furthermore, we define the *tangent spaces* to the $\mathcal{R}(\mathtt{X})$-orbit of f:

$$L\mathcal{R}_1(\mathtt{X}) \cdot f = \Theta_1(\mathtt{X}) \cdot f,\ L\mathcal{R}(\mathtt{X}) \cdot f = L\mathcal{R}_e(\mathtt{X}) \cdot f = \Theta(\mathtt{X}) \cdot f.$$

The $\mathcal{R}(\mathtt{X})$-codimension is given by $d(f, \mathcal{R}(\mathtt{X})) = \dim_{\mathbb{R}}(\mathcal{E}_4 / L\mathcal{R}(\mathtt{X}) \cdot f)$.

Proposition 4.2. [5] *Let $f : (\mathbb{R}^4, 0) \to (\mathbb{R}, 0)$ be a smooth germ and $h_1, \dots, h_r$ be homogeneous polynomials of degree $k+1$ with the property that*

$$\mathcal{M}_4^{k+1} \subset L\mathcal{R}_1(X) \cdot f + sp\{h_1, \dots, h_r\} + \mathcal{M}_4^{k+2}.$$

Then any germ g with $j^k f(0) = j^k g(0)$ is $\mathcal{R}_1(X)$-equivalent to a germ of the form

$$f + \sum_{i=1}^{r} u_i h_i + \phi,$$

where $\phi \in \mathcal{M}_4^{k+2}$. The vector subspace $sp\{h_1, \dots, h_r\}$ is called a complete $(k+1)$-$\mathcal{R}(X)$-transversal of f.

Corollary 4.3. [5] *The following hold:*

(i) *If $\Theta_1(X) \cdot f + \mathcal{M}_4^{k+2} \supset \mathcal{M}_4^{k+1}$, then f is k-$\mathcal{R}(X)$-determined;*
(ii) *If every vector field in $\Theta(X)$ vanishes at the origin and $\Theta(X) \cdot f + \mathcal{M}_4^{k+2} \supset \mathcal{M}_4^{k+1}$, then f is $(k+1)$-$\mathcal{R}(X)$-determined.*

Definition 4.4. A germ of a smooth 1-parameter family of functions

$$F : (\mathbb{R}^4 \times \mathbb{R}, (0,0)) \to (\mathbb{R}, 0)$$

with $F(0,t) = 0$ for t small is said to be k-$\mathcal{R}(\mathtt{X})$-trivial if there exists a germ of a 1-parameter family of diffeomorphisms $H : (\mathbb{R}^4 \times \mathbb{R}, (0,0)) \to (\mathbb{R}^4, 0)$, with H_t preserving $\mathtt{X}$, such that $H(x,0) = 0$, $H(0,t) = 0$ (for small t) and

$$F(H(x,t),t) = F(x,0) + \psi(x,t)$$

for some $\psi \in \mathcal{M}_4^{k+1} \subset \mathcal{E}_5$. If $\psi \equiv 0$, then F is said to be $\mathcal{R}(\mathtt{X})$-trivial.

The next result about trivial families will be needed.

Proposition 4.5. [5] *Let $F : (\mathbb{R}^4 \times \mathbb{R}, (0,0)) \to (\mathbb{R}, 0)$ be a smooth family of functions such that $F(0,t) = 0$ for t small enough. Also, let $\xi_1, \dots, \xi_p$ be vector fields in $\Theta(\mathtt{X})$ that vanish at the origin. Then, the family F is k-$\mathcal{R}(\mathtt{X})$-trivial if $\frac{\partial F}{\partial t} \in \langle \xi_1(F), \dots, \xi_p(F)\rangle + \mathcal{M}_4^{k+1} \subset \mathcal{E}_5$.*

Two families of germs of functions F and $G : (\mathbb{R}^4 \times \mathbb{R}^a, (0,0)) \to (\mathbb{R}, 0)$ are $P - \mathcal{R}^+(\mathtt{X})$-equivalent if there exist a germ of a diffeomorphism $\Psi : (\mathbb{R}^4 \times \mathbb{R}^a, (0,0)) \to (\mathbb{R}^4 \times \mathbb{R}^a, (0,0))$ preserving $(\mathtt{X} \times \mathbb{R}^a, (0,0))$ and of the form $\Psi(x,u) = (\alpha(x,u), \psi(x,u))$ and a germ $c : (\mathbb{R}^a, 0) \to \mathbb{R}$ such that $G(x,u) = F(\Psi(x,u)) + c(u)$.

A family F is said to be an $\mathcal{R}^+(\mathtt{X})$-versal deformation of $F_0(x) = F(x,0)$ if any other deformation G of F_0 can be written in the form $G(x,u) = F(\Psi(x,u)) + c(u)$ for some germs of smooth mappings Ψ and c as above with Ψ not necessarily a germ of diffeomorphism.

Proposition 4.6. [5] *A deformation $F : (\mathbb{R}^4 \times \mathbb{R}^a, (0,0)) \to (\mathbb{R}, 0)$ of a germ of function f on $\mathtt{X}$ is $\mathcal{R}^+(\mathtt{X})$-versal if and only if*

$$L\mathcal{R}_e(\mathtt{X}) \cdot f + \mathbb{R}.\{1, \dot{F}_1, \dots, \dot{F}_a\} = \mathcal{E}_4,$$

where $\dot{F}_i(x) = \frac{\partial F}{\partial u_i}(x,0)$.

Theorem 4.7. *Let $\mathtt{X}$ be the germ of the $\mathcal{A}$-model surface parametrised by $f(x,y) = (x, xy, y^2, y^3)$. Then, any germ of a $\mathcal{R}(\mathtt{X})$-finitely determined submersion in $\mathcal{M}_4$ with $\mathcal{R}(\mathtt{X})$-codimension $\leqslant 3$ is $\mathcal{R}(\mathtt{X})$-equivalent to one of the germs in Table 1.*

Table 1. Germs of submersions in $\mathcal{M}_4$ of $\mathcal{R}(\mathtt{X})$-codimension $\leqslant 3$

Normal form	$d(f, \mathcal{R}(\mathtt{X}))$	$\mathcal{R}(\mathtt{X})$-*versal deformation*
X	0	X
$\pm Z \pm X^2$	1	$\pm Z \pm X^2 + a_1 X$
$\pm Z + X^3$	2	$\pm Z + X^3 + a_1 X + a_2 X^2$
$\pm Z \pm X^4$	3	$\pm Z \pm X^3 + a_1 X + a_2 X^2 + a_3 X^3$
Y	2	$Y + a_1 X + a_2 Z$
$\pm W \pm X^2$	3	$\pm W \pm X^2 + a_1 X + a_2 Y + a_3 Z$

Proof. We shall consider the vector fields in Proposition 4.1. The linear change of coordinates in $\mathcal{R}(\mathtt{X})$ obtained by integrating the 1-jets of the vector fields in $\Theta(\mathtt{X})$ are:

$$\begin{aligned} &\eta_1 = (e^\alpha X, e^\alpha Y, Z, W),\ \alpha \in \mathbb{R}, && \eta_2 = (X, Y, Z + \alpha Y, W),\ \alpha \neq 0,\\ &\eta_3 = (X, e^\alpha Y, e^{2\alpha} Z, e^{3\alpha} W),\ \alpha \in \mathbb{R}, && \eta_4 = (X + \alpha Y, Y, Z, W),\ \alpha \neq 0,\\ &\eta_5 = (X + \alpha Z, Y + \alpha W, Z, W),\ \alpha \neq 0, && \eta_6 = (X, Y, Z + \alpha W, W),\ \alpha \neq 0,\\ &\eta_7 = (X + \alpha W, Y, Z, W),\ \alpha \neq 0, && \eta_8 = (-X, -Y, Z, W). \end{aligned}$$

Consider the non zero 1-jet $g = aX + bY + cZ + dW$. If $a \neq 0$, after changes of coordinates ($\eta_i,\ i = 4, 5, 7, 1, 8$, in this order) we get X. If $a = 0 \neq c$, (using $\eta_i,\ i = 2, 6, 3$) we get $\pm Z$. If $a = c = 0 \neq b$, (using $\eta_i,\ i = 5, 1, 8$) we have Y. At last, if $a = b = c = 0 \neq d$, using η_3, we have W.

(i) Consider the 1-jet $g = X$. This case is the most simple. Notice that every vector field $\xi_i \in \Theta(\mathtt{X})$ vanishes at the origin and $\mathcal{M}_4 \subset \Theta(\mathtt{X}) \cdot g + \mathcal{M}_4^2$, so g is 1-$\mathcal{R}(\mathtt{X})$-determined by Corollary 4.3. Also,

$$\mathcal{R}(\mathtt{X})\text{-}cod(g) = \dim_{\mathbb{R}}(\mathcal{M}_4/\Theta(\mathtt{X}) \cdot g) = 0.$$

(ii) Consider the 1-jet $g = \pm Z$. For $k \geqslant 2$, the complete k-$\mathcal{R}(\mathtt{X})$-transversal of g is given by $\pm Z + \delta X^k$. If $\delta \neq 0$, using η_1 we get $g_k = \pm Z + (-1)^{k+1} X^k$. For g_k,

$$\mathcal{M}_4^k \subset \Theta(\mathtt{X}) \cdot g_k + \mathcal{M}_4^{k+1},$$

that is, g_k is k-$\mathcal{R}(\mathtt{X})$-determined and $\mathcal{R}(\mathtt{X})$-$cod(g_k) = k-1$.

(iii) Now, consider the 1-jet $g = Y$. The complete 2-$\mathcal{R}(\mathtt{X})$-transversal of g is given by

$$g = Y + \beta X^2 + \gamma Z^2 + \delta XZ.$$

Consider g as a 1-parameter family of germs of functions parametrised by γ. Then $\partial g/\partial \gamma = Z^2 \in \langle \xi_1(g), \ldots, \xi_{13}(g)\rangle + \mathcal{M}_4^3$. So, by Proposition 4.5, g is equivalent to $Y + \beta X^2 + \delta XZ$. In a similar way, we can prove that considering g a family parametrised by δ and then by β, we have g equivalent to Y. Moreover, $g = Y$ is 2-$\mathcal{R}(\mathtt{X})$-determined, since $\mathcal{M}_4^2 \subset \Theta(\mathtt{X}) \cdot g + \mathcal{M}_4^3$ and $\mathcal{R}(\mathtt{X})$-$cod(g) = 2$.

(iv) The last 1-jet is $g = W$. Now, the complete 2-$\mathcal{R}(\mathtt{X})$ transversal is

$$g = \pm W + \alpha X^2 + \beta Z^2 + \gamma XY + \delta XZ.$$

Considering g a 1-parameter family of germs of functions parametrised by β, it is possible to show that it 2-$\mathcal{R}(\mathtt{X})$-trivial and so g is equivalent to $\pm W + \alpha X^2 + \gamma XY + \delta XZ$. At this point, we split the study in two cases. If $\alpha \neq 0$, using again the triviality result, we show that the germ is equivalent to $\pm W \pm X^2$ (after using η_1). Besides, g is now 2-$\mathcal{R}(\mathtt{X})$-determined and $\mathcal{R}(\mathtt{X})$-$cod(g) = 3$. However, when $\alpha = 0$, the germs obtained have stratum codimension greater than 3 and will not be considered here.

Therefore, we conclude the proof. □

4.2. **Contact with hyperplanes.** The following result gives us a generic normal form up to order 3 for any surface whose local parametrisation is $\mathcal{A}$-equivalent to the singularity I_1.

Theorem 4.8. *Let $f_1 : (\mathbb{R}^2, 0) \to (\mathbb{R}^4, 0)$ be a map germ $\mathcal{A}$-equivalent to $f(x, y) = (x, xy, y^2, y^3)$. Then, there are smooth change of coordinates in the source and isometries in the target that make f_1 equivalent to*

$$\left(x, xy, \sum_{i+j=2,3} b_{ij}x^iy^j, c_{20}x^2 + \sum_{i+j=3} c_{ij}x^iy^j\right) + O(4),$$

with $b_{ij}, c_{ij} \in \mathbb{R}$ and $b_{02}c_{03} \neq 0$.

Proof. In [1], it is proved that I_1 is $\mathcal{R}^2 \times \mathcal{O}(4)$-equivalent to

$$(x, y) \mapsto \left(x, xy + a_{03}y^3, \sum_{i+j=2,3} b_{ij}x^iy^j, c_{20}x^2 + \sum_{i+j=3} c_{ij}x^iy^j\right) + O(4),$$

with $b_{02}, c_{03} \neq 0$. In order to obtain the desired normal form, we have to eliminate $a_{03}y^3$. Consider the change T and the angle $\theta = \arctan(a_{03}/c_{03})$, such that

$$(\sin\theta, \cos\theta) = (a_{03}, c_{03})/\sqrt{a_{03}^2 + c_{03}^2}:$$

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{pmatrix}.$$

Hence, we obtain

$$\left(x, \cos\theta xy - \sin\theta(c_{20}x^2 + c_{30}x^3 + c_{21}x^2y + c_{12}xy^2), \sum_{i+j=2,3} b_{ij}x^iy^j, \bar{c}_{20}x^2 + \sum_{i+j=3} \bar{c}_{ij}x^iy^j\right).$$

To eliminate the monomials x^2, x^3, x^2y and xy^2 from the second coordinate, take the change in the source given by:

$$x \mapsto x' = x \text{ and } y \mapsto y' = y + \frac{\sin\theta}{\cos\theta}(c_{20}x + c_{30}x^2 + c_{21}xy + c_{12}y^2).$$

Therefore, we have

$$\left(x, \cos\theta xy, \sum_{i+j=2,3} a_{ij}x^iy^j, \bar{c}_{20}x^2 + \sum_{i+j=3} \bar{c}_{ij}x^iy^j\right) + O(4).$$

Finally, a change of coordinates in the source provides the generic normal form. □

Given a corank 1 surface $M \subset \mathbb{R}^4$ at p, locally parametrised by the normal form in Theorem 4.8, we can deduce some information: The plane E_p is the YZ-plane, the umbilic curvature is given by $\kappa_u(p) = 2|c_{20}|$ and the tangent cone C_pM is the XZ-plane.

Let $M \subset \mathbb{R}^4$ be a corank 1 surface locally parametrised by a map germ $\mathcal{A}$-equivalent to I_1. The *family of height functions* of M is given by

$$H : M \times \mathbb{S}^3 \to \mathbb{R},\ H(p, v) = \langle p, v\rangle.$$

Fixing $v \in \mathbb{S}^3$, the singularities of the height function $h_v(p) = H(p, v)$ measures the contact of M with the hyperplane orthogonal to v, denoted by Γ_v. This contact is also described by the one obtained using the fibers $\{g = 0\}$ from Theorem 4.7, where g appears in the proof of Theorem 4.7. Using a local parametrisation of M given by Theorem 4.8, we have

$$h_v(x, y) = xv_1 + xyv_2 + \sum_{i+j=2,3} b_{ij}x^iy^jv_3 + c_{20}x^2v_4 + \sum_{i+j=3} c_{ij}x^iy^jv_4,$$

for $v = (v_1, v_2, v_3, v_4) \in \mathbb{S}^3$.

The height function h_v is singular at the origin if and only if $v_1 = 0$. Geometrically, this means that Γ_v contains T_pM. Hence, if $v_1 \neq 0$, h_v is regular and the fiber Γ_v is transversal to C_pM and contains E_p. This contact is also described by the contact of the zero fiber of $g_1 = X$ with the model surface $\mathtt{X}$.

Consider $S \subset \mathbb{R}^4$ the associated regular surface of M, as done before (see Theorem 3.8). Given a binormal direction of M, $\nu \in N_pM$, $\mathbf{u}$ will denote the corresponding asymptotic direction (which is also an asymptotic direction of S). Furthermore, τ is the torsion of the normal section of the surface S tangent to the asymptotic direction $\mathbf{u}$. Let $CS(\varepsilon)$ be the canal hypersurface of S. We denote by $\mathcal{C}$ the curve of cuspidal edge points of the Gauss map of $CS(\varepsilon)$.

Proposition 4.9. *Let $v = (0, v_2, v_3, 0)$ with $v_3 \neq 0$. The hyperplane Γ_v is tangent to T_pM and transversal to C_pM and E_p. The height function h_v can have singularities of type $A^{\pm}_{k-1}$, $k = 2, 3, 4$ which are modeled by the contact of the zero fibre of the submersions*

$$g_{2k} = \pm Z + (-1)^{k+1}X^k$$

with the model surface $\mathtt{X}$ (i.e. modeled by the composition of the submersions with the parametrisation of the model surface), respectively. It has a singularity of type A_1 (Morse) if and only if $v \in N_pM$ is not a binormal direction. For more degenerate singularities, this configuration has three possibilities:

(i) *If p is a hyperbolic point, the singularity is of type A_2 iff v is a binormal direction of M and $\tau \neq 0$. Finally, the height function has an A_3 singularity iff v is a binormal direction, $\tau = 0$ and the asymptotic direction $\boldsymbol{u}$ of S is transversal to the curve $\mathcal{C}$ of cuspidal edge points of the Gauss map. See Table 2.*

(ii) *If p is a parabolic point, h_v has singularity of type A_2 iff v is a binormal direction of M and the associated asymptotic direction $\boldsymbol{u}$ is transversal to the parabolic curve of S. The singularity is of type A_3 iff v is a binormal direction and $\boldsymbol{u}$ is tangent to the parabolic curve of S with first order contact.*

(iii) *If p is elliptic, the height function can only have singularity of type A_1.*

Proof. The proof follows from Theorem 2.2 and Theorem 3.8 since both surfaces M and S have the same height function. However we will present some calculations for the case p hyperbolic, that is, $b_{20} > 0$. Let $v = (0, v_2, v_3, 0)$ with $v_3 \neq 0$. For the normal form in Theorem 4.8, E_p is the YZ-plane and the tangent cone C_pM is the XZ-plane. Hence, Γ_v is transversal to E_p and C_pM. So this situation is modeled by the zero fiber of $g = \pm Z + (-1)^{k+1}X^k$, $k = 2, 3, 4$ and the model surface $\mathtt{X}$.

Taking $v = (0, v_2, 1, 0)$, the height function is given by

$$h_v(x, y) = (b_{11} + v_2)xy + b_{20}x^2 + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3,$$

where $b_{02} > 0$. The determinant of the Hessian matrix of h_v is given by

$$\det(\mathcal{H}(h_v(x, y))) = 4b_{20}b_{02} - (v_2 + b_{11})^2.$$

So, h_v has a singularity of type A_1 (Morse) if and only if, $v_2 \neq -b_{11} \pm 2\sqrt{b_{20}b_{02}}$, which is equivalent to v not being a binormal direction (see [1]).

The conditions for h_v to have a singularity of type A_2 are: v is a binormal direction and

$$b_{30} \mp \frac{b_{21}\sqrt{b_{20}b_{02}}}{b_{02}} + \frac{b_{12}b_{20}}{b_{02}} \mp \frac{b_{03}b_{20}\sqrt{b_{20}b_{02}}}{b_{02}^2} \neq 0.$$

On the other hand, the kernel of the Hessian of the height function h_v with

$$v = (0, v_2, 1, 0) = (0, -b_{11} \pm 2\sqrt{b_{20}b_{02}}, 1, 0)$$

is the asymptotic direction $\mathbf{u} = (u_1, \mp\sqrt{b_{20}b_{02}}u_1/b_{02})$. The normal section along this asymptotic direction can be parametrised by

$$\begin{aligned}\gamma(u_1) \;&= (u_1, \mp\tfrac{\sqrt{b_{20}b_{02}}}{b_{02}}u_1, \mp\tfrac{\sqrt{b_{20}b_{02}}}{b_{02}}u_1^2, (2b_{20} \mp \tfrac{b_{11}\sqrt{b_{20}b_{02}}}{b_{02}})u_1^2 \\ &+(b_{30} \mp \tfrac{b_{21}\sqrt{b_{20}b_{02}}}{b_{02}} + \tfrac{b_{12}b_{20}}{b_{02}} \mp b_{03}(\tfrac{\sqrt{b_{20}b_{02}}}{b_{02}})^3)u_1^3) + O(4).\end{aligned}$$

Consider the rotation on the target given by the matrix

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\theta = \arctan(\pm\frac{\sqrt{b_{20}b_{02}}}{b_{02}})$. Therefore $\sin\theta \mp \frac{\sqrt{b_{20}b_{02}}}{b_{02}}\cos\theta = 0$, and since $b_{20}, b_{02} > 0$, we have $\cos\theta \pm \frac{\sqrt{b_{20}b_{02}}}{b_{02}}\sin\theta \neq 0$. Let $\tilde{\gamma}$ be the curve obtained by rotating γ with the previous rotation. The second component of $\tilde{\gamma}$ is zero, so we can consider it as curve in $\mathbb{R}^3$. The torsion of $\tilde{\gamma}$ (and hence of γ) is given by

$$\tau(0) = \frac{\frac{\mp 12\sqrt{b_{20}b_{02}}}{b_{02}}(\cos\theta \pm \frac{\sqrt{b_{20}b_{02}}}{b_{02}}\sin\theta)}{||\tilde{\gamma}'(0) \times \tilde{\gamma}''(0)||^2}\left(b_{30} \mp \frac{b_{21}\sqrt{b_{20}b_{02}}}{b_{02}} + \frac{b_{12}b_{20}}{b_{02}} \mp \frac{b_{03}b_{20}\sqrt{b_{20}b_{02}}}{b_{02}^2}\right).$$

Hence, $\tau(0) \neq 0$ if and only if $b_{30} \mp \frac{b_{21}\sqrt{b_{20}b_{02}}}{b_{02}} + \frac{b_{12}b_{20}}{b_{02}} \mp \frac{b_{03}b_{20}\sqrt{b_{20}b_{02}}}{b_{02}^2} \neq 0$, which is precisely the condition to have an A_2 singularity. $\square$

The singularities of the height function h_v at a hyperbolic point are presented in Table 2. For each possibility of $v \in \mathbb{S}^3$ we give the relative position of Γ_v, E_p and C_pM, in addition to the submersion whose contact of the zero fibre with the model surface $\mathtt{X}$ models the singularity type.

Table 2. Types of singularities of h_v (hyperbolic point)

Vector	Singularity type	submersion
$v=(1,0,0,0)$ $E_p \subset \Gamma_v \pitchfork T_pM, C_pM$	submersion	$g_1 = X$
$v=(0,v_2,v_3,0)$ $\Gamma_v \pitchfork E_p, C_pM$	$A_1 \Leftrightarrow v$ is not binormal $A_2 \Leftrightarrow v$ is binormal and $\tau \neq 0$ $A_3 \Leftrightarrow v$ is binormal, $\tau = 0$ and $\mathbf{u} \pitchfork \mathcal{C}$.	$g_{2k} = \pm Z + (-1)^{k+1}X^k$ $k = 2,3,4$
$v=(0,v_2,0,0)$ $C_pM \subset \Gamma_v \pitchfork E_p$	A_1	$g_3 = Y$
$v=(0,0,0,v_4)$ $E_p, C_pM \subset \Gamma_v$,	$A_2 \Leftrightarrow \kappa_u(p) \neq 0$	$g_4 = \pm W \pm X^2$

τ is the torsion of the normal section along an asymptotic direction which is given in the proof of Proposition 4.9.

Corollary 4.10. *The hyperplane Γ_v is an osculating hyperplane if and only if it is transversal to E_p and the height function has singularity of type $A_{\geqslant 2}$.*

Proposition 4.11. *Let $v = (0, v_2, 0, 0)$, $v_2 \neq 0$, the hyperplane Γ_v contains the tangent cone C_pM and is transversal to E_p. The height function has singularity of type A_1, which is described by the contact of the zero fiber of the submersion $g_3 = Y$ with the model surface $\mathtt{X}$.*

Proof. When $v = (0, v_2, 0, 0)$, $v_2 \neq 0$, we can take $v = (0, 1, 0, 0)$ and the height function is given by $h_v(x, y) = xy + O(4)$, whose singularity is of type A_1. $\square$

Proposition 4.12. *Let $v = (0, 0, 0, v_4)$, $v_4 \neq 0$. The hyperplane Γ_v contains both E_p and C_pM. The height function h_v has singularity of type $A_{\geq 2}$, which is described by the contact of the zero fiber of the submersion $g_4 = \pm W \pm X^2$ with the model surface $\mathtt{X}$ if and only if $\kappa_u(p) \neq 0$.*

Proof. Taking $v = (0, 0, 0, 1)$, the height function is given by

$$h_v(x, y) = c_{20}x^2 + \sum_{i+j=3} c_{ij}x^i y^j + O(4).$$

It has singularity of type $A_{\geq 2}$ if and only if $c_{20} \neq 0$, which is equivalent to $\kappa_u(p) = 2|c_{20}| \neq 0$. $\square$

References

[1] P. Benedini Riul, R. Oset Sinha and M. A. S. Ruas *The geometry of corank* 1 *surfaces in* $\mathbb{R}^4$. Q. J. Math. 70 (2019), no. 3, 767–795. DOI: 10.1093/qmath/hay064

[2] J. W. Bruce and A. C. Nogueira *Surfaces in* $\mathbb{R}^4$ *and duality.* Quart. J. Math. Oxford Ser. 49 (1998), 433–443.

[3] J. W. Bruce and R. M. Roberts *Critical points of functions on analytic varieties.* Topology, 27 (1) (1988), 57–90. DOI: 10.1016/0040-9383(88)90007-9

[4] J. W. Bruce and F. Tari *Families of surfaces in* $\mathbb{R}^4$. Proc. Edinb. Math. Soc. (2) 45 (2002), no. 1, 181–203.
[5] J. W. Bruce and J. M. West *Functions on a crosscap. Math. Proc. Cambridge Philos. Soc.* 123 (1998), 19–39.
[6] J. N. Damon *Topological triviality and versality for subgroups of* $\mathcal{A}$ *and* $\mathcal{K}$. Amer. Math. Soc. 75 (1988), no. 389.
[7] F. S. Dias and F. Tari *On the geometry of the cross-cap in Minkowski 3- space and binary differential equations*. Tohoku Math. J. (2) 68, (2016), no.2, 293–328.
[8] T. Fukui and M. Hasegawa *Fronts of Whitney umbrella - a differential geometric approach via blowing up*. J. Singul. 4 (2012), 35–67. DOI: 10.5427/jsing.2012.4c
[9] R. Garcia, D. K. H. Mochida, M. C. Romero Fuster and M. A. S. Ruas *Inflection points and topology of surfaces in 4-space*. Trans. Amer. Math. Soc. 352 (2000), 3029–3043.
[10] M. Hasegawa, A. Honda, K. Naokawa, M. Umehara and K. Yamada *Intrinsic invariants of cross caps*. Selecta Math. (N.S.) 20 (2014), no. 3, 769–785.
[11] M. Hasegawa, A. Honda, K. Naokawa, K. Saji, M. Umehara and K. Yamada *Intrinsic properties of surfaces with singularities*. Internat. J. Math. 26 (2015), no. 4, 1540008, 34 pp.
[12] S. Izumiya, M. C. Romero Fuster, M. A. S. Ruas and F. Tari *Differential Geometry from Singularity Theory Viewpoint*. World Scientific Publishing Co Pte Ltd, Singapore (2015). DOI: 10.1142/9108
[13] C. Klotz, O. Pop and J. H. Rieger *Real double-points of deformations of* $\mathcal{A}$*-simple map-germs from* $\mathbb{R}^n$ *to* $\mathbb{R}^{2n}$. Math. Proc. Camb. Phil. Soc. (2007), 142-341.
[14] M. Kokubu, W. Rossman, K. Saji, M. Umehara and K. Yamada, *Singularities of flat fronts in hyperbolic space*. Pacific J. Math. 221 (2005), no. 2, 303–351.
[15] J. A. Little *On singularities of submanifolds of higher dimensional Euclidean spaces*. Ann. Mat. Pura Appl. 83 (4) (1969), 261–335.
[16] L. F. Martins and J. J. Nuño-Ballesteros, *Contact properties of surfaces in* $\mathbb{R}^3$ *with corank* 1 *singularities*. Tohoku Math. J. 67 (2015), 105–124.
[17] L. F. Martins and K. Saji, *Geometric invariants of cuspidal edges*. Canadian J. Math 68 (2016), no. 2, 445–462. DOI: 10.4153/cjm-2015-011-5
[18] L. F. Martins and K. Saji, *Geometry of cuspidal edges with boundary*. Topology and its applications, v. 234 (2018), 209–219. DOI: 10.1016/j.topol.2017.11.024
[19] D. K. H. Mochida, M. C. Romero Fuster and M. A. S. Ruas, *The geometry of surfaces in* 4*-space from a contact viewpoint*.
[20] D. K. H. Mochida, M. C. Romero Fuster and M. A. S. Ruas, *Osculating hyperplanes and asymptotic directions of codimension two submanifolds of Euclidean spaces*. Geom. Dedicata 77 (1999), 305–315.
[21] K. Naokawa, M. Umehara and K. Yamada, *Isometric deformations of cuspidal edges*. Tohoku Math. J. 68, (2016), 73–90. DOI: 10.2748/tmj/1458248863
[22] J. J. Nuño-Ballesteros and F. Tari, *Surfaces in* $\mathbb{R}^4$ *and their projections to* 3*-spaces*. Proc. Roy. Soc. Edinburgh Sect. A, 137 (2007), 1313–1328.
[23] R. Oset Sinha and K. Saji *On the geometry of folded cuspidal edges*. Rev. Mat. Complut. 31 (2018), no. 3, 627–650. DOI: 10.1007/s13163-018-0257-6
[24] R. Oset Sinha and F. Tari, *Projections of surfaces in* $\mathbb{R}^4$ *to* $\mathbb{R}^3$ *and the geometry of their singular images*. Rev. Mat. Iberoam. 32 (2015), no. 1, 33–50.
[25] R. Oset Sinha and F. Tari, *On the flat geometry of the cuspidal edge*. Osaka J. Math. 55 (2018), no. 3, 393–421.
[26] M. C. Romero Fuster, *Semiumbilics and geometrical dynamics on surfaces in* 4*-spaces*. Real and complex singularities, Contemp. Math., 354, Amer. Math. Soc., Providence, RI. (2004) 259–276.
[27] M. C. Romero Fuster, M. A. S. Ruas and F. Tari, *Asymptotic curves on surfaces in* $\mathbb{R}^5$. Communications in Contemporary Maths. 10 (2008), 1–27.
[28] K. Saji, M. Umehara, and K. Yamada, *The geometry of fronts*. Ann. of Math (2) 169 (2009), 491–529. DOI: 10.4007/annals.2009.169.491

P. Benedini Riul, Instituto de Ciências Matemáticas e de Computação - USP, Av. Trabalhador são-carlense, 400 - Centro, CEP: 13566-590 - São Carlos - SP, Brazil
Email address: pedro.benedini.riul@gmail.com

R. Oset Sinha, Departament de Matemàtiques, Universitat de València, Campus de Burjassot, 46100 Burjassot, Spain
Email address: raul.oset@uv.es

Journal *of* Singularities
Volume 21 (2020), 15-29

Proc. of Geometric and Algebraic Singularity Theory, Będlewo, 2017
DOI: 10.5427/jsing.2020.21b

FRONTS OF CONTROL-AFFINE SYSTEMS IN $\mathbb{R}^3$

ILYA BOGAEVSKY

To Goo Ishikawa on the occasion of his sixtieth birthday

Abstract. We consider a control-affine system in three-dimensional space with control parameters belonging to a two-dimensional disk and study its fronts evolving from a point for small times. We prove that generically the Legendrian lifts of such fronts have standard singularities and there are only two principally different typical cases — hyperbolic and elliptic.

Introduction

The ends of local time-optimal trajectories of a control system that start at a given point form its front depending on time. We consider control-affine systems in three-dimensional space with control parameters belonging to a two-dimensional disk and study singularities of their fronts for small times.

If our system is linear-control then it defines a sub-Riemannian structure and its fronts are described in [1] in the case that the sub-Riemannian structure is contact. For such a typical system the fronts have infinite number of swallowtails at any neighborhood of the initial point. Therefore their structure is complicated but it becomes much more simpler from the viewpoint of contact geometry. Namely, let us consider the Legendrian surface consisting of all contact elements being tangent to a considered front and cooriented outside. According to our result this submanifold is smooth except two points lying over the initial point. Moreover, these singularities are standard for all contact sub-Riemannian structures — not only for typical ones. It means that all of them have the same normal form with respect to contact diffeomorphisms of the ambient space.

A considered control-affine system can have hyperbolic and elliptic points introduced in [6]. The sets formed by them are open always and its union is dense for a typical system. In particular, a linear-control system cannot have hyperbolic points at all and is elliptic exactly at the points where the corresponding sub-Riemannian structure is contact.

According to the present paper the Legendrian surface consisting of all contact elements being tangent to a front and cooriented outside is homeomorphic to the two-dimensional sphere and has the following singularities.

If the initial point is elliptic then the considered Legendrian surface is smooth outside two points where it has singularities E_2. If the initial point is hyperbolic then the considered Legendrian surface is smooth outside two disjoint segments, where it has singularities H_1 at their inner points and H_2 at their four ends. All singularities with the same name (E_2, H_1, or H_2) are equivalent to each other with respect to contact diffeomorphisms of the ambient space. In particular, their normal forms do not contain continuous invariants.

Non-typical examples of instant fronts of elliptic (left) and hyperbolic (right) points are shown in Fig. 1. (These figures are published in [7] and [6] respectively.)

Partially supported by RFBR-16-01-00766.

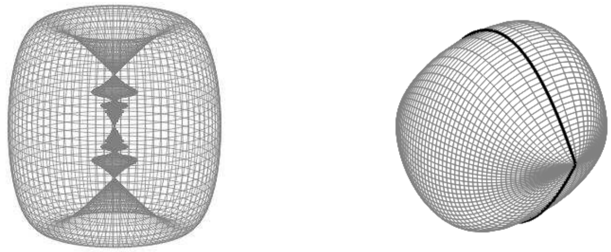


Figure 1. Non-typical examples of instant fronts of elliptic (left) and hyperbolic (right) points

1. Definitions

1.1. **Instant fronts of control-affine systems in $\mathbb{R}^3$.** We consider a control-affine system in $\mathbb{R}^3$ with control parameters $u = (u_1, u_2)$:

$$\dot{\mathbf{x}} = \xi_0(\mathbf{x}) + u_1\xi_1(\mathbf{x}) + u_2\xi_2(\mathbf{x}), \quad u_1^2 + u_2^2 \leq 1 \tag{1}$$

as a family of vector fields in $\mathbb{R}^3$ depending on u. Here $\mathbf{x} \in \mathbb{R}^3$, $(\mathbf{x}, \dot{\mathbf{x}}) \in T^*\mathbb{R}^3$, and ξ_0, ξ_1, ξ_2 are bounded smooth[1] vector fields on $\mathbb{R}^3$ such that the vectors $\xi_1(\mathbf{x})$ and $\xi_2(\mathbf{x})$ are linearly independent at any point $\mathbf{x} \in \mathbb{R}^3$.

Definition. A Lipschitzian mapping $\varphi : [0, T] \to \mathbb{R}^3$, $T > 0$ is called a *trajectory* of the control-affine system (1) if there exist measurable functions $\tilde{u}_1, \tilde{u}_2 : [0, T] \to \mathbb{R}$ such that the equations

$$\frac{d\varphi}{dt} = \xi_0(\varphi(t)) + \tilde{u}_1(t)\ \xi_1(\varphi(t)) + \tilde{u}_2(t)\ \xi_2(\varphi(t)), \quad \tilde{u}_1^2(t) + \tilde{u}_2^2(t) \leq 1$$

hold for almost all $t \in [0, T]$.

Definition. The ends $\varphi(T)$ of all trajectories $\varphi : [0, T] \to \mathbb{R}^3$ of the system (1) starting at a given point $\varphi(0) = \mathbf{x}_0$ form the *attainable set* of the point $\mathbf{x}_0 \in \mathbb{R}^3$ for the time T:

$$\mathcal{A}_{\mathbf{x}_0}(T) = \left\{\mathbf{x} \in \mathbb{R}^3 \mid \exists\ \varphi \text{ s. t. } \varphi(0) = \mathbf{x}_0,\ \varphi(T) = \mathbf{x}\right\}.$$

Its boundary is denoted by $\partial\mathcal{A}_{\mathbf{x}_0}(T)$.

Definition. If a trajectory $\varphi : [0, T] \to \mathbb{R}^3$ of the system (1) satisfies the condition

$$\varphi(T) \in \partial\mathcal{A}_{\varphi(0)}(T)$$

then it is called *geometrically optimal*.

Remark. According to Filippov's theorem (Theorem 10.1 in [2]) the attainable set $\mathcal{A}_{\mathbf{x}_0}(T)$ is compact. Therefore its boundary $\partial\mathcal{A}_{\mathbf{x}_0}(T) \subseteq \mathcal{A}_{\mathbf{x}_0}(T)$ consists of the ends $\varphi(T)$ of all geometrically optimal trajectories $\varphi : [0, T] \to \mathbb{R}^3$ starting at the point $\varphi(0) = \mathbf{x}_0$.

[1] "Smooth" means "infinitely smooth" everywhere.

Definition. A trajectory $\varphi : [0,T] \to \mathbb{R}^3$ of the system (1) is called *locally geometrically optimal* if there exists $\delta > 0$ such that

$$\varphi(t) \in \partial\mathcal{A}_{\varphi(t_0)}(t-t_0) \quad \forall\, t_0, t \in [0,T] : t_0 < t < t_0 + \delta.$$

Remark. It is well known that any geometrically optimal trajectory $\varphi : [0,T] \to \mathbb{R}^3$ of the system (1) satisfies the condition

$$\varphi\,(t) \in \partial\mathcal{A}_{\varphi(t_0)}(t-t_0) \quad \forall\, t_0, t \in [0,T] : t_0 < t.$$

In particular, $\varphi : [0,T] \to \mathbb{R}^3$ is locally geometrically optimal.

Definition. The closure of the set formed by the ends $\varphi(T)$ of all locally geometrically optimal trajectories $\varphi : [0,T] \to \mathbb{R}^3$ starting at a given point $\varphi(0) = \mathbf{x}_0$ is called its *instant front* $\mathcal{F}_{\mathbf{x}_0}(T)$ for the time T.

Remark. By definition, $\mathcal{F}_{\mathbf{x}_0}(T) \supseteq \partial\mathcal{A}_{\mathbf{x}_0}(T)$.

1.2. **Relativistic viewpoint: hyperbolic and elliptic points.** Let us consider the space-time $\mathbb{R}^{3+1}$ and fix a point $m = (\mathbf{x}, 0) \in \mathbb{R}^{3+1}$. The control-affine system (1) defines a hyperplane

$$\Pi(m) = \langle \Xi_0(m), \Xi_1(m), \Xi_2(m) \rangle_{\mathbb{R}} \subset T_m\mathbb{R}^{3+1}$$

where

$$\Xi_0 = (\xi_0, 1), \quad \Xi_1 = (\xi_1, 0), \quad \Xi_2 = (\xi_2, 0)$$

are vector fields on $\mathbb{R}^{3+1}$. This hyperplane contains the cone

$$C(m) = \left\{ v_0\, \Xi_0(m) + v_1\, \Xi_1(m) + v_2\, \Xi_2(m) \mid v_0^2 - v_1^2 - v_2^2 = 0 \right\} \subset \Pi(m)$$

formed by all directions belonging to the control-affine system (1) such that $u_1^2 + u_2^2 = 1$.

Let Π be locally defined as the field of 0-spaces of some non-zero 1-form θ on $\mathbb{R}^{3+1}$. The restriction $d\theta|_{\Pi(m)}$ is an antisymmetric 2-form in the three-dimensional vector space $\Pi(m)$. Its kernel

$$k(m) = \ker d\theta|_{\Pi(m)} \subset \Pi(m)$$

has dimension 1 or 3 and is defined by the field Π, i. e. does not depend on the choice of a non-zero 1-form θ.

Definition. Let $m = (\mathbf{x}, 0)$ and the kernel $k(m)$ be one-dimensional. If the kernel $k(m)$ lies in the inner part of the complement of the cone $C(m)$, then the point $\mathbf{x}$ is called *elliptic*. If the kernel $k(m)$ lies in the outer part of the complement of the cone $C(m)$, then the point $\mathbf{x}$ is called *hyperbolic*. If the kernel $k(m)$ belongs to the cone $C(m)$ itself, then the point $\mathbf{x}$ is called *parabolic*. All these cases are shown in Fig. 2.

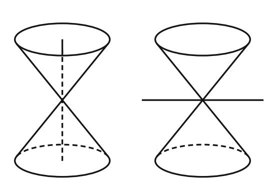


Figure 2. Elliptic, hyperbolic, and parabolic points

Remark. In the present paper parabolic points are not studied.

Example H. All points of the control-affine system

$$\dot{x} = u_1, \quad \dot{y} = u_2, \quad \dot{z} = y, \quad u_1^2 + u_2^2 \le 1$$

are hyperbolic. Here

- $\xi_0 = (0,0,y)$, $\xi_1 = (1,0,0)$, $\xi_2 = (0,1,0)$;
- $\Pi = \{v_0(0,0,y,1) + v_1(1,0,0,0) + v_2(0,1,0,0)\}$;
- $\theta = y\,dt - dz$, $d\theta = dy \wedge dt$, $d\theta|_\Pi = dv_2 \wedge dv_0$;
- $k = \{v_0 = v_2 = 0\} \subset \Pi$;
- $C = \left\{v_0^2 - v_1^2 - v_2^2 = 0\right\}$.

The instant fronts of these control-affine system are diffeomorphic to the shown in Fig. 1 on the right.

Example E. All points of the control-affine system

$$\dot{x} = u_1, \quad \dot{y} = u_2, \quad \dot{z} = u_1 y, \quad u_1^2 + u_2^2 \le 1$$

are elliptic. Here

- $\xi_0 = 0$, $\xi_1 = (1,0,y)$, $\xi_2 = (0,1,0)$;
- $\Pi = \{v_0(0,0,0,1) + v_1(1,0,y,0) + v_2(0,1,0,0)\}$;
- $\theta = y\,dx - dz$, $d\theta = dy \wedge dx$, $d\theta|_\Pi = dv_2 \wedge dv_1$;
- $k = \{v_1 = v_2 = 0\} \subset \Pi$;
- $C = \left\{v_0^2 - v_1^2 - v_2^2 = 0\right\}$.

The instant fronts of these control-affine system are diffeomorphic to the shown in Fig. 1 on the left.

1.3. **Stratified Legendrian submanifolds.**

Definition. A stratified submanifold of a contact space is called *Legendrian* if it is the closure of the smooth Legendrian submanifold being the union of its strata of maximal dimension.

Let $\mathbb{R}^5$ be a contact space with coordinates (P_1, P_2, Q_1, Q_2, U), the origin

$$O = \{P_1 = P_2 = Q_1 = Q_2 = U = 0\},$$

and the contact structure defined as the field of 0-spaces of the contact form

$$\Theta = \frac{1}{2}P\,dQ - \frac{1}{2}Q\,dP - dU.$$

The following stratified submanifolds are Legendrian:

- $\mathcal{H}_1 = \left\{2P_1 \ln P_1^2 + Q_1 = Q_2 = U + P_1^2 = 0\right\}$ where $P_1 \ln P_1^2 = 0$ if $P_1 = 0$;
- $\mathcal{H}_2 = \left\{P_1 = A^2,\ P_2 = AB,\ Q_1 = B^2,\ Q_2 = 2AB\ln A^2,\ U = A^2B^2/2\right\}$ where $A, B \in \mathbb{R}$ are parameters and $A \ln A^2 = 0$ if $A = 0$;
- $\mathcal{E}_2 = \left\{P_1 + iQ_1 = Ue^{i\left(\psi - \frac{1}{U}\right)}, Q_2 + iP_2 = Ue^{i\left(\psi + \frac{1}{U}\right)}, U \ge 0\right\}$ where $i = \sqrt{-1}$, $\psi \in \mathbb{R} \bmod 2\pi\mathbb{Z}$ is a parameter, and $Ue^{i\left(\psi \pm \frac{1}{U}\right)} = 0$ if $U = 0$.

The submanifold $\mathcal{H}_1$ consists of three connected smooth strata: the two surfaces distinguished by the inequalities $P_1 \gtrless 0$ and the line $\mathcal{H}_1^1 = \{P_1 = Q_1 = Q_2 = U = 0\}$.

The submanifold $\mathcal{H}_2$ appears in [4] (Chapter 8) and consists of three connected smooth strata: the surface distinguished by the conditions $A \ne 0$, the open ray

$$\mathcal{H}_2^1 = \{P_1 = P_2 = Q_2 = U = 0, Q_1 > 0\}$$

distinguished by the conditions $A = 0$, $B \ne 0$, and the origin O distinguished by the conditions $A = B = 0$.

The submanifold $\mathcal{E}_2$ consists of two connected smooth strata: the cylinder distinguished by the conditions $U > 0$ and the origin O distinguished by the conditions $U = 0$.

Definition. We say that a two-dimensional stratified Legendrian submanifold Λ of a contact space has a singularity H_1, H_2, or E_2 at a point $\lambda \in \Lambda$ if its germ (Λ, λ) is contact diffeomorphic to the germ $(\mathcal{H}_1, O)$, $(\mathcal{H}_2, O)$, or $(\mathcal{E}_2, O)$ respectively.

For instance, it is clear that the stratified Legendrian submanifold $\mathcal{H}_1$ has a singularity H_1 not only at the origin O but at any point of its stratum $\mathcal{H}_1^1$ as well. Besides, the stratified Legendrian submanifold $\mathcal{H}_2$ has singularities H_1 at all points of its stratum $\mathcal{H}_2^1$ — it is shown in [5] .

2. Main result

Let $ST^*\mathbb{R}^n$ be the space of cooriented contact elements in $\mathbb{R}^n$ with the standard contact structure and $\pi : ST^*\mathbb{R}^n \to \mathbb{R}^n$ be the natural projection. (A cooriented contact element in $\mathbb{R}^n$ is a pair $([\mathbf{p}]; \mathbf{x})$ consisting of a point $\mathbf{x} \in \mathbb{R}^n$ and a ray $[\mathbf{p}] = \{\kappa \mathbf{p} \mid \kappa > 0\}$ generated by a non-zero covector $\mathbf{p} \in T^*_{\mathbf{x}}\mathbb{R}^n \cong \mathbb{R}^{n*}$).

Definition. The image $\pi(\Lambda)$ is called the *front* of a stratified Legendrian submanifold Λ.

Theorem 1. *Let $\mathbf{x}_0$ be any hyperbolic or elliptic point of the control-affine system (1). Then there exists $\delta > 0$ such that for any $T \in (0, \delta)$ the instant front $\mathcal{F}_{\mathbf{x}_0}(T)$ is the front of some stratified Legendrian submanifold of $ST^*\mathbb{R}^3$ denoted by $\mathcal{L}_{\mathbf{x}_0}(T)$ and satisfying the following conditions:*

- *$\mathcal{L}_{\mathbf{x}_0}(T)$ is homeomorphic to the two-dimensional sphere;*
- *in the hyperbolic case $\mathcal{L}_{\mathbf{x}_0}(T)$ is smooth outside two disjoint segments and has singularities H_1 at inner their points and H_2 at their four ends;*
- *in the elliptic case $\mathcal{L}_{\mathbf{x}_0}(T)$ is smooth outside two points where it has singularities E_2.*

Remark. Theorem 1 claims the existence of stratified Legendrian submanifolds $\mathcal{L}_{\mathbf{x}_0}(T)$ satisfying the indicated conditions. The submanifolds $\mathcal{L}_{\mathbf{x}_0}(T)$ themselves are explicitly constructed in Subsection 3.1.

3. Proofs

3.1. **Construction of $\mathcal{L}_{\mathbf{x}_0}(T)$.** Let $ST^*\mathbb{R}^{3+1}$ be the space of cooriented contact elements $([\mathbf{p}, s]; \mathbf{x}, t)$ in the space-time $\mathbb{R}^{3+1}$ with the standard contact structure and $\pi : ST^*\mathbb{R}^{3+1} \to \mathbb{R}^{3+1}$ be the natural projection where $[\mathbf{p}, s] = \{\kappa(\mathbf{p}, s) \mid \kappa > 0\}$ is the open ray generated by a non-zero covector $(\mathbf{p}, s) \in T^*_{\mathbf{x},t}\mathbb{R}^{3+1} \cong \mathbb{R}^{3+1*}$.

Following Section 12.1 in [2] let us construct the Hamiltonian

$$h(\mathbf{p}; \mathbf{x}) = \max_{u_1^2+u_2^2 \le 1} \langle \mathbf{p}, \xi_0(\mathbf{x}) + u_1\xi_1(\mathbf{x}) + u_2\xi_2(\mathbf{x})\rangle$$

$$= \langle \mathbf{p}, \xi_0(\mathbf{x})\rangle + \sqrt{\langle \mathbf{p}, \xi_1(\mathbf{x})\rangle^2 + \langle \mathbf{p}, \xi_2(\mathbf{x})\rangle^2}$$

associated with the control-affine system (1). The Hamiltonian h defines the singular hypersurface

$$\Sigma = \left\{([\mathbf{p}, s]; \mathbf{x}, t) \in ST^*\mathbb{R}^{3+1} \mid h(\mathbf{p}; \mathbf{x}) + s = 0\right\}$$

$$= \left\{\left(\langle \mathbf{p}, \xi_0(\mathbf{x})\rangle + s\right)^2 = \langle \mathbf{p}, \xi_1(\mathbf{x})\rangle^2 + \langle \mathbf{p}, \xi_2(\mathbf{x})\rangle^2,\ \langle \mathbf{p}, \xi_0(\mathbf{x})\rangle + s \le 0\right\},$$

its singularities form the smooth 4-dimensional submanifold:

$$\Sigma^4 = \left\{([\mathbf{p}, s]; \mathbf{x}, t) \in ST^*\mathbb{R}^{3+1} \mid \langle \mathbf{p}, \xi_0(\mathbf{x})\rangle + s = \langle \mathbf{p}, \xi_1(\mathbf{x})\rangle = \langle \mathbf{p}, \xi_2(\mathbf{x})\rangle = 0\right\}.$$

The smooth stratum $\Sigma \setminus \Sigma^4$ (as a hypersurface in a contact space) consists of its characteristics. Such a characteristic satisfies the equations

$$\frac{d\mathbf{p}}{dt} = -\partial_{\mathbf{x}} h(\mathbf{p};\mathbf{x}), \quad \frac{d\mathbf{x}}{dt} = \partial_{\mathbf{p}} h(\mathbf{p};\mathbf{x}), \quad h(\mathbf{p};\mathbf{x}) + s = 0$$

and its projection to the space-time is the graph of a locally geometrically optimal trajectory according to Proposition 12.1 and Section 17.1 in [2].

Definition. The *world* stratified Legendrian submanifold of a point $\mathbf{x}_0 \in \mathbb{R}^3$ is the closure of the union of all characteristics Γ of $\Sigma \setminus \Sigma^4$ passing through $\pi^{-1}(\mathbf{x}_0, 0)$:

$$\Lambda_{\mathbf{x}_0} = \bigcup_{\pi(\Gamma)\ni(\mathbf{x}_0,0)} \Gamma \subset ST^*\mathbb{R}^{3+1}.$$

Let $\tau : ST^*\mathbb{R}^{3+1} \to \mathbb{R}$ be the time function sending $([\mathbf{p}, s]; \mathbf{x}, t) \mapsto t$ and $\varrho : \Sigma \to ST^*\mathbb{R}^3$ be the projection sending $([\mathbf{p}, s]; \mathbf{x}, t) \mapsto ([\mathbf{p}]; \mathbf{x})$ which is correctly defined because Σ does not contain contact elements with $\mathbf{p} = 0$ and $s \neq 0$. The *instant* stratified Legendrian submanifold of the point $\mathbf{x}_0$ at a time T

$$\mathcal{L}_{\mathbf{x}_0}(T) = \varrho\left(\Lambda_{\mathbf{x}_0} \cap \tau^{-1}(T)\right) \subset ST^*\mathbb{R}^3$$

is the projection of the section of the world stratified Legendrian submanifold with the isochrone $\tau = T$.

3.2. **Arnold's singularities of Σ.** For any point $(\mathbf{x}_0, t_0) \in \mathbb{R}^{3+1}$ the fiber $\pi^{-1}(\mathbf{x}_0, t_0)$ contains exactly two singularities of Σ: the contact elements $([\mathbf{p}, s]; \mathbf{x}_0, t_0)$ distinguished by the conditions

$$\langle \mathbf{p}, \xi_0(\mathbf{x}_0)\rangle + s = \langle \mathbf{p}, \xi_1(\mathbf{x}_0)\rangle = \langle \mathbf{p}, \xi_2(\mathbf{x}_0)\rangle = 0.$$

In other words, they are exactly the hyperplane $\Pi(\mathbf{x}_0, t_0)$ introduced in Subsection 1.2 with two possible coorientations and denoted as $\Pi^+(\mathbf{x}_0, t_0)$ and $\Pi^-(\mathbf{x}_0, t_0)$.

Let $O = \Pi^+(\mathbf{x}_0, t_0)$ or $O = \Pi^-(\mathbf{x}_0, t_0)$. Then in a neighborhood of O there exist local coordinates $(P_1, P_2, P_3, Q_1, Q_2, Q_3, U)$ such that the contact structure is given as the field of 0-spaces of the contact form

$$\Theta = \frac{1}{2} P\, dQ - \frac{1}{2} Q\, dP - dU \tag{2}$$

and:

- $\Sigma = \left\{P_1 Q_1 - P_2^2 = 0,\ P_1 + Q_1 \geq 0\right\}$ if $\mathbf{x}_0$ is a hyperbolic point of the control-affine system (1);
- $\Sigma = \left\{P_1^2 + Q_1^2 - P_2^2 = 0,\ P_2 \geq 0\right\}$ if $\mathbf{x}_0$ is an elliptic point of the control-affine system (1).

This fact follows directly from [3] where the equations $P_1 Q_1 - P_2^2 = 0$ and $P_1^2 + Q_1^2 - P_2^2 = 0$ appear as normal forms of degeneracy hypersurfaces for symbols of systems of partial differential equations.

Example H. For the hyperbolic control-affine system

$$\dot{x} = u_1, \quad \dot{y} = u_2, \quad \dot{z} = y, \quad u_1^2 + u_2^2 \leq 1$$

from Example H of Subsection 1.2 we get

$$\langle \mathbf{p}, \xi_1(\mathbf{x})\rangle = p, \quad \langle \mathbf{p}, \xi_2(\mathbf{x})\rangle = q, \quad \langle \mathbf{p}, \xi_0(\mathbf{x})\rangle + s = ry + s.$$

Hence in the affine chart $r = -1$

$$\Sigma = \left\{p^2 + q^2 = (-y + s)^2,\ -y + s \leq 0\right\}$$

and

$$p\,dx + q\,dy - dz + s\,dt = 0$$

is the contact structure. Let

$$U = 2z - qy - px - st$$

and

$$\begin{array}{rclrclrcl} P_1 & = & q - s + y, & P_2 & = & p, & P_3 & = & -q - s + t, \\ Q_1 & = & -q - s + y, & Q_2 & = & 2x, & Q_3 & = & q - s - t. \end{array}$$

In these coordinates

$$\Sigma = \left\{ P_1 Q_1 - P_2^2 = 0,\ P_1 + Q_1 \geq 0 \right\},$$

$$\pi^{-1}(0) = \{x = y = z = t = 0\} = \{Q_1 = P_3, Q_2 = 0, Q_3 = P_1, U = 0\},$$

and the contact structure is given by the equation $\Theta = 0$.

Example E. For the elliptic control-affine system

$$\dot{x} = u_1, \quad \dot{y} = u_2, \quad \dot{z} = u_1 y, \quad u_1^2 + u_2^2 \leq 1$$

from Example E of Subsection 1.2 we get

$$\langle \mathbf{p}, \xi_1(\mathbf{x}) \rangle = p + ry, \quad \langle \mathbf{p}, \xi_2(\mathbf{x}) \rangle = q, \quad \langle \mathbf{p}, \xi_0(\mathbf{x}) \rangle + s = s.$$

Hence in the affine chart $r = -1$

$$\Sigma = \left\{ (p - y)^2 + q^2 = s^2,\ s \leq 0 \right\}$$

and

$$p\,dx + q\,dy - dz + s\,dt = 0$$

is the contact structure. Let

$$U = 2z - qy - px - st$$

and

$$\begin{array}{rclrclrcl} P_1 & = & p - y, & P_2 & = & -s, & P_3 & = & q - x, \\ Q_1 & = & q, & Q_2 & = & -t, & Q_3 & = & p. \end{array}$$

In these coordinates

$$\Sigma = \left\{ P_1^2 + Q_1^2 - P_2^2 = 0,\ P_2 \geq 0 \right\},$$

$$\pi^{-1}(0) = \{x = y = z = t = 0\} = \{Q_1 = P_3, Q_2 = 0, Q_3 = P_1, U = 0\},$$

and the contact structure is given by the equation $\Theta = 0$.

3.3. **Contact vector fields.** A vector field $\vec{K}$ in a contact space is called *contact* if it preserves the contact structure. If the contact structure is given as the field of 0-spaces of a contact form Θ then $K = \Theta(\vec{K})$ is called the *generating function* of $\vec{K}$. We will use the following well known facts:

- $\vec{K}$ is uniquely defined by its generating function $K = \Theta(\vec{K})$;
- $\vec{K}$ is tangent to the hypersurface $\{K = 0\}$ and its characteristics;
- $\vec{K}$ is tangent to a smooth Legendrian submanifold L if and only if $K|_L = 0$.

In our case (2)

$$\vec{K} = \left\{ \begin{array}{rcrcrcr} \dot{P} & = & -\partial_Q K & - & P\,\partial_U K/2 & & \\ \dot{Q} & = & \partial_P K & & & - & Q\,\partial_U K/2 \\ \dot{U} & = & -K & + & P\,\partial_P K/2 & + & Q\,\partial_Q K/2 \end{array} \right. . \tag{3}$$

In particular,

$$\vec{K}(O) = 0 \quad \Longleftrightarrow \quad K(O) = 0 \quad \text{and} \quad d_O K|_{\{dU=0\}} = 0 \tag{4}$$

where $d_O K$ is the differential of the generating function at O and $\{dU = 0\}$ is the contact hyperplane at O.

3.4. **Topology of $\Lambda_{\mathbf{x}_0}$.** If $K = P_1Q_1 - P_2^2$ the formulas (3) give:

$$\dot{P}_1 = -P_1, \ \ \dot{Q}_1 = Q_1, \ \ \dot{P}_2 = 0, \ \ \dot{Q}_2 = -2P_2, \ \ \dot{P}_3 = \dot{Q}_3 = \dot{U} = 0.$$

According to Subsections 3.2 and 3.3 in the hyperbolic case a characteristic of the smooth stratum $\Sigma \setminus \Sigma^4$ is tangent to this contact vector field.

In particular, $P_2 = \text{const}$ along the characteristics. A characteristic with $P_2 \neq 0$ lies in the smooth stratum $\Sigma \setminus \Sigma^4$. In the limit case $P_2 = 0$ we get $P_1Q_1 = 0$, $Q_2 = \text{const}$, $P_3 = \text{const}$, $Q_3 = \text{const}$, $U = \text{const}$, $P_1 + Q_1 \geq 0$. This curve intersects the stratum Σ^4 as $P_1 = Q_1 = 0$ and is not smooth at the intersection point. Such curves and characteristics of $\Sigma \setminus \Sigma^4$ with $P_2 \neq 0$ are called *characteristics of* Σ.

If $K = P_1^2/2 + Q_1^2/2 - P_2^2/2$ the formulas (3) give:

$$\dot{P}_1 = -Q_1, \ \ \dot{Q}_1 = P_1, \ \ \dot{P}_2 = 0, \ \ \dot{Q}_2 = -P_2, \ \ \dot{P}_3 = \dot{Q}_3 = \dot{U} = 0,$$

According to Subsections 3.2 and 3.3 in the elliptic case a characteristic of the smooth stratum $\Sigma \setminus \Sigma^4$ is tangent to this contact vector field.

In particular, $P_2 = \text{const}$. The characteristics with $P_2 > 0$ lie in the smooth stratum $\Sigma \setminus \Sigma^4$. In the limit case $P_2 = 0$ we get a line $P_1 = Q_1 = 0$, $P_3 = \text{const}$, $Q_3 = \text{const}$, $U = \text{const}$ which lies in the stratum Σ^4. Such lines and characteristics of $\Sigma \setminus \Sigma^4$ with $P_2 \neq 0$ are called *characteristics of* Σ.

Characteristics of Σ satisfy the existence–uniqueness–continuity property: any point of Σ belongs a locally unique characteristic which depends continuously on the point.

Lemma 1. *The Legendrian submanifold $\Lambda_{\mathbf{x}_0}$ in some neighborhood of $(\mathbf{x}_0, 0)$ is homeomorphic to the cylinder over the two-dimensional sphere if $\mathbf{x}_0$ is hyperbolic or elliptic point of the control-affine system (1).*

Proof. The Legendrian submanifold is the union of all characteristics of Σ intersecting the set

$$\Sigma \cap \pi^{-1}(\mathbf{x}_0, 0) = \left\{[\mathbf{p}, s] \in ST^*_{\mathbf{x}_0,0}\mathbb{R}^{3+1} \mid h(\mathbf{p}; \mathbf{x}) + s = 0\right\},$$

which is homeomorphic to the two-dimensional sphere. But in some neighborhood of $(\mathbf{x}_0, 0)$ characteristics of Σ satisfy the existence–uniqueness–continuity property. □

3.5. **Basic Lemmas.** Let $\mathbb{R}^7$ be a contact space with coordinates $(P_1, P_2, P_3, Q_1, Q_2, Q_3, U)$, its contact structure be defined as the field of 0-spaces of the contact form (2), and Σ be one of the two hypersurfaces:

$$\Sigma = \left\{P_1Q_1 - P_2^2 = 0\right\} \quad \text{or} \quad \Sigma = \left\{P_1^2 + Q_1^2 - P_2^2 = 0\right\}.$$

The hypersurface consists of the two smooth strata:

$$\Sigma^4 = \{P_1 = Q_1 = P_2 = 0\}$$

and $\Sigma \setminus \Sigma^4$. Let $O \in \Sigma^4$ be the origin $P = Q = U = 0$ and $\mathfrak{L}$ be the space of the germs (L, O) at the origin of all smooth Legendrian submanifolds L that pass through the origin and are transversal to Σ^4. In particular,

$$(L_0, O) \in \mathfrak{L}, \quad L_0 = \{Q_1 = P_3, Q_2 = 0, Q_3 = P_1, U = 0\}.$$

Lemma 2. *The space $\mathfrak{L}$ is arcwise connected and P_2, P_3, Q_3 are coordinates on any $(L, O) \in \mathfrak{L}$.*

Proof. A germ (L, O) of a Legendrian submanifold at the origin is transversal to Σ^4 if and only if the restrictions of the differentials dP_1, dQ_1, and dP_2 to the tangent plane $T_O L$ are linearly independent. Hence:

$$T_O L = \begin{cases} dQ_2 &= a_{11}\, dP_1 + a_{12}\, dQ_1 + a_{13}\, dP_2 \\ dP_3 &= a_{21}\, dP_1 + a_{22}\, dQ_1 + a_{23}\, dP_2 \\ dQ_3 &= a_{31}\, dP_1 + a_{32}\, dQ_1 + a_{33}\, dP_2 \\ dU &= 0 \end{cases} .$$

But the tangent plane $T_O L$ is a Lagrangian subspace of the contact hyperplane $dU = 0$ endowed with a linear symplectic form $d\Theta|_{\Theta=0} = dP \wedge dQ$; and the condition

$$dP \wedge dQ\big|_{T_O L} = 0, \quad dP \wedge dQ\big|_{T_O L} = (1 + a_{21}a_{32} - a_{22}a_{31})\, dP_1 \wedge dQ_1$$
$$+ (-a_{11} + a_{21}a_{33} - a_{23}a_{31})\, dP_1 \wedge dP_2 + (-a_{12} + a_{22}a_{33} - a_{23}a_{32})\, dQ_1 \wedge dP_2$$

gives

$$a_{21}a_{32} - a_{22}a_{31} = -1, \quad a_{11} = a_{21}a_{33} - a_{23}a_{31}, \quad a_{12} = a_{22}a_{33} - a_{23}a_{32}.$$

These three equalities show that the space formed by all tangent planes $T_O L$ such that $(L, 0) \in \mathfrak{L}$ is homotopically equivalent to a circle and, in particular, arcwise connected. But two germs of Legendrian submanifolds at the origin with the same tangent plane can be connected by a continuous path consisting of germs having the same tangent plane. Hence the space $\mathfrak{L}$ is arcwise connected.

The equality $a_{21}a_{32} - a_{22}a_{31} = -1$ implies that the restrictions of the differentials dP_2, dP_3, and dQ_3 to the tangent plane $T_O L$ are linearly independent. So P_2, P_3, Q_3 are coordinates on $(L, O) \in \mathfrak{L}$. □

LEMMA 3. *For any $(L_1, O) \in \mathfrak{L}$ there exists a local contact diffeomorphism h_1 such that $(L_1, O) = h_1(L_0, O)$ and $h_1(\Sigma) = \Sigma$.*

Proof. According to Lemma 2 we can include the Legendrian germs (L_0, O) and (L_1, O) into a family $(L_\varepsilon, O) \in \mathfrak{L}$ where $\varepsilon \in [0, 1]$, $L_\varepsilon = k_\varepsilon(L_0)$, and k_ε is a smooth family of contact diffeomorphisms such that $k_\varepsilon(O) = O$ for all $\varepsilon \in [0, 1]$. Let

$$\vec{K}_\varepsilon(k_\varepsilon e) = \frac{d}{d\varepsilon}\, k_\varepsilon e, \quad e \in \mathbb{R}^7, \quad \vec{K}_\varepsilon(O) = 0$$

be a contact vector field which depends smoothly on ε.

Let $K_\varepsilon = \Theta(\vec{K}_\varepsilon)$. According to Lemma 2 in some neighborhood $\mathcal{U}_O$ of the origin P_2, P_3, and Q_3 are coordinates on L_ε for any $\varepsilon \in [0, 1]$. Therefore there exists a unique function $H_\star : [0, 1] \times \mathcal{U}_O \to \mathbb{R}$ depending only on ε, P_2, P_3, Q_3 such that

$$H_\varepsilon\big|_{L_\varepsilon} = K_\varepsilon\big|_{L_\varepsilon}. \tag{5}$$

Let $\vec{H}_\varepsilon$ be the contact vector field defined by the formulas (3) where $K = H_\varepsilon$.

First of all, let us show that $\vec{H}_\varepsilon(O) = 0$. Indeed, according to (4) $K_\varepsilon(O) = 0$ and $d_O K_\varepsilon|_{\{dU=0\}} = 0$ because $\vec{K}_\varepsilon(O) = 0$. Hence $H_\varepsilon(O) = 0$ and $d_O H_\varepsilon = 0$ because L_ε is tangent to the hyperplane $\{dU = 0\}$. So according to (4) $\vec{H}_\varepsilon(O) = 0$.

Now we can define a family of local contact diffeomorphisms h_ε depending on $\varepsilon \in [0, 1]$ such that

$$\vec{H}_\varepsilon(h_\varepsilon e) = \frac{d}{d\varepsilon}\, h_\varepsilon e \quad \forall\, e \in \mathcal{V}_O,$$

where $\mathcal{V}_O$ is a neighborhood of the origin. Indeed, it is possible because $\vec{H}_\varepsilon(O) = 0$. Besides, the equality $\vec{H}_\varepsilon(O) = 0$ implies that $h_\varepsilon(O) = O$.

The formulas (3) imply that the coordinate functions P_1, P_2, and Q_1 are first integrals of the contact vector field $\vec{H}_\varepsilon$ because its generating function $\vec{H}_\varepsilon$ does not depend on P_1, Q_1, Q_2, and U. Hence the contact vector field $\vec{H}_\varepsilon$ is tangent to Σ^4 and $\Sigma \setminus \Sigma^4$. Therefore $h_\varepsilon(\Sigma) = \Sigma$ for all $\varepsilon \in [0,1]$.

The equality (5) implies that for any $\varepsilon \in [0,1]$ the vector field $\vec{H}_\varepsilon - \vec{K}_\varepsilon$ is tangent to $L_\varepsilon = k_\varepsilon(L_0)$. So $h_\varepsilon(L_0) = k_\varepsilon(L_0)$ for all $\varepsilon \in [0,1]$.

Therefore $(L_\varepsilon, O) = h_\varepsilon(L_0, O)$ and $h_\varepsilon(\Sigma) = \Sigma$ for all $\varepsilon \in [0,1]$. In particular, it holds for $\varepsilon = 1$. ☐

3.6. **Local normal forms of $\Lambda_{\mathbf{x}_0}$.** Lemma 3 implies the following

LEMMA 4. *Let $O = \Pi^+(\mathbf{x}_0, 0)$ or $O = \Pi^-(\mathbf{x}_0, 0)$. Then in a neighborhood of O there exist local coordinates $(P_1, P_2, P_3, Q_1, Q_2, Q_3, U)$ such that:*

- *the contact structure is given as the field of 0-spaces of the contact form*
$$\Theta = \frac{1}{2} P\,dQ - \frac{1}{2} Q\,dP - dU;$$
- $\pi^{-1}(\mathbf{x}_0, 0) = \{Q_1 = P_3, Q_2 = 0, Q_3 = P_1, U = 0\}$;
- *if $\mathbf{x}_0$ is a hyperbolic point then*
$$\Sigma = \left\{P_1Q_1 - P_2^2 = 0,\ P_1 + Q_1 \geq 0\right\} \quad \textit{and} \quad \Lambda_{\mathbf{x}_0} = \Lambda_+^H \cup \Lambda_-^H$$
where
$$\Lambda_+^H = \begin{cases} P_1 = a^2b^2, & Q_1 = c^2, \\ P_2 = abc, & Q_2 = 2abc\ln a^2, \quad U = 0, \\ P_3 = a^2c^2, & Q_3 = b^2, \end{cases}$$
$$\Lambda_-^H = \begin{cases} P_1 = b^2, & Q_1 = a^2c^2, \\ P_2 = abc, & Q_2 = -2abc\ln a^2, \quad U = 0, \\ P_3 = c^2, & Q_3 = a^2b^2, \end{cases}$$
$a \in [0,1]$, $b, c \in \mathbb{R}$ are parameters, and $a \ln a^2 = 0$ if $a = 0$;
- *if $\mathbf{x}_0$ is an elliptic point then*
$$\Sigma = \left\{P_1^2 + Q_1^2 - P_2^2 = 0,\ P_2 \geq 0\right\} \quad \textit{and} \quad \Lambda_{\mathbf{x}_0} = \Lambda^E$$
where
$$\Lambda^E = \begin{cases} P_2 \geq 0, & \\ P_1 + iQ_1 = P_2 e^{i\left(\psi - \frac{Q_2}{2P_2}\right)}, & U = 0, \\ Q_3 + iP_3 = P_2 e^{i\left(\psi + \frac{Q_2}{2P_2}\right)}, & \end{cases}$$
$i = \sqrt{-1}$, $\psi \in \mathbb{R} \mod 2\pi\mathbb{Z}$ is a parameter, and $P_2 e^{i\left(\psi \pm \frac{Q_2}{2P_2}\right)} = 0$ if $P_2 = 0$.

REMARK. Examples H and E of coordinates from Lemma 4 are given in Subsection 3.2.

Proof. According to Subsection 3.2 and Lemma 3 in a neighborhood of O there exist local coordinates $(P_1, P_2, P_3, Q_1, Q_2, Q_3, U)$ such that:

- the contact structure is given by the equation $\Theta = 0$;
- $\pi^{-1}(\mathbf{x}_0, 0) = \{Q_1 = P_3, Q_2 = 0, Q_3 = P_1, U = 0\}$;
- if $\mathbf{x}_0$ is a hyperbolic point then $\Sigma = \left\{P_1Q_1 - P_2^2 = 0,\ P_1 + Q_1 \geq 0\right\}$;
- if $\mathbf{x}_0$ is an elliptic point then $\Sigma = \left\{P_1^2 + Q_1^2 - P_2^2 = 0,\ P_2 \geq 0\right\}$.

Let us consider the following parameterizations of $\Sigma \cap \pi^{-1}(\mathbf{x}_0, 0)$:

- $P_1 = b^2$, $Q_1 = c^2$, $P_2 = bc$, $Q_2 = 0$, $P_3 = c^2$, $Q_3 = b^2$, $U = 0$ (where $b, c \in \mathbb{R}$ are parameters and $\mathbf{x}_0$ is hyperbolic);
- $P_2 \geq 0$, $P_1 + iQ_1 = P_2 e^{i\psi}$, $Q_2 = 0$, $Q_3 + iP_3 = P_2 e^{i\psi}$, $U = 0$ (where $\psi \in \mathbb{R} \mod 2\pi\mathbb{Z}$ is a parameter and $\mathbf{x}_0$ is elliptic).

According to Subsection 3.4 the characteristics of Σ have parameterizations (with a real parameter σ) satisfying the differential equations:

- $\frac{dP_1}{d\sigma} = -P_1$, $\frac{dQ_1}{d\sigma} = Q_1$, $\frac{dP_2}{d\sigma} = 0$, $\frac{dQ_2}{d\sigma} = -2P_2$, $\frac{dP_3}{d\sigma} = \frac{dQ_3}{d\sigma} = \frac{dU}{d\sigma} = 0$,
  if $\mathbf{x}_0$ is hyperbolic;
- $\frac{dP_1}{d\sigma} + i\frac{dQ_1}{d\sigma} = i(P_1 + iQ_1)$, $\frac{dP_2}{d\sigma} = 0$, $\frac{dQ_2}{d\sigma} = -P_2$, $\frac{dP_3}{d\sigma} = \frac{dQ_3}{d\sigma} = \frac{dU}{d\sigma} = 0$,
  if $\mathbf{x}_0$ is elliptic.

Therefore the characteristics passing through $\Sigma \cap \pi^{-1}(\mathbf{x}_0, 0)$ are given by the equations:

- $P_1 = b^2 e^{-\sigma}$, $Q_1 = c^2 e^{\sigma}$, $P_2 = bc$, $Q_2 = -2bc\sigma$, $P_3 = c^2$, $Q_3 = b^2$, $U = 0$,
  if $\mathbf{x}_0$ is hyperbolic;
- $P_2 \geq 0$, $P_1 + iQ_1 = P_2 e^{i(\psi+\sigma)}$, $Q_2 = -P_2\sigma$, $Q_3 + iP_3 = P_2 e^{i\psi}$, $U = 0$,
  if $\mathbf{x}_0$ is elliptic.

Here $\sigma \in \mathbb{R}$ is a parameter along the characteristics.

In the hyperbolic case for $\sigma \geq 0$ we get the formulas for Λ_+^H from Lemma 4 changing $c \mapsto ce^{-\sigma/2}$ and setting $a = e^{-\sigma/2}$.

In the hyperbolic case for $\sigma \leq 0$ we get the above formulas for Λ_-^H from Lemma 4 changing $b \mapsto be^{\sigma/2}$ and setting $a = e^{\sigma/2}$.

In the elliptic case we get the formulas for Λ^E changing $\psi \mapsto \psi - \sigma/2$ and setting $\sigma = -Q_2/P_2$. $\square$

3.7. **Singularities of $\Lambda_{\mathbf{x}_0}$.**

Definition. We say that a three-dimensional stratified Legendrian submanifold Λ of a contact space has a singularity H_1, H_2, or E_2 at a point $\lambda \in \Lambda$ if its germ (Λ, λ) is contact diffeomorphic to the germ $(\mathcal{H}_1 \times \mathbb{R}, O)$, $(\mathcal{H}_2 \times \mathbb{R}, O)$, or $(\mathcal{E}_2 \times \mathbb{R}, O)$ respectively.

Lemma 5. *The Legendrian submanifold* $\Lambda_+^H \cup \Lambda_-^H$

(1) *has singularities* H_1 *if*

$$P_1 = P_2 = P_3 = Q_2 = U = 0, \quad Q_1 > 0, \quad Q_3 > 0,$$

or

$$P_2 = Q_1 = Q_2 = Q_3 = U = 0, \quad P_1 > 0, \quad P_3 > 0;$$

(2) *has singularities* H_2 *if*

$$P_1 = P_2 = P_3 = Q_2 = U = 0, \quad Q_1 = 0, \quad Q_3 > 0,$$

or

$$P_1 = P_2 = P_3 = Q_2 = U = 0, \quad Q_1 > 0, \quad Q_3 = 0,$$

or

$$P_2 = Q_1 = Q_2 = Q_3 = U = 0, \quad P_1 = 0, \quad P_3 > 0,$$

or

$$P_2 = Q_1 = Q_2 = Q_3 = U = 0, \quad P_1 > 0, \quad P_3 = 0;$$

(3) *has more complicated singularity if*

$$P_1 = P_2 = P_3 = Q_1 = Q_2 = Q_3 = U = 0;$$

(4) *is smooth at the other points.*

Proof. The Legendrian submanifold $\Lambda^H_+ \cup \Lambda^H_-$ has singularities if and only if $a = 0$ in the formulas of Lemma 4. It gives the set of singularities of Λ^H_+:

$$P_1 = P_2 = P_3 = Q_2 = U = 0, \quad Q_1 \geq 0, \quad Q_3 \geq 0;$$

and the set of singularities of Λ^H_-:

$$P_2 = Q_1 = Q_2 = Q_3 = U = 0, \quad P_1 \geq 0, \quad P_3 \geq 0;$$

proving the item 4 from Lemma 5. Let us consider the following transformations:

- $a \mapsto a$, $b \mapsto \kappa b$, $c \mapsto c$, $\kappa > 0$,
  $(P_1, P_2, P_3, Q_1, Q_2, Q_3, U) \mapsto (\kappa^2 P_1, \kappa P_2, P_3, Q_1, \kappa Q_2, \kappa^2 Q_3, \kappa^2 U)$;
- $a \mapsto a$, $b \mapsto b$, $c \mapsto \kappa c$, $\kappa > 0$,
  $(P_1, P_2, P_3, Q_1, Q_2, Q_3, U) \mapsto (P_1, \kappa P_2, \kappa^2 P_3, \kappa^2 Q_1, \kappa Q_2, Q_3, \kappa^2 U)$;
- $a \mapsto a$, $b \mapsto c$, $c \mapsto b$,
  $(P_1, P_2, P_3, Q_1, Q_2, Q_3, U) \mapsto (P_3, P_2, P_1, Q_3, Q_2, Q_1, U)$;
- $a \mapsto a$, $b \mapsto b$, $c \mapsto c$,
  $(P_1, P_2, P_3, Q_1, Q_2, Q_3, U) \mapsto (Q_3, P_2, Q_1, P_3, -Q_2, P_1, -U)$.

All of them preserve the contact structure and the Legendrian submanifold $\Lambda^H_+ \cup \Lambda^H_-$. Besides, these transformations divide the set of singularities of $\Lambda^H_+ \cup \Lambda^H_-$ into the three orbits mentioned in the items 1–3 of Lemma 5. In particular, we prove its item 3.

The point $P_1 = P_2 = P_3 = Q_1 = Q_2 = U = 0$, $Q_3 = 1$ belongs to Λ^H_+. Let us consider its section with $Q_3 = 1$. Then $b = 1$ or $b = -1$ but these conditions define the same submanifold:

$$\begin{cases} P_1 = a^2, & Q_1 = c^2, \\ P_2 = ac, & Q_2 = 2ac\ln a^2, \quad U = 0. \\ P_3 = a^2c^2, & Q_3 = 1, \end{cases}$$

The form Θ defines the contact structure

$$\frac{1}{2}\left(P_1\, dQ_1 + P_2\, dQ_2 - Q_1\, dP_1 - Q_2\, dP_2\right) - d\frac{P_3}{2} = 0$$

in the plane $Q_3 = 1$, $U = 0$ and our section is Legendrian. Denoting $A = a$, $B = c$, $U = P_3/2$ we get the Legendrian submanifold $\mathcal{H}_2$ from Subsection 1.3 and prove the item 2 of Lemma 5.

But the stratified Legendrian submanifold $\mathcal{H}_2$ has singularities H_1 if $A = 0$ and $B \neq 0$ that is shown in [5]. It proves the item 1 of Lemma 5. ☐

LEMMA 6. *The Legendrian submanifold* Λ^E

(1) *has singularities* E_2 *if*

$$P_1 = P_2 = P_3 = Q_1 = Q_3 = U = 0, \quad Q_2 \neq 0;$$

(2) *has more complicated singularity if*

$$P_1 = P_2 = P_3 = Q_1 = Q_2 = Q_3 = U = 0;$$

(3) *is smooth at the other points.*

Proof. The Legendrian submanifold Λ^E has singularities if and only if $P_2 = 0$ in the formulas of Lemma 4. It gives the set of singularities of Λ^E:

$$P_1 = P_2 = P_3 = Q_1 = Q_3 = U = 0;$$

and proves the item 3 from Lemma 6. Let us consider the following transformations:

- $(P_1, P_2, P_3, Q_1, Q_2, Q_3, U) \mapsto (\kappa P_1, \kappa P_2, \kappa P_3, \kappa Q_1, \kappa Q_2, \kappa Q_3, \kappa^2 U)$, $\kappa > 0$;
- $(P_1, P_2, P_3, Q_1, Q_2, Q_3, U) \mapsto (Q_3, P_2, Q_1, P_3, -Q_2, P_1, -U)$.

All of them preserve the contact structure and the Legendrian submanifold Λ^E. Besides, these transformations divide the set of singularities of Λ^E into the two orbits mentioned in the items 1, 2 of Lemma 6. In particular, we prove its item 2.

Let us consider the section of Λ^E with $Q_2 = 2$:

$$\Lambda^E = \begin{cases} P_2 & \geq & 0, \\ P_1 + iQ_1 & = & P_2 e^{i\left(\psi - \frac{1}{P_2}\right)}, \\ Q_3 + iP_3 & = & P_2 e^{i\left(\psi + \frac{1}{P_2}\right)}, \end{cases} \quad U = 0,$$

The form Θ defines the contact structure

$$\frac{1}{2}\left(P_1\, dQ_1 + P_3\, dQ_3 - Q_1\, dP_1 - Q_3\, dP_3\right) - dP_2 = 0$$

in the plane $Q_3 = 1$, $U = 0$ and our section is Legendrian. After obvious renaming $P_2 \mapsto U$, $P_3 \mapsto P_2$, $Q_3 \mapsto Q_2$ we get the Legendrian submanifold $\mathcal{E}_2$ from Subsection 1.3 and prove the item 1 of Lemma 6. □

3.8. **Time function τ.** Here we prove some conditions which have to be satisfied by the time function τ in the coordinates from Lemma 4.

LEMMA 7. *Let $O = \Pi^+(\mathbf{x}_0, 0)$ or $O = \Pi^-(\mathbf{x}_0, 0)$ and $d_O\tau$ be the differential of the time function τ at O. Then in the coordinates from Lemma 4*

$$d_O\tau = \gamma_1(dQ_1 - dP_3) + \gamma_2\, dQ_2 + \gamma_3(dQ_3 - dP_1) + \gamma_0\, dU$$

where

- $\gamma_1\gamma_3 > \gamma_2^2$ *if* $\mathbf{x}_0$ *is hyperbolic;*
- $\gamma_2^2 > \gamma_1^2 + \gamma_3^2$ *if* $\mathbf{x}_0$ *is elliptic.*

Proof. The equality

$$d_O\tau = \gamma_1(dQ_1 - dP_3) + \gamma_2\, dQ_2 + \gamma_3(dQ_3 - dP_1) + \gamma_0\, dU$$

follows from the conditions

$$\pi^{-1}(\mathbf{x}_0, 0) = \{Q_1 = P_3, Q_2 = 0, Q_3 = P_1, U = 0\} \subset \tau^{-1}(0).$$

Let us prove the inequalities $\gamma_1\gamma_3 > \gamma_2^2$ and $\gamma_2^2 > \gamma_1^2 + \gamma_3^2$.

The Legendrian submanifold $\pi^{-1}(\mathbf{x}_0, 0) \subset ST^*\mathbb{R}^{3+1}$ is situated in the isochrone $\tau^{-1}(0)$ and consists of its characteristics: the lines $([\mathbf{p}, \cdot]; \mathbf{x}_0, 0)$ with $\mathbf{p} \neq 0$ and the two points $([0, \pm 1]; \mathbf{x}_0, 0)$. In an affine neighborhood of O the hypersurface $\Sigma \cap \pi^{-1}(\mathbf{x}_0, 0)$ is a half-cone. It turns out that one of the two half-characteristics of the isochrone $\tau^{-1}(0)$ starting at O lies inside of this half-cone.

Indeed, let us choose local coordinates (x, y, z) in a neighborhood of $\mathbf{x}_0 \in \mathbb{R}^3$ such that $\xi_1(\mathbf{x}_0) = (1, 0, 0)$, $\xi_2(\mathbf{x}_0) = (0, 1, 0)$, and $\xi_0(\mathbf{x}_0) = (a_0, b_0, c_0)$. Then according to Subsection 3.1 we get that in the coordinates (p, q, r, s) that are dual to (x, y, z, t):

$$\Sigma \cap \pi^{-1}(\mathbf{x}_0, 0) = \left\{ [p, q, r, s] \;\middle|\; a_0 p + b_0 q + c_0 r + s + \sqrt{p^2 + q^2} = 0 \right\}$$

and $O = [0, 0, 1, -c_0]$ or $O = [0, 0, -1, c_0]$. So, we can take the affine neighborhood $r = 1$ or $r = -1$ respectively. It is clear that in each case the ray

$$p = q = 0, \quad \pm c_0 + s < 0$$

is situated inside of the half-cone $\left\{ a_0 p + b_0 q \pm c_0 + s + \sqrt{p^2 + q^2} = 0 \right\}$.

But according to Subsection 3.3 the characteristics of $\tau^{-1}(0)$ are tangent to the contact vector field $\vec{\tau}$ defined by the formulas (3) for $K=\tau$. Hence one of the two vectors $\pm\vec{\tau}(0)$ where

$$\vec{\tau}(0)=\left\{\dot{P}_1=\gamma_1,\ \dot{P}_2=\gamma_2,\ \dot{P}_3=\gamma_3,\ \dot{Q}_1=\gamma_3,\ \dot{Q}_2=0,\ \dot{Q}_3=\gamma_1,\ \dot{U}=0\right\}$$

must lie inside of the half-cone

$$\Sigma\cap\pi^{-1}(\mathbf{x}_0,0)=\Sigma\cap\{Q_1=P_3,Q_2=0,Q_3=P_1,U=0\}\,.$$

It means that

- $\dot{P}_1\dot{Q}_1-\dot{P}_2^{\,2}=\gamma_1\gamma_3-\gamma_2^2>0$ if $\mathbf{x}_0$ is hyperbolic and
- $\dot{P}_1^{\,2}+\dot{Q}_1^{\,2}-\dot{P}_2^{\,2}=\gamma_1^2+\gamma_3^2-\gamma_2^2<0$ if $\mathbf{x}_0$ is elliptic.

□

3.9. **Proof of Theorem 1.** According to Lemma 1 in some neighborhood of $(\mathbf{x}_0,0)$ the Legendrian submanifold $\Lambda_{\mathbf{x}_0}$ is homeomorphic to the cylinder over the two-dimensional sphere, the elements of the cylinder are characteristics of Σ. But an isochrone $\tau^{-1}(T)$ is transversal to these characteristics because their projections are the graphs of trajectories of the control-affine system (1). It proves that $\mathcal{L}_{\mathbf{x}_0}(T)$ is homeomorphic to the two-dimensional sphere.

In neighborhoods of two contact elements $\Pi^+(\mathbf{x}_0,0)$ or $\Pi^-(\mathbf{x}_0,0)$ the Legendrian submanifold $\Lambda_{\mathbf{x}_0}$ has singularities described in Lemmas 4, 5, and 6.

In the hyperbolic case Theorem 1 follows from Lemma 5. Namely, singularities H_1 form two quadrants described in Lemma 5. But one and only one of them lies in the domain $\tau>0$ according to Lemma 7.

In the elliptic case Theorem 1 follows from Lemma 6. Namely, singularities E_2 form two rays described in Lemma 6. But one and only one of them lies in the domain $\tau>0$ according to Lemma 7.

4. Appendix

Theorem 1 implies that for enough small $T>0$ the stratified Legendrian submanifolds $\mathcal{L}_{\mathbf{x}_0}(T)$ are reduced to a normal form $\mathcal{L}^H$ in the hyperbolic case and to a normal form $\mathcal{L}^E$ in the elliptic case. Here we give explicit formulas for $\mathcal{L}^H$ based on [6] and for $\mathcal{L}^E$ based on [7]. The fronts of the stratified Legendrian submanifolds $\mathcal{L}^E$ and $\mathcal{L}^H$ are shown in Fig. 1 on the left and the right respectively.

Normal form $\mathcal{L}^H$:

$$\mathcal{L}^H=\left\{(p:q:r;x,y,z)\in ST^*\mathbb{R}^3\;\middle|\;p=\frac{4\alpha\beta}{(1+\alpha^2)(1+\beta^2)},\quad q=\frac{1-\beta^2}{1+\beta^2},\right.$$
$$\left.r=\frac{1-\alpha^2}{1+\alpha^2},\quad x=\Phi(\alpha)\frac{2\beta}{1+\beta^2},\quad y=\frac{1-\beta^2}{1+\beta^2},\quad z=\Psi(\alpha)\frac{2\beta^2}{(1+\beta^2)^2}\right\}$$

where $\alpha,\beta\in\mathbb{R}\cup\{\infty\}$ are parameters,

$$\Phi(\alpha)=-\frac{\alpha\ln\alpha^2}{1-\alpha^2},\quad \Psi(\alpha)=\frac{1-\alpha^4+2\alpha^2\ln\alpha^2}{(1-\alpha^2)^2},$$

$$\Phi(0)=\Phi(\infty)=\Psi(1)=\Psi(-1)=0,\quad \Phi(1)=-\Phi(-1)=\Psi(0)=-\Psi(\infty)=1.$$

Normal form $\mathcal{L}^E$:

$$\mathcal{L}^E = L^E \cup \{\mathcal{P}^+, \mathcal{P}^-\}, \quad \mathcal{P}^\pm = (0:0:\pm 1; 0,0,0) \in ST^*\mathbb{R}^3,$$

$$L^E = \left\{ (p:q:r;x,y,z) \in ST^*\mathbb{R}^3 \;\middle|\; p = \cos r\, \cos\phi, \quad q = \cos r\, \sin\phi, \right.$$

$$\left. x = \frac{2\,\sin r\, \cos\phi}{r}, \quad y = \frac{2\,\sin r\, \sin\phi}{r}, \quad z = \frac{2r - \sin 2r}{2r^2} \right\}$$

where $\phi \in \mathbb{R} \mod 2\pi\mathbb{Z}$ is a parameter.

References

[1] A. A. Agrachev. Exponential mappings for contact sub-Riemannian structures. *J. Dynamical and Control Systems*, 2:321–358, 1996. DOI: 10.1007/bf02269423

[2] A. A. Agrachev and Y. L. Sachkov. *Control Theory from the Geometric Viewpoint*, volume 87 of *Encycl. Math. Sci.* Springer, 2004.

[3] V. I. Arnold. On the interior scattering of waves, defined by hyperbolic variational principles. *J. Geom. Phys.*, 5(3):305–315, 1988.

[4] V. I. Arnold. *Singularities of Caustics and Wave Fronts*. Springer Netherlands, Dordrecht, 1990.

[5] I. A. Bogaevsky. New singularities and perestroikas of fronts of linear waves. *Moscow Math. J.*, 3(3):807–821, 2003. DOI: 10.17323/1609-4514-2003-3-3-807-821

[6] I. A. Bogaevsky. Sub-Lorentzian structures in $\mathbb{R}^4$: left-invariance and conformal normal forms. *Journal of Dynamical and Control Systems*, 24(3):371–389, 2018. DOI: 10.1007/s10883-018-9396-9

[7] A. M. Vershik and V. Y. Gershkovich. Nonholonomic dynamical systems, geometry of distributions and variational problems. In *Dynamical systems. VII*, volume 16 of *Encycl. Math. Sci.*, pages 1–81. Springer, 1994. Translation from Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Fundam. Napravleniya 16, 5–85 (1987).

Lomonosov Moscow State University, Faculty of Mechanics and Mathematics, Leninskie Gory, Moscow GSP-1, 119991, Russia

Scientific Research Institute for System Analysis of RAS, Moscow, Russia

Journal *of* Singularities
Volume 21 (2020), 30-49

Proc. of Geometric and Algebraic Singularity Theory, Będlewo, 2017
DOI: 10.5427/jsing.2020.21c

FRAMED SURFACES AND ONE-PARAMETER FAMILIES OF FRAMED CURVES IN EUCLIDEAN 3-SPACE

TOMONORI FUKUNAGA AND MASATOMO TAKAHASHI

Dedicated to Professor Goo Ishikawa on the occasion of his 60th birthday

Abstract. In this paper, we consider two objects as surfaces with singular points in Euclidean 3-space. One is the class of framed surfaces and the other is that of one-parameter families of framed curves. The basic invariants of a framed surface or the curvature of a one-parameter family of framed curves determine the surface and the moving frame up to congruence. We give relations between framed surfaces and one-parameter families of framed curves. In particular, a surface with corank one singularities can be considered as a one-parameter family of framed curves at least locally. Moreover, we give concrete examples of such surfaces with singular points described as one-parameter families of framed curves.

1. Introduction

Recently, differential geometry of curves and surfaces with singular points is extensively investigated (for instance, see [3, 4, 5, 6, 7, 8, 11, 13, 14, 17, 19, 20, 21, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34]). All non-singular surfaces are locally diffeomorphic to each other. Therefore, a diffeomorphism on the target breaks down the differential geometry on surfaces in this sense.

In [34, 6], a normal form of cross caps is given by using a parameter change on the source and an isometry (a rotation) on the target. Moreover, normal forms of cuspidal edges, swallowtails and cuspidal cross caps are given in [20, 29, 24], respectively. By using the normal forms, they obtain $SO(3)$ invariants and give differential geometric properties of surfaces with singular points by using the invariants.

We treat surfaces with singular points, that is, singular surfaces more directly. As a way to study surfaces with singular points in Euclidean 3-space, we consider two approaches. One is to consider framed surfaces and the other is to use one-parameter families of framed curves. We give relations between these two objects.

A framed surface is a surface in Euclidean 3-space with a moving frame (cf. [10]). Framed surfaces may have singular points. By using the moving frames, the basic invariants and the curvatures of framed surfaces are introduced in [10].

On the other hand, a framed curve is a curve in Euclidean 3-space with a moving frame (cf. [12]). Framed curves may have singular points. Therefore, we may consider one-parameter families of framed curves as surfaces with singular points. In [27], the authors have considered one-parameter families of framed curves in order to define an envelope of a family of space curves. By using the moving frame, the curvature of a one-parameter family of framed curves is

2010 *Mathematics Subject Classification.* 58K05, 57R45, 53A55.

Key words and phrases. framed surface, one-parameter family of framed curves, singularity.

The authors would like to thank Professor Goo Ishikawa for his constant encouragement. The authors would also like to thank the referee for helpful comments that improved the original manuscript. The first author was partially supported by JSPS KAKENHI Grant Number JP 15K17457 and the second author was partially supported by JSPS KAKENHI Grant Number JP 17K05238.

introduced in [27]. We review the theories for framed surfaces, framed curves and one-parameter families of framed curves in §2. The basic invariants of a framed surface or the curvature of a one-parameter family of framed curves determine the surface and the moving frame up to congruence. We give relations between framed surfaces and one-parameter families of framed curves in §3. We then prove that surfaces with corank one singularities can be considered as one-parameter families of framed curves at least locally (Theorem 4.1). As concrete examples of one-parameter families of framed curves, we give surfaces with first kind singularities (for example, cuspidal edges and cuspidal cross caps), second kind singularities (for example, swallowtails) and cross caps by using normal forms in §4. In general, non-degenerate singular points are also of corank one. Moreover, $\mathcal{A}$-simple singularities of a map from a 2-dimensional manifold to a 3-dimensional one are also of corank one, see [22]. Hence, it is possible to treat map germs of non-degenerate singular points and $\mathcal{A}$-simple singularities as one-parameter families of framed curves.

All maps and manifolds considered in this paper are differentiable of class C^∞.

2. Previous results

Let $\mathbb{R}^3$ be the 3-dimensional Euclidean space equipped with the inner product

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3,$$

where $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$. The norm of $\boldsymbol{a}$ is given by $|\boldsymbol{a}| = \sqrt{\boldsymbol{a}\cdot\boldsymbol{a}}$ and the vector product is given by

$$\boldsymbol{a}\times\boldsymbol{b} = \det\begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is the canonical basis of $\mathbb{R}^3$. Let S^2 be the unit sphere in $\mathbb{R}^3$, that is,

$$S^2 = \{\boldsymbol{a} \in \mathbb{R}^3 | |\boldsymbol{a}| = 1\}.$$

We denote the 3-dimensional smooth manifold $\{(\boldsymbol{a}, \boldsymbol{b}) \in S^2 \times S^2 | \boldsymbol{a}\cdot\boldsymbol{b} = 0\}$ by Δ.

Let U be a simply connected domain in $\mathbb{R}^2$ and I be an interval in $\mathbb{R}$. We quickly review the theories of framed surfaces, framed curves and one-parameter families of framed curves.

2.1. **Framed surfaces in Euclidean 3-space.** A framed surface in Euclidean 3-space is a smooth surface with a moving frame.

Definition 2.1. We say that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ is a *framed surface* if

$$\boldsymbol{x}_u(u, v)\cdot\boldsymbol{n}(u, v) = 0, \boldsymbol{x}_v(u, v)\cdot\boldsymbol{n}(u, v) = 0$$

for all $(u, v) \in U$, where $\boldsymbol{x}_u(u, v) = (\partial\boldsymbol{x}/\partial u)(u, v)$ and $\boldsymbol{x}_v(u, v) = (\partial\boldsymbol{x}/\partial v)(u, v)$. We say that $\boldsymbol{x} : U \to \mathbb{R}^3$ is a *framed base surface* if there exists $(\boldsymbol{n}, \boldsymbol{s}) : U \to \Delta$ such that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a framed surface.

By definition, a framed base surface is a frontal. For definition and properties of frontals see [1, 2, 30]. On the other hand, a frontal is a framed base surface at least locally.

We denote $\boldsymbol{t}(u, v) = \boldsymbol{n}(u, v) \times \boldsymbol{s}(u, v)$. Then $\{\boldsymbol{n}(u, v), \boldsymbol{s}(u, v), \boldsymbol{t}(u, v)\}$ is a moving frame along $\boldsymbol{x}(u, v)$. Thus, we have the following systems of differential equations:

$$\begin{pmatrix} \boldsymbol{x}_u \\ \boldsymbol{x}_v \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} \boldsymbol{s} \\ \boldsymbol{t} \end{pmatrix}, \tag{1}$$

$$\begin{pmatrix} \boldsymbol{n}_u \\ \boldsymbol{s}_u \\ \boldsymbol{t}_u \end{pmatrix} = \begin{pmatrix} 0 & e_1 & f_1 \\ -e_1 & 0 & g_1 \\ -f_1 & -g_1 & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{s} \\ \boldsymbol{t} \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{n}_v \\ \boldsymbol{s}_v \\ \boldsymbol{t}_v \end{pmatrix} = \begin{pmatrix} 0 & e_2 & f_2 \\ -e_2 & 0 & g_2 \\ -f_2 & -g_2 & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{s} \\ \boldsymbol{t} \end{pmatrix}, \tag{2}$$

where $a_i, b_i, e_i, f_i, g_i : U \to \mathbb{R}, i = 1, 2$ are smooth functions, which we call *basic invariants* of the framed surface. We denote the matrices in the equalities (1) and (2) by $\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2$, respectively. We also call the matrices $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$ *basic invariants* of the framed surface $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$. Note that (u, v) is a singular point of $\boldsymbol{x}$ if and only if $\det \mathcal{G}(u, v) = 0$.

Considering the integrability conditions $\boldsymbol{x}_{uv} = \boldsymbol{x}_{vu}$ and $\mathcal{F}_{2,u} - \mathcal{F}_{1,v} = \mathcal{F}_1\mathcal{F}_2 - \mathcal{F}_2\mathcal{F}_1$, the basic invariants should satisfy the following conditions:

$$(3) \qquad \begin{cases} a_{1v} - b_1 g_2 = a_{2u} - b_2 g_1, \\ b_{1v} - a_2 g_1 = b_{2u} - a_1 g_2, \\ a_1 e_2 + b_1 f_2 = a_2 e_1 + b_2 f_1, \end{cases} \qquad \begin{cases} e_{1v} - f_1 g_2 = e_{2u} - f_2 g_1, \\ f_{1v} - e_2 g_1 = f_{2u} - e_1 g_2, \\ g_{1v} - e_1 f_2 = g_{2u} - e_2 f_1. \end{cases}$$

Definition 2.2. Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}), (\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}}) : U \to \mathbb{R}^3 \times \Delta$ be framed surfaces. We say that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ and $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}})$ are *congruent as framed surfaces* if there exist a constant rotation $A \in SO(3)$ and a translation $\boldsymbol{a} \in \mathbb{R}^3$ such that

$$\widetilde{\boldsymbol{x}}(u, v) = A(\boldsymbol{x}(u, v)) + \boldsymbol{a}, \widetilde{\boldsymbol{n}}(u, v) = A(\boldsymbol{n}(u, v)), \widetilde{\boldsymbol{s}}(u, v) = A(\boldsymbol{s}(u, v)),$$

for all $(u, v) \in U$.

We have the existence and uniqueness theorems for framed surfaces in terms of basic invariants (cf. [10]).

Theorem 2.3 (Existence Theorem for framed surfaces)**.** *Let U be a simply connected domain in $\mathbb{R}^2$ and let $a_i, b_i, e_i, f_i, g_i : U \to \mathbb{R}, i = 1, 2$ be smooth functions with the integrability conditions (3). Then, there exists a framed surface $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ whose associated basic invariants coincide with a_i, b_i, e_i, f_i, g_i, $i = 1, 2$.*

Theorem 2.4 (Uniqueness Theorem for framed surfaces)**.** *Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}), (\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}}) : U \to \mathbb{R}^3 \times \Delta$ be framed surfaces with basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2), (\widetilde{\mathcal{G}}, \widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2)$, respectively. Then $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ and $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}})$ are congruent as framed surfaces if and only if the basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$ and $(\widetilde{\mathcal{G}}, \widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2)$ coincide.*

Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface with basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$. We consider rotations of the vectors $\boldsymbol{s}, \boldsymbol{t}$. We denote

$$\begin{pmatrix} \boldsymbol{s}^\theta(u, v) \\ \boldsymbol{t}^\theta(u, v) \end{pmatrix} = \begin{pmatrix} \cos\theta(u, v) & -\sin\theta(u, v) \\ \sin\theta(u, v) & \cos\theta(u, v) \end{pmatrix} \begin{pmatrix} \boldsymbol{s}(u, v) \\ \boldsymbol{t}(u, v) \end{pmatrix},$$

where $\theta : U \to \mathbb{R}$ is a smooth function. Then $\boldsymbol{n} \times \boldsymbol{s}^\theta = \boldsymbol{t}^\theta$ and $\{\boldsymbol{n}, \boldsymbol{s}^\theta, \boldsymbol{t}^\theta\}$ is also a moving frame along $\boldsymbol{x}$. It follows that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}^\theta)$ is a framed surface. We call the frame $\{\boldsymbol{n}, \boldsymbol{s}^\theta, \boldsymbol{t}^\theta\}$ a *rotation frame* by θ of the framed surface $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$. We denote by $(\mathcal{G}^\theta, \mathcal{F}_1^\theta, \mathcal{F}_2^\theta)$ the basic invariants of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}^\theta)$. By a direct calculation, we have the following.

Proposition 2.5. *Under the above notations, the relations between the basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$ and $(\mathcal{G}^\theta, \mathcal{F}_1^\theta, \mathcal{F}_2^\theta)$ are given by*

$$\mathcal{G}^\theta = \mathcal{G} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} a_1\cos\theta - b_1\sin\theta & a_1\sin\theta + b_1\cos\theta \\ a_2\cos\theta - b_2\sin\theta & a_2\sin\theta + b_2\cos\theta \end{pmatrix},$$

$$\mathcal{F}_1^\theta = \begin{pmatrix} 0 & e_1\cos\theta - f_1\sin\theta & e_1\sin\theta + f_1\cos\theta \\ -e_1\cos\theta + f_1\sin\theta & 0 & g_1 - \theta_u \\ -e_1\sin\theta - f_1\cos\theta & -g_1 + \theta_u & 0 \end{pmatrix},$$

$$\mathcal{F}_2^\theta = \begin{pmatrix} 0 & e_2\cos\theta - f_2\sin\theta & e_2\sin\theta + f_2\cos\theta \\ -e_2\cos\theta + f_2\sin\theta & 0 & g_2 - \theta_v \\ -e_2\sin\theta - f_2\cos\theta & -g_2 + \theta_v & 0 \end{pmatrix}.$$

2.2. **Framed curves in Euclidean 3-space.** A framed curve in Euclidean 3-space is a smooth curve with a moving frame.

Definition 2.6. We say that $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ is a *framed curve* if $\dot\gamma(t) \cdot \nu_1(t) = 0$ and $\dot\gamma(t) \cdot \nu_2(t) = 0$ for all $t \in I$. We say that $\gamma : I \to \mathbb{R}^3$ is a *framed base curve* if there exists $(\nu_1, \nu_2) : I \to \Delta$ such that (γ, ν_1, ν_2) is a framed curve.

We denote $\boldsymbol{\mu}(t) = \nu_1(t) \times \nu_2(t)$. Then $\{\nu_1(t), \nu_2(t), \boldsymbol{\mu}(t)\}$ is a moving frame along the framed base curve $\gamma(t)$ in $\mathbb{R}^3$ and we have the Frenet type formula,

$$\begin{pmatrix} \dot\nu_1(t) \\ \dot\nu_2(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) & m(t) \\ -\ell(t) & 0 & n(t) \\ -m(t) & -n(t) & 0 \end{pmatrix} \begin{pmatrix} \nu_1(t) \\ \nu_2(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}, \ \dot\gamma(t) = \alpha(t)\boldsymbol{\mu}(t)$$

where $\ell(t) = \dot\nu_1(t) \cdot \nu_2(t)$, $m(t) = \dot\nu_1(t) \cdot \boldsymbol{\mu}(t), n(t) = \dot\nu_2(t) \cdot \boldsymbol{\mu}(t)$ and $\alpha(t) = \dot\gamma(t) \cdot \boldsymbol{\mu}(t)$. We call the mapping (ℓ, m, n, α) *the curvature of the framed curve* (γ, ν_1, ν_2). Note that t_0 is a singular point of γ if and only if $\alpha(t_0) = 0$.

Definition 2.7. Let $(\gamma, \nu_1, \nu_2), (\widetilde\gamma, \widetilde\nu_1, \widetilde\nu_2) : I \to \mathbb{R}^3 \times \Delta$ be framed curves. We say that (γ, ν_1, ν_2) and $(\widetilde\gamma, \widetilde\nu_1, \widetilde\nu_2)$ are *congruent as framed curves* if there exist a constant rotation $A \in SO(3)$ and a translation $\boldsymbol{a} \in \mathbb{R}^3$ such that $\widetilde\gamma(t) = A(\gamma(t)) + \boldsymbol{a}$, $\widetilde\nu_1(t) = A(\nu_1(t))$ and $\widetilde\nu_2(t) = A(\nu_2(t))$ for all $t \in I$.

We have the existence and uniqueness theorems for framed curves in terms of the curvatures (cf. [12]).

Theorem 2.8 (Existence Theorem for framed curves)**.** *Let $(\ell, m, n, \alpha) : I \to \mathbb{R}^4$ be a smooth mapping. Then, there exists a framed curve $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ whose curvature is given by (ℓ, m, n, α).*

Theorem 2.9 (Uniqueness Theorem for framed curves)**.** *Let*

$$(\gamma, \nu_1, \nu_2), (\widetilde\gamma, \widetilde\nu_1, \widetilde\nu_2) : I \to \mathbb{R}^3 \times \Delta$$

be framed curves with curvatures $(\ell, m, n, \alpha), (\widetilde\ell, \widetilde m, \widetilde n, \widetilde\alpha)$, respectively. Then (γ, ν_1, ν_2) and $(\widetilde\gamma, \widetilde\nu_1, \widetilde\nu_2)$ are congruent as framed curves if and only if the curvatures (ℓ, m, n, α) and $(\widetilde\ell, \widetilde m, \widetilde n, \widetilde\alpha)$ coincide.

As a special case of a framed curve, let us consider a spherical Legendre curve, for details see [31]. We say that $(\gamma, \nu) : I \to \Delta$ is a spherical Legendre curve if $\dot\gamma(t) \cdot \nu(t) = 0$ for all $t \in I$. We call γ a frontal and ν a dual of γ.

We define $\boldsymbol{\mu}(t) = \gamma(t) \times \nu(t)$. Then $\boldsymbol{\mu}(t) \in S^2$, $\gamma(t) \cdot \boldsymbol{\mu}(t) = 0$ and $\nu(t) \cdot \boldsymbol{\mu}(t) = 0$ for all $t \in I$. It follows that $\{\gamma(t), \nu(t), \boldsymbol{\mu}(t)\}$ is a moving frame along the frontal $\gamma(t)$.

Let $(\gamma, \nu) : I \to \Delta$ be a spherical Legendre curve. Then we have

$$\begin{pmatrix} \dot{\gamma}(t) \\ \dot{\nu}(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & m(t) \\ 0 & 0 & n(t) \\ -m(t) & -n(t) & 0 \end{pmatrix} \begin{pmatrix} \gamma(t) \\ \nu(t) \\ \boldsymbol{\mu}(t) \end{pmatrix},$$

where $m(t) = \dot{\gamma}(t) \cdot \boldsymbol{\mu}(t)$ and $n(t) = \dot{\nu}(t) \cdot \boldsymbol{\mu}(t)$.

We say that the pair of functions (m, n) is the curvature of the spherical Legendre curve $(\gamma, \nu) : I \to \Delta$.

2.3. **One-parameter families of framed curves in Euclidean 3-space.** We consider one-parameter families of framed curves in Euclidean 3-space. Let $(\gamma, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ be a smooth mapping, where U is a simply connected domain in $\mathbb{R}^2$.

Definition 2.10. We say that $(\gamma, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ is a *one-parameter family of framed curves with respect to* u (respectively, *with respect to* v) if $(\gamma(\cdot, v), \nu_1(\cdot, v), \nu_2(\cdot, v))$ is a framed curve for each v (respectively, $(\gamma(u, \cdot), \nu_1(u, \cdot), \nu_2(u, \cdot))$ is a framed curve for each u).

If $(\gamma, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ is a one-parameter family of framed curves with respect to u, then we denote $\boldsymbol{\mu}(u, v) = \nu_1(u, v) \times \nu_2(u, v)$. It follows that $\{\nu_1(u, v), \nu_2(u, v), \boldsymbol{\mu}(u, v)\}$ is a moving frame along $\gamma(u, v)$. We have the Frenet type formula.

$$\begin{aligned}
\begin{pmatrix} \nu_{1u}(u,v) \\ \nu_{2u}(u,v) \\ \boldsymbol{\mu}_u(u,v) \end{pmatrix} &= \begin{pmatrix} 0 & \ell(u,v) & m(u,v) \\ -\ell(u,v) & 0 & n(u,v) \\ -m(u,v) & -n(u,v) & 0 \end{pmatrix} \begin{pmatrix} \nu_1(u,v) \\ \nu_2(u,v) \\ \boldsymbol{\mu}(u,v) \end{pmatrix}, \\
\begin{pmatrix} \nu_{1v}(u,v) \\ \nu_{2v}(u,v) \\ \boldsymbol{\mu}_v(u,v) \end{pmatrix} &= \begin{pmatrix} 0 & L(u,v) & M(u,v) \\ -L(u,v) & 0 & N(u,v) \\ -M(u,v) & -N(u,v) & 0 \end{pmatrix} \begin{pmatrix} \nu_1(u,v) \\ \nu_2(u,v) \\ \boldsymbol{\mu}(u,v) \end{pmatrix}, \\
\gamma_u(u,v) &= \alpha(u,v)\boldsymbol{\mu}(u,v), \\
\gamma_v(u,v) &= P(u,v)\nu_1(u,v) + Q(u,v)\nu_2(u,v) + R(u,v)\boldsymbol{\mu}(u,v),
\end{aligned}$$

where

$$\begin{aligned}
\ell(u,v) &= \nu_{1u}(u,v) \cdot \nu_2(u,v), & m(u,v) &= \nu_{1u}(u,v) \cdot \boldsymbol{\mu}(u,v), \\
n(u,v) &= \nu_{2u}(u,v) \cdot \boldsymbol{\mu}(u,v), & \alpha(u,v) &= \gamma_u(u,v) \cdot \boldsymbol{\mu}(u,v), \\
L(u,v) &= \nu_{1v}(u,v) \cdot \nu_2(u,v), & M(u,v) &= \nu_{1v}(u,v) \cdot \boldsymbol{\mu}(u,v), \\
N(u,v) &= \nu_{2v}(u,v) \cdot \boldsymbol{\mu}(u,v), & P(u,v) &= \gamma_v(u,v) \cdot \nu_1(u,v), \\
Q(u,v) &= \gamma_v(u,v) \cdot \nu_2(u,v), & R(u,v) &= \gamma_v(u,v) \cdot \boldsymbol{\mu}(u,v).
\end{aligned}$$

By $\gamma_{uv}(u,v) = \gamma_{vu}(u,v)$, $\nu_{1uv}(u,v) = \nu_{1vu}(u,v)$, $\nu_{2uv}(u,v) = \nu_{2vu}(u,v)$ and $\boldsymbol{\mu}_{uv}(u,v) = \boldsymbol{\mu}_{vu}(u,v)$, we have the integrability condition

$$\begin{aligned}
L_u(u,v) &= M(u,v)n(u,v) - N(u,v)m(u,v) + \ell_v(u,v), \\
M_u(u,v) &= N(u,v)\ell(u,v) - L(u,v)n(u,v) + m_v(u,v), \\
N_u(u,v) &= L(u,v)m(u,v) - M(u,v)\ell(u,v) + n_v(u,v), \\
P_u(u,v) &= Q(u,v)\ell(u,v) + R(u,v)m(u,v) - \alpha(u,v)M(u,v), \\
Q_u(u,v) &= -P(u,v)\ell(u,v) + R(u,v)n(u,v) - \alpha(u,v)N(u,v), \\
R_u(u,v) &= -P(u,v)m(u,v) - Q(u,v)n(u,v) + \alpha_v(u,v)
\end{aligned} \tag{4}$$

for all $(u,v) \in U$.

We call the mapping $(\ell, m, n, \alpha, L, M, N, P, Q, R)$ satisfying the integrability condition (4) the *curvature of the one-parameter family of framed curves with respect to* u of (γ, ν_1, ν_2).

Definition 2.11. Let $(\gamma, \nu_1, \nu_2), (\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2) : U \to \mathbb{R}^3 \times \Delta$ be one-parameter families of framed curves with respect to u. We say that (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ are *congruent as one-parameter families of framed curves* if there exist a constant rotation $A \in SO(3)$ and a translation $\boldsymbol{a} \in \mathbb{R}^3$ such that $\widetilde{\gamma}(u, v) = A(\gamma(u, v)) + \boldsymbol{a}$, $\widetilde{\nu}_1(u, v) = A(\nu_1(u, v))$ and $\widetilde{\nu}_2(u, v) = A(\nu_2(u, v))$ for all $(u, v) \in U$.

We also have the existence and uniqueness theorems for one-parameter families of framed curves in terms of curvatures (cf. [27]).

Theorem 2.12 (Existence Theorem for one-parameter families of framed curves)**.** *Let $(\ell, m, n, \alpha, L, M, N, P, Q, R) : I \to \mathbb{R}^{10}$ be a smooth mapping satisfying the integrability condition (4). Then, there exists a one-parameter family of framed curves with respect to u, $(\gamma, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ whose curvature is given by $(\ell, m, n, \alpha, L, M, N, P, Q, R)$.*

Theorem 2.13 (Uniqueness Theorem for one-parameter families of framed curves)**.** *Let*

$$(\gamma, \nu_1, \nu_2), (\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2) : U \to \mathbb{R}^3 \times \Delta$$

be one-parameter families of framed curves with respect to u with curvatures

$$(\ell, m, n, \alpha, L, M, N, P, Q, R), (\widetilde{\ell}, \widetilde{m}, \widetilde{n}, \widetilde{\alpha}, \widetilde{L}, \widetilde{M}, \widetilde{N}, \widetilde{P}, \widetilde{Q}, \widetilde{R}),$$

respectively. Then (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ are congruent as one-parameter families of framed curves if and only if the curvatures $(\ell, m, n, \alpha, L, M, N, P, Q, R)$ and $(\widetilde{\ell}, \widetilde{m}, \widetilde{n}, \widetilde{\alpha}, \widetilde{L}, \widetilde{M}, \widetilde{N}, \widetilde{P}, \widetilde{Q}, \widetilde{R})$ coincide.

Let $(\gamma, \nu_1, \nu_2) : U \to \mathbb{R}^3 \times \Delta$ be a one-parameter family of framed curves with respect to u with curvature $(\ell, m, n, \alpha, L, M, N, P, Q, R)$. For the normal plane of $\gamma(u, v)$, spanned by $\nu_1(t, \lambda)$ and $\nu_2(t, \lambda)$, there are other frames by rotations. We define $(\nu_1^\theta(u, v), \nu_2^\theta(u, v)) \in \Delta$ by

$$\begin{pmatrix} \nu_1^\theta(u, v) \\ \nu_2^\theta(u, v) \end{pmatrix} = \begin{pmatrix} \cos\theta(u, v) & -\sin\theta(u, v) \\ \sin\theta(u, v) & \cos\theta(u, v) \end{pmatrix} \begin{pmatrix} \nu_1(u, v) \\ \nu_2(u, v) \end{pmatrix},$$

where $\theta : U \to \mathbb{R}$ is a smooth function. Then $(\gamma, \nu_1^\theta, \nu_2^\theta) : U \to \mathbb{R}^3 \times \Delta$ is also a one-parameter family of framed curves with respect to u and

$$\boldsymbol{\mu}^\theta(u, v) = \nu_1^\theta(u, v) \times \nu_2^\theta(u, v) = \nu_1(u, v) \times \nu_2(u, v) = \boldsymbol{\mu}(u, v).$$

Proposition 2.14. *Under the above notation, the curvature*

$$(\ell^\theta, m^\theta, n^\theta, \alpha^\theta, L^\theta, M^\theta, N^\theta, P^\theta, Q^\theta, R^\theta)$$

of $(\gamma, \nu_1^\theta, \nu_2^\theta)$ is given by

$$\begin{aligned}&(\ell - \theta_u, m\cos\theta - n\sin\theta, m\sin\theta + n\cos\theta, \alpha, L - \theta_v, M\cos\theta - N\sin\theta,\\ &M\sin\theta + N\cos\theta, P\cos\theta - Q\sin\theta, P\sin\theta + Q\cos\theta, R).\end{aligned}$$

We call the moving frame $\{\nu_1^\theta(u, v), \nu_2^\theta(u, v), \boldsymbol{\mu}(u, v)\}$ the rotated frame along $\gamma(u, v)$ by $\theta(u, v)$.

We also have similar results for the case of one-parameter families of framed curves with respect to v.

3. Relations between framed surfaces and one-parameter families of framed curves

3.1. **Framed surfaces as one-parameter families of framed curves.** Let

$$(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$$

be a framed surface with basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$. We denote $\boldsymbol{t} = \boldsymbol{n} \times \boldsymbol{s}$. We give conditions for the framed surface to be a one-parameter family of framed curves.

Lemma 3.1. *Under the above notations, we have the following.*

(1) $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ *is a one-parameter family of framed curves with respect to* u *if and only if* $a_1(u, v) = 0$ *for all* $(u, v) \in U$.

(2) $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{t})$ *is a one-parameter family of framed curves with respect to* u *if and only if* $b_1(u, v) = 0$ *for all* $(u, v) \in U$.

(3) $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ *is a one-parameter family of framed curves with respect to* v *if and only if* $a_2(u, v) = 0$ *for all* $(u, v) \in U$.

(4) $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{t})$ *is a one-parameter family of framed curves with respect to* v *if and only if* $b_2(u, v) = 0$ *for all* $(u, v) \in U$.

Proof. (1) If $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a one-parameter family of framed curves with respect to u, then $\boldsymbol{x}_u(u, v) \cdot \boldsymbol{n}(u, v) = 0$ and $\boldsymbol{x}_u(u, v) \cdot \boldsymbol{s}(u, v) = 0$ for all $(u, v) \in U$. Since $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a framed surface, the condition $\boldsymbol{x}_u(u, v) \cdot \boldsymbol{n}(u, v) = 0$ holds. Hence, the condition $\boldsymbol{x}_u(u, v) \cdot \boldsymbol{s}(u, v) = 0$ for all $(u, v) \in U$ is equivalent to $a_1(u, v) = 0$ for all $(u, v) \in U$.

The other cases can be proved similarly. □

Proposition 3.2. *Under the above notations, we have the following.*

(1) *Suppose that there exist smooth functions* $k_1, k_2 : U \to \mathbb{R}$ *such that*

$$(k_1(u, v), k_2(u, v)) \neq (0, 0)$$

and

$$k_1(u, v)a_1(u, v) + k_2(u, v)b_1(u, v) = 0$$

for all $(u, v) \in U$. *Then there exist smooth functions* $\theta, \varphi : U \to \mathbb{R}$ *such that* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}^\theta)$ *and* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{t}^\varphi)$ *are one-parameter families of framed curves with respect to* u.

(2) *Suppose that there exist smooth functions* $k_1, k_2 : U \to \mathbb{R}$ *such that*

$$(k_1(u, v), k_2(u, v)) \neq (0, 0)$$

and $k_1(u, v)a_2(u, v) + k_2(u, v)b_2(u, v) = 0$ *for all* $(u, v) \in U$. *Then there exist smooth functions* $\theta, \varphi : U \to \mathbb{R}$ *such that* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}^\theta)$ *and* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{t}^\varphi) : U \to \mathbb{R}^3 \times \Delta$ *are one-parameter families of framed curves with respect to* v.

Proof. (1) We take a smooth function $\theta : U \to \mathbb{R}$ which satisfies the condition

$$(\cos\theta(u, v), \sin\theta(u, v)) = \left(\frac{k_1(u, v)}{\sqrt{k_1^2(u, v) + k_2^2(u, v)}}, \frac{-k_2(u, v)}{\sqrt{k_1^2(u, v) + k_2^2(u, v)}} \right).$$

Then by Proposition 2.5,

$$\begin{aligned} a_1^\theta(u, v) &= a_1(u, v)\cos\theta(u, v) - b_1(u, v)\sin\theta(u, v) \\ &= \frac{1}{\sqrt{k_1^2(u, v) + k_2^2(u, v)}} \left(k_1(u, v)a_1(u, v) + k_2(u, v)b_1(u, v)\right) \\ &= 0. \end{aligned}$$

By Lemma 3.1 (1), $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}^\theta)$ is a one-parameter family of framed curves with respect to u. Moreover, we take a smooth function $\varphi : U \to \mathbb{R}$ which satisfies the condition

$$(\cos\varphi(u,v), \sin\varphi(u,v)) = \left(\frac{k_2(u,v)}{\sqrt{k_1^2(u,v)+k_2^2(u,v)}}, \frac{k_1(u,v)}{\sqrt{k_1^2(u,v)+k_2^2(u,v)}}\right).$$

Then by Proposition 2.5,

$$\begin{aligned} b_1^\varphi(u,v) &= a_1(u,v)\sin\varphi(u,v) + b_1(u,v)\cos\varphi(u,v) \\ &= \frac{1}{\sqrt{k_1^2(u,v)+k_2^2(u,v)}}\left(k_1(u,v)a_1(u,v) + k_2(u,v)b_1(u,v)\right) \\ &= 0. \end{aligned}$$

By Lemma 3.1 (2), $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{t}^\varphi)$ is a one-parameter family of framed curves with respect to u.

(2) We can prove the assertion by a similar calculation. □

We give a relation between basic invariants of a framed surface and the curvature of the one-parameter family of framed curves under a condition.

Proposition 3.3. *Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface with basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$. Suppose that $a_1(u,v) = 0$ for all $(u,v) \in U$. Then the curvature of the one-parameter family of framed curves with respect to u of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ is given by*

$$\begin{aligned} &(\ell(u,v), m(u,v), n(u,v), \alpha(u,v), L(u,v), M(u,v), N(u,v), P(u,v), Q(u,v), R(u,v)) \\ &= (e_1(u,v), f_1(u,v), g_1(u,v), b_1(u,v), e_2(u,v), f_2(u,v), g_2(u,v), 0, a_2(u,v), b_2(u,v)). \end{aligned}$$

Proof. By definitions of basic invariants and the curvature, we have

$$\begin{aligned} \ell(u,v) &= \boldsymbol{n}_u(u,v)\cdot\boldsymbol{s}(u,v) = e_1(u,v), & m(u,v) &= \boldsymbol{n}_u(u,v)\cdot\boldsymbol{t}(u,v) = f_1(u,v), \\ n(u,v) &= \boldsymbol{s}_u(u,v)\cdot\boldsymbol{t}(u,v) = g_1(u,v), & \alpha(u,v) &= \boldsymbol{x}_u(u,v)\cdot\boldsymbol{t}(u,v) = b_1(u,v), \\ L(u,v) &= \boldsymbol{n}_v(u,v)\cdot\boldsymbol{s}(u,v) = e_2(u,v), & M(u,v) &= \boldsymbol{n}_v(u,v)\cdot\boldsymbol{t}(u,v) = f_2(u,v), \\ N(u,v) &= \boldsymbol{s}_v(u,v)\cdot\boldsymbol{t}(u,v) = g_2(u,v), & P(u,v) &= \boldsymbol{x}_v(u,v)\cdot\boldsymbol{n}(u,v) = 0, \\ Q(u,v) &= \boldsymbol{x}_v(u,v)\cdot\boldsymbol{s}(u,v) = a_2(u,v), & R(u,v) &= \boldsymbol{x}_v(u,v)\cdot\boldsymbol{n}(u,v) = b_2(u,v). \end{aligned}$$

□

We give examples of framed surfaces which are not a one-parameter family of framed curves with respect to u nor v as follows.

Example 3.4. Let $\boldsymbol{x} : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$\boldsymbol{x}(u,v) = \begin{cases} \left(e^{-\frac{1}{u^2}-\frac{1}{v^2}}\cos\frac{1}{u^2}\cos\frac{1}{v^2}, e^{-\frac{1}{u^2}-\frac{1}{v^2}}\sin\frac{1}{u^2}\sin\frac{1}{v^2}, 0\right) & (u,v \neq 0), \\ (0,0,0) & (u = 0 \text{ or } v = 0). \end{cases}$$

Then $\boldsymbol{x}$ is a smooth mapping. Moreover, if we take $\boldsymbol{n}(u,v) = (0,0,1)$ and $\boldsymbol{s}(u,v) = (1,0,0)$, then $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : \mathbb{R}^2 \to \mathbb{R}^3 \times \Delta$ is a framed surface.

Next, we show that $\boldsymbol{x}$ is not a one-parameter family of framed curves with respect to u nor v. If $u, v \neq 0$, then we have

$$\begin{aligned} \boldsymbol{x}_u(u,v) &= \frac{2e^{-\frac{1}{u^2}-\frac{1}{v^2}}}{u^3}\left(\left(\cos\frac{1}{u^2}+\sin\frac{1}{u^2}\right)\cos\frac{1}{v^2}, \left(\sin\frac{1}{u^2}-\cos\frac{1}{u^2}\right)\sin\frac{1}{v^2}, 0\right), \\ \boldsymbol{x}_v(u,v) &= \frac{2e^{-\frac{1}{u^2}-\frac{1}{v^2}}}{v^3}\left(\left(\cos\frac{1}{v^2}+\sin\frac{1}{v^2}\right)\cos\frac{1}{u^2}, \left(\sin\frac{1}{v^2}-\cos\frac{1}{v^2}\right)\sin\frac{1}{u^2}, 0\right). \end{aligned}$$

For $v \in \mathbb{R}$ with $\cos(1/v^2)\sin(1/v^2) \neq 0$, $\lim_{u\to 0+0} \boldsymbol{x}_u(u,v)/|\boldsymbol{x}_u(u,v)|$ does not exist. Hence there does not exist $(\nu_1^u, \nu_2^u) : \mathbb{R}^2 \to \Delta$ such that $(\boldsymbol{x}, \nu_1^u, \nu_2^u)$ is a one-parameter family of framed curves with respect to u (cf. [9]). Also, for $u \in \mathbb{R}$ with $\cos(1/u^2)\sin(1/u^2) \neq 0$, $\lim_{v\to 0+0} \boldsymbol{x}_v(u,v)/|\boldsymbol{x}_v(u,v)|$ does not exist. Hence, there does not exist $(\nu_1^v, \nu_2^v) : \mathbb{R}^2 \to \Delta$ such that $(\boldsymbol{x}, \nu_1^v, \nu_2^v)$ is a one-parameter family of framed curves with respect to v. In particular, $\boldsymbol{x}$ is not a one-parameter family of framed base curves with respect to u nor v around $(0,0)$.

A singular point of a mapping $\boldsymbol{x} : U \to \mathbb{R}^3$ is a $D_4^\pm$ singularity if $\boldsymbol{x}$ at the point is $\mathcal{A}$-equivalent (equivalent by diffeomorphisms of the source and of the target) to the map germ $(u,v) \mapsto (uv, u^2 \pm 3v^2, u^2 v \pm v^3)$ at $(0,0)$ (cf. [2, 28]).

Example 3.5 ($D_4^\pm$ singularity)**.** Let $\boldsymbol{x}^\pm : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$\boldsymbol{x}^\pm(u,v) = (uv, u^2 \pm 3v^2, u^2 v \pm v^3).$$

Define $\boldsymbol{n} : \mathbb{R}^2 \to S^2$ by $\boldsymbol{n}(u,v) = (2u, v, -2)/\sqrt{4u^2+v^2+4}$. Since $\boldsymbol{x}_u^\pm(u,v) = (v, 2u, 2uv)$ and $\boldsymbol{x}_v^\pm(u,v) = (u, \pm 6v, u^2 \pm 3v^2)$, $\boldsymbol{x}_u^\pm(u,v)\cdot\boldsymbol{n}(u,v) = \boldsymbol{x}_v^\pm(u,v)\cdot\boldsymbol{n}(u,v) = 0$ for all $(u,v) \in \mathbb{R}^2$. It follows that $(\boldsymbol{x}^\pm, \boldsymbol{n}) : \mathbb{R}^2 \to \mathbb{R}^3 \times S^2$ is a Legendre immersion. However, $\boldsymbol{x}^\pm$ are not one-parameter families of framed base curves with respect to u nor v around $(0,0)$.

We give an example of a framed surface which is also a one-parameter family of framed curves with respect to u and v, respectively.

Example 3.6. Let m_1, n_1, k_1, m_2, n_2 and k_2 be positive integers with

$$m_1 = n_1 + k_1 \quad \text{and} \quad m_2 = n_2 + k_2.$$

Let $\boldsymbol{x} : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$\boldsymbol{x}(u,v) = \left(\frac{1}{n_1}u^{n_1}, \frac{1}{m_1}u^{m_1} + \frac{1}{n_2}v^{n_2}, \frac{1}{m_2}v^{m_2}\right).$$

Define $(\boldsymbol{n}, \boldsymbol{s}) : \mathbb{R}^2 \to \Delta$ by

$$\boldsymbol{n}(u,v) = \frac{(u^{k_1}v^{k_2}, -v^{k_2}, 1)}{\sqrt{u^{2k_1}v^{2k_2} + v^{2k_2} + 1}},\ \boldsymbol{s}(u,v) = \frac{(1, u^{k_1}, 0)}{\sqrt{u^{2k_1}+1}}.$$

Since

$$\boldsymbol{x}_u(u,v) = (u^{n_1-1}, u^{m_1-1}, 0) = u^{n_1-1}(1, u^{k_1}, 0),\ \boldsymbol{x}_v(u,v) = (0, v^{n_2-1}, v^{m_2-1}) = v^{n_2-1}(0, 1, v^{k_2}),$$

we have $\boldsymbol{x}_u(u,v)\cdot\boldsymbol{n}(u,v) = \boldsymbol{x}_v(u,v)\cdot\boldsymbol{n}(u,v) = 0$ for all $(u,v) \in \mathbb{R}^2$. It follows that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a framed surface. If $n_1, n_2 > 1$, then $(0,0)$ is a corank 2 singular point of $\boldsymbol{x}$. Moreover, define $(\nu_1^u, \nu_2^u) : \mathbb{R}^2 \to \Delta$ and $(\nu_1^v, \nu_2^v) : \mathbb{R}^2 \to \Delta$ by

$$\nu_1^u(u,v) = \frac{(-u^{k_1}, 1, 0)}{\sqrt{u^{2k_1}+1}},\ \nu_2^u(u,v) = (0,0,1),\ \nu_1^v(u,v) = \frac{(0, -v^{k_2}, 1)}{\sqrt{v^{2k_2}+1}},\ \nu_2^v(u,v) = (1,0,0).$$

Then $(\boldsymbol{x}, \nu_1^u, \nu_2^u)$ and $(\boldsymbol{x}, \nu_1^v, \nu_2^v)$ are one-parameter families of framed curves with respect to u and v, respectively.

3.2. **One-parameter families of framed curves as framed surfaces.** First, we consider a one-parameter family of framed curves with respect to u. We give conditions for the surface to be a framed base surface. In this section, we use the following notations. Let

$$(\boldsymbol{x}, \nu_1^u, \nu_2^u) : U \to \mathbb{R}^3 \times \Delta$$

be a one-parameter family of framed curves with respect to u with curvature

$$(\ell^u, m^u, n^u, \alpha^u, L^u, M^u, N^u, P^u, Q^u, R^u).$$

Lemma 3.7. *Under the above notations, we have the following.*
(1) $(\boldsymbol{x}, \nu_1^u, \nu_2^u) : U \to \mathbb{R}^3 \times \Delta$ *is a framed surface if and only if* $P^u(u,v) = 0$ *for all* $(u,v) \in U$.
(2) $(\boldsymbol{x}, \nu_2^u, \nu_1^u) : U \to \mathbb{R}^3 \times \Delta$ *is a framed surface if and only if* $Q^u(u,v) = 0$ *for all* $(u,v) \in U$.

Proof. (1) Since $(\boldsymbol{x}, \nu_1^u, \nu_2^u)$ is a one-parameter family of framed curves with respect to u, we have $\boldsymbol{x}_u(u,v) \cdot \nu_1^u(u,v) = 0$ for all $(u,v) \in U$. Since $\boldsymbol{x}_v(u,v) \cdot \nu_1^u(u,v) = P^u(u,v)$, $(\boldsymbol{x}, \nu_1^u, \nu_2^u)$ is a framed surface if and only if $P^u(u,v) = 0$ for all $(u,v) \in U$.

(2) We can prove the assertion by a similar calculation. $\square$

Proposition 3.8. *Under the above notations, suppose that there exist smooth functions* $k_1, k_2 : U \to \mathbb{R}$ *such that* $(k_1(u,v), k_2(u,v)) \neq (0,0)$ *and* $k_1(u,v)P^u(u,v) + k_2(u,v)Q^u(u,v) = 0$ *for all* $(u,v) \in U$. *Then there exist smooth functions* $\theta, \varphi : U \to \mathbb{R}$ *such that* $(\boldsymbol{x}, \nu_1^{u,\theta}, \nu_2^{u,\theta})$ *and* $(\boldsymbol{x}, \nu_2^{u,\varphi}, \nu_1^{u,\varphi}) : U \to \mathbb{R}^3 \times \Delta$ *are framed surfaces.*

Proof. We take a smooth function $\theta : U \to \mathbb{R}$ which satisfies the condition

$$(\cos\theta(u,v), \sin\theta(u,v)) = \left(\frac{k_1(u,v)}{\sqrt{k_1^2(u,v) + k_2^2(u,v)}}, \frac{-k_2(u,v)}{\sqrt{k_1^2(u,v) + k_2^2(u,v)}} \right).$$

Then by Proposition 2.14,

$$\begin{aligned} P^{u,\theta}(u,v) &= P^u(u,v)\cos\theta(u,v) - Q^u(u,v)\sin\theta(u,v) \\ &= \frac{1}{\sqrt{k_1^2(u,v) + k_2^2(u,v)}} (k_1(u,v)P^u(u,v) + k_2(u,v)Q^u(u,v)) \\ &= 0. \end{aligned}$$

By Lemma 3.7 (1), $(\boldsymbol{x}, \nu_1^{u,\theta}, \nu_2^{u,\theta})$ is a framed surface. Moreover, we take a smooth function $\varphi : U \to \mathbb{R}$ which satisfies the condition

$$(\cos\varphi(u,v), \sin\varphi(u,v)) = \left(\frac{k_2(u,v)}{\sqrt{k_1^2(u,v) + k_2^2(u,v)}}, \frac{k_1(u,v)}{\sqrt{k_1^2(u,v) + k_2^2(u,v)}} \right).$$

Then by Proposition 2.14,

$$\begin{aligned} Q^{u,\theta}(u,v) &= P^u(u,v)\sin\theta(u,v) + Q^u(u,v)\cos\theta(u,v) \\ &= \frac{1}{\sqrt{k_1^2(u,v) + k_2^2(u,v)}} (k_1(u,v)P^u(u,v) + k_2(u,v)Q^u(u,v)) \\ &= 0. \end{aligned}$$

By Lemma 3.7 (2), $(\boldsymbol{x}, \nu_2^{u,\varphi}, \nu_1^{u,\varphi})$ is a framed surface. $\square$

Next, we consider one-parameter families of framed curves with respect to u and v. We give conditions for the surface to be a framed base surface.

Let $(\boldsymbol{x}, \nu_1^u, \nu_2^u) : U \to \mathbb{R}^3 \times \Delta$ be a one-parameter family of framed curves with respect to u and $(\boldsymbol{x}, \nu_1^v, \nu_2^v) : U \to \mathbb{R}^3 \times \Delta$ be a one-parameter family of framed curves with respect to v, respectively. We denote $\boldsymbol{\mu}^u = \nu_1^u \times \nu_2^u$ and $\boldsymbol{\mu}^v = \nu_1^v \times \nu_2^v$.

Proposition 3.9. *Under the above notations, we have the following.*

(1) *Suppose that* $\boldsymbol{\mu}^u(u,v)$ *and* $\boldsymbol{\mu}^v(u,v)$ *are linearly independent for all* $(u,v) \in U$, *that is, if* $k_1(u,v)\boldsymbol{\mu}^u(u,v) + k_2(u,v)\boldsymbol{\mu}^v(u,v) = 0$ *for all* $(u,v) \in U$, *where* $k_1, k_2 : U \to \mathbb{R}$ *are smooth functions, then* $(k_1(u,v), k_2(u,v)) = (0,0)$ *for all* $(u,v) \in U$. *Then there exists a smooth mapping* $(\boldsymbol{n}, \boldsymbol{s}) : U \to \Delta$ *such that* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ *is a framed surface.*

(2) *Suppose that* $\boldsymbol{\mu}^u(u,v)$ *and* $\boldsymbol{\mu}^v(u,v)$ *are linearly dependent for all* $(u,v) \in U$, *that is, there exist smooth functions* $k_1, k_2 : U \to \mathbb{R}$ *such that* $(k_1(u,v), k_2(u,v)) \neq (0,0)$ *and*

$$k_1(u,v)\boldsymbol{\mu}^u(u,v) + k_2(u,v)\boldsymbol{\mu}^v(u,v) = 0$$

for all $(u,v) \in U$. *Then there exists a smooth mapping* $(\boldsymbol{n}, \boldsymbol{s}) : U \to \Delta$ *such that* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ *is a framed surface.*

Proof. (1) Since $\boldsymbol{\mu}^u(u,v)$ and $\boldsymbol{\mu}^v(u,v)$ are linearly independent, we can define the smooth mapping $(\boldsymbol{n}, \boldsymbol{s}) : U \to \Delta$ by

$$\boldsymbol{n}(u,v) = \frac{\boldsymbol{\mu}^u(u,v) \times \boldsymbol{\mu}^v(u,v)}{|\boldsymbol{\mu}^u(u,v) \times \boldsymbol{\mu}^v(u,v)|},\ \boldsymbol{s}(u,v) = \boldsymbol{\mu}^u(u,v).$$

It follows that

$$\begin{aligned}
\boldsymbol{x}_u(u,v) \cdot \boldsymbol{n}(u,v) &= \alpha^u(u,v)\boldsymbol{\mu}^u(u,v) \cdot (\boldsymbol{\mu}^u(u,v) \times \boldsymbol{\mu}^v(u,v)/|\boldsymbol{\mu}^u(u,v) \times \boldsymbol{\mu}^v(u,v)|) = 0,\\
\boldsymbol{x}_v(u,v) \cdot \boldsymbol{n}(u,v) &= \alpha^v(u,v)\boldsymbol{\mu}^v(u,v) \cdot (\boldsymbol{\mu}^u(u,v) \times \boldsymbol{\mu}^v(u,v)/|\boldsymbol{\mu}^u(u,v) \times \boldsymbol{\mu}^v(u,v)|) = 0.
\end{aligned}$$

Moreover, $\boldsymbol{n}(u,v) \cdot \boldsymbol{s}(u,v) = (\boldsymbol{\mu}^u(u,v) \times \boldsymbol{\mu}^v(u,v)/|\boldsymbol{\mu}^u(u,v) \times \boldsymbol{\mu}^v(u,v)|) \cdot \boldsymbol{\mu}^u(u,v) = 0$. Therefore, $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ is a framed surface.

(2) By the assumption and $\boldsymbol{\mu}^u(u,v), \boldsymbol{\mu}^v(u,v) \in S^2$, if $k_1(p) = 0$ (respectively, $k_2(p) = 0$), then $k_2(p) = 0$ (respectively, $k_1(p) = 0$). It follows that $k_1(u,v) \neq 0$ and $k_2(u,v) \neq 0$ for all $(u,v) \in U$. Then we have $\boldsymbol{\mu}^v(u,v) = \pm\boldsymbol{\mu}^u(u,v)$. We define the smooth mapping $(\boldsymbol{n}, \boldsymbol{s}) : U \to \Delta$ by $\boldsymbol{n}(u,v) = \nu_1^u(u,v)$, $\boldsymbol{s}(u,v) = \boldsymbol{\mu}^u(u,v)$. Then $\boldsymbol{x}_u(u,v) \cdot \boldsymbol{n}(u,v) = 0$ and

$$\boldsymbol{x}_v(u,v) \cdot \boldsymbol{n}(u,v) = \alpha^v(u,v)\boldsymbol{\mu}^v(u,v) \cdot \nu_1^u(u,v) = \pm\alpha^v(u,v)\boldsymbol{\mu}^u(u,v) \cdot \nu_1^u(u,v) = 0.$$

Moreover, $\boldsymbol{n}(u,v) \cdot \boldsymbol{s}(u,v) = \nu_1^u(u,v) \cdot \boldsymbol{\mu}^u(u,v) = 0$. Therefore, $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ is a framed surface. □

We give an example of a one-parameter family of framed curves with respect to u and v which is not a framed base surface.

Example 3.10 (A cross cap)**.** Let $\boldsymbol{x} : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $\boldsymbol{x}(u,v) = (u+v, (u+v)v, v^2)$. Note that $\boldsymbol{x}$ is diffeomorphic to the cross cap $\widetilde{\boldsymbol{x}}(u,v) = (u, uv, v^2)$ by using the parameter change $\phi(u,v) = (u+v, v)$. Since $\boldsymbol{x}_u(u,v) = (1, v, 0)$, if we consider the smooth mapping $(\nu_1^u, \nu_2^u) : \mathbb{R}^2 \to \Delta$ defined by

$$\nu_1^u(u,v) = \frac{(-v, 1, 0)}{\sqrt{1+v^2}},\ \nu_2^u(u,v) = (0,0,1),$$

then $\boldsymbol{x}_u(u,v) \cdot \nu_1^u(u,v) = 0$, $\boldsymbol{x}_u(u,v) \cdot \nu_2^u(u,v) = 0$ and $\nu_1^u(u,v) \cdot \nu_2^u(u,v) = 0$ for all $(u,v) \in \mathbb{R}^2$. Hence, $(\boldsymbol{x}, \nu_1^u, \nu_2^u)$ is a one-parameter family of framed curves with respect to u. Moreover, since $\boldsymbol{x}_v(u,v) = (1, u+2v, 2v)$, if we consider the smooth mapping $(\nu_1^v, \nu_2^v) : \mathbb{R}^2 \to \Delta$ defined by

$$\nu_1^v(u,v) = \frac{(-(u+2v), 1, 0)}{\sqrt{1+(u+2v)^2}},\ \nu_2^v(u,v) = \frac{(2v, 2v(u+2v), -1-(u+2v)^2)}{\sqrt{(1+(u+2v)^2)(1+(u+2v)^2+4v^2)}},$$

then $\boldsymbol{x}_v(u,v) \cdot \nu_1^v(u,v) = 0$, $\boldsymbol{x}_v(u,v) \cdot \nu_2^v(u,v) = 0$ and $\nu_1^v(u,v) \cdot \nu_2^v(u,v) = 0$ for all $(u,v) \in \mathbb{R}^2$. Hence, $(\boldsymbol{x}, \nu_1^v, \nu_2^v)$ is a one-parameter family of framed curves with respect to v. However, the cross cap is not a frontal at $(0,0)$ (cf. [6]). Hence $\boldsymbol{x}$ is not a framed base surface. Since

$$\boldsymbol{\mu}^u(u,v) = \frac{(1, v, 0)}{\sqrt{1+v^2}},\ \boldsymbol{\mu}^v(u,v) = -\frac{(1, u+2v, 2v)}{\sqrt{1+(u+2v)^2+4v^2}},$$

the conditions in Proposition 3.9 are not satisfied around $(0,0)$.

4. Surfaces with corank one singular points

We consider surfaces with corank one singular points from the view point of one-parameter families of framed curves.

If $(0,0)$ is a corank one singular point of $\boldsymbol{x}$, then

$$\boldsymbol{x}(u,v) = (u, f(u,v), g(u,v)) \quad \text{or} \quad \boldsymbol{x}(u,v) = (v, f(u,v), g(u,v))$$

around $(0,0)$ by using a parameter change (a one-parameter parameter change).

Theorem 4.1. *Let $\boldsymbol{x} : U \to \mathbb{R}^3$ be a smooth mapping and $p \in U$ be a corank one singular point. Suppose that $\boldsymbol{x}$ is given by the form $\boldsymbol{x}(u,v) = (u, f(u,v), g(u,v))$.*

(1) There exists a smooth mapping $(\nu_1^u, \nu_2^u) : U \to \Delta$ such that $(\boldsymbol{x}, \nu_1^u, \nu_2^u)$ is a one-parameter family of framed curves with respect to u.

(2) If there exist smooth functions $k_1, k_2 : U \to \mathbb{R}$ such that $(k_1(u,v), k_2(u,v)) \neq (0,0)$ and $k_1(u,v)f_v(u,v) + k_2(u,v)g_v(u,v) = 0$ for all $(u,v) \in U$, then there exists a smooth mapping $(\nu_1^v, \nu_2^v) : U \to \Delta$ such that $(\boldsymbol{x}, \nu_1^v, \nu_2^v)$ is a one-parameter family of framed curves with respect to v. Conversely, if there exists a smooth mapping $(\nu_1^v, \nu_2^v) : U \to \Delta$ such that $(\boldsymbol{x}, \nu_1^v, \nu_2^v)$ is a one-parameter family of framed curves with respect to v, then there exist smooth function germs $k_1, k_2 : (U,p) \to \mathbb{R}$ such that $(k_1(u,v), k_2(u,v)) \neq (0,0)$ and $k_1(u,v)f_v(u,v) + k_2(u,v)g_v(u,v) = 0$ around p.

Proof. (1) Since $\boldsymbol{x}_u(u,v) = (1, f_u(u,v), g_u(u,v))$, we consider smooth mappings

$$\nu_1^u(u,v) = \frac{(-f_u(u,v), 1, 0)}{\sqrt{1 + f_u^2(u,v)}}, \ \nu_2^u(u,v) = \frac{(-g_u(u,v), -f_u(u,v)g_u(u,v), 1 + f_u^2(u,v))}{\sqrt{(1 + f_u^2(u,v) + g_u^2(u,v))(1 + f_u^2(u,v))}}.$$

By a direct calculation, we have

$$\boldsymbol{x}_u(u,v) \cdot \nu_1^u(u,v) = 0, \quad \boldsymbol{x}_u(u,v) \cdot \nu_2^u(u,v) = 0, \ \text{and} \quad \nu_1^u(u,v) \cdot \nu_2^u(u,v) = 0$$

for all $(u,v) \in U$. Hence, $(\boldsymbol{x}, \nu_1^u, \nu_2^u) : U \to \mathbb{R}^3 \times \Delta$ is a one-parameter family of framed curves with respect to u.

(2) Since $\boldsymbol{x}_v(u,v) = (0, f_v(u,v), g_v(u,v))$, we consider smooth mappings

$$\nu_1^v(u,v) = \frac{(0, k_1(u,v), k_2(u,v))}{\sqrt{k_1^2(u,v) + k_2^2(u,v)}}, \ \nu_2^v(u,v) = (1,0,0).$$

By a direct calculation, we have

$$\boldsymbol{x}_v(u,v) \cdot \nu_1^v(u,v) = 0, \quad \boldsymbol{x}_v(u,v) \cdot \nu_2^v(u,v) = 0, \ \text{and} \quad \nu_1^v(u,v) \cdot \nu_2^v(u,v) = 0$$

for all $(u,v) \in U$. Hence, $(\boldsymbol{x}, \nu_1^v, \nu_2^v) : U \to \mathbb{R}^3 \times \Delta$ is a one-parameter family of framed curves with respect to v.

Conversely, suppose that $(\boldsymbol{x}, \nu_1^v, \nu_2^v) : U \to \mathbb{R}^3 \times \Delta$ is a one-parameter family of framed curves with respect to v. We denote

$$\nu_1^v(u,v) = (\nu_{11}^v(u,v), \nu_{12}^v(u,v), \nu_{13}^v(u,v))$$

and $\nu_2^v(u,v) = (\nu_{21}^v(u,v), \nu_{22}^v(u,v), \nu_{23}^v(u,v))$. It follows that

$$\begin{aligned}
\boldsymbol{x}_v(u,v) \cdot \nu_1^v(u,v) &= \nu_{12}^v(u,v)f_v(u,v) + \nu_{13}^v(u,v)g_v(u,v) = 0, \\
\boldsymbol{x}_v(u,v) \cdot \nu_2^v(u,v) &= \nu_{22}^v(u,v)f_v(u,v) + \nu_{23}^v(u,v)g_v(u,v) = 0.
\end{aligned}$$

If $(\nu_{12}^v(p), \nu_{13}^v(p)) \neq (0,0)$, then $(\nu_{12}^v(u,v), \nu_{13}^v(u,v)) \neq (0,0)$ around p. If we consider $(k_1, k_2) = (\nu_{12}^v, \nu_{13}^v)$, then the condition is satisfied. On the other hand, if $(\nu_{12}^v(p), \nu_{13}^v(p)) = (0,0)$, then $\nu_1^v(p) = (\pm 1, 0, 0)$. Since $\nu_1^v(p) \cdot \nu_2^v(p) = 0$, we have $(\nu_{22}^v(p), \nu_{23}^v(p)) \neq (0,0)$. It follows that $(\nu_{22}^v(u,v), \nu_{23}^v(u,v)) \neq (0,0)$ around p. If we consider $(k_1, k_2) = (\nu_{22}^v, \nu_{23}^v)$, then the condition is satisfied. $\square$

Remark 4.2. Suppose that $\boldsymbol{x}: U \to \mathbb{R}^3$ is given by $\boldsymbol{x}(u,v) = (u, f(u,v), g(u,v))$ and there exists a smooth mapping $(\nu_1^v, \nu_2^v): U \to \Delta$ such that $(\boldsymbol{x}, \nu_1^v, \nu_2^v)$ is a one-parameter family of framed curves with respect to v. Then $(f,g): U \to \mathbb{R}^2$ is a one-parameter family of frontal curves with respect to v around $p \in U$. For definition and properties of one-parameter families of frontal curves (Legendre curves) see [16, 32]. Conversely, if $(f,g): U \to \mathbb{R}^2$ is a one-parameter family of frontal curves with respect to v, then there exists a smooth mapping $(\nu_1^v, \nu_2^v): U \to \Delta$ such that $(\boldsymbol{x}, \nu_1^v, \nu_2^v)$ is a one-parameter family of framed curves with respect to v by Theorem 4.1. Also see [18].

Proposition 4.3. (1) *Let* $(\boldsymbol{x}, \nu_1^u, \nu_2^u): U \to \mathbb{R}^3 \times \Delta$ *be given by* $\boldsymbol{x}(u,v) = (u, f(u,v), g(u,v))$,

$$\nu_1^u(u,v) = \frac{(-f_u(u,v), 1, 0)}{\sqrt{1+f_u^2(u,v)}},\ \nu_2^u(u,v) = \frac{(-g_u(u,v), -f_u(u,v)g_u(u,v), 1+f_u^2(u,v))}{\sqrt{(1+f_u^2(u,v)+g_u^2(u,v))(1+f_u^2(u,v))}}.$$

Then the curvature of the one-parameter family of framed curves with respect to u, $(\boldsymbol{x}, \nu_1^u, \nu_2^u)$ *is given by*

$$\begin{aligned}
\ell^u(u,v) &= \nu_{1u}^u(u,v) \cdot \nu_2^u(u,v) = \frac{f_{uu}(u,v)g_u(u,v)}{(1+f_u^2(u,v))\sqrt{1+f_u^2(u,v)+g_u^2(u,v)}},\\
m^u(u,v) &= \nu_{1u}^u(u,v) \cdot \boldsymbol{\mu}^u(u,v) = \frac{-f_{uu}(u,v)}{\sqrt{(1+f_u^2(u,v))(1+f_u^2(u,v)+g_u^2(u,v))}},\\
n^u(u,v) &= \nu_{2u}^u(u,v) \cdot \boldsymbol{\mu}^u(u,v) = \frac{-g_{uu}(u,v)+f_{uu}(u,v)f_u(u,v)g_u(u,v)-f_u^2(u,v)g_{uu}(u,v)}{(1+f_u^2(u,v)+g_u^2(u,v))\sqrt{1+f_u^2(u,v)}},\\
\alpha^u(u,v) &= \boldsymbol{x}_u(u,v) \cdot \boldsymbol{\mu}^u(u,v) = \sqrt{1+f_u^2(u,v)+g_u^2(u,v)},\\
L^u(u,v) &= \nu_{1v}^u(u,v) \cdot \nu_2^u(u,v) = \frac{f_{uv}(u,v)g_u(u,v)}{(1+f_u^2(u,v))\sqrt{1+f_u^2(u,v)+g_u^2(u,v)}},\\
M^u(u,v) &= \nu_{1v}^u(u,v) \cdot \boldsymbol{\mu}^u(u,v) = \frac{-f_{uv}(u,v)}{\sqrt{(1+f_u^2(u,v))(1+f_u^2(u,v)+g_u^2(u,v))}},\\
N^u(u,v) &= \nu_{2v}^u(u,v) \cdot \boldsymbol{\mu}^u(u,v) = \frac{-g_{uv}(u,v)+f_{uv}(u,v)f_u(u,v)g_u(u,v)-f_u^2(u,v)g_{uv}(u,v)}{(1+f_u^2(u,v)+g_u^2(u,v))\sqrt{1+f_u^2(u,v)}},\\
P^u(u,v) &= \boldsymbol{x}_v(u,v) \cdot \nu_1^u(u,v) = \frac{f_v(u,v)}{\sqrt{1+f_u^2(u,v)}},\\
Q^u(u,v) &= \boldsymbol{x}_v(u,v) \cdot \nu_2^u(u,v) = \frac{-f_u(u,v)g_u(u,v)f_v(u,v)+g_v(u,v)+f_u^2(u,v)g_v^2(u,v)}{\sqrt{(1+f_u^2(u,v))(1+f_u^2(u,v)+g_u^2(u,v))}},\\
R^u(u,v) &= \boldsymbol{x}_v(u,v) \cdot \boldsymbol{\mu}^u(u,v) = \frac{f_u(u,v)f_v(u,v)+g_u(u,v)g_v(u,v)}{\sqrt{1+f_u^2(u,v)+g_u^2(u,v)}}.
\end{aligned}$$

(2) *Suppose that there exist smooth functions* $k_1, k_2: U \to \mathbb{R}$ *such that* $(k_1(u,v), k_2(u,v)) \neq (0,0)$ *and* $k_1(u,v)f_v(u,v) + k_2(u,v)g_v(u,v) = 0$ *for all* $(u,v) \in U$. *Let* $(\boldsymbol{x}, \nu_1^v, \nu_2^v): U \to \mathbb{R}^3 \times \Delta$ *be given by* $\boldsymbol{x}(u,v) = (u, f(u,v), g(u,v))$,

$$\nu_1^v(u,v) = \frac{(0, k_1(u,v), k_2(u,v))}{\sqrt{k_1^2(u,v)+k_2^2(u,v)}},\ \nu_2^v(u,v) = (1,0,0)$$

Then the curvature of the one-parameter family of framed curves with respect to v, $(\boldsymbol{x}, \nu_1^v, \nu_2^v)$ is given by

$$\begin{aligned}
\ell^v(u,v) &= \nu_{1v}^v(u,v) \cdot \nu_2^v(u,v) = 0,\\
m^v(u,v) &= \nu_{1v}^v(u,v) \cdot \boldsymbol{\mu}^v(u,v) = \frac{k_{1v}(u,v)k_2(u,v) - k_{2v}(u,v)k_1(u,v)}{(k_1^2(u,v) + k_2^2(u,v))},\\
n^v(u,v) &= \nu_{2v}^v(u,v) \cdot \boldsymbol{\mu}^v(u,v) = 0,\\
\alpha^v(u,v) &= \boldsymbol{x}_v(u,v) \cdot \boldsymbol{\mu}^v(u,v) = \frac{k_2(u,v)f_v(u,v) - k_1(u,v)g_v(u,v)}{\sqrt{k_1^2(u,v) + k_2^2(u,v)}},\\
L^v(u,v) &= \nu_{1u}^v(u,v) \cdot \nu_2^v(u,v) = 0,\\
M^v(u,v) &= \nu_{1u}^v(u,v) \cdot \boldsymbol{\mu}^v(u,v) = \frac{k_{1u}(u,v)k_2(u,v) - k_{2u}(u,v)k_1(u,v)}{(k_1^2(u,v) + k_2^2(u,v))},\\
N^v(u,v) &= \nu_{2u}^v(u,v) \cdot \boldsymbol{\mu}^v(u,v) = 0,\\
P^v(u,v) &= \boldsymbol{x}_u(u,v) \cdot \nu_1^v(u,v) = \frac{f_u(u,v)k_1(u,v) + g_u(u,v)k_2(u,v)}{\sqrt{k_1^2(u,v) + k_2^2(u,v)}},\\
Q^v(u,v) &= \boldsymbol{x}_u(u,v) \cdot \nu_2^v(u,v) = 1,\\
R^v(u,v) &= \boldsymbol{x}_u(u,v) \cdot \boldsymbol{\mu}^v(u,v) = \frac{f_u(u,v)k_2(u,v) - g_u(u,v)k_1(u,v)}{\sqrt{k_1^2(u,v) + k_2^2(u,v)}}.
\end{aligned}$$

Proof. (1) By definition, we have

$$\boldsymbol{\mu}^u(u,v) = \nu_1^u(u,v) \times \nu_2^u(u,v) = \frac{(1, f_u(u,v), g_u(u,v))}{\sqrt{1 + f_u^2(u,v) + g_u^2(u,v)}}.$$

By a direct calculation, we have the curvature.

(2) By definition, we have

$$\boldsymbol{\mu}^v(u,v) = \nu_1^v(u,v) \times \nu_2^v(u,v) = \frac{(0, k_2(u,v), -k_1(u,v))}{\sqrt{k_1^2(u,v) + k_2^2(u,v)}}.$$

By a direct calculation, we have the curvature. □

For the surface $\boldsymbol{x}(u,v) = (u, f(u,v), g(u,v))$, if we consider a parameter change

$$\phi(u,v) = (u+v, v),$$

then we have $\boldsymbol{x} \circ \phi(u,v) = (u+v, \tilde{f}(u,v), \tilde{g}(u,v))$. Then we have the following corollary.

Corollary 4.4. *Let $\boldsymbol{x} : U \to \mathbb{R}^3$ be a smooth mapping given by the form*

$$\boldsymbol{x}(u,v) = (u+v, f(u,v), g(u,v)).$$

Then there exist smooth mappings $(\nu_1^u, \nu_2^u) : U \to \Delta$ and $(\nu_1^v, \nu_2^v) : U \to \Delta$ such that $(\boldsymbol{x}, \nu_1^u, \nu_2^u)$ and $(\boldsymbol{x}, \nu_1^v, \nu_2^v)$ are one-parameter families of framed curves with respect to u and v, respectively.

By a similar calculation of Theorem 4.1 (2), we also have the following result (cf. [23, Proposition 3.4]).

Proposition 4.5. *Let $\boldsymbol{x} : U \to \mathbb{R}^3$ be a smooth mapping and $p \in U$ be a corank one singular point. Suppose that $\boldsymbol{x}$ is given by the form $\boldsymbol{x}(u,v) = (u, f(u,v), g(u,v))$. Then there exist smooth functions $k_1, k_2 : U \to \mathbb{R}$ such that $(k_1(u,v), k_2(u,v)) \neq (0,0)$ and*

$$k_1(u,v)f_v(u,v) + k_2(u,v)g_v(u,v) = 0$$

for all $(u,v) \in U$ if and only if there exists a smooth mapping $\boldsymbol{n} : U \to S^2$ such that $(\boldsymbol{x}, \boldsymbol{n})$ is a Legendre surface.

Proof. Suppose that $k_1(u,v)f_v(u,v)+k_2(u,v)g_v(u,v)=0$ for all $(u,v)\in U$. Since
$$\boldsymbol{x}_u(u,v)=(1,f_u(u,v),g_u(u,v))$$
and $\boldsymbol{x}_v(u,v)=(0,f_v(u,v),g_v(u,v))$, we define $\boldsymbol{n}:U\to S^2$ by
$$\boldsymbol{n}(u,v)=\frac{(-k_1(u,v)f_u(u,v)-k_2(u,v)g_u(u,v),k_1(u,v),k_2(u,v))}{\sqrt{(k_1(u,v)f_u(u,v)+k_2(u,v)g_u(u,v))^2+k_1^2(u,v)+k_2^2(u,v)}}.$$
Then $\boldsymbol{x}_u(u,v)\cdot\boldsymbol{n}(u,v)=0$ and $\boldsymbol{x}_v(u,v)\cdot\boldsymbol{n}(u,v)=0$ for all $(u,v)\in U$. Hence, $(\boldsymbol{x},\boldsymbol{n})$ is a Legendre surface.

Conversely, suppose that $(\boldsymbol{x},\boldsymbol{n}):U\to\mathbb{R}^3\times S^2$ is a Legendre surface. We denote
$$\boldsymbol{n}(u,v)=(n_1(u,v),n_2(u,v),n_3(u,v)).$$
By definition, we have
$$\begin{aligned}\boldsymbol{x}_u(u,v)\cdot\boldsymbol{n}(u,v)&=n_1(u,v)+f_u(u,v)n_2(u,v)+g_u(u,v)n_3(u,v)=0,\\ \boldsymbol{x}_v(u,v)\cdot\boldsymbol{n}(u,v)&=f_v(u,v)n_2(u,v)+g_v(u,v)n_3(u,v)=0.\end{aligned}$$
If $n_2(u,v)=n_3(u,v)=0$, then $n_1(u,v)=0$. It contradicts the fact that $\boldsymbol{n}(u,v)\in S^2$. Hence $(n_2(u,v),n_3(u,v))\neq(0,0)$ for all $(u,v)\in U$ and $f_v(u,v)n_2(u,v)+g_v(u,v)n_3(u,v)=0$. □

By Theorem 4.1 (2) and Proposition 4.5, we have the following corollary.

Corollary 4.6. *Let $\boldsymbol{x}:(U,p)\to\mathbb{R}^3$ be a smooth mapping germ and p be a corank one singular point. Suppose that $\boldsymbol{x}$ is given by the form $\boldsymbol{x}(u,v)=(u,f(u,v),g(u,v))$. The following are equivalent:*

(1) *There exists a smooth mapping germ $(\nu_1^v,\nu_2^v):(U,p)\to\Delta$ such that $(\boldsymbol{x},\nu_1^v,\nu_2^v)$ is a one-parameter family of framed curves with respect to v.*

(2) *There exists a smooth mapping germ $\boldsymbol{n}:(U,p)\to S^2$ such that $(\boldsymbol{x},\boldsymbol{n})$ is a Legendre surface.*

(3) *There exists a smooth mapping germ $(\boldsymbol{n},\boldsymbol{s}):(U,p)\to\Delta$ such that $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})$ is a framed surface.*

We consider concrete examples of one-parameter families of framed curves. We give cuspidal edges, swallowtails and cuspidal cross caps which are generic singularities of frontals. Since these are frontals, they are also framed surfaces at least locally. Moreover, we consider cross caps and ruled surfaces as one-parameter families of framed curves.

We say that a singular point of a mapping $\boldsymbol{x}:U\to\mathbb{R}^3$ is a *cuspidal edge* (respectively, *swallowtail, cuspidal cross cap* or *cross cap*) if $\boldsymbol{x}$ at the point is $\mathcal{A}$-equivalent to the map germ $(u,v)\mapsto(u,v^2,v^3)$ (respectively, $(u,4v^3+2uv,3v^4+uv^2),(u,v^2,uv^3)$ or (u,uv,v^2)) at $(0,0)$.

Let $\boldsymbol{x}:U\to\mathbb{R}^3$ be the frontal of a Legendre surface $(\boldsymbol{x},\boldsymbol{n})$, where U is a domain in $\mathbb{R}^2$. We define the discriminant function $\lambda:U\to\mathbb{R}$ by $\lambda(u,v)=\det(\boldsymbol{x}_u,\boldsymbol{x}_v,\boldsymbol{n})(u,v)$ where (u,v) is a coordinate system on U. When a singular point p of $\boldsymbol{x}$ is non-degenerate, that is, $d\lambda(p)\neq0$, there exists a smooth parametrization $\delta(t):(-\varepsilon,\varepsilon)\to U$, $\delta(0)=p$ of the singular set $S(\boldsymbol{x})$. We call the curve $\delta(t)$ the singular curve of $\boldsymbol{x}$. Moreover, there exists a smooth vector field $\eta(t)$ along δ satisfying that $\eta(t)$ generates $\ker d\boldsymbol{x}_{\delta(t)}$.

Remark 4.7. If a singular point p is non-degenerate of $(\boldsymbol{x},\boldsymbol{n})$, then p is also of corank one. Hence $\boldsymbol{x}$ is a one-parameter family of framed base curves around p.

A non-degenerate singular point p is called of *first kind* (respectively, of *second kind*) if $\eta\lambda(p)\neq0$ (respectively, $\eta\lambda(p)=0$ and $\eta\eta\lambda(p)\neq0$), see [29, 21].

Now we define a function $\phi_x(t)$ on $(-\epsilon,\epsilon)$ by $\phi_x(t)=\det((\boldsymbol{x}\circ\delta)',\boldsymbol{n}\circ\delta,d\boldsymbol{n}(\eta))(t)$. Using these notations, we have the following result (see [15] for example).

Theorem 4.8 ([4], [17])**.** *Let $(\boldsymbol{x}, \boldsymbol{n}) : U \to \mathbb{R}^3$ be a Legendre surface and $p \in U$ be a non-degenerate singular point of $\boldsymbol{x}$. Then the following assertions hold.*

(1) *If $\eta\lambda(p) \neq 0$, then $\boldsymbol{x}$ to be a front near p if and only if $\phi_x(0) \neq 0$ holds.*

(2) *The map germ $\boldsymbol{x}$ at p is $\mathcal{A}$-equivalent to the cuspidal edge if and only if $\boldsymbol{x}$ to be front near p and $\eta\lambda(p) \neq 0$ hold.*

(3) *The map germ $\boldsymbol{x}$ at p is $\mathcal{A}$-equivalent to the swallowtail if and only if $\boldsymbol{x}$ to be front near p and $\eta\lambda(p) = 0$ and $\eta\eta\lambda(p) \neq 0$ hold.*

(4) *The map germ $\boldsymbol{x}$ at p is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $\eta\lambda(p) \neq 0$, $\phi_x(0) = 0$ and $\phi_x'(0) \neq 0$ hold.*

Here, $\eta\lambda : U \to \mathbb{R}$ means the directional derivative of λ by the vector field $\tilde{\eta}$, where $\tilde{\eta}$ is an extended vector field of η to U.

4.1. **First kind singularities.** We consider first kind singularities. A normal form of the first kind singularities is given in [24].

Proposition 4.9 (R. Oset Sinha, K. Saji [24])**.** *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a frontal with a normal unit vector field ν. Let 0 be a singular point of the first kind. Then there exist a coordinate system (u, v) on $(\mathbb{R}^2, 0)$ and an isometry germ $\Phi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ satisfying that*

$$\Phi \circ f(u, v) = \left(u, a(u) + \frac{v^2}{2}, b_0(u) + b_1(u)v^2 + b_2(u)v^3 + b_3(u, v)v^4\right),$$

where a, b_0, b_1, b_2, b_3 be smooth functions satisfying that $a(0) = a'(0) = b_0(0) = b_0'(0) = b_1(0) = 0$.

By using Proposition 4.9, we have the following.

Proposition 4.10. *Let $\boldsymbol{x} : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be given by $\boldsymbol{x}(u, v) = \Phi \circ f(u, v)$ in Proposition 4.9. Then there exist smooth mappings $(\nu_1^u, \nu_2^u) : (\mathbb{R}^2, 0) \to \Delta$ and $(\nu_1^v, \nu_2^v) : (\mathbb{R}^2, 0) \to \Delta$ such that $(\boldsymbol{x}, \nu_1^u, \nu_2^u)$ and $(\boldsymbol{x}, \nu_1^v, \nu_2^v) : (\mathbb{R}^2, 0) \to \mathbb{R}^3 \times \Delta$ are one-parameter families of framed curve germs with respect to u and v, respectively.*

Proof. By Theorem 4.1 (1), there exists a smooth mapping $(\nu_1^u, \nu_2^u) : (\mathbb{R}^2, 0) \to \Delta$ such that $(\boldsymbol{x}, \nu_1^u, \nu_2^u)$ is a one-parameter family of framed curve germs with respect to u.

We denote

$$\begin{aligned} f(u, v) &= a(u) + \frac{v^2}{2}, \\ g(u, v) &= b_0(u) + b_1(u)v^2 + b_2(u)v^3 + b_3(u, v)v^4. \end{aligned}$$

Then $f_v(u, v) = v$ and $g_v(u, v) = 2b_1(u)v + 3b_2(u)v^2 + b_{3v}(u, v)v^4 + 4b_3(u, v)v^3$. Hence, if we consider $k_1(u, v) = 2b_1(u) + 3b_2(u)v + b_{3v}(u, v)v^3 + 4b_3(u, v)v^2$ and $k_2(u, v) = -1$, then $(k_1(u, v), k_2(u, v)) \neq (0, 0)$ and $k_1(u, v)f_v(u, v) + k_2(u, v)g_v(u, v) = 0$. By Theorem 4.1 (2), there exists a smooth mapping $(\nu_1^v, \nu_2^v) : (\mathbb{R}^2, 0) \to \Delta$ such that $(\boldsymbol{x}, \nu_1^v, \nu_2^v)$ is a one-parameter family of framed curve germs with respect to v. □

We treat cuspidal edges and cuspidal cross caps as concrete examples of the first kind singularities in the following. A normal form of the cuspidal cross cap is given in [24]. They consider folding mappings. Here we give the following normal form similarly to cuspidal edges in [20].

Theorem 4.11. (1) **[L. Martins, K. Saji** [20]**]** *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a cuspidal edge germ. Then there exist a diffeomorphism germ $\phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ and isometry germ $\Phi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ satisfying that*

$$\Phi \circ f \circ \phi(u, v) = \left(u, \frac{a_{20}}{2}u^2 + \frac{a_{30}}{6}u^3 + \frac{1}{2}v^2, \frac{b_{20}}{2}u^2 + \frac{b_{12}}{2}uv^2 + \frac{b_{03}}{6}v^3\right) + h(u, v)$$

$(b_{03} \neq 0, b_{20} \geq 0)$, *where*

$$h(u,v) = (0, u^4 h_1(u), u^4 h_2(u) + u^2 v^2 h_3(u) + u v^3 h_4(u) + v^5 h_5(u,v)),$$

with $h_1(u), h_2(u), h_3(u), h_4(u), h_5(u,v)$ *are smooth functions.*

(2) *Let* $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ *be a cuspidal cross cap germ. Then there exist a diffeomorphism germ* $\phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ *and isometry germ* $\Phi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ *satisfying that*

$$\begin{aligned} \Phi \circ f \circ \phi(u,v) \quad = \quad & \Bigg(u, \frac{a_{20}}{2}u^2 + \frac{a_{30}}{6}u^3 + \frac{a_{40}}{24}u^4 + \frac{1}{2}v^2, \\ & \quad \frac{b_{20}}{2}u^2 + \frac{b_{30}}{6}u^3 + \frac{b_{40}}{24}u^4 + \frac{b_{12}}{2}uv^2 + \frac{b_{13}}{6}uv^3 + \frac{b_{04}}{24}v^4\Bigg) + h(u,v), \end{aligned}$$

$(b_{13} \neq 0, b_{20} \geq 0)$, *where*

$$h(u,v) = (0, u^5 h_1(u), u^5 h_2(u) + u^3 v^2 h_3(u) + u^2 v^3 h_4(u,v) + v^5 h_5(v)),$$

with $h_1(u), h_2(u), h_3(u), h_4(u,v), h_5(v)$ *are smooth functions.*

Proof. (2) Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a cuspidal cross cap germ and ν be a unit normal of f. By using the same method in [20], we may assume that a null vector field η is given by the form ∂_v on $S(f)$ and the singular curve $\delta(t)$ is given by the form $(t, 0)$. Moreover, we may assume that

$$f(u,v) = (u, a_1(u) + v^2/2, b_1(u) + v^2 b_2(u) + v^3 b_3(u,v)), \tag{5}$$

where a_1, b_1, b_2 and b_3 are smooth functions, $a_1(0) = a_1'(0) = b_1(0) = b_1'(0) = b_2(0) = 0$ and a_1' means the derivation of a_1 with respect to u for example. By a direct calculation, we obtain $\nu(u,v) = \mathcal{N}(u,v)\widetilde{\nu}(u,v)$ where

$$\begin{aligned} \widetilde{\nu}(u,v) \quad = \quad & (a_1'(u)(2b_2(u) + 3vb_3(u,v) + v^2 b_{3,v}(u,v)) - (b_1'(u) + v^2 b_2'(u) + v^3 b_{3,u}(u,v)), \\ & -(2b_2(u) + 3vb_3(u,v) + v^2 b_{3,v}(u,v)), 1) \end{aligned}$$

and $\mathcal{N}(u,v) = 1/|\widetilde{\nu}(u,v)|$. Then $\phi_f(t) = \det(f(t,0), \nu(t,0), \nu_v(t,0)) = 3\mathcal{N}(t,0)b_3(t,0)$. Since f is not a front and Theorem 4.8 (1), we have $\phi_f(0) = 3b_3(0,0) = 0$, that is, $b_3(0,0) = 0$. Moreover, under this condition, $\phi_f'(t) = 3\mathcal{N}(t,0)b_{3,u}(t,0)$. Since f is a cuspidal cross cap germ and Theorem 4.8 (4), we have $\phi_f'(0) = 3\mathcal{N}(0,0)b_{3,u}(0,0) \neq 0$, that is, $b_{3,u}(0,0) \neq 0$. Hence, we have $b_3(u,v) = ua_4(u,v) + b_4(v)$, where a_4 and b_4 are smooth functions, $a_4(0,0) \neq 0$ and $b_4(0) = 0$. Substituting this equation to (5), we have

$$f(u,v) = (u, a_1(u) + v^2/2, b_1(u) + v^2 b_2(u) + uv^3 a_4(u,v) + v^3 b_4(v)),$$

where $a_1(0) = a_1'(0) = b_1(0) = b_1'(0) = b_2(0) = b_4(0) = 0$ and $a_4(0,0) \neq 0$. By rotations $(u,v) \mapsto (-u,-v)$ and $(x,y,z) \mapsto (-x,y,z)$, we may assume $b_1''(0) \geq 0$. Summarizing up the above argument, we have the normal form of cuspidal cross cap. □

By Corollary 4.6, or by using Theorem 4.11, we have the following.

Proposition 4.12. *Let* $\boldsymbol{x} : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ *be given by* $\boldsymbol{x}(u,v) = \Phi \circ f \circ \phi(u,v)$ *in Theorem 4.11 (1) or (2). Then there exist smooth mappings* $(\nu_1^u, \nu_2^u) : (\mathbb{R}^2, 0) \to \Delta$ *and* $(\nu_1^v, \nu_2^v) : (\mathbb{R}^2, 0) \to \Delta$ *such that* $(\boldsymbol{x}, \nu_1^u, \nu_2^u)$ *and* $(\boldsymbol{x}, \nu_1^v, \nu_2^v) : (\mathbb{R}^2, 0) \to \mathbb{R}^3 \times \Delta$ *are one-parameter families of framed curve germs with respect to* u *and* v*, respectively.*

4.2. **Second kind singularities.**

Proposition 4.13 (K. Saji [29]). *For any functions g and h satisfying $g_{vvv}(0,0) > 0$, $g(0,0) = h(0,0) = 0$, $g_u(0,0) - g_{vv}(0,0) = 0$, $h_u(0,0) - h_{vv}(0,0) = 0$ and $h_{vvv}(0,0) = 0$,*

$$f(u,v) = \left(u, \left(\frac{v^2}{2} - u\right) g_{vv}(u,v) - v g_v(u,v) + g(u,v),\right.$$
$$\left.\left(\frac{v^2}{2} - u\right) h_{vv}(u,v) - v h_v(u,v) + h(u,v)\right)$$

is a frontal satisfying that 0 is a singular point of the second kind, and $f_u(0,0) = (1,0,0)$, a null vector field $\eta = \partial_v$, the singular set $S(f) = \{v^2/2 - u = 0\}$. Moreover, if $h_{vvvv}(0,0) \neq 0$, then 0 is a swallowtail. Conversely, for any singular point of second kind p of a frontal $f : U \to \mathbb{R}^3$, there exists a coordinate system (u,v) on U, and an orientation preserving isometry Φ on $\mathbb{R}^3$ such that $\Phi \circ f(u,v)$ can be written in the above form.

By using Proposition 4.13, we have the following.

Proposition 4.14. *Let $\boldsymbol{x} : U \to \mathbb{R}^3$ be given by $\boldsymbol{x}(u,v) = \Phi \circ f(u,v)$ in Proposition 4.13. Then there exist smooth mappings $(\nu_1^u, \nu_2^u) : U \to \Delta$ and $(\nu_1^v, \nu_2^v) : U \to \Delta$ such that $(\boldsymbol{x}, \nu_1^u, \nu_2^u)$ and $(\boldsymbol{x}, \nu_1^v, \nu_2^v) : U \to \mathbb{R}^3 \times \Delta$ are one-parameter families of framed curve germs with respect to u and v around p, respectively.*

Proof. By Theorem 4.1 (1), there exists a smooth mapping $(\nu_1^u, \nu_2^u) : U \to \Delta$ such that $(\boldsymbol{x}, \nu_1^u, \nu_2^u)$ is a one-parameter family of framed curve germs with respect to u.

By a direct calculation, we have

$$\boldsymbol{x}_v(u,v) = \left(0, \left(\frac{v^2}{2} - u\right) g_{vvv}(u,v), \left(\frac{v^2}{2} - u\right) h_{vvv}(u,v)\right).$$

Since $g_{vvv}(0,0) > 0$, we have $g_{vvv}(u,v) \neq 0$ around $p \in U$. Hence, if we consider $(k_1(u,v), k_2(u,v)) = (-h_{vvv}(u,v), g_{vvv}(u,v))$, then $(k_1(u,v), k_2(u,v)) \neq (0,0)$ and

$$k_1(u,v) f_v(u,v) + k_2(u,v) g_v(u,v) = 0.$$

By Theorem 4.1 (2), there exists a smooth mapping $(\nu_1^v, \nu_2^v) : U \to \Delta$ such that $(\boldsymbol{x}, \nu_1^v, \nu_2^v)$ is a one-parameter family of framed curve germs with respect to v around p. □

4.3. **Cross caps.** The cross cap map germ is not a frontal. However, the generic singularities from 2-dimensional manifolds to 3-dimensional one are cross caps. In [6, 34, 11], they investigate cross caps from the view point of differential geometry.

Proposition 4.15 (J. M. West [34], T. Fukui, M. Hasegawa [6]). *Let $g : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a smooth map with a cross cap at $(0,0)$. Then there are a rotation $T : \mathbb{R}^3 \to \mathbb{R}^3$ and a diffeomorphism $\phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ so that*

$$T \circ g \circ \phi(u,v) = \left(u, uv + B(v) + O(u,v)^{k+1}, \sum_{j=2}^{k} A_j(u,v) + O(u,v)^{k+1}\right) \ (k \geq 3),$$

where

$$B(v) = \sum_{i=3}^{k} \frac{b_i}{i!} v^i \quad \textit{and} \quad A_j(u,v) = \sum_{i=0}^{j} \frac{a_{i,j-i}}{i!(j-i)!} u^i v^{j-i} \ \textit{with } a_{02} \neq 0.$$

By Theorem 4.1 (1), we have the following.

Proposition 4.16. *Let $\boldsymbol{x} : U \to \mathbb{R}^3$ be given by $\boldsymbol{x}(u,v) = T \circ g \circ \phi(u,v)$ in Proposition 4.15. Then there exists a smooth mapping $(\nu_1^u, \nu_2^u) : U \to \Delta$ such that $(\boldsymbol{x}, \nu_1^u, \nu_2^u) : U \to \mathbb{R}^3 \times \Delta$ is a one-parameter family of framed curve germs with respect to u.*

Moreover, the $\mathcal{A}$-simple singularities of a map from a 2-dimensional manifold to a 3-dimensional one are also of corank one, see [22]. These are also one-parameter families of framed base curves.

4.4. **Ruled surfaces.** We consider ruled surfaces as follows. Let $\gamma : I \to \mathbb{R}^3$ be a smooth curve and $(\delta, \nu) : I \to \Delta$ a spherical Legendre curve with the curvature (m, n), see §2.2 (cf. [31]). We define a *ruled surface* $\boldsymbol{x} : \mathbb{R} \times I \to \mathbb{R}^3$ by $\boldsymbol{x}(u,v) = \gamma(v) + u\delta(v)$. We denote $\boldsymbol{\mu}(v) = \delta(v) \times \nu(v)$.

Since ruled surfaces are constructed by a one-parameter family of straight lines, these are one-parameter families of framed curves.

Proposition 4.17. *Under the above notations, there exists a smooth mapping*

$$(\nu_1^u, \nu_2^u) : \mathbb{R} \times I \to \Delta$$

such that $(\boldsymbol{x}, \nu_1^u, \nu_2^u)$ is a one-parameter family of framed curves with respect to u with the curvature

$$\begin{aligned}(\ell^u(u,v), m^u(u,v), n^u(u,v), \alpha^u(u,v), L^u(u,v), M^u(u,v), N^u(u,v), P^u(u,v), Q^u(u,v), R^u(u,v))\\ = (0,0,0,1,n(v),0,-m(v),\dot\gamma(v)\cdot\nu(v),\dot\gamma(v)\cdot\boldsymbol{\mu}(v)+um(v),\dot\gamma(v)\cdot\delta(v)).\end{aligned}$$

Proof. Since $\boldsymbol{x}_u(u,v) = \delta(v)$, if we take $\nu_1^u(u,v) = \nu(v)$, $\nu_2^u(u,v) = \boldsymbol{\mu}(v)$, then

$$(\boldsymbol{x}, \nu_1^u, \nu_2^u) : \mathbb{R} \times I \to \mathbb{R}^3 \times \Delta$$

is a one-parameter family of framed curves with respect to u. By a direct calculation, we have the curvature. □

References

[1] V. I. Arnol'd, *Singularities of Caustics and Wave Fronts.* Mathematics and Its Applications **62** Kluwer Academic Publishers, 1990.

[2] V. I. Arnol'd, S. M. Gusein-Zade, A. N. Varchenko, *Singularities of Differentiable Maps vol.* I. Birkhäuser, 1986.

[3] J. W. Bruce, P. J. Giblin, *Curves and Singularities. A geometrical introduction to singularity theory. Second edition.* Cambridge University Press, Cambridge, 1992. DOI: 10.1017/cbo9781139172615

[4] S. Fujimori, K. Saji, M. Umehara, K. Yamada, Singularities of maximal surfaces. *Math. Z.* **259** (2008), 827–848. DOI: 10.1007/s00209-007-0250-0

[5] T. Fukui, Local differential geometry of cuspidal edge and swallowtail. *To appear in Osaka Math. J.*.

[6] T. Fukui, M. Hasegawa, Fronts of Whitney umbrella-a differential geometric approach via blowing up. *J. Singul.* **4** (2012), 35–67. DOI: 10.5427/jsing.2012.4c

[7] T. Fukunaga, M. Takahashi, Evolutes of fronts in the Euclidean plane. *J. Singul.* **10** (2014), 92–107.

[8] T. Fukunaga, M. Takahashi, Evolutes and involutes of frontals in the Euclidean plane. *Demonstr. Math.* **48** (2015), 147–166. DOI: 10.1515/dema-2015-0015

[9] T. Fukunaga, M. Takahashi, Existence conditions of framed curves for smooth curves. J. Geom. **108** (2017), 763–774 DOI: 10.1007/s00022-017-0371-5

[10] T. Fukunaga, M. Takahashi, Framed surfaces in the Euclidean space. *Bull. Braz. Math. Soc. (N.S.)* **50**(2019), 37-65. DOI: 10.1007/s00574-018-0090-z

[11] M. Hasegawa, A. Honda, K. Naokawa, M. Umehara, K. Yamada, Intrinsic invariants of cross caps. *Selecta Math. (N.S.)* **20** (2014), 769–785.

[12] S. Honda, M. Takahashi, Framed curves in the Euclidean space. *Adv. Geom.* **16** (2016), 265–276.

[13] G. Ishikawa, *Singularities of Curves and Surfaces in Various Geometric Problems.* CAS Lecture Notes 10, Exact Sciences, 2015.

[14] S. Izumiya, M. C. Romero-Fuster, M. A. S. Ruas, F. Tari, *Differential Geometry from a Singularity Theory Viewpoint.* World Scientific Pub. Co Inc. 2015.DOI: 10.1142/9108

[15] S. Izumiya, K. Saji, The mandala of Legendrean dualities for pseudo-spheres in Lorentz-Minkowski space and "flat" spacelike surfaces. *J. Singl.* **2** (2010), 92–127.DOI: 10.5427/jsing.2010.2g

[16] Y. Kabata, M. Takahashi, One-parameter families of Legendre curves and plane line congruence. *Preprint.* (2019).

[17] M. Kokubu, W. Rossman, K. Saji, M. Umehara, K. Yamada, Singularities of flat fronts in hyperbolic space. *Pacific J. Math.* **221** (2005), 303–351.DOI: 10.2140/pjm.2005.221.303

[18] W. L. Marar, J. J. Nuño-Ballesteros, Slicing corank 1 map germs from $\mathbb{C}^2$ to $\mathbb{C}^3$. *Q. J. Math.* **65** (2014), 1375–1395. DOI: 10.1093/qmath/hat061

[19] L. Martins, J. J. Nuño-Ballesteros, Contact properties of surfaces in $\mathbb{R}^3$ with corank 1 singularities. *Tohoku Math. J.* **67** (2015), 105–124.DOI: 10.2748/tmj/1429549581

[20] L. Martins, K. Saji, Geometric invariants of cuspidal edges. *Canad. J. Math.* **68** (2016), 445–462. DOI: 10.4153/cjm-2015-011-5

[21] L. Martins, K. Saji, M. Umehara, K. Yamada, Behavior of Gaussian curvature and mean curvature near non-degenerate singular points on wave fronts. *Geometry and topology of manifolds*, Springer Proc. Math. Stat, **154** (2016), 247–281. DOI: 10.1007/978-4-431-56021-0_14

[22] D. Mond, On the classification of germs of maps from $\mathbb{R}^2$ to $\mathbb{R}^3$. *Proc. London Math. Soc.* **50** (1985), 333–369.

[23] J. J. Nuño-Ballesteros, Unfolding plane curves with cusps and nodes. *Proc. Roy. Soc. Edinburgh Sect. A* **145** (2015), 161–174.

[24] R. Oset Sinha, K. Saji, On the geometry of the folded cuspidal edges. *Rev. Mat. Complut.* **31**(2018), 627-650. DOI: 10.1007/s13163-018-0257-6

[25] R. Oset Sinha, F. Tari, Projections of surfaces in $\mathbb{R}^4$ to $\mathbb{R}^3$ and the geometry of their singular images. *Rev. Mat. Iberoam.* **31** (2015), 33–50.

[26] R. Oset Sinha, F. Tari, Flat geometry of cuspidal edges. *Osaka J. Math.***55** (2018), 393-421.

[27] D. Pei, M. Takahashi, H. Yu, Envelopes of one-parameter families of framed curves. *J. Geom.***110**. (2019), Art. 48, 31 pp.

[28] K. Saji, Criteria for D_4 singularities of wave fronts. *Tohoku Math. J.* **63** (2011), 137–147.DOI: 10.2748/tmj/1303219939

[29] K. Saji, Normal form of the swallowtail and its applications. *Internat. J. Math.* **29** (2018), 1850046, 17 pp.DOI: 10.1142/s0129167x18500465

[30] K. Saji, M. Umehara, K. Yamada, The geometry of fronts. *Ann. of Math. (2)* **169** (2009), 491–529.DOI: 10.4007/annals.2009.169.491

[31] M. Takahashi, Legendre curves in the unit spherical bundle over the unit sphere and evolutes. *Contemp. Math.* **675** (2016), 337–355.DOI: 10.1090/conm/675/13600

[32] M. Takahashi, Envelopes of Legendre curves in the unit tangent bundle over the Euclidean plane. *Result in Math.* **71** (2017), 1473–1489. DOI:10.1007/s00025-016-0619-7.

[33] K. Teramoto, Parallel and dual surfaces of cuspidal edges. *Differential Geom. Appl.* **44** (2016), 52–62.

[34] J. M. West, The differential geometry of the cross cap. Ph.D. thesis, The University of Liverpool, 1995.

Tomonori Fukunaga, Fukuoka Institute of Technology, Fukuoka 811-0295, Japan
Email address: `fukunaga@fit.ac.jp`

Masatomo Takahashi, Muroran Institute of Technology, Muroran 050-8585, Japan
Email address: `masatomo@mmm.muroran-it.ac.jp`

Journal *of* Singularities
Volume 21 (2020), 50-83

Proc. of Geometric and Algebraic Singularity Theory, Będlewo, 2017
DOI: 10.5427/jsing.2020.21d

JET BUNDLES ON GORENSTEIN CURVES AND APPLICATIONS

LETTERIO GATTO AND ANDREA T. RICOLFI

Dedicated to Professor Goo Ishikawa, on the occasion of his 60th birthday

Abstract. In the last twenty years a number of papers appeared aiming to construct locally free replacements of the sheaf of principal parts for families of Gorenstein curves. The main goal of this survey is to present to the widest possible mathematical audience a catalogue of such constructions, discussing the related literature and reporting on a few applications to classical problems in Enumerative Algebraic Geometry.

0. Introduction

The purpose of this expository paper is to present a catalogue of locally free replacements of the sheaves of *principal parts* for (families of) Gorenstein curves. In the smooth category, locally free sheaves of principal parts are better known as *jet bundles*, understood as those locally free sheaves whose transition functions reflect the transformation rules of the partial derivatives of a local section under a change of local coordinates (more details in Section 1.4). Being a natural globalisation of the fundamental notion of Taylor expansion of a function in a neighborhood of a point, jet bundles are ubiquitous in Mathematics. They proved powerful tools for the study of deformation theories within a wide variety of mathematical situations and have a number of purely algebraic incarnations: besides the aforementioned *principal parts* of a quasi-coherent sheaf [28] we should mention, for instance, the theory of *arc spaces* on algebraic varieties [10, 40], introduced by Nash in [44] to deal with resolutions of singular loci of singular varieties.

The issue we want to cope with in this survey is that sheaves of principal parts of vector bundles defined on a singular variety X are not locally free. Roughly speaking, the reason is that the analytic construction carried out in the smooth category, based on gluing local expressions of sections together with their partial derivatives, up to a given order, is no longer available. Indeed, around singular points there are no local parameters with respect to which one can take derivatives. This is yet another way of saying that the sheaf Ω^1_X of sections of the cotangent bundle is not locally free at the singular points.

If C is a projective reduced singular curve, it is desirable, in many interesting situations, to dispose of a notion of global derivative of a regular section. If the singularities of C are mild, that is, if they are *Gorenstein*, locally free substitutes of the classical principal parts can be constructed by exploiting a natural derivation $\mathscr{O}_C \to \omega_C$, taking values in the dualising sheaf, which by the Gorenstein condition is an *invertible* sheaf. This allows one to mimic the usual procedure adopted in the smooth category. Related constructions have recently been reconsidered by A. Patel and A. Swaminathan in [46], under the name of *sheaves of invincible parts*, motivated by the classical problem of counting hyperflexes in one-parameter families of plane curves. Besides *loc. cit.*, locally free jets on Gorenstein curves have been investigated by a number of authors,

The first author was partially supported by INDAM-GNSAGA and by PRIN “Geometria sulle varietà algebriche”. The visit of the first author to Stavanger and of the second author to Torino was supported by the grant “Finanziamento Diffuso della Ricerca” no. 53_RBA17GATLET by Politecnico di Torino and “Progetto di Eccellenza del Dipartimento di Scienze Matematiche”, 2018–2022 no. E11G18000350001.

starting about twenty years ago [35, 36, 34, 18, 25]. The reader can consult [19, 26, 20], and the references therein, for several applications.

0.1. **The role of jet bundles in Algebraic Geometry.** The importance of jet extensions of line bundles in algebraic geometry emerges from their ability to provide the proper flexible framework where to formulate and solve elementary but classical enumerative questions, such as:

(i) How many flexes does a plane curve possess?
(ii) How many members in a generic pencil of plane curves have a hyperflex?
(iii) How many fibres in a one-parameter family of curves of genus 3 are hyperelliptic?
(iv) What is the class, in the rational Picard group of $\overline{M}_g$, the moduli space of stable curves of genus g, of the closure of the locus of smooth curves possessing a special Weierstrass point?

We will touch upon each of these problems in this survey report.

0.2. **Wronskian sections over Gorenstein curves.** A theory of *ramification points of linear systems on Gorenstein curves* was proposed in 1984 by C. Widland in his Ph.D. thesis, also exposed in a number of joint papers with Robert F. Lax [53, 52]. The dualising sheaf ω_C on an integral curve C, first defined by Rosenlicht [49] via residues on the normalisation $\widetilde{C}$, can be realised as the sheaf of regular differentials on C, as explained by Serre in [50, Ch. 4 § 3]. There is a natural map $\Omega^1_C \to \omega_C$ allowing one to define a derivation $\mathrm{d}\colon \mathscr{O}_C \to \omega_C$, by composition with the universal derivation $\mathscr{O}_C \to \Omega^1_C$. Differentiating local regular functions by means of this composed differential allowed Widland [51] and Lax to define a global *Wronskian section* associated to a linear system on a Gorenstein curve C, coinciding with the classical one for smooth curves.

As a quick illustration of how such construction works, consider a plane curve $\iota\colon C \hookrightarrow \mathbb{P}^2$ of degree d, carrying the degree d line bundle $\mathscr{O}_C(1) = \iota^*\mathscr{O}_{\mathbb{P}^2}(1)$. The Wronskian by Widland and Lax vanishes along all the flexes of C, but also at singular points. The total order of vanishing equals the number of flexes on a smooth curve of the same degree. For example, if C is an irreducible nodal plane cubic, the Wronskian associated to the bundle $\mathscr{O}_C(1)$ would vanish at three smooth flexes, but also at the node with multiplicity 6. If C were cuspidal, the Wronskian would vanish at the unique smooth flex, and at the cusp with multiplicity 8. In all cases the "total number" (which is 9) of inflection points is conserved.

In sum, the Wronskian defined by Widland and Lax is able to recover the classical Plücker formula counting smooth flexes on singular curves, but within a framework that is particularly suited to deal with degeneration problems, provided one learns how to extend it to families. For families of smooth curves, as pointed out by Laksov [33], the Wronskian section of a relative line bundle should be thought of as the determinant of a map from the pullback of the Hodge bundle to a jet bundle. The theory by Widland and Lax, however, was lacking a suitable notion of jet bundles for Gorenstein curves, as Ragni Piene [47] remarked in her AMS review of [53]:

> "This (Widland and Lax) Wronskian is a section of the line bundle
>
> $$L^{\otimes s} \otimes \omega_C^{\otimes (s-1)s/2},$$
>
> where $s := \dim H^0(X, L)$. They define the section locally and show that it patches. (In the classical case in which X is smooth, one easily defines the Wronskian globally, by using the $(s-1)$st sheaf of principal parts on X of L. To do this in the present case, one would need a generalisation of these sheaves, where ω plays the role of Ω^1_X. Such a generalisation is known only for $s = 2$.)"

These generalisations are nowadays available in the aforementioned references. In the last two sections we will present a few applications and open questions arising from the use of such an extended notion of jet bundles for one-parameter families of stable curves.

0.3. **Overview of contents.** In the first section we describe the construction of principal parts, jet bundles (with a glimpse on an abstract construction by Laksov and Thorup) and *invincible parts* by Patel and Swaminathan. In Section 2 we describe two applications of locally free replacements: the enumeration of hyperflexes in families of plane curves via automatic degeneracies [46], and the determination of the class of the stable hyperelliptic locus in genus 3 [19]. In Section 3 we define ramification points of linear systems on smooth curves; we introduce the classical Wronskian section attached to a linear system and state the associated Brill–Segre formula. In Section 4 we describe a generalisation to Gorenstein curves, due to Lax and Widland. In Section 5 we review the main ingredients needed in the computation of the class in $\mathrm{Pic}(\overline{M}_g)\otimes\mathbb{Q}$ of the locus of curves possessing a special Weierstrass point as in [26]. In Section 6 we propose a few examples and some natural but still open questions.

Conventions. All schemes are noetherian and defined over $\mathbb{C}$. Any scheme X comes equipped with a sheaf of $\mathbb{C}$-algebras $\mathscr{O}_X$. If $U \subset X$ is an open subset in the Zariski (resp. analytic) topology, then $\mathscr{O}_X(U)$ is the ring of regular (resp. holomorphic) functions on U. A *curve* is a reduced, purely 1-dimensional scheme of finite type over $\mathbb{C}$. We denote by K_C the canonical line bundle of a smooth curve C. In the presence of singularities, we will write ω_C for the dualising sheaf. We denote by Ω^1_π the sheaf of relative Kähler differentials on a (flat) family of curves $\pi\colon X \to S$.

Acknowledgment. Both authors are grateful to the anonymous referee for carefully reading the paper and for providing valuable comments, that definitely improved the shape of the paper in terms of clarity and readability. The first author is also indebted to Professor Stanisław (Staszek) Janeczko for encouraging support. The second author wishes to thank Max-Planck Institut für Mathematik for support.

This paper is dedicated to Professor Goo Ishikawa, on the occasion of the celebration (Goo '60) of his sixtieth birthday, wishing him many more years of new beautiful theorems.

1. Principal parts, jets and invincible parts

This first section is devoted to recall the definition and properties of the sheaves of principal parts and to introduce a couple of related constructions: jets of vector bundles, especially those of rank 1, and the Patel–Swaminathan invincible parts. We start by giving the general idea of jets, which blends their analytic construction with the algebraic presence of the dualising sheaf.

These constructions lead to the technique of locally free replacements of principal parts for families of curves with at worst Gorenstein singularities. They are intended to deal with degenerations of ramification points of linear systems in one parameter families of curves of fixed arithmetic genus. In fact, in Section 2 we shall give two applications to see the theory in action: the count of hyperflexes in a pencil, as performed in [46], and the determination of the class of the stable hyperelliptic locus in genus 3, as worked out by Esteves [19].

1.1. **The idea of jets.** Our guiding idea is the following ansatz, which we shall implement below only in the case of algebraic curves. Let X be a (not necessarily smooth) complex algebraic variety of dimension r. If X is not smooth, the sheaf of differentials Ω^1_X is not locally free. Even in this case it is possible to construct, in a purely algebraic fashion, the sheaf of principal parts (see Section 1.3) attached to any quasi-coherent sheaf $\mathscr{M}$. If X is singular, this sheaf is not locally free (even if $\mathscr{M}$ is locally free), and this makes harder its use even to solve elementary enumerative problems. But suppose one has an $\mathscr{O}_X$-module homomorphism $\phi\colon \Omega^1_X \to \mathscr{M}$, where $\mathscr{M}$ is a locally free sheaf of rank $r = \dim X$. This induces a derivation $\mathrm{d}\colon \mathscr{O}_X \to \mathscr{M}$ obtained by composing ϕ with the universal derivation $\mathscr{O}_X \to \Omega^1_X$ attached to X. Let $P \in X$ be a point and U an open neighborhood of P trivialising $\mathscr{M}$, that is,

$$\mathscr{M}(U) = \mathscr{O}(U) \cdot m_1 \oplus \cdots \oplus \mathscr{O}(U) \cdot m_r.$$

Such a trivialisation allows one to define partial derivatives with respect to the generators $m_1, \ldots, m_r \in \mathscr{M}(U)$. In the smooth case, and taking $\mathscr{M} = \Omega^1_X$, these generators can just be taken to be the differentials of a local system of parameters around P. Following an idea essentially due to Lax and Widland, one defines for each $f \in \mathscr{O}(U)$ its "partial derivatives" $d_i f \in \mathscr{O}(U)$ by means of the relation

$$\mathrm{d}\, f = \sum_{i=1}^{r} d_i f \cdot m_i$$

in $\mathscr{M}(U)$. Iterating this process in the obvious way, one can define higher order partial derivatives (with respect to $m_1, \ldots, m_r$), and thus jet bundles, precisely as in the smooth category.

1.2. **Dualising sheaves.** This technical section can be skipped at a first reading. It will be applied below in special cases only, but it is important because it puts the subject in the perspective of new applications.

Any proper flat family of curves $\pi\colon X \to S$ has a dualising complex $\omega^{\cdot}_\pi := \pi^! \mathscr{O}_S$. Here $\pi^!$ is the right adjoint to $R\pi_*$. The cohomology sheaf of the dualising complex

$$\omega_\pi = h^{-1}(\omega^{\cdot}_\pi),$$

in degree -1 (where 1 is the relative dimension of π) is called the *relative dualising sheaf* of the family. Its formation commutes with arbitrary base change; for instance, we have

$$\omega_\pi\big|_{X_s} = \omega_{X_s}$$

for $X_s = \pi^{-1}(s)$ a fibre of π.

Example 1.1. Let $\pi\colon X \to S$ be a local complete intersection morphism. This means that there is a factorisation $\pi\colon X \to Y \to S$ with $i\colon X \to Y$ a regular immersion and $Y \to S$ a smooth morphism. Then one can compute the dualising sheaf of π as

$$\omega_\pi = \det(\mathscr{I}/\mathscr{I}^2)^\vee \otimes_{\mathscr{O}_X} i^* \det \Omega^1_{Y/S}, \tag{1.1}$$

where $\mathscr{I} \subset \mathscr{O}_Y$ is the ideal sheaf of X in Y. Every curve in a smooth surface is a local complete intersection scheme. For instance, if $i\colon C \hookrightarrow \mathbb{P}^2$ is a plane curve of degree d, the ideal sheaf of i is $\mathscr{O}_{\mathbb{P}^2}(-d)$ and so (1.1) yields

$$\omega_C = \mathscr{O}_C(d) \otimes_{\mathscr{O}_C} i^* \det \Omega^1_{\mathbb{P}^2} = \mathscr{O}_C(d-3).$$

Definition 1.2. A (proper) $\mathbb{C}$-scheme X is said to be *Cohen–Macaulay* if its dualising complex $\omega^{\cdot}_X$ is quasi-isomorphic to a sheaf. When this sheaf, necessarily isomorphic to ω_X, is invertible, X is called *Gorenstein*.

For a proper flat morphism $\pi\colon X \to S$, the relative dualising sheaf ω_π is invertible precisely when π has Gorenstein fibres.

1.3. **Principal parts.** Sheaves of principal parts were introduced in [28, Ch. 16.3]. Let $\pi\colon X \to S$ be a morphism of schemes, $\mathscr{I}$ the ideal sheaf of the diagonal $\Delta\colon X \to X \times_S X$ and denote by $\Omega^1_\pi = \Delta^*(\mathscr{I}/\mathscr{I}^2)$ the sheaf of relative Kähler differentials. Let p and q denote the projections $X \times_S X \to X$, and denote by $\Delta_k \subset X \times_S X$ the closed subscheme defined by $\mathscr{I}^{k+1}$, for every $k \geq 0$. Then, for every quasi-coherent $\mathscr{O}_X$-module E, the sheaf

$$P^k_\pi(E) \coloneqq p_*\left(q^*E \otimes \mathscr{O}_{\Delta_k}\right)$$

is quasi-coherent and is called the k-th *sheaf of principal parts* associated to the pair (π, E). When $S = \operatorname{Spec} \mathbb{C}$ we simply write $P^k(E)$ instead of $P^k_\pi(E)$.

Proposition 1.3. *Let $\pi\colon X \to S$ be a smooth morphism, E a quasi-coherent $\mathscr{O}_X$-module. The sheaves of principal parts fit into right exact sequences*

$$E \otimes \operatorname{Sym}^k \Omega^1_\pi \to P^k_\pi(E) \to P^{k-1}_\pi(E) \to 0$$

for every $k \geq 1$. If E is locally free then the sequence is exact on the left, and $P^k_\pi(E)$ is locally free for all $k \geq 0$.

Proof. Consider the short exact sequence

$$0 \to \mathscr{I}^k/\mathscr{I}^{k+1} \to \mathscr{O}_{\Delta_k} \to \mathscr{O}_{\Delta_{k-1}} \to 0.$$

Tensoring it with q^*E gives an exact sequence

$$q^*E \otimes \mathscr{I}^k/\mathscr{I}^{k+1} \xrightarrow{\epsilon} q^*E \otimes \mathscr{O}_{\Delta_k} \to q^*E \otimes \mathscr{O}_{\Delta_{k-1}} \to 0. \tag{1.2}$$

The sheaf $q^*E \otimes \mathscr{I}^k/\mathscr{I}^{k+1}$ is supported on the diagonal $\Delta_0 \subset X \times_S X$, and the same is true for its quotient $\mathscr{Q} \coloneqq (q^*E \otimes \mathscr{I}^k/\mathscr{I}^{k+1})/\ker \epsilon \subset q^*E \otimes \mathscr{O}_{\Delta_k}$. Since $p|_{\Delta_0}$ is an isomorphism, we have $R^ip_*\mathscr{F} = 0$ for all $i > 0$ and all sheaves $\mathscr{F}$ supported on Δ_0. Therefore, applying p_* to (1.2) we obtain

$$p_*\left(q^*E \otimes \mathscr{I}^k/\mathscr{I}^{k+1}\right) \to P^k_\pi(E) \to P^{k-1}_\pi(E) \to R^1p_*\mathscr{Q} = 0, \tag{1.3}$$

which is the required exact sequence, since

$$\begin{aligned} p_*\left(q^*E \otimes \mathscr{I}^k/\mathscr{I}^{k+1}\right) &= \Delta^*\left(q^*E \otimes \mathscr{I}^k/\mathscr{I}^{k+1}\right) \\ &= \Delta^*q^*E \otimes \Delta^*\left(\mathscr{I}^k/\mathscr{I}^{k+1}\right) \\ &= E \otimes \Delta^* \operatorname{Sym}^k\left(\mathscr{I}/\mathscr{I}^2\right) \\ &= E \otimes \operatorname{Sym}^k \Omega^1_\pi. \end{aligned}$$

We used smoothness of π to ensure that $\mathscr{I}$ is locally generated by a regular sequence. This allowed us to make the identification $\mathscr{I}^k/\mathscr{I}^{k+1} = \operatorname{Sym}^k(\mathscr{I}/\mathscr{I}^2)$ in the third equality above. If E is locally free, then (1.2) is exact on the left, and the same is true for (1.3), so that local freeness of $P^k_\pi(E)$ follows by induction exploiting the resulting short exact sequence and the base case provided by $P^0_\pi(E) = E$. $\square$

Example 1.4. Suppose $\pi\colon X \to S$ is smooth. Then there is a splitting $P^1_\pi(\mathscr{O}_X) = \mathscr{O}_X \oplus \Omega^1_\pi$. For an arbitrary vector bundle E, the splitting of the first order bundle of principal parts usually fails even when S is a point. In fact, in this case, the splitting is equivalent to the vanishing of the *Atiyah class* of E, which by definition is the extension class

$$A(E) \in \operatorname{Ext}^1_X(E, E \otimes \Omega^1_X)$$

attached to the short exact sequence of Proposition 1.3 taken with $k = 1$. But the vanishing of the Atiyah class is known to be equivalent to the existence of a holomorphic connection on E.

Note that for every quasi-coherent sheaf E on X one has a canonical map

$$\nu\colon \pi^*\pi_*E \to p_*q^*E \to P^k_\pi(E), \tag{1.4}$$

where the first one is an isomorphism when π is flat, and the second one comes from applying $p_*(q^*E \otimes -)$ to the surjection $\mathscr{O} \twoheadrightarrow \mathscr{O}_{\Delta_k}$.

Example 1.5. To illustrate the classical way of dealing with bundles of principal parts, we now compute the number δ of singular fibres in a general pencil of hypersurfaces of degree d in $\mathbb{P}^n$. This calculation will be used in Subsection 2.1.2. The number δ is nothing but the degree of the discriminant hypersurface in the space of degree d forms on $\mathbb{P}^n$, which in turn is the degree of

$$c_n(P^1(\mathscr{O}_{\mathbb{P}^n}(d))) \in A^n(\mathbb{P}^n).$$

By Proposition 1.3, the bundle $P^1(\mathscr{O}_{\mathbb{P}^n}(d))$ is an extension of $\mathscr{O}_{\mathbb{P}^n}(d)$ by $\Omega^1_{\mathbb{P}^n}(d)$. The Euler sequence

$$0 \to \Omega^1_{\mathbb{P}^n} \to \mathscr{O}_{\mathbb{P}^n}(-1)^{n+1} \to \mathscr{O}_{\mathbb{P}^n} \to 0$$

twisted by $\mathscr{O}_{\mathbb{P}^n}(d)$ says that the same is true for the bundle $\mathscr{O}_{\mathbb{P}^n}(d-1)^{n+1}$. Then the Whitney sum formula implies that

$$c(P^1(\mathscr{O}_{\mathbb{P}^n}(d))) = c(\mathscr{O}_{\mathbb{P}^n}(d-1)^{n+1}) = (1+(d-1)\zeta)^{n+1},$$

where $\zeta \in A^1(\mathbb{P}^n)$ is the hyperplane class. Computing the n-th Chern class gives

$$\delta = (n+1)\cdot(d-1)^n. \tag{1.5}$$

1.4. **Jet bundles.** Let $\pi\colon X \to S$ be a quasi-projective local complete intersection morphism of constant relative dimension $d \geq 0$. Let Ω^1_π be the sheaf of relative differentials, and Ω^d_π its d-th exterior power. Then there exists a canonical morphism $\Omega^d_\pi \to \omega_\pi$ restricting to the identity over the smooth locus of π (see Corollary 4.13 in [39, Section 6.4] for a proof). The construction goes as follows. Let $X \to Y \to S$ be a factorisation of π, with $i : X \to Y$ a regular immersion with ideal $\mathscr{I} \subset \mathscr{O}_Y$ and $Y \to S$ smooth. The exact sequence

$$\mathscr{I}/\mathscr{I}^2 \to i^*\Omega^1_{Y/S} \to \Omega^1_\pi \to 0$$

induces a canonical map

$$\mu_Y\colon \Omega^d_\pi \otimes \det \mathscr{I}/\mathscr{I}^2 \to i^* \det \Omega^1_{Y/S}.$$

According to (1.1), tensoring μ_Y with the dual of $\det \mathscr{I}/\mathscr{I}^2$ gives a morphism $\Omega^d_\pi \to \omega_\pi$. It is not difficult to see that this map does not depend on the choice of the factorisation.

A natural morphism of sheaves $\Omega^1_\pi \to \omega_\pi$, restricting to the identity on the smooth locus of π, exists for arbitrary flat families $\pi\colon (X, x_0) \to (S, 0)$ of germs of reduced curves [1, Prop. 4.2.1]. More generally, the results in [17, Sec. 4.4] show that a natural morphism

$$\phi\colon \Omega^d_\pi \to \omega_\pi, \tag{1.6}$$

can be constructed for every flat morphism $\pi\colon X \to S$ of relative dimension d over a reduced base S (and over a field of characteristic zero).

We now apply this construction to flat families $\pi\colon X \to S$ of *Gorenstein* curves (so for $d = 1$), taking advantage of the invertibility of ω_π in order to construct locally free jets. When dealing with such families, we will therefore assume to be working over a reduced base, which will be enough for all our applications. Composing ϕ with the exterior derivative homomorphism $\mathrm{d}\colon \mathscr{O}_X \to \Omega^1_\pi$ attached to the family gives an $\mathscr{O}_S$-linear derivation

$$\mathrm{d}_\pi\colon \mathscr{O}_X \to \omega_\pi. \tag{1.7}$$

For every integer $k \geq 0$ and line bundle L on X, there exists a vector bundle

$$J^k_\pi(L) \tag{1.8}$$

of rank $k+1$ on X, called the *k-th jet extension* of L relative to the family π. We refer to [26, Section 2] for its detailed construction in the case of stable curves. The same construction (as well as the proof of Proposition 1.7 below) extends to any family of Gorenstein curves as one only uses the map $\Omega^1_\pi \to \omega_\pi$ and the invertibility of the relative dualising sheaf. The bundle (1.8) depends on the derivation d_π (although we do not emphasise it in the notation), and formalises the idea of taking derivatives (with respect to d_π) of sections of L along the fibres of π. It can be thought of as a holomorphic, or algebraic, analogue of the $\mathcal{C}^\infty$ bundle of coefficients of the Taylor expansion of the smooth functions on a differentiable manifold. When $S = \operatorname{Spec}\mathbb{C}$ we simply write $J^k(L)$.

We now sketch the construction of the jet bundle (1.8). Suppose we have an open covering $\mathcal{U} = \{\, U_\alpha \,\}$ of X, trivialising ω_π and L at the same time, with generators $\epsilon_\alpha \in \omega_\pi(U_\alpha)$ and $\psi_\alpha \in L(U_\alpha)$ respectively over the ring of functions on U_α. Then for every non constant global section $\lambda \in H^0(X, L)$ we can write

$$\lambda|_{U_\alpha} = \rho_\alpha \cdot \psi_\alpha \in L(U_\alpha)$$

for certain functions $\rho_\alpha \in \mathscr{O}_X(U_\alpha)$. Define operators $D^i_\alpha \colon \mathscr{O}_{U_\alpha} \to \mathscr{O}_{U_\alpha}$ inductively for $i \geq 0$, by letting $D^0_\alpha(\rho_\alpha) = \rho_\alpha$ and by the relation

$$\mathrm{d}_\pi(D^{i-1}_\alpha(\rho_\alpha)) = D^i_\alpha(\rho_\alpha)\cdot \epsilon_\alpha.$$

It is then an easy technical step to show that over the intersection $U_{\alpha\beta} = U_\alpha \cap U_\beta$, the $(k+1)$-vectors $(D^i_\alpha(\rho_\alpha))^T$ and $(D^i_\beta(\rho_\beta))^T$ differ by a matrix $M_{\alpha\beta} \in \mathrm{GL}_{k+1}(\mathscr{O}_{U_{\alpha\beta}})$, and that in fact the data $\{M_{\alpha\beta}\}$ define a 1-cocycle with respect to $\mathcal{U}$. The verification of this fact uses that d_π is a derivation. The upshot is that the vectors $(D^i_\alpha(\rho_\alpha))$ glue to a global section

$$D^k\lambda \tag{1.9}$$

of a well defined vector bundle $J^k_\pi(L)$. Moreover, the bundle obtained comes with a natural $\mathbb{C}$-linear morphism

$$\delta\colon \mathscr{O}_X \to J^k_\pi(L) \tag{1.10}$$

such that if $J^k_\pi(L)|_{U_\alpha}$ is free with basis $\{\, \epsilon_{\alpha,i} : 0 \leq i \leq k \,\}$, then δ is defined on this open patch by $f \mapsto \sum_{i=0}^k D^i_\alpha(f)\cdot \epsilon_{\alpha,i}$.

Example 1.6. When S is a point, X is a smooth projective curve, L is the cotangent bundle Ω^1_X with the exterior derivative $\mathrm{d}\colon \mathscr{O}_X \to \Omega^1_X$, the $\mathbb{C}$-linear map (1.10) reduces to the "Taylor expansion" truncated at order k. More precisely, let $U \subset X$ be an open subset (trivialising $\omega_X = \Omega^1_X$) with local coordinate x. Then we can take $\epsilon = \mathrm{d}\,x \in \Omega^1_X(U)$ as an $\mathscr{O}_X(U)$-linear generator, and $\{\, \mathrm{d}\,x^i : 0 \leq i \leq k \,\}$ can be taken as a basis of $J^k(\Omega^1_X)|_U$. The restriction $\delta|_U$ of (1.10) then takes the form

$$f \mapsto \sum_{i=0}^k \frac{1}{i!}\frac{\partial^i f}{\partial x^i}\,\mathrm{d}\,x^i,$$

where the denominator $1/i!$ is there for cosmetic reasons. The cocycle condition that the above coefficients need to satisfy is equivalent to the *chain rule* for holomorphic functions.

Computations in intersection theory involving jet bundles often rely on the application of the following key result.

Proposition 1.7 ([26, Prop. 2.5]). *Let $\pi\colon X \to S$ be a flat family of Gorenstein curves. Then, for every $k \geq 1$ and line bundle L on X, there is an exact sequence of vector bundles*

$$0 \to L \otimes \omega_\pi^{\otimes k} \to J^k_\pi(L) \to J^{k-1}_\pi(L) \to 0. \tag{1.11}$$

LEMMA 1.8. *Let $\pi\colon X \to S$ be a flat family of Gorenstein curves with smooth locus $U \subset X$, let L be a line bundle on X, and fix an integer $k \geq 0$. Then*

$$J^k_\pi(L)\big|_U = P^k_\pi(L)\big|_U.$$

PROOF. The derivation $\mathrm{d}_\pi\colon \mathscr{O}_X \to \omega_\pi$ defined in (1.7) and used to define the k-jets restricts to the universal derivation $\mathrm{d}\colon \mathscr{O}_U \to \Omega^1_{U/S}$ over the smooth locus U. But jet bundles taken with respect to the universal derivation agree with principal parts in the smooth case, as one can verify directly from their construction; see also [35, Section 4.11] for a reference. $\square$

1.4.1. *The approach of Laksov and Thorup.* Laksov and Thorup [35] generalised the construction of (1.10) in the following sense. Given an S-scheme X and a quasi-coherent $\mathscr{O}_X$-module $\mathscr{M}$ admitting an $\mathscr{O}_S$-linear derivation $\mathrm{d}\colon \mathscr{O}_X \to \mathscr{M}$, they constructed for all $k \geq 0$ an $\mathscr{O}_S$-algebra

$$\mathcal{J}^k = \mathcal{J}^k_{\mathscr{M},\mathrm{d}}$$

over X, along with an algebra map $\delta\colon \mathscr{O}_X \to \mathcal{J}^k$ generalising the one constructed in (1.10). The sheaf $\mathcal{J}^k$ is called the k-th *algebra of jets*. It is quasi-coherent, and of finite type whenever $\mathscr{M}$ is. For every $\mathscr{O}_X$-module $\mathscr{L}$, one can consider the $\mathscr{O}_X$-module

$$\mathcal{J}^k(\mathscr{L}) = \mathcal{J}^k \otimes_{\mathscr{O}_X} \mathscr{L}$$

of *$\mathscr{L}$-twisted jets*. They fit into exact sequences

$$\mathscr{L} \otimes \mathscr{M}^{\otimes k} \to \mathcal{J}^k(\mathscr{L}) \to \mathcal{J}^{k-1}(\mathscr{L}) \to 0,$$

that are left exact whenever $\mathscr{M}$ is S-flat. The construction carried out in [35] works over fields of arbitrary characteristic and is completely intrinsic, in particular it avoids the technical step of verifying the cocycle condition.

1.4.2. *Arc spaces.* The study of arc spaces (also called jet schemes) was initiated by Nash [44] in the 60's in the context of Singularity Theory. Arcs on algebraic varieties received a lot of attention more recently since Kontsevich's lecture [32]. See for instance the papers by Denef–Loeser [10, 9] and Looijenga [40]. An arc of order n on a variety X based at point P is a morphism

$$\alpha\colon \operatorname{Spec} \mathbb{C}[t]/t^{n+1} \to X$$

sending the closed point to P. The reader may correctly think of it as the expression of a germ of complex curve considered together with its first n derivatives. For instance if $n = 1$, one obtains the classical notion of tangent space at a point. These maps form an algebraic variety $\mathcal{L}_n(X)$, and the inverse limit $\mathcal{L}(X) = \lim \mathcal{L}_n(X)$ is the full *arc space* of X, an infinite type scheme whose $\mathbb{C}$-points correspond to morphisms $\operatorname{Spec} \mathbb{C}[\![t]\!] \to X$. Kontsevich invented *Motivic Integration* in order to prove that smooth birational Calabi–Yau manifolds have the same Hodge numbers; he constructed a *motivic measure* on $\mathcal{L}(X)$, which can be thought of as the analogue of the p-adic measure used earlier by Batyrev to show that smooth birational Calabi–Yau manifolds have equal Betti numbers. Other remarkable notions introduced by Denef–Loeser are the *motivic Milnor fibre* and the *motivic vanishing cycle*; the latter is the motivic incarnation of the perverse sheaf of vanishing cycles attached to a regular (holomorphic) function $U \to \mathbb{C}$. This theory has a wide variety of applications in Singularity Theory, but it has also proven successful in Algebraic Geometry, for instance in the study of degenerations of abelian varieties via motivic zeta functions [29].

1.5. **Invincible parts.** An elegant approach to the problem of locally free replacements of principal parts has been proposed by Patel and Swaminathan in their recent report [46]. Their construction is formally more adherent to the purely algebraic definition of principal parts as described in Section 1.3. To perform the construction they restrict to certain families of curves according to the following:

Definition 1.9. Let $\pi\colon X \to S$ be a proper flat morphism of pure Gorenstein curves. Then π is called an *admissible family* if the locus $\Gamma \subset X$ over which π is not smooth has codimension at least 2.

Let $\pi\colon X \to S$ be an admissible family with X and S smooth, irreducible varieties, and assume $\dim S = 1$. Let E be a vector bundle on the total space X. Patel and Swaminathan define the k-th order *sheaf of invincible parts* associated to (π, E) as the double dual sheaf

$$P^k_\pi(E)^{\vee\vee}.$$

This intrinsic construction is related to the gluing procedure (giving rise to jets) described in Section 1.4, via the following observation.

Proposition 1.10. *Let $\pi\colon X \to S$ be an admissible family of Gorenstein curves, with X and S smooth irreducible varieties and $\dim S = 1$. Let L be a line bundle on X. Then the sheaf of invincible parts $P^k_\pi(L)^{\vee\vee}$ agrees with the jet bundle $J^k_\pi(L)$ of (1.8).*

Proof. The vector bundle $J^k_\pi(L)$ restricted to the smooth locus $U = X \setminus \Gamma$ of π agrees with $P^k_\pi(L)|_U$ by Lemma 1.8. But by [46, Prop. 10], $P^k_\pi(L)^{\vee\vee}$ is the unique locally free sheaf with this property. □

2. Two applications

2.1. **Counting flexes via automatic degeneracies.** In this section we report on one of the main applications of the *sheaves of invincible parts* that motivated the research by Patel and Swaminathan. In particular, we wish to describe the application of their theory of *automatic degeneracies* to the enumeration of hyperflexes in general pencils of plane curves. A *hyperflex* on a plane curve $C \subset \mathbb{P}^2$ is a point on the normalisation $P \in \widetilde{C}$ such that for some line $\ell \subset \mathbb{P}^2$ we have $\mathrm{ord}_P(\nu^*\ell) \geq 4$, where $\nu\colon \widetilde{C} \to C$ is the normalisation map. The general plane curve of degree $d > 1$ has no hyperflexes, but one expects to find a finite number of hyperflexes in a pencil. One has the following classical result.

Proposition 2.1. *In a general pencil of plane curves of degree d, exactly*

$$6(d-3)(3d-2)$$

will have hyperflexes.

Remark 2.2. Note that this number vanishes for $d = 3$. This should be expected, for in a general pencil of plane cubics all fibres are irreducible, but a cubic possessing a hyperflex is necessarily reducible.

A proof of Proposition 2.1 via principal parts can be found in [16]. A different approach, via relative Hilbert schemes, has been taken by Ran [48]. In [46], the authors apply their theory of automatic degeneracies to give a new proof of Proposition 2.1. More precisely, after a suitable Chern class calculation, which we review below in the language of jet bundles, the authors subtract the individual contribution of each node in the pencil to get the desired answer. Let us note that it is extremely useful to have an explicit function (see Subsection 2.1.1 below) computing the "correction term" one has to take into account while performing a Chern class/Porteous calculation over a family of curves containing singular members.

2.1.1. *Automatic degeneracies.* Given a (proper, non-smooth) morphism of Gorenstein curves $X \to S$, the associated sheaves of principal parts are not locally free, but the jets constructed out of the derivation (1.7) are locally free. To answer questions on the inflectionary behavior of the family $X \to S$, the classical strategy is to set up a suitable Porteous calculation and compute the degree of the appropriate Chern classes of the jet bundles. However, inflection points are by definition smooth points, and singularities in the fibres X_s tend to "attract" inflection points as limits; so one has to excise the contribution to this Porteous calculation coming from the singular points of the fibres. This problem was tackled in [46], where the authors propose a theoretical solution, working nicely at least under certain assumptions. More precisely, the authors are able to attach to any germ $f \in \mathbb{C}[\![x, y]\!]$ of a plane curve singularity a function

$$\mathrm{AD}(f)\colon \mathbb{N} \to \mathbb{N}, \qquad m \mapsto \mathrm{AD}^m(f),$$

whose value at $m \in \mathbb{N}$ they call the *m-th order automatic degeneracy* associated to f. As explained in [46, Remark 18], the function $\mathrm{AD}(f)$ is an *analytic invariant* of the germ f. We refer the reader to [46, Section 5] for an algorithmic approach to the computation of the values of this function.

Given a 1-parameter admissible family $X \to S$ of curves where the singularity $f = 0$ appears in a fibre, the number $\mathrm{AD}^m(f)$ is the correction term one has to take into account in the Porteous calculation aimed at enumerating m-th order inflection points on $X \to S$. The authors determine this function in the nodal case by proving [46, Theorem 24] the formula

$$\mathrm{AD}^m(xy) = \binom{m+1}{4}. \tag{2.1}$$

It remains an open problem to compute the function $\mathrm{AD}(f)$ for other singularities, although in *loc. cit.* a few computations for a specific m are carried out, for instance

$$\mathrm{AD}^4\left(y^2 - x^3\right) = 10$$

for the cusp singularity.

2.1.2. *The count of hyperflexes.* Let $X \subset \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^1$ be a generic pencil of plane curves of degree d. It can be realised explicitly as follows. Let us choose two general plane curves C_1 and C_2 of degree d, the generators of the pencil. Their intersection will consist of d^2 reduced points. Blowing up these points gives

$$\pi\colon X \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^1.$$

Consider the line bundle $L_d = b^*\mathscr{O}_{\mathbb{P}^2}(d)$, where $b\colon X \to \mathbb{P}^2$ is the blow up map. The number we are after is

$$\int_X c_2(J^3_\pi(L_d)) - \binom{5}{4} \cdot \delta,$$

where $\delta = 3(d-1)^2$ is the number of nodes computed in (1.5) and the binomial coefficient computes the automatic degeneracy of a node, using (2.1) with $m = 4$. This number is determined by the Chern classes

$$\eta = c_1(\omega_\pi), \qquad \zeta = c_1(L_d).$$

Using the exact sequences of Proposition 1.7 we get

$$c_2(J^3_\pi(L_d)) = 11\eta^2 + 18\eta\zeta + 6\zeta^2.$$

It is easy to see that $\zeta^2 \in A^2(X)$ has degree 1. Exploiting that $E^2 = -d^2$, one can check that η^2 has degree $3d^2 - 12d + 9$. Finally, $\eta\zeta$ has degree $2d - 3$. The difference

$$11(3d^2 - 12d + 9) + 18(2d - 3) + 6 - 5 \cdot 3(d - 1)^2 = 6(d - 3)(3d - 2)$$

computes the number of hyperflexes prescribed by Proposition 2.1.

2.2. **The stable hyperelliptic locus in genus 3, after Esteves.** In this section we will see the sheaves of principal parts and the technique of locally free replacements in action to solve a concrete problem. The results in this section hold over an algebraically closed field k of characteristic different from 2. Consider the moduli space M_3 of smooth, projective, connected curves of genus 3. A hyperelliptic curve of genus 3 is a $2:1$ branched covering of the projective line with 8 ramification points.

Let $H \subset M_3$ be the divisor parametrising hyperelliptic curves, and let $\overline{H}$ be its closure in the Deligne–Mumford moduli space $\overline{M}_3$ of stable curves. The vector space $\mathrm{Pic}(M_3) \otimes \mathbb{Q}$ is generated by the Hodge class λ (pulled back from $\overline{M}_3$), whereas $\mathrm{Pic}(\overline{M}_3) \otimes \mathbb{Q}$ is generated by λ, δ_0 and δ_1, with δ_i denoting the boundary classes on $\overline{M}_3$. A proof of the following theorem, expressing the classes $[H]$ and $[\overline{H}]$ in terms of the above generators, can be found in [30].

THEOREM 2.1. *One has*

$$[H] = 9\lambda \tag{2.2}$$

and

$$[\overline{H}] = 9\lambda - \delta_0 - 3\delta_1. \tag{2.3}$$

Formula (2.2) also follows from Mumford's relation [43, p. 314]. Below is a quick description of how Esteves [19, Thm. 1] proves formula (2.3).

2.2.1. *Smooth curves.* Let $\pi\colon \mathcal{C} \to S$ be a smooth family of genus 3 curves. We constructed in (1.4) a natural map of vector bundles $\nu\colon \pi^*\pi_*\Omega^1_\pi \to P^1_\pi(\Omega^1_\pi)$ on $\mathcal{C}$, where the source has rank 3 and the target has rank 2. Assuming the general fibre is not hyperelliptic, it turns out that the top degeneracy scheme D of ν (supported on points P such that $\nu|_P$ is not onto) has the expected codimension, namely 2. Then Porteous formula applies and gives $[D] = c_2(P^1_\pi(\Omega^1_\pi) - \pi^*\pi_*\Omega^1_\pi) \cap [\mathcal{C}]$. Pushing this identity down to S, and observing that there are 8 Weierstrass points on a hyperelliptic curve of genus 3, one gets, after a few calculations, the relation $8h_\pi = 72\lambda_\pi$, proving the formula for $[H]$.

2.2.2. *Stable curves.* Let now $\mathfrak{X} \to S$ be a family of stable curves of genus 3, which for simplicity we assume general from the start. This means S is smooth and 1-dimensional, the general fibre of π is smooth and the finitely many singular fibres have only one singularity. One can see that only two types of singularities can appear in the fibres: a uninodal irreducible curve $Z \subset \mathfrak{X}$, or a reducible curve $X \cup_N Y \subset \mathfrak{X}$ consisting of a genus 1 curve X meeting a genus 2 curve transversally at the node N. It is also harmless to assume there is exactly one singular fibre of each type.

After replacing the sheaf of differentials Ω^1_π with the (invertible) dualising sheaf ω_π, Esteves obtains, via a certain pushout construction, a natural map of vector bundles

$$\overline{\nu}\colon \pi^*\pi_*\omega_\pi \to P^1_\pi(\omega_\pi) \to \mathscr{F}$$

where, as before, the source has rank 3 and the target has rank 2. Note that the middle sheaf, the sheaf $P^1_\pi(\omega_\pi)$ of principal parts, is not locally free because of the presence of singularities. However, by construction, the restriction of $\overline{\nu}$ to the smooth locus recovers the old map ν from the previous paragraph. Unfortunately, one cannot apply Porteous formula directly here, because this time the top degeneracy scheme of $\overline{\nu}$ has the wrong dimension, as it contains the elliptic component X.

The way out is to replace ω_π by its twist $L = \omega_\pi \otimes \mathscr{O}_{\mathfrak{X}}(-X)$.[1] Repeating the pushout construction gives the diagram

$$\begin{tikzcd} 0 \arrow[r] & L \otimes \Omega^1_\pi \arrow[r] \arrow[d, "\mathrm{id}\otimes\phi"'] & P^1_\pi(L) \arrow[r] \arrow[d] & L \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L \otimes \omega_\pi \arrow[r] & \mathscr{F}' \arrow[r] & L \arrow[r] & 0 \end{tikzcd}$$

where ϕ is as in (1.6). The map of vector bundles

$$\nu' : \pi^*\pi_*L \to P^1_\pi(L) \to \mathscr{F}'$$

has now top degeneracy scheme of the expected dimension. It can be characterised as follows.

Proposition 2.3 ([19, Prop. 2]). *The top degeneracy scheme D' of ν' consists of:*

(1) *the 8 Weierstrass points of each smooth hyperelliptic fibre, each with multiplicity 1;*
(2) *the node of Z, with multiplicity 1;*
(3) *the node $N = X \cap Y$, with multiplicity 2;*
(4) *the 3 points $A \in X \setminus \{N\}$ such that $2A = 2N$, each with multiplicity 1;*
(5) *the 6 Weierstrass points of Y, each with multiplicity 1.*

The multiplicities tell us how much the points we do not want to count actually contribute. Esteves then proves [19, Prop. 3] the crucial relation $\pi_*[D'] = 72\lambda_\pi - 7\delta_{0,\pi} - 7\delta_{1,\pi}$. Subtracting the unwanted contributions (2) – (5) with the indicated multiplicities on both sides, one gets the relation

$$8\overline{h}_\pi = 72\lambda_\pi - 8\delta_{0,\pi} - 24\delta_{1,\pi},$$

thus proving the formula for $[\overline{H}]$ in Theorem 2.1.

3. Ramification points on Riemann surfaces

In order to make clear that, at least from the point of view of ramification points of linear systems, Gorenstein curves almost behave as if they were smooth, it is probably useful to quickly introduce the notion of ramification loci of linear systems in the classical case of compact Riemann surfaces, which correspond, in the algebraic category, to smooth projective curves.

3.1. **Ramification loci of Linear Systems.** A linear system on a smooth curve C of genus g is a pair (L, V), where L is a line bundle and $V \subset H^0(C, L)$ is a linear subspace. If L has degree d and $\dim V = r + 1$, one refers to (L, V) as a g^r_d on C. When $V = H^0(C, L)$ the linear system is called complete. For instance the complete linear system attached to K_C is the canonical linear system. Every g^r_d defines a rational map

$$\varphi_V : C \dashrightarrow \mathbb{P}V, \qquad P \mapsto (v_0(P) : v_1(P) : \cdots : v_r(P)),$$

where $(v_0, \ldots, v_r)$ is a $\mathbb{C}$-basis of V. The closure of the image of φ_V is a projective curve, not necessarily smooth, of arithmetic genus $g + \delta$ where δ is a measure of the singularities of the image, that may be also rather nasty. See Proposition 4.8 in the next section for the (local) meaning of the number δ. The rational map φ_V turns into a morphism if (L, V) has no *base point*, that is, for all $P \in C$ there is a section $v \in V$ not vanishing at P. If moreover the map separates points, in the sense that for all pairs P_1, $P_2 \in C$ there is a section vanishing at P_i and

[1] A similar technique involving twisting by suitable divisors will be exploited in Section 5.2.

not at P_j, then the map is an embedding and the image itself is smooth of the same geometric genus as C. For most curves a basis $(\omega_0, \dots, \omega_{g-1})$ of $H^0(C, K_C)$ is enough to embed C in $\mathbb{P}^{g-1}$. The curves for which the canonical morphism is not an embedding are called *hyperelliptic*. They can be embedded in $\mathbb{P}^{3g-4}$ by means of a basis of $K_C^{\otimes 2}$.

We now define what it means for a section $v \in V \setminus 0$ to vanish at a point $P \in C$ to a given order. This is a crucial concept in the theory of ramification (or inflectionary behavior) of linear systems. Observe that, given a point $P \in C$, any section $v \in V$ defines an element v_P in the stalk L_P via the maps

$$V \subset H^0(C, L) \to L_P.$$

Definition 3.1. Let $v \in V \setminus 0$ be a section, $P \in C$ a point. We define

$$\operatorname{ord}_P v \coloneqq \dim_{\mathbb{C}} L_P / v_P \in \mathbb{N}$$

to be the *order of vanishing* of v at P.

Definition 3.2. Let (L, V) be a g^r_d. A point $P \in C$ is said to be a *ramification point* of (L, V) if there exists a section $v \in V \setminus 0$ such that $\operatorname{ord}_P v \geq r + 1$. A ramification point of the canonical linear system $(K_C, H^0(C, K_C))$ is called a *Weierstrass point*.

Example 3.3. Let $\iota \colon C \hookrightarrow \mathbb{P}^2$ be a smooth plane quartic. Then C has genus 3 and the complete linear system attached to $K_C = \iota^* \mathscr{O}_{\mathbb{P}^2}(1)$ is the linear system cut out by lines. Therefore the Weierstrass points of C are precisely the *flexes*. It is known classically that there are 24 of them. We take the opportunity here to recall that flexes of plane quartics are geometrically very relevant: their configuration in the plane determines and is determined by the smooth quartic. See the work of Pacini and Testa [45] for this exciting story.

Example 3.4. The g^2_4 on $\mathbb{P}^1$ determined by

$$V = \mathbb{C} \cdot x_0 x_1^3 \oplus \mathbb{C} \cdot x_1^4 \oplus \mathbb{C} \cdot x_0^4 \in G(3, H^0(\mathscr{O}_{\mathbb{P}^1}(4)))$$

defines the morphism $\varphi_V \colon \mathbb{P}^1 \to \mathbb{P}^2$ given by

$$(x_0 : x_1) \mapsto (x_0 x_1^3 : x_1^4 : x_0^4).$$

In the coordinates x, y and z on $\mathbb{P}^2$, the image of φ_V is the plane quartic curve $x^4 - y^3 z = 0$. The curve possesses a unique triple point at $P \coloneqq (0 : 0 : 1)$ and a hyperflex at the point $Q \coloneqq (0 : 1 : 0)$, as it is clear from the local equation $x^4 - z = 0$ (the tangent is $z = 0$). An elementary Hessian calculation shows that Q has multiplicity[2] 2 in the count of flexes of C. Then, by Example 3.3, any reasonable theory of Weierstrass points on singular curves should assign the "weight" 22 to the triple point P, in order to reach the total number of flexes of a quartic curve. See Example 4.11 for the same calculation in terms of the Wronskian (cf. also Remark 4.12 for the relationship between the Hessian and the Wronskian at smooth points).

In fact, the curve C can be easily smoothed in a pencil

$$x^4 - y^3 z + t \cdot L(x, y) z^3 = 0,$$

where $L(x, y) = ax + by$ is a general linear form. An easy check, based on the computation of the Jacobian ideal, shows that the generic fibre of the pencil is a smooth quartic having a hyperflex at the point $(0 : 1 : 0)$. Then there must be exactly 22 smooth flexes that for $t = 0$ collapse at the point $P = (0 : 0 : 1)$. According to the theory of Widland and Lax, sketched in Section 4, the triple point is a singular Weierstrass point of the curve, thought of as a Gorenstein curve of arithmetic genus 3.

[2]We will soon interpret this *multiplicity* as *ramification weight*, see (3.2).

3.2. **Gap sequences and weights.** Let $P \in C$ be an arbitrary point, (L, V) a linear system, and assume $0 < r < d$. For $i \geq 0$, let us denote by

$$V(-iP) \subset V$$

the subspace of sections vanishing at P with order at least i. Note that $V(-(d+1)P) = 0$. If

$$\dim V(-(i-1)P) > \dim V(-iP),$$

then i is called a *gap* of (L, V) at P. It is immediate to check that in the descending filtration

$$V \supseteq V(-P) \supseteq V(-2P) \supseteq \cdots \supseteq V(-(r+1)P) \supseteq \cdots \supseteq V(-dP) \supseteq 0 \tag{3.1}$$

there are exactly $r + 1 = \dim V$ gaps. Note that 1 is not a gap at P if and only if P is a base point of V.

Definition 3.5. The *gap sequence* of (L, V) at $P \in C$ is the sequence

$$\alpha_{L,V}(P) : \alpha_1 < \alpha_2 < \cdots < \alpha_{r+1}$$

consisting of the gaps of (L, V) at P, ordered increasingly.

For a generic point on C, the gap sequence is $(1, 2, \ldots, r+1)$, meaning that the dimension jumps in (3.1) occur as early as possible. Equivalent to the gap sequence is the *vanishing sequence*, whose i-th term is $\alpha_i - i$. The *ramification weight* of (L, V) at P is the sum

$$\mathsf{wt}_{L,V}(P) = \sum_i (\alpha_i - i). \tag{3.2}$$

One may rephrase the condition that P is a ramification point for (L, V) in the following equivalent ways:

(i) $V(-(r+1)P) \neq 0$, that is, $(r+1)P$ is a *special divisor* on C;
(ii) the gap sequence of (L, V) at P is not $(1, 2, \ldots, r+1)$;
(iii) the vanishing sequence of (L, V) at P is not $(0, 0, \ldots, 0)$;
(iv) the ramification weight $\mathsf{wt}_{L,V}(P)$ is strictly positive.

According to (i), $P \in C$ is a Weierstrass point if and only if $h^0(K_C(-gP)) > 0$.

Definition 3.6. Weierstrass points of weight one are called *normal*, or *simple*. On a general curve of genus at least 3 these are the only Weierstrass points to be found. Those of weight at least two are usually called *special* (or *exceptional*) Weierstrass points.

The locus in $\overline{M}_g$ of curves possessing special Weierstrass points has been studied by Cukierman and Diaz. We review the core computations in the subject in Section 5.

3.3. **Total ramification weight and Brill–Segre formulas.** The notion of ramification point of a linear system (L, V) recalled in Definition 3.2 relies on the notion of order of vanishing of a section of L. This compact algebraic definition can be phrased also in the following way, which was used for the first time by Laksov [33] to study ramification points of linear systems on curves in arbitrary characteristic. There exists a map

$$D^r : C \times V \to J^r(L), \qquad (P, v) \mapsto D^r v(P), \tag{3.3}$$

where $D^r v \in H^0(C, J^r(L))$ is the section defined in (1.9), and whose vanishing at P is equivalent to the condition $\operatorname{ord}_P v \geq r + 1$ of Definition 3.2. The map D^r is a map of vector bundles of the same rank $r + 1$, so it is locally represented by an $(r+1) \times (r+1)$ matrix. The condition $D^r v(P) = 0$ then says that (3.3) drops rank at P. This in turn means that P is a zero of the *Wronskian section*

$$\mathbb{W}_V := \det D^r \in H^0\left(C, \bigwedge^{r+1} J^r(L)\right)$$

attached to (L, V). The *total ramification weight* of (L, V), namely the total number of ramification points (counted with multiplicities), is

$$\mathsf{wt}_V := \deg \bigwedge^{r+1} J^r(L) = \sum_P \mathsf{wt}_{L,V}(P).$$

It can be computed by means of the short exact sequence

$$0 \to L \otimes K_C^{\otimes r} \to J^r(L) \to J^{r-1}(L) \to 0,$$

reviewed in Proposition 1.7. By induction, one obtains a canonical identification

$$\bigwedge^{r+1} J^r(L) = L^{\otimes r+1} \otimes K_C^{r(r+1)/2}.$$

Using that $\deg K_C = 2g - 2$, one finds the *Brill–Segre formula*

$$\mathsf{wt}_V = (r+1)d + (g-1)r(r+1) \tag{3.4}$$

attached to (L, V). For instance, since $h^0(C, K_C) = g$, the number of Weierstrass points (counted with multiplicities) is easily computed as

$$\mathsf{wt}_{K_C} = \deg \bigwedge^{g} J^{g-1}(K_C) = (g-1)g(g+1). \tag{3.5}$$

For $g = 3$, (3.5) gives the 24 flexes on a plane quartic, as in Example 3.3.

4. Ramification points on Gorenstein curves

The study of Weierstrass points on singular curves is mainly motivated by degeneration problems. For instance it is a well known result of Diaz [13, Appendix 2, p. 60] that the node of an irreducible uninodal curve of arithmetic genus g can be seen as a limit of $g(g-1)$ Weierstrass points on nearby curves. In this section we review the Lax and Widland construction of the Wronskian section attached to a linear system on a Gorenstein curve.

The key idea is to define derivatives of local regular functions in the extended sense sketched at the beginning of Section 1. One exploits the natural map $\Omega_C^1 \to \omega_C$ (see the references in Section 1.4 for its construction), where ω_C is *invertible* by the Gorenstein condition. The dualising sheaf is explicitly described by means of regular differentials on C. Thanks to this extended definition of differential Widland and Lax are able to attach a Wronskian section to each linear system on C, as we shall show in Section 4.2, after a few preliminaries aimed to reinterpret the Gorenstein condition of Definition 1.2 in local analytic terms. In the last year some progress has been done also in the direction of linear systems on non-Gorenstein curves, essentially thanks to the investigations of R. Vidal-Martins. See e.g. [41] and references therein. As for Gorenstein curves we should mention the clever way to deform monomial curves due to Contiero and Stöhr [2] to compute dimension of moduli spaces of curves possessing a Weierstrass point with prescribed numerical semigroup.

4.1. **The analytic Gorenstein condition.** Let C be a Cohen–Macaulay curve. Its dualising sheaf ω_C has the properties

$$H^0(C, \mathscr{O}_C) = H^1(C, \omega_C)^\vee, \qquad H^1(C, \mathscr{O}_C) = H^0(C, \omega_C)^\vee. \tag{4.1}$$

Recall that $g := p_a(C) := h^1(C, \mathscr{O}_C)$ is the *arithmetic genus* of C. For smooth curves we have $\Omega_C^1 = \omega_C$. But if C is singular, the sheaf Ω_C^1 is no longer locally free and it does not coincide with ω_C. The dualising sheaf itself may or may not be locally free: the curves for which it is are the *Gorenstein curves*.

Example 4.1. All local complete intersection curves are Gorenstein. This includes curves embedded in smooth surfaces as well as the *stable curves* of Deligne–Mumford. Note that, by the adjunction formula, a plane curve $\iota\colon C \hookrightarrow \mathbb{P}^2$ of degree d has canonical bundle $\omega_C = \iota^*\mathscr{O}_{\mathbb{P}^2}(d-3)$, clearly a line bundle. See also Example 1.1 for a relative, more general formula.

The dualising sheaf ω_C of a reduced curve C was first defined by Rosenlicht [49] in terms of *residues* on the normalisation of C. For a Gorenstein curve, this sheaf has a very simple local description. In [50, Section IV.10], to which we refer the reader for further details, it is shown that the stalk $\omega_{C,P}$ is the module of *regular differentials* at P. We now recall an analytic criterion allowing one to check local freeness of ω_C.

Let $\nu\colon \widetilde{C} \to C$ be the normalisation of an integral curve C, and let $S \subset C$ be its singular locus. The canonical morphism $\mathscr{O}_C \to \nu_*\mathscr{O}_{\widetilde{C}}$ is injective, with quotient a finite length sheaf supported on S. We denote by

$$\delta_P := \dim_{\mathbb{C}} \widetilde{\mathscr{O}}_{C,P}/\mathscr{O}_{C,P} \tag{4.2}$$

the fibre dimension of this finite sheaf at a point $P \in C$. Clearly $\delta_P > 0$ if and only if $P \in S$. This number is an *analytic* invariant of singularities [50, p. 59]. The sum $\sum_P \delta_P = p_a(C) - p_a(\widetilde{C})$ is the number δ quickly mentioned in Section 3. Another local measure of singularities is the conductor ideal.

Definition 4.2. Let B be the integral closure of an integral domain A. The *conductor ideal* of $A \subset B$ is the largest ideal $I \subset A$ that is an ideal of B, that is, the set of elements $a \in A$ such that $a \cdot B \subset A$. Let C be an integral curve, $P \in C$ a point. We denote by $\mathfrak{c}_P \subset \mathscr{O}_{C,P}$ the conductor ideal of $\mathscr{O}_{C,P} \subset \widetilde{\mathscr{O}}_{C,P}$. Define the number

$$n_P := \dim_{\mathbb{C}} \widetilde{\mathscr{O}}_{C,P}/\mathfrak{c}_P.$$

For instance if $\widetilde{\mathscr{O}}_{C,P} = \mathscr{O}_{C,P}$ then $\mathfrak{c}_P = \widetilde{\mathscr{O}}_{C,P}$ and n_P vanishes in this case. We wish to recall the following characterisation.

Proposition 4.3 ([50, Proposition IV.7]). *An integral projective curve C is Gorenstein if and only if $n_P = 2\delta_P$ for all $P \in C$.*

In other words, the numerical condition $n_P = 2\delta_P$ guarantees that the sheaf of regular differentials is invertible at P.

Example 4.4. Let P be the origin $(0,0)$ of the affine cuspidal plane cubic $y^2 - x^3 = 0$. Then $\mathscr{O}_{C,P} = \mathbb{C}[t^2,t^3]_{(t^2,t^3)}$. The normalisation is the local ring $\widetilde{\mathscr{O}}_{C,P} = \mathbb{C}[t]_{(t)}$. In this case the conductor is the localisation of the conductor of the subring $\mathbb{C}[t^2,t^3] \subset \mathbb{C}[t]$. Since

$$\mathbb{C}[t^2,t^3] = \mathbb{C} + \mathbb{C}t^2 + \mathbb{C}t^3 + t^2\mathbb{C}[t],$$

the conductor is the ideal (t^2,t^3), and its extension in $\widetilde{\mathscr{O}}_{C,P}$ is (t^2). Then $n_P = \dim_{\mathbb{C}} \mathbb{C}[t]/t^2 = 2$, and $\delta_P = \dim_{\mathbb{C}} \mathbb{C}[t]/\mathbb{C}[t^2,t^3] = 1$. Thus P is a Gorenstein singularity. Having this point as its only singularity, the cuspidal curve is a Gorenstein curve of arithmetic genus 1.

Example 4.5. Let C be the complex rational curve defined by the parametric equations $X = t^3, Y = t^4, Z = t^5$. Then C is the spectrum (the set of prime ideals) of the ring $\mathbb{C}[t^3,t^4,t^5]$. Clearly the origin $P = (0,0,0)$ of $\mathbb{A}^3$ is a singular point of C. One has that

$$\mathscr{O}_{C,P} = \mathbb{C}[t^3,t^4,t^5]_{(t^3,t^4,t^5)}$$

is not a Gorenstein singularity: the conductor of $\mathbb{C}[t^3,t^4,t^5]_{(t^3,t^4,t^5)} \subset \mathbb{C}[t]_{(t)}$ is $t^3\mathbb{C}[t]_{(t)}$. Thus $n_P = 3$, an odd number, and C cannot be Gorenstein at P.

4.2. **The Wronskian section after Widland–Lax.** We now explain the construction, due to Widland and Lax, of the Wronskian attached to a linear system (L, V) on a Gorenstein curve. For simplicity we shall stick to the case of *integral* (reduced, as usual, and irreducible) curves to avoid coping with linear systems possessing non zero sections identically vanishing along an irreducible component. For example if $X \cup Y$ is a uninodal reducible curve of arithmetic genus g the space of global sections of the dualising sheaf has dimension g but there are non-zero sections vanishing identically along X (or on Y). However if one considers a linear system on a reducible curve that is not degenerate on any component, then everything goes through just as in the irreducible case.

If $P \in C$ is a singular point on an (integral) curve C, the maximal ideal $\mathfrak{m}_P \subset \mathscr{O}_{C,P}$ is not principal and so there is no local parameter whose differential would be able to freely generate $\Omega^1_{C,P}$. But we can still consider the natural map $\Omega^1_C \to \omega_C$ (cf. Section 1.4) and its composition

$$\mathrm{d}\colon \mathscr{O}_C \to \omega_C$$

with the universal derivation $\mathscr{O}_C \to \Omega^1_C$.

Let now (L, V) be a g^r_d on the (Gorenstein) curve C, and let P be any point (smooth or not). Let $(v_0, v_1, \ldots, v_r)$ be a basis of V. Then $v_{i,P}$, the image of v_i in the stalk L_P, is of the form $v_{i,P} = f_i \cdot \psi_P$ where $f_i \in \mathscr{O}_{C,P}$ and ψ_P generates L_P over $\mathscr{O}_{C,P}$. Letting σ_P be a generator of $\omega_{C,P}$ over $\mathscr{O}_{C,P}$, one can define regular functions $f_i', f_i^{(2)}, \ldots, f_i^{(r)} \in \mathscr{O}_{C,P}$ through the identities

$$\mathrm{d}\, f_i = f_i' \cdot \sigma_P, \qquad \mathrm{d}\, f_i^{(j-1)} = f^{(j)} \cdot \sigma_P$$

in $\omega_{C,P}$, for each $i = 0, 1, \ldots, r$ (cf. also Section 1.4). If P were a smooth point, one could take $\sigma_P = \mathrm{d}\, z$, where z is a generator of the maximal ideal $\mathfrak{m}_P \subset \mathscr{O}_{C,P}$, thus recovering the classical situation.

Definition 4.6. The Widland–Lax (WL) *Wronskian* around $P \in C$ is the determinant

$$\mathsf{WL}_{V,\sigma_P} = \begin{vmatrix} f_0 & f_1 & \cdots & f_r \\ f_0' & f_1' & \cdots & f_r' \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(r)} & f_1^{(r)} & \cdots & f_r^{(r)} \end{vmatrix} \in \mathscr{O}_{C,P}. \tag{4.3}$$

A point P is said to be a V-*ramificaton* point (or also a V-Weierstrass point) if $\mathsf{WL}_{V,\sigma_P}(P) = 0$, that is, if $\mathrm{ord}_P\, \mathsf{WL}_{V,\sigma_P} > 0$.

Our next task will be to show that the germ (4.3), as well as its vanishing at P, does not depend on the choice of the generators ψ_P and σ_P of L_P and $\omega_{C,P}$ respectively; then we will use the explicit description of ω_C in the previous section to check that singular points are V-ramification points with high weight.

So if ϕ_P and τ_P are others generators, then $v_i = g_i \phi_P$ and $\mathrm{d}\, g^{(i-1)} = g^{(i)} \tau_P$. Let $\psi_P = \ell_P \phi_P$ and $\sigma_P = k_P \tau_P$. Then a straightforward exercise shows that

$$\mathsf{WL}_{V,\sigma_P} = \ell_P^{r+1} k_P^{r(r+1)/2} \cdot \mathsf{WL}_{V,\tau_P}.$$

This proves at once that the vanishing is well defined and that all the sections WL_{V,σ_P} patch together to give a global section

$$\mathsf{WL}_{V,P} \in H^0\left(C, L^{\otimes r+1} \otimes \omega_C^{\otimes r(r+1)/2}\right).$$

If $f \in \mathscr{O}_{C,P}$ is any germ, according to Definition 3.1 one has

$$\operatorname{ord}_P f = \dim_{\mathbb{C}} \frac{\mathscr{O}_P}{f \cdot \mathscr{O}_P} = \dim_{\mathbb{C}} \frac{\widetilde{\mathscr{O}}_P}{f \cdot \widetilde{\mathscr{O}}_P} = \sum_{Q \in \nu^{-1}(P)} \operatorname{ord}_Q f, \tag{4.4}$$

where in the last equality f is seen as an element of $\mathscr{O}_{\widetilde{C},Q}$ via $\mathscr{O}_{C,P} \subset \widetilde{\mathscr{O}}_{C,P} \subset \mathscr{O}_{\widetilde{C},Q}$.

Definition 4.7. Let $P \in C$. Define the *V-weight* of P and *total V-ramification weight* as

$$\mathsf{wt}_V(P) := \operatorname{ord}_P \mathsf{WL}_{V,P}, \qquad \mathsf{wt}_V := \sum_{P \in C} \mathsf{wt}_V(P).$$

According to (4.4), one can compute the V-weight at P as

$$\mathsf{wt}_V(P) = \sum_{Q \in \nu^{-1}(P)} \operatorname{ord}_Q \mathsf{WL}_{V,P}.$$

Proposition 4.8. *Let (L, V) be a g^r_d on a Gorenstein curve C of arithmetic genus g. Then*

$$\mathsf{wt}_V = (r+1)d + (g-1)r(r+1). \tag{4.5}$$

Moreover, for all $P \in C$, the inequality

$$\mathsf{wt}_V(P) \geq \delta_P r(r+1) \tag{4.6}$$

holds, with δ_P as defined in (4.2). That is, singular points have "high weight".

In particular if $L = \omega_C$, one has that $\mathsf{wt}_{\omega_C}(P) \geq \delta_P g(g-1)$. Proposition 4.8 in [38] relies on an explicit description of the generator of the dualising sheaf around the singularities, that we shall review below just to provide a few examples illustrating the situation. The verification we offer here makes evident how the theory by Lax and Widland offers the right framework to study the classical Plücker formulas in terms of degenerations.

Proof of Proposition 4.8. Formula (4.5) is clear. Let now P be a singular point of C and $\nu_P \colon \widetilde{C}_P \to C$ be the partial normalisation of C around a singular point P. Then $\widetilde{C}_P$ is Gorenstein of arithmetic genus $g - \delta_P$. Consider the linear system $(\widetilde{V}, \nu_P^* L)$, where $\widetilde{V}$ is spanned by $\nu_P^* v_0, \nu_P^* v_1, \ldots, \nu_P^* v_r$. It is a g^r_d on $\widetilde{C}_P$. Applying the formula (4.5) for the total weight to $\widetilde{V}$, we find

$$\mathsf{wt}_{\widetilde{V}} = (r+1)d + (g-1-\delta_P)r(r+1) = \mathsf{wt}_V - \delta_P r(r+1).$$

The $\widetilde{V}$-Weierstrass points on $\widetilde{C}_P$ are the same as the V-Weierstrass points on C. Then the difference counts the minimum weight of the singular point P with respect to (L, V). □

In general $\mathsf{wt}_V(P) = \delta_P r(r+1) + E(P)$. The correction $E(P)$ is called the *extraweight*. It is zero if no point of $\nu_P^{-1}(P)$ is a ramification point of the linear system $(\widetilde{V}, \nu_P^* L)$.

Example 4.9. If $P \in C$ is a cusp, one has $\delta_P = 1$, hence its weight is at least $r(r+1)$. However the vanishing sequence of $\widetilde{V}$ at the preimage of P in the normalisation is $0, 2, \ldots, r+1$. It follows that

$$\mathsf{wt}_V(P) = r(r+1) + r = r(r+2).$$

If $L = \omega_C$ then $\mathsf{wt}_{\omega_C}(P) = g^2 - 1$.

Before offering a few examples of how the WL Wronskian works concretely in computations, we recall the following fact.

Proposition 4.10 ([24, p. 362]). *Let τ be a local section of $\Omega^1_{\widetilde{C}}$ and let τ_Q its image in the stalk $\Omega^1_{\widetilde{C},Q}$. Assume that $\Omega^1_{\widetilde{C},Q} = \mathscr{O}_{\widetilde{C},Q}\cdot\tau_Q$ for all $Q\in\nu^{-1}(P)$ and that h generates the conductor in each local ring $\mathscr{O}_{\widetilde{C},Q}$. Then τ/h generates $\omega_{C,P}$ over $\mathscr{O}_{C,P}$.*

Example 4.11. Let us revisit from Example 3.4 the rational irreducible quartic plane curve given by $x^4-y^3z=0$ in homogeneous coordinates x, y, z on $\mathbb{P}^2$. It is Gorenstein of arithmetic genus 3 with $\omega_C=\mathscr{O}_{\mathbb{P}^2}(1)|_C$. It has a triple point at $P\coloneqq(0:0:1)$ and a hyperflex at $Q\coloneqq(0:1:0)$, i.e. a Weierstrass point of weight 2. To see that the Weierstrass weight at Q is 2 one may argue by writing down the Wronskian of a basis of holomorphic differentials adapted at Q (i.e. $\omega_0=\mathrm{d}\,t$, $\omega_1=t\,\mathrm{d}\,t$ and $\omega_2=t^4\,\mathrm{d}\,t$). The vanishing sequence is $0,1,4$ (equivalently, the gap sequence is $1,2,5$) so the weight is 2.

In the chart $z\neq 0$, $V=H^0(C,\omega_C)$ is spanned by $(t^3,t^4,1)$, which are nothing but the parametric equations mapping $\mathbb{P}^1$ onto the quartic. One has

$$\mathscr{O}_{C,P}=\mathbb{C}+\mathbb{C}\cdot t^3+\mathbb{C}\cdot t^4+t^6\cdot\mathbb{C}[t]_{(t)},\qquad n_P=6,\qquad \delta_P=3.$$

According to Proposition 4.8, P is a Weierstrass point with weight at least $\delta_P\cdot 3(3-1)=18$. The exact weight can be directly computed through the Wronskian as follows. The preimage of P through the normalisation map is just one point $\widetilde{P}$. Then $\mathrm{d}\,t$ generates $\Omega^1_{\widetilde{C},\widetilde{P}}$ and therefore $\sigma=\mathrm{d}\,t/t^6$ is a regular differential at P. A basis of the space of regular differentials at P is then given by

$$(\sigma,t^3\sigma,t^4\sigma),$$

so the Wronskian is

$$\begin{vmatrix}1 & t^3 & t^4\\ 0 & 3t^8 & 4t^9\\ 0 & 24t^{13} & 36t^{14}\end{vmatrix}\in t^{22}\cdot\mathbb{C}[t].$$

It follows that P is a Weierstrass point of weight 22, as anticipated in Example 3.4. Together with the hyperflex at Q, one fills the total weight, 24, of a Gorenstein curve of genus 3. The example shows that the point P has extraweight $E(P)=4$. This can also be computed by looking at the vanishing sequence of the linear system $\widetilde{V}$, generated by $(1,t^3,t^4)$. Clearly the vanishing sequence is $0,3,4$, whose weight is 4, as predicted by the calculation above.

The output of this example is of course in agreement with the classical fact that the Hessian of the given plane curve cuts the singular points and the flexes. In this case the Hessian cuts indeed the singular point with multiplicity 22 and the hyperflex Q with multiplicity 2.

Remark 4.12. A local calculation shows that the Hessian of a plane curve cutting the inflection points with respect the linear system of lines follows by the vanishing of the Wronskians at those points (at least when they are smooth).

Example 4.13. The previous example was rather easy because we have dealt with a unibranch singularity (that is, $\nu^{-1}(P)$ consisted of just one point). To illustrate the behavior of the WL Wronskian with multibranch singularities, let C be the plane cubic $x^3+x^2z-y^2z=0$. It has a unique singular point, the node $P\coloneqq(0:0:1)$. The curve C is Gorenstein of arithmetic genus 1. Let us compute its V-weight, where V denotes the complete linear system $H^0(C,\mathscr{O}_{\mathbb{P}^2}(1)|_C)$. Clearly the coordinate functions x, y and z form a basis of V. They can be expressed by means of a local parameter t on the normalisation $\nu\colon\mathbb{P}^1\to C$. In the open set $z\neq 1$, indeed, C has parametric equations $x=t^2-1$ and $y=t(t^2-1)$. The preimage of the point P via ν are $Q_1\coloneqq(t-1)$ and $Q_2\coloneqq(t+1)$ thought of as points of Spec $\mathbb{C}[t]$. One has that $\mathscr{O}_{C,P}=\mathbb{C}+(t^2-1)\cdot\widetilde{\mathscr{O}}_{C,P}$, thus the conductor is (t^2-1). Since $\mathrm{d}\,t$ generates both $\Omega^1_{Q_1}$ and $\Omega^1_{Q_2}$,

$\sigma_P := \mathrm{d}\,t/(t^2-1)$ generates the dualising sheaf $\omega_{C,P}$. Let

$$\sigma_{Q_1} := \frac{\mathrm{d}\,t}{t-1} \qquad \text{and} \qquad \sigma_{Q_2} := \frac{\mathrm{d}\,t}{t+1}.$$

Then one has

$$\mathsf{wt}_V(P) = \operatorname{ord}_P \mathsf{WL}_{V,\sigma_P} = \operatorname{ord}_{Q_1} \mathsf{WL}_{V,\sigma_{Q_1}} + \operatorname{ord}_{Q_2} \mathsf{WL}_{V,\sigma_{Q_2}}.$$

We shall show that $\operatorname{ord}_{Q_1} \mathsf{WL}_{V,\sigma_{Q_1}} = 3$, By symmetry, the same will hold for $\operatorname{ord}_{Q_2} \mathsf{WL}_{V,\sigma_{Q_2}}$, showing that the weight of P as a singular ramification point is 6 as expected. For simplicity, let us put $z = t-1$. In this new coordinate the basis of ν^*L near Q_1 is given by $v_0 := z(z+2)$, $v_1 := z(z^2+3z+2)$ and $v_3 = 1$. The conductor is generated by z near Q_1. Then the WL-Wronskian near Q_1 is:

$$\mathsf{WL}_{V,\sigma_{Q_1}} = \begin{vmatrix} z^2+2z & z^3+3z^2+2z & 1 \\ 2z^2+2z & 3z^2+6z^2+2z & 0 \\ 4z^2+2z & 6z^2+12z^2+2z & 0 \end{vmatrix} = z^3(3z+4) \in z^3 \cdot \mathbb{C}[z]$$

as desired. The computations around Q_2 are similar and then P is a singular ramification point of weight 6.

5. The class of special Weierstrass points

5.1. **Introducing the main characters.** Let M_g be the moduli space of smooth projective curves of genus $g \geq 2$. It is a normal quasi-projective variety of dimension $3g-3$. Let

$$M_g \subset \overline{M}_g$$

be its Deligne–Mumford compactification via *stable* curves. It is a projective variety with orbifold singularities. Thus, its Picard group with rational coefficients is as well-behaved as the Picard group of a smooth variety. The boundary $\overline{M}_g \setminus M_g$ is a union of divisors $\Delta_i \subset \overline{M}_g$, each obtained as the image of the clutching morphism

$$\overline{M}_{i,1} \times \overline{M}_{g-i,1} \to \overline{M}_g,$$

defined by glueing two stable 1-pointed curves (X,x) and (Y,y) identifying the markings x and y. By a general point of Δ_i we shall mean a curve that lies in the image of the open part $M_{i,1} \times M_{g-i,1}$. Note that i ranges from 0 to $[g/2]$, with $i = 0$ corresponding to irreducible uninodal curves. We use the standard notation δ_i for the class of Δ_i in $\operatorname{Pic}(\overline{M}_g) \otimes \mathbb{Q}$, and we always assume $i \leq g-i$.

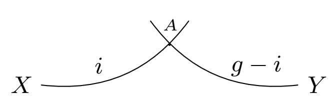


FIGURE 1. A general element of the boundary divisor $\Delta_i \subset \overline{M}_g$.

This section aims to sketch the calculation of the class in $\operatorname{Pic}(\overline{M}_g) \otimes \mathbb{Q}$ of the closure in $\overline{M}_g$ of the locus of points in M_g corresponding to curves possessing a special Weierstrass point. Recall from Definition 3.6 that a Weierstrass point (WP, for short) is *special* if its weight as a zero of the Wronskian is strictly bigger than 1. Let us define

$$\mathsf{wt}(k) := \{\, [C] \in M_g \mid C \text{ has a WP with weight at least } k \,\}. \tag{5.1}$$

Let $M_{g,1}$ be the space of 1-pointed smooth curves, and $\overline{M}_{g,1}$ be the moduli space of stable 1-pointed curves. Borrowing standard notation from the literature, define the "vertical" loci

$$\begin{aligned}\mathrm{V}\mathbb{D}_{g-1} &:= \{\, [C,P] \in M_{g,1} \mid P \text{ is a WP whose first non-gap is } g-1 \,\}\\ \mathrm{V}\mathbb{D}_{g+1} &:= \left\{\, [C,P] \in M_{g,1} \;\middle|\; \text{there is } \sigma \in H^0(C,K_C) \text{ such that } D^g\sigma(P) = 0 \,\right\}.\end{aligned}$$

Taking their images along the forgetful morphism $M_{g,1} \to M_g$ we get the subvarieties $\mathbb{D}_{g-1}$ and $\mathbb{D}_{g+1}$ of M_g, respectively. Diaz [13, Section 7] and Cukierman [6, Section 5] were able to determine the classes

$$\left[\overline{\mathbb{D}}_{g\pm 1}\right] \in \operatorname{Pic}(\overline{M}_g) \otimes \mathbb{Q}.$$

The main observation of [26] is that while computing the classes of $\overline{\mathbb{D}}_{g\pm 1}$ is quite hard, the computation of their sum is quite straightforward. Let

$$\overline{\mathsf{Vwt}(2)} \subset \overline{M}_{g,1}$$

be the closure of the locus of points $[C,P] \in M_{g,1}$ such that P is a special Weierstrass point on C, namely a zero of the Wronskian of order bigger than 1. The goal is to globalise the notion of Wronskian to families possessing singular fibres. This will be achieved through jet extensions of the relative dualising sheaf defined on a family of stable curves. Using (a) the invertibility of the relative dualising sheaf and (b) the locally free replacement of the principal part sheaves for such families, everything goes through via a standard Chern class calculation, as we show below. We warn the reader that our computation is not performed on the entire moduli space but just on 1-parameter families of stable curves with smooth generic fibre, in order to avoid delicate foundational issues regarding the geometry of the moduli space of curves.

5.2. **Special Weierstrass points.** Let $\pi' \colon \mathcal{C}' \to T$ be a (proper, flat) family of stable curves over a smooth projective curve T, such that $\mathcal{C}'$ is a smooth surface, with smooth generic fibre $\mathcal{C}'_\eta$. In particular, by the compactness of T, the fibre $\mathcal{C}'_t$ is smooth for all but finitely many $t \in T$. If the family is general, the singular fibres are general curves of type Δ_i. The general fibre of type Δ_0 is an irreducible uninodal curve of arithmetic genus g. Let $\pi \colon \mathcal{C} \to T$ be the family one gets by blowing up all the nodes of the irreducible singular curves. The irreducible nodal fibres get replaced by curves of the form $C \cup L$, where C is a smooth irreducible curve of genus $g-1$ and L is a smooth rational curve, intersecting C transversally at two points (the preimages of the node through the blow up map). The rational component L is the exceptional divisor which contracts onto the node by blow down. From now on we shall work with the new family

$$\pi \colon \mathcal{C} \to T,$$

where all the singular fibres are reducible.

As for all families of stable curves, the dualising sheaf ω_π is invertible, and its pushforward

$$\mathbb{E}_\pi := \pi_*\omega_\pi$$

is a rank g vector bundle on T, called the *Hodge bundle* (of the family). Its fibre over $t \in T$ computes

$$H^0(\mathcal{C}_t, \omega_\pi|_{\mathcal{C}_t}) = H^0(\mathcal{C}_t, \omega_{\mathcal{C}_t}).$$

If $\mathcal{C}_{t_0} = X \cup_A Y$ is a uninodal reducible curve of type Δ_i, one has a splitting

$$H^0(\mathcal{C}_0, \omega_{\mathcal{C}_0}) = H^0(X, K_X(A)) \oplus H^0(Y, K_Y(A)). \tag{5.2}$$

A Weierstrass point on the generic fibre is a ramification point of the complete linear series attached to $K_{\mathcal{C}_\eta} := \omega_\pi|_{\mathcal{C}_\eta}$. So it must belong to the degeneracy locus of the map of rank g vector bundles

$$
\begin{tikzcd}
\pi^*\mathbb{E}_\pi \arrow[rr, "\mathsf{D}^{g-1}"] \arrow[dr, dash] & & J^{g-1}_\pi(\omega_\pi) \arrow[dl, dash] \\
& \mathcal{C} \arrow[d, "\pi"] & \\
& T &
\end{tikzcd}
$$

The zero locus of the determinant map $\bigwedge^g \mathsf{D}^{g-1}$ may be identified with a section $\mathbb{W}_\pi$ of the line bundle

$$\mathscr{L} := \bigwedge^g J^{g-1}_\pi(\omega_\pi) \otimes \pi^* \bigwedge^g \mathbb{E}_\pi^\vee.$$

The vanishing locus of this section cuts the Weierstrass points on the generic fibre. Moreover $\mathbb{W}_\pi$ identically vanishes on the reducible fibres $\mathcal{C}_t$ of type Δ_i for $1 \le i \le [g/2]$. Indeed, the identification (5.2) shows that there exist nonzero regular differentials on $\mathcal{C}_t$ vanishing identically on either component. Moreover $\mathbb{W}_\pi$ identically vanishes on the rational components L gotten by blowing up the nodes of the original irreducible nodal fibres.

A local computation due to Cukierman [6, Proposition 2.0.8] (but see also [7] for an alternative way of computing), determines the order of vanishing of $\mathbb{W}_\pi$ along each component of the reducible fibres of π. Let $F \subset \mathcal{C}$ be the Cartier divisor corresponding to the zero locus of $\mathbb{W}_\pi$ along the singular fibres. Then, letting Z_η be the cycle representing $\overline{Z(\mathbb{W}_\pi|_{\mathcal{C}_\eta})} \subset \mathcal{C}$, one has

$$[Z(\mathbb{W}_\pi)] = \mathsf{Z}_\eta + F.$$

One can view $\overline{Z(\mathbb{W}_\pi|_{\mathcal{C}_\eta})}$ as the zero locus of the Wronskian section "divided out" by the local equations of the components of the singular fibres. More precisely, $\mathbb{W}_\pi$ induces a section $\widetilde{\mathbb{W}}_\pi$ of the line bundle $\mathscr{L}(-F)$, which coincides with $\mathbb{W}_\pi$ away from F. Therefore we have

$$\mathsf{Z}_\eta = c_1\left(\bigwedge^g J^{g-1}_\pi(\omega_\pi)\right) - \pi^* c_1(\mathbb{E}_\pi) - F = \frac{1}{2} g(g+1) c_1(\omega_\pi) - \pi^*\lambda_\pi - F \tag{5.3}$$

where $\lambda_\pi := c_1(\mathbb{E}_\pi)$ denotes, as is customary, the first Chern class of the Hodge bundle of the family. From now on, we use the (standard) notation $K_\pi := c_1(\omega_\pi)$.

Remark 5.1. Intersecting the class (5.3) with a fibre $\mathcal{C}_t$, one gets

$$\mathsf{Z}_\eta \cdot \mathcal{C}_t = \frac{1}{2} g(g+1) K_\pi \cdot \mathcal{C}_t - \pi^*\lambda_\pi \cdot \mathcal{C}_t - F \cdot \mathcal{C}_t.$$

But the second and third products vanish because $\mathcal{C}_t$ is linearly equivalent to the generic fibre (and the intersection of two fibres is zero), whereas the first term corresponds to a divisor of degree $(g-1)g(g+1)$ on $\mathcal{C}_t$. In the case where t corresponds to a singular fibre, the degree of this divisor would be the total weight of the *limits* of Weierstrass points on that fibre.

The issue is now to detect and compute the class of the locus of special Weierstrass points in the fibres of π. Since the family π may have singular fibres, the traditional version of principal parts would not help unless one decided to focus on open sets where they are locally free. This is for example the approach followed in [6]. However, using the locally free replacement provided by jet bundles, we can now consider the "derivative" $D\widetilde{\mathbb{W}}_\pi$ of the section $\widetilde{\mathbb{W}}_\pi \in H^0(\mathcal{C}, \mathscr{L}(-F))$, where $\mathscr{L}(-F))$ denotes the twist $\mathscr{L} \otimes_{\mathscr{O}_\mathcal{C}} \mathscr{O}_\mathcal{C}(-F)$. The derivative $D\widetilde{\mathbb{W}}_\pi$ is a global holomorphic section of the rank two bundle

$$J^1_\pi(\mathscr{L}(-F)).$$

By abuse of notation let us write simply $\overline{\mathsf{Vwt}(2)}$ for the locus $\overline{\mathsf{Vwt}(2)}_\pi \subset \mathcal{C}$ defined by the zero locus of $D\widetilde{\mathbb{W}}_\pi$.

Definition 5.2. Let C_0 be any stable curve of arithmetic genus $g \geq 2$. A point $P_0 \in C_0$ is said to be a limit of a (special) Weierstrass point if there exists a family $\mathfrak{X} \to \operatorname{Spec} \mathbb{C}[\![t]\!]$ such that $\mathfrak{X}_\eta$ is smooth, $\mathfrak{X}_0$ is semistably equivalent to C_0 and there is a (special) Weierstrass point P_η such that $P_0 \in \overline{P_\eta}$.

It turns out that $\overline{\mathsf{Vwt}(2)}$ is the locus of special Weierstrass points on smooth fibres of π. In fact if the family $\mathcal{C} \to T$ is general, then only singular fibres of the codimension 1 boundary strata of $\overline{M}_g$ occur. If $X \cup_A Y$ is a general member of Δ_i, one may assume that A is not a Weierstrass point neither for X nor for Y. Then if $P_0 \in X \subset X \cup_A Y$ is a limit of a special Weierstrass point it must be a special ramification point of $K_X((g-i+1)P)$ by [7, Theorem 5.1]. But by [8], for a general curve X and for each $j \geq 0$, there are only finitely many pairs $(P,Q) \in X \times X$ such that Q is a special ramification point of the linear system $K_X((j+1)P)$. See also Example 6.4 below.

It follows that the locus $\overline{\mathsf{Vwt}(2)}$ is zero dimensional. Indeed, the special Weierstrass points have the expected codimension 2 in general family of smooth curves. Its class is given by the top (that is, second) Chern class of $J^1_\pi(\mathscr{L}(-F))$. Explicitly, we have

$$\left[\overline{\mathsf{Vwt}(2)}\right] = c_2\left(J^1_\pi\left(\omega_\pi^{\otimes g(g+1)/2} \otimes \pi^* \bigwedge^g \mathbb{E}_\pi^\vee(-F)\right)\right). \tag{5.4}$$

By the Whitney sum formula applied to the short exact sequence

$$0 \to \omega_\pi \otimes \mathscr{L}(-F) \to J^1_\pi(\mathscr{L}(-F)) \to \mathscr{L}(-F) \to 0,$$

and recalling that (5.3) is computing precisely $c_1(\mathscr{L}(-F))$, one finds

$$\left[\overline{\mathsf{Vwt}(2)}\right] = \left(\frac{1}{2}g(g+1)K_\pi - \pi^*\lambda_\pi - F\right)\left(\frac{1}{2}g(g+1)K_\pi + K_\pi - \pi^*\lambda_\pi - F\right).$$

Thus in $A^2(\mathcal{C})$ we find

$$\left[\overline{\mathsf{Vwt}(2)}\right] = \frac{1}{4}g(g+1)(g^2+g+2)K_\pi^2 - (g^2+g+1)(K_\pi(F+\pi^*\lambda_\pi)) + F^2,$$

where we have used $(\pi^*\lambda_\pi)^2 = 0 = F \cdot \pi^*\lambda_\pi$. We want to compute the pushforward

$$\begin{aligned}\pi_*\left[\overline{\mathsf{Vwt}(2)}\right] = {} & \frac{1}{4}g(g+1)(g^2+g+2)\pi_*K_\pi^2 \\ & - (g^2+g+1)\big(\pi_*(K_\pi \cdot F) + \pi_*(K_\pi \cdot \pi^*\lambda_\pi)\big) + \pi_*F^2.\end{aligned} \tag{5.5}$$

The reason why we are interested in the class (5.5) is that if $g \geq 4$ the degree of π restricted to $\mathsf{Vwt}(2)$ is 1. Therefore, if we let

$$\overline{\mathsf{wt}(2)} \subset T$$

be the locus of points parametrising fibres possessing special Weierstrass points, then its class is given by (5.5). The reason why for $g \geq 4$ the degree of π is 1, is because of the following important result, obtained by combining results by Coppens [3] and Diaz [11].

THEOREM 5.1. *If a general curve of genus $g \geq 4$ has a special Weierstrass point, then all the other points are normal.*

To complete the computation, let $F_i \subset \mathcal{C}$ be the (vertical) divisor corresponding to the zero locus of the Wronskian along the singular fibres of type type Δ_i, for $1 \le i \le [g/2]$. Thus $F = \sum_{i=1}^{[g/2]} F_i$ and clearly we have $F_{i_1} \cdot F_{i_2} = 0$ for $i_1 \neq i_2$. Moreover, we have decompositions

$$F_i := \sum_j F_{ij}, \qquad F_{ij} = m_i X_j + m_{g-i} Y_j,$$

with each F_{ij} supported on a fibre $X_j \cup_{A_j} Y_j$ of type Δ_i. Recall that the notation means that X_j and Y_j have genus i and $g-i$ respectively, and they meet transversally at the (unique) node A_j. The multiplicities m_i (resp. m_{g-i}) with which $\mathbb{W}_\pi$ vanishes along X_j (resp. Y_j) only depend on i. Using that $-Y_j^2 = -X_j^2 = X_j \cdot Y_j = [A_j] \in A^2(\mathcal{C})$, it is easy to check that

$$F_{ij}^2 = \left(2m_i m_{g-i} - m_i^2 - m_{g-i}^2\right)[A_j].$$

To compute (5.5), we will apply the projection formula $\pi_*(\pi^*\alpha \cdot \beta) = \alpha \cdot \pi_*\beta$. The pushforward $\pi_* K_\pi^2$ is by definition the tautological class $\kappa_1 \in A^1(T)$. Define

$$\delta_{i,\pi} := \sum_j \pi_*[A_j] \in A^1(T).$$

This is the class of the points corresponding to singular fibres of type Δ_i. We have the following equalities in $A^1(T)$:

$$\begin{aligned}
\pi_*(K_\pi \cdot \pi^*\lambda_\pi) &= \pi_* K_\pi \cdot \lambda_\pi = (2g-2)\lambda_\pi \\
\pi_*(K_\pi \cdot F_{ij}) &= m_i \pi_*(K_\pi \cdot X_j) + m_{g-i}\pi_*(K_\pi \cdot Y_j) \\
&= (m_i(2i-1) + m_{g-i}(2(g-i)-1)) \cdot \pi_*[A_j] \\
&= (2(im_i + (g-i)m_{g-i}) - m_i - m_{g-i}) \cdot \pi_*[A_j].
\end{aligned}$$

Substituting the above equalities in (5.5) we obtain

$$\pi_*\left[\overline{\mathsf{Vwt}(2)}\right] = \frac{1}{4}g(g+1)(g^2+g+2)\kappa_1 - 2(g^2+g+1)(g-1)\lambda_\pi - c_0\delta_{0,\pi} - \sum_{i=1}^{[g/2]} c_i \delta_{i,\pi} \tag{5.6}$$

where δ_0 is the class of the locus in T of type Δ_0 (irreducible uninodal), c_0 is a coefficient to be determined and

$$c_i = (g^2+g+1)\left(2(im_i + (g-i)m_{g-i}) - m_i - m_{g-i}\right) + 2m_i m_{g-i} - m_i^2 - m_{g-i}^2. \tag{5.7}$$

Now one uses one of the most fundamental relations between tautological classes. The class $\kappa_{1,\pi}$ and λ_π are not independent, as they are related by

$$\kappa_{1,\pi} = 12\lambda_\pi - \sum_i \delta_{i,\pi}.$$

This is a consequence of the Grothendieck–Riemann–Roch formula, as explained for instance in [42]. Thus formula (5.6) can be simplified into

$$\begin{aligned}
\pi_*\left[\overline{\mathsf{Vwt}(2)}\right] = &\left(3g(g+1)(g^2+g+2) - 2(g^2+g+1)(g-1)\right)\lambda_\pi \\
&- \sum_{i=0}^{[g/2]}\left(c_i + \frac{1}{4}g(g+1)(g^2+g+2)\right)\delta_{i,\pi},
\end{aligned} \tag{5.8}$$

which, after renaming coefficients, becomes

$$\pi_*\left[\overline{\mathsf{Vwt}(2)}\right] = \left(2 + 6g + 9g^2 + 4g^3 + 3g^4\right)\lambda_\pi - a_0\delta_0 - \sum_{i=1}^{[g/2]} b_i \delta_{i,\pi}. \tag{5.9}$$

Clearly the expression (5.9) is not complete: one still needs to determine the coefficients a_0 and b_i. Computing b_i amounts to finding the explicit expressions for m_i, for all $1 \leq i \leq [g/2]$. This has been done by Cukierman in his doctoral thesis (but see [7, Proposition 6.3] for an alternative slightly more conceptual, although probably longer, proof).

THEOREM 5.2 ([6, Prop. 2.0.8]). *The multiplicities m_i with which the Wronskian $\mathbb{W}_\pi$ vanishes along X_j (of genus i), are given by:*

$$m_i = \binom{g-i+1}{2}. \tag{5.10}$$

The way Cukierman proves Theorem 5.2 is the following. He considers a family $f\colon \mathfrak{X} \to S$ of curves of genus g parametrised by $S = \operatorname{Spec} \mathbb{C}[\![t]\!]$, with smooth generic fibre and special fibre semistably equivalent to a uninodal reducible curve $X \cup_A Y$ with components of genus i and $g-i$ respectively. After checking that $f_*\omega_f \otimes k(0)$ is isomorphic to $H^0(K_X(A)) \oplus H^0(K_Y(A))$, he constructs suitable global bases of $f_*\omega_f$ such that the first elements are non degenerate on one component and vanish on the other. He then computes the relative Wronskian using such bases and finds the multiplicity displayed in (5.10). All the technical details are in [6].

Granting Theorem 5.2, we can now compute the right hand side of (5.9). We need to substitute the expressions (5.10) into the constant c_i defined in (5.7). This finally gives (see also [26] for more computational details)

$$b_i = (g^3 + 3g^2 + 2g + 2)i(g-i). \tag{5.11}$$

We still have to determine a_0. To this end, we use the following argument, due to Harris and Mumford [31]. Consider the simple elliptic pencil $x_0E_1 + x_1E_2$, where E_1 and E_2 are two plane cubics intersecting transversally at 9 points. Let $\mathcal{S}$ be the blow-up of $\mathbb{P}^2$ at the intersection points. This gives an elliptic fibration

$$\epsilon\colon \mathcal{S} \to \mathbb{P}^1 \tag{5.12}$$

with nine sections (the exceptional divisors of the blown up points). Let Σ_1 be any one of them. Then consider a general curve C of genus $g-1$, and choose a constant section $P\colon C \to C \times C$. Construct the family $\phi\colon \mathcal{F}_1 \to \mathbb{P}^1$, by gluing $C \times C$ and $\mathcal{S}$, by identifying Σ_1 with P. The fibre over a point $t \in \mathbb{P}^1$ is the union $C \cup E_t$, with C meeting $E_t = \epsilon^{-1}(t)$ transversally at a single point. In other words, what varies in the family is just the j-invariant of the elliptic curve.

THEOREM 5.3 ([13, Lemma 7.2]). *The fibres of $\phi\colon \mathcal{F}_1 \to \mathbb{P}^1$ contain no limits of special Weierstrass points, that is, $\phi_*[\overline{\mathsf{Vwt}(2)}] = 0$.*

Harris and Mumford computed the degrees of λ, δ_0 and δ_1 to be, respectively: 1, 12 and -1. Taking degrees on both sides of (5.9), with ϕ taking the role of π, we get the (numerical) relation

$$0 = \int_{\mathbb{P}^1} \phi_* \left[\overline{\mathsf{Vwt}(2)}\right] = (2 + 6g + 9g^2 + 4g^3 + 3g^4) \cdot 1 - a_0 \cdot 12 + b_1 \cdot 1.$$

Given the expression of b_1 computed in (5.11), one obtains

$$a_0 = \frac{1}{6}g(g+1)(2g^2 + g + 3).$$

We have therefore reconstructed the proof of the following result.

THEOREM 5.4 ([26, Theorem 5.1]). *Let $\pi\colon \mathcal{C} \to T$ be a family of stable curves of genus $g \geq 4$ with smooth generic fibre. Then the class in $A^1(T)$ of the locus of points whose fibres possess a special Weierstrass point is*

$$\pi_*\left[\overline{\mathsf{Vwt}(2)}\right] = \left(2+6g+9g^2+4g^3+3g^4\right)\lambda_\pi - \frac{1}{6}g(g+1)(2g^2+g+3)\delta_0 - \sum_{i=1}^{[g/2]}(g^3+3g^2+2g+2)i(g-i)\delta_i. \tag{5.13}$$

Remark 5.3. Let now $[\overline{\mathsf{wt}(2)}]$ be the class in $A^1(T)$ of the locus of points of T corresponding to fibres carrying special Weierstrass points. By Theorem 5.1, for $g \geq 4$ one has

$$\left[\overline{\mathsf{wt}(2)}\right] = \deg(\pi)\left[\pi(\overline{\mathsf{Vwt}(2)})\right] = \left[\pi(\overline{\mathsf{Vwt}(2)})\right] = \pi_*\left[\overline{\mathsf{Vwt}(2)})\right],$$

because $\deg(\pi) = 1$. We may conclude that for $g \geq 4$, the right hand side of (5.13) is the expression of the class $[\overline{\mathsf{wt}(2)}]$.

5.3. **Low genus.** We observe that formula (5.13) holds for genus 1, 2 and 3 as well, and actually recovers classical relations among tautological classes.

5.3.1. *Genus* 1. Recall the elliptic fibration ϵ from (5.12). No member of the pencil (either a smooth or rational plane cubic) possesses Weierstrass points. In particular there are no special Weierstrass points. Then $[\overline{\mathsf{wt}(2)}] = 0$. Setting $g = 1$ in (5.13) one obtains the relation

$$12\lambda - \delta_0 = 0, \tag{5.14}$$

expressing the classical fact that $\epsilon\colon \mathcal{S} \to \mathbb{P}^1$ has 12 irreducible nodal fibres. Indeed, the degree of λ on this pencil is 1, as the relative dualising sheaf restricted to the section $\Sigma_1 \subset \mathcal{S}$ is $\mathscr{O}_{\mathcal{S}}(-\Sigma_1)|_{\Sigma_1}$, which has degree $-\Sigma_1^2 = 1$.

5.3.2. *Genus* 2. A curve of genus 2 is hyperelliptic: it is a ramified double cover of the projective line. The Riemann–Hurwitz formula gives 6 ramification points which are the Weierstrass points. All these ramification points are simple. This means that if $\mathcal{C} \to T$ is a family of curves of genus 2, then

$$0 = \left[\overline{\mathsf{wt}(2)}\right] = 130\lambda - 13\delta_0 - 26\delta_1. \tag{5.15}$$

This recovers the well known relation $10\lambda - \delta_0 - 2\delta_1 = 0$, discussed in [43], showing that the classes $\lambda, \delta_0, \delta_1$ are not independent in $\mathrm{Pic}(\overline{M}_2) \otimes \mathbb{Q}$. See [5] for the generalisation and [20] for the interpretation of the Cornalba and Harris formula generalising (5.15) in the rational Picard group of moduli spaces of stable hyperelliptic curves.

5.3.3. *Genus* 3. In genus 3 the hyperelliptic locus is contained in $\mathrm{V}\mathsf{wt}(2)$. Since each hyperelliptic curve of genus 3 has 8 Weierstrass points, the map π restricted to it has degree greater than 1. Since each hyperelliptic Weierstrass point has weight 3, a local check performed carefully in [12] shows that the degree of π restricted to $\mathrm{V}H_3$ is 16. On the other hand it is known (see e. g. [14]) that each genus 3 curve possessing a hyperflex has only one such. So the degree of π restricted to $\mathcal{H}$, the hyperflex locus, is 1 and then for $g = 3$ formula (5.13) can be correctly written as

$$16 \cdot [\overline{H}_3] + [\mathcal{H}] = \left[\overline{\mathsf{wt}(2)}\right] = 452\lambda - 48\delta_0 - 124\delta_1.$$

The calculation $[\overline{H}_3] = 9\lambda - \delta_0 - 3\delta_1$ was already reviewed in Section 2.2. Then, the class of the curves possessing a hyperflex is given by

$$[\mathcal{H}] = 308\lambda - 32\delta_0 - 82\delta_1. \tag{5.16}$$

Example 5.4. Consider a pencil of plane quartic curves with smooth generic fibre. Since it has no reducible fibres, the degree of δ_1 is zero on this family. The degree of δ_0 is 27 while the degree of λ is 3. Then in a pencil of plane quartics one finds precisely $308 \cdot 3 - 32 \cdot 27 = 60$ hyperflexes, as predicted by Proposition 2.1 using the automatic degeneracy formula by Patel and Swaminathan.

6. Further examples and open questions

The purpose of this section is to show how the theory of Weierstrass points on Gorenstein curves may help to interpret some phenomenologies that naturally occur in the geometry and intersection theory of the moduli space of curves.

6.1. The Examples.

Example 6.1. Let $\pi\colon \mathfrak{X} \to S := \operatorname{Spec} \mathbb{C}[\![t]\!]$ be a family of stable curves, such that

(1) $\mathfrak{X}$ is a smooth surface analytically equivalent to $xy - t = 0$,
(2) $\mathfrak{X}_\eta$ is a smooth curve of genus g, and
(3) $\mathfrak{X}_0$ is a stable uninodal curve, union of a smooth curve X of genus $g-1$ intersecting transversally an elliptic curve E at a point A, that is, $\mathfrak{X}_0 = X \cup_A E$.

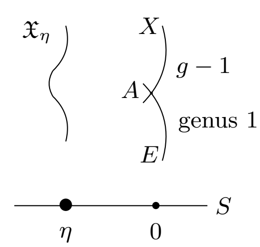


Figure 2. A family of stable curves degenerating to a general member of $\Delta_1 \subset \overline{M}_g$.

One says that $P_0 \in \mathfrak{X}_0 \setminus \{A\}$ is a *limit of a Weierstrass point* if, possibly after a base change, there is a rational section $P\colon S \to \mathfrak{X}$ such that P_η is a Weierstrass point on $\mathfrak{X}_\eta$. The limit of Weierstrass points are very well understood for reducible curves of compact type, by means of many investigations due to Eisenbud, Harris and their school. In fact several classical references (see e.g. [13, 15]) show that

(a) if $P_0 \in E$, then $P_0 \neq A$ is a ramification point of the linear system $\mathscr{O}(gA)$. Applying the Brill–Segre formula (3.4), the total weight $\mathtt{wt}_V$ of the ramification points of the linear system $V = H^0(E, \mathscr{O}(gA))$ is g^2, including the point A. Thus there are at most $g^2 - 1$ Weierstrass points on the smooth generic fibre degenerating to the elliptic component. All the ramification points of V are simple, as one can check via the sequence of dimensions

$$\dim V \geq \dim V(-A) \geq \cdots \geq \dim V(-gA) \geq \dim V(-(g+1)A) = 0.$$

(b) If $P_0 \in X \setminus \{A\}$ is a limit of a Weierstrass point, then it is a ramification point of the linear system $W := H^0(X, K_X(2A))$. Applying the Brill–Segre formula (3.4) once more, by replacing $r+1$ by g and d by $2g-2$, one obtains

$$\mathsf{wt}_W = 2g(g-1) + (g-2)g(g-1) = (g-1)(2g+g^2-2g) = g^2(g-1).$$

The point A contributes with weight $g-1$ (as one easily checks by looking at its vanishing sequence) and thus there are at most $(g-1)^2(g+1)$ Weierstrass points on $\mathfrak{X}_\eta$ degenerating to X.

It follows that no more than

$$(\mathsf{wt}_V - 1) + (\mathsf{wt}_W - g + 1) = \mathsf{wt}_V + \mathsf{wt}_W - g = g^3 - g$$

Weierstrass points on $\mathfrak{X}_\eta$ can degenerate to $\mathfrak{X}_0$. Since the total weight of the Weierstrass points of $\mathfrak{X}_\eta$ is g^3-g, it follows that all the ramification points of the linear systems V and W are indeed limits of Weierstrass points. There are exactly g^2-1 distinct Weierstrass points degenerating on E and a total weight of $(g-1)^2(g+1)$ Weierstrass points on $\mathfrak{X}_\eta$ degenerating on X. Moreover, the counting argument shows that the node A is not a limit. Notice that g^2-1 is the weight of a cuspidal curve of arithmetic genus g, according to Example 4.9. This is not a coincidence.

The situation just described is related to the behavior of a family of smooth genus g curves, degenerating to a cuspidal curve of arithmetic genus g. The relative dualising sheaf coincides with the canonical sheaf on smooth fibres. The Weierstrass points of the smooth fibres degenerate to the Weierstrass points on the special fibre (with respect to the dualising sheaf), including the cusp, and the cusp has weight g^2-1 in the sense of Widland and Lax. Let us now show how to construct a model of the original family contracting the elliptic curve to a cusp. The idea is to consider $\omega_\pi(-X)$, the dualising sheaf twisted by $-X$ (a Cartier divisor, due to the smoothness hypothesis on $\mathfrak{X}$). We have

$$\pi_*\omega_\pi(-X) \otimes \mathbb{C}(0) \cong H^0(\mathfrak{X}_0, \omega_\pi(-X)|_{\mathfrak{X}_0}).$$

Now observe that $h^0(\mathfrak{X}_0, \omega_\pi(-X)|_{\mathfrak{X}_0}) \geq g = h^0(X, \omega_X(2A))$. But the restriction map

$$H^0(\mathfrak{X}_0, \omega_\pi(-X)|_{\mathfrak{X}_0}) \to H^0(X, \omega_X(2A)), \qquad \sigma \mapsto \sigma|_X, \tag{6.1}$$

is injective. Indeed, if $\sigma|_X = 0$ then $\sigma(A) = 0$, that is, $\sigma|_E \in H^0(\mathscr{O}_E(-A)) = 0$. Thus $\sigma = 0$, which implies that the (6.1) is an isomorphism. Now the sheaf $\mathscr{M} := \pi_*\omega_\pi(-X)$ maps the family $\pi\colon \mathfrak{X} \to S$ in $\mathbb{P}(\pi_*\omega_\pi(-X))$, i.e. we have the following diagram:

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "\phi_{\mathscr{M}}"] \arrow[d, "\pi"'] & \mathbb{P}(\pi_*\omega_\pi(-X)) \arrow[dl] \\ S & \end{tikzcd}$$

The generic fibre $\mathfrak{X}_\eta$ is mapped by $\phi_{\mathscr{M}}$ isomorphically onto its canonical image, a geometrically smooth curve of genus g, whereas the special fibre is a cuspidal curve having a cusp in A, and the elliptic component of $\mathfrak{X}_0$ is contracted to A by $\phi_{\mathscr{M}}$. In fact, since the restriction of such a map to E has degree 0, one has $\phi_{\mathscr{M}}(Q) = \phi_{\mathscr{M}}(A)$ for all $Q \in E$. Then there are g^2-1 Weierstrass points degenerating onto the cusp: this number equals the weight of the cusp as a Weierstrass point with respect to the dualising sheaf.

Example 6.2. As another illustration of the same phenomenology, consider the classical case of a pencil of cubics, for instance

$$\mathcal{C}_t: \quad zy^2 - x^3 - tyz^2 = 0.$$

The generic fibre $\mathcal{C}_t$ is smooth. It has 9 flexes, as classically known. But $\mathcal{C}_0$ has only one smooth flex at $F \coloneqq (0 : 1 : 0)$. Thus the remaining flexes collapse to the cusp $P \coloneqq (1 : 0 : 0)$, as is visible by considering the normalisation. The Weierstrass points with respect to the linear system of lines can be detected via the Wronskian determinant by Widland and Lax. It predicts that the cusp has weight 8. The cubic $\mathcal{C}_0$ is the image of the map $(x_0^3, x_0x_1^2, x_1^3)\colon \mathbb{P}^1 \to \mathbb{P}^2$. In the open affine set $x_0 = 1$, it is just the map $t \to (t^2, t^3)$. Notice that $\mathrm{d}\,t$ is a regular differential at P of $\mathbb{A}^1 \subset \mathbb{P}^1$ and then $\sigma \coloneqq \mathrm{d}\,t/t^2$ generates the dualising sheaf at the cusp (where (t^2) is the conductor of $\mathscr{O}_P \subset \widetilde{\mathscr{O}}_P$). One has

$$(t^n)'\sigma \coloneqq \mathrm{d}(t^n) = nt^{n-1}\,\mathrm{d}\,t = nt^{n+1}\frac{\mathrm{d}\,t}{t^2} = nt^{n+1}\sigma$$

from which $(t^n)' = nt^{n+1}$. The Wronskian around the point P is then given by

$$\begin{vmatrix} 1 & t^2 & t^3 \\ 0 & (t^2)' & (t^3)' \\ 0 & (t^2)'' & (t^3)'' \end{vmatrix} = \begin{vmatrix} 1 & t^2 & t^3 \\ 0 & 2t^3 & 3t^4 \\ 0 & 6t^4 & 12t^5 \end{vmatrix} \in t^8 \cdot \mathbb{C}[t].$$

Example 6.3. In [15], Eisenbud and Harris study limits of Weierstrass points on a nodal reducible curve C which is the union of a curve X of genus $g - i$ together with $1 \le i \le g$ elliptic tails, a curve of arithmetic genus g. More precisely, if $\mathfrak{X} \to S$ has smooth generic fibre $\mathfrak{X}_\eta$ and $\mathfrak{X}_0$ is semistably equivalent to C, then each elliptic tail carries $g^2 - 1$ limits of Weierstrass points on nearby smooth curves: these are in turn the ramification points of the linear systems $\mathscr{O}_{E_j}(A_j)$, where A_j is the intersection point $X \cap E_j$. The remaining Weierstrass points of $\mathfrak{X}_\eta$ degenerate on smooth points of X. The theory predicts that if $P_0 \in X$ is a limit of a Weierstrass point $P_\eta \in \mathfrak{X}_\eta$, then it is a ramification point of a linear system $V \in G(g, H^0(K_X(2A_1 + \cdots + 2A_i))$ such that A_i is a base point of $V(-A_1 - \cdots - A_i)$. If $\widehat{X}$ is the i-cuspidal curve got by making each A_j into a cusp, as explained in [50], then $V = \langle \nu^*\omega_1, \ldots, \nu^*\omega_g\rangle$, where $(\omega_1, \ldots, \omega_g)$ is a basis of $H^0(\widehat{X}, \omega_{\widehat{X}})$ and $\nu\colon X \to \widehat{X}$ is the normalisation. This linear system coincides with the one induced by the dualising sheaf of the irreducible curve with i cusps that X normalises.

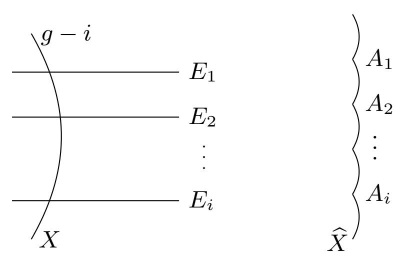


Figure 3. Stable reduction of a degeneration to a cuspidal curve.

Example 6.4. Let C be a smooth complex curve of genus $g - 1 \ge 1$ and let $\widehat{C} \to C$ be a family of cuspidal curves parametrised by C itself contructed as follows. If $Q \in C$ is a point, the fibre $\widehat{C}_Q$ is the cuspidal curve obtained from C by creating a cusp at the point Q, that is, the cuspidal curve associated to the modulus $2Q$ in the sense of [50, p. 61]. In other words, $\widehat{C}_Q$ is the curve such that $\mathscr{O}_{\widehat{C}_Q,P} = \mathscr{O}_{C,P}$ if $P \ne Q$, whilst $\mathscr{O}_{\widehat{C}_Q,Q}$ is the subring of $\mathscr{O}_{C,Q}$ of the regular functions whose derivatives vanish at Q. One wonders which fibres of the family carry special Weierstrass points (with respect to the dualising sheaf) away from the cusp $\{Q\}$. Let $\nu : C \to \widehat{C}_Q$ be the normalisation of $\widehat{C}_Q$. Then $\nu^*\omega_{\widehat{C}_Q} = K_C(2Q)$ and then the special ramification points, but Q, of

$\widehat{C}_Q$ are the special ramification points of the linear system $K_C(2Q)$. For general Q, one cannot expect to find any such point. So, solving the problem amounts to finding the locus SW_1 of all the pairs $(P,Q) \in C \times C$ such that P is a special ramification point of $K_C(2Q)$. The number $N(g)$ of such pairs is obtained by putting $i = 1$ in [8, formula (20)]:

$$N(g) := \int_{C\times C} [\mathrm{SW}_1] = 6g^4 + 14g^3 + 10g^2 - 14g - 16.$$

Notice that $N(1) = 0$, because a rational cuspidal curve of arithmetic genus 1 (i.e. a plane cuspidal cubic) has no hyperflexes.

Example 6.5. Example 6.4 can be interpreted within the geometrical framework of moduli space of stable curves as follows. Let $\mathcal{C} \to X$ be a family such that $\mathcal{C}_Q$ is the curve $X \cup_{Q\sim 0} E$, where $(E, 0)$ is an elliptic curve. Then $P_0 \in X$ is a limit of a special Weierstrass point if and only if it is a special Weierstrass point of the linear system $K_C(2P)$. This fact has been generalised first of all in [7]: if $X \cup_A Y$ is a uninodal stable curve of arithmetic genus g union of a smooth curve of genus i and a smooth curve of genus $g - i$ then $P_0 \in X$ is limit of a special Weierstrass point on $\mathfrak{X}_\eta$ if and only if either P_0 is a ramification point of the linear system $K_X((g_Y + 1)A)$ or P_0 is a ramification point of the linear system $K_X((g_Y + 2)A)$ and A is a Weierstrass point for the component Y. In case Y is an elliptic curve, i.e. without Weierstrass points, the limits on X are solely the ramification points of $K_X(2P)$, as claimed.

Example 6.6. The first example not immediately treated by the theory of Eisenbud and Harris is that of a family $\mathfrak{X} \to S$ of curves of genus 3 such that the special fibre $\mathfrak{X}_0$ is the union of two elliptic curves intersecting transversally at two points A_1 and A_2 (the "banana curve").

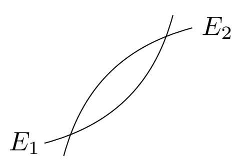


FIGURE 4. The banana curve: an example of a genus 3 curve carrying a 1-parameter family of limits of Weierstrass points.

In this case each point on each component can be limit of Weierstrass points, in the sense that for each point P_0, say in E_1, there exists a smoothing family $\mathfrak{X} \to S$ such that P_0 is limit of a Weierstrass point of a curve of genus 3. All the Weierstrass points distribute themselves in twelve points on E_1 and twelve points on E_2. Esteves and Medeiros prove in [21] that the variety of limit canonical system of the "banana curve" is parametrised by $\mathbb{P}^1$.

Indeed each $P_0 \in E_i$ determines uniquely a point in the pencil of linear systems

$$V \in G(3, H^0(\mathscr{O}(2A_1 + 2A_2))$$

which contains $H^0(\mathscr{O}(A_1 + A_2))$. Thus for each component there is a $12 : 1$ ramified covering $E_i \to \mathbb{P}^1$ and the (fixed) ramification points are the limits of special Weierstrass points on nearby smooth curves. Also this example may be interpreted in terms of the theory of Widland and Lax (see [4] for details). In fact the linear system V_{P_0} defined on E_1 maps E_1 to a plane quartic with a tacnodal singularity ($\delta_A = 2$, local analytic equation $(y - x^2)^2 = 0$) at the coincident images of A_1 and A_2 . Then the limits of Weierstrass points on E_1 are precisely the smooth flexes, while the information about the Weierstrass points degenerating on the other components is lost in the tacnode. Notice that according the theory of Widland and Lax a tacnode must have weight at least $\delta \cdot 3 \cdot 2 = 2 \cdot 3 \cdot 2 = 12$.

6.2. **Open Questions.**

6.2.1. *Porteous Formula with excess.* Consider the loci

$$\begin{aligned}\mathsf{wt}(2) &:= \{\, [C] \in M_g \mid C \text{ has a special Weierstrass point} \,\}\,,\\ \mathbb{D}_{g-1} &:= \{\, [C] \in M_g \mid C \text{ has a special Weierstrass point of type } g-1 \,\}\,,\\ \mathbb{D}_{g+1} &:= \{\, [C] \in M_g \mid C \text{ has a special Weierstrass point of type } g+1 \,\}\,.\end{aligned}$$

Although $\mathsf{wt}(2)$ is clearly equal to the set-theoretic union $\mathbb{D}_{g-1} \cup \mathbb{D}_{g+1}$, it is not obvious that

$$\left[\overline{\mathsf{wt}(2)}\right] = \left[\overline{\mathbb{D}}_{g-1}\right] + \left[\overline{\mathbb{D}}_{g+1}\right].$$

This is the main result of [26]. Within the general framework discussed in Section 5, consider the maps of vector bundles

$$\begin{tikzcd}[column sep=small] \pi^*\mathbb{E}_\pi \arrow[rr, "\mathsf{D}^{g-2}"] \arrow[dr, dash] & & J^{g-2}_\pi\omega_\pi \arrow[dl, dash] \\ & \mathcal{C} \arrow[d] & \\ & T & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] \pi^*\mathbb{E}_\pi \arrow[rr, "\mathsf{D}^{g}"] \arrow[dr, dash] & & J^{g}_\pi\omega_\pi \arrow[dl, dash] \\ & \mathcal{C} \arrow[d] & \\ & T & \end{tikzcd}$$

The loci $\overline{\mathbb{D}}_{g-1}$ and $\overline{\mathbb{D}}_{g+1}$ are in fact in the degeneracy loci of the above maps; however these maps degenerate identically along the special singular fibre which are divisors of $\mathcal{C}$. So, to compute the class of the loci of $\overline{\mathbb{D}}_{g-1}$ and $\overline{\mathbb{D}}_{g+1}$ one should dispose of a Porteous formula with excess, generalising the residual formula for top Chern classes as in [23, Example 14.1.4]. To our knowledge, such formulas are not known up to now.

6.2.2. *Computing automatic degeneracies.* It is an interesting problem, already raised in [46], to compute the function $\mathrm{AD}^m(f)$ of automatic degeneracies (as discussed in Section 2.1.1) for more complicated plane curve singularities than the node. Some results for low values of m have already been obtained in *loc. cit.* For instance it would be very useful to be able to determine the function $\mathrm{AD}(f)$ for cusps, ordinary triple points, tacnodes.

6.2.3. *Porteous formula for Coherent sheaves.* To study situations like 6.2.1 but avoiding the locally free replacement of the principal parts, S. Diaz proposed in [14] a Porteous formula for maps of coherent sheaves. This was a question asked by Harris and Morrison in [30]. The purpose is that of getting rid of two issues at once: excess contributions, and the lack of local freeness of principal parts of the dualising sheaf at singularities. Diaz's theory is nice and elegant. However the main example he proposes is the computation of the hyperelliptic locus in genus 3, which Esteves computed as sketched in Section 2.2, again using locally free substitute of principal parts. It would be interesting to work out more examples to extract all the potential of Diaz' extension of Porteous' formula for coherent sheaves.

6.2.4. *Dimension estimates.* Recall the definition (5.1) of $\mathsf{wt}(k)$. In [27] it is proven that for $g \geq 4$ the locus $\mathsf{wt}(3)$ of curves possessing a special Weierstrass point of weight at least 3 has the expected codimension 2. It is a hard problem to determine the irreducible components of $\mathsf{wt}(k)$ and their dimensions. For instance Eisenbud and Harris prove that if $k \leq [g/2]$ then $\mathsf{wt}(k)$ has at least one irreducible component of the expected codimension k. In general, however, the problem is widely open. It would be natural to conjecture that $\mathsf{wt}(k) \subset M_g$ has the expected codimension k if $g \gg 0$, but there is really no rigorous evidence to support such a guess.

6.2.5. *Computing new classes.* Only a handful of classes of geometrically defined loci of higher codimension in $\overline{M}_g$ have been computed. For instance Faber and Pandharipande have determined the class of the hyperelliptic locus in $\overline{M}_4$ via stable maps [22]. Let $\mathcal{C} \to S$ be a family of stable curves of genus $g \geq 5$ parametrised by a smooth complete surface S. Many singular fibres belonging to boundary strata of $\overline{M}_g$ of higher codimension can occur. If $\pi\colon \mathfrak{X} \to S$ is a family of stable curves of genus 4 parameterised by a complete scheme of dimension at least 2, then Faber and Pandharipande are able to compute the locus of points in S corresponding to hyperelliptic fibres. Esteves and Abreu (private communication) are able to compute the class $[\overline{H}_4]$ using the same method we discussed in Section 2.2. However it seems a hard problem to determine the class in $A_{3g-5}(\overline{M}_g)$ (already for $g = 4$) of the locus $\overline{\mathsf{wt}(3)}$. This would be the push forward of the third Chern class of

$$J^2_\pi\left(\omega_\pi^{g(g+1)/2} \otimes \bigwedge^g \mathbb{E}_\pi^\vee\right),$$

where J^2_π is the locally free replacement constructed in the previous sections. Unfortunately, one has no control on the degree of the restriction of π to the irreducible components of $\overline{\mathsf{Vwt}(3)}$. In genus 4 this locus should contain, with some multiplicity, the hyperelliptic locus, the (nonempty) locus of curves possessing a Weierstrass point with gap sequence $(1, 2, 3, 7)$ and the (nonempty) locus of curves possessing a Weierstrass point with gap sequence $(1, 2, 4, 7)$. These loci all have the expected codimension 2 (by [37]), but as far as we know their multiplicities in $\overline{\mathsf{wt}(3)}$ are not known.

References

1. Ragnar-Olaf Buchweitz and Gert-Martin Greuel, *The Milnor number and deformations of complex curve singularities*, Invent. Math. **58** (1980), no. 3, 241–281. DOI: 10.1007/bf01390254
2. André Contiero and Karl-Otto Stöhr, *Upper bounds for the dimension of moduli spaces of curves with symmetric Weierstrass semigroups*, J. Lond. Math. Soc. (2) **88** (2013), no. 2, 580–598. DOI: 10.1112/jlms/jdt034
3. Marc Coppens, *The number of Weierstrass points on some special curves. I*, Arch. Math. (Basel) **46** (1986), no. 5, 453–465. DOI: 10.1007/bf01210786
4. Marc Coppens and Letterio Gatto, *Limit Weierstrass schemes on stable curves with 2 irreducible components*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **12** (2001), 205–228 (2002). DOI: 10.4171/rlm
5. Maurizio Cornalba and Joe Harris, *Divisor classes associated to families of stable varieties, with applications to the moduli space of curves*, Annales scientifiques de l'École normale supérieure **21** (1988), no. 3, 455–475. DOI: 10.24033/asens.1564
6. Fernando Cukierman, *Families of Weierstrass points*, Duke Math. J. **58** (1989), no. 2, 317–346. DOI: 10.1215/s0012-7094-89-05815-8
7. Caterina Cumino, Eduardo Esteves, and Letterio Gatto, *Limits of special Weierstrass points*, Int. Math. Res. Pap. IMRP (2008), no. 2, Art. ID rpn001, 65.
8. ______, *Special ramification loci on the double product of a general curve*, Q. J. Math. **59** (2008), no. 2, 163–187. DOI: 10.1093/qmath/ham032
9. Jan Denef and François Loeser, *Germs of arcs on singular algebraic varieties and motivic integration*, Inventiones Mathematicae **135** (1999), no. 1, 201–232. DOI: 10.1007/s002220050284
10. ______, *Geometry on arc spaces of algebraic varieties*, 3rd European congress of mathematics (ECM), Barcelona, Spain, July 10–14, 2000. Volume I, Basel: Birkhäuser, 2001, pp. 327–348. DOI: 10.1007/978-3-0348-8268-2_19
11. Steven Diaz, *Tangent spaces in moduli via deformations with applications to Weierstrass points*, Duke Math. J. **51** (1984), no. 4, 905–922. DOI: 10.1215/s0012-7094-84-05140-8
12. ______, *Deformations of exceptional Weierstrass points*, Proc. Am. Math. Soc. **96** (1986), 7–10.
13. ______, *Exceptional Weierstrass points and the divisor on moduli space that they define*, Mem. Amer. Math. Soc. **56** (1986), 7–10.
14. ______, *Porteous's formula for maps between coherent sheaves*, Michigan Math. J. **52** (2004), no. 3, 507–514. DOI: 10.1307/mmj/1100623410

15. David Eisenbud and Joe Harris, *Existence, decomposition, and limits of certain Weierstrass points*, Invent. Math. **87** (1987), no. 3, 495–515. DOI: 10.1007/bf01389240
16. ______, *3264 and all that – a second course in Algebraic Geometry*, Cambridge University Press, Cambridge, 2016. DOI: 10.1017/cbo9781139062046
17. Fouad El Zein, *Complexe dualisant et applications à la classe fondamentale d'un cycle*, Bull. Soc. Math. France Mém. (1978), no. 58, 93. DOI: 10.24033/msmf.242
18. Eduardo Esteves, *Wronski algebra systems on families of singular curves*, Ann. Sci. École Norm. Sup. (4) **29** (1996), no. 1, 107–134. DOI: 10.24033/asens.1736
19. ______, *The stable hyperelliptic locus in genus 3: an application of Porteous formula*, J. Pure Appl. Algebra **220** (2016), no. 2, 845–856. DOI: 10.1016/j.jpaa.2015.07.020
20. Eduardo Esteves and Letterio Gatto, *A geometric interpretation and a new proof of a relation by Cornalba and Harris*, Comm. Algebra **31** (2003), no. 8, 3753–3770, Special issue in honor of Steven L. Kleiman. DOI: 10.1081/agb-120022441
21. Eduardo Esteves and Nivaldo Medeiros, *Limit canonical systems on curves with two components*, Inventiones Mathematicae **149** (2002), no. 2, 267–338. DOI: 10.1007/s002220200211
22. Carel Faber and Rahul Pandharipande, *Relative maps and tautological classes*, Journal of the European Mathematical Society (2005), 13–49. DOI: 10.4171/jems/20
23. William Fulton, *Intersection theory*, Springer, 1984.
24. Arnaldo García and Robert F. Lax, *Weierstrass weight of Gorenstein singularities with one or two branches*, Manuscripta Math. **81** (1993), no. 3-4, 361–378. DOI: 10.1007/bf02567864
25. Letterio Gatto, *Weight sequences versus gap sequences at singular points of Gorenstein curves.*, Geom. Dedicata **54** (1995), no. 3, 267–300 (English). DOI: 10.1007/bf01265343
26. ______, *On the closure in $\overline{M}_g$ of smooth curves having a special Weierstrass point.*, Math. Scand. **88** (2001), no. 1, 41–71 (English). DOI: 10.7146/math.scand.a-14313
27. Letterio Gatto and Fabrizio Ponza, *Derivatives of Wronskians with applications to families of special Weierstrass points*, Trans. Amer. Math. Soc. **351** (1999), no. 6, 2233–2255.
28. Alexandre Grothendieck, *Éléments de géométrie algébrique. IV. étude locale des schémas et des morphismes de schémas IV*, Inst. Hautes Études Sci. Publ. Math. (1967), no. 32, 361. DOI: 10.1007/bf02684747
29. Lars Halvard Halle and Johannes Nicaise, *Motivic zeta functions for degenerations of abelian varieties and Calabi-Yau varieties*, Zeta functions in algebra and geometry, Contemp. Math., vol. 566, Amer. Math. Soc., Providence, RI, 2012, pp. 233–259. DOI: 10.1090/conm/566/11223
30. Joe Harris and Ian Morrison, *Moduli of curves*, Graduate Texts in Mathematics, vol. 187, Springer-Verlag, New York, 1998.
31. Joe Harris and David Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math. **67** (1982), no. 1, 23–88, With an appendix by William Fulton. DOI: 10.1007/bf01393371
32. Maxim Kontsevich, *Lecture at Orsay*, December 1995.
33. Dan Laksov, *Weierstrass points on curves*, Young tableaux and Schur functors in algebra and geometry (Toruń, 1980), Astérisque, vol. 87, Soc. Math. France, Paris, 1981, pp. 221–247.
34. Dan Laksov and Anders Thorup, *Weierstrass points and gap sequences for families of curves*, Ark. Mat. **32** (1994), no. 2, 393–422. DOI: 10.1007/bf02559578
35. ______, *The algebra of jets*, Michigan Math. J. **48** (2000), 393–416, Dedicated to William Fulton on the occasion of his 60th birthday. DOI: 10.1307/mmj/1030132726
36. ______, *Wronski systems for families of local complete intersection curves*, Comm. Algebra **31** (2003), no. 8, 4007–4035, Special issue in honor of Steven L. Kleiman. DOI: 10.1081/agb-120022452
37. Robert F. Lax, *Weierstrass points of the universal curve*, Math. Ann. **216** (1975), 35–42. DOI: 10.1007/bf02547970
38. ______, *On the distribution of Weierstrass points on singular curves*, Israel J. Math. **57** (1987), no. 1, 107–115. DOI: 10.1007/bf02769464
39. Qing Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, Oxford, 2002, Translated from the French by Reinie Erné, Oxford Science Publications.
40. Eduard Looijenga, *Motivic measures*, Séminaire Bourbaki. Volume 1999/2000. Exposés 865–879, Paris: Société Mathématique de France, 2002, pp. 267–297, ex.
41. Renato Vidal Martins, *Trigonal non-Gorenstein curves*, Journal of Pure and Applied Algebra **209** (2007), no. 3, 873–882. DOI: 10.1016/j.jpaa.2006.08.010
42. David Mumford, *Stability of projective varieties*, Enseignement Math. (2) **23** (1977), no. 1-2, 39–110.
43. ______, *Towards an enumerative geometry of the moduli space of curves*, pp. 271–328, Birkhäuser Boston, Boston, MA, 1983. DOI: 10.1007/978-1-4757-9286-7_12
44. John F. Nash, *Arc structure of singularities*, Duke Math. J. **81** (1995), no. 1, 31–38 (1996). MR 1381967

45. Marco Pacini and Damiano Testa, *Recovering plane curves of low degree from their inflection lines and inflection points*, Israel J. Math. **195** (2013), no. 1, 283–316. DOI: 10.1007/s11856-012-0129-6
46. Anand Patel and Ashvin Swaminathan, *Inflectionary Invariants for Isolated Complete Intersection Curve Singularities*, 2019. arχiv: 1705.08761v3
47. Ragni Piene, MR1038736, Review of *Weierstrass Points on Goresntein Curves,* by C. Widland and R. F. Lax, `https://mathscinet.ams.org/mathscinet/pdf/1038736.pdf`, 2017.
48. Ziv Ran, *Tautological module and intersection theory on Hilbert schemes of nodal curves*, Asian J. Math. **17** (2013), no. 2, 193–263. DOI: 10.4310/ajm.2013.v17.n2.a1
49. Maxwell Rosenlicht, *Equivalence relations on algebraic curves*, Ann. of Math. (2) **56** (1952), 169–191.
50. Jean-Pierre Serre, *Algebraic groups and class fields*, Graduate Texts in Mathematics, vol. 117, Springer-Verlag, New York, 1988, Translated from the French.
51. Carl Widland, *Weierstrass points on Gorenstein curves*, ProQuest LLC, Ann Arbor, MI, 1984, Thesis (Ph.D.) – Louisiana State University and Agricultural & Mechanical College.
52. Carl Widland and Robert F. Lax, *Gap sequences at a singularity*, Pacific J. Math. **150** (1990), no. 3, 111–122. DOI: 10.2140/pjm.1991.150.111
53. ———, *Weierstrass points on Gorenstein curves*, Pacific J. Math. **150** (1990), no. 3, 111–122. DOI: 10.2140/pjm.1990.142.197

Letterio Gatto, Politecnico di Torino
Email address: `letterio.gatto@polito.it`

Andrea T. Ricolfi, Max-Planck-Institut für Mathematik
Email address: `atricolfi@mpim-bonn.mpg.de`

Journal *of* Singularities
Volume 21 (2020), 84-96

Proc. of Geometric and Algebraic Singularity Theory, Będlewo, 2017
DOI: 10.5427/jsing.2020.21e

REFLEXION MAPS AND GEOMETRY OF SURFACES IN $\mathbb{R}^4$

P.J. GIBLIN, S. JANECZKO, AND M.A.S. RUAS

Abstract. In this article we introduce new affinely invariant points—'special parabolic points'—on the parabolic set of a generic surface M in real 4-space, associated with symmetries in the 2-parameter family of reflexions of M in points of itself. The parabolic set itself is detected in this way, and each arc is given a sign, which changes at the special points, where the family has an additional degree of symmetry. Other points of M which are detected by the family of reflexions include inflexion points of real and imaginary type, and the first of these is also associated with sign changes on the parabolic set. We show how to compute the special points globally for the case where M is given in Monge form and give some examples illustrating the birth of special parabolic points in a 1-parameter family of surfaces. The tool we use from singularity theory is the contact classification of certain symmetric maps from the plane to the plane and we give the beginning of this classification, including versal unfoldings which we relate to the geometry of M.

1. Introduction

In a previous article [6] the first two authors studied families of local reflexion maps on surfaces in $\mathbb{R}^3$ and their bifurcation sets, in particular showing that certain special parabolic points, not related to the flat geometry of the surface, are detected by the structure of the corresponding bifurcation set. These special parabolic, or A_2^* points, arose also in earlier work on centre symmetry sets of surfaces [7]. Although the definition of the reflexion maps is local the bifurcation sets could be extended over the whole surface, producing curves connecting the special parabolic points. In this article we extend some of these results to surfaces in $\mathbb{R}^4$, again studying local reflexions and bifurcation sets of familites of contact maps. In the present situation we need to study the contact between two surfaces in $\mathbb{R}^4$ and this is measured by a map (germ) $\mathbb{R}^2, 0 \to \mathbb{R}^2, 0$. The appropriate equivalence relation to measure contact is $\mathcal{K}$-equivalence (see [10]) and therefore the bifurcation set of a family of contact maps must be constructed according to this equivalence relation, taking into account the inherent $\mathbb{Z}_2$-symmetry of the contact maps.

We find new 'special parabolic points' on a surface in $\mathbb{R}^4$, which are of two types, 'elliptic' and 'hyperbolic', and are in some ways analogues of the special parabolic points encountered in $\mathbb{R}^3$; the local structure of the bifurcation sets is also similar to the 3-dimensional case. For a surface in $\mathbb{R}^4$ however there are more special kinds of points and the bifurcation set of our family of contact functions displays different structures at these. We have not so far found a natural interpretation of a *global* bifurcation set, connecting special parabolic points and other points through the hyperbolic and elliptic regions of the surface.

In §2 we derive the family of reflexion maps and explain our interpretation of the bifurcation set of such a family. The abstract classification which we need is given in Theorem 3.1 and the application to surfaces in $\mathbb{R}^4$ occupies the remainder of §3. We find the bifurcation set germ at parabolic points, at the two types of special parabolic points, and at inflexion points of real and imaginary type. In particular we show that arcs of the parabolic set between these various

2010 *Mathematics Subject Classification.* 52A05, 57R45.

special points can be given a sign, which changes in a well-defined way at the special points. Identifying the local structure of the bifurcation sets requires that we are able to check versal unfolding conditions and we give the criteria for these to hold in each case.

The above calculations are done with a surface M in Monge form at the origin. In §4 we show how to compute the special parabolic points on a whole surface given in Monge form. The special parabolic points are found as the intersection of the parabolic set with another curve in M and we find an explicit formula for this curve, given in Appendix A but applied to several examples in §4. An example, adapted from [4], shows the birth of special parabolic points on a loop of the parabolic set created in a generic 1-parameter family of surfaces—an elliptic island in a hyperbolic sea. Immediately after the moment that the island appears it has no special parabolic points but two of these, of the same type, can be born as the island grows larger. Between the two the sign of the parabolic set changes.

Finally in §5 we give some concluding remarks and open problems.

2. Families of contact maps

Consider a surface M in $\mathbb{R}^4$, with coordinates (a,b,c,d), parametrized by

$$\gamma(x,y) = (f(x,y), g(x,y), x, y),$$

where we shall assume that the 1-jets of f and g at $(x,y)=(0,0)$ are zero. Let (p,q) be the parameters of a fixed point on the surface. Reflecting a point $\gamma(p+x,q+y)$ of M in the point $\gamma(p,q)$ gives $2\gamma(p,q)-\gamma(p+x,q+y)$, so that reflecting M in $\gamma(p,q)$ gives the surface M^* through $\gamma(p,q)$ with parametrization $\mathbb{R}^2 \to \mathbb{R}^4$:

$$(x,y) \mapsto (2f(p,q) - f(p+x,q+y),\ 2g(p,q) - g(p+x,q+y),\ p-x,\ q-y).$$

Thus $x = y = 0$ returns the point $\gamma(p,q)$. Composing this parametrization with the map $\mathbb{R}^4 \to \mathbb{R}^2$ defined by $(a,b,c,d) \mapsto (f(c,d)-a,\ g(c,d)-b)$, for which the inverse image of $(0,0)$ is equal to M, gives the following map (germ) $F_{(p,q)} : \mathbb{R}^2,(0,0) \to \mathbb{R}^2,(0,0)$, whose $\mathcal{K}$-class measures the contact between M and M^* at $\gamma(p,q)$ (see [10]).

$$\begin{aligned} F_{(p,q)}(x,y) &= (f(p+x,q+y) + f(p-x,q-y) - 2f(p,q), \\ &\quad\ g(p+x,q+y) + g(p-x,q-y) - 2g(p,q)). \end{aligned} \tag{1}$$

When we include the parameters p,q we write $F(x,y,p,q)$. Note that

$$F(x,y,p,q) \equiv F(-x,-y,p,q):$$

for each (p,q) the map $F_{(p,q)}$ is symmetric with respect to the reflexion $(x,y) \to (-x,-y)$.

Thus F is a family of symmetric mappings $\mathbb{R}^2 \to \mathbb{R}^2$, with variables x,y parametrized by p,q. We investigate the bifurcation set of this family, the fundamental definition of which is

$$\begin{aligned} \mathcal{B}_F \quad = \quad \{(p,q): &\text{ there exist } (x,y) \text{ such that } F_{(p,q)} \text{ has an unstable} \\ &\text{singularity at } x,y \text{ with respect to } \mathcal{K} \text{ equivalence} \\ &\text{of maps symmetric in the above sense}\}. \end{aligned}$$

In [6] the corresponding bifurcation set of a family F of real-valued *functions* was analysed by studying the critical set of F. Here we need to work directly with $\mathcal{K}$-equivalence of maps, where the critical set does not play so significant a role, and we adopt a different approach.

At $(p,q)=(0,0)$ the contact map is

$$F_{(0,0)}(x,y) = (f(x,y) + f(-x,-y),\ g(x,y) + g(-x,-y)), \tag{2}$$

which is twice the even part of (f,g), but we shall sometimes ignore the factor 2. Thus the conditions on M needed for the classification of $F_{(0,0)}$ involve only the *even degree* terms of f,g;

however the conditions for the family F with parameters p, q to give a $\mathcal{K}$-versal unfolding will involve also the odd degree terms.

We work within the set of maps $h : \mathbb{R}^2 \to \mathbb{R}^2$ which are symmetric by reflexion in the origin: $h(x, y) = h(-x, -y)$. To do this we use the basis $u = x^2, v = xy, w = y^2$ for all functions of two variables which are symmetric with respect to this symmetry and study map germs $H : \mathbb{R}^3 \to \mathbb{R}^2$ with coordinates (u, v, w) in $\mathbb{R}^3$, up to $\mathcal{K}$-equivalence preserving the homogeneous variety (cone) $V : v^2 = uw$. (In fact for us this is a half-cone since $u = x^2$ and $w = y^2$ are non-negative, but for classification purposes we may assume that the whole cone is preserved.) We write ${}_V\mathcal{K}$-equivalence for this equivalence of germs $H : \mathbb{R}^3, (0, 0, 0) \to \mathbb{R}^2, (0, 0)$. We shall work with ${}_V\mathcal{K}$-versal unfoldings and construct bifurcation diagrams for these in a sense we now explain.

For a given germ H, the ${}_V\mathcal{K}$ equivalence will preserve the intersection $H^{-1}(0) \cap V$ up to local diffeomorphism of $\mathbb{R}^3$, and indeed will preserve the multiplicity of intersection of the curve $H^{-1}(0)$ with the cone V. As the map H varies in a family the multiplicity will change and furthermore intersection points of multiplicity > 1 may move away from the origin; these points nevertheless form part of the 'contact data' of $H^{-1}(0)$ and V since they represent unstable mappings. Except in one case, described below, all the contact data are concentrated at the origin.

Definition 2.1. *The strata of our bifurcation set are those points in the versal unfolding space for which the contact data consisting of the multiplicity of contact between $H^{-1}(0, 0)$ and V in an arbitrarily small neighbourhood of the origin in $\mathbb{R}^3$ are constant.*

The idea is best illustrated by an example, which will arise in §3.5 below. Consider the family of maps $H_{\lambda,\mu}(u, v, w) = (v, u - w^3 + \lambda w + \mu w^2)$. For any (λ, μ), $H^{-1}_{\lambda,\mu}(0)$ lies in the plane $v = 0$ with coordinates (u, w), and $V : v^2 = uw$ intersects this plane in the two lines $u = 0, w = 0$ (for real solutions for x, y we require indeed $u \geq 0$ and $w \geq 0$). We therefore examine how the curve $u - w^3 + \lambda w + \mu w^2 = 0$ in the (u, w) plane meets the two coordinate axes. Intersection with the axis $w = 0$ gives only the origin. Intersection with the axis $u = 0$ requires $w(-w^2 + \mu w + \lambda) = 0$ which gives tangency at the origin when $\lambda = 0$, so that in the (λ, μ) plane the axis $\lambda = 0$, apart from the origin, is one stratum of the bifurcation set. The total contact between $H^{-1}_{\lambda,\mu}(0, 0)$ and V at the origin is 3. The origin $\lambda = \mu = 0$ is a separate stratum since the contact there between $H^{-1}_{\lambda,\mu}(0, 0)$ and V is 4. There is also a double root of $-w^2 + \mu w + \lambda = 0$ at $w = \frac{1}{2}\mu$ when $\mu^2 + 4\lambda = 0$, resulting in ordinary tangency between $H^{-1}_{\lambda,\mu}(0, 0)$ and V at $(u, w) = (0, \frac{1}{2}\mu)$. This gives a stratum $\mu^2 + 4\lambda = 0$ of the bifurcation set, with $\mu \geq 0$ since $w = y^2 \geq 0$, which intersects every neighbourhood of $(0, 0)$ in the plane of the unfolding parameters (λ, μ). The various possibilities are sketched in Figure 1 where the intersection number between $H_{\lambda,\mu} = 0$ and V is indicated against each intersection point. For real solutions (x, y) we require these intersection points to be in the quadrant $u \geq 0, w \geq 0$ of the (u, w) plane. The resulting bifurcation set is also drawn in Figure 1, with four strata of positive codimension in the (λ, μ) plane.

3. Classification of the contact maps up to ${}_V\mathcal{K}$-equivalence

We consider map germs $\mathbb{R}^3 \to \mathbb{R}^2$, with coordinates u, v, w in the source ($u = x^2$, $v = xy$, $w = y^2$ as above), under contact equivalence which preserves the homogeneous variety $V : uw - v^2 = 0$. Vector fields generating those tangent to this variety are given by the Euler vector field and the three hamiltonian vector fields:

$$u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w},\ 2v\frac{\partial}{\partial u} + w\frac{\partial}{\partial v},\ u\frac{\partial}{\partial v} + 2v\frac{\partial}{\partial w},\ u\frac{\partial}{\partial u} - w\frac{\partial}{\partial w}. \tag{3}$$

The tangent space to the ${}_V\mathcal{K}$ orbit at $H(u, v, w)$ is $dH(\theta_V) + H^*(m_2)\mathcal{E}_3^2$, where dH is the jacobian matrix of H and θ_V is the $\mathcal{E}_3$ module generated by the above vector fields.

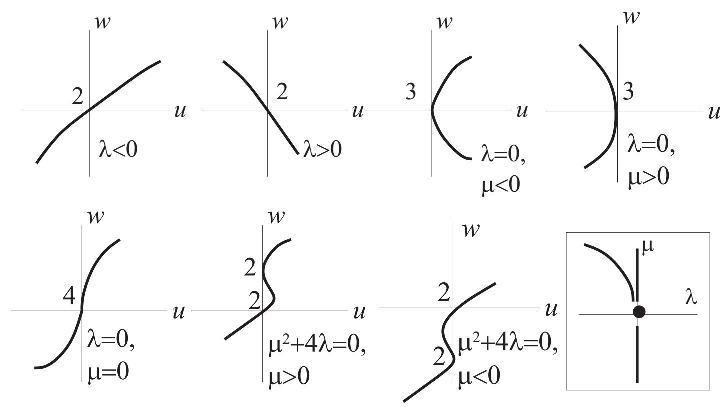


FIGURE 1. The unstable intersections between the curve $v = 0,\ u = w^3 - \lambda w - \mu w^2$ and the cone $V : v^2 = uw$ for various values of λ, μ. These give 0- and 1-dimensional strata of the bifurcation set of the family $H(u, v, w) = (v, u - w^3 + \mu w + \lambda w^2)$, shown in the boxed diagram at bottom right. Intersections corresponding to real values of (x, y) are in the quadrant $u \geq 0, w \geq 0$ of the u, w-plane.

The classification which we need is summarized in Theorem 3.1, which is proved by the method of complete transversals [3] and the finite determinacy theorem for ${}_V\mathcal{K}$ equivalence [5]. Comments on this classification and application to our geometrical situation are in the remainder of this section. (We remark here that a different but related classification of maps involving only odd degree terms is obtained in [9].)

Theorem 3.1. *The abstract classification of map germs $H : \mathbb{R}^3 \to \mathbb{R}^2$ up to $\mathcal{K}$-equivalence preserving the half-cone $V : v^2 - uw = 0,\ u \geq 0,\ w \geq 0$ starts with the classes given in Table 1. The classes of symmetric germs $h : \mathbb{R}^2 \to \mathbb{R}^2$, where $h(x, y) = h(-x, -y)$, up to $\mathcal{K}$-equivalence preserving the symmetry are obtained by replacing u, v, w by x^2, xy, y^2 respectively.* □

type	*normal form*	${}_V\mathcal{K}$ *codimension*	*versal unfolding*	*geometry*
(H)	(w, u)	*0*	—	*hyperbolic point*
(E)	$(u - w, v)$	*0*	—	*elliptic point*
(P)	$(v, u \pm w^2)$	*1*	$(0, \lambda w)$	*ordinary parabolic point*
(SP)	$(v, u \pm w^3)$	*2*	$(0, \lambda w + \mu w^2)$	*special parabolic point*
(IR)	$(v, u^2 + 2buw \pm w^2)$ $b^2 \neq 1$ *for* $+$	*3*	$(0, buw + \lambda u + \mu w)$	*inflexion of real type*
(II)	$(u + w, ku^2 + uv)$ *or* $(u + w, uv + kv^2)$	*3* *3*	$(0, ku^2 + \lambda u + \mu v)$ $(0, kv^2 + \lambda u + \mu v)$	*inflexion of imaginary type*

TABLE 1. The lowest codimension singularities in the ${}_V\mathcal{K}$ classification of map germs $\mathbb{R}^3, (0, 0, 0) \to \mathbb{R}^2, (0, 0)$.

We shall see that the moduli b and k in the normal forms above do not affect the geometry of the situation. Note that the two forms $(v, u \pm w^2)$ are not equivalent since $u \geq 0$ so we cannot replace u by $-u$. The same applies to the two forms $(v, u \pm w^3)$. Note that the germs (P) and (SP) are the first two in a sequence $(v, u \pm w^k),\ k \geq 2$, distinguished by the contact between the zero-set of the germ and the cone $V : v^2 - uw = 0$.

The contact maps are invariant under affine transformations of the space $\mathbb{R}^4$ in which our surface M lies, so that we may first put M in a standard form at the origin in (a,b,c,d)-space. We can assume the tangent plane at the origin is the (c,d)-plane and the quadratic terms $f_2,\ g_2$ of $f,\ g$ are reduced by the action of $GL(2,\mathbb{R})\times GL(2,\mathbb{R})$ on pairs of binary quadratic forms to a standard form. Finally a linear transformation of $\mathbb{R}^4$ reparametrizes M as $(x,y)\mapsto(f,g,x,y)$ where now f and g have their quadratic parts in standard form. See for example [4, pp. 182–183] for the standard forms of 2-jets of surfaces in $\mathbb{R}^4$.

There is a convenient way to recognize the types (P) and (SP) of the contact map

$$(u,v,w)\mapsto(C_1(u,v,w),C_2(u,v,w)),$$

which will be useful below.

Lemma 3.2. *In each case the zero set $C_1=C_2=0$ in $\mathbb{R}^3$ is a smooth curve at the origin and (P): has exactly 2-point contact (ordinary tangency) with the cone $V:v^2=uw$ at the origin, (SP): has exactly 3-point contact with the cone V at the origin.* □

3.1. **First stable case: hyperbolic point.** A standard form for the 2-jet of the surface at a hyperbolic point is $(y^2,\ x^2,\ x,\ y)$, or in a less reduced form $(f_{11}xy+f_{02}y^2,\ g_{20}x^2,\ x,\ y)$ where $f_{02}\neq0, g_{20}\neq0$. The contact map at the origin of $\mathbb{R}^4$, ignoring the factor 2 in (2), has 1-jet $F_1=(f_{11}v+f_{02}w,\ g_{20}u)$ (or just (w,u) in the reduced form). This is ${}_V\mathcal{K}$-stable and is the case where the kernel of the linear map $F_1:\mathbb{R}^3\to\mathbb{R}^2$ intersects the cone $V\subset\mathbb{R}^3$ only in the origin. The bifurcation set germ is empty.

3.2. **Second stable case: elliptic point.** A standard form for the 2-jet of the surface is $(x^2-y^2,\ xy,\ x,\ y)$, or in a less reduced form $(f_{20}x^2+f_{02}y^2,\ g_{11}xy,x,y)$, $f_{20}f_{02}<0, g_{11}\neq0$ as in [4]. This corresponds to 1-jet $F_1=(f_{20}u+f_{02}w,\ g_{11}v)$ (or $(u-w,\ v)$ in reduced form). This is ${}_V\mathcal{K}$-stable and it is the case where the kernel of the linear map $F_1:\mathbb{R}^3\to\mathbb{R}^2$ intersects the cone V in two distinct generators. The bifurcation set germ is empty.

3.3. **Codimension 1 case: ordinary parabolic point.** A standard form of the 2-jet of M at a parabolic point, up to affine transformations of $\mathbb{R}^4$, is

$$(f_{11}xy,\ g_{20}x^2,\ x,\ y),$$

where $f_{11}\neq0,\ g_{20}\neq0$. The corresponding 1-jet in (u,v,w) coordinates is (v,u) from the abstract classification, with gives 2-jet $(v,u\pm w^2)$ which is 2-${}_V\mathcal{K}$-determined. The two cases, with signs $\pm$, are not equivalent. Note that with 1-jet (v,u) the kernel of the linear map from $\mathbb{R}^3$ to $\mathbb{R}^2$, $(u,v,w)\mapsto(v,u)$, is along the w-axis, which is a generator of the cone V.

For the contact map $F_{(0,0)}$ we obtain $(f_{11}xy,g_{20}x^2\pm g_{04}y^4)$, provided the coefficient g_{04} of y^4 is nonzero, with two cases according as $g_{20}g_{04}>0$ or <0. (It can be checked that in reducing to this form the coefficients of $(0,x^2)$ and $(0,y^4)$ are not changed, in particular the final values are not influenced by the coefficients in the polynomial f, provided of course that $f_{11}\neq0$.) The coefficient of y^2 in the expansion of the second component of $F(x,y,p,q)$ is $2g_{12}p+6g_{03}q$; thus provided $g_{12}\neq0$ or $g_{03}\neq0$ the family (1) with parameters p,q gives a versal unfolding (note that these are odd degree terms of $g(x,y)$). We call such points, where the expansion of M at the origin has the 2-jet $(f_{11}xy,\ g_{20}x^2,\ x,\ y)$ and

$$f_{11}\neq0,\ g_{20}\neq0,\ g_{04}\neq0,\ g_{12}\text{ or }g_{03}\neq0, \tag{4}$$

ordinary parabolic points of M. The last condition is equivalent to the smoothness of the parabolic set of M at the origin (see below) but the condition $g_{04}\neq0$ does not arise from the flat geometry of M and is analogous to the condition found in [6] for an 'ordinary' (A_2) point of the parabolic set of $M\subset\mathbb{R}^3$.

A standard result is that the global equation of the parabolic set of a surface M in the form $(f(x,y), g(x,y), x, y)$ is

$$(f_{xx}g_{yy} - f_{yy}g_{xx})^2 = 4(f_{xy}g_{yy} - f_{yy}g_{xy})(f_{xx}g_{xy} - f_{xy}g_{xx}). \tag{5}$$

This can be proved by considering the 3-parameter family of height functions at any point of M, say $H(x,y,\lambda,\mu,\nu) = \lambda f(x,y) + g(x,y) + \mu x + \nu y$ or $H(x,y,\lambda,\mu,\nu) = f(x,y) + \lambda g(x,y) + \mu x + \nu y$ and writing down the condition that there is a unique direction $(\lambda, 1, \mu, \nu)$ or $(1, \lambda, \mu, \nu)$ with the height function having a non-Morse singularity, that is $H_x = H_y = H_{xx}H_{yy} - H_{xy}^2 = 0$. (All normal vectors to M have one of these two forms.) We note below in §3.4 that the formula also follows from our analysis of contact functions.

In the present case the lowest terms in the equation of the parabolic set at the origin are, from (5), $16f_{11}^2 g_{20}(g_{12}x + 3g_{03}y)$, so that the parabolic set is smooth at the origin if and only if g_{12} or g_{03} is nonzero: the last condition of (4). We can unambiguously label smooth segments of the parabolic set with the sign $+$ or $-$ according as, with 2-jet of (f,g) equal to $(f_{11}xy,\ g_{20}x^2)$, both coefficients being nonzero, the product $g_{20}g_{04}$ of the coefficients of $(0, x^2)$ and $(0, y^4)$ is > 0 or < 0. We shall see below when the sign of the parabolic set changes.

For the bifurcation set, we consider the map $(u,v,w) \mapsto (v,\ u \pm w^2 + \lambda w)$ and the multiplicity of the zero set of this in an arbitrarily small neighbourhood of the origin. Since $v = 0$ the intersection lies in the (u,w) plane, at points of the u- and w-axes. The curve $u = \mp w^2 - \lambda w$ is tangent to the w axis if and only if $\lambda = 0$ and then the multiple value of w is 0 so the tangency is at the origin. In the geometrical case of a surface, as above, the condition $\lambda = 0$ is replaced by $2g_{12}p + 6g_{03}q = 0$, which is the tangent line to the parabolic set at the origin. Thus the germ of the bifurcation diagram in the (p,q) parametrization plane of the surface consists of the tangent line to the parabolic set:

Proposition 3.3. *At an point of the parabolic set satisfying (4) the bifurcation set $\mathcal{B}$ is locally exactly the parabolic set. We can give a sign to each such point of the parabolic set by the sign of $g_{20}g_{04}$ when the 2-jet of (f,g) is reduced to $(f_{11}xy,\ g_{20}x^2)$.*

Points off the parabolic set have stable contact maps, in fact they are elliptic or hyperbolic points as in §§3.1 and 3.2.

3.4. **Formulas for loci of types (P) and (SP) in Table 3.1.** We can use the criterion in Lemma 3.2 to obtain the equation (5) for the parabolic set on a general surface in Monge form, and then find an additional equation which holds at special parabolic points. We shall use these in §4 to analyse some examples of special parabolic points.

For the contact map (1) at the point of M with parameters p, q write f_{11} for $f_{xx}(p,q)$, f_{12} for $f_{xy}(p,q)$, f_{1222} for $f_{xyyy}(p,q)$ and so on. Then the 2-jet of the first component of the contact map $F = F_{(p,q)}$ in terms of u, v, w is (taking into account the factor 2 which automatically arises)

$$C_1(u,v,w) = (f_{11}u + 2f_{12}v + f_{22}w) +$$
$$\tfrac{1}{12}\left(f_{1111}u^2 + 4f_{1112}uv + 6f_{1122}uw + 4f_{1222}vw + f_{2222}w^2\right),$$

with a similar formula for the second component.

We can now solve the equations $C_1 = C_2 = 0$ for say u and v in terms of w up to order 2, and substitute in the equation $v^2 = uw$ of the cone V to obtain the order of contact of the zero set of C with V. The condition for the order of contact to be at least 2, that is the condition for the coefficient of w^2 after substitution to be zero, then works out at exactly (5) where f_{xx} appears as f_{11} and so on.

The additional condition for the contact to be of order at least 3, that is for the coefficient of w^3 also to be zero, is naturally more complicated and requires solving for u and v as above to a higher order. But it is possible to use this condition in explicit examples and it is stated in appendix A. This formula is used in examples in §4.

3.5. **Codimension 2 case: special parabolic point.** This degeneracy occurs for the abstract map $\mathbb{R}^3 \to \mathbb{R}^2$ when the coefficient of w^2 in §3.3 equals zero but there is a nonzero coefficient of w^3. The kernel of the 1-jet map $\mathbb{R}^3 \to \mathbb{R}^2$, $(u, v, w) \mapsto (v, u)$ is still 1-dimensional and along a generator of the cone V. The bifurcation set of the abstract germ in the case $(v, u - w^3)$ was analysed in §2 and is illustrated in Figure 1. The other case, $(v, u + w^3)$, is similar and the full picture of the bifurcation set is in Figure 2.

In our geometrical situation, on the surface M the above degeneracy corresponds to a parabolic point with the 2-jet of (f, g) being $(f_{11}xy,\ g_{20}x^2)$ and $g_{04} = 0$. The additional condition which ensures that the contact singularity is no more degenerate is $g_{13}^2 - 4g_{20}g_{06} \neq 0$, that is the even degree terms $g_{20}x^2 + g_{13}xy^3 + g_{06}y^6$ do not form a perfect square. (This condition remains unchanged when the higher terms of f are eliminated, in particular the condition to avoid further degeneracy does not involve the higher degree terms of f.) We call these *special parabolic points*[1]. The further condition that in the family of contact maps the parameters p, q give a versal unfolding is $5g_{12}g_{05} - 3g_{03}g_{14} \neq 0$.

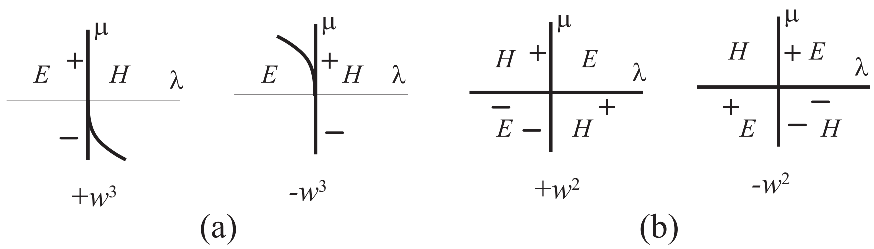


FIGURE 2. (a) The bifurcation set of the unfolding
$$(v,\ u \pm w^3 + \lambda w + \mu w^2) = (xy,\ x^2 \pm y^6 + \lambda y^2 + \mu y^4),$$
as in §3.5 (special parabolic points), with $+w^3$ on the left and $-w^3$ on the right. The bifurcation set in each case consists of a germ of the μ-axis and a half parabola. In the geometrical situation the μ-axis corresponds to the parabolic set of M and the sign, $+$ or $-$, against this axis is the sign attached to that segment of the parabolic set as in §3.3. Further E and H refer to the parts of the (λ, μ) plane which correspond with elliptic and hyperbolic points of M, respectively, using the normal forms of §§3.1,3.2. The left-hand figure of (a) corresponds with $4g_{20}g_{06} - g_{13}^2 > 0$ and the right-hand figure with $4g_{20}g_{06} - g_{13}^2 < 0$.
(b) Similarly for the bifurcation set of $(v, u^2 \pm w^2 + \lambda u + \mu w)$ as in §3.6 (inflexions of real type), corresponding to $4g_{40}g_{04} - g_{22}^2 > 0$ on the left and < 0 on the right in the geometrical situation.

The two cases, distinguished by the sign of $g_{13}^2 - 4g_{20}g_{06}$ in the geometrical situation, differ as to the region of M, elliptic or hyperbolic, in which the 'half parabola' branch of $\mathcal{B}$ lies. Figure 2(a) shows the two cases. Furthermore, at points along the parabolic set, the local expansion of the surface has $g_{04} \neq 0$ and g_{04} changes sign at special parabolic points. Thus if we label points of the parabolic set by $+$ or $-$ then the sign changes at special parabolic points. See Figure 2(a).

Summing up the conclusions of this section:

---

[1]In the case of a surface in $\mathbb{R}^3$ they had an alternative name, "A_2^* points", referring to the fact that the contact between the surface and its tangent plane at any parabolic point is a function of type A_2, but this notation is not appropriate here.

Proposition 3.4. *A parabolic point of M, with the 2-jet of (f,g) in the form $(f_{11}xy,\ g_{20}x^2)$ is called a* special parabolic point *if the coefficient g_{04} of y^4 in g is zero and $g_{13}^2-4g_{20}g_{06}\neq 0$. The sign attached to ordinary parabolic points close to this one, as in Proposition 3.3, changes at a special parabolic point. Provided $5g_{12}g_{05}-3g_{03}g_{14}\neq 0$ the p,q parameters versally unfold the contact singularity in the family F as in (1) and the bifurcation set is the union of the parabolic set and a "half-parabola" lying in the hyperbolic or elliptic region according to the sign of $4g_{20}g_{06}-g_{13}^2$, as in Figure 2(a).*

We do not know whether there is any significance attached to the sign of $5g_{12}g_{05}-3g_{03}g_{14}$.

3.6. **First codimension 3 case: inflexions of real type.** The 2-jet of (f,g) at inflexion points of real type (also called real inflexions or umbilic points) on M has the form $(f_{11}xy,\ 0)$, where $f_{11}\neq 0$.

The abstract map $\mathbb{R}^3\to\mathbb{R}^2,\ (u,v,w)\mapsto(v,0)$ has a 2-dimensional kernel which intersects the cone V along two generators. The abstract normal form is $(v,u^2+2buw\pm w^2)$ where the second component should not be a perfect square, that is $b^2\neq\pm1$ (for the $-$ case this is no restriction). An abstract ${}_V\mathcal{K}$-versal unfolding is given by $(v,u^2+2buw\pm w^2+\lambda u+\mu w)$, that is b is a smooth modulus in this case. The bifurcation set $\mathcal{B}$ is found by considering the contact of the curve $u^2+2buw\pm w^2+\lambda u+\mu w=0$ with the u and w axes in the (u,w) plane. The condition for tangency comes to $\mu=0$ or $\lambda=0$, irrespective of the sign in the normal form and the value of b. Thus $\mathcal{B}$ consists of the complete λ and μ axes (not half-axes), and does not depend on the modulus b. Note that although $uw=v^2$ on the cone V our map germs are defined on $\mathbb{R}^3$ and not just on the cone, so we cannot use left-equivalence to remove the modulus term $2buw$.

Remark 3.5. We do not know if b has any geometrical significance. However, taking the two components of the map $(v,u^2+2buw\pm w^2)$, the intersection of the cone V with the plane $v=0$ gives two lines in the plane, $u=0$ and $w=0$, and the second component gives two more lines which are real when $b^2>\pm1$ (no restriction for the $-$ sign). The cross-ratio of these four lines will be responsible for the existence of a smooth modulus.

The contact singularity for $\lambda=0,\mu\neq0$ or $\mu=0,\lambda\neq0$ is equivalent to that for a parabolic point as in §3.3. Thus the two crossing branches of $\mathcal{B}$ represent, in our geometrical situation, the parabolic set on M. Indeed at a generic inflexion of real type the parabolic set does have a transverse self-crossing. Furthermore, as λ passes through zero the normal form for the contact singularity at a parabolic point changes from the $+$ case to the $-$ case or vice versa; similarly when μ passes through zero. So the sign attached to the parabolic set changes along each branch of $\mathcal{B}$ at an inflexion point of real type.

In the geometrical situation, on the surface M the bifurcation set divides the surface locally into four regions, two opposite regions being hyperbolic and two elliptic. The configuration corresponding to the two normal forms is shown in Figure 2(b). The condition to avoid further degeneracy is $g_{22}^2-4g_{40}g_{04}\neq0$ and the condition for p,q in the family of contact maps to versally unfold the singularity is $9g_{30}g_{03}-g_{12}g_{21}\neq0$. Perhaps surprisingly, this latter condition is the same as that for an inflexion point of real type to be $\mathcal{R}^+$ versally unfolded by the family of height functions. (See[2] [8, Prop.7.9, p.224].) As above, the bifurcation set consists of the two intersecting branches of the parabolic set, and passing through the crossing point on either branch the "sign" of the parabolic set, as in §3.3, changes. See Figure 2.

[2]Translating notation from this to our notation we have $a_{20}=f_{20}=0, a_{02}=f_{02}=0, a_{21}=f_{21}$, $b_{30}=g_{30}, b_{31}=g_{21}, b_{32}=g_{12}, b_{33}=g_{03}$. The condition in [8] for a versally unfolded D_4 then reduces to our $\frac{1}{2}f_{11}(-9g_{30}g_{03}+g_{21}g_{12})\neq0$. Of course we do not have a D_4 singularity, that is the nondegeneracy of the degree 3 terms of g does not apply. Instead we have a nondegeneracy condition on the degree 4 terms of g.

Proposition 3.6. *At a generic inflexion point of real type on M the ${}_V\mathcal{K}$ bifurcation set of the family of contact maps consists of the two branches of the parabolic set through the inflexion point. The sign as in §3.3 changes along each branch. See Figure 2(b). With 2-jet of (f, g) equal to $(f_{11}xy,\ 0)$, where $f_{11} \neq 0$, the conditions are $g_{22}^2 - 4g_{40}g_{04} \neq 0$ and $9g_{30}g_{03} - g_{12}g_{21} \neq 0$.*

3.7. **Second codimension 3 case: inflexion point of imaginary type.** The 2-jet of (f, g) at inflexion points of imaginary type on M (also called imaginary inflexions or umbilic points) has the form $(f_{20}x^2 + f_{02}y^2,\ 0)$, where $f_{20}f_{02} > 0$.

The abstract map $\mathbb{R}^3 \to \mathbb{R}^2$ has kernel of the linear part $(u+w,\ 0)$, a plane meeting the cone V only in the origin, and reduces to the abstract normal form $H(u, v, w) = (u+w,\ au^2+2buv+cv^2)$, subject to the conditions $b^2 - ac \neq 0$ and also $4b^2 + (a-c)^2 \neq 0$, that is b and $a - c$ are not both 0. This time there is no explicit requirement that a, c are nonzero, indeed $a = c = 0, b \neq 0$ gives a 2-${}_V\mathcal{K}$-determined germ.

We can however reduce to two alternative normal forms, as in Table 1, as follows. Applying the four vector fields (3) to dH the quadratic form $\phi(u, v) = au^2 + 2buv + cv^2$ can be changed to any linear combination of ϕ and $\psi(u, v) = u\phi_v - v\phi_u = bu^2 + (c - a)uv - bv^2$, provided the conditions above are not violated. Using $b\phi + c\psi$ we can obtain $ku^2 + uv$ for some k, provided $2b^2+c(c-a) \neq 0$, and using $b\phi-a\psi$ we can obtain $uv+kv^2$ for some k provided $2b^2-a(c-a) \neq 0$. If both these reductions fail then it is easy to check that $a = c$ and $b = 0$ which violates the original condition on ϕ.

Remark 3.7. We do not know whether this remaining smooth modulus k has any geometrical significance. However, as in the real inflexion case (Remark 3.5), a smooth modulus is to be expected in view of the presence of four concurrent lines in the intersection of the cone V and the zero-set of the map $(u, v, w) \mapsto (u + w, ku^2 + uv)$, to take one of the above alternatives. Setting $u + w = z$, the equation $uw = v^2$ becomes $u(z - u) = v^2$ and setting $z = 0$ we have four lines in this plane, $u^2 + v^2 = 0$ and $u(ku + v) = 0$. Of course the first pair of these lines are never real.

A ${}_V\mathcal{K}$ versal unfolding is given by

$$(u + w,\ ku^2 + uv + \lambda u + \mu v + \nu u^2) \quad \text{or} \quad (u + w,\ uv + kv^2 + \lambda u + \mu v + \nu v^2),$$

where k is a smooth modulus. There are no restrictions on the value of k; in particular it can be 0. The ${}_V\mathcal{K}$ bifurcation set $\mathcal{B}$ in this case consists of the origin only in the (λ, μ)-plane since $u + w = 0$ is possible only for $x = y = 0$, hence $u = v = w = 0$.

In the geometrical case we require $g_{31}^2 - 4g_{40}g_{22} \neq 0$, and $g_{31},\ f_{20}g_{22} - f_{02}g_{40}$ are not both zero. For p, q in the family of contact maps to versally unfold the singularity we require[3]

$$g_{21}^2 - 3g_{12}g_{30} \neq 0.$$

The inflexion points of imaginary type are isolated points of the parabolic set of M. They also lie on the curve on M defined by the vanishing of the *normal curvature* κ of M. This is the same as saying that the curvature ellipse collapses to a segment (and so has zero area). See [2, pp. 9, 17]. Points of the $\kappa = 0$ curve on M other than the inflexions of imaginary type are not distinguished by the family of reflexion maps since in general $\kappa = 0$ is not an affine invariant of M.

Proposition 3.8. *At an inflexion point of imaginary type on M, with 2-jet of (f, g) equal to $(f_{20}x^2 + f_{02}y^2,\ 0)$, where $f_{20}f_{02} > 0$, the ${}_V\mathcal{K}$ bifurcation set consists of the point only. A*

[3]This is not the same condition as that in [8, Prop.7.9, p.224] which in our notation becomes $f_{02}(3g_{30}g_{12} - g_{21}^2) + f_{20}(3g_{21}g_{03} - g_{12}^2) \neq 0$.

generic point of this kind is an isolated point of the parabolic set of M. The conditions are $g_{31}^2 - 4g_{40}g_{22} \neq 0$, g_{22}, $g_{40} - g_{22}$ are not both zero and $g_{21}^2 - 3g_{12}g_{30} \neq 0$.

4. Examples

In this section we show how to calculate special parabolic points in practice over a whole surface M given in Monge form.

A good source of examples where something interesting is happening is [4, pp.189-90]. In these examples the parabolic set undergoes a transition as M changes in a 1-parameter family, so that a loop appears (either an elliptic island in a hyperbolic sea or vice versa), or a crossing on the parabolic set separates in a Morse transition. In fact from our point of view the examples of [4] are not quite generic since at special parabolic points, when these exist, our family of contact maps does not versally unfold the singularity according to the criterion of §3.5. However this is easily remedied by additing an extra term to one of the defining equations.

For us it is not generic for a crossing or isolated point on the parabolic set to be in addition a special parabolic point, since special parabolic points are isolated on the parabolic set. Thus when a loop of parabolic points appears on M the loop will generically have no special parabolic points on it but these can develop as the loop expands, as the examples show. We can check numerically that the sign of the parabolic curve, in the sense of §3.3, changes at a special point, and we can calculate the type of the special point, as defined in §3.5.

Example 4.1. Consider the family of surfaces given in Monge form by $f(x,y) = xy + y^3$, $g(x,y) = x^2 + x^2y^2 + xy^3 - \frac{1}{2}y^4 + \frac{1}{30}y^5 + \mu y^2$, where the term in y^5 is added to the formula in [4, p.189] (with $\lambda = -\frac{1}{2}$) to make the special points generic from the family of reflexion maps, and small negative values of the parameter μ produce a loop on the parabolic set. Figure 3 illustrates the formation of two special points on the parabolic set as μ becomes more negative.

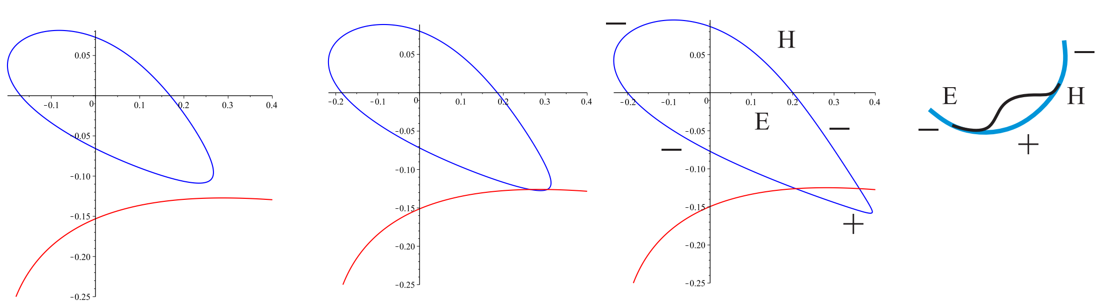


Figure 3. The parameter plane of the curve of Example 4.1 near the origin $x = y = 0$ for, left to right, $\mu = -\frac{1}{35}$, $\mu = -\frac{1}{29}$, $\mu = -\frac{1}{25}$. The figure shows a loop on the parabolic set and the additional curve whose intersections with the parabolic set give special points, as in §3.4, §A. Two special points appear at about $\mu = -\frac{1}{29}$. The signs of the parabolic set arcs are marked in the third figure and the elliptic region E and the hyperbolic region H. The right-hand figure is a schematic representation of the germ of a “semi-lips” which joins the two bifurcation sets of the special parabolic points immediately after their creation. Note that this is consistent with Figure 2(a) with the $-w^3$ sign.

We can calculate the type of the special points, and the sign of the parabolic curve on either side of them, as follows, where the calculations are necessarily numerical rather than exact. Having calculated numerically the parameter values (p, q) of a special point, that is where the two curves in Figure 3 intersect we 'move the origin' to this point. This re-parametrizes M near $(f(p,q), g(p,q), p, q)$ as the set of points $(f(x'+p,\ y'+q)-f(p,q),\ g(x'+p,\ y'+q)-g(p,q),\ x',\ y')$ where (x', y') are the new coordinates in the parameter plane, with origin at $x = p, y = q$. We can now proceed to reduce the quadratic terms of this parametrization to $(x'y', x'^2)$, ignoring any linear terms which can be removed by a global affine transformation of $\mathbb{R}^4$. Having done this, we can apply the formulas of §3.5 to determine the type of special parabolic point (elliptic or hyperbolic) and to check that it is nondegenerate and that the family of contact maps is versally unfolded. All these calculations are straightforward and were performed in MAPLE. The same method can be used at an ordinary parabolic point to determine whether it is positive or negative in the sense of §3.3.

For the example above we find that the special parabolic points are both elliptic, that is the germ of the bifurcation set is inside the elliptic island of M. We find that after reduction of the quadratic terms of f, g the conditions $g_{04} = 0,\ 4g_{20}g_{06} - g_{13}^2 < 0,\ 5g_{12}g_{05} - 3g_{03}g_{14} \neq 0$ in the notation of §3.5, all hold at both special points. The latter condition does not hold without the addition of the term in y^5 to g.

We also find that the sign of the parabolic points on the loop is negative for small μ before the special points appear; this is to be expected since the sign of y^4 in $g(x,y)$ is < 0. The arc of the parabolic set between the special points consists of positive parabolic points.

Example 4.2. A second example, also adapted from [4], is provided by

$$f(x,y) = xy + y^3,\ g(x,y) = x^2 + x^2y - 3x^2y^2 + 3y^4 + y^5 + \mu y^2.$$

See Figure 4 for an illustration. Calculation as above stows that the special parabolic point is elliptic and is versally unfolded by the family of contact maps so that the bifurcation set is as described in §3.5. Also, the signs of the parabolic set are as in the figure. Note that this transition on the parabolic set via a self-crossing is not to be confused with the inflexion point of real type as in §3.6.

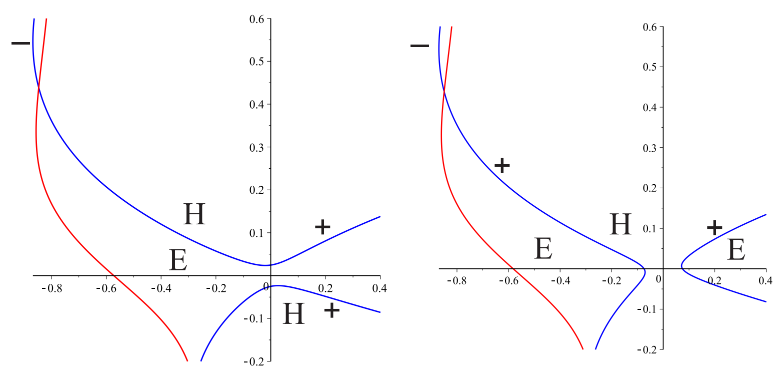


FIGURE 4. The parabolic set in the parameter plane for Example 4.2, with (left) $\mu = -\frac{1}{60}$ and (right) $\mu = \frac{1}{60}$. The special parabolic points where the two curves meet are of elliptic type; H stands for the hyperbolic region, E for the elliptic region and $+, -$ refer to the sign of these sections of the parabolic set, computed using the method explained above.

5. Concluding remarks

We have shown how the family of contact maps by reflexion in points of a surface M in $\mathbb{R}^4$ identifies the parabolic set of M and also some special but still smooth points of the parabolic set which are not part of the flat geometry of M but are affine invariants of M. We do not know of a different characterisation of these points.

In [6] it was possible to extend the bifurcation set of the family of contact maps on a surface M in $\mathbb{R}^3$ to a global bifurcation set, even though it was not entirely clear what geometrical significance this had away from the parabolic set on M. In the present case, for M a surface in $\mathbb{R}^4$, we do not know of any reasonable way to make the bifurcation set global.

Because of the sign attached to points of the parabolic set which changes at special parabolic points and also at self-crossings of the parabolic set, it is possible to formulate some statements about the numbers of special points. For instance, on a smooth closed loop of the parabolic set there must be an even number of special parabolic points (possibly zero). Similarly on a figure-eight component of the parabolic set there must be the same parity of special parabolic points on each loop.

It is possible in principle to extend the explicit calculations of special parabolic points, as in §4, to the case when the surface is parametrized in a general way, as

$$(A(x,y), B(x,y), C(x,y), D(x,y)).$$

However there is a significant difficulty in writing down the contact map, as in (1) which is valid for the case $C(x,y)=x, D(x,y)=y$, without an expression for M as the zero set of a submersion $\mathbb{R}^4 \to \mathbb{R}^2$. We need to construct the contact map from parametrizations of both M and its reflexion M^* in a point of M. Extension to a general parametrization would allow us to examine examples such as those in [1]. Even more challenging is the explicit calculation of the contact map for a surface which is given in implicit form as the zero set of a submersion.

Appendix A. The additional formula for the locus of special parabolic points

Consider a surface in Monge form $(f(x,y), g(x,y), x, y)$. For our purposes it does not matter whether f, g have linear terms since they can be removed by a global affine transformation of $\mathbb{R}^4$ which will not affect the parabolic curves or special parabolic points. The additional condition, besides (5), for a point with parameters (p,q) to be a special parabolic point, is as follows. We use the notation of §3.4.

Let

$$\Theta_1 = f_{12}g_{22} - f_{22}g_{12},\ \Theta_2 = f_{11}g_{22} - f_{22}g_{11},\ \Theta_3 = f_{11}g_{12} - f_{12}g_{11}$$
$$\Phi_1 = f_{11}g_{11}g_{22} - 2f_{11}g_{12}^2 + 2f_{12}g_{11}g_{12} - f_{22}g_{11}^2$$
$$\Phi_2 = f_{11}g_{11}f_{22} - 2f_{12}^2g_{11} + 2f_{11}f_{12}g_{12} - f_{11}^2g_{22}$$

Then the condition is

$$\Theta_1^2\Phi_1 f_{1111} - 2\Theta_1\Theta_2\Phi_1 f_{1112} + 6\Theta_1\Theta_3\Phi_1 f_{1122} - 2\Theta_2\Theta_3\Phi_1 f_{1222}$$
$$+\Theta_3^2\Phi_1 f_{2222} + \Theta_1^2\Phi_2 g_{1111} - 2\Theta_1\Theta_2\Phi_2 g_{1112} + 6\Theta_1\Theta_3\Phi_2 g_{1122}$$
$$-2\Theta_2\Theta_3\Phi_2 g_{1222} + \Theta_3^2\Phi_2 g_{2222} = 0.$$

In the case that $f_{11}=0, f_{12}\neq 0, f_{22}=0, g_{11}\neq 0, g_{12}=0, g_{22}=0$ this reduces to $g_{2222}=0$, as we expect from §3.5 where the condition appears as $g_{04}=0$ when we are working at the origin.

Acknowledgements We are very grateful to Victor Goryunov for helpful conversations, in particular for suggesting the correct definition of the bifurcation set in our case. Also, the first two authors acknowledge the hospitality and support of the University of São Paulo at São Carlos in 2014; the first author acknowledges the hospitality and support of the Center for Advanced Studies at the Technical University of Warsaw; and the second and third authors acknowledge the hospitality and support of the University of Liverpool, the London Mathematical Society and the UK research council EPSRC during a visit in 2016.

References

[1] R. Alves Garcia, D.K.H. Mochida, Maria del Carmen Romero-Fuster, M.A. Soares Ruas, "Inflection points and topology of surfaces in 4-space", *Trans. Amer. Math. Soc* 352 (2000), 3029–3043. DOI: 10.1090/s0002-9947-00-02404-1

[2] J. Basto-Gonçalves, "Local geometry of surfaces in $\mathbb{R}^4$", 2013, ICCS.pdf (Accessed 17 January 2020.)

[3] J.W.Bruce, N.P.Kirk and A. duPlessis, ' Complete transversals and the classification of singularities', *Nonlinearity* 10 (1997), 253–275. DOI: 10.1088/0951-7715/10/1/017

[4] J.W.Bruce and F.Tari, "Families of surfaces in $\mathbb{R}^4$", *Proc. Edinburgh Math. Soc* 45 (2002), 181–203. DOI: 10.1017/S0013091500000213

[5] J.N.Damon, 'The unfolding and determinacy theorems for subgroups of $\mathcal{A}$ and $\mathcal{K}$', *Memoirs of the American Math. Soc.* 306 (1984). Electronic ISBN: 978-1-4704-0719-3

[6] P.J.Giblin and S.Janeczko, "Bifurcation sets of families of functions on surfaces in $\mathbb{R}^3$", *Proc. Royal Soc. Edinburgh* 147A (2017), 337-352. DOI: 10.1017/S0308210516000184

[7] P.J.Giblin and V.M.Zakalyukin, "Recognition of centre symmetry set singularities", *Geometriae Dedicata* 130 (2007), 43–58. DOI: 10.1007/s10711-007-9204-2

[8] S.Izumiya, M.del C.Romero Fuster, M.A.S.Ruas, F.Tari, *Differential Geometry from a Singularity Theory Viewpoint*, World Scientific 2016, ISBN 978-981-4590-44-0. DOI: 10.1142/9108

[9] M.Manoel, P.de M.Rios and W.Domitrz, 'The Wigner caustic on shell and singularities of odd functions', *J.Geometry and Physics* 71 (2013), 58–72. DOI: 10.1016/j.geomphys.2013.04.005

[10] J.Montaldi, "On contact between submanifolds", *Michigan Math. J.* 33 (1986), 195–199. DOI: 10.1307/mmj/1029003348

Peter J. Giblin, Department of Mathematical Sciences, The University of Liverpool, Liverpool L69 7ZL, UK

Email address: `pjgiblin@liv.ac.uk`

Stanisław Janeczko, Faculty of Mathematics and Information Science, Warsaw University of Technology, Pl. Politechniki 1, 00-661 Warsaw, Poland; and Institute of Mathematics Polish Academy of Sciences, Sniadeckich 8, 00-656 Warsaw, Poland

Email address: `S.Janeczko@mini.pw.edu.pl`

Maria Aparecida Ruas, ICMC, USP, Avenida Trabalhador São-Carlense, 400 Centro, CEP: 13566-590, São Carlos SP, Brazil

Email address: `maasruas@icmc.usp.br`

Journal *of* Singularities
Volume 21 (2020), 97-118

Proc. of Geometric and Algebraic Singularity Theory, Będlewo, 2017
DOI: 10.5427/jsing.2020.21f

EQUIDISTANTS FOR FAMILIES OF SURFACES

PETER GIBLIN AND GRAHAM REEVE

Abstract. For a smooth surface in $\mathbb{R}^3$ this article investigates certain affine equidistants, that is loci of points at a fixed ratio between points of contact of parallel tangent planes (but excluding ratios 0 and 1 where the equidistant contains one or other point of contact). The situation studied occurs generically in a 1-parameter family, where two parabolic points of the surface have parallel tangent planes at which the unique asymptotic directions are also parallel. The singularities are classified by regarding the equidistants as critical values of a 2-parameter unfolding of maps from $\mathbb{R}^4$ to $\mathbb{R}^3$. In particular, the singularities that occur near the so-called 'supercaustic chord', joining the two special parabolic points, are classified. For a given ratio along this chord either one or three special points are identified at which singularities of the equidistant become more special. Many of the resulting singularities have occurred before in the literature in abstract classifications, so the article also provides a natural geometric setting for these singularities, relating back to the geometry of the surfaces from which they are derived.

1. Introduction

A smooth closed surface in affine 3-space will contain pairs of points at which the affine tangent planes are parallel; indeed the tangent plane at a given point may be parallel to that at several other points if the surface is non-convex. Associated with these pairs of points, and the chords joining them, there are a number of affinely invariant constructions. The *affine equidistants* are the loci of points at a fixed ratio $\lambda : 1-\lambda$ along the chords, and the *centre symmetry set* is the envelope of the chords, which can be locally empty. These constructions have been examined from the point of view of singularity theory in the last few years by several authors; there are many connexions with earlier work such as the 'Wigner caustic' of Berry [2] which, for a curve in the plane, is the equidistant corresponding to a ratio $\lambda = \frac{1}{2}$, that is the midpoints of the parallel tangent chords, and the bifurcations of central symmetry of Janeczko [11]. Notable among recent studies is the work of Domitrz and his co-authors, for example [3].

A generic surface M in affine 3-space will generically have pairs of points at which the tangent planes are parallel and for which both points in the pair are parabolic points of M. For the locus of parabolic points of M is generically a 1-dimensional set, a union of smooth curves, and requiring parallel tangent planes imposes two conditions on a pair of points of this set, so that a finite number of solutions can be expected. In this article we investigate one possible local degeneration of this generic situation by requiring also that the unique asymptotic directions coincide at such a pair of parabolic points with parallel tangent planes. For this to occur the surface M must be contained in a smoothly varying family M_ε of surfaces. Since our investigation is local we shall in fact consider two surface patches M_0 and N_0 which vary in a 1-parameter family $M_\varepsilon, N_\varepsilon$. A similar degeneracy was investigated for plane curves in [6]; we sometimes call it a 'supercaustic' situation. This term is defined in §2.3.

2010 *Mathematics Subject Classification.* 57R45, 53A05.

Key words and phrases. affine equidistant, surface family in 3-space, critical set, map germ 4-space to 3-space.

We find the values $\lambda \neq 0, 1$ for which the ratio $\lambda : 1-\lambda$ determines an equidistant at which the structure undergoes a qualitative change. There are one or three of these values, depending on the relative orientation of M_0 and N_0. One 'degenerate' value always exists and results in a high codimension singularity; we are able to give a partial analysis of this case. When the other two values exist we call them *special values* (Definition 2.6), and a complete analysis is given.

The article is organized as follows. In §2 we introduce the family of surfaces we shall work with (§2.1), and the maps which we shall classify up to $\mathcal{A}$-equivalence to study the equidistants (§2.2, §2.3). We also show how some of the conditions that arise later can be interpreted geometrically in terms of a scaled reflexion map (§2.4, Definition 2.5). In §3 we find normal forms of maps up to $\mathcal{A}$-equivalence that generate the equidistants: they are the sets of critical values of these maps. We examine in that section general values of the ratio (Generic Case 1.1) and the two 'special' values (Special Case 1.2), leaving the 'degenerate' value (Degenerate Case 2) to §4.

The main results are contained in Proposition 3.2 and the accompanying Figure 1 for Generic Case 1.1; Proposition 3.4 and the accompanying Figure 4 for Special Case 1.2, and Table 1 in §4.6 for Degenerate Case 2.

2. The general setup

2.1. **A generic family of surfaces.** Consider the parabolic set P (assumed to be a nonempty smooth curve) of a generic smooth closed surface M in $\mathbb{R}^3$. We can expect generically to find a finite number of pairs of distinct points on P for which the tangent planes to M are parallel, since the two points give us two degrees of freedom and it is two conditions for the tangent planes to be parallel. However it will not be generically true that the unique asymptotic directions at such a pair of points are parallel. For that we require a 1-parameter family of surfaces and it is this situation which we study here.

Our considerations are local, and also affinely invariant. For this situation we have two surfaces, M_ε and N_ε, varying in a 1-parameter family; using a family of affine transformations of $\mathbb{R}^3$ (coordinates (x, y, z)) we can assume that the origin lies on M_ε, that the origin is a parabolic point of M_ε and that the unique asymptotic direction there is always along the y-axis, for all ε close to 0. Further we can assume that the point $(0, 0, 1)$ lies on N_ε for all small ε and that for $\varepsilon = 0$ this point is parabolic, has horizontal tangent plane parallel to the (x, y)-plane, and has unique asymptotic direction parallel to the y-axis. We realise this setup by the surfaces

$$\begin{aligned} M_\varepsilon : z = f(x,y,\varepsilon) &= f_{20}x^2 + f_{300}x^3 + f_{210}x^2y + f_{120}xy^2 + f_{030}y^3 + \dots \\ &+ \varepsilon\left(f_{301}x^3 + f_{211}x^2y + \dots\right) + \varepsilon^2\left(f_{302}x^3 + \dots\right) + \dots, \end{aligned} \tag{1}$$

$$\begin{aligned} N_\varepsilon : z = 1 + g(x,y,\varepsilon) &= 1 + g_{20}x^2 + g_{300}x^3 + g_{210}x^2y + g_{120}xy^2 + g_{030}y^3 + \dots \\ &+ \varepsilon\left(g_{101}x + g_{011}y + g_{201}x^2 + g_{111}xy + g_{021}y^2 + \dots\right) \\ &+ \varepsilon^2\left(g_{102}x + g_{012}y + \dots\right) + \dots. \end{aligned} \tag{2}$$

For terms other than f_{20}, g_{20}, subscripts ijk indicate that the corresponding monomial is $\varepsilon^k x^i y^j$.

We make the following assumptions about these expansions.

Assumptions 2.1. (i) $f_{20} \neq 0, g_{20} \neq 0$, that is neither M_0 nor N_0 is umbilic at its basepoint $(0, 0, 0)$ or $(0, 0, 1)$.

(ii) $f_{030} \neq 0, g_{030} \neq 0$, that is the parabolic curves of M_0 at the origin and N_0 at $(0, 0, 1)$ are smooth and not tangent to the asymptotic directions there (i.e. these points are not cusps of Gauss). We shall take $f_{030} > 0$ without loss of generality, and we sometimes write

$$f_{030} = f_3^2,\ g_{030} = \pm g_3^2$$

when a definite sign is needed, to avoid square roots appearing in the formulas.

2.2. **Family of maps for the equidistants.** The λ-equidistant for a fixed ε is the locus of points in $\mathbb{R}^3$ of the form $(1-\lambda)\boldsymbol{p}+\lambda\boldsymbol{q}$ where $\boldsymbol{p}\in M_\varepsilon, \boldsymbol{q}\in N_\varepsilon$ and the tangent planes to M_ε at $\boldsymbol{p}$ and N_ε at $\boldsymbol{q}$ are parallel.

We always assume $\lambda\neq 0, \lambda\neq 1$ in what follows.

We use $s=(s_1,s_2)$ as parameters on M_ε and similarly $t=(t_1,t_2)$ for N_ε; we have a 2-parameter family of maps $\mathbb{R}^4\to\mathbb{R}^3$:

$$\mathbb{R}^4\times\mathbb{R}^2\to\mathbb{R}^3,\ (s,t,\varepsilon,\lambda)\mapsto(1-\lambda)(s_1,s_2,f(s_1,s_2,\varepsilon))+\lambda(t_1,t_2,1+g(t_1,t_2,\varepsilon)). \tag{3}$$

Then it is straightforward to check that, for fixed ε and λ, the set of critical values of this map is the λ-equidistant of M_ε and N_ε. We are therefore interested in this family of maps up to $\mathcal{A}$-equivalence. We make the change of variables

$$(1-\lambda)s_1+\lambda t_1=u_1,\ (1-\lambda)s_2+\lambda t_2=u_2,\ \text{and write } \lambda=\lambda_0+\alpha,$$

replacing t_1 and t_2, to rewrite (3) as a map of the form (for any $\lambda_0\neq 0,1$)

$$H:\mathbb{R}^4\times\mathbb{R}^2\to\mathbb{R}^3,\ H(s_1,s_2,u_1,u_2,\varepsilon,\alpha)=(u_1,u_2,h(s_1,s_2,u_1,u_2,\varepsilon,\lambda_0+\alpha)). \tag{4}$$

regarded as a 2-parameter unfolding of the map $H_0(s_1,s_2,u_1,u_2,0,\lambda_0)$. Therefore we have the following.

Proposition 2.2. *The λ-equidistant for fixed ε is the set of points $(u_1,u_2,h)\in\mathbb{R}^3$ for which $\partial h/\partial s_1=\partial h/\partial s_2=0$. For fixed λ the union of all the equidistants, spread out in $\mathbb{R}^4$, the planar sections of which are the $\varepsilon=$ constant equidistants, is the set of points $(u_1,u_2,h,\varepsilon)\in\mathbb{R}^4$ for which the same conditions $\partial h/\partial s_1=\partial h/\partial s_2=0$ hold.*

2.3. **Maps and supercaustics.** Let $\phi:\mathbb{R}^4\to\mathbb{R}^2$ be given, for fixed λ and ε, by

$$\phi(s_1,s_2,u_1,u_2)=(h_{s_1},h_{s_2}),$$

subscripts denoting partial derivatives as usual. Then the corresponding equidistant, given by $\phi^{-1}(0,0)$, is singular when there is a kernel vector of $d\phi$ with image under dH equal to $\mathbf{0}$, these being evaluated at a point of $\phi^{-1}(0,0)$. This requires that

$$\text{rank } J<4 \text{ where } J=\begin{pmatrix} h_{s_1s_1} & h_{s_1s_2} & h_{s_1u_1} & h_{s_1u_2}\\ h_{s_2s_1} & h_{s_2s_2} & h_{s_2u_1} & h_{s_2u_2}\\ 0 & 0 & h_{u_1} & h_{u_2}\\ 0&0&1&0\\ 0&0&0&1\end{pmatrix},$$

that is $h_{s_1s_1}h_{s_2s_2}=h^2_{s_1s_2}$. The singular points of the equidistant for fixed λ and ε are therefore

$$\{(u_1,u_2,h(s_1,s_2,u_1,u_2))\ :\ h_{s_1}=h_{s_2}=h_{s_1s_1}h_{s_2s_2}-h^2_{s_1s_2}=0\}. \tag{5}$$

We note here that, for fixed ε, the 'centre symmetry set' of the pair of surfaces M,N [8], which is the locus of singular points of the equidistants for varying λ, is given by the same formula (5) where h is now a function of s_1,s_2,u_1,u_2,λ but with ε still fixed.

It is possible that some singular points of the equidistant arise from singularities of the critical set itself in $\mathbb{R}^4$. In our case this requires, for fixed λ and ε, that the top two rows of the above matrix J are dependent. Indeed, evaluating these rows at $(s_1,s_2,u_1,u_2,\lambda,\varepsilon)=(0,0,0,0,\lambda,0)$ the second row is entirely zero. This means that, for all λ, but $\varepsilon=0$, the critical set itself is singular at the origin of $\mathbb{R}^4$.

Definition 2.3. In the above situation, the λ-axis is called a *supercaustic*; see [6]. The whole of this axis maps to singular points of the equidistants.

Remark 2.4. This depends crucially on the special nature of our surfaces, with not only parallel tangent planes at parabolic points of M_0 and N_0 but also the asymptotic directions at those points being parallel. If instead we assume that the asymptotic directions are distinct (without loss of generality we can take them along the x and y axes) then the top two rows of J become independent for $s_1 = s_2 = u_1 = u_2 = \varepsilon = 0$ and arbitrary λ. In fact, writing g_{020} for the coefficient of y^2 in the parametrization of N_0 and putting $g_{20} = 0$ these rows become

$$\begin{pmatrix} 2(1-\lambda)f_{20} & 0 & 0 & 0 \\ 0 & \frac{2g_{020}(1-\lambda)^2}{\lambda} & 0 & -\frac{2g_{020}(1-\lambda)}{\lambda} \end{pmatrix}.$$

In this case the 'supercaustic' is empty.

2.4. **Scaled reflexion map and contact.** Consider the affine map $\mathcal{S} : \mathbb{R}^3 \to \mathbb{R}^3$ given by $\mathcal{S}(x, y, z) = (\mu x, \mu y, \mu(z-1))$ where $\mu = \frac{\lambda}{\lambda-1} \neq 0$. This leaves the point $(0, 0, \lambda)$ fixed and maps $(0, 0, 1)$ to the origin. We can measure the contact between $\mathcal{S}(N_0)$ and M_0 by composing the parametrization of $\mathcal{S}(N_0)$ given by $(\mu x, \mu y, \mu g(x, y, 0))$ with the equation of M_0, say

$$Z - f(X, Y, 0) = 0.$$

Definition 2.5. The *scaled contact map* is the contact map germ

$$K : \mathbb{R}^2, (0, 0) \to \mathbb{R}, 0,\ K(x, y) = \mu g(x, y, 0) - f(\mu x, \mu y, 0),\ \mu = \frac{\lambda}{\lambda - 1} \quad \text{as above.}$$

We shall find this contact map useful in interpreting the conditions which arise from ε-families of equidistants as ε passes through 0.

The 2-jet of K is $K_2(x, y) = \mu(g_{20} - \mu f_{20})x^2$ so that in our situation K is always non-Morse; it has corank 1 and is of type A_k at $(0, 0)$ for some k, provided $f_{20}\lambda + g_{20}(1 - \lambda) \neq 0$ (when this fails we call this the 'Degenerate Case 2'; see §4). The coefficient of y^3 in K is $\mu(g_{030} - \mu^2 f_{030})$ so that K is then of type exactly A_3 provided $f_{030}\lambda^2 - g_{030}(1-\lambda)^2 \neq 0$. If f_{030}, g_{030} are nonzero and have opposite signs then of course this coefficient can never be zero.

Definition 2.6. Assume as above that $f_{20}\lambda + g_{20}(1 - \lambda) \neq 0$. When f_{030}, g_{030} have the same sign (without loss of generality, positive), and the above coefficient $f_{030}\lambda^2 - g_{030}(1 - \lambda)^2$ of y^3 is zero, then we refer to the two resulting values of λ as *special values*. Writing $f_{030} = f_3^2, g_{030} = g_3^2$ where we may take $f_3 > 0, g_3 > 0$, these special values of λ are $\dfrac{g_3}{g_3 \pm f_3}$. (We shall usually assume $f_3 \neq g_3$ to avoid one of the special values 'going to infinity'.) These special values of λ give rise to what we shall call Special Case 1.2. This is examined in detail in §3.2.

When λ has a special value, say $\dfrac{g_3}{g_3 + f_3}$, the condition for K to have exactly type A_3 at $(0, 0)$ works out to be

$$(6)\quad (4g_{040}g_{20} - g_{120}^2)f_3^4 + 4g_{040}f_{20}f_3^3g_3 + 2f_{120}g_{120}f_3^2g_3^2 + 4f_{040}g_{20}f_3g_3^3 + (4f_{040}f_{20} - f_{120}^2)g_3^4 \neq 0.$$

This condition will be satisfied by a generic pair of surfaces M_0, N_0. With the other special value the signs in front of the coefficients of $f_3^3g_3$ and $f_3g_3^3$ both change to minus.

When the quadratic terms of the contact map K vanish identically, that is when

$$f_{20}\lambda + g_{20}(1 - \lambda) = 0,$$

the cubic terms will in general be nondegenerate and K will generically have type $D_4^{\pm}$, that is $\mathcal{R}$-equivalent to $x^3 \pm xy^2$. The polynomial in the coefficients of f and g which distinguishes the two cases is rather complicated but, remarkably, it has a different interpretation which we give in §4 in the context of self-intersections of the equidistant. See Remark §4.3.

3. The equidistants: normal forms

For a general study of the equidistants we need to expand the function h in (4) using the parametrizations (1) and (2). We begin with $\varepsilon = 0$ and write, for a fixed λ,

$$H_{0\lambda}(s,u) = (u, h_{0\lambda}(s,u)) = H(s,u,0,\lambda).$$

The coefficient of $s_1^i s_2^j u_1^k u_2^\ell$ in $h_{0\lambda}$ will be written $c_{ijk\ell}$. We find:

The 2-jet of $h_{0\lambda}$ at $s=u=0$ is $(1-\lambda)(\lambda f_{20} + (1-\lambda)g_{20})s_1^2 - 2g_{20}\frac{1-\lambda}{\lambda}s_1u_1$.

Note that the coefficient of s_1u_1 is nonzero.

The main subdivision is between those λ for what $\lambda f_{20} + (1-\lambda)g_{20}$ is nonzero (Generic Case 1) or zero (Degenerate Case 2). We cover the Generic Case here and the Degenerate Case in §4 below.

Case 1 $\lambda f_{20} + (1-\lambda)g_{20} \neq 0$. From §2.4 this is also the condition for the contact function K to have type A_k for some k.

We can now redefine the variable s_1 ('completing the square') to eliminate all terms containing s_1 besides s_1^2 in $h_{0\lambda}$. The coefficient of s_2^3 then becomes

$$c_{0300} = \frac{1-\lambda}{\lambda^2}(f_{030}\lambda^2 - g_{030}(1-\lambda)^2).$$

3.1. **The general values of λ. Generic Case 1.1** $c_{0300} \neq 0$, that is, $Q \neq 0$ where

$$Q = f_{030}\lambda^2 - g_{030}(1-\lambda)^2. \tag{7}$$

From §2.4 this is also the condition for the contact function K to have type A_2 and that λ is not a *special value*.

Consider the 3-jet of $H_{0\lambda}$. There are six degree 3 monomials which do not involve s_1 and which do involve s_2 (any monomial in u_1, u_2 alone can be eliminated by a 'left-change' of coordinates). We still have the freedom to change coordinates in s_2 (involving s_2, u_1, u_2) and in u_1, u_2 (involving u_1, u_2 only). Using only the first of these the terms in $s_2^2u_1$ and $s_2^2u_2$ can be eliminated, leaving

$$\left(u_1, u_2, (1-\lambda)(\lambda f_{20} + (1-\lambda)g_{20})s_1^2 + c_{0300}s_2^3 + s_2\left(c_{0120}u_1^2 + c_{0111}u_1u_2 + c_{0102}u_2^2\right)\right). \tag{8}$$

(The coefficients $c_{ijk\ell}$ need to be updated to take account of the substitutions.) The quadratic form in u_1 and u_2 can be diagonalised, eliminating the term in $s_2u_1u_2$ so that, scaling s_1, the last coordinate in $\mathbb{R}^3$ and s_2, we have 3-jet, say

$$(u_1, u_2, s_1^2 + s_2^3 + as_2u_1^2 + bs_2u_2^2).$$

Suppose that the quadratic form in parentheses in (8) is not a perfect square, that is

$$c_{0111}^2 - 4c_{0120}c_{0102} \neq 0.$$

Then a and b above are nonzero. The condition for this is $R \neq 0$ where

$$R = f_{20}^2 f_{030}\left(g_{120}^2 - 3g_{210}g_{030}\right) - g_{20}^2 g_{030}\left(f_{120}^2 - 3f_{210}f_{030}\right). \tag{9}$$

Since this condition does not involve λ it will be satisfied by a generic pair of surfaces M_0, N_0. Note that the condition separates into a quantity for M_0 unequal to the same quantity for N_0.

Proposition 3.1. *The condition $R \neq 0$ can also be interpreted as saying that the images under the Gauss map of the parabolic curves on M_0 and N_0 have ordinary tangency (that is, 2-point contact) in the Gauss sphere. These images are smooth by Assumptions 2.1.*

Proof The parabolic curves on the two surfaces are given by

$$f_{xx}f_{yy} - f_{xy}^2 = 0 \quad \text{and} \quad g_{xx}g_{yy} - g_{xy}^2 = 0$$

for M_0 and N_0 respectively. The surface M_0 has a parabolic point at the origin and N_0 has a parabolic point at $(0, 0, 1)$ and since they have parallel asymptotic directions at these points the images of the respective parabolic curves under the Gauss map are tangent. We shall use the modified Gauss maps, that is $(x, y) \mapsto (X, Y) = (f_x, f_y)$ and similarly for g. By a direct calculation, for M_0 the image of the parabolic curve, parametrized by x, under the modified Gauss map has an equation, up to terms in X^2, of the form

$$Y = \frac{3f_{030}f_{210} - f_{120}^2}{12f_{20}^2 f_{030}} X^2$$

with a similar result for N_0. The coefficients of X^2 are unequal, that is the images have ordinary tangency, if and only if the condition R above is nonzero. $\square$

Further scaling allows this case to be reduced to

$$H_{0\lambda}(s, u) = (u_1, u_2, s_1^2 + s_2^3 \pm s_2u_1^2 \pm s_2u_2^2), \tag{10}$$

where the $\pm$ signs are independent, but by interchanging u_1 and u_2 we reduce to three cases, as follows.

Proposition 3.2. *The normal form* (10) *is as follows, using the notation of* (7) *and* (9). *See Figure 1.*
Subcase 1.1.1 *(positive definite):* $H_{0\lambda}(s, u) = (u_1, u_2, s_1^2 + s_2^3 + s_2u_1^2 + s_2u_2^2)$.
The condition for this is $f_{030}g_{030} < 0$ *and* $QR > 0$. *Bearing in mind the assumptions 2.1 the latter condition is equivalent to* $R > 0$. *This subcase will also be referred to as* A_2^{++}.
Subcase 1.1.2 *(negative definite):* $H_{0\lambda}(s, u) = (u_1, u_2, s_1^2 + s_2^3 - s_2u_1^2 - s_2u_2^2)$.
The condition for this is $f_{030}g_{030} > 0$ *and* $QR > 0$. *This subcase will also be referred to as* A_2^{--}
Subcase 1.1.3 *(indefinite):* $H_{0\lambda}(s, u) = (u_1, u_2, s_1^2 + s_2^3 + s_2u_1^2 - s_2u_2^2)$.
The condition for this is $QR < 0$. *In the case when* $f_{030}g_{030} < 0$ *the condition becomes* $R < 0$. *This subcase will also be referred to as* A_2^{+-}, $\square$

The values of f_{030}, g_{030} and R are fixed by the two surfaces M_0 and N_0. However, assuming $f_{030}g_{030} > 0$, special values of λ exist at which Q as in (7) is zero. Then, as λ passes through such a special value, the normal form changes between negative definite and indefinite, so that the family of equidistants, for ε passing through 0, changes accordingly.

Using standard techniques it can be checked that (10) is 3-$\mathcal{A}$-determined, and that an $\mathcal{A}_e$-versal unfolding is given by adding a multiple of $(0, 0, s_2)$ to the above normal form:

$$H_{\varepsilon\lambda}(s, u) = (u_1, u_2, s_1^2 + s_2^3 \pm s_2u_1^2 \pm s_2u_2^2 + \varepsilon s_2). \tag{11}$$

In terms of the original surfaces the coefficient of εs_2 is $-g_{011}(1 - \lambda)$, and therefore we require $g_{011} \neq 0$ for a versal unfolding by the parameter ε.

Remark 3.3. It is interesting to relate the above classification to that of the regions on M and N which contribute to the pairs of parallel tangent planes (compare Prop.2.4 and Figure 3 of [5]). A schematic diagram of the common regions for M and N on the Gauss sphere is given in Figure 2 below. The relationship between these and the classification of Proposition 3.2 is as follows.
Subcase 1.1.1 (positive definite, $f_{030}g_{030} < 0$ and $R > 0$): This is (d).
Subcase 1.1.2 (negative definite, $f_{030}g_{030} > 0$ and $QR > 0$): This is (ac).
Subcase 1.1.3 (indefinite): This can arise in two ways, as either (ac) or (b)

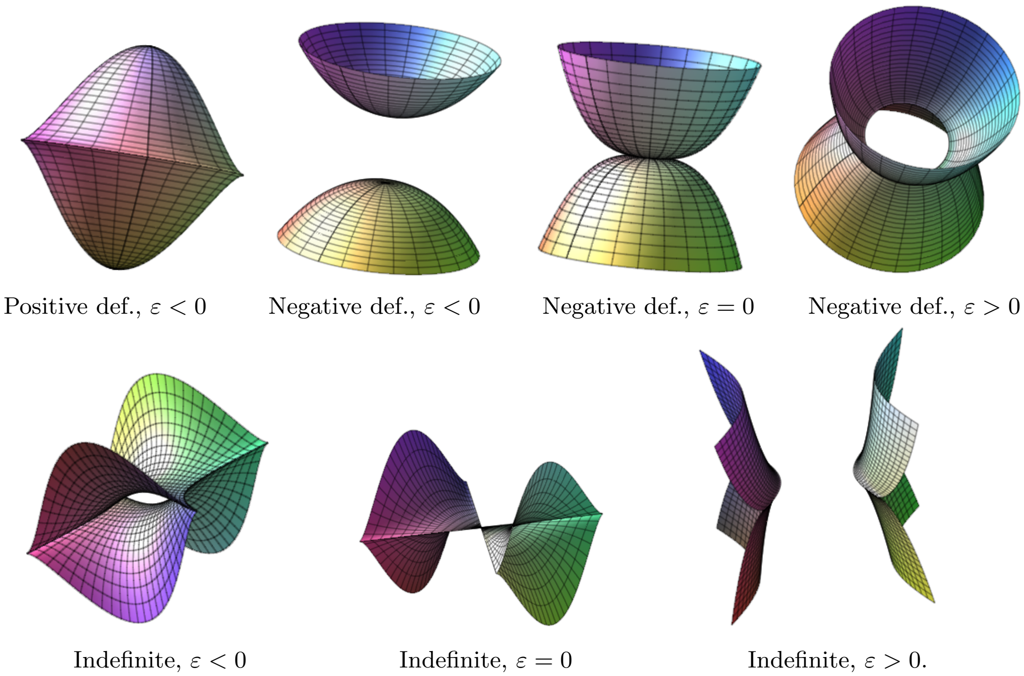


FIGURE 1. The various subcases of Proposition 3.2: Positive definite (for $\varepsilon > 0$ the equidistant is empty and for $\varepsilon < 0$ has a compact cuspidal edge); 1.1.2 Negative definite, where for $\varepsilon > 0$ there is a compact cuspidal edge; 1.1.3 Indefinite, which has two cuspidal edges for $\varepsilon \neq 0$ that form a crossing when $\varepsilon = 0$.

(ac) when $f_{030}g_{030} > 0$ and $QR < 0$,
(b) when $f_{030}g_{030} < 0$ and $R < 0$.

Let us call a pair of points, one from M_ε and the other from N_ε, at which the tangent planes are parallel, 'mates'. Consider for example the top left diagram of Figure 2 and assume that the upper curve is the image of the parabolic curve of N_ε in the Gauss sphere. Each point above this curve is the image of two points of N_ε and two points of M_ε giving altogether four mates. Each point on the upper curve is the image of two points of M_ε and a single parabolic point of N_ε which is a mate for both of them. On the surface M_ε itself there will be a region close to the base-point $(0, 0, 0)$ consisting of those points of M_ε with at least one mate, and usually two mates, on N_ε—a region 'doubly covered by mates on N_ε'. This region will have a local boundary corresponding in the way just described to the parabolic curve on N_ε. Turning to the upper right diagram of Figure 2 the hatched region representing mates now contains a segment of the parabolic curve of M_ε. On the surface N_ε this will result in a closed loop on the boundary of the region of points having mates on M_ε. The situation on the surfaces themselves is illustrated schematically in Figure 3.

3.2. **The 'special values' of λ. Special Case 1.2** $c_{0300} = 0$, that is λ has one of the two special values as in §2.4. Note that this requires f_{030} and g_{030} to have the same sign, which we take as positive, and write $f_{030} = f_3^2, g_{030} = g_3^2$ where $f_3 > 0, g_3 > 0$.

This case will be examined by choosing one of the special values for λ given by $c_{0300} = 0$, namely $\lambda = \dfrac{g_3}{g_3 + f_3}$. We can eliminate the terms in $s_2u_2^2$ and $s_2u_1u_2$ by a substitution of the

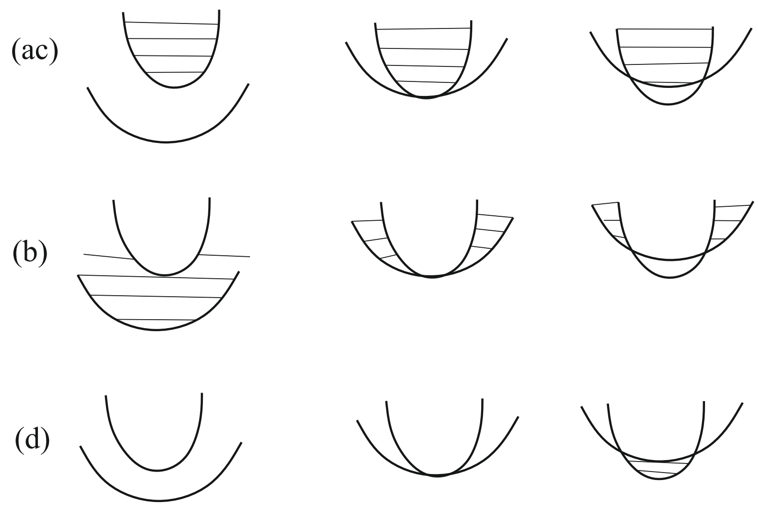


FIGURE 2. Schematic diagrams of the images of the Gauss map for the surfaces M_ε and N_ε. The curves represent the parabolic curves of these surfaces, along which the Gauss map has a fold, and the hatched regions represent the regions where the images of the Gauss maps of M_ε and N_ε intersect, that is the regions of the Gauss sphere representing parallel normals (or parallel tangent planes). Left to right of each row shows varying ε, with the middle diagram $\varepsilon = 0$, and the three possible cases are labelled (ac), (b), (d) as described in the text, to accord with Figure 3 in [5]. Note that the two curves for $\varepsilon = 0$ have ordinary tangency—see Remark 3.1.

form $s_2 = s_2' + au_1 + bu_2$, assuming only the condition $\lambda f_{20} + (1-\lambda)g_{20} \neq 0$ of Generic Case 1. The coefficient of $s_2^2u_2$ then becomes $3f_2^2 \neq 0$ and the remaining degree 3 terms in $h_{0\lambda}$, namely $s_2^2u_1, s_2^2u_2$ and $s_2u_1^2$ can therefore be reduced to the last two by redefining u_2, at the same time making the coefficient of $s_2^2u_2$ equal to 1. The 3-jet of $H_{0\lambda}$ is now of the form (scaling s_1)

$$(u_1, u_2, \pm s_1^2 + s_2^2u_2 + c_{0120}s_2u_1^2),$$

where the updated c_{0120} is nonzero if and only if $R \neq 0$ as in (9), and for generic M_0, N_0 this will be satisfied.

Passing to the 4-jet of $H_{0\lambda}$, we can first remove all monomials divisible by s_1 besides $\pm s_1^2$ by completing the square, and then eliminate all degree 4 monomials besides s_2^4 and $s_2^3u_1$, without adding any new monomials of degree 3. This can be done, for example, by substitutions of the form $s_2 = s_2' +$ quadratic terms in s_2', u_1', u_2', $u_1 = u_1' +$ quadratic terms in u_1', u_2', and similarly for u_2. A left change of coordinates will then restore the first two components of $H_{0\lambda}$ to (u_1, u_2).

The 4-jet is now reduced to

$$\left(u_1, u_2, \pm s_1^2 + s_2^2u_2 + c_{0120}s_2u_1^2 + c_{0400}s_2^4 + c_{0310}s_2^3u_1\right).$$

This is 4-$\mathcal{A}$-determined provided all the coefficients are nonzero. The coefficient c_{0400} is nonzero if and only if the 'exactly A_3 contact condition' (6) holds. Unfortunately we do not know a geometrical criterion for the coefficient of $s_2^2u_2$ to be nonzero; it involves only the coefficients in the functions f, g which define the surfaces M_0, N_0.

Scaling reduces all but the coefficient of s_1^2 to 1 and we summarize this discussion as follows.

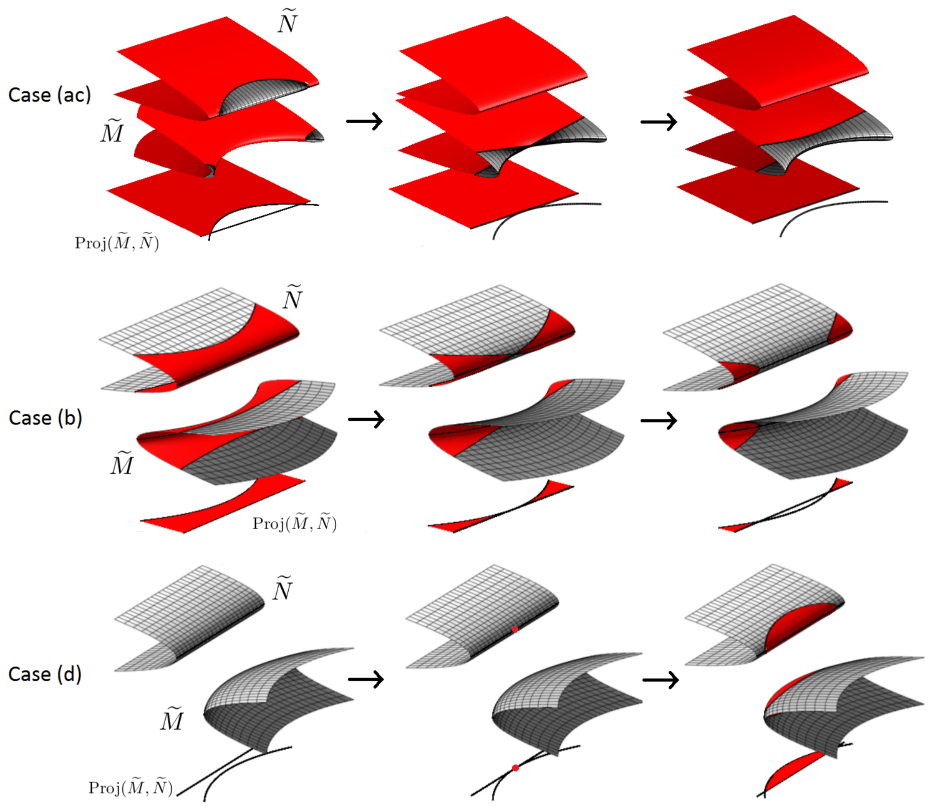


FIGURE 3. In this diagram, the Gauss map of the surfaces M_ε and N_ε is represented by vertical projection and the surfaces in this schematic representation are labelled $\widetilde{M}, \widetilde{N}$. The rows and columns are arranged as in Figure 2. See the above text for further explanation.

Proposition 3.4. *For Special Case 1.2, that is $f_{030} = f_3^2, g_{030} = g_3^2$, a special value of $\lambda = g_3/(g_3 \pm f_3)$ (Definition 2.6 or $Q = 0$ as in (7)) but $\lambda f_{20} + (1-\lambda)g_{20} \neq 0$, the function $H_{0\lambda}$ reduces under $\mathcal{A}$-equivalence to the normal form*

$$H_{0\lambda}(s_1, s_2, u_1, u_2) = \left(u_1, u_2, \pm s_1^2 + s_2^2 u_2 + s_2 u_1^2 + s_2^4 + s_2^3 u_1 + (p s_2 + q s_2^3)\right), \tag{12}$$

provided the geometrical conditions $R \neq 0$ (9), and 'exactly A_3-contact' (6) hold, together with a third condition on M_0, N_0 which will be generically satisfied. The terms $ps_2 + qs_2^3$ in brackets represent an $\mathcal{A}_e$-versal unfolding provided the geometrical condition $g_{011} \neq 0$ in (1) holds. See Figure 4 for a 'clock diagram' of the equidistants in the (p, q)-plane. □

A similar normal form, without the fourth variable s_1, but with an additional ambiguity of sign, occurs as 4_2^2 in [12]; see also [9]. The sign in front of s_1^2 will not affect our results since the critical set of $H_{0\lambda}$ has $s_1 = 0$. The versal unfolding condition means that as ε changes through 0 the normal to N tilts in a direction with a nonzero component along the y-axis, which is the asymptotic direction at $\varepsilon = 0$.

When λ moves away from a special value then, in (12), p remains at 0 while q becomes small and nonzero. We can then reduce (12) as in Generic Case 1.1, as follows. The 3-jet of (12) becomes $(u_1, u_2, s_1^2+s_2^2u_2+s_2u_1^2+qs_2^3)$ with $q \neq 0$. Replacing s_2 by ms_2+nu_2 where $3qn+1=0$ and $qm^3=1$, and then removing terms in the third component involving only u_1, u_2, reduces this to

$$\left(u_1, u_2, s_1^2 + \frac{1}{q^{1/3}}s_2u_1^2 - \frac{1}{3q^{4/3}}s_2u_2^2 + s_2^3\right).$$

The product of terms in front of $s_2u_1^2$ and $s_2u_2^2$ therefore has the sign of $-q$ and hence changes as q passes through 0. Furthermore it is not possible for both signs to be positive. We deduce the following.

Corollary 3.5. *Moving λ through a special value $\lambda = g_3/(g_3 \pm f_3)$ but keeping $\varepsilon = 0$ the type of equidistant always changes between Subcase 1.1.2 (negative definite) and Subcase 1.1.3 (indefinite) as in Proposition 3.2. It is not possible to realize the positive definite Subcase 1.1.1.*

Figure 4 shows a typical way in which equidistants near to a special value evolve as λ and ε change.

3.3. **Some further details of Special Case 1.2.** We take $\lambda_0 = \frac{g_3}{g_3+f_3}$ as a special value, assuming $f_{20} \neq 0, g_{20} \neq 0, f_3 > 0, g_3 > 0, \lambda_0 f_{20} + (1-\lambda_0)g_{20} \neq 0$, i.e. $f_{20}g_3 + g_{20}f_3 \neq 0$, and also $R \neq 0$ (9) hold. We write $\lambda = \lambda_0 + \alpha$ for nearby values, and examine the full versal unfolding $\widetilde{H}$ of H, as follows.

Thus the family of equidistants can be reduced to

$$\widetilde{H}(s_1, s_2, u_1, u_2, p, q) = \left(u_1, u_2, \pm s_1^2 + s_2^2u_2 + s_2u_1^2 + s_2^4 + s_2^3u_1 + ps_2 + qs_2^3\right) = (u_1, u_2, \tilde{h}), \tag{13}$$

say, where p, q are unfolding parameters that are closely related to ε, α respectively.

As an aid to understanding the equidistants for (ε, α) close to $(0,0)$ we can calculate the loci in the (p,q)-plane at which the structure of the singular set or the self-intersection set on the equidistant changes.

(1) **Singular set** For fixed p, q the singular set is the image under $\widetilde{H}$ of the set of points (using suffices for partial derivatives)

$$(0, s_2, u_1, u_2) \text{ such that } \tilde{h}_{s_2} = \tilde{h}_{s_2s_2} = 0.$$

Eliminating u_2, the equations reduce to

$$u_1^2 - 3s_2^2u_1 + (p - 3s_2^2q - 8s_2^3) = 0,$$

and the condition for this to have real solutions for u_1 is

$$9s_2^4 + 32s_2^3 + 12qs_2^2 - 4p \geq 0.$$

We are therefore interested in finding the pairs (p,q) for which there is a change in the number of real intervals in the set of s_2 satisfying this inequality. This will occur when the discriminant with respect to s_2 vanishes, and that gives a locus of the form

$$p = 0 \text{ or } p = \tfrac{1}{16}q^3 + \tfrac{9}{1024}q^4 + \dots. \tag{14}$$

See Figure 5.

(2) **Self-intersection locus** Suppose $(0, s_{21}, u_1, u_2)$ and $(0, s_{22}, u_1, u_2)$ are both in the critical set of $\widetilde{H}$ ($h_{s_1} = 0$ gives $s_1 = 0$) and have the same image under $\widetilde{H}$. Then with a little more trouble we can eliminate the u variables and obtain a condition in s_{21}, s_{22} alone. It is slightly more convenient to write $s_{21} = v_1 + v_2,\ s_{22} = v_1 - v_2$; then in fact we require $v_1(4v_1^3 + 16v_1^2 + 8qv_1 + p + q^2) \geq 0$. The number of v_1-intervals on which this holds will change when the discriminant with respect to v_1 vanishes. One case here gives

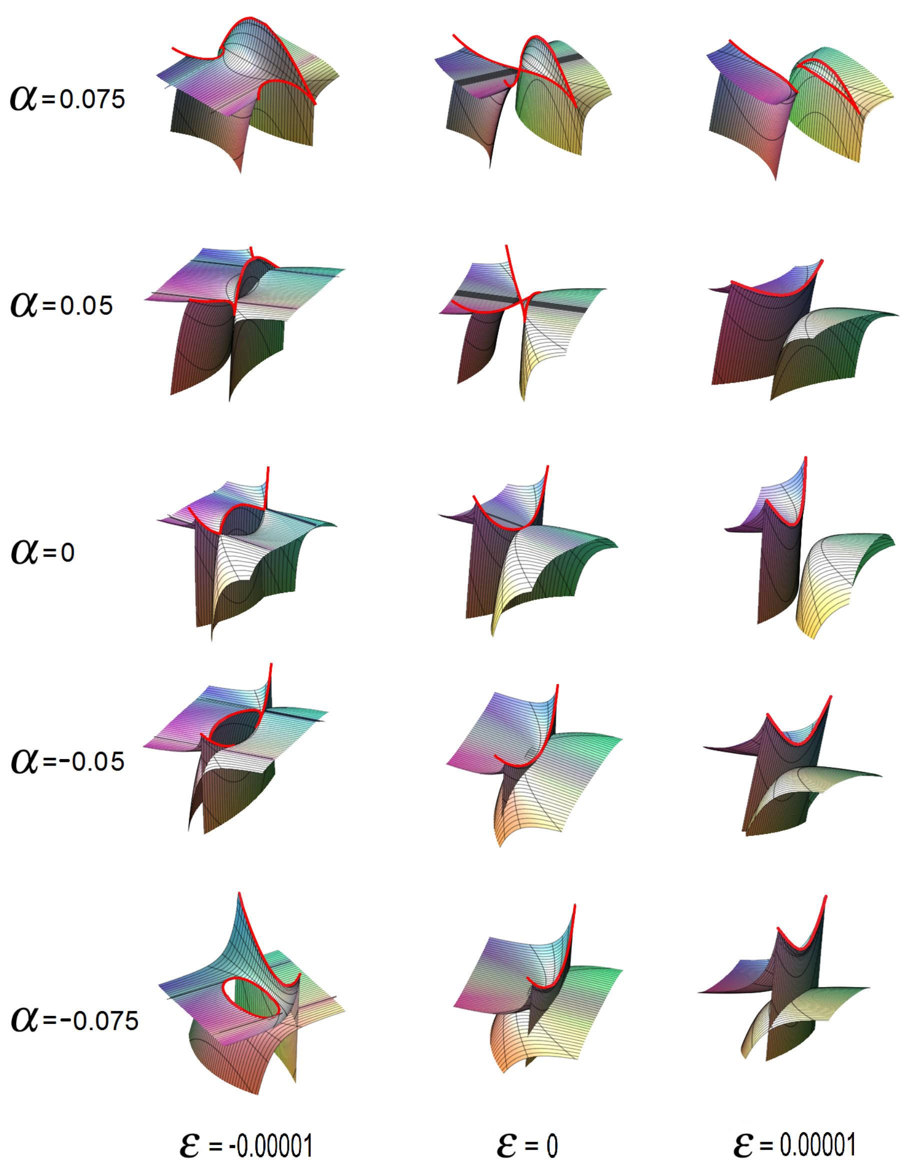


FIGURE 4. Special Case 1.2. A typical 'clock diagram' of equidistants close to a special value of $\lambda_0 = g_3/(g_3 \pm f_3)$. The vertical axis represents $\lambda = \lambda_0 + \alpha$ and the horizontal axis the parameter ε in the family of surfaces.

the same condition as (i) above, but we are concerned with the remaining possibility: taking into account that v_1, v_2 must both have real solutions the locus in the (p, q)-plane is

$$p = -q^2, \ q \geq 0, \tag{15}$$

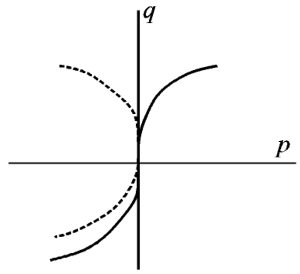


FIGURE 5. Special Case 1.2. A schematic drawing of two curves in the p, q-plane at which the structure of the equidistant in the family (13) changes, either because the cuspidal edge set changes (solid curve, together with the q-axis) or the self-intersection set changes (dashed curve).

where of course the double root is $v_1 = 0$, that is $s_{22} = -s_{21}$. (The other potential double root when $p = -q^2$ leads to $q = 2$ and is therefore not relevant to a neighbourhood of the origin in the (p, q)-plane.) See Figure 5.

4. Degenerate Case 2

In this section we give some details of Degenerate Case 2, that is $\lambda f_{20} + (1-\lambda)g_{20} = 0$. This gives a unique value of λ, namely $\lambda = \dfrac{g_{20}}{g_{20} - f_{20}}$. (If $f_{20} = g_{20}$ then, using $\lambda f_{20} + (1-\lambda)g_{20} = 0$, it follows that $f_{20} = g_{20} = 0$, contrary to our assumptions.) Thus whatever surfaces M_0, N_0 we start with there will be an equidistant which falls into this case. It turns out to be a rich area for investigation; here we shall give some invariants which help to separate out the many subcases. One of these invariants classifies the effect of changing λ slightly from the degenerate value, while preserving the geometrical situation of two surfaces with parallel tangent planes at parabolic points where the asymptotic directions are parallel, that is $\varepsilon = 0$ in (1), (2). See Proposition 4.1.

4.1. **A normal form for Degenerate Case 2.** The 2-jet of $H_{0\lambda}$ is now $(u_1, u_2, 2f_{20}s_1u_1)$. Writing the third component as $u_1(s_1 + \text{h.o.t.})+$ terms independent of u_1 and then using the bracketed expression to redefine s_1 we can eliminate u_1 from the higher terms. Then replacing s_2 by an expression of the form $s_2 + au_2$ we can remove the degree 3 terms $s_1u_2^2$ and $s_1s_2u_2$. When this is done, the coefficient of $s_2^2u_2$ becomes $3g_{030}f_{20}^2/g_{20}^2 \neq 0$ and the coefficient of $s_2u_2^2$ becomes $3f_{20}g_{030}(g_{20} - f_{20})/g_{20}^2 \neq 0$. We shall also assume that the coefficient of s_1^3 is nonzero to avoid further degeneration. We can now use scaling to reduce the 3-jet of $H_{0\lambda}$ to

$$\left(u_1,\ u_2,\ s_1u_1 + s_1^3 + s_2^2u_2 + s_2u_2^2 + bs_1^2s_2 + cs_1^2u_2 + ds_1s_2^2 + es_2^3\right),$$

for coefficients b, c, d, e. The 4-jet can then be reduced by similar arguments, including scaling, to

$$(u_1, u_2, h) = \left(u_1, u_2, s_1u_1 + s_1^3 + s_2^2u_2 + s_2u_2^2 + bs_1^2s_2 + cs_1^2u_2 + ds_1s_2^2 + es_2^3 + s_1^4 + (ps_2 + qs_1^2)\right),$$

provided the coefficient of s_1^4 is nonzero: this and the 4-$\mathcal{A}$-determinacy of this 4-jet hold generically, by standard calculations. The terms in brackets, $ps_1 + qs_1^2$, represent an $\mathcal{A}$-versal unfolding of this germ. We have not been able to reduce the number of coefficients b, c, d, e. We shall work with (4.1) as a 'normal form' and when appropriate interpret the coefficients in terms of the surfaces M_0, N_0.

The equidistant for M_0, N_0 and $\lambda = g_{20}/(g_{20} - f_{20})$ is then locally diffeomorphic to the image under (4.1) of the set $\{(s_1, s_2, u_1, u_2) : h_{s_1} = h_{s_2} = 0\}$. Here, $h_{s_1} = 0$ defines u_1 as a smooth function of the other three variables, while $h_{s_2} = 0$ can be written

$$\frac{\partial h}{\partial s_2} = (s_2 + u_2)^2 + bs_1^2 + 2ds_1s_2 + (3e - 1)s_2^2 = (s_2 + u_2)^2 - T(s_1, s_2) = 0, \tag{16}$$

say where T is a quadratic form in s_1, s_2 which we shall assume to be nondegenerate, that is $d^2 - b(3e - 1) \neq 0$.

4.2. **Plotting the equidistants.** It is also useful to rewrite the equation of the quadric cone C, given by $h_{s_2} = 0$, where $p = q = 0$ in (4.1), and provided $b \neq 0$, as

$$C: \quad (s_2 + u_2)^2 + b\left(s_1 + \frac{d}{b}s_2\right)^2 + \left(\frac{3be - b - d^2}{b}\right)s_2^2 = 0. \tag{17}$$

Note that this is a single point at the origin if and only if all coefficients are > 0 (since the first one is > 0), that is

$$b > 0, \ d^2 + b - 3be < 0;$$

compare Proposition 4.1.

The equidistant (for $p = q = 0$) is the image of C under the map $\mathbb{R}^3 \to \mathbb{R}^3$ given by

$$(s_1, s_2, u_2) \mapsto (u_1, u_2, \overline{h}(s_1, s_2, u_2))$$

where on the right-hand side u_1 is expressed in terms of s_1, s_2, u_2 using $h_{s_1} = 0$ and this is substituted into h, giving the function $\overline{h}$.

We can find a 'good' parametrization of the equidistant by using coordinates (x_1, x_2, s_2) and writing (17) as

$$x_1^2 + bx_2^2 + ks_2^2, \text{ where } k = \tfrac{3be-b-d^2}{b}, \ x_1 = s_2 + u_2, \ x_2 = s_1 + \tfrac{d}{b}s_2.$$

Thus the substitution to use in $\overline{h}$ is $u_2 = x_1 - s_2, \ s_1 = x_2 - \frac{d}{b}s_2$. The equidistant is then plotted as follows.

(1) If $b > 0$ and C is not a single point then $k < 0$ (i.e. $d^2 + b - 3be > 0$) and we write
$$x_1^2 + bx_2^2 = (-k)s_2^2,$$
so that for any $(x_1, x_2) \neq (0, 0)$ we have two distinct values for s_2: there is no restriction on the values of x_1, x_2. We use x_1, x_2 as parameters and the two 'halves' of C are given by the two values of s_2.

(2) If $b < 0, k > 0$ (i.e. $d^2 + b - 3be > 0$) then we similarly write $x_1^2 + ks_2^2 = (-b)x_2^2$, so that for any $(x_1, s_2) \neq (0, 0)$ we have two distinct values for x_2. Here x_1, s_2 are used as parameters.

(3) Finally if $b < 0, k < 0$ (i.e. $d^2 + b - 3be < 0$) then we write $x_1^2 = (-b)x_2^2 + (-k)s_2^2$ and for any $(x_2, s_2) \neq (0, 0)$ we have two distinct values for x_1. Here x_2, s_2 are used as parameters.

For values of (p, q) other than $(0, 0)$ the equation of C acquires an extra term $-p$ on the right-hand side, thus creating a hyperboloid of one or two sheets (or an ellipsoid when C is a single point). In fact the hyperboloid has one sheet when $bkp > 0$, that is $(d^2 + b - 3be)p < 0)$, and two sheets when $bkp < 0$, that is $(d^2 + b - 3be)p > 0)$. In the two-sheet situation the same method as above plots the equidistant, without restrictions on the values of the parameters. In the one-sheet situation the points in the parameter plane lie outside an ellipse, the 'waist' of the hyperboloid. This ellipse is given in the three situations above by $x_1^2 + bx_2^2 = -p, \ x_1^2 + ks_2^2 = -p$ and $(-b)x_2^2 + (-k)s_2^2 = p$ respectively. In the situation where C is a single point, and $p < 0$, the points in the parameter plane lie inside an ellipse. In all situations, q does not affect the hyperboloid or ellipsoid, but of course its value affects the function $\overline{h}$.

4.3. **Nearby non-special values of λ.** Here, we examine the effect of adding in the term qs_1^2 in (4.1). This represents changing λ from the value $g_{20}/(g_{20} - f_{20})$ to a nearby value, which will be of the type considered in Generic Case 1.1, provided the coefficient e of s_2^3 in (4.1) is nonzero, and to avoid further degeneracy we shall assume this to be true. We determine here, in terms of b, c, d, e, which subcase of Proposition 3.2 is obtained, and then refer this back to the surfaces M_0, N_0. (The subcase does not depend on the sign of q in the added term qs_1^2.) To do this we reduce (4.1), with $p = 0$ but with qs_1^2 present, to the normal form found above for Generic Case 1.1, by making the 'left' and 'right' changes of coordinates as sketched above. We can restrict attention for this to the terms of (4.1) of degree ≤ 3 since the Generic Case 1.1 germ is 3-$\mathcal{A}$-determined. Thus we start by redefining s_1 ('completing the square') to change the degree 2 terms to s_1^2, remove the terms in u_1, u_2 only, remove the remaining terms besides s_1^2 that are divisible by s_1 and then redefine s_2 by adding suitable multiples of u_1 and u_2. The result of this is to reduce the 3-jet of (4.1) by $\mathcal{A}$-equivalence to the form

$$\left(u_1,\ u_1,\ qs_1^2 + es_2^3 + \frac{s_2}{12eq^2}\left((3be - d^2)u_1^2 + 4qdu_1u_2 + 4q^2(3e-1)u_2^2\right)\right).$$

The discriminant of the quadratic form in u_1, u_2 is $(d^2 + b - 3be)/3eq^2$, so this form is definite if and only if $e(b + d^2 - 3be) < 0$. Scaling so that the terms in s_1^2, s_2^3 have coefficients equal to 1 multiplies the quadratic form in u_1, u_2 by $(q^2e)^{-1/3}$, and from this we deduce the following, where (i) and (ii) are derived by direct calculations from the parametrizations of M_0 and N_0.

Proposition 4.1. *The normal form (4.1) for Degenerate Case 2, with $p = 0$ but q nonzero and small, corresponding to a small change in λ, gives the following subcases of Generic Case 1.1 (general λ):*
Subcase 1.1.1 (positive definite, $++$): $e > \frac{1}{3}$ and $d^2 + b - 3be < 0$,
Subcase 1.1.2 (negative definite, $--$): $e < \frac{1}{3}$ and $e(d^2 + b - 3be) < 0$,
Subcase 1.1.3 (indefinite, $+-$): $e(d^2 + b - 3be > 0$.

In terms of the surfaces M_0, N_0,
(i) *When $f_{030}g_{030} > 0$, so $f_{030} = f_3^2$, $g_{030} = g_3^2$, $e < \frac{1}{3}$ and has the sign of $f_{20}g_3^2 - g_{20}f_3^2$ while $d^2 + b - 3be$ has the sign of $-R$ as in (9).*
(ii) *When $f_{030}g_{030} < 0$, so $f_{030} = f_3^2$, $g_{030} = -g_3^2$, $e > \frac{1}{3}$ and $d^2 + b - 3be$ has the sign of R.*

4.4. **Invariants distinguishing subcases of Degenerate Case 2.** We shall use the following:

(1) The number of cuspidal edges on the equidistant for $p = q = 0$, which can be 0, 2 or 4 (see below);
(2) The number of self-intersection curves on the equidistant for $p = q = 0$, which can be 0, 1, 2 or 3 (see §4.5);
(3) The subcase of Generic Case 1.1 given in Proposition 4.1 which is obtained by changing λ slightly.

This might give $3 \times 4 \times 3 = 36$ subcases but fortunately many of these combinations cannot be realized. We shall give values of b, c, d, e realizing of all possible subcases in §4.6, Table 1 below.

For given values of these invariants, the interval in which e lies, either $e < 0$ or $0 < e < \frac{1}{3}$ or $e > \frac{1}{3}$ could in principle affect the equidistant but so far as we are aware the basic geometrical structure—the qualitative nature of the equidistant—is not affected.

The number of cuspidal edges, that is 1-dimensionial singular sets, on the equidistant, can be calcuated as follows. We can regard $h_{s_2} = 0$, as in §4.2 above, as the equation of a quadric cone C in $\mathbb{R}^3$ with coordinates (s_1, s_2, u_2). The quadric cone C is nondegenerate since T in (16) is a nondegenerate quadratic form, and consists of the origin alone if and only if T is negative definite (that is, $d^2 < b(3e-1)$ and $b > 0$), otherwise it is a real cone, or equivalently a real nonsingular conic in $\mathbb{R}P^2$.

When T is not negative definite, the equidistant therefore has two 'branches', which are the images of the two halves of the cone; these branches may intersect (apart from at the origin) and will generally themselves be singular. Writing the equation of C more briefly as $\gamma(s_1, s_2, u_2) = 0$, the singular set of the equidistant is the image of certain curves on C, given by the additional equation

$$\overline{h}_{s_1}\gamma_{s_2} - \overline{h}_{s_2}\gamma_{s_1} = 0.$$

(This can be written in terms of h itself as $h_{s_1s_1}h_{s_2s_2} - h^2_{s_1s_2} = 0$.) The lowest terms of the left hand side are of degree 2 in s_1, s_2, u_2 and therefore give another conic C_2 in $\mathbb{R}P^2$. The equation of C_2 is in fact

$$(b^2 - 3d)s_1^2 + (bd - 9e)s_1s_2 - (cd + 3)s_1u_2 + (d^2 - 3be)s_2^2 - (3ce + b)s_2u_2 - cu_2^2 = 0.$$

This meets the nonsingular conic $\gamma = 0$ in 0, 2 or 4 real points. (The conic C_2 cannot in fact be a single point: examination of the matrix of the above quadratic form in variables s_1, s_2, u_2 defining C_2 shows that its determinant is always ≤ 0 so the quadratic form cannot be positive definite, and negative definiteness is also ruled out by examining the signs of the other leading minors. The leading 1×1 minor cannot be < 0 at the same time as the leading 2×2 minor is > 0.) There are therefore 0, 2 or 4 curves through the origin on C whose images are the singular points, the cuspidal edges, of the equidistant. These cuspidal edges pass through the origin, lying on both 'sheets' of the equidistant.

The number of cuspidal edges can be calculated for example by substituting

$$(s_1, s_2, u_2) = (mt, nt, t)$$

in the equations of C and C_2, taking out the factor t^2 and finding the common solutions of the two resulting quadratic equations in m, n. Eliminating one of m, n gives a degree 4 equation in the other and there are standard algebraic techniques for computing the number of real solutions of a quartic equation—or for given (b, c, d, e) we can solve numerically. The results for the Classes I-X are given in Table 1 below.

4.5. **Self-intersections of the equidistant in Degenerate Case 2.** We start with the normal form (4.1) in §4, namely

$$(u_1, u_2, h) = \left(u_1,\ u_2,\ s_1u_1 + s_1^3 + s_2^2u_2 + s_2u_2^2 + bs_1^2s_2 + cs_1^2u_2 + ds_1s_2^2 + es_2^3 + s_1^4 + ps_2 + qs_1^2\right),$$

subject to the critical set conditions $h_{s_1} = h_{s_2} = 0$. We include the unfolding terms $ps_2 + qs_1^2$ though we are particularly interested in the self-intersections for $p = q = 0$. We can immediately solve $h_{s_1} = 0$ for u_1:

$$u_1 = -2bs_1s_2 - 2cs_1u_2 - ds_2^2 - 3s_1^2 - 4s_1^3 - 2qs_1,$$

so that the equations which state that two domain points (s_1, s_2, u_1, u_2) and say (t_1, t_2, u_1, u_2) have the same image take the following form.
(SI1): the above formula for u_1 gives the same answer for both domain points;
(SI2): the formula for h above gives the same answer for both domain points;
(SI3): $h_{s_2}(s_1, s_2, u_1, u_2) = 0$; and
(SI4): $h_{t_2}(t_1, t_2, u_1, u_2) = 0$.

It is convenient to make the substitution $s_1 = x_1 + y_1, t_1 = x_1 - y_1, s_2 = x_2 + y_2, t_2 = x_2 - y_2$, so that the 'trivial solution' $s_1 = t_1, s_2 = t_2$ becomes $y_1 = y_2 = 0$. Furthermore replacing y_1 by $-y_1$ and y_2 by $-y_2$ interchanges (s_1, s_2) and (t_1, t_2), that is interchanges the two domain points (s_1, s_2, u_1, u_2) and (t_1, t_2, u_1, u_2) with the same image in $\mathbb{R}^3$ under the normal form map (4.1). With this substitution the equations become say (SI1′), etc., and we use (SI3′)-(SI4′) to solve for u_2:

$$u_2 = -\frac{bx_1y_1 + dx_1y_2 + dx_2y_1 + 3ex_2y_2}{y_2},$$

where the denominator y_2 is harmless since it is easy to check that if $y_2 = 0$ then the other equations imply that $y_1 = 0$ too. *Note that this expression does not involve p, q.*

We can solve (SI1′) for x_2:

$$x_2 = \frac{bcx_1y_1^2 + cdx_1y_1y_2 - bx_1y_2^2 - 6x_1^2y_1y_2 - 2y_1^3y_2 - 3x_1y_1y_2 - qy_1y_2}{-cdy_1^2 - 3cey_1y_2 + by_1y_2 + dy_2^2}.$$

This time we may need to investigate the vanishing of the denominator, but assuming the denominator is nonzero and substituting for x_2 we find that the equation (SI2′)-y_2((SI3′)+(SI4′)) reduces to

$$\text{SI5} : by_1^2y_2 + dy_1y_2^2 + ey_2^3 + 4x_1y_1^3 + y_1^3 = 0. \tag{18}$$

This is to be treated as the equation of a surface in 3-space (x_1, y_1, y_2) which contains the x_1-axis, since $(x_1, 0, 0)$ is always a solution. The surface will have a certain number of 'sheets' passing through the origin, equal to the number of values of k which make the first coordinate zero in the following parametriztion of SI5 by k and y_1.

$$\left(-\frac{ek^3 + dk^2 + bk + 1}{4},\ y_1,\ ky_1\right). \tag{19}$$

If $y_1 = 0$ in (18), then $y_2 = 0$ and x_1 is arbitrary; and indeed, being cubic in k, (19) gives all points $(x_1, 0, 0)$, possibly for more than one (real) k. If $y_1 \neq 0$ then we solve (18) for x_1 and writing $y_2 = ky_1$ produces the given value $-\frac{1}{4}(ek^3 + dk^2 + bk + 1)$ for x_1. Conversely, every point (19) satisfies (18) by substitution. Hence (19) parametrizes the complete surface (18). Two examples are shown in Figure 6.

Note that the surface (18) and the parametrization (19) are independent of the unfolding parameters p, q.

Proposition 4.2. *The number of smooth real sheets of the surface (18) through the origin in (x_1, y_1, y_2)-space is 1 or 3 according as*

$$27e^2 + 2b(2b^2 - 9d)e + d^2(4d - b^2) > 0 \textit{ or } < 0 \textit{ respectively.}$$

This number is therefore the maximum number of self-intersection branches of the equidistant, for any p, q. If $b^2 < 3d$ then the displayed expression is > 0 for all values of e.

Proof This is a matter of calculating the discriminant of the cubic polynomial ek^3+dk^2+bk+1 in k, and the discriminant $16(b^2 - 3d)^3$ of the displayed quadratic polynomial in e. The sheets will be smooth provided the cubic in k has no repeated root, that is provided the discriminant is nonzero. □

Remark 4.3. In §2.4 we noted that, in the current Degenerate Case 2, the sign of a certain polynomial in the coefficients of the two surfaces M_0, N_0 determines whether the 'scaled contact map' has type D_4^+ or D_4^-. By reducing to normal form as in §3 we can re-express this polynomial in terms of the coefficients b, c, d, e of the normal form. When this is done, we find that the condition for one (resp. three) sheets as in the above proposition coincides with the condition for D_4^+ (resp. D_4^-) in the scaled contact map. We do not know the full significance of this fact.

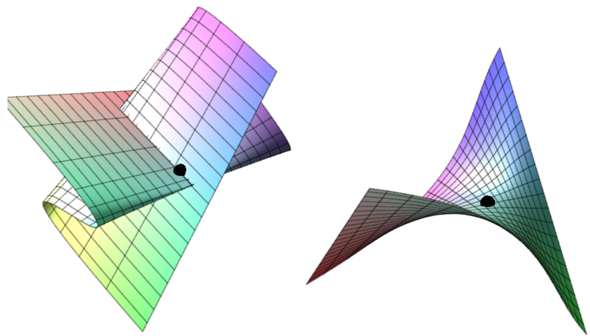


FIGURE 6. The surface given by (18) or (19), for (left) $b = 8, c = -4, d = -3, e = -1$, with three smooth sheets through the origin, which is marked by a black dot; (right) $b = -8, c = 4, d = -3, e = -1$, with one smooth sheet. (See Proposition 4.2.) These are respectively Class III and Class IX in Table 1 below. Note that in the first of these there are nevertheless only two self-intersection curves of the equidistant for $p = q = 0$, using the criterion of Proposition 4.6. In fact the picture for Class II is very similar to the left-hand figure, but there is only one self-intersection curve of the equidistant for $p = q = 0$.

Substituting $x_1 = -\frac{1}{4}(ek^3 + dk^2 + bk + 1)$ and $y_2 = ky_1$ in one of the conditions on x_1, y_1, y_2 not fully used yet (for example, SI2$'$) we obtain a single equation in y_1, k (involving now p and q) which determines the branches of the self-intersection set of the equidistant. We are interested in values of k close to a zero k_0 of the polynomial $ek^3 + dk^2 + bk + 1$, so we now write $k = k_0 + z$ say where z, as well as y_1, p, q, will be small. Since k_0 satisfies a cubic equation we can express k_0^3 in terms of k_0 and k_0^2, namely as $k_0^3 = (-dk_0^2 - bk_0 - 1)/e$, and therefore all higher powers of k_0 can be expressed in terms of k_0, k_0^2 as well.

Definition 4.4. For a chosen value of k_0, the polynomial in y_1, z, p, q just formed, the zero set of which determines the solutions to (SI1)-(SI4) or their equivalents (SI1$'$)-(SI4$'$), and hence determines the points corresponding to self-intersections of the equidistant, will be called $L(k_0)$. In the special case $p = q = 0$, we shall write $L_0(k_0)$ for the polynomial in y_1 and z.

We deduce the following; the statements 2-5 are easily checked by direct calculation.

Proposition 4.5. (1) *For each real root k_0 of $ek^3 + dk^2 + bk + 1 = 0$ one smooth sheet of the surface (18) is parametrized by (y_1, z) and the points which correspond to self-intersections on the equidistant for any p, q are given by the additional equation $L(k_0) = 0$.*

(2) *The polynomials $L(k_0)$ and $L_0(k_0)$ contain only the powers y_1^2 and y_1^4 of y_1. For any p, q the zero-set of $L(k_0)$ is symmetric about the y_1-axis in the (y_1, z)-plane.*

(3) *The other variable z occurs to powers ≤ 14 in $L(k_0)$. The coefficient of z^{14} is in fact $27e^5(3e - 1)$ which will not be zero since $e = 0, \frac{1}{3}$ are excluded values.*

(4) *The linear part of $L(k_0)$ has the form constant $\times p$. The nonzero quadratic terms are in y_1^2, z^2, zp, zq and q^2.*

(5) *The 2-jet of $L_0(k_0)$ has the form $c_0y_1^2 + c_2z^2$.*

The last statement above implies that, for $p = q = 0$, a given sheet of the surface (18), that is a given value of k_0, will correspond to a branch of the self-intersection set of the equidistant if and only of c_0, c_2 have opposite signs. When $c_0c_2 > 0$ there is only an isolated point at $y_1 = z = 0$. When $c_0c_2 < 0$ the two real branches of the set $L_0(k_0) = 0$ (forming a crossing at the origin $y_1 = z = 0$) will give only one branch of the self-intersection set because, as noted above, replacing y_1 by $-y_1$, and hence $y_2 = ky_1$ by $-y_2 = k(-y_1)$, merely interchanges the domain points contributing to the self-intersection.

Each of c_0, c_2 is quadratic in k_0; multiplying them gives an expression of degree 4 which can be reduced to degree 2 again using the equation $ek^3 + dk^2 + bk + 1 = 0$. Writing the resulting quadratic expression as $N = N_0(b, c, d, e) + N_1(b, c, d, e)k_0 + N_2(b, c, d, e)k_0^2$ we have the following, which is used to determine the number of self-intersection branches of the equidistant in the ten classes of Table 1.

Proposition 4.6. *The number of real branches of the self-intersection set of the equidistant for $p = q = 0$ is the number of solutions $k = k_0$ of $ek^3 + dk^2 + bk + 1 = 0$ at which the quadratic N is < 0.*

As (p, q) moves away from $(0, 0)$ we can still trace the zero set of $L(y_0)$ in the (y_1, z)-plane. An isolated point may disappear or open into a symmetric loop, which represents a self-intersection of the equidistant having two endpoints, if the loop crosses the y_1-axis, and a closed self-intersection curve if it does not. A crossing will become a 'hyperbola'; if it crosses the y_1-axis then the corresponding self-intersection curve will have two endpoints and if not then it will be an unbroken arc. This is illustrated in the next section.

4.6. **Examples.** Considering different realizable values of the three invariants in §4.4, we have the ten classes of equidistant given in Table 1. It is also possible in some of these classes to allow values of e in different ranges $e < 0,\ 0 < e < \frac{1}{3},\ e > \frac{1}{3}$ but this does not appear to affect the equidistant in any qualitative way. We can compute the curves in the (p, q)-plane alomg which the cusp edges or the self-intersection curves on the equidistant underfgo a qualitative change. (The ten cases of the table in fact have ten distinct configurations of these curves.)

Class	Cusp edges	self-int	Subcase (Prop. 4.1)	b	c	d	e
I	0	0	$++$	8	4	-3	1
II	0	1	$+-$	8	-4	-3	$\frac{1}{6}$
III	0	2	$--$	8	-4	-3	-1
IV	2	0	$+-$	-13	6	-3	-5
V	2	1	$--$	1	2	3	-1
VI	2	2	$+-$	8	4	-3	$\frac{1}{6}$
VII	2	3	$--$	-13	-6	1	$\frac{1}{6}$
VIII	2	3	$+-$	-8	4	1	$\frac{1}{6}$
IX	4	1	$+-$	-8	4	-3	-1
X	4	3	$+-$	-8	6	-3	10

TABLE 1. Ten distinct classes of Case 2, giving all possible realizations of the three invariants of §4.4, and examples of values of b, c, d, e which realize these invariants. The fourth column refers to the 'non-special' type which results from changing λ slightly from the degenerate value.

We shall now give more detail on Case II of the table, showing how the cuspidal edges and self-intersections of the equidistant evolve as (p, q) in (4.1) makes a circuit of the origin. Figure 8 shows the transformations in the cuspidal edge as (p, q) moves in such a circuit and Figure 9 gives schematic diagrams of the corresponding equidistants, indicating their self-intersections and cusp

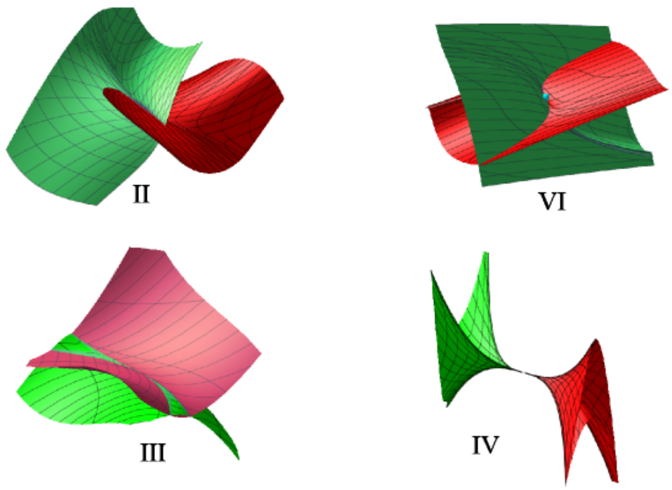


FIGURE 7. Cases II, III, IV and VI from Table 1, for $p = q = 0$. The origin is marked for Case VI, where there are two very narrow swallowtails passing through the origin, contributing two cusp edges and one self-intersection, and the other self-intersection is visible where the sheets pass through one-another.

edges. We use the following labelling on these figures to indicate transitions (perestroikas) in the structure of the equidistant.

Notation 4.7. $A_2^{++}, A_2^{--}, A_2^{+-}$ refer to Subcases 1.1.1, 1.1.2 and 1.1.3, as in Proposition 3.2. The corresponding transitions have also been described as 'Zeldovich's pancakes' or 'flying saucers', 'the death of a compact component of an edge', and 'the hyperbolic transformation of an edge', respectively. See also [9, 10].

A_3^+, A_3^- refer to the 'swallowtail-lips' and 'swallowtail-beaks' singularity respectively.

D_4^- refers to the 'pyramid' singularity (and D_4^+ would similarly be the 'purse' singularity).

$TA_1^{3,1}$, called such in [10, 1] (see also [9]) refers to the situation where three smooth sheets of the equidistant are pairwise transversal to each other, but the curve of intersection of any two of them is tangent to the third sheet at the moment of bifurcation.

5. Conclusion and further work

There have been many recent studies of singularities of (affine) equidistants of surfaces. For a single equidistant of a fixed surface, the generic singularities are A_1, A_2, A_3 (see for example [8, 4]); for a fixed surface, but allowing the ratio λ defining the equidistant to vary, the generic singularities are now A_1 (smooth surface), A_2 (cusp edge), A_3 (swallowtail), $A_3^{\pm}$ (swallowtail beaks/lips transition), A_4 (butterfly) and also $D_4^{\pm}$ (purse/pyramid) (compare [7]). The context of the present paper is to extend this to 1-parameter families of surfaces, the parameter in the family being ε in our notation, so that there are now two parameters to consider, λ and ε. The particular degeneracy in the ε family studied here comes from a 'supercaustic chord',

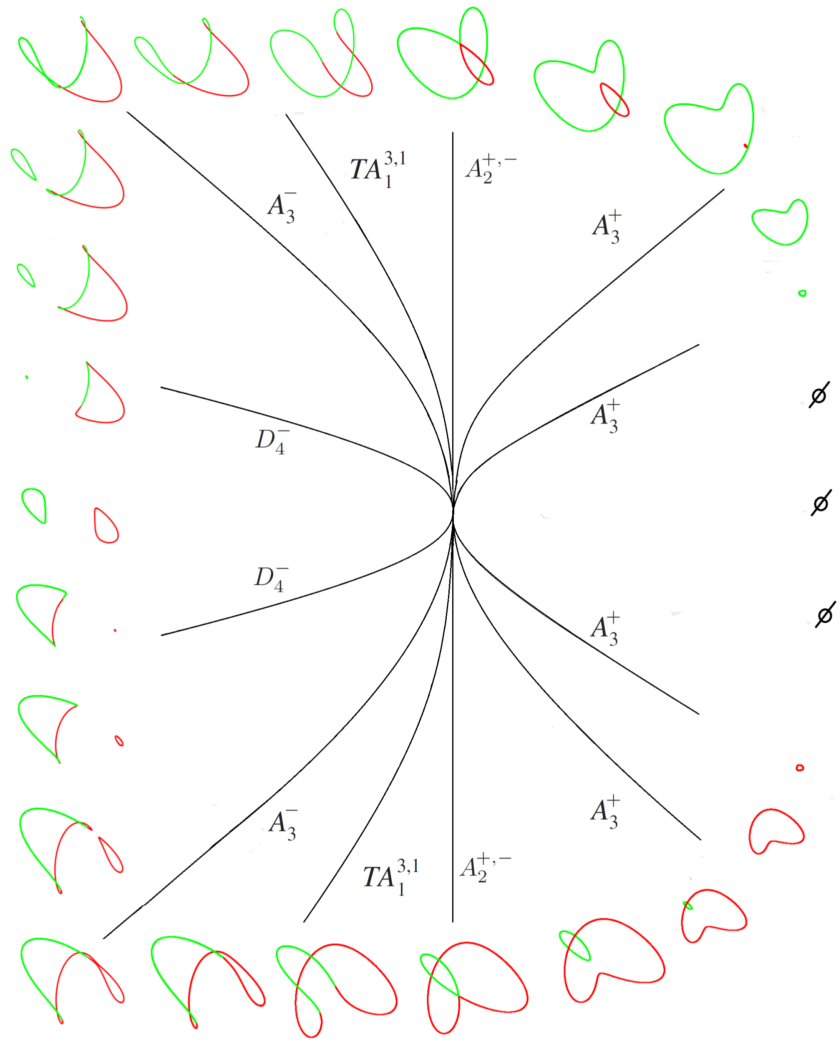


FIGURE 8. Pre-images of the cuspidal edges on the equidistants in Class II of Table 1 for unfolding parameters (p, q) making a circuit of the origin. The colours correspond to either the two parts of a hyperboloid of two sheets as in §4.2 or to the two parts into which a hyperboloid of one sheet is cut by the plane through the 'waist'. For the labelling of transitions, see Notation 4.7.

that is a chord joining two parabolic points with parallel tangent planes and parallel asymptotic directions. This occurs generically only in a 1-parameter family of surfaces. Along such a chord there may be special values of λ where singularities become more degenerate, depending on the relative local geometry of the surface patches at the ends of the chord. When two such special values exist (our Case 1.2) this corresponds to the intersection of an A_3 stratum with the supercaustic. In addition, there always exists a value of λ, which we call the degenerate Case 2. This corresponds to the intersection of a D_4 stratum with the supercaustic, and we

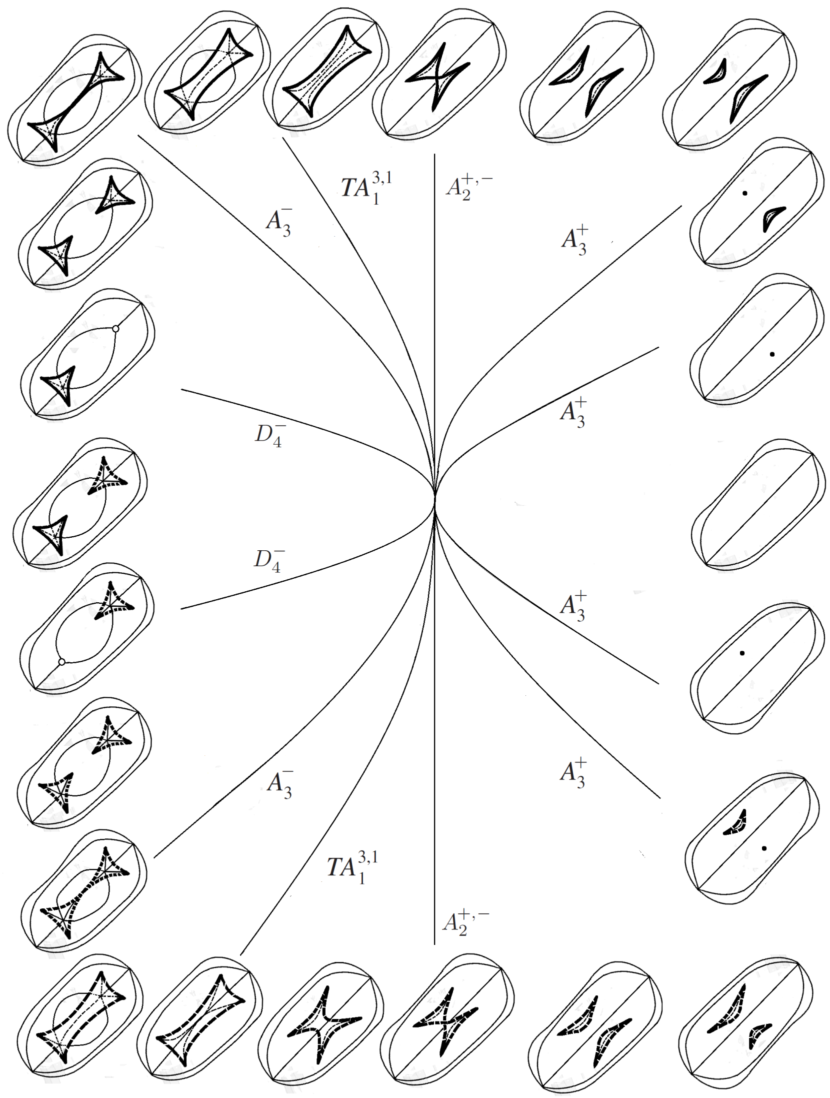


FIGURE 9. Schematic diagram of the equidistants for Class II of Table 1, with the unfolding parameters (p, q) making a circuit of the origin. The figure shows cuspidal edges (thick lines) and self-intersections (thin lines) with solid and dashed curves indicating visibility from one direction. For the labelling, see Notation 4.7.

elucidate ten geometrically distinct cases. Our paper also gives a natural geometric setting for many singularity types which belong to the list of corank 1 maps from $\mathbb{R}^3$ to $\mathbb{R}^3$ ([12, 9]), with the addition of a quadratic term in the extra variable which does not affect the critical set. The cases where equidistants are defined by $\lambda = 0$ or 1 remain to be studied.

A second natural 1-parameter family of surfaces is derived from the 'tangential' case in which two surface pieces share a common tangent plane (see for example [8]); here boundary singularities occur in the generic case, so that making one contact point parabolic in a 1-parameter family will introduce additional boundary singularities. The full adjacency diagram for singularities of equidistants of 1-parameter families of surfaces, not restricted to the supercaustic case, also remains to be found.

Acknowledgement We are grateful to Aleksandr Pukhlikov for helpful discussions on calculating self-intersections.

References

[1] Suliman Alsaeed, *Local Invariants of Fronts in 3-Manifolds*, PhD thesis, University of Liverpool, 2014. AlsaeedSul_Nov2014_2006756.pdf
[2] M. V. Berry, 'Semi-classical mechanics in phase space: a study of Wigner's function', *Philos. Trans. Royal Soc. London* 287 (1977), 237–271. DOI: 10.1098/rsta.1977.0145
[3] W. Domitrz, M. Manoel and P. de M. Rios, 'The Wigner caustic on shell and singularities of odd functions', *J. Geometry and Physics* 71 (2013), 58–72. DOI: 10.1016/j.geomphys.2013.04.005
[4] W. Domitrz, P. de M. Rios and M. A. S. Ruas, 'Singularities of affine equidistants: projections and contacts', *J. Singularities* 10 (2014), 67–81. DOI: 10.5427/jsing.2014.10d
[5] Peter Giblin and Graham Reeve, 'Centre symmetry sets of families of plane curves', *Demonstratio Math.* 48 (2015), 167–192. DOI: 10.1515/dema-2015-0016
[6] Peter Giblin and Graham Reeve, 'Equidistants and their duals for families of plane curves' *Advanced Studies in Pure Mathematics, (Singularities in Generic Geometry)* 78 (2018), 251–272. DOI: 10.2969/aspm/07810251
[7] P. J. Giblin and V. M. Zakalyukin, 'Singularities of centre symmetry sets', *Proc. London Math. Soc.* 90 (2005), 132–166. DOI: 10.1112/s0024611504014923
[8] Peter J. Giblin and Vladimir M. Zakalyukin, 'Recognition of centre symmetry set singularities', *Geometriae Dedicata* 130 (2007), 43–58. DOI: 10.1007/s10711-007-9204-2
[9] Victor Goryunov, 'Local invariants of maps between 3-manifolds', *J. Topology* 6, (2013), 757-776. DOI: 10.1112/jtopol/jtt015
[10] Victor Goryunov and Suliman Alsaeed, 'Local Invariants of Framed Fronts in 3-Manifolds', *Arnold Math J.* 211 (2015), 211–232. DOI: 10.1007/s40598-015-0016-4
[11] S. Janeczko, 'Bifurcations of the center of symmetry', *Geom. Dedicata* 60 (1996), 9–16. DOI: 10.1007/bf00150864
[12] W. L. Marar and F. Tari, 'On the geometry of simple germs of co-rank 1 maps from $\mathbb{R}^3$ to $\mathbb{R}^3$', *Math. Proc. Cambridge Philos. Soc.* 119 (1996), 469–481. DOI: 10.1017/S030500410007434X
[13] Graham M. Reeve and Vladimir M. Zakalyukin, 'Propagations from a space curve in three space with indicatrix a surface', *Journal of Singularities* 6 (2012), 131–145. DOI: 10.5427/jsing.2012.6k

Peter Giblin, Department of Mathematical Sciences, The University of Liverpool, Liverpool L69 7ZL, UK
Email address: pjgiblin@liv.ac.uk

Graham Reeve, Department of Mathematics and Computer Science, Liverpool Hope University, Liverpool L16 9JD, UK
Email address: reeveg@hope.ac.uk

Journal *of* Singularities
Volume 21 (2020), 119-131

Proc. of Geometric and Algebraic Singularity Theory, Będlewo, 2017
DOI: 10.5427/jsing.2020.21g

ON THE COLENGTH OF FRACTIONAL IDEALS

E. M. N. DE GUZMÁN AND A. HEFEZ

Abstract. The main goal of this paper is to give a recursive formula for the colength of a fractional ideal in terms of some maximal points of its value set and of its projections. The fractional ideals are relative to a class of rings called admissible, a more general class of one dimensional local rings that contains those of algebroid curves. For fractional ideals of such rings with two or three minimal primes, a closed formula for the colength is provided.

1. Introduction

The computation of the colength of a fractional ideal of a ring of an irreducible algebroid plane curve in terms of its value set was known since the work of Gorenstein in the fifties of last century, at least (cf. [6]). Such computation was performed for a larger class of analytically reduced but reducible rings by D'Anna in [2, §2], where colengths of fractional ideals and lengths of maximal saturated chains in their sets of values are related. D'Anna's method requires the knowledge of many elements in the set of values, a disadvantage that would be desirable to overcome to increase computational efficiency. In fact, in the particular case of an algebroid curve with two branches, Barucci, D'Anna and Fröberg, in [1], were able to give an explicit formula for the colength of a given fractional ideal in terms of some maximal points of its value set.

Local rings of algebroid curves and the class studied by D'Anna in [2] belong to the larger class of *admissible rings* considered in this paper. By such a ring, we mean a one dimensional, local, noetherian, Cohen-Macaulay, analytically reduced and residually rational ring such that the cardinality of its residue field is sufficiently large (see [8] for more details). For simplicity and without loss of generality (cf. [2, §1]), we will also assume that our rings are complete with respect to the topology induced by the maximal ideal. In such case, a sufficiently large residue field means that its cardinality is greater than or equal to the number r of minimal primes of the ring.

One of our main results, Theorem 10, gives a recursive formula on the number r for the colength of a fractional ideal in a complete admissible ring. The important feature is that the computation requires only few special points of the value set, namely, its relative maximal points and those of its projections. The other main result is Corollary 20 that provides a closed formula for the colength in the case of three minimal primes. It is worth noting that such a closed formula for three minimal primes is not a straightforward consequence of the recursive formula established in Theorem 10, since its proof demands a careful analysis of the geometry of the maximal points of the value set.

The outline of the paper is as follows. Section 2 collects some preliminaries and notation regarding the general background of the article. Section 3 is concerned with the definition of

---

2010 *Mathematics Subject Classification.* 13H10, 14H20.

Key words and phrases. Admissible rings, Algebroid curves, Fractional ideals, Value sets of ideals, Colength of ideals.

The first author was supported by a fellowship from CAPES, while the second one was partially supported by the CNPq Grant 307873/2016-1.

value sets, recalling three useful analog properties to ones obtained for semigroups of values by Delgado and Garcia (cf. [3] and [5]). Section 4 introduces and analyzes different kinds of maximal points in the value set to get enough tools to pass to Section 5 that is eventually concerned with the announced recursive formula for the colength of fractional ideals in admissible rings. To ease the comparison with the previous results due to Barucci, D'Anna and Fröberg, we first analyze their recipe for $r = 2$, while we devote Section 5.2 to the case $r \geq 3$. The closed formula for $r = 3$ is finally dealt with in Section 6 where a fine detailed analysis of the geometry of the maximal points is offered in a series of lemmas, culminating with Lemma 18 that unavoidably leads, after the case by case analysis, the statement and proof of Theorem 19 that confirms a conjectural formula by M. Hernandes (cf. [7]).

2. General background

In this section we refer to [2] for our unproved statements. Let $\wp_1, \ldots, \wp_r$ be the minimal primes of an admissible complete ring R. We will use the notation $I = \{1, \ldots, r\}$. We set $R_i = R/\wp_i$ and will denote by $\pi_i \colon R \to R_i$ the canonical surjection. Since R is reduced, we have an injective homomorphism

$$\begin{array}{rcl} \pi\colon R & \hookrightarrow & R_1 \times \cdots \times R_r \\ h & \mapsto & (\pi_1(h), \ldots, \pi_r(h)). \end{array}$$

More generally, if $J = \{j_1 < \cdots < j_s\}$ is any subset of I, we may consider $R_J = R/\bigcap_{i=1}^{s} \wp_{j_i}$ and will denote by $\pi_J \colon R \longrightarrow R_J$ the natural surjection.

We will denote by $\mathcal{K}$ the total ring of fractions of R and when $J \subset I$ we denote by $\mathcal{K}_J$ the total ring of fractions of the ring R_J. Notice that $R_I = R$ and $\mathcal{K}_I = \mathcal{K}$. If $J = \{i\}$, then $R_{\{i\}}$ is equal to the above defined domain R_i whose field of fractions will be denoted by $\mathcal{K}_i$. Let $\widetilde{R}$ be the integral closure of R in $\mathcal{K}$ and $\widetilde{R}_J$ be that of R_J in $\mathcal{K}_J$. One has that $\widetilde{R}_J \simeq \widetilde{R}_{j_1} \times \cdots \times \widetilde{R}_{j_s}$, which in turn is the integral closure of $R_{j_1} \times \cdots \times R_{j_s}$ in its total ring of fractions. We have the following diagram:

$$\begin{array}{ccc} \mathcal{K}_J & \simeq & \mathcal{K}_{j_1} \times \cdots \times \mathcal{K}_{j_s} \\ \uparrow & & \uparrow \\ \widetilde{R}_J & \simeq & \widetilde{R}_{j_1} \times \cdots \times \widetilde{R}_{j_s} \\ \uparrow & & \uparrow \\ R_J & \hookrightarrow & R_{j_1} \times \cdots \times R_{j_s} \end{array}$$

Since each $\widetilde{R_i}$ is a DVR, with a valuation denoted by v_i, one has that $\mathcal{K}_i$ is a valuated field with the extension of the valuation v_i which is denoted by the same symbol. This allows one to define the value map

$$\begin{array}{rcl} v\colon \mathcal{K} \setminus Z(\mathcal{K}) & \to & \mathbb{Z}^r \\ h & \mapsto & (v_1(\pi_1(h)), \ldots, v_r(\pi_r(h))), \end{array}$$

where π_i here denotes the projection $\mathcal{K} \to \mathcal{K}_i$, which is the extension of the previously defined projection map $\pi_i \colon R \to R_i$ and $Z(\mathcal{K})$ stands for the set of zero divisors of $\mathcal{K}$.

An R-submodule $\mathcal{I}$ of $\mathcal{K}$ will be called a *regular fractional ideal* of R if it contains a regular element of R and there is a regular element d in R such that $d\mathcal{I} \subset R$.

Since $d\mathcal{I}$ is an ideal of R, which is a noetherian ring, one has that $\mathcal{I} \subset \mathcal{K}$ is a nontrivial fractional ideal if and only if it contains a regular element of R and it is a finitely generated R-module.

Examples of fractional ideals of R are R itself, $\widetilde{R}$, the conductor $\mathcal{C}$ of $\widetilde{R}$ in R, or any ideal of R or of $\widetilde{R}$ that contains a regular element. Also, if $\mathcal{I}$ is a regular fractional ideal of R, then for all

$\emptyset \neq J \subset I$ one has that $\pi_J(\mathcal{I})$ is a regular fractional ideal of R_J, where, this time, $\pi_J : \mathcal{K} \to \mathcal{K}_J$ denotes the natural projection.

3. Value sets

If $\mathcal{I}$ is a regular fractional ideal of R, we define the *value set* of $\mathcal{I}$ as being

$$E = v(\mathcal{I} \setminus Z(\mathcal{K})) \subset \mathbb{Z}^r.$$

If $J = \{j_1 < \cdots < j_s\} \subset I$, then we denote by pr_J the projection $\mathbb{Z}^r \to \mathbb{Z}^s$,

$$(\alpha_1, \ldots, \alpha_r) \mapsto (\alpha_{j_1}, \ldots, \alpha_{j_s}).$$

Let us define

$$E_J = v(\pi_J(\mathcal{I}) \setminus Z(\mathcal{K}_J)).$$

If $j \in J = \{j_1, \ldots, j_t, \ldots j_s\} \subset I$, with $j_t = j$, for $\alpha = (\alpha_{j_1}, \ldots, \alpha_{j_s}) \in E_J$, then we define

$$\widetilde{\mathrm{pr}}_j(\alpha) = \alpha_{j_t} = \alpha_j.$$

We will consider on $\mathbb{Z}^r$ the product order $\leq$ and will write $(a_1, \ldots, a_r) < (b_1, \ldots, b_r)$ when $a_i < b_i$, for all $i = 1, \ldots, r$.

Value sets of fractional ideals have the following fundamental analog properties to those of semigroups of values described by Garcia for $r = 2$ in [5] and by Delgado for $r > 2$ in [3] (see also [2] or [1]):

Property (A). *If $\alpha = (\alpha_1, \ldots, \alpha_r)$ and $\beta = (\beta_1, \ldots, \beta_r)$ belong to E, then*

$$min(\alpha, \beta) = (min(\alpha_1, \beta_1), \ldots, min(\alpha_r, \beta_r)) \in E.$$

Property (B). *If $\alpha = (\alpha_1, \ldots, \alpha_r), \beta = (\beta_1, \ldots, \beta_r)$ belong to E, $\alpha \neq \beta$ and $\alpha_i = \beta_i$ for some $i \in \{1, \ldots, r\}$, then there exists $\gamma \in E$ such that $\gamma_i > \alpha_i = \beta_i$ and $\gamma_j \geq min\{\alpha_j, \beta_j\}$ for each $j \neq i$, with equality holding if $\alpha_j \neq \beta_j$.*

Property (C). *There exist $\alpha \in \mathbb{Z}^r$ and $\gamma \in \mathbb{N}^r$ such that*

$$\gamma + \mathbb{N}^r \subset E \subset \alpha + \mathbb{Z}^r.$$

Properties (A) and (C) allow one to conclude that there exist a unique $m_E = (m_1, \ldots, m_r)$ such that $\beta_i \geq m_i$, $i = 1, \ldots, r$, for all $(\beta_1, \ldots, \beta_r) \in E$ and a unique least element $\gamma \in E$ with the property that $\gamma + \mathbb{N}^r \subset E$. This element is what we call the conductor of E and will denote it by $c(E)$.

Observe that one always has

$$c(E_J) \leq \mathrm{pr}_J(c(E)), \quad \forall\, J \subset I.$$

One has the following result:

Lemma 1. *If $\mathcal{I}$ is a fractional ideal of R and $\emptyset \neq J \subset I$, then $\mathrm{pr}_J(E) = E_J$.*

Proof. One has obviously that $\mathrm{pr}_J(E) \subset E_J$. On the other hand, let $\alpha_J \in E_J$. Take $h \in \mathcal{I}$ such that $v_J(\pi_J(h)) = \alpha_J$. If $h \notin Z(K)$ we are done. Otherwise, choose any $h' \in \mathcal{I} \setminus Z(K)$ such that $\mathrm{pr}_J(v(h')) > \alpha_J$, which exists since E has a conductor. Hence, $v_J(h + h') = \alpha_J$, proving the other inclusion. $\square$

4. Maximal points

We now introduce the important notion of a *fiber* of an element $\alpha \in E$ with respect to a subset $J \subset I$ that will play a central role in what follows.

Definition 1. Given $A \subset \mathbb{Z}^r$, $\alpha \in \mathbb{Z}^r$ and $\emptyset \neq J \subset I$, we define

$$F_J(A, \alpha) = \{\beta \in A; \mathrm{pr}_J(\beta) = \mathrm{pr}_J(\alpha) \text{ and } \mathrm{pr}_{I \setminus J}(\beta) > \mathrm{pr}_{I \setminus J}(\alpha)\},$$

$$\overline{F}_J(A, \alpha) = \{\beta \in A; \mathrm{pr}_J(\beta) = \mathrm{pr}_J(\alpha), \text{ and } \mathrm{pr}_{I \setminus J}(\beta) \geq \mathrm{pr}_{I \setminus J}(\alpha)\},$$

The set $F(A, \alpha) = \bigcup_{i=1}^{r} F_{\{i\}}(A, \alpha)$ will be called the *fiber* of α in A.

The sets $F_{\{i\}}(A, \alpha)$ and $\overline{F}_{\{i\}}(A, \alpha)$ will be denoted simply by $F_i(A, \alpha)$ and $\overline{F}_i(A, \alpha)$. Notice that $F_I(\mathbb{Z}^r, \alpha) = \overline{F}_I(\mathbb{Z}^r, \alpha) = \{\alpha\}$.

Definition 2. $\alpha \in A$ is called a *maximal point* of A, if $F(A, \alpha) = \emptyset$.

This means that there is no element in A with one coordinate equal to the corresponding coordinate of α and the others bigger.

From now on, E will denote the value set of the regular fractional ideal $\mathcal{I}$ of R. From the fact that E has a minimum m and a conductor $\gamma = c(E)$, one has immediately that all maximal points of E are in the limited region $\{(x_1, \ldots, x_r) \in \mathbb{Z}^r;\ m_i \leq x_i < \gamma_i,\ \ i = 1, \ldots, r\}$. This implies that E has finitely many maximal points.

Definition 3. We will say that a maximal point α of E is an *absolute maximal* if $F_J(E, \alpha) = \emptyset$ for every $J \subset I$, $J \neq I$. If a maximal point α of E is such that $F_J(E, \alpha) \neq \emptyset$, for every $J \subset I$ with $\#J \geq 2$, then α will be called a *relative maximal* of E.

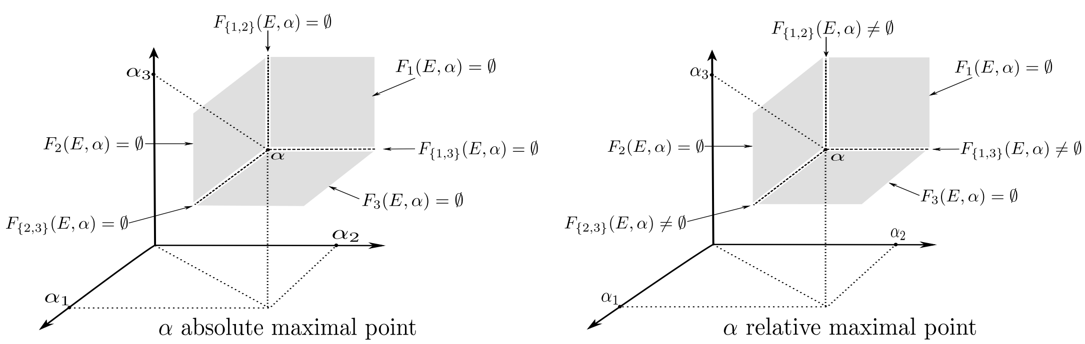


FIGURE 1. Maximal points

In the case where $r = 2$, the notions of maximal, relative maximal and absolute maximal coincide. For $r = 3$ we may only have relative maximals or absolute maximals, but in general there will be several types of maximals.

We will denote by $M(E)$, $RM(E)$ and $AM(E)$ the sets of maximals, of relative maximals and absolute maximals of the set E, respectively.

The importance of the relative maximals is attested by the theorem below that says that the set $RM(E)$ determines E in a combinatorial sense as follows:

Theorem 2 (generation)**.** *Let $\alpha \in \mathbb{Z}^r$ be such that $p_J(\alpha) \in E_J$ for all $J \subset I$ with $\#J = r - 1$. Then*

$$\alpha \in E \Longleftrightarrow \alpha \notin F(\mathbb{Z}^r, \beta),\ \forall \beta \in RM(E).$$

We will omit the proof since this result is a slight modification of [3, Theorem 1.5] with essentially the same proof.

The following two lemmas give us characterizations of the relative and absolute maximal points that will be useful in Section 4.

Lemma 3. *Given a value set $E \subset \mathbb{Z}^r$ and $\alpha \in \mathbb{Z}^r$ with the following properties:*

i) *there is $i \in I$ such that $F_i(E, \alpha) = \emptyset$,*

ii) *$F_{i,j}(E, \alpha) \neq \emptyset$ for all $j \in I \setminus \{i\}$.*

Then α is a relative maximal of E.

Proof. Follows the same steps as the proof of [3, Lemma 1.3] □

Lemma 4. *Given a value set $E \subset \mathbb{Z}^r$ and $\alpha \in E$, assume that there exists an index $i \in I$ such that $F_J(E, \alpha) = \emptyset$ for every $J \subsetneq I$ with $i \in J$. Then α is an absolute maximal of E.*

Proof. We have to prove that $F_K(E, \alpha) = \emptyset$ for all $K \subset I$ with $i \notin K$.

Assume, by reductio ad absurdum, that there exists some $K \subset I$ with $i \notin K$ such that $F_K(E, \alpha) \neq \emptyset$. Let β be an element in $F_K(E, \alpha)$, then $\beta_k = \alpha_k$, $\forall k \in K$ and $\beta_j > \alpha_j$, for all $j \notin K$. Applying Property (B) for α, β and any index $k' \in K$, we have that there exists $\theta \in E$ such that $\theta_{k'} > \beta_{k'} = \alpha_{k'}$, $\theta_l \geq \min\{\alpha_l, \beta_l\}$, $\forall l \neq k'$ and $\theta_j = \alpha_j$ for all $j \notin K$. If $B = (I \setminus K) \cup \{l \in K, \theta_l = \alpha_l\}$, then we have $\theta \in F_B(E, \alpha)\,(\neq \emptyset)$, with $i \in B$, which is a contradiction. □

5. Colengths of fractional ideals

Let R be a complete admissible ring and let $\mathcal{J} \subset \mathcal{I}$ two regular fractional ideals of R with value sets D and E, respectively. Since $\mathcal{J} \subset \mathcal{I}$, one has that $D \subset E$, hence $c(E) \leq c(D)$. Our aim in this section is to find a formula for the length $\ell_R(\mathcal{I}/\mathcal{J})$ of $\mathcal{I}/\mathcal{J}$ as R-modules, called the colength of $\mathcal{J}$ with respect to $\mathcal{I}$, in terms of the value sets D and E.

The motivation comes from the case $r = 1$, that is, when R is a domain. In this case, as observed by Gorenstein [6], one can easily show that

$$\ell_R(\mathcal{I}/\mathcal{J}) = \#(E \setminus D).$$

When $r > 1$, then $E \setminus D$ is not finite anymore.

For $\alpha \in \mathbb{Z}^r$ and $\mathcal{I}$ a fractional ideal of R, with value set E, we define

$$\mathcal{I}(\alpha) = \{h \in \mathcal{I};\ v(h) \geq \alpha\}.$$

It is clear that if $m_E = \min E$, then $\mathcal{I}(m_E) = \mathcal{I}$.

One has the following result:

Proposition 5. ([1, Proposition 2.7]) *Let $\mathcal{J} \subseteq \mathcal{I}$ be two fractional ideals of R, with value sets D and E, respectively, then*

$$\ell_R\left(\frac{\mathcal{I}}{\mathcal{J}}\right) = \ell_R\left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)}\right) - \ell_R\left(\frac{\mathcal{J}}{\mathcal{J}(\gamma)}\right),$$

for sufficiently large $\gamma \in \mathbb{N}^r$ (for instance, if $\gamma \geq c(D)$).

If $e_i \in \mathbb{Z}^r$ denotes the vector with zero entries except the i-th entry which is equal to 1, then the following result will give us an effective way to calculate colengths of ideals.

Proposition 6. [2, Proposition 2.2] *If $\alpha \in \mathbb{Z}^r$, then we have*

$$\ell_R\left(\frac{\mathcal{I}(\alpha)}{\mathcal{I}(\alpha + e_i)}\right) = \begin{cases} 1, & if\ \overline{F}_i(E, \alpha) \neq \emptyset, \\ \\ 0, & otherwise. \end{cases}$$

So, to compute, for instance, $\ell_R\left(\dfrac{\mathcal{I}}{\mathcal{I}(\gamma)}\right)$, one may take a chain

$$m_E = \alpha^0 \leq \alpha^1 \leq \cdots \leq \alpha^m = \gamma,$$

where $\alpha^j \in \mathbb{Z}^r$ and $\alpha^j - \alpha^{j-1} \in \{e_i,\ i = 1, \ldots, r\}$, and then using Proposition 6 by observing that

$$\ell_R\left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)}\right) = \ell_R\left(\frac{\mathcal{I}(\alpha^0)}{\mathcal{I}(\gamma)}\right) = \sum_{j=1}^{m} \ell_R\left(\frac{\mathcal{I}(\alpha^{j-1})}{\mathcal{I}(\alpha^j)}\right).$$

D'Anna in [2] showed that $\ell_R\left(\dfrac{\mathcal{I}}{\mathcal{I}(\gamma)}\right)$ is equal to the length n of a saturated chain

$$m_E < \alpha^0 < \alpha^1 < \cdots < \alpha^n = \gamma$$

in E. The drawback of this result is that one has to know all points of E in the hypercube with opposite vertices m_E and γ.

The fact that E is determined by its projections E_J and its relative maximal points, suggests that $\ell_R\left(\dfrac{\mathcal{I}}{\mathcal{I}(\gamma)}\right)$ can be computed in terms of these data. In fact, this will be done in Theorem 1 below.

In what follows we will denote ℓ_R simply by ℓ.

5.1. **Case r=2.** This simplest case was studied by Barucci, D'Anna and Fröberg in [1] and we reproduce it here because it gives a clue on how to proceed in general.

Let $\alpha^0 = m_E$ and consider the chain in $\mathbb{Z}^2$

$$\alpha^0 \leq \cdots \leq \alpha^m = \gamma = (\gamma_1, \gamma_2) \geq c(E)$$

such that

$$\alpha^0 = (\alpha_1^0, \alpha_2^0),\ \alpha^1 = (\alpha_1^0 + 1, \alpha_2^0), \ldots, \alpha^s = (\gamma_1, \alpha_2^0),$$

$$\alpha^{s+1} = (\gamma_1, \alpha_2^0 + 1),\ \alpha^{s+2} = (\gamma_1, \alpha_2^0 + 2), \ldots, \alpha^m = (\gamma_1, \gamma_2),$$

and consider the following sets

$$L_1 = \{\alpha^0, \alpha^1, \ldots, \alpha^s\} \ \text{ and } \ L_2 = \{\alpha^s, \alpha^{s+1}, \ldots, \alpha^m\}.$$

By Proposition 6, we have

$$\begin{aligned}\ell\left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)}\right) \ &= \ \#L_1 - \#\{\alpha \in L_1;\ \overline{F}_1(E,\alpha) = \emptyset\} + \\ &\ \ \#L_2 - \#\{\alpha \in L_2;\ \overline{F}_2(E,\alpha) = \emptyset\}.\end{aligned}$$

Now, because of our choice of L_1, denoting by $\mathcal{G}(E_i)$ the set of gaps of E_i in the interval $(\min(E_i), +\infty)$, we have that

$$\forall\, \alpha \in L_1,\ \overline{F}_1(E,\alpha) = \emptyset \iff \mathrm{pr}_1(\alpha) \in \mathcal{G}(E_1),$$

hence

$$\#\{\alpha \in L_1;\ \overline{F}_1(E,\alpha) = \emptyset\} = \#\mathcal{G}(E_1).$$

Observe that not all $\alpha \in L_2$ with $\overline{F}_2(E,\alpha) = \emptyset$ are such that $\mathrm{pr}_2(\alpha) \in \mathcal{G}(E_2)$, hence

$$\#\{\alpha \in L_2;\ \overline{F}_2(E,\alpha) = \emptyset\} = \#\mathcal{G}(E_2) - \xi,$$

where ξ is the number of α in L_2 with $\mathrm{pr}_2(\alpha) \in E_2$ and $\overline{F}_2(E,\alpha) = \emptyset$. But, such α are in one-to-one correspondence with the maximal points of E, hence $\xi = \#M(E)$.

Putting all this together, we get

Proposition 7. *If $\gamma \geq c(E)$, then*

$$\ell\left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)}\right) = (\gamma_1 - \alpha_1^0) + (\gamma_2 - \alpha_2^0) - \#\mathcal{G}(E_1) - \#\mathcal{G}(E_2) - \#M(E). \tag{1}$$

5.2. **Case $r \geq 3$.** Let us assume that $\mathcal{I}$ is a fractional ideal of R, where R has r minimal primes. Let

$$m_E = \alpha^0 \leq \alpha^1 \leq \cdots \leq \alpha^m = \gamma \geq c(E),$$

be the chain in $\mathbb{Z}^r$, given by the union of the following paths (see Figure 2, for $r = 3$):

$$L_1 \colon \alpha^0, \alpha^1 = \alpha^0 + e_1, \ldots, \alpha^{s_1} = \alpha^0 + (\gamma_1 - \alpha_1^0)e_1 = (\gamma_1, \alpha_2^0, \ldots, \alpha_r^0),$$

$$\ldots$$

$$L_r \colon \alpha^{s_{r-1}} = (\gamma_1, \ldots, \gamma_{r-1}, \alpha_r^0), \alpha^{s_{r-1}+1} = \alpha^{s_{r-1}} + e_r, \ldots, \alpha^m = \gamma.$$

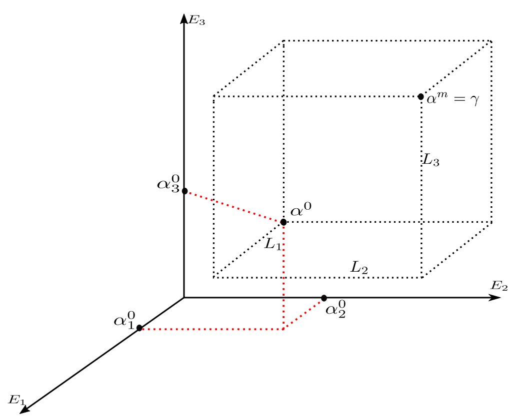


FIGURE 2. The chain for $r = 3$

For $i \in I$, let us define $[1, i] = [1, i+1) = \{1, \ldots, i\}$. We will need the following result:

Lemma 8. *For any $\alpha \in L_1 \cup \ldots \cup L_{r-1}$, and for $i \in [1, r)$, one has*

$$\overline{F}_i(E, \alpha) \neq \emptyset \iff \overline{F}_i(E_{[1,r)}, \mathrm{pr}_{[1,r)}(\alpha)) \neq \emptyset.$$

Proof. ($\Rightarrow$) This is obvious.
($\Leftarrow$) Suppose that

$$(\theta_1, \ldots, \theta_{r-1}) \in \overline{F}_i(E_{[1,r)}, \mathrm{pr}_{[1,r)}(\alpha)) \neq \emptyset.$$

Since by Lemma 1 one has that $\mathrm{pr}_{[1,r)}(E) = E_{[1,r)}$, then there exists $\theta = (\theta_1, \ldots, \theta_{r-1}, \theta_r) \in E$. Since $\alpha \in L_i$ for some $i = 1, \ldots, r-1$, it follows that $\alpha_r = \alpha_r^0$. Then one cannot have $\theta_r < \alpha_r = \alpha_r^0$, because otherwise

$$(\alpha_1^0, \ldots, \alpha_{r-1}^0, \theta_r) = \min(\alpha^0, \theta) \in E,$$

which is contradiction, since α^0 is the minimum of E. Hence $\theta_r \geq \alpha_r$, so $\theta \in \overline{F}_i(E, \alpha)$, and the result follows. $\square$

Lemma 8 allows us to write:

$$\ell\left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)}\right)=\ell\left(\frac{\pi_{[1,r)}(\mathcal{I})}{\pi_{[1,r)}(\mathcal{I})(\mathrm{pr}_{[1,r)}(\gamma))}\right)+(\gamma_r-\alpha_r^0)-\#\{\alpha\in L_r;\ \overline{F}_r(E,\alpha)=\emptyset\}. \tag{2}$$

Hence to get an inductive formula for $\ell\left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)}\right)$, we only have to compute

$$\#\{\alpha\in L_r;\ \overline{F}_r(E,\alpha)=\emptyset\},$$

and for this we will need the following lemma.

Lemma 9. *Let $\alpha\in\mathbb{Z}^r$, then $\overline{F}_j(E,\alpha)=\emptyset$ if and only if either $\alpha_j\in\mathcal{G}(E_j)$ or there exist some $J\subseteq I$ with $\{j\}\subsetneq J$ and a relative maximal β of E_J such that $\widetilde{\mathrm{pr}}_j(\beta)=\alpha_j$ and $\widetilde{\mathrm{pr}}_i(\beta)<\alpha_i$, for all $i\in J$, $i\neq j$.*

Proof. ($\Leftarrow$) (We prove more, since it is enough to assume β is any maximal of E_J) It is obvious that if $\alpha_j\in\mathcal{G}(E_j)$, then $\overline{F}_j(E,\alpha)=\emptyset$. Let us now assume that there exist $J\subset I$, with $\{j\}\subsetneq J$ and $\beta\in M(E_J)$, such that $\widetilde{\mathrm{pr}}_j(\beta)=\alpha_j$ and $\widetilde{\mathrm{pr}}_i(\beta)<\alpha_i$, for all $i\in J$, $i\neq j$.

Suppose by reductio ad absurdum that $\overline{F}_j(E,\alpha)\neq\emptyset$. Let $\theta\in\overline{F}_j(E,\alpha)$, that is, $\theta_j=\alpha_j$ and $\theta_i\geq\alpha_i,\forall i\in J\setminus\{j\}$. Now since, $\forall i\in J, i\neq j$,

$$\widetilde{\mathrm{pr}}_j(\mathrm{pr}_J(\theta))=\theta_j=\alpha_j=\widetilde{\mathrm{pr}}_j(\beta)\text{ and }\widetilde{\mathrm{pr}}_i(\mathrm{pr}_J(\theta))=\theta_i\geq\alpha_i>\widetilde{\mathrm{pr}}_i(\beta),$$

then $\mathrm{pr}_J(\theta)\in F_j(E_J,\beta)$, which contradicts the assumption that $\beta\in M(E_J)$.
($\Rightarrow$) Since $\overline{F}_j(E,\alpha)=\emptyset$ implies $F_j(E,\alpha)=\emptyset$, the proof follows the same lines as the proof of [4, Theorem 1.5]. ☐

Going back to our main calculation, by Lemma 9, if $\alpha\in L_r$ is such that $\overline{F}_r(E,\alpha)=\emptyset$, then either $\alpha_r\in\mathcal{G}(E_r)$, or there exist a subset J of $I=\{1,\dots,r\}$, with $\{r\}\subsetneq J$, and $\beta\in RM(E_J)$, with $\widetilde{\mathrm{pr}}_r(\beta)=\alpha_r$ and $\widetilde{\mathrm{pr}}_i(\beta)<\alpha_i$ for $i\in J, i\neq r$.

Notice that for $\alpha\in L_r$ one has $\alpha_i=\gamma_i$ for $i\neq r$, so the condition $\widetilde{\mathrm{pr}}_i(\beta)<\alpha_i$ for $i\in J$, $i\neq r$ is satisfied, since $\beta\in M(E_J)$. So, we have a bijection

$$\{\alpha\in L_r;\ \overline{F}_r(E,\alpha)=\emptyset\}\quad\longleftrightarrow\quad\mathcal{G}(E_r)\cup\bigcup_{\{r\}\subsetneq J\subseteq I}\widetilde{\mathrm{pr}}_r(RM(E_J)).$$

Since for all J, with $\{r\}\subsetneq J\subseteq I$, the sets $\mathcal{G}(E_r)$ and $\widetilde{\mathrm{pr}}_r(RM(E_J))$ are disjoint, it follows that

$$\#\{\alpha\in L_r;\overline{F}_r(E,\alpha)=\emptyset\}=\#\mathcal{G}(E_r)+\#\left(\bigcup_{\{r\}\subsetneq J\subset I}\widetilde{\mathrm{pr}}_r(RM(E_J))\right). \tag{3}$$

Let us define

$$\Theta_1=0,\quad\text{and}\quad\Theta_i=\#\bigcup_{\{i\}\subsetneq J\subseteq[1,i]}\widetilde{\mathrm{pr}}_i RM(E_J)),\quad 2\leq i\leq r.$$

Now, putting together Equations (2) and (3), we get the following recursive formula:

Theorem 10. *Let $\mathcal{I}$ be a fractional ideal of a ring R with r minimal primes with values set E. If $\gamma\geq c(E)$, then*

$$\ell\left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)}\right)=\ell\left(\frac{\pi_{[1,r)}(\mathcal{I})}{\pi_{[1,r)}(\mathcal{I})(\mathrm{pr}_{[1,r)}(\gamma))}\right)+(\gamma_r-\alpha_r^0)-\#\mathcal{G}(E_r)-\Theta_r \tag{4}$$

Corollary 11. *With the same hypotheses as in Theorem 10, one has the formula*

$$\ell\left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)}\right)=\sum_{i=1}^{r}\left(\gamma_i-\alpha_i^0-\#\mathcal{G}(E_i)-\Theta_i\right).$$

6. A closed formula for $r=3$

In this section, we provide a nicer formula than Equation (4), when $r=3$. To simplify notation, for any $J\subset I=\{1,2,3\}$, we will denote by RM_J, AM_J and M_J the sets $RM(E_J), AM(E_J)$ and $M(E_J)$, respectively. Notice also that if $\#J=2$, then $RM_J=AM_J=M_J$.

From Formulas (1) and (2), for $\gamma\geq c(E)$, one has

$$\begin{aligned}\ell\left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)}\right) &= (\gamma_1-\alpha_1^0)-\#\mathcal{G}(E_1)+(\gamma_2-\alpha_2^0)-\#\mathcal{G}(E_2)-\#M_{\{1,2\}}+\\ &\quad (\gamma_3-\alpha_3^0)-\#\{\alpha\in L_3;\ \overline{F}_3(E,\alpha)=\emptyset\}.\end{aligned}$$

We will use the following notation:

$$L_3'=\{\alpha\in L_3;\overline{F}_3(E,\alpha)=\emptyset\}.$$

Now, from Lemma 9, the points $\alpha=(\alpha_1,\alpha_2,\alpha_3)\in L_3'$ are such that $\alpha_3\in\mathcal{G}(E_3)$ or they are associated to maximal points of either $E_{\{1,3\}}$, $E_{\{2,3\}}$, or E with last coordinate equal to α_3. So, we have

$$\#L_3'=\#\mathcal{G}(E_3)+\#M_{\{1,3\}}+\#M_{\{2,3\}}+\#RM-\eta, \tag{5}$$

where η is some correcting term which will take into account the eventual multiple counting of maximals having the same last coordinate.

To compute η we will analyze in greater detail the geometry of maximal points.

If $\alpha,\beta\in M$ with $\alpha_3=\beta_3$, then $\alpha_1\neq\beta_1$ and $\alpha_2\neq\beta_2$. If $\alpha_1<\beta_1$, then necessarily $\beta_2<\alpha_2$.

We say that two relative (respectively, absolute) maximals α and β of E with $\alpha_3=\beta_3$ and $\alpha_1<\beta_1$ are *adjacent*, if there is no $(\theta_1,\theta_2,\alpha_3)$ in RM (respectively, in AM) with $\alpha_1<\theta_1<\beta_1$ and $\beta_2<\theta_2<\alpha_2$.

We will describe below the geometry of the maximal points of E

Lemma 12. *If $\alpha\in AM$, then one of the following three conditions is verified:*

(i) *there exist two adjacent relative maximals β and θ of E such that $\mathrm{pr}_{\{1,3\}}(\beta)=\mathrm{pr}_{\{1,3\}}(\alpha)$ and $\mathrm{pr}_{\{2,3\}}(\theta)=\mathrm{pr}_{\{2,3\}}(\alpha)$;*
(ii) *there exists $\beta\in RM$ such that $\mathrm{pr}_{\{1,3\}}(\beta)=\mathrm{pr}_{\{1,3\}}(\alpha)$ and $\mathrm{pr}_{\{2,3\}}(\alpha)\in M_{\{2,3\}}$, or $\mathrm{pr}_{\{2,3\}}(\beta)=\mathrm{pr}_{\{2,3\}}(\alpha)$ and $\mathrm{pr}_{\{1,3\}}(\alpha)\in M_{\{1,3\}}$;*
(iii) *$\mathrm{pr}_{\{1,3\}}(\alpha)\in M_{\{1,3\}}$ and $\mathrm{pr}_{\{2,3\}}(\alpha)\in M_{\{2,3\}}$.*

Proof. Let $\alpha=(\alpha_1,\alpha_2,\alpha_3)\in AM$, then $F(E,\alpha)=\emptyset$. We consider the following sets:

$$R_1=\{\beta\in\mathbb{Z}^3;\ \beta_3=\alpha_3,\beta_1>\alpha_1,\beta_2<\alpha_2\}$$

and

$$R_2=\{\theta\in\mathbb{Z}^3;\ \theta_3=\alpha_3,\theta_1<\alpha_1,\theta_2>\alpha_2\}.$$

Then there are four possibilities:

$$R_1\cap E\neq\emptyset \text{ and } R_2\cap E\neq\emptyset,\quad R_1\cap E\neq\emptyset \text{ and } R_2\cap E=\emptyset.$$

$$R_1\cap E=\emptyset \text{ and } R_2\cap E\neq\emptyset,\quad R_1\cap E=\emptyset \text{ and } R_2\cap E=\emptyset.$$

Suppose $R_1 \cap E \neq \emptyset$ and $R_2 \cap E \neq \emptyset$. Choose $\beta \in R_1 \cap E$ and $\theta \in R_2 \cap E$, such that $\alpha_2 - \beta_2$ and $\alpha_1 - \theta_1$ are as small as possible. Then by Property (A), we have $\min(\alpha,\beta), \min(\alpha,\theta) \in E$. Obviously $\mathrm{pr}_{\{1,3\}}(\beta) = \mathrm{pr}_{\{1,3\}}(\alpha)$ and $\mathrm{pr}_{\{2,3\}}(\theta) = \mathrm{pr}_{\{2,3\}}(\alpha)$. Moreover, according to Lemma 3, these are relative maximals because $F_3(E, \min(\alpha,\beta))$ and $F_3(E, \min(\alpha,\theta))$ are empty and the sets $F_{\{1,3\}}(E, \min(\alpha,\beta))$, $F_{\{1,3\}}(E, \min(\alpha,\theta))$, $F_{\{2,3\}}(E, \min(\alpha,\beta))$ and $F_{\{2,3\}}(E, \min(\alpha,\theta))$ are nonempty. It follows that $\min(\alpha,\beta)$ and $\min(\alpha,\theta)$ are adjacent relative maximals.

Suppose $R_1 \cap E \neq \emptyset$ and $R_2 \cap E = \emptyset$. Choose $\beta \in R_1 \cap E$ such that $\alpha_2 - \beta_2$ is as small as possible, then, as we argued above, we have that $\min(\alpha,\beta) \in RM$ and $\mathrm{pr}_{\{1,3\}}(\beta) = \mathrm{pr}_{\{1,3\}}(\alpha)$. Moreover, as $R_2 \cap E = \emptyset$, it follows that $\mathrm{pr}_{\{2,3\}}(\alpha) \in M_{\{2,3\}}$.

The case $R_1 \cap E = \emptyset$ and $R_2 \cap E \neq \emptyset$ is similar to the above one, giving us the second possibility in (ii).

Suppose $R_1 \cap E = \emptyset$ and $R_2 \cap E = \emptyset$. It is obvious that

$$\mathrm{pr}_{\{1,3\}}(\alpha) \in M_{\{1,3\}} \quad \text{and} \quad \mathrm{pr}_{\{2,3\}}(\alpha) \in M_{\{2,3\}}.$$

□

Given two points $\theta^1, \theta^2 \in \mathbb{Z}^3$ such that $\mathrm{pr}_3(\theta^1) = \mathrm{pr}_3(\theta^2)$, we will denote by $\mathcal{R}(\theta^1, \theta^2)$ the parallelogram determined by the coplanar points $\theta^1, \theta^2, \min(\theta^1,\theta^2)$ and $\max(\theta^1,\theta^2)$. We have the following result:

Corollary 13. *Let $\theta^1, \theta^2 \in AM$ be such that $\mathrm{pr}_3(\theta^1) = \mathrm{pr}_3(\theta^2)$. Then one has*

$$\mathcal{R}(\theta^1, \theta^2) \cap RM \neq \emptyset.$$

Proof. Because $\theta^1, \theta^2 \in AM$, it follows immediately that (iii) of Lemma 12 cannot happen, therefore, the existence of the relative maximal is ensured by (i) or (ii). □

Lemma 14. *If β and β' are adjacent relative maximals, with $\beta_3 = \beta'_3$, then $\max(\beta, \beta')$ is an absolute maximal of E.*

Proof. We may suppose that $\beta_1 > \beta'_1$ and $\beta_2 < \beta'_2$. As β and β' are adjacent, we have that $F_{\{1,3\}}(E,\beta) \cap F_{\{2,3\}}(E,\beta') \neq \emptyset$, because otherwise, take $\alpha^1 \in F_{\{1,3\}}(E,\beta)$, with α^1_2 the greatest possible and $\alpha^2 \in F_{\{2,3\}}(E,\beta')$, with α^2_1 the greatest possible. From Lemma 4 it follows that α^1 and α^2 are absolute maximals of E, then by Corollary 13 there exists a relative maximal in the region $\mathcal{R}(\alpha^1, \alpha^2)$, this contradicts the fact that β and β' are adjacent relative maximals.

Then, effectively, $F_{\{1,3\}}(E,\beta) \cap F_{\{2,3\}}(E,\beta') = \{\max(\beta,\beta')\}$, which is an absolute maximal. □

Recall that the elements in L'_3 are of the form $(\gamma_1, \gamma_2, \alpha_3)$, with $\alpha^0_3 \leq \alpha_3 \leq \gamma_3$.

Lemma 15. *Let $\alpha \in L'_3$ be such that*

$$\alpha_3 \in (\widetilde{\mathrm{pr}}_3(M_{\{1,3\}}) \setminus \widetilde{\mathrm{pr}}_3(M_{\{2,3\}})) \cap \mathrm{pr}_3(RM) \quad \text{or} \quad \alpha_3 \in (\widetilde{\mathrm{pr}}_3(M_{\{2,3\}}) \setminus \widetilde{\mathrm{pr}}_3(M_{\{1,3\}})) \cap \mathrm{pr}_3(RM).$$

Then there are the same number of relative as absolute maximals in E with third coordinate equal to α_3.

Proof. We assume that $\alpha_3 \in (\widetilde{\mathrm{pr}}_3(M_{\{1,3\}}) \setminus \widetilde{\mathrm{pr}}_3(M_{\{2,3\}})) \cap \mathrm{pr}_3(RM)$, since the other case is analogous.

Since $\alpha_3 \in \mathrm{pr}_3(RM)$, we may assume that there are $s\,(\geq 1)$ relative maximals $\beta^1, \ldots, \beta^s$ in E with third coordinate equal to α_3. We may suppose that $\beta^1_1 < \beta^2_1 < \cdots < \beta^s_1$, so the β^i's are successively adjacent relative maximals, hence, by lemma 14, we have that

$$\max(\beta^1, \beta^2), \ldots, \max(\beta^{s-1}, \beta^s) \in AM.$$

This shows that there are at least $s - 1$ absolute maximals in E with third coordinate α_3.

Now as $\mathrm{pr}_3(\alpha) \in \widetilde{\mathrm{pr}}_3(M_{\{1,3\}})$, then there is a $(\eta_1^1, \alpha_3) \in M_{\{1,3\}}$ with $\eta_1^1 \leq \alpha_1 \ (= \gamma_1)$, because $c(E_{\{1,3\}}) \leq \mathrm{pr}_{\{1,3\}}(c(E)) = (\gamma_1, \gamma_3)$. Because of our hypothesis, the elements δ in the fiber $F_{\{1,3\}}(E, \beta^s)$ are such that $\beta_1^s < \delta_1 \leq \eta_1^1$. But we must have $\delta_1 = \eta_1^1$, because, otherwise, there would be a point $\eta^1 = (\eta_1^1, \eta_2^1, \alpha_3) \in \mathrm{pr}_{\{1,3\}}^{-1}(\eta_1^1, \alpha_3)$, with $\eta_2^1 < \beta_2^s$, and a point $\eta^2 \in F_{\{2,3\}}(E, \beta^s)$ with $\eta_1^2 < \eta_1^1$ and $\eta_2^2 = \beta_2^s$. These η^1 and η^2 are absolute maximals, due to Lemma 4, then from Corollary 13, there would exist a relative maximal in the region $\mathcal{R}(\eta^1, \eta^2)$, which contradicts the fact that we have s relative maximals. This implies that $(\beta_1^s, \eta_2^1, \alpha_3)$ is an absolute maximal of E.

We have to show that there are no other absolute maximals. If such maximal existed, then one of the three conditions in Lemma 12 would be satisfied. Obviously conditions (i) and (iii) cannot be satisfied, but neither condition (ii) can be satisfied, because otherwise $\alpha_3 \in \widetilde{\mathrm{pr}}_3(M_{\{2,3\}})$, which is a contradiction. ☐

Lemma 16. *Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in L_3'$ be such that $\alpha_3 \in \big(\widetilde{\mathrm{pr}}_3(M_{\{1,3\}}) \cap \widetilde{\mathrm{pr}}_3(M_{\{2,3\}})\big) \setminus \mathrm{pr}_3(RM)$, then there exists one and only one absolute maximal of E with third coordinate equal to α_3.*

Proof. As
$$\alpha_3 \in \widetilde{\mathrm{pr}}_3(M_{\{1,3\}}) \cap \widetilde{\mathrm{pr}}_3(M_{\{2,3\}}),$$
then there exist $(\beta_1^1, \alpha_3) \in M_{\{1,3\}}$ and $(\beta_2^2, \alpha_3) \in M_{\{2,3\}}$ such that $\beta_1^1 < \alpha_1 (= \gamma_1)$ and $\beta_2^2 < \alpha_2 (= \gamma_2)$, because one always has that $c(E_{\{i,j\}}) \leq \mathrm{pr}_{\{i,j\}}(c(E))$.

Consider the element $\theta = (\beta_1^1, \beta_2^2, \alpha_3)$. If $\theta \in E$, since it is easy to verify that $F_J(E, \theta) = \emptyset$ for $3 \in J \subsetneq \{1, 2, 3\}$, it follows by Lemma 4 that θ is an absolute maximal of E, which is unique in view of Corollary 13 and the hypothesis that $\alpha_3 \notin \mathrm{pr}_3(RM)$.

If $\theta \notin E$, then take $\theta_1 = (\beta_1^1, \delta_2^1, \alpha_3) \in \mathrm{pr}_{\{1,3\}}^{-1}(\beta_1^1, \alpha_3) \cap E$, and
$$\theta_2 = (\delta_1^2, \beta_2^2, \alpha_3) \in \mathrm{pr}_{\{2,3\}}^{-1}(\beta_2^2, \alpha_3) \cap E.$$
We have that $\delta_1^2 < \beta_1^1$ and $\delta_2^1 < \beta_2^2$, because otherwise $\theta \in E$ or, (β_1^1, α_3) and/or (β_2^2, α_3) would not be maximals of $E_{\{1,3\}}$ and/or $E_{\{2,3\}}$. Choose δ_1^2 and δ_2^1 the greatest possible, then it is easy to verify that $F_J(E, \theta_i) = \emptyset$ for $i = 1, 2$ and $3 \in J \subsetneq \{1, 2, 3\}$. Hence from Lemma 4, θ_1 and θ_2 are absolute maximals of E, therefore from Corollary 13 there would be a relative maximal of E with third coordinate equal to α_3, which is a contradiction. ☐

Lemma 17. *Let $\alpha \in L_3'$ be such that $\alpha_3 \in \widetilde{\mathrm{pr}}_3(M_{\{1,3\}}) \cap \widetilde{\mathrm{pr}}_3(M_{\{2,3\}}) \cap \mathrm{pr}_3(RM)$. If there exist s relative maximals with third coordinate equal to α_3, then there exist $s+1$ absolute maximals with third coordinate equal to α_3.*

Proof. Following the proof of Lemma 15, we have $s-1$ absolute maximals obtained by taking the maximum of each pair of adjacent relative maximals. The conditions $\alpha_3 \in \widetilde{\mathrm{pr}}_3(M_{\{1,3\}})$ and $\alpha_3 \in \widetilde{\mathrm{pr}}_3(M_{\{2,3\}})$ give us two extra absolute maximals, and the same argument used there, shows that there are no other. ☐

Lemma 18. *Let $\alpha \in L_3'$ be such that $\alpha_3 \in \mathrm{pr}_3(RM) \setminus \big(\widetilde{\mathrm{pr}}_3(M_{\{1,3\}}) \cup \widetilde{\mathrm{pr}}_3(M_{\{2,3\}})\big)$. If there exist s relative maximals with third coordinate equal to α_3, then we have $s-1$ absolute maximals with third coordinate equal to α_3.*

Proof. The arguments used in the proofs of the last two lemmas give us the result. ☐

Going back to Formula (5), we want to calculate η. From Lemma 9 we can ensure that $\alpha \in L_3' = \{\alpha \in L_3;\ \overline{F}_3(E, \alpha) = \emptyset\} \setminus \mathcal{G}(E_3)$, only if α falls into one of the following five cases:

(i) $\alpha_3 \in (\widetilde{\mathrm{pr}}_3(M_{\{1,3\}}) \setminus \widetilde{\mathrm{pr}}_3(M_{\{2,3\}})) \cap \mathrm{pr}_3(RM)$.

If there exist such α, then they are related to a unique element of $M_{\{1,3\}}$ and if there are s_1 relative maximals with third coordinate α_3, then in our formula α was counted s_1+1 times. By Lemma 15 we know that there exist s_1 absolute maximals of E with third coordinate α_3. So, we subtract s_1 from our counting to partially correct the formula.

(ii) $\alpha_3 \in (\widetilde{\mathrm{pr}}_3(M_{\{2,3\}}) \setminus \widetilde{\mathrm{pr}}_3(M_{\{1,3\}})) \cap \mathrm{pr}_3(RM)$.

Analogously to (i), α is related to a unique element of $M_{\{2,3\}}$ and if there are s_2 relative maximals with third coordinate α_3, then α was counted s_2+1 times in the formula. Again, by Lemma 15 we know that there are s_2 absolute maximals of E with third coordinate α_3. So, we subtract s_2 from our counting to partially correct the formula.

(iii) $\alpha_3 \in \big(\widetilde{\mathrm{pr}}_3(M_{\{1,3\}}) \cap \widetilde{\mathrm{pr}}_3(M_{\{2,3\}})\big) \setminus \mathrm{pr}_3(RM)$.

In this case, α is related to a unique elements in $M_{\{1,3\}}$ and in $M_{\{2,3\}}$, so in the formula we are counting α twice. By Lemma 16 there is a unique absolute maximal of E with third coordinate α_3 such that its projections $\mathrm{pr}_{\{1,3\}}$ and $\mathrm{pr}_{\{2,3\}}$ are in $M_{\{1,3\}}$ and $M_{\{1,3\}}$, respectively. So, we correct partially the formula by subtracting 1, which corresponds to this unique absolute maximal.

(iv) $\alpha_3 \in \widetilde{\mathrm{pr}}_3(M_{\{1,3\}}) \cap \widetilde{\mathrm{pr}}_3(M_{\{2,3\}}) \cap \mathrm{pr}_3(RM)$.

In this case, α is related to a unique element of $M_{\{1,3\}}$, to a unique element of $M_{\{2,3\}}$ and, let us say, s_3 elements of RM, so in our counting, α was counted s_3+2 times. By Lemma 17 there exist s_3+1 absolute maximals of E with third coordinate α_3. In this case, the correcting term is s_3+1, equal to the number of these absolute maximals.

(v) $\alpha_3 \in \mathrm{pr}_3(RM) \setminus \big(\widetilde{\mathrm{pr}}_3(M_{\{1,3\}}) \cup \widetilde{\mathrm{pr}}_3(M_{\{2,3\}})\big)$.

In this case, α is related with, let us say, s_4 elements of RM with third coordinate equal to α_3, so we are counting it s_4 times. By Lemma 18 there exist s_4-1 absolute maximals with third coordinate α_3. This is exactly the correcting term we must apply to our formula.

Observe that the above cases exhaust all absolute maximals of E, implying the following result conjectured by M. E. Hernandes after having analyzed several examples (cf. [7]):

Theorem 19. *Let R be an admissible ring with three minimal primes and let $\mathcal{I}$ be a fractional ideal of R with values set E. If $\gamma \geq c(E)$, then*

$$\ell\left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)}\right) = \sum_{i=1}^{r}\big((\gamma_i-\alpha_i^0) - \#\mathcal{G}(E_i)\big) - \sum_{1\leq i<j\leq 3}\#M_{\{i,j\}} - \#RM + \#AM.$$

Corollary 20. *Let $\mathcal{J} \subseteq \mathcal{I}$ be two fractional ideals of an admissible ring R, with three minimal primes. Denote by E and D, respectively, the value sets of $\mathcal{I}$ and $\mathcal{J}$. Then*

$$\begin{aligned}\ell_R\left(\frac{\mathcal{I}}{\mathcal{J}}\right) = {} & \sum_{i=1}^{3}\big((\beta_i^0-\alpha_i^0) + (\#\mathcal{G}(D_i) - \#\mathcal{G}(E_i))\big) + \\ & \sum_{1\leq i<j\leq 3}\#M_{\{i,j\}}(D) - \sum_{1\leq i<j\leq 3}\#M_{\{i,j\}}(E) + \\ & \#RM(D) - \#RM(E) + \#AM(E) - \#AM(D),\end{aligned}$$

where $\alpha^0 = \min(E)$ and $\beta^0 = \min(D)$.

References

[1] Barucci, V.; D'anna, M.; Fröberg, R., *The Semigroup of Values of a One-dimensional Local Ring with two Minimal Primes*, Communications in Algebra, 28(8), pp 3607-3633 (2000). DOI: 10.1080/00927870008827044

[2] D'anna, M., *The Canonical Module of a One-dimensional Reduced Local Ring*, Communications in Algebra, 25, pp 2939-2965 (1997). DOI: 10.1080/00927879708826033

[3] Delgado de la Mata, F., *The semigroup of values of a curve singularity with several branches*, Manuscripta Math. 59, pp 347-374 (1987). DOI: 10.1007/bf01174799
[4] Delgado de la Mata, F., *Gorenstein curves and symmetry of the semigroup of values*, Manuscripta Math. 61, pp 285-296 (1988). DOI: 10.1007/bf01258440
[5] Garcia, A., *Semigroups associated to singular points of plane curves*, J. Reine. Angew. Math. 336, 165-184 (1982). DOI: 10.1515/crll.1982.336.165
[6] Gorenstein, D., *An arithmetic theory of adjoint plane curves*, Trans. Amer. Math. Soc. 72, pp 414-436 (1952). DOI: 10.1090/s0002-9947-1952-0049591-8
[7] Hernandes, M. E., *Private communication*
[8] Korell, P.; Schulze, M.; Tozzo, L., *Duality on value semigroups*, J. Commut. Alg. 11, no. 1, 81-129, (2019). DOI: 10.1216/JCA-2019-11-1-81

E. M. N. de Guzmán, Departamento de Matemática Aplicada, Universidade Federal Fluminense
Email address: e.marcavillaca@gmail.com

A. Hefez, Departamento de Matemática Aplicada, Universidade Federal Fluminense
Email address: ahefez@id.uff.br

Journal *of* Singularities
Volume 21 (2020), 132-148

Proc. of Geometric and Algebraic Singularity Theory, Będlewo, 2017
DOI: 10.5427/jsing.2020.21h

DUALITY OF SINGULARITIES FOR FLAT SURFACES IN EUCLIDEAN SPACE

ATSUFUMI HONDA

Dedicated to Professor Goo Ishikawa on the occasion of his 60th birthday.

Abstract. In this paper, we shall discuss the duality of singularities for a class of flat surfaces in Euclidean space. After introducing the definition of the conjugate of a tangent developable, we show that, if a tangent developable admits a swallowtail, its conjugate has a cuspidal cross cap. Similarly, we prove that the conjugate of a tangent developable having cuspidal S_1^+ singularities has cuspidal butterflies, and that cuspidal beaks have self-duality. We also show that cuspidal edges do not possess such a property, by exhibiting an example of a tangent developable with cuspidal edges whose conjugate has 5/2-cuspidal edges. Finally, we prove that conjugates of complete flat fronts with embedded ends cannot be complete flat fronts.

1. Introduction

We denote Euclidean 3-space by $\boldsymbol{R}^3$. It is well-known that, for a minimal surface

$$f = (x_1, x_2, x_3) : M \to \boldsymbol{R}^3,$$

its coordinate functions x_j $(j = 1, 2, 3)$ are harmonic functions on M. Then, the harmonic conjugates $x_j^\sharp$ $(j = 1, 2, 3)$ define another minimal surface $f^\sharp = (x_1^\sharp, x_2^\sharp, x_3^\sharp)$, which is called the *conjugate* minimal surface. Similarly, for maximal surfaces in the Lorentz-Minkowski 3-space $\boldsymbol{L}^3$, we can define the conjugate. Since the only complete maximal surfaces are spacelike planes [2], we need to consider maximal surfaces with singular points. Umehara–Yamada [23] introduced a class of maximal surfaces with admissible singularities called *maxfaces*, which satisfy the following property so-called the *duality of singularities*:

Fact 1.1 ([23, 4])**.** *Let $f : M \to \boldsymbol{L}^3$ be a maxface, $f^\sharp : M \to \boldsymbol{L}^3$ its conjugate, and $p \in M$ a singular point. Then, f at p is $\mathcal{A}$-equivalent to the cuspidal edge* (*resp. swallowtail, cuspidal cross cap*) *if and only if $f^\sharp$ at p is $\mathcal{A}$-equivalent to the cuspidal edge* (*resp. cuspidal cross cap, swallowtail*)*.*

The property as in Fact 1.1 is called the *duality of singularities*. Let S_1^3 (resp. S^3) be the de Sitter 3-space (resp. the 3-sphere) of constant sectional curvature 1. Also, let H_1^3 (resp. H^3) be the anti-de Sitter 3-space (resp. the hyperbolic 3-space) of constant sectional curvature -1, and Q^3 be the 3-lightcone. It is known that such a duality of singularities holds for various classes of surfaces as follows:

- timelike minimal surfaces (so-called *minfaces*) in $\boldsymbol{L}^3$ [21] (cf. [1]),
- spacelike surfaces of non-zero constant mean curvature in $\boldsymbol{L}^3$ [7],
- spacelike surfaces of constant mean curvature 1 in S_1^3 [4],
- timelike surfaces of constant mean curvature 1 in H_1^3 [24],

2010 *Mathematics Subject Classification.* Primary 53C42; Secondary 57R45.
Key words and phrases. flat surface, flat front, frontal, singular point, duality of singularities.
This work was supported by the Grant-in-Aid for Young Scientists (B) No. 16K17605 from JSPS.

- spacelike surfaces of zero extrinsic curvature in S^3_1, H^3 and Q^3 [13],
- surfaces of zero extrinsic curvature in S^3 [12].

We remark that such a duality is known for more degenerate singularities, such as cuspidal beaks, cuspidal butterflies and cuspidal S_1^- singularities ([12, 18]).

In this article, we shall study the duality of singularities in the case of flat surfaces with singularities in $\boldsymbol{R}^3$. Murata–Umehara [17] investigated the global properties of flat surfaces with singularities called *flat fronts* (cf. Fact 2.1). In particular, they proved that complete flat fronts with non-empty singular sets must be *tangent developables*. Ishikawa [10] investigated the singularities of tangent developables from the view point of the (real) *projective geometry*. In particular, Ishikawa [10] used the Scherbak's dual curves [20] in the dual projective space to define the dual tangent developables, and proved the duality of singularities. For more details, see [10, 11] (cf. [3]). However, to the best of the author's knowledge, there was no notion like the conjugate of flat surfaces $\boldsymbol{R}^3$ in the setting of *Euclidean geometry*. Thus, we shall find a suitable definition of the conjugate of flat fronts which satisfy the duality of singularities.

This paper is organized as follows. In Section 2, we review some basic facts on flat fronts, singularities of frontals in $\boldsymbol{R}^3$, and frontals in the 2-sphere S^2. Then, in Section 3, after reviewing a-orientable admissible developable frontals introduced by Murata–Umehara [17], we apply the criteria for cuspidal cross caps to such developable frontals. Comparing the condition for swallowtails and that for cuspidal cross caps, we give a definition of the conjugates for tangent developables (Definition 3.6, cf. Corollary 3.9). In Section 4, applying the criteria for other singularities (cuspidal beaks, cuspidal butterfly, cuspidal $S_1^{\pm}$, 5/2-cuspidal edge) to such tangent developables (cf. Propositions 4.2, 4.4, 4.6 and 4.9), we obtain the duality of singularities (Theorem 4.10). In the case of the cuspidal edge, we exhibit an example which does not satisfy the desired duality (see Example 4.11). Finally, in Section 5, we glance a global property of such conjugate operation, by proving that the conjugate of a complete flat front with embedded ends cannot be a complete flat front (Proposition 5.1).

2. Preliminaries

We denote by $\boldsymbol{R}^3$ the Euclidean 3-space. Let M be a connected smooth 2-manifold and

$$f : M \longrightarrow \boldsymbol{R}^3$$

a smooth map. A point $p \in M$ is called a *singular point* if f is not an immersion at p. Otherwise, we say p a *regular point*. Denote by $S(f)$ $(\subset M)$ the singular set. If $S(f)$ is empty, we call f a (regular) surface. In this case, at least locally, we can take a smooth unit normal vector field ν along f, that is, for every point $p \in M$, there exist an open neighborhood U of p and a smooth map $\nu : U \to S^2$ such that

$$df_q(\boldsymbol{v}) \cdot \nu(q) = 0 \quad \text{holds for each } q \in U \text{ and } \boldsymbol{v} \in T_qM, \tag{2.1}$$

where the dot '$\cdot$' is the canonical inner product on $\boldsymbol{R}^3$ and S^2 is the unit sphere

$$S^2 := \{\boldsymbol{x} \in \boldsymbol{R}^3 \,;\, \boldsymbol{x} \cdot \boldsymbol{x} = 1\}.$$

2.1. **Flat fronts.** A smooth map $f : M \to \boldsymbol{R}^3$ is called a *frontal* if, for each point $p \in M$, there exist a neighborhood U of p and a smooth map $\nu : U \to S^2$ which satisfies (2.1). Such a ν is called the unit normal vector field or the Gauss map of f. If ν can be defined throughout M, f is called *co-orientable*. If $(L :=)$ $(f, \nu) : U \to \boldsymbol{R}^3 \times S^2$ gives an immersion, f is called a *wave front* (or a *front*, for short).

A front f with a unit normal ν is called *flat* if $\operatorname{rank}(d\nu) \le 1$ on M. Denote by $ds^2 := df \cdot df$ the first fundamental form of f. In the case that f is regular, f is flat as a front if and only if

f is flat as a regular surface (namely, the Gaussian curvature K of ds^2 is identically zero $K = 0$ on M).

A smooth map $f : M \to \boldsymbol{R}^3$ is called *complete* if there exists a symmetric covariant tensor T on M with compact support such that $ds^2 + T$ gives a complete Riemannian metric on M. If f is complete and the singular set $S(f)$ is non-empty, then $S(f)$ must be compact.

Murata-Umehara [17] proved the following.

Fact 2.1 ([17]). *Let $\boldsymbol{\xi} : S^1 \to S^2$ be a regular curve without inflection points, and $\alpha = a(t)dt$ a 1-form on $S^1 = \boldsymbol{R}/2\pi\boldsymbol{Z}$ such that $\int_{S^1} \boldsymbol{\xi}\,\alpha = 0$ holds. Then, $f : S^1 \times \boldsymbol{R} \to \boldsymbol{R}^3$ defined by*

$$f(t,v) := \sigma(t) + v\,\boldsymbol{\xi}(t) \qquad \left(\sigma(t) := \int_0^t a(\tau)\,\boldsymbol{\xi}(\tau)d\tau\right) \tag{2.2}$$

is a complete flat front with non-empty singular set. Conversely, let $f : M \to \boldsymbol{R}^3$ be a complete flat front defined on a connected smooth 2-manifold M. If the singular set $S(f)$ of f is not empty, then f is umbilic-free, co-orientable, M is diffeomorphic to $S^1 \times \boldsymbol{R}$, and f is given by (2.2)*. Moreover, if the ends of f are embedded, f has at least four singular points other than cuspidal edges.*

For the definition of umbilic points, see [17] (cf. [5, 6, 8]). The final statement of Fact 2.1 may be regarded as a variant of four vertex theorem for plane curves.

2.2. **Singularities of frontals.** Fix a smooth 2-manifold M and take two points $p_i \in M$ $(i = 1,2)$. Let $f_i : (M, p_i) \to (\boldsymbol{R}^3, f(p_i))$ $(i = 1,2)$ be two map germs. We say f_1 is $\mathcal{A}$-*equivalent* to f_2 if there exist diffeomorphism germs

$$\varphi : (M,p_1) \to (M,p_2) \quad \text{and} \quad \Phi : (\boldsymbol{R}^3, f(p_1)) \to (\boldsymbol{R}^3, f(p_2))$$

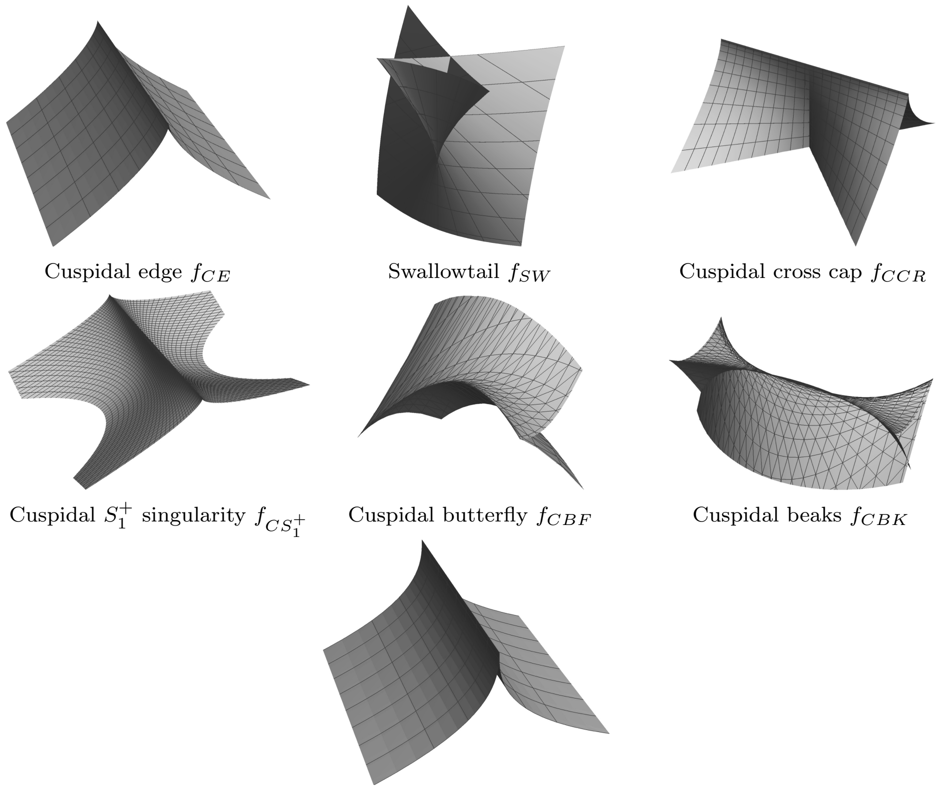
such that $\Phi \circ f_1 \circ \varphi^{-1} = f_2$. We set f_{CE}, f_{SW}, f_{CCR}, f_{CBK}, f_{CBF}, $f_{CS_k^\pm}$, f_{rCE} to be the germs from $(\boldsymbol{R}^2, 0)$ to $(\boldsymbol{R}^3, 0)$ given by:

$$\begin{aligned}
f_{CE}(u,v) &:= (u, v^2, v^3),\\
f_{SW}(u,v) &:= (4u^3 + 2uv, 3u^4 + u^2v, -v),\\
f_{CCR}(u,v) &:= (u, v^2, uv^3),\\
f_{CBK}(u,v) &:= (v, -2u^3 + uv^2, -3u^4 + u^2v^2),\\
f_{CBF}(u,v) &:= (u, 5v^4 + 2uv, 4v^5 + uv^2 - u^2),\\
f_{CS_k^\pm}(u,v) &:= (u, v^2, v^3(u^{k+1} \pm v^2)),\\
f_{rCE}(u,v) &:= (u, v^2, v^5),
\end{aligned} \tag{2.3}$$

respectively, where k is a positive integer. We call the map germ f_{CE} (resp. f_{SW}, f_{CCR}, f_{CBK}, f_{CBF}, $f_{CS_k^\pm}$, f_{rCE}) the *cuspidal edge* (resp. *swallowtail, cuspidal cross cap, cuspidal beaks, cuspidal butterfly, cuspidal $S_k^\pm$ singularity, 5/2-cuspidal edge*).

Kokubu–Rossman–Saji–Umehara–Yamada [15] gave a useful criteria for cuspidal edge and swallowtail. Similar useful criteria for other singularities are given in the following: [4] for cuspidal cross cap (cf. Fact 3.4); [14] for cuspidal beaks (cf. Fact 4.1); [13] for cuspidal butterfly (cf. Fact 4.3); [19] for cuspidal $S_k^\pm$ singularity (cf. Fact 4.5); and [9] for 5/2-cuspidal edge (cf. Fact 4.8). To state such criteria, we shall review some basic notions for frontals.

Let $f : M \to \boldsymbol{R}^3$ be a frontal with the (locally defined) unit normal ν. Take a point $p \in M$. Let $(U; u, v)$ be a coordinate neighborhood of p. We call $\lambda := \det(f_u, f_v, \nu)$ the *signed area density function*. Remark that p is a singular point of f if and only if $\lambda(p) = 0$. If $d\lambda(p) \neq 0$, a singular



Cuspidal edge f_{CE} Swallowtail f_{SW} Cuspidal cross cap f_{CCR}

Cuspidal S_1^+ singularity $f_{CS_1^+}$ Cuspidal butterfly f_{CBF} Cuspidal beaks f_{CBK}

5/2-cuspidal edge f_{rCE}

FIGURE 1. The images of standard models of the singularities (f_{CE}, f_{SW}, f_{CCR}, $f_{CS_1^+}$, f_{CBF}, f_{CBK}, f_{rCE}) given in (2.3).

point p is called *non-degenerate*. We remark that if p is non-degenerate, then $\operatorname{rank}(df)_p = 1$ holds. By the implicit function theorem, there exists a regular curve $\gamma(t)$ ($|t| < \varepsilon$) on the uv-plane such that $\gamma(0) = p$ and the image of γ coincides with the singular point set $S(f)$ near p, where $\varepsilon > 0$. We call $\gamma(t)$ the *singular curve* and $\gamma' = d\gamma/dt$ the singular direction. Then, there exists a non-zero smooth vector field $\zeta(t)$ along $\gamma(t)$ such that $\zeta(t)$ is a null vector (i.e., $df(\zeta(t)) = 0$) for each t. Such a vector field $\zeta(t)$ is called a *null vector field*. On the other hand, a non-vanishing smooth vector field $\zeta = \zeta(u, v)$ on U so that $\zeta|_{S(f)}$ gives a kernel direction of f is also called a *null vector field*. We set the functions $\delta(t)$ and $\psi_{ccr}(t)$ as

$$\delta(t) := \det\left(\gamma'(t), \zeta(t)\right), \quad \psi_{ccr}(t) := \det\left((f \circ \gamma)'(t), (\nu \circ \gamma)(t), d\nu(\zeta(t))\right), \tag{2.4}$$

respectively. Later we use these functions in the criteria for various singularity types (cf. Facts 3.4, 4.1, 4.3, 4.5 and 4.8).

2.3. **Frontals in 2-sphere.** Let J be an open interval of $\boldsymbol{R}$. A smooth map $\boldsymbol{\xi} : J \to S^2$ is called a *frontal* if there exists a smooth unit vector field $\boldsymbol{n}$ along $\boldsymbol{\xi}$ such that $\boldsymbol{\xi}' \cdot \boldsymbol{n} = 0$ holds. We call $\boldsymbol{n}$ the *unit normal vector field* or the *spherical dual*. The pair $(\boldsymbol{\xi}, \boldsymbol{n})$ gives a Legendre curve in the unit tangent bundle

$$T_1 S^2 = \{(p, v) \in S^2 \times S^2\, ;\, p \cdot v = 0\}$$

with respect to the canonical contact structure. Since $\boldsymbol{\xi} \cdot \boldsymbol{n}' = 0$, there exist smooth 1-forms ρ, ω such that

$$d\boldsymbol{\xi} = \rho\, \boldsymbol{\eta}, \qquad d\boldsymbol{n} = -\omega\, \boldsymbol{\eta}, \tag{2.5}$$

where we set $\boldsymbol{\eta} := \boldsymbol{n} \times \boldsymbol{\xi}$. Then, the frame $\mathcal{F}(t) := \{\boldsymbol{\xi}(t), \boldsymbol{\eta}(t), \boldsymbol{n}(t)\}$ satisfies

$$\mathcal{F}^{-1} d\mathcal{F} = \begin{pmatrix} 0 & -\rho & 0 \\ \rho & 0 & -\omega \\ 0 & \omega & 0 \end{pmatrix}, \tag{2.6}$$

where we used the identity $d\boldsymbol{\eta} = -\rho\, \boldsymbol{\xi} + \omega\, \boldsymbol{n}$. Conversely, the following holds.

Fact 2.2 ([22, Theorem 2.5])**.** *Let ρ, ω be smooth 1-forms on an interval J. Then, there exists a frontal $\boldsymbol{\xi} : J \to S^2$ with the spherical dual $\boldsymbol{n}$ such that (2.5) holds.*

Therefore, we may conclude that there exists a one-to-one correspondence between frontals with spherical duals and pairs of smooth 1-forms. We call the pair of 1-forms (ρ, ω) the *data* of the frontal $\boldsymbol{\xi} : J \to S^2$.

3. Conjugates of tangent developables

In this section, comparing the criteria for swallowtail and cuspidal cross cap, we give a definition of the conjugates of developable frontals.

3.1. **Developable frontals.** Let J be an open interval including $0 \in J$. Take 1-forms α, β on J and a frontal $\boldsymbol{\xi} : J \to S^2$ with the spherical dual $\boldsymbol{n}$. Then a smooth map $f : J \times \boldsymbol{R} \to \boldsymbol{R}^3$ defined by

$$f(t, v) := \sigma(t) + v\, \boldsymbol{\xi}(t) \qquad \left(\sigma(t) := \int_0^t (\alpha\, \boldsymbol{\xi} + \beta\, \boldsymbol{\eta}),\ \boldsymbol{\eta} := \boldsymbol{n} \times \boldsymbol{\xi}\right) \tag{3.1}$$

is a co-orientable frontal in $\boldsymbol{R}^3$ so that $\nu(t, v) := \boldsymbol{n}(t)$ is a unit normal. We shall call $f(t, v)$ an *a-orientable admissible developable frontal* and v is called the *asymptotic parameter*. The quadruple of the 1-forms $(\alpha, \beta, \rho, \omega)$ is independent of the choice of the parameter t on J as a 1-dimensional manifold, which we call the *data* of $f(t, v)$. Here, (ρ, ω) is the data corresponding to a frontal $\boldsymbol{\xi}$ in S^2 with the spherical dual $\boldsymbol{n}$ (cf. (2.5)).

Remark 3.1. We remark that Murata–Umehara defined a-orientable admissible developable frontals in [17, Definition 2.3], where 'a-orientable' means 'asymptotically orientable', see [17, page 289]. They gave a representation formula in [17, Theorem 2.8]. Our definition is based on [17, Theorem 2.8].

If f is a cylinder, then $\boldsymbol{\xi} : J \to S^2$ is a constant map, that is, $r(t) = 0$ holds for all $t \in J$, where $\rho = r(t)dt$. We call a point $p_0 = (t_0, v_0)$ a *cylindrical point* of $f(t, v)$ if $\boldsymbol{\xi}'(t_0) = 0$ (i.e., $r(t_0) = 0$) holds[1]. We denote by $S_c(f)$ (resp. $S_{nc}(f)$) the set of cylindrical singular points (resp. non-cylindrical singular points).

Lemma 3.2 (cf. [17, Proposition 2.16])**.** *Let $f(t, v)$ be an a-orientable admissible developable frontal whose data is given by $(\alpha, \beta, \rho, \omega) = (a(t)\, dt, b(t)\, dt, r(t)\, dt, w(t)\, dt)$. Then, a point*

$$p_0 = (t_0, v_0) \in J \times \boldsymbol{R}$$

is a singular point of f if and only if $b(t_0) + v_0\, r(t_0) = 0$. Moreover,

- *f is a front at a singular point $p_0 = (t_0, v_0)$ if and only if $w(t_0) \neq 0$.*

[1] Cylindrical singular points are *linear singular points* in the sense of [17, Definition 2.15].

- *$p_0 = (t_0, v_0)$ is a cylindrical singular point of f if and only if $b(t_0) = r(t_0) = 0$. Such a $p_0 \in S_c(f)$ is non-degenerate if and only if $b'(t_0) + v_0\, r'(t_0) \neq 0$. Setting*

$$\gamma_c(v) := (t_0, v), \qquad \zeta_c(v) := \partial_t - a(t_0)\,\partial_v, \tag{3.2}$$

we have that $\gamma_c(v)$ is a singular curve passing through $\gamma_c(v_0) = p_0$, and $\zeta_c(v)$ is a null vector field along $\gamma_c(v)$. Moreover, we have (cf. (2.4))

$$\delta_c(v) := \det\left(\gamma_c'(v), \zeta_c(v)\right) = -1. \tag{3.3}$$

- *$p_0 = (t_0, v_0)$ is a non-cylindrical singular point of f if and only if $r(t_0) \neq 0$ and $v_0 = -b(t_0)/r(t_0)$. Such a $p_0 \in S_{nc}(f)$ is non-degenerate, and setting*

$$\gamma_{nc}(t) := \left(t, -\frac{b(t)}{r(t)}\right), \qquad \zeta_{nc}(t) := \partial_t - a(t)\,\partial_v, \tag{3.4}$$

we have that $\gamma_{nc}(t)$ is a singular curve passing through $\gamma_{nc}(t_0) = p_0$, and $\zeta_{nc}(t)$ is a null vector field along $\gamma_{nc}(t)$. Moreover, we have (cf. (2.4))

$$\delta_{nc}(t) := \det\left(\gamma_{nc}'(t), \zeta_{nc}(t)\right) = -a(t) + \left(\frac{b(t)}{r(t)}\right)'. \tag{3.5}$$

Proof. By (2.5), we have

$$f_t = a(t)\,\boldsymbol{\xi}(t) + (b(t) + v\,r(t))\,\boldsymbol{\eta}(t), \quad f_v = \boldsymbol{\xi}(t). \tag{3.6}$$

So, the signed area density function λ is given by

$$\lambda = \det(f_t, f_v, \nu) = (b(t) + v\,r(t))\det\left(\boldsymbol{\eta}(t), \boldsymbol{\xi}(t), \boldsymbol{n}(t)\right) = -b(t) - v\,r(t). \tag{3.7}$$

Thus, we have $S(f) = \{(t, v) \in J \times \boldsymbol{R}\,;\, b(t) + v\,r(t) = 0\}$ and

$$-\lambda_t = b'(t) + v\,r'(t), \qquad -\lambda_v = r(t). \tag{3.8}$$

On the singular set $S(f)$, $f_t - a(t) f_v = 0$ holds. Thus, setting $\zeta(t, v) := \partial_t - a(t)\partial_v$, we have $df(\zeta) = 0$ at a singular point p_0. Since f is front at $p_0 \in S(f)$ if and only if

$$(dL)_{p_0} = ((df)_{p_0}, (d\nu)_{p_0})$$

is injective, this condition is equivalent to $(d\nu)_{p_0}(\zeta) \neq 0$. Since $-d\nu(\zeta) = -\boldsymbol{n}' = w\boldsymbol{\eta}$, f is front at $p_0 \in S(f)$ if and only if $w(t_0) \neq 0$.

If p_0 is cylindrical, $r(t_0) = 0$ holds. Thus, p_0 is a cylindrical singular point if and only if $r(t_0) = 0$ and $b(t_0)\,(= b(t_0) + v_0\,r(t_0)) = 0$. By (3.8), p_0 is non-degenerate if and only if $b'(t_0) + v_0\,r'(t_0) \neq 0$. In this case, $\gamma_c(v)$ given in (3.2) is a singular curve passing through $\gamma_c(v_0) = p_0$. By (3.6), $f_t - a(t_0)\,f_v = 0$ holds along $\gamma_c(v)$, and hence we have $\zeta_c(v)$ given in (3.2) is a null vector field along $\gamma_c(v)$.

If p_0 is a non-cylindrical singular point, $r(t_0) \neq 0$ holds. By (3.8), p_0 must be non-degenerate. Then $\gamma_{nc}(t)$ given in (3.4) is a singular curve passing through $\gamma_{nc}(t_0) = p_0$. By (3.6),

$$f_t - a(t)\,f_v = 0$$

holds along $\gamma_{nc}(t)$, and hence we have $\zeta_{nc}(t)$ given in (3.4) is a null vector field along $\gamma_{nc}(t)$. $\square$

As we seen in Lemma 3.2, the cylindrical and non-cylindrical singular sets, $S_c(f)$ and $S_{nc}(f)$, are written as

$$S_c(f) = \{(t, v) \in J \times \boldsymbol{R}\,;\, b(t) = r(t) = 0\}\,, \tag{3.9}$$

$$S_{nc}(f) = \left\{(t, v) \in J \times \boldsymbol{R}\,;\, r(t) \neq 0,\ v = -\frac{b(t)}{r(t)}\right\}, \tag{3.10}$$

respectively.

Murata–Umehara [17] applied the criteria for cuspidal edge and swallowtail given in [15] to developable frontals as follows:

Fact 3.3 ([17, Proposition 2.16])**.** *Let $f(t,v)$ be an a-orientable admissible developable frontal whose data is given by $(\alpha,\beta,\rho,\omega)=(a(t)\,dt, b(t)\,dt, r(t)\,dt, w(t)\,dt)$. Then, a point*

$$p_0=(t_0,v_0)\in J\times \boldsymbol{R}$$

is a singular point of f if and only if $b(t_0)+v_0\,r(t_0)=0$. Moreover, for a singular point $p_0=(t_0,v_0)$ of f, we have that

- *f at p_0 is $\mathcal{A}$-equivalent to the cuspidal edge if and only if*

$$r(t_0)\neq 0,\quad a(t_0)\neq \left(\frac{b(t)}{r(t)}\right)'\Bigg|_{t=t_0},\quad w(t_0)\neq 0,$$

or

$$r(t_0)=0,\quad b'(t_0)+v_0\,r'(t_0)\neq 0,\quad w(t_0)\neq 0.$$

- *f at p_0 is $\mathcal{A}$-equivalent to the swallowtail if and only if*

$$r(t_0)\neq 0,\quad a(t_0)= \left(\frac{b(t)}{r(t)}\right)'\Bigg|_{t=t_0},\quad a'(t_0)\neq \left(\frac{b(t)}{r(t)}\right)''\Bigg|_{t=t_0},\quad w(t_0)\neq 0. \tag{3.11}$$

We can observe that swallowtails never appear on the cylindrical singular set $S_c(f)$.

3.2. **Cuspidal cross cap.** Here we review the criterion for the cuspidal cross cap given by Fujimori–Saji–Umehara–Yamada [4].

Fact 3.4 (Criterion for cuspidal cross cap [4])**.** *Let $f:U\to \boldsymbol{R}^3$ be a frontal defined on a domain U of $\boldsymbol{R}^2$, with the unit normal ν, and $p\in U$ a non-degenerate singular point of f. And let $\gamma(t)$ be a singular curve such that $\gamma(0)=p$, $\zeta(t)$ a null vector field, $\delta(t)$ and $\psi_{ccr}(t)$ be the functions defied by (2.4). Then, the map germ f at p is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $\delta(0)\neq 0$, $\psi_{ccr}(0)=0$ and $\psi'_{ccr}(0)\neq 0$.*

Now, we shall apply Fact 3.4 to a-orientable admissible developable frontals.

Proposition 3.5. *Let $f(t,v)$ be an a-orientable admissible developable frontal whose data is given by $(\alpha,\beta,\rho,\omega)=(a(t)\,dt, b(t)\,dt, r(t)\,dt, w(t)\,dt)$. For a singular point $p_0=(t_0,v_0)$ of f, we have that f at p_0 is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if*

$$r(t_0)\neq 0,\quad a(t_0)\neq \left(\frac{b(t)}{r(t)}\right)'\Bigg|_{t=t_0},\quad w(t_0)=0,\quad w'(t_0)\neq 0. \tag{3.12}$$

Proof. First, assume that $p_0\in S_{nc}(f)$. By Lemma 3.2, $\gamma_{nc}(t)$ is a singular curve passing through $\gamma_{nc}(t_0)=p_0$, and $\zeta_{nc}(t)$ is a null vector field along $\gamma_{nc}(t)$, where $\gamma_{nc}(t)$ and $\zeta_{nc}(t)$ are given by (3.4). Let $\delta_{nc}(t)$ be the function given in (3.5). By Lemma 3.2, the function δ given in (2.4) coincides with $\delta_{nc}(t)$. On the other hand, setting $\hat{\gamma}_{nc}(t):=f(\gamma_{nc}(t))$, we have

$$\hat{\gamma}'_{nc}(t)=-\delta_{nc}(t)\boldsymbol{\xi}(t).$$

Hence, the function ψ_{ccr} given in (2.4) is

$$\psi_{ccr}(t)=-\delta_{nc}(t)\det\left(\boldsymbol{\xi}(t),\,\boldsymbol{n}(t),\,\boldsymbol{n}'(t)\right)=-\delta_{nc}(t)w(t). \tag{3.13}$$

Therefore, f at p_0 is $\mathcal{A}$-equivalent to cuspidal cross cap if and only if (3.12) holds.

Next, we shall prove that, if $p_0\in S_c(f)$, f at p_0 cannot be $\mathcal{A}$-equivalent to the cuspidal cross cap. By Lemma 3.2, $\gamma_c(v)$ is a singular curve passing through $\gamma_c(v_0)=p_0$, and $\zeta_c(v)$ is a null vector field along $\gamma_c(v)$, where $\gamma_c(v)$ and $\zeta_c(v)$ are given by (3.2). By Lemma 3.2, the function δ given in (2.4) is identically -1. On the other hand, setting $\hat{\gamma}_c(v) := f(\gamma_c(v))$, we have $\hat{\gamma}_c'(v) = \boldsymbol{\xi}(t_0)$. Hence, the function ψ_{ccr} given in (2.4) is

$$\psi_{ccr}(v) = \det\left(\boldsymbol{\xi}(t),\, \boldsymbol{n}(t_0),\, w(t_0)\boldsymbol{\eta}(t_0)\right) = w(t_0). \tag{3.14}$$

Therefore, $\psi_{ccr}(v_0) = 0$ and $\psi_{ccr}'(v_0) \neq 0$ do not occur at the same time. Thus, f at p_0 cannot be $\mathcal{A}$-equivalent to the cuspidal cross cap. $\square$

3.3. **Observation and definition.** For an a-orientable admissible developable frontal $f = f(t,v)$, we would like to find its conjugate $f^\sharp$ which satisfies the so-called *duality of singularities* as in Fact 1.1 in the introduction.

We shall compare the condition (3.11) for swallowtail and that (3.12) for cuspidal cross cap. If $\beta = b(t)\,dt$ is identically zero, (3.11) is equivalent to

$$r(t_0) \neq 0, \quad a(t_0) = 0, \quad a'(t_0) \neq 0, \quad w(t_0) \neq 0, \tag{3.15}$$

and (3.12) is equivalent to

$$r(t_0) \neq 0, \quad a(t_0) \neq 0, \quad w(t_0) = 0, \quad w'(t_0) \neq 0. \tag{3.16}$$

Thus, for an a-orientable admissible developable frontal $f = f(t,v)$ with the data $(\alpha, 0, \rho, \omega)$, if we set $f^\sharp$ to be the a-orientable admissible developable frontal whose data is given by

$$(\alpha^\sharp, 0, \rho^\sharp, \omega^\sharp) := (\omega, 0, \rho, \alpha),$$

we have that *f at p is $\mathcal{A}$-equivalent to the swallowtail if and only if $f^\sharp$ at p is $\mathcal{A}$-equivalent to the cuspidal cross cap.* Namely, f and $f^\sharp$ satisfy the duality of singularities.

A-orientable admissible developable frontals with $\beta = 0$ are *tangent developables*. In fact, when $\beta = 0$, f given in (3.1) is written as

$$f(t,v) := \sigma(t) + v\,\boldsymbol{\xi}(t) \qquad \left(\sigma(t) := \int_0^t \alpha\,\boldsymbol{\xi},\ \boldsymbol{\eta} := \boldsymbol{n} \times \boldsymbol{\xi}\right). \tag{3.17}$$

Since $\sigma'(t)$ and $\boldsymbol{\xi}(t)$ are linearly dependent, we may conclude that f is a tangent developable.

Definition 3.6 (A-tangent developable). We call an a-orientable admissible developable frontal with $\beta = 0$ an *a-tangent developable*. For an a-tangent developable f, the triplet of the 1-forms (α, ρ, ω) is also called the *data*. Then, the a-tangent developable $f^\sharp$ whose data is given by $(\alpha^\sharp, \rho^\sharp, \omega^\sharp) := (\omega, \rho, \alpha)$ is called the *conjugate* of f.

We remark that, by Lemma 3.2 and $\rho = \rho^\sharp$, the singular set of an a-tangent developable f coincides with that of the conjugate $f^\sharp$ of f, namely $S(f) = S(f^\sharp) = \{(t,v) \in J \times \boldsymbol{R}\,;\, v\,r(t) = 0\}$ holds. In the case that the a-tangent developable $f = f(t,v)$ is defined on $M := S^1 \times \boldsymbol{R}$, the domain of the conjugate $f^\sharp$ is the universal covering $\tilde{M} = \boldsymbol{R}^2$ of M.

Remark 3.7. An a-orientable admissible developable frontal without cylindrical points is an a-tangent developable. If (t,v) is non-cylindrical, by changing the parameter $v \mapsto v - b(t)/r(t)$, f can be written as

$$f = \sigma(t) + \left(v - \frac{b(t)}{r(t)}\right)\boldsymbol{\xi}(t) = \tilde{\sigma}(t) + v\,\boldsymbol{\xi}(t).$$

Here we set $\tilde{\sigma}(t) := \sigma(t) - (b(t)/r(t))\boldsymbol{\xi}(t)$, which satisfies that $\tilde{\sigma}'(t)$ and $\boldsymbol{\xi}(t)$ are linearly dependent.

Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be the canonical orthonormal basis of $\boldsymbol{R}^3$, namely, $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) = \mathit{Id}$, where Id is the identity matrix $\mathit{Id} := \mathrm{diag}(1,1,1)$. The procedure of constructing the a-tangent developable from a given data (α, ρ, ω) is as follows:

- Take $\mathcal{F}_0 \in \mathrm{SO}(3)$ arbitrarily.

- Let $\mathcal{F} = \mathcal{F}(t)$ be a solution of (2.6) with the initial value $\mathcal{F}(t_0) = \mathcal{F}_0$.
- Setting $\boldsymbol{\xi}(t) := \mathcal{F}(t)\boldsymbol{e}_1$, then,

$$f(t, v) = \sigma(t) + v\,\boldsymbol{\xi}(t) \qquad \left(\sigma(t) := \int_{t_0}^{t} \alpha\,\boldsymbol{\xi}\right)$$

  is an a-tangent developable whose data is given by (α, ρ, ω) such that $\boldsymbol{n}(t) := \mathcal{F}(t)\boldsymbol{e}_3$ is a unit normal.

Taking account of the data of the conjugate $(\alpha^\sharp, \rho^\sharp, \omega^\sharp) = (\omega, \rho, \alpha)$, we have the following.

Lemma 3.8. *Let $f = f(t, v)$ be the a-tangent developable defined on $J \times \boldsymbol{R}$ whose data is given by (α, ρ, ω). Fix $t_0 \in J$. Take a solution $\mathcal{F}^\sharp = \mathcal{F}^\sharp(t)$ of the following initial value problem*

$$(\mathcal{F}^\sharp)^{-1} d\mathcal{F}^\sharp = \begin{pmatrix} 0 & -\rho & 0 \\ \rho & 0 & -\alpha \\ 0 & \alpha & 0 \end{pmatrix}, \qquad \mathcal{F}^\sharp(t_0) = Id. \tag{3.18}$$

Then setting $\boldsymbol{\xi}^\sharp(t) := \mathcal{F}^\sharp(t)\boldsymbol{e}_1$, the conjugate $f^\sharp$ is given by

$$f^\sharp(t, v) = \sigma^\sharp(t) + v\,\boldsymbol{\xi}^\sharp(t) \qquad \left(\sigma^\sharp(t) := \int_{t_0}^{t} \omega\,\boldsymbol{\xi}^\sharp\right) \tag{3.19}$$

such that the data of $f^\sharp$ is given by $(\alpha^\sharp, \rho^\sharp, \omega^\sharp) := (\omega, \rho, \alpha)$, and $\boldsymbol{n}^\sharp(t) := \mathcal{F}^\sharp(t)\boldsymbol{e}_3$ gives a unit normal of $f^\sharp$.

By [17, Proposition 2.16] (cf. Fact 3.3) and Proposition 3.5, we have the following:

Corollary 3.9. *Let $f(t, v)$ be an a-tangent developable whose data is given by*

$$(\alpha, \rho, \omega) = (a(t)\,dt, r(t)\,dt, w(t)\,dt).$$

Take a singular point $p_0 = (t_0, v_0) \in S(f)$. Then,

- *f at p_0 is $\mathcal{A}$-equivalent to the cuspidal edge if and only if*

$$v_0 = 0, \quad r(t_0) \neq 0, \quad a(t_0) \neq 0, \quad w(t_0) \neq 0,$$

  or

$$v_0 \neq 0, \quad r(t_0) = 0, \quad r'(t_0) \neq 0, \quad w(t_0) \neq 0.$$

- *f at p_0 is $\mathcal{A}$-equivalent to the swallowtail if and only if*

$$v_0 = 0, \quad r(t_0) \neq 0, \quad a(t_0) = 0, \quad a'(t_0) \neq 0, \quad w(t_0) \neq 0. \tag{3.20}$$

- *f at p_0 is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if*

$$v_0 = 0, \quad r(t_0) \neq 0, \quad a(t_0) \neq 0, \quad w(t_0) = 0, \quad w'(t_0) \neq 0. \tag{3.21}$$

In particular, f at p_0 is $\mathcal{A}$-equivalent to the swallowtail if and only if $f^\sharp$ at p_0 is $\mathcal{A}$-equivalent to the cuspidal cross cap, where $f^\sharp$ is the conjugate of f.

As an example, we calculate the conjugate of the standard swallowtail.

Example 3.10 (Conjugate of the standard swallowtail). Let f_{SW} be the standard swallowtail given in (2.3). By a parameter change $(u, v) \mapsto (t, y - 6t^2)$, we have

$$f_{SW}(t, y) = (-8t^3, -3t^4, 6t^2) + y\,(2t, t^2, -1).$$

Thus, setting $v := y\sqrt{1 + 4t^2 + t^4}$, f_{SW} is an a-tangent developable $f_{SW}(t, v) = \sigma(t) + v\,\boldsymbol{\xi}(t)$, where

$$\sigma(t) := (-8t^3, -3t^4, 6t^2), \qquad \boldsymbol{\xi}(t) := \frac{1}{\sqrt{1 + 4t^2 + t^4}}(2t, t^2, -1).$$

Since $\sigma'(t) = -12t\sqrt{1+4t^2+t^4}\,\boldsymbol{\xi}(t)$, we have

$$a(t) = -12t\sqrt{1+4t^2+t^4}. \tag{3.22}$$

Then the spherical dual $\boldsymbol{n}(t)$ of $\boldsymbol{\xi}(t)$ and $\boldsymbol{\eta}(t) = \boldsymbol{n}(t) \times \boldsymbol{\xi}(t)$ are given by

$$\boldsymbol{n}(t) = \frac{1}{\sqrt{1+t^2+t^4}}\left(t, -1, t^2\right),$$
$$\boldsymbol{\eta}(t) = \frac{1}{\sqrt{1+4t^2+t^4}\sqrt{1+t^2+t^4}}\left(1-t^4, t+2t^3, 2t+t^3\right),$$

respectively. Hence we have

$$r(t) = \frac{2\sqrt{1+t^2+t^4}}{1+4t^2+t^4}, \qquad w(t) = -\frac{\sqrt{1+4t^2+t^4}}{1+t^2+t^4}, \tag{3.23}$$

where $r(t) = \boldsymbol{\xi}'(t)\cdot\boldsymbol{\eta}(t)$, $w(t) = -\boldsymbol{n}'(t)\cdot\boldsymbol{\eta}(t)$.

Then, applying Lemma 3.8 with $(\alpha, \rho, \omega) = (a(t)\,dt, r(t)\,dt, w(t)\,dt)$ and $t_0 = 0$, we obtain the conjugate $f^{\sharp}_{SW}(t, v)$, where $a(t)$, $r(t)$, $w(t)$ are given by (3.22) and (3.23), respectively (cf. Figure 2).

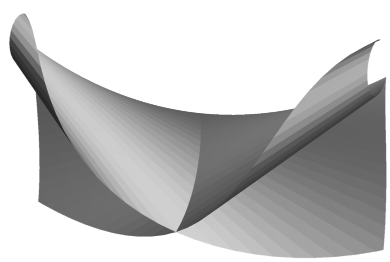


FIGURE 2. The a-tangent developable $f^{\sharp}_{SW}$ which is the conjugate of the standard swallowtail f_{SW} given by (2.3) (cf. Figure 1). By Corollary 3.9, we have that $f^{\sharp}_{SW}$ at $(t, v) = (0, 0)$ is $\mathcal{A}$-equivalent to the cuspidal cross cap. This figure is plotted by integrating (3.18) and (3.19) numerically.

4. Other singularities

Here, we shall write down the criteria for other singularities (cuspidal beaks, cuspidal butterfly, cuspidal $S_1^{\pm}$ singularity, 5/2-cuspidal edge) on a-tangent developables in terms of their data.

4.1. **Cuspidal beaks.**
First, we review the criterion for the cuspidal beaks given by Izumiya–Saji–Takahashi [14].

Fact 4.1 (Criterion for cuspidal beaks [14])**.** *Let $f : U \to \boldsymbol{R}^3$ be a front defined on a domain U of $\boldsymbol{R}^2$ with the unit normal ν. Also let $p \in U$ be a singular point of f and ζ a null vector field. Then, the map germ f at p is $\mathcal{A}$-equivalent to the cuspidal beaks if and only if* $\operatorname{rank}(df)_p = 1$, $d\lambda(p) = 0$, $\det \operatorname{Hess}\lambda(p) < 0$ *and* $\zeta\zeta\lambda(p) \neq 0$ *hold.*

Applying Fact 4.1 to a-tangent developables, we have the following.

Proposition 4.2. *Let $f(t,v)$ be an a-tangent developable whose data is given by*

$$(\alpha, \rho, \omega) = (a(t)\,dt, r(t)\,dt, w(t)\,dt).$$

Then, for a singular point $p_0 = (t_0, v_0)$ of f, we have that f at p_0 is $\mathcal{A}$-equivalent to the cuspidal beaks if and only if

$$v_0 = 0, \quad r(t_0) = 0, \quad r'(t_0) \neq 0, \quad a(t_0) \neq 0, \quad w(t_0) \neq 0. \tag{4.1}$$

Proof. We remark that for any singular point p_0 of f, $\operatorname{rank}(df)_{p_0} = 1$ holds (cf. (3.6)). Hence, by Fact 4.1, f at p_0 is $\mathcal{A}$-equivalent to the cuspidal beaks if and only if $d\lambda(p_0) = 0$, $\det \mathrm{Hess}\lambda(p_0) < 0$, $\zeta\zeta\lambda(p_0) \neq 0$, and f is a front at p_0. By (3.8), $d\lambda(p_0) = 0$ if and only if $r(t_0) = 0$ (i.e., p_0 is cylindrical) and $v_0\, r'(t_0) = 0$. Since the signed area density function λ is given by

$$\lambda(t, v) = -v\, r(t)$$

(cf. (3.7)), we have

$$\det \mathrm{Hess}\lambda = \det \begin{pmatrix} \lambda_{tt} & \lambda_{tv} \\ \lambda_{tv} & \lambda_{vv} \end{pmatrix} = -\lambda_{tv}^2 = -(r')^2.$$

Hence, $\det \mathrm{Hess}\lambda(p_0) < 0$ if and only if $r'(t_0) \neq 0$. As we see in the proof of Lemma 3.2, $\zeta(t,v) := \partial_t - a(t)\partial_v$ gives a null vector field. Since $\zeta\lambda = v\, r'(t) - a(t)r(t)$, we have

$$\zeta^2\lambda = v\, r''(t) - a'(t)r(t) - 2a(t)r'(t). \tag{4.2}$$

Therefore, f at p_0 is $\mathcal{A}$-equivalent to the cuspidal beaks if and only if (4.1) holds. □

4.2. **Cuspidal butterfly.** Next, we review the criterion for the cuspidal butterfly given by Izumiya–Saji [13].

Fact 4.3 (Criterion for cuspidal butterfly [13])**.** *Let $f : U \to \boldsymbol{R}^3$ be a front defined on a domain U of $\boldsymbol{R}^2$ with the unit normal ν. Take a non-degenerate singular point $p \in U$ of f. Let $\gamma(t)$ be a singular curve such that $\gamma(0) = p$ and $\zeta(t)$ a null vector field. Then, the map germ f at p is $\mathcal{A}$-equivalent to the cuspidal butterfly if and only if $\delta(0) = \delta'(0) = 0$ and $\delta''(0) \neq 0$ hold.*

Applying Fact 4.3 to a-tangent developables, we have the following.

Proposition 4.4. *Let $f(t,v)$ be an a-tangent developable whose data is given by*

$$(\alpha, \rho, \omega) = (a(t)\,dt, r(t)\,dt, w(t)\,dt).$$

Then, for a singular point $p_0 = (t_0, v_0)$ of f, we have that f at p_0 is $\mathcal{A}$-equivalent to the cuspidal butterfly if and only if

$$v_0 = 0, \quad r(t_0) \neq 0, \quad a(t_0) = a'(t_0) = 0, \quad a''(t_0) \neq 0, \quad w(t_0) \neq 0. \tag{4.3}$$

Proof. By (4.2), we have

$$\zeta^3\lambda = v\, r'''(t) - a''(t)r(t) - 3a'(t)r'(t) - 3a(t)r''(t). \tag{4.4}$$

Hence, by Fact 4.3, f at p_0 is $\mathcal{A}$-equivalent to the cuspidal butterfly if and only if

(i) f is a front at $p_0 = (t_0, v_0)$ (i.e., $w(t_0) \neq 0$),
(ii) $p_0 = (t_0, v_0)$ is non-degenerate (i.e., $r(t_0) \neq 0$ or $r(t_0) = 0$, $v_0\, r'(t_0) \neq 0$),
(iii) $v_0\, r(t_0) = 0$,
(iv) $v_0\, r'(t_0) - a(t_0)r(t_0) = 0$,
(v) $v_0\, r''(t_0) - a'(t_0)r(t_0) - 2a(t_0)r'(t_0) = 0$,
(vi) $v_0\, r'''(t_0) - a''(t_0)r(t_0) - 3a'(t_0)r'(t_0) - 3a(t_0)r''(t_0) \neq 0$.

If we assume that $p_0 = (t_0, v_0)$ is cylindrical (i.e., $r(t_0) = 0$), the condition (i) implies $v_0\, r'(t_0) \neq 0$. This contradicts the condition (iv), $v_0\, r'(t_0) = 0$. Thus, we have $r(t_0) \neq 0$. Then, we can check that the conditions (i)–(vi) are equivalent to (4.3). □

4.3. **Cuspidal $S_1^{\pm}$ singularity.** Now, we review the criterion for the cuspidal $S_1^{\pm}$ singularity given by Saji [19].

Fact 4.5 (Criterion for cuspidal $S_1^{\pm}$ singularity [19])**.** *Let $f : U \to \boldsymbol{R}^3$ be a frontal defined on a domain U of $\boldsymbol{R}^2$ with the unit normal ν. Take a non-degenerate singular point $p \in U$ of f. Let $\gamma(t)$ be a singular curve such that $\gamma(0) = p$ and ζ a null vector field. Then, the map germ f at p is $\mathcal{A}$-equivalent to the cuspidal S_1^+ singularity* (*resp. the cuspidal S_1^- singularity*) *if and only if the following* (i)-(iv) *hold:*

(i) $\delta(0) \neq 0$,

(ii) $\psi_{ccr}(0) = \psi'_{ccr}(0) = 0$ *and*

$$(d_1 :=)\, \psi''_{ccr}(0) \neq 0, \tag{4.5}$$

(iii) *there exist a regular curve $c : (-\varepsilon, \varepsilon) \to U$ and $\ell \in \boldsymbol{R}$ such that $c(0) = p$, $c'(0)$ is parallel to $\zeta(0)$, $\hat{c}''(0) \neq 0$, $\hat{c}'''(0) = \ell\, \hat{c}''(0)$ and*

$$(d_2 :=)\ \det\Big(df_p(\xi_p),\, \hat{c}''(0),\, 3\hat{c}^{(5)}(0) - 10\ell\, \hat{c}^{(4)}(0)\Big) \neq 0 \tag{4.6}$$

hold, where $\hat{c} := f \circ c$ and $\xi_p := \gamma'(0)$,

(iv) *the product $d_1 d_2$ is positive* (*resp. negative*)*, where d_1, d_2 are given by* (4.5)*,* (4.6)*, respectively. Here, we choose ζ and c so that $c'(0)$ points the same direction as the null vector $\zeta(0)$ and that $\{\gamma'(0), \zeta(0)\}$ is positively oriented.*

Applying Fact 4.5 to a-tangent developables, we have the following.

Proposition 4.6. *Let $f(t, v)$ be an a-tangent developable whose data is given by*

$$(\alpha, \rho, \omega) = (a(t)\, dt, r(t)\, dt, w(t)\, dt).$$

Then, for a singular point $p_0 = (t_0, v_0)$ of f, we have that f at p_0 is $\mathcal{A}$-equivalent to the cuspidal S_1^+ singularity if and only if

$$v_0 = 0, \quad r(t_0) \neq 0, \quad a(t_0) \neq 0, \quad w(t_0) = w'(t_0) = 0, \quad w''(t_0) \neq 0. \tag{4.7}$$

Remark 4.7. It is known that, by Ishikawa's theorem [10], developable surfaces do not admit any cuspidal $S_k^{\pm}$ singularities for $k > 1$. We also remark that, by Mond [16] and Saji [19, Theorem 4.1], tangent developable surfaces of a regular space curve do not admit cuspidal S_1^- singularity, as in the following proof.

Proof of Proposition 4.6. We first show that p_0 is non-cylindrical. If we assume $p_0 = (t_0, v_0)$ is cylindrical, we have $\gamma_c(v) = (t_0, v)$ is a singular curve passing through $\gamma_c(v_0) = p_0$. Then the function ψ_{ccr} defined as (2.4) is given by $\psi_{ccr}(v) = w(t_0)$ (cf. (3.14)). Thus,

$$\psi_{ccr}(v_0) = \psi'_{ccr}(v_0) = 0$$

and $\psi''_{ccr}(v_0) \neq 0$ do not occur at the same time. Therefore, p_0 must be non-cylindrical.

Since $r(t_0) \neq 0$ and $0 = \lambda(t_0, v_0) = -v_0\, r(t_0)$, we have $v_0 = 0$. Then, $\gamma_{nc}(t) = (t, 0)$ is a singular curve passing through $\gamma_{nc}(t_0) = p_0$, and $\zeta_{nc}(t) = \partial_t - a(t)\partial_v$ is a null vector field along $\gamma_{nc}(t)$. Then we have $\delta(t) = \det(\gamma'_{nc}(t), \zeta_{nc}(t)) = -a(t)$ (cf. (3.5)). Thus, the condition (i) of the criterion in Fact 4.5 implies $a(t_0) \neq 0$.

Now, assume that $a(t_0) < 0$, namely, $\{\gamma'_{nc}(t_0), \zeta_{nc}(t_0)\}$ is positively oriented. The function ψ_{ccr} defined as (2.4) is given by $\psi_{ccr}(t) = a(t)w(t)$ (cf. (3.13)). Thus, under the condition (i), the condition (ii) in Fact 4.5 implies $w(t_0) = w'(t_0) = 0$ and $w''(t_0) \neq 0$ hold. The constant d_1 in

(4.5) is given by $d_1 = a(t_0)w''(t_0)$. With respect to the condition (iii) in Fact 4.5, by a parallel translation of $\boldsymbol{R}^3$, we may assume that $\sigma(t_0) = 0$ without loss of generality. Then, setting

$$c(\tau) := (\tau, -\varphi(\tau)) \qquad \left(\varphi(\tau) := \frac{\sigma(\tau)\cdot\boldsymbol{\xi}(t_0)}{\boldsymbol{\xi}(\tau)\cdot\boldsymbol{\xi}(t_0)}\right), \tag{4.8}$$

we have $c(t_0) = p_0$. Differentiating $\varphi(\tau)$, we have that $c'(t_0) = \zeta_{nc}(t_0)$. Since

$$\hat{c}''(t_0) = -a(t_0)\rho(t_0)\boldsymbol{\eta}(t_0)$$

and $\hat{c}'''(t_0) = -(2a(t_0)\rho'(t_0) + a'(t_0)\rho(t_0))\boldsymbol{\eta}(t_0)$ under the conditions (i) and (ii) in Fact 4.5, we have

$$\hat{c}'''(t_0) = \ell\,\hat{c}''(t_0) \qquad \left(\ell := \frac{2a(t_0)\rho'(t_0) + a'(t_0)\rho(t_0)}{a(t_0)\rho(t_0)}\right).$$

Moreover, by a direct calculation, we can check that $\hat{c}^{(4)}(t_0)$ is a constant multiple of $\boldsymbol{\eta}(t_0)$ and

$$\hat{c}^{(5)}(t_0) = k_1\boldsymbol{\eta}(t_0) - 4a(0)\rho(0)\omega''(0)\boldsymbol{n}(t_0)$$

holds, where $k_1 \in \boldsymbol{R}$ is a constant. Thus, the constant d_2 in (4.6) is given by

$$\begin{aligned} d_2 &= \det\left(a(t_0)\boldsymbol{\xi}(t_0),\ -a(t_0)r(t_0)\boldsymbol{\eta}(t_0),\ -4a(0)r(t_0)w''(t_0)\boldsymbol{n}(t_0)\right) \\ &= 12a(t_0)^3 r(t_0)^2 w''(t_0). \end{aligned}$$

Hence, under the conditions (i) and (ii) in Fact 4.5, the condition (iii) is always satisfied.

In the case of $a(t_0) > 0$, we take the null vector field as $\zeta_{nc}(t) := -\partial_t + a(t)\partial_v$ and the curve $c(\tau)$ as $c(\tau) := (-\tau, \varphi(\tau))$, where $\varphi(\tau)$ is given by (4.8). Then, by a similar calculation as above, the constant d_1 in (4.5) is given by $d_1 = -a(t_0)w''(t_0)$, and the constant d_2 in (4.6) is $d_2 = -12a(t_0)^3 r(t_0)^2 w''(t_0)$. Therefore, regardless of the sign of $a(t_0)$, we have

$$d_1 d_2 = 12a(t_0)^4 r(t_0)^2 w''(t_0)^2 > 0.$$

Thus, Fact 4.5 implies that any a-tangent developable does not admit cuspidal S_1^- singularities, and that f at $p_0 = (t_0, v_0)$ is $\mathcal{A}$-equivalent to the cuspidal S_1^+ singularity if and only if (4.7) holds. $\square$

4.4. **5/2-cuspidal edge.** Finally, we review the criterion for the 5/2-cuspidal edge given in [9].

Fact 4.8 (Criterion for 5/2-cuspidal edge [9]). *Let $f : U \to \boldsymbol{R}^3$ be a frontal defined on a domain U of $\boldsymbol{R}^2$ with the unit normal ν. Take a non-degenerate singular point $p \in U$ of f. Let $\gamma(t)$ ($|t| < \varepsilon$) be a singular curve such that $\gamma(0) = p$ and ζ a null vector field. Then, the map germ f at p is $\mathcal{A}$-equivalent to the 5/2-cuspidal edge if and only if the following* (i)-(iii) *hold:*

- (i) $\delta(0) \neq 0$,
- (ii) $\det\left(\hat{\gamma}',\, \zeta^2 f,\, \zeta^3 f\right)\big|_{(u,v)=\gamma(t)} = 0$ *holds for each* $t \in (-\varepsilon, \varepsilon)$,
- (iii) $\det\left(\hat{\gamma}'(0),\, \bar{\zeta}^2 f(p),\, 3\bar{\zeta}^5 f(p) - 10C\,\bar{\zeta}^4 f(p)\right) \neq 0$.

Here $\bar{\zeta}$ is a special null vector field such that

$$\hat{\gamma}'(0)\cdot\bar{\zeta}^2 f(p) = \hat{\gamma}'(0)\cdot\bar{\zeta}^3 f(p) = 0, \qquad \bar{\zeta}^3 f(p) = C\,\bar{\zeta}^2 f(p), \tag{4.9}$$

where $C \in \boldsymbol{R}$ is a constant.

We remark that if $\hat{\gamma}'(0)\cdot\bar{\zeta}^2 f(p) = \hat{\gamma}'(0)\cdot\bar{\zeta}^3 f(p) = 0$ holds, then there exists a constant $C \in \boldsymbol{R}$ which satisfies $\bar{\zeta}^3 f(p) = C\,\bar{\zeta}^2 f(p)$. Applying Fact 4.8 to a-tangent developables, we have the following.

Proposition 4.9. *Let $f(t,v)$ be an a-tangent developable whose data is given by*

$$(\alpha, \rho, \omega) = (a(t)\,dt, r(t)\,dt, w(t)\,dt).$$

Then, for a singular point $p_0 = (t_0, v_0)$ of f, we have that f at p_0 is $\mathcal{A}$-equivalent to the $5/2$-cuspidal edge if and only if

$$v_0 \neq 0, \quad r'(t_0) \neq 0, \quad r(t_0) = w(t_0) = 0, \quad \left(\frac{w'}{r'}\right)'\bigg|_{t=t_0} \neq -\frac{2a(t_0)w'(t_0)}{v_0 r'(t_0)}. \tag{4.10}$$

Proof. We first show that p_0 is cylindrical. If we assume that $p_0 = (t_0, v_0)$ is non-cylindrical, we have that $r(t_0) \neq 0$ and $v_0 = 0$. As we have seen in Lemma 3.2, $\gamma_{nc}(t) = (t, 0)$ is a singular curve passing through $\gamma_{nc}(t_0) = p_0$ and $\zeta_{nc}(t) = \partial_t - a(t)\partial_v$ is a null vector field. Since the function δ defined as (2.4) is given by $\delta_{nc}(t) = -a(t)$ (cf. (3.5)), the condition (i) in Fact 4.8 is equivalent to $a(t_0) \neq 0$. On the other hand, since $\zeta_{nc}^2 f(\gamma_{nc}(t)) = -a(t)r(t)\boldsymbol{\eta}(t)$ and

$$\zeta_{nc}^3 f(\gamma_{nc}(t)) \quad = \quad 2a(t)r(t)^2\boldsymbol{\xi}(t) \quad - \quad (r(t)a'(t) + 2a(t)m'(t))\,\boldsymbol{\eta}(t) \quad + \quad 2a(t)r(t)w(t)\boldsymbol{n}(t),$$

we have that the condition (ii) in Fact 4.8 is equivalent to $w(t) = 0$ for all t. Then, setting

$$\bar{\zeta} := \left(1 - \frac{r(t_0)}{a(t_0)^2}v^2\right)\partial_t - a(t)\partial_v,$$

we have $\bar{\zeta}^2 f(p_0) = -a(t_0)r(t_0)\boldsymbol{\eta}(t_0)$ and $\bar{\zeta}^3 f(p_0) = -(a'(t_0)m(t_0) + 2a(t_0)m'(t_0))\boldsymbol{\eta}(t_0)$. Hence, $\bar{\zeta}$ is a null vector field satisfying (4.9) with the constant

$$C := (a'(t_0)m(t_0) + 2a(t_0)m'(t_0))/(a(t_0)m(t_0)).$$

Then, by a direct calculation, we have $\bar{\zeta}^4 f(p_0),\, \bar{\zeta}^5 f(p_0) \in \mathrm{Span}(\boldsymbol{\xi}(t_0), \boldsymbol{\eta}(t_0))$, which implies

$$\det\left(df(\gamma_{nc}'(0)),\, \bar{\zeta}^2 f(p_0),\, 3\bar{\zeta}^5 f(p_0) - 10C\,\bar{\zeta}^4 f(p_0)\right) = 0.$$

Hence, p_0 must be cylindrical.

As we have seen in Lemma 3.2, a non-degenerate cylindrical singular point $p_0 = (t_0, v_0)$ satisfies $r(t_0) = 0$, $r'(t_0) \neq 0$, $v_0 \neq 0$. Then, $\gamma_c(v) = (t_0, v)$ is a singular curve passing through $\gamma_c(v_0) = p_0$ and $\zeta_c = \partial_t - a(t)\partial_v$ is a null vector field. Since the function δ defined as (2.4) is given by $\delta_c(v) = -1$ (cf. (3.3)), the condition (i) in Fact 4.8 is always satisfied. On the other hand, since $\zeta_c^2 f(\gamma_c(v)) = vm'(t_0)\boldsymbol{\eta}(t_0)$ and

$$\zeta_c^3 f(\gamma_c(v)) = (-2a(t_0)m'(t_0) + vm''(t_0))\,\boldsymbol{\eta}(t_0) - 2vw(t_0)m'(t_0)\boldsymbol{n}(t_0),$$

we have that the condition (ii) in Fact 4.8 is equivalent to $w(t_0) = 0$. Then, $\zeta_c = \partial_t - a(t)\partial_v$ is a null vector field satisfying (4.9) with the constant $C := (v_0 m''(t_0) - 2a(t_0)m'(t_0))/(vm'(t_0))$. By a direct calculation, we have

$$\begin{aligned}
&\det\left(df(\gamma_c'(0)),\, \zeta_c^2 f(p_0),\, 3\zeta_c^5 f(p_0) - 10C\,\zeta_c^4 f(p_0)\right) \\
&\qquad = -12v_0 m'(t_0)\left(2a(t_0)m'(t_0)w'(t_0) - v_0 m''(t_0)w'(t_0) + v_0 m'(t_0)w''(t_0)\right).
\end{aligned}$$

Hence, by Fact 4.8, we have that f at $p_0 = (t_0, v_0)$ is $\mathcal{A}$-equivalent to the $5/2$-cuspidal edge if and only if (4.10) holds. $\square$

	Criteria
Cuspidal edge	$v_0=0,\ r(t_0)\neq 0,\ a(t_0)\neq 0,\ w(t_0)\neq 0$ or $v_0\neq 0,\ r(t_0)=0,\ r'(t_0)\neq 0,\ w(t_0)\neq 0$
Swallowtail	$v_0=0,\ r(t_0)\neq 0,\ a(t_0)=0,\ a'(t_0)\neq 0,\ w(t_0)\neq 0$
Cuspidal cross cap	$v_0=0,\ r(t_0)\neq 0,\ a(t_0)\neq 0,\ w(t_0)=0,\ w'(t_0)\neq 0$
Cuspidal beaks	$v_0=0,\ r(t_0)=0,\ r'(t_0)\neq 0,\ a(t_0)\neq 0,\ w(t_0)\neq 0$
Cuspidal butterfly	$v_0=0,\ r(t_0)\neq 0,\ a(t_0)=a'(t_0)=0,$ $a''(t_0)\neq 0,\ w(t_0)\neq 0$
Cuspidal S_1^+ singularity	$v_0=0,\ r(t_0)\neq 0,\ a(t_0)\neq 0,$ $w(t_0)=w'(t_0)=0,\ w''(t_0)\neq 0$
5/2-cuspidal edge	$v_0\neq 0,\ r'(t_0)\neq 0,\ r(t_0)=w(t_0)=0,$ $\left.\left(\dfrac{w'}{r'}\right)'\right\rvert_{t=t_0}\neq -\dfrac{2a(t_0)w'(t_0)}{v_0 r'(t_0)}$

TABLE 1. The criterion for singularities of a-tangent developables. See Corollary 3.9, Propositions 4.2, 4.4, 4.6 and 4.9.

4.5. **Duality of singularities.** Here, we give a summary of the criterion for singularities of a-tangent developables. Let $f(t,v)$ be an a-tangent developable defined on $J\times\boldsymbol{R}$ whose data is given by $(\alpha,\rho,\omega)=(a(t)\,dt, r(t)\,dt, w(t)\,dt)$. In Corollary 3.9, Propositions 4.2, 4.4, 4.6 and 4.9, we proved that the singularity type of the germ f at $p_0=(t_0,v_0)\in J\times\boldsymbol{R}$ is determined by the data as in Table 1.

Since the conjugate $f^\sharp$ of an a-tangent developable f is given by the data

$$(\alpha^\sharp,\rho^\sharp,\omega^\sharp):=(\omega,\rho,\alpha),$$

exchanging the roles α and ω we have the following.

Theorem 4.10 (Duality of singularities for a-tangent developables)**.** *Let $f: M\to\boldsymbol{R}^3$ be an a-tangent developable, $f^\sharp$ the conjugate of f, and $p_0\in M$ a singular point, where $M:=J\times\boldsymbol{R}$. Then, f at p_0 is $\mathcal{A}$-equivalent to the swallowtail* (*resp. cuspidal cross cap, cuspidal beaks, cuspidal butterfly, cuspidal S_1^+ singularity*) *if and only if $f^\sharp$ at p_0 is $\mathcal{A}$-equivalent to the cuspidal cross cap* (*resp. swallowtail, cuspidal beaks, cuspidal S_1^+ singularity, cuspidal butterfly*)*.*

In the case of the cuspidal edge, there exist examples which do not satisfy the desired duality of singularities.

Example 4.11. Let $f=f(t,v)$ be an a-tangent developable whose data is given by

$$(\alpha,\rho,\omega)=(2t\,dt, t\,dt, dt).$$

By Corollary 3.9, f at $(0,v)$ is cuspidal edge for $v\neq 0$ (see Figure 3). The conjugate $f^\sharp=f^\sharp(t,v)$ of f is given by the data $(\alpha^\sharp,\rho^\sharp,\omega^\sharp)=(dt, t\,dt, 2t\,dt)$. By Proposition 4.9, $f^\sharp$ at $(0,v)$ is 5/2-cuspidal edge for $v\neq 0$ (see Figure 4).

5. CONJUGATE OF COMPLETE FLAT FRONTS

Finally, we observe a global behavior of the conjugate operations among a-tangent developables.

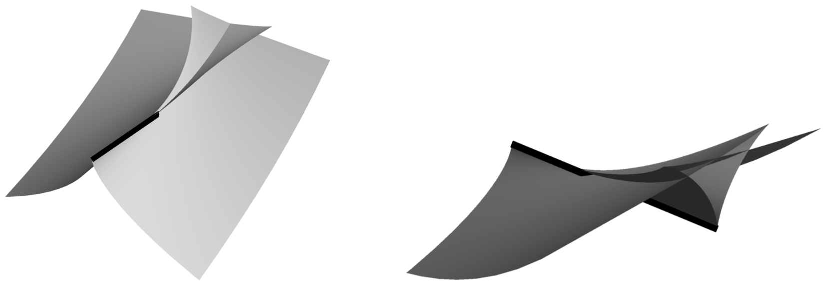


FIGURE 3. The image of the a-tangent developable $f = f(t, v)$ whose data is given by $(\alpha, \rho, \omega) = (2t\,dt, t\,dt, dt)$. By Corollary 3.9, we have that f at $(0, v)$ is $\mathcal{A}$-equivalent to the cuspidal edge for $v \neq 0$. This figure is plotted by integrating (2.6) and (3.17) numerically. The black line is the image of the cylindrical singular set $S_c(f) = \{(0, v)\,;\, v \neq 0\}$.

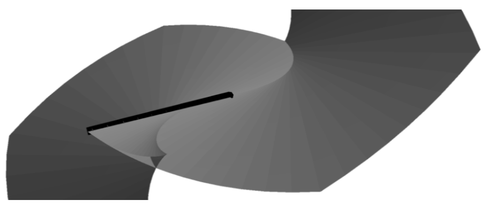


FIGURE 4. The image of the conjugate $f^\sharp = f^\sharp(t, v)$ of the a-tangent developable with the data $(\alpha, \rho, \omega) = (2t\,dt, t\,dt, dt)$. Since the data of $f^\sharp$ is given by $(\alpha^\sharp, \rho^\sharp, \omega^\sharp) = (dt, t\,dt, 2t\,dt)$, Proposition 4.9 yields that f at $(0, v)$ is $\mathcal{A}$-equivalent to the 5/2-cuspidal edge for $v \neq 0$. This figure is plotted by integrating (3.18) and (3.19) numerically. The black line is the image of the cylindrical singular set $S_c(f^\sharp) = \{(0, v)\,;\, v \neq 0\}$.

Proposition 5.1. *Let $f : M \to \boldsymbol{R}^3$ be an a-tangent developable such that f is a complete flat front with embedded ends, where $M := S^1 \times \boldsymbol{R}$. Then, the conjugate $f^\sharp$ of f is not a front. In particular, the conjugate of a complete flat front with embedded ends cannot be a complete flat front.*

Proof. Let (α, ρ, ω) be the data of f. By Fact 2.1, f has at least four singular points other than cuspidal edges. In fact, if we denote by $\alpha = a(t)\,dt$, it is proved in [17, pp. 311–312] that $a(t)$ changes signs at least four times on S^1. Since the data of $f^\sharp$ is given by $(\alpha^\sharp, \rho^\sharp, \omega^\sharp) := (\omega, \rho, \alpha)$, and $f^\sharp$ is front if and only if $\omega^\sharp$ never vanishes, we have that $f^\sharp$ cannot be a front. □

Acknowledgements. The author would like to thank the referee, Professors Goo Ishikawa, Kentaro Saji, Masaaki Umehara and Kotaro Yamada for their valuable comments.

REFERENCES

[1] S. Akamine, *Behavior of the Gaussian curvature of timelike minimal surfaces with singularities*, Hokkaido Math. J. **48** (2019), 537–568. DOI: 10.14492/hokmj/1573722017

[2] E. Calabi, *Examples of the Bernstein problem for some nonlinear equations*, Proc. Symp. Pure Math. **15** (1970), 223–230.

[3] J.P. Cleave, *The form of the tangent developable at points of zero torsion on space curves*, Math. Proc. Cambridge Philos. Soc. **88** (1980), 403–407. DOI: 10.1017/s0305004100057753

[4] S. Fujimori, K. Saji, M. Umehara, K. Yamada, *Singularities of maximal surfaces*, Math. Z. **259** (2008), 827–848. DOI: 10.1007/s00209-007-0250-0

[5] A. Honda, *Weakly complete wave fronts with one principal curvature constant*, Kyushu J. Math. **70** (2016), 217–226. DOI: 10.2206/kyushujm.70.217

[6] A. Honda, *Isometric immersions with singularities between space forms of the same positive curvature*, J. Geom. Anal. **27** (2017), 2400–2417. DOI: 10.1007/s12220-017-9765-8

[7] A. Honda, *Duality of singularities for spacelike CMC surfaces*, Kobe J. Math. **34** (2017), 1–11.

[8] A. Honda, *Complete flat fronts as hypersurfaces in Euclidean space*, Proc. Japan Acad. Ser. A Math. Sci. **94** (2018), 25–30. DOI: 10.3792/pjaa.94.25

[9] A. Honda, M. Koiso and K. Saji, *Fold singularities on spacelike CMC surfaces in Lorentz-Minkowski space*, Hokkaido Math. J. **47** (2018), 245–267. DOI: 0.14492/hokmj/1529308818

[10] G. Ishikawa, *Determinacy of the envelope of the osculating hyperplanes to a curve*, Bull. London Math. Soc. **25** (1993), 603–610.

[11] G. Ishikawa, *Developable of a curve and determinacy relative to osculation-type*, Quart. J. Math. Oxford **46** (1995), 437–451.

[12] S. Izumiya, T. Nagai and K. Saji, *Great circular surfaces in the three-sphere*, Diff. Geom. Appl. **29** (2011), 409–425.

[13] S. Izumiya and K. Saji, *The mandala of Legendrian dualities for pseudo-spheres in Lorentz-Minkowski space and "flat" spacelike surfaces*, J. Singul. **2** (2010), 92–127. DOI: 10.5427/jsing.2010.2g

[14] S. Izumiya, K. Saji and M. Takahashi, *Horospherical flat surfaces in Hyperbolic 3-space*, J. Math. Soc. Japan **62** (2010), 789–849. DOI: 10.2969/jmsj/06230789

[15] M. Kokubu, W. Rossman, K. Saji, M. Umehara and K. Yamada, *Singularities of flat fronts in hyperbolic 3-space*, Pacific J. Math. **221** (2005), 303–351. DOI: 10.2140/pjm.2005.221.303

[16] D. Mond, *On the tangent developable of a space curve*, Math. Proc. Cambridge Philos. Soc. **91** (1982), 351–355.

[17] S. Murata and M. Umehara, *Flat surfaces with singularities in Euclidean 3-space*, J. Differential Geom. **82** (2009), 279–316. DOI: 10.4310/jdg/1246888486

[18] Y. Ogata and K. Teramoto, *Duality between cuspidal butterflies and cuspidal S_1^- singularities on maximal surfaces*, Note Mat. **38** (2018), 115–130.

[19] K. Saji, *Criteria for cuspidal S_k singularities and their applications*, Journal of Gökova Geometry Topology **4** (2010), 67–81.

[20] O.P. Scherbak, *Projectively dual space curves and Legendre singularities*, Trudy Tbiliss. Univ. **232**–**233** (1982), 280–336.

[21] H. Takahashi, *Timelike minimal surfaces with singularities in three-dimensional spacetime* (Japanese), Master thesis, Osaka University, 2012.

[22] M. Takahashi, *Legendre curves in the unit spherical bundle over the unit sphere and evolutes*, Real and complex singularities, 337–355, Contemp. Math., 675, Amer. Math. Soc., Providence, RI, 2016. DOI: 10.1090/conm/675/13600

[23] M. Umehara and K. Yamada, *Maximal surfaces with singularities in Minkowski space*, Hokkaido Math. J. **35** (2006), 13–40. DOI: 10.14492/hokmj/1285766302

[24] M. Yasumoto, *Weierstrass-type representations for timelike surfaces*, Singularities in generic geometry, 449–469, Adv. Stud. Pure Math., 78, Math. Soc. Japan, Tokyo, 2018. DOI: 10.2969/aspm/07810449

Atsufumi Honda, Department of Applied Mathematics, Faculty of Engineering, Yokohama National University, 79-5 Tokiwadai, Hodogaya, Yokohama 240-8501, Japan

Email address: `honda-atsufumi-kp@ynu.ac.jp`

Journal *of* Singularities
Volume 21 (2020), 149-166

Proc. of Geometric and Algebraic Singularity Theory, Będlewo, 2017
DOI: 10.5427/jsing.2020.21i

RECOGNITION PROBLEM OF FRONTAL SINGULARITIES

GOO ISHIKAWA

ABSTRACT. A natural class of mappings, frontal mappings, is explained from both geometric and algebraic aspects. Several results on the recognition of frontal singularities, in particular, cuspidal edges, folded umbrellas, swallowtails, Mond singularities, Shcherbak singularities, and their openings are surveyed.

1. INTRODUCTION

This is a survey article on recognition problem of frontal singularities.

First we explain the recognition problem of singularities and its significance.

Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ and $f' : (\mathbb{R}^n, a') \to (\mathbb{R}^m, b')$ be smooth (= C^∞) map-germs. Then f and f' are called $\mathscr{A}$*-equivalent* or *diffeomorphic* if there exist diffeomorphism-germs $\sigma : (\mathbb{R}^n, a) \to (\mathbb{R}^n, a')$ and $\tau : (\mathbb{R}^m, b) \to (\mathbb{R}^m, b')$ such that the diagram

$$\begin{array}{ccc} (\mathbb{R}^n, a) & \xrightarrow{f} & (\mathbb{R}^m, b) \\ \downarrow \sigma & & \downarrow \tau \\ (\mathbb{R}^n, a') & \xrightarrow{f'} & (\mathbb{R}^m, b') \end{array}$$

commutes. By a *singularity* of smooth mappings, we mean an $\mathscr{A}$*-equivalence class* of map-germs.

Suppose that we investigate "singularities" of mappings belonging to some given class. Then the recognition problem of singularities may be understood as the following dual manners:

Problem: Given two map-germs f and f', *belonging to the given class,* determine, *as easily as possible* whether f and f' are equivalent or not.

Problem: Given a singularity, find criteria to determine as easy as possible whether a map-germ f *belonging to some class* has (= falls into) the given singularity or not.

Importance of the recognition problem of singularities can be explained as follows.

Once we establish a classification list of singularities in a *situation A*, we will face (at least) *two kinds of needs*:

1. Given a map-germ in the same *situation A*, we want to know which singularity is it in the list.

2. For another *situation B*, we want to know how similar is the classification list of singularities as *A* or not.

In both cases, we need to recognize the singularities, *as easily as possible*, by *as many as possible* criteria. For applications of singularity theory, it is indispensable to recognize singularities and to solve classification problems in various situations.

The recognition problem of singularities of smooth map-germs has been treated by the many mathematicians, motivated by differential geometry and other wide area, and its solutions are supposed to have many applications.

In fact most of known results of recognition of singularities are found under the motivation of geometric studies of singularities appearing in Euclid geometry and various Klein geometries ([21, 3, 19]).

2010 *Mathematics Subject Classification.* Primary 57R45; Secondary 58K50, 53A07, 53D12, 53C50.

Key words and phrases. Jacobi ideal, kernel field, Jacobi module, opening, ramification module, Lorentzian manifold.

The author was supported by JSPS KAKENHI No.15H03615 and No.15K13431.

Example 1.1. (Singularities in non-Euclidean geometry) The following is a diagram representing the history of non-Euclidean geometry found in the reference [26]:

$$\begin{array}{ccc} \text{Euclid geometry} & \to & \text{Riemann geometry} \\ \downarrow & & \downarrow \\ \text{Klein geometry} & \to & \text{Cartan geometry} \end{array}$$

Then it would be natural to ask
Problem: How are the classification results of singularities in Euclid geometry (resp. in Klein geometry) valid in Riemann geometry (resp. in Cartan geometry)?
In other words,
Problem: Do the classifications of singularities in flat ambient spaces work also for "curved" ambient spaces?

In fact, we applied the several results of recognition ([21, 3]), for instance, to the generic classification of singularities of improper affine spheres and of surfaces of constant Gaussian curvature ([13]), and moreover, to the classification of generic singularities appearing in tangent surfaces which are ruled by geodesics in general Riemannian spaces ([17, 18]). See also §6.

In this paper we will pay our attention to the class of mappings, *frontal mappings*, which is introduced and studied in §2. Then we survey several recognition theorems on them in §3. Note that the recognitions of fronts or frontals $(\mathbb{R}^n, a) \to \mathbb{R}^m$ are studied by many authors ([21, 3, 24, 25, 20]).

To show the theorems given in §3, we introduce the notion of *openings*, relating it with that of frontals, in §4. See also [9, 10]. In fact, in §4, we observe that *any frontal singularity is an opening of a map-germ from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$ (Lemma 4.3).

Then we naturally propose:
Problem: Study the *recognition problem* of frontals from the *recognition results* on map-germs $(\mathbb{R}^n, a) \to \mathbb{R}^n$, $(n = m)$, combined with the viewpoint of *openings*.

In this paper, in connection with the above problems, we specify *geometrically* several frontal singularities which we are going to treat (Example 2.2). Then we solve the recognition problem of such singularities, in §3, giving explicit normal forms. In fact we combine the recognition results on $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ by K. Saji (∼2010) and several arguments on openings, which was implicitly performed for the classification of singularities of tangent surfaces (tangent developables) by the author (∼1995) over twenty years, the idea of which traces back to the author's master thesis [5]. We prove recognition theorems in §5.

In the last section §6, as an application of our solutions of recognition problem of frontal singularities, we announce the classification of singularities appearing in tangent surfaces of generic null curves which are ruled by null geodesics in general Lorentz 3-manifolds ([14, 16]), mentioning related recognition results and open problems.

In this paper, all manifolds and mappings are assumed to be of class C^∞ unless otherwise stated.

The author truly thanks to the organisers for giving him the chance to write this paper down and he deeply thanks to anonymous referees for their helpful comments to improve the paper.

2. Frontal singularities

Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ be a map-germ. Suppose $n \leq m$.

Then f is called a *frontal map-germ* or a *frontal* in short, if there exists a smooth (C^∞) family of n-planes $\widetilde{f}(t) \subseteq T_{f(t)}\mathbb{R}^m$ along f, $t \in (\mathbb{R}^n, a)$, i.e. there exists a smooth lift $\widetilde{f} : (\mathbb{R}^n, a) \to \mathrm{Gr}(n, T\mathbb{R}^m)$ satisfying the "*integrality condition*"

$$T_t f(T_t\mathbb{R}^n) \subset \widetilde{f}(t) \ (\subset T_{f(t)}\mathbb{R}^m),$$

for any $t \in \mathbb{R}^n$ nearby a, such that $\pi \circ \widetilde{f} = f$:

$$\begin{tikzcd} & \mathrm{Gr}(n, T\mathbb{R}^m) \arrow[d, "\pi"] \\ (\mathbb{R}^n, a) \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & (\mathbb{R}^m, b). \end{tikzcd}$$

Here $\mathrm{Gr}(n, T\mathbb{R}^m)$ is the *Grassmann bundle* consisting of n-planes $V \subset T_x\mathbb{R}^m (x \in \mathbb{R}^m)$ with the canonical projection $\pi(x,V) = x$, and $T_t f : T_t\mathbb{R}^n \to T_{f(t)}\mathbb{R}^m$ is the differential of f at $t \in (\mathbb{R}^n, a)$.

Then $\widetilde{f}$ is called a *Legendre lift* or an *integral lift* of the frontal f. Actually $\widetilde{f}$ is an integral mapping to the canonical or contact distribution on $\mathrm{Gr}(n, T\mathbb{R}^m)$ (cf. [8]).

Example 2.1. (1) Any *immersion* is a frontal. In fact then the Legendre lift is given by $\widetilde{f}(t) := T_t f(T_t\mathbb{R}^n)$.

(2) Any map-germ $(\mathbb{R}^n, a) \to (\mathbb{R}^n, b), (n = m)$ is a frontal. In fact the Legendre lift is given by $\widetilde{f}(t) := T_{f(t)}\mathbb{R}^n$.

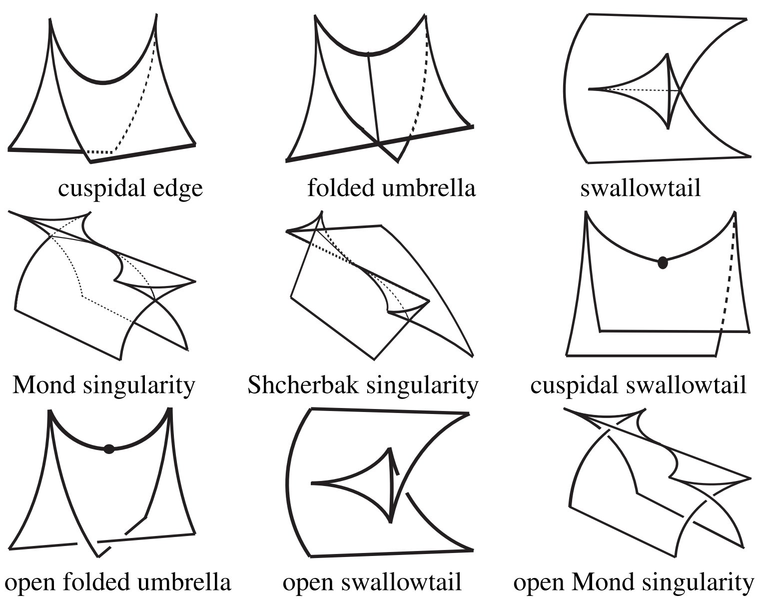
(3) Any *constant map-germ* is a frontal. In fact we can take any lift $\widetilde{f}$ of f.

(4) Any *wave-front* $(\mathbb{R}^n, a) \to (\mathbb{R}^{n+1}, b)$, that is a Legendre projection of a Legendre submanifold in $\mathrm{Gr}(n, T\mathbb{R}^{n+1}) = PT^*\mathbb{R}^{n+1}$, is a frontal. Take the inclusion of the Legendre submanifold as the Legendre lift.

Example 2.2. (Singularities of tangent surfaces) Let $\gamma : (\mathbb{R}, 0) \to \mathbb{R}^m$ be a curve-germ in Euclidean space. Then the tangent surface $\mathrm{Tan}(\gamma) : (\mathbb{R}^2, 0) \to \mathbb{R}^m$ is defined as the ruled surface generated by tangent lines along the curve. Suppose γ is of type $\mathbf{L} = (\ell_1, \ell_2, \ell_3, \ldots,), (1 \leq \ell_1 < \ell_2 < \ell_3 < \cdots)$, i.e.

$$\gamma(t) = (t^{\ell_1} + \cdots,\ t^{\ell_2} + \cdots,\ t^{\ell_3} + \cdots,\ \ldots)$$

for a system of affine coordinates of $\mathbb{R}^m$ centered at $\gamma(0)$. Then it is known that the singularity of $\mathrm{Tan}(\gamma)$ is *uniquely determined* by the type $\mathbf{L}$ and called cuspidal edge (*CE*) if $\mathbf{L} = (1, 2, 3, \ldots)$, folded umbrella (*FU*) or cuspidal cross cap (*CCC*) if $(1, 2, 4)$, swallowtail (*SW*) if $(2, 3, 4)$, Mond (*MD*) or cuspidal beaks (*CB*) if $(1, 3, 4)$, Shcherbak (*SB*) if $(1, 3, 5)$, cuspidal swallowtail (*CS*) if $(3, 4, 5)$, open folded umbrella (*OFU*) if $(1, 2, 4, 5, \ldots)$, open swallowtail (*OSW*) if $(2, 3, 4, 5, \ldots)$, open Mond (*OMD*) or open cuspidal beaks (*OCB*) if $(1, 3, 4, 5, \ldots)$ (see [8]).



In general, a frontal $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ is called a *front* if f has an immersive Legendre lift $\widetilde{f}$.

Let $\mathscr{E}_a := \{h : (\mathbb{R}^n, a) \to \mathbb{R}\}$ denote the $\mathbb{R}$-algebra of smooth function-germs on $(\mathbb{R}^n, a)$.
Denote by Γ the set of subsets $I \subseteq \{1, 2, \dots, m\}$ with $\#(I) = n$. For a map-germ

$$f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b), n \leq m$$

and $I \in \Gamma$, we set $D_I = \det(\partial f_i/\partial t_j)_{i \in I, 1 \leq j \leq n}$. Then *Jacobi ideal* J_f of f is defined as the ideal generated in $\mathscr{E}_a$ by all n-minor determinants D_I $(I \in \Gamma)$ of Jacobi matrix $J(f)$ of f. Then we have:

Lemma 2.3. (Criterion of frontality) *Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ be a map-germ. If f is a frontal, then the Jacobi ideal J_f of f is principal, i.e. it is generated by one element. In fact J_f is generated by D_I for some $I \in \Gamma$. Conversely, if J_f is principal and the singular locus*

$$S(f) = \{t \in (\mathbb{R}^n, a) \mid \operatorname{rank}(T_t f : T_t\mathbb{R}^n \to T_{f(t)}\mathbb{R}^m) < n\}$$

of f is nowhere dense in $(\mathbb{R}^n, a)$, then f is a frontal.

Proof: Let f be a frontal and $\widetilde{f}$ be a Legendre lift of f. Take $I_0 \in \Gamma$ such that $\widetilde{f}(a)$ projects isomorphically by the projection $\mathbb{R}^m \to \mathbb{R}^n$ to the components belonging to I_0. Let $(p_I)_{I \in \Gamma}$ be the Plücker coordinates of $\widetilde{f}$. Then $p_{I_0}(a) \neq 0$. This implies that for any $I \in \Gamma$, there exists $h_I \in \mathscr{E}_a$ such that $D_I = h_I D_{I_0}$. Set $\lambda = D_{I_0}$. Then the Jacobi ideal J_f is generated by λ.

Conversely suppose J_f is generated by one element $\lambda \in \mathscr{E}_a$. Since J_f is generated by λ, we have that there exists $k_I \in \mathscr{E}_a$ for any $I \in \Gamma$ such that $D_I = k_I\lambda$. Since $\lambda \in J_f$, there exists $\ell_I \in \mathscr{E}_a$ for any $I \in \Gamma$ such that $\lambda = \sum_{I \in \Gamma} \ell_I D_I$. Therefore $(1 - \sum_{I \in \Gamma} \ell_I k_I)\lambda = 0$. Suppose $(\ell_I k_I)(a) = 0$ for any $I \in \Gamma$. Then $1 - \sum_{I \in \Gamma} \ell_I k_I$ is a unit and therefore $\lambda = 0$. Thus we have $J_f = 0$. This contradicts to the assumption that $S(f)$ is nowhere dense. Hence there exists $I_0 \in \Gamma$ such that $(\ell_{I_0} k_{I_0})(a) \neq 0$. Then $k_{I_0}(a) \neq 0$. Therefore J_f is generated by D_{I_0}. Hence $D_I = h_I D_{I_0}$ for any $I \in \Gamma$ with $h_{I_0}(a) = 1$. Then the Legendre lift $\widetilde{f}$ on $\mathbb{R}^n \setminus S(f)$ extends to $(\mathbb{R}^n, a)$, which is given by the Plücker coordinates $(h_I)_{I \in \Gamma}$. □

Example 2.4. Define $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ by $f(t_1, t_2) := (\varphi(t_1),\ \varphi(t_1)t_2,\ \varphi(-t_1))$, where the C^∞ function $\varphi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ is given by $\varphi(t) = \exp(-1/t^2)(t \geq 0), 0(t \leq 0)$. Then the Jacobi ideal J_f is generated by $\varphi'(t_1)\varphi(t_1)$ and therefore J_f is principal and $J_f \neq 0$. However f is not a frontal. In fact, for $t_1 > 0$, $(T_{(t_1,t_2)}f)(T_{(t_1,t_2)}\mathbb{R}^2)$ is given by the plane $dx_3 = 0$ and for $t_1 < 0$, $(T_{(t_1,t_2)}f)(T_{(t_1,t_2)}\mathbb{R}^2)$ contains the x_3-axis. Therefore f can not be a frontal.

Corollary 2.5. *Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ be a map-germ. Suppose f is analytic and $J_f \neq 0$. Then f is a frontal if and only if J_f is a principal ideal.*

Proof: By Lemma 2.3, if f is frontal, then J_f is principal. If J_f is principal and $J_f \neq 0$, then $D_I \neq 0$ for some $I \in \Gamma$. Since f is analytic, $S(f)$ is nowhere dense. Thus by Lemma 2.3, f is a frontal. □

Example 2.6. Define $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^4, 0)$ by $f(t_1, t_2, t_3) := (t_1^3,\ t_1^2 t_2,\ t_1 t_2^2,\ t_2^3)$. The germ f parametrizes the cone over a non-degenerate cubic in $P(\mathbb{R}^4) = \mathbb{R}P^3$. Then f is analytic and $J_f = 0$ is principal. However f is not a frontal.

Definition 2.7. Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ be a frontal. Then a generator $\lambda \in \mathscr{E}_a$ of J_f is called a *Jacobian* (or a *singularity identifier*) of f, which is uniquely determined from f up to multiplication of a unit in $\mathscr{E}_a$.

The singular locus $S(f)$ of a frontal f is given by the zero-locus of the Jacobian λ of f.

Definition 2.8. (Proper frontals) A frontal $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ is called *proper* if the singular locus $S(f)$ is nowhere dense in $(\mathbb{R}^n, a)$.

Remark 2.9. Our naming "proper" is a little confusing since its usage is different from the ordinary meaning of properness (inverse images of any compact is compact). Our condition that the singular locus S_f is nowhere dense is easy to handle for the local study of mappings.

Lemma 2.10. *Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ be a proper frontal or $n = m$. Then f has a unique Legendre lift $\widetilde{f} : (\mathbb{R}^n, a) \to \mathrm{Gr}(n, T\mathbb{R}^m)$.*

Proof: On the regular locus $\mathbb{R}^n \setminus S(f)$, there is the unique Legendre lift $\widetilde{f}$ defined by $\widetilde{f}(t) := (T_t f)(T_t\mathbb{R}^n)$. Let f be a proper frontal. Then $\mathbb{R}^n \setminus S(f)$ is dense in $(\mathbb{R}^n, a)$. Therefore the extension of $\widetilde{f}(t)$ is unique. Let $n = m$. Then the unique lift $\widetilde{f}$ is defined by $\widetilde{f}(t) = T_{f(t)}\mathbb{R}^m$ (Example 2.1 (2)). □

Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ be a frontal (resp. a proper frontal) and $\widetilde{f} : (\mathbb{R}^n, a) \to \mathrm{Gr}(n, T\mathbb{R}^m)$ a Legendre lift of f. Recall that $\widetilde{f}(t), (t \in (\mathbb{R}^n, a))$ is an n-plane field along f. In particular $\widetilde{f}(a) \subseteq T_b\mathbb{R}^m$.

Definition 2.11. A system $(x_1, \dots, x_n, x_{n+1}, \dots, x_m)$ of local coordinates of $\mathbb{R}^m$ centered at b is called *adapted* to $\widetilde{f}$ (or, to f) if

$$\widetilde{f}(a) = \left\langle \left(\frac{\partial}{\partial x_1}\right)_b, \dots, \left(\frac{\partial}{\partial x_n}\right)_b \right\rangle_{\mathbb{R}}$$

$$(= \{v \in T_b\mathbb{R}^m \mid dx_{n+1}(v) = 0, \dots, dx_m(v) = 0\}).$$

Clearly we have

Lemma 2.12. *Any frontal $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ has an adapted system of local coordinates on $(\mathbb{R}^m, b)$. In fact any system of local coordinates on $(\mathbb{R}^m, b)$ is modified into an adapted system of local coordinates by a linear change of coordinates.*

Remark 2.13. For an adapted system of coordinates $(x_1, \dots, x_n, x_{n+1}, \dots, x_m)$ of f, the Jacobian λ is given by the ordinary Jacobian $\frac{\partial(f_1, \dots, f_n)}{\partial(t_1, \dots, t_n)}$, where $f_i = x_i \circ f$.

Example 2.14. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be given by

$$(u, t) \mapsto (x_1, x_2, x_3) = (t + u,\ t^3 + 3t^2u,\ t^4 + 4t^3u),$$

which is the tangent surface, Mond surface, of the curve $t \mapsto (t, t^3, t^4)$.

Then the Jacobi matrix $J(f)$ of f is given by

$$J(f) = \begin{pmatrix} 1 & 1 \\ 3t^2 & 3t^2 + 6tu \\ 4t^3 & 4t^3 + 12t^2u \end{pmatrix},$$

and its minors are calculated as

$$\begin{cases} D_{12} = 6tu, \\ D_{13} = 12t^2u = 2t(6tu), \\ D_{23} = 12t^4u = 2t^3(6tu), \end{cases}$$

Then the Jacobi ideal J_f is generated by $\lambda = tu$. Therefore f is a proper frontal with

$$S(f) = \{(u, t) \mid tu = 0\}.$$

The unique Legendre lift $\widetilde{f} : (\mathbb{R}^2, 0) \to \mathrm{Gr}(2, T\mathbb{R}^3)$ of f is given, via the Plücker coordinates of fibre components,

$$D_{12}/D_{12} = 1,\ D_{13}/D_{12} = 2t,\ D_{23}/D_{12} = 2t^3.$$

The system of coordinates (x_1, x_2, x_3) is adapted for f in the example.

3. Recognition of several frontal singularities

To give our recognition results we need the notion of "kernel fields" in addition to that of Jacobians of frontals.

Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ be a map-germ. We denote by $\mathscr{V}_a$ the $\mathscr{E}_a$-module of vector fields over $(\mathbb{R}^n, a)$ and set

$$\mathscr{N}_f := \{\eta \in \mathscr{V}_a \mid \eta f_i \in J_f,\ (1 \leq i \leq m)\},$$

which is an $\mathscr{E}_a$-submodule of $\mathscr{V}_a$.

Note that, if $\eta \in \mathscr{N}_f$, then $\eta(t) \in \mathrm{Ker}(T_t f : T_t\mathbb{R}^n \to T_{f(t)}\mathbb{R}^m)$ for any $t \in S(f)$. Moreover note that, if $\lambda \in J_f$, then $\lambda \cdot \mathscr{V}_a \subseteq \mathscr{N}_f$.

A map-germ $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ is called *of corank k* if $\dim_{\mathbb{R}} \mathrm{Ker}(T_a f : T_a\mathbb{R}^n \to T_b\mathbb{R}^m) = k$.

Then we have

Lemma 3.1. *Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ be a map-germ of corank 1. Then $\mathscr{N}_f / J_f \cdot \mathscr{V}_a$ is a free $\mathscr{E}_a$-module of rank 1, i.e. $\mathscr{N}_f / J_f \cdot \mathscr{V}_a$ is isomorphic to $\mathscr{E}_a$ as $\mathscr{E}_a$-modules by $[\eta] \to 1$, for some $\eta \in \mathscr{N}_f$.*

Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ be a frontal of corank 1 and λ_f the Jacobian of f (Definition 2.7). Then by Lemma 3.1, $\mathscr{N}_f / \lambda_f \cdot \mathscr{V}_a$ is a free module of rank 1.

Definition 3.2. A vector field η over $(\mathbb{R}^n, a)$ is called a *kernel field* (or a *null field*) of f if η generates the free $\mathscr{E}_a$-module $\mathscr{N}_f / \lambda_f \cdot \mathscr{V}_a$.

Remark 3.3. The notion of null fields is introduced first in [21].

Proof of Lemma 3.1: Since f is of corank 1, f is $\mathscr{A}$-equivalent to a map-germ $(\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$ of form

$$g = (t_1, \ldots, t_{n-1}, \varphi_n(t), \ldots, \varphi_m(t)).$$

Note that $\mathscr{N}_f / J_f \mathscr{V}_a$ is isomorphic to $\mathscr{N}_g / J_f \mathscr{V}_0$. Moreover the Jacob ideal of g is generated by

$$\partial\varphi_n(t)/\partial t_n, \ldots, \partial\varphi_m(t)/\partial t_n.$$

Let $\eta = \sum_{i=1}^n \eta_i \partial/\partial t_i \in \mathscr{V}_0$. Then $\eta \in \mathscr{N}_g$ if and only if $\eta_1, \ldots, \eta_{n-1} \in J_g$. Therefore $\mathscr{N}_g / J_g \mathscr{V}_0$ is freely generated by $\partial/\partial t_n$. Thus we have that $\mathscr{N}_f / J_f \cdot \mathscr{V}_a$ is a free $\mathscr{E}_a$-module of rank 1, $\square$

Now we start to give our recognition theorems on the frontal singularities introduced in Example 2.2. To begin with, we recall the following fundamental recognition result due to Saji ([24]), which is a reformulation of Whitney's original results in [27] for parts (1) and (2).

Theorem 3.4. (Saji[24]) *Let $f : (\mathbb{R}^2, a) \to (\mathbb{R}^2, b)$ be a frontal map-germ of corank 1. Then, for the Jacobian λ and the kernel field η of f, we have*

(1) f is $\mathscr{A}$-equivalent to the fold, i.e. to $(t_1, t_2) \mapsto (t_1, t_2^2)$, if and only if $(\eta\lambda)(a) \neq 0$.

(2) f is $\mathscr{A}$-equivalent to Whitney's cusp, i.e. to $(t_1, t_2) \mapsto (t_1, t_2^3 + t_1 t_2)$, if and only if

$$(d\lambda)(a) \neq 0, (\eta\lambda)(a) = 0, (\eta\eta\lambda)(a) \neq 0.$$

(3) f is $\mathscr{A}$-equivalent to bec à bec (beak-to-beak), $(t_1, t_2) \mapsto (t_1, t_2^3 + t_1 t_2^2)$, if and only if λ has an indefinite Morse critical point at a and $(\eta\eta\lambda)(a) \neq 0$.

Remark 3.5. Each condition (1), (2), (3) of Theorem 3.4 is independent of the choice of λ and η, and depends only on $\mathscr{J}$-equivalence class of f which is introduced in Definition 4.13. In fact, if $\mathscr{J}_{f' \circ \sigma} = \mathscr{J}_f$, then f' satisfies the condition for $\lambda' = \lambda \circ \sigma^{-1}$ and $\eta' = (T\sigma)\eta \circ \sigma^{-1}$. (See §4).

Remark 3.6. For a map-germ $f : (\mathbb{R}^2, a) \to (\mathbb{R}^2, b)$ of corank 1, the condition $(d\lambda)(a) \neq 0$ is equivalent to that the Jacobian is $\mathscr{K}$-equivalent to the germ $(t_1,t_2) \mapsto t_1$ at the origin. The condition that λ has an indefinite Morse critical point at a is equivalent to that λ is $\mathscr{K}$-equivalent to the germ $(t_1,t_2) \mapsto t_1t_2$ at the origin.

Remark 3.7. For plane to plane map-germs, the fold (resp. Whitney cusp, bec à bec) is characterized as a"tangent map" of a planar curve of type $(1,2)$ (resp. $(2,3)$, $(1,3)$), which is ruled by tangent lines to the curve ([8, 15]).

Let $f : (\mathbb{R}^2, a) \to (\mathbb{R}^m, b), (m \geq 3)$ be a proper frontal of corank 1. We wish to recognize the singularity, i.e. $\mathscr{A}$-equivalence class of f by the Jacobian $\lambda = \lambda_f$ and the kernel field $\eta = \eta_f$. Moreover we wish to recognize the singularity of f as an opening of a plane-to-plane map-germ. To realize this, we will use an adapted system of coordinates $(x_1, x_2, x_3, \ldots, x_m)$ for f and set $f_i = x_i \circ f$. Note that we mention several conditions to recognize singularities in terms of adapted coordinates, however the conditions are, of course, independent of the choice of an adapted coordinates, and therefore any system of adapted coordinates can be taken to simplify the checking of a suitable condition.

In general, we use the following notation:

Definition 3.8. For a germ of vector field $\eta \in \mathscr{V}_a$ over $(\mathbb{R}^n, a)$ and a function-germ $h \in \mathscr{E}_a$ on $(\mathbb{R}^n, a)$, the *vanishing order* $\mathrm{ord}_a^\eta(h)$ of the function h at the point a for the vector-field η is defined by

$$\mathrm{ord}_a^\eta(h) := \inf\{i \in \mathbb{N} \cup \{0\} \mid (\eta^i h)(a) \neq 0\}.$$

Then we characterize the cuspidal edge as an opening of fold map-germ:

Theorem 3.9. (Recognition of cuspidal edge) *For a frontal $f : (\mathbb{R}^2, a) \to (\mathbb{R}^3, b)$ of corank 1, the following conditions are equivalent to each other:*

(1) *f is $\mathscr{A}$-equivalent to the cuspidal edge (CE).*
(1') *f is $\mathscr{A}$-equivalent to the germ $(t_1,t_2) \mapsto (t_1, t_2^2, t_2^3)$.*
(2) *f is a front and $\eta\lambda(a) \neq 0$.*
(3) *$\eta\lambda(a) \neq 0$ and $\mathrm{ord}_a^\eta(f_3) = 3$, for an adapted system of coordinates (x_1, x_2, x_3) of $(\mathbb{R}^3, b)$.*

Theorem 3.9 is generalized by

Theorem 3.10. (Recognition of embedded cuspidal edge) *For a frontal $f : (\mathbb{R}^2, a) \to (\mathbb{R}^m, b), 3 \leq m$ of corank 1, the following conditions are equivalent to each other:*

(1) *f is $\mathscr{A}$-equivalent to the cuspidal edge, i.e. the tangent surface to a curve of type $(1,2,3,\ldots)$.*
(1') *f is $\mathscr{A}$-equivalent to the germ $(t_1,t_2) \mapsto (t_1, t_2^2, t_2^3, 0, \ldots, 0)$.*
(2) *f is a front and $\eta\lambda(a) \neq 0$.*
(3) *$\eta\lambda(a) \neq 0$ and $\mathrm{ord}_a^\eta(f_i) = 3$ for some $i, 3 \leq i \leq m$, for an adapted system of coordinates $(x_1, x_2, x_3, \ldots, x_m)$ of $(\mathbb{R}^m, b)$.*

The following is a recognition of the folded umbrella due to the theory of openings:

Theorem 3.11. (Recognition of folded umbrella (cuspidal cross cap)) *Let $f : (\mathbb{R}^2, a) \to (\mathbb{R}^3, b)$ be a frontal of corank 1. The following conditions are equivalent to each other:*

(1) *f is $\mathscr{A}$-equivalent to the folded umbrella (FU), i.e. the tangent surface to a curve of type $(1,2,4)$.*
(1') *f is $\mathscr{A}$-equivalent to the germ $(t_1,t_2) \mapsto (t_1,\ t_2^2,\ t_1t_2^3)$.*
(2) *$\eta\lambda(a) \neq 0, (\eta^3 f_3)(a) = 0$ and $(d\lambda \wedge d(\eta^3 f_3))(a) \neq 0$.*

Remark 3.12. It is already known another kind of recognition of folded umbrella by [3].

As for cases of higher codimension, we have

Theorem 3.13. (Recognition of open folded umbrella (open cuspidal cross cap))
Let $f : (\mathbb{R}^2, a) \to (\mathbb{R}^m, b), (m \geq 4)$ *be a frontal of corank* 1*. Then the following conditions are equivalent to each other:*

(1) f *is* $\mathscr{A}$*-equivalent to the open folded umbrella, i.e. the tangent surface to a curve of type* $(1, 3, 4, 5, \dots)$.

(1') f *is* $\mathscr{A}$*-equivalent to the germ* $(t_1, t_2) \to (t_1,\ t_2^2,\ t_1 t_2^3,\ t_2^5,\ 0, \dots, 0)$.

(2) $(\eta\lambda)(a) \neq 0$, $(\eta^3 f_k)(a) = 0, (3 \leq k \leq m)$*, and there exist* $3 \leq i < j \leq m$ *and* $A \in \mathrm{GL}(2, \mathbb{R})$ *such that, setting* $(f_i, f_j)A = (f'_3, f'_4)$, $(d\lambda \wedge \eta^3 f'_3)(a) \neq 0, (d\lambda \wedge \eta^3 f'_4)(a) = 0, (\eta^5 f'_4)(a) \neq 0$.

As for openings of Whitney's cusp mapping, we have

Theorem 3.14. (Recognition of swallowtail) *Let* $f : (\mathbb{R}^2, a) \to (\mathbb{R}^3, b)$ *be a frontal of corank* 1*. Then the following conditions are equivalent to each other:*

(1) f *is* $\mathscr{A}$*-equivalent to the swallowtail (SW), i.e. the tangent surface to a curve of type* $(2, 3, 4)$.

(1') f *is* $\mathscr{A}$*-equivalent to the germ* $(t_1, t_2) \mapsto (t_1,\ t_2^3 + t_1 t_2,\ \frac{3}{4} t_2^4 + \frac{1}{2} t_1 t_2^2)$.

(2) f *is a front,* $(d\lambda)(a) \neq 0$ *and* $\mathrm{ord}_a^\eta(\lambda) = 2$.

(3) λ *is* $\mathscr{K}$*-equivalent to the germ* $(t_1, t_2) \mapsto t_1$ *at* 0*,* $\mathrm{ord}_a^\eta(\lambda) = 2$ *and* $\mathrm{ord}_a^\eta(f_3) = 4$*, for an adapted system of coordinates* (x_1, x_2, x_3).

As for cases of higher codimension, we have

Theorem 3.15. (Recognition of open swallowtail) *Let* $f : (\mathbb{R}^2, a) \to (\mathbb{R}^m, b)$ *be a frontal of corank* 1 *with* $m \geq 4$*. Then the following conditions are equivalent to each other:*

(1) f *is* $\mathscr{A}$*-equivalent to the open swallowtail, i.e. the tangent surface to a curve of type* $(2, 3, 4, 5, \dots)$.

(1') f *is* $\mathscr{A}$*-equivalent to the germ* $(t_1, t_2) \mapsto (t_1,\ t_2^3 + t_1 t_2,\ \frac{3}{4} t_2^4 + \frac{1}{2} t_1 t_2^2,\ \frac{3}{5} t_2^5 + \frac{1}{3} t_1 t_2^3,\ 0, \dots)$.

(2) *The Jacobian* λ *is* $\mathscr{K}$*-equivalent to the germ* $(t_1, t_2) \mapsto t_1$ *at the origin,* $\mathrm{ord}_a^\eta(\lambda) = 2$*,* $(\eta^3 f_i)(a) = 0, (3 \leq k \leq m)$*, and there exist* $3 \leq i < j \leq m$ *and* $A \in \mathrm{GL}(2, \mathbb{R})$ *such that, setting* $(f_i, f_j)A = (f'_3, f'_4)$, $\mathrm{ord}_a^\eta(f'_3) = 4$, $\mathrm{ord}_a^\eta(f'_4) = 5$.

Remark 3.16. Though we treat the open swallowtail as the singularity appeared in tangent surfaces, first it appeared as a singularity of Lagrangian varieties and geometric solutions of differential systems ([1, 4]). The open swallowtail and open folded umbrella appear also in the context of frontal-symplectic versality (Example 12.3 of [12]).

As for openings of bec à bec mapping, we have

Theorem 3.17. (Recognition of Mond singularity (cuspidal beaks), (1)(2) [19]) *Let* $f : (\mathbb{R}^2, a) \to (\mathbb{R}^3, b)$ *be a frontal of corank* 1*. Then the following conditions are equivalent to each other:*

(1) f *is* $\mathscr{A}$*-equivalent to Mond singularity (cuspidal beaks), i.e. the tangent surface to a curve of type* $(1, 3, 4)$.

(1') f *is* $\mathscr{A}$*-equivalent to the germ* $(t_1, t_2) \mapsto (t_1,\ t_2^3 + t_1 t_2^2,\ \frac{3}{4} t_2^4 + \frac{2}{3} t_1 t_2^3)$.

(2) f *is a front,* λ *is* $\mathscr{K}$*-equivalent* $t_1 t_2$ *at the origin, and* $\mathrm{ord}_a^\eta(\lambda) = 2$.

(3) λ *is* $\mathscr{K}$*-equivalent* $t_1 t_2$ *at the origin,* $\mathrm{ord}_a^\eta(\lambda) = 2$ *and* $\mathrm{ord}_a^\eta(f_3) = 4$.

Moreover we have:

Theorem 3.18. (Recognition of open Mond singularities (open cuspidal beaks)) *Let* $f : (\mathbb{R}^2, a) \to (\mathbb{R}^m, b)$ *be a frontal of corank* 1 *with* $m \geq 4$*. Then the following conditions are equivalent to each other:*

(1) f *is* $\mathscr{A}$*-equivalent to the open Mond singularity, i.e. the tangent surface to a curve of type* $(1, 3, 4, 5, \dots)$.

(1') f *is* $\mathscr{A}$*-equivalent to the germ* $(t_1, t_2) \mapsto (t_1,\ t_2^3 + t_1 t_2^2,\ \frac{3}{4} t_2^4 + \frac{2}{3} t_1 t_2^3,\ \frac{3}{5} t_2^5 + \frac{1}{2} t_1 t_2^4,\ \dots)$.

(2) λ *is* $\mathscr{K}$*-equivalent to* $(t_1, t_2) \mapsto t_1 t_2$ *at the origin,* $\mathrm{ord}_a^\eta(\lambda) = 2$*,* $(\eta^3 f_i)(a) = 0, (3 \leq k \leq m)$*, and there exist* $3 \leq i \neq j \leq m$ *and* $A \in \mathrm{GL}(2, \mathbb{R})$ *such that, setting* $(f_i, f_j)A = (f'_3, f'_4)$, $\mathrm{ord}_a^\eta(f'_3) = 4$, $\mathrm{ord}_a^\eta(f'_4) = 5$.

To conclude this section, we give the result on recognition of Shcherbak singularity:

Theorem 3.19. (Recognition of Shcherbak singularity) *Let $f : (\mathbb{R}^2, a) \to (\mathbb{R}^3, b)$ be a frontal of corank 1. Then the following conditions are equivalent to each other:*

(1) *f is $\mathscr{A}$-equivalent to Shcherbak singularity, i.e. the tangent surface to a curve of type* $(1,3,5)$.

(1') *f is $\mathscr{A}$-equivalent to the germ $(t_1,t_2) \mapsto (t_1,\ t_2^3 + t_1 t_2^2,\ \frac{3}{5}t_2^5 + \frac{1}{2}t_1 t_2^4)$ at the origin.*

(2) *λ is $\mathscr{K}$-equivalent to the germ $(t_1,t_2) \mapsto t_1 t_2$ at the origin,* $\mathrm{ord}_a^\eta(\lambda) = 2$, $\mathrm{ord}_c^\eta(f_3) \geq 4$ *for any point c on a component of the singular locus $S(f)$, and* $\mathrm{ord}_a^\eta(f_3) = 5$.

Note that Shcherbak singularity necessarily has the $(2,5)$ cuspidal-edge along one component of the singular locus, while it has the ordinary $(2,3)$ cuspidal edge along another component.

4. Frontals and openings

To understand the frontal singularities and to prove the results in the previous section, we introduce the notion of openings and make clear its relation to frontal singularities (see also [11]).

Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ be a frontal (resp. a proper frontal) and $\widetilde{f} : (\mathbb{R}^n, a) \to \mathrm{Gr}(n, T\mathbb{R}^m)$ any Legendre lift of f. Let

$$(x_1, \dots, x_n, x_{n+1}, \dots, x_m)$$

be an adapted system of coordinates to $\widetilde{f}$ (resp. to f) (Definition 2.11). Then, setting $f_i = x_i \circ f, 1 \leq i \leq m$, we have

$$df_i = h_{i1}df_1 + h_{i2}df_2 + \cdots + h_{in}df_n,\ (n+1 \leq i \leq m)$$

for some $h_{ij} \in \mathscr{E}_a, h_{ij}(a) = 0, n+1 \leq i \leq m, 1 \leq j \leq n$.

Definition 4.1. In general, for a map-germ $f = (f_1, \dots, f_m) : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$, we define the $\mathscr{E}_a$-submodule

$$\mathscr{J}_f := \sum_{j=1}^m \mathscr{E}_a df_j = \mathscr{E}_a d(f^* \mathscr{E}_b)$$

of the $\mathscr{E}_a$-module of differential 1-forms Ω_a^1 on $(\mathbb{R}^n, a)$. We would like to call $\mathscr{J}_f$ the *Jacobi module* of f.

Note that $\mathscr{J}_f$ is determined by the Jacobi matrix $J(f)$ of f. Returning to our original situation, we define the following key notion:

Definition 4.2. We call a map-germ $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ an *opening* of a map-germ

$$g : (\mathbb{R}^n, a) \to (\mathbb{R}^n, g(a))$$

if f is of the form $(g_1, \dots, g_n, f_{n+1}, \dots, f_m)$ with $df_j \in \mathscr{J}_g, (n+1 \leq j \leq m)$ via a system of local coordinates of $(\mathbb{R}^m, b)$.

Then we observe the following:

Lemma 4.3. *Any frontal $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ is an opening of $g := (f_1, \dots, f_n) : (\mathbb{R}^n, a) \to (\mathbb{R}^n, g(a))$ via adapted coordinates to a Legendre lift of f. Conversely, any opening of a map-germ*

$$g : (\mathbb{R}^n, a) \to (\mathbb{R}^n, g(a))$$

is a frontal. An opening of g is a proper frontal if and only if g is proper, i.e. $S(g)$ is nowhere dense.

Proof: The first half is clear. To see the second half, let $f = (g_1, \dots, g_n, f_{n+1}, \dots, f_m)$ be an opening of g. Then

$$df_i = h_{i1}df_1 + h_{i2}df_2 + \cdots + h_{in}df_n,\ (n+1 \leq i \leq m)$$

for some $h_{ij} \in \mathscr{E}_a, n+1 \leq i \leq m, 1 \leq j \leq n$. Then a Legendre lift $\widetilde{f} : (\mathbb{R}^n, a) \to \mathrm{Gr}(n, T\mathbb{R}^m)$ is given, via Grassmannian coordinates of the fiber, by

$$t \mapsto (f(t), \begin{pmatrix} E_n \\ H(t) \end{pmatrix}),$$

where E_n is the $n \times n$ unit matrix and $H(t)$ is given by the $(m-n) \times n$-matrix $(h_{ij}(t))$. Therefore f is a frontal. Note that an adapted system of coordinates for f is given by $(x_1, \dots, x_n, \widetilde{x}_{n+1}, \dots, \widetilde{x}_m)$ with $\widetilde{x}_i = x_i - \sum_{j=n+1}^m h_{ij}(a)x_j$ $(n+1 \leq i \leq m)$. The last statement follows clearly. □

Here we recall one of key notion for our approach to the recognition problem of frontal singularities.

Definition 4.4. ([8]) An opening

$$f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b), f = (g; f_{n+1}, \dots, f_m),$$

of a map-germ $g : (\mathbb{R}^n, a) \to (\mathbb{R}^n, g(a))$ is called a *versal opening* if, for any $h \in \mathscr{E}_a$ with $dh \in \mathscr{J}_g$, there exist $k_0, k_1, \dots, k_{m-n} \in \mathscr{E}_{\mathbb{R}^n, g(a)}$ such that

$$h = g^*(k_0) + g^*(k_1)f_{n+1} + \cdots + g^*(k_{m-n})f_m.$$

We will use the following result which is proved in Proposition 6.9 of [8].

Theorem 4.5. *Any two versal openings $f, f' : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ (having the same target dimension) of a map-germ g are $\mathscr{A}$-equivalent to each other.*

Recall, for a map-germ $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$, we have defined $\mathscr{J}_f = \mathscr{E}_a d(f^*\mathscr{E}_b)$ (Definition 4.1).

Lemma 4.6. (1) *Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$, $f' : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b')$ be map-germs. If f and f' are $\mathscr{L}$-equivalent, i.e. if there exists a diffeomorphism-germ $\tau : (\mathbb{R}^m, b) \to (\mathbb{R}^m, b')$ such that $f' = \tau \circ f$, then $\mathscr{J}_f = \mathscr{J}_{f'}$.*

(2) *Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$, $f' : (\mathbb{R}^n, a') \to (\mathbb{R}^m, b)$ be map-germs. If f and f' are $\mathscr{R}$-equivalent, i.e. if there exists a diffeomorphism-germ $\sigma : (\mathbb{R}^n, a) \to (\mathbb{R}^n, a')$ such that $f' = f \circ \sigma$, then $\sigma^*(\mathscr{J}_f) = \mathscr{J}_{f'}$.*

Proof: (1) Since $f^*\mathscr{E}_b = f'^*\mathscr{E}_{b'}$, we have $\mathscr{J}_f = \mathscr{E}_a d(f^*\mathscr{E}_b) = \mathscr{E}_a d(f'^*\mathscr{E}_{b'}) = \mathscr{J}_{f'}$.

(2) Since $f'^*\mathscr{E}_b = \sigma^*(f^*\mathscr{E}_b)$, we have

$$\mathscr{J}_{f'} = \mathscr{E}_{a'} d(f'^*\mathscr{E}_b) = \mathscr{E}_{a'} d(\sigma^*(f^*\mathscr{E}_b)) = \sigma^*\mathscr{E}_a \sigma^* d(f^*\mathscr{E}_b) = \sigma^*(\mathscr{E}_a d(f^*\mathscr{E}_b)) = \sigma^*(\mathscr{J}_f).$$

□

The equality of Jacobi modules $\mathscr{J}_f$ has a simple meaning:

Lemma 4.7. *Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$, $f' : (\mathbb{R}^n, a) \to (\mathbb{R}^{m'}, b')$ be map-germs.*

Then the following conditions (i), (ii) *are equivalent:*

(i) *The Jacobi module $\mathscr{J}_f = \mathscr{J}_{f'}$.*

(ii) *There exist an $m' \times m$-matrix P and an $m \times m'$-matrix Q with entries in $\mathscr{E}_a$ such that the Jacobi matrix $J(f') = PJ(f)$ and $J(f) = QJ(f')$.*

In particular, (i) *implies that the Jacobi ideal $J_f = J_{f'}$.*

Moreover, if the target dimension $m = m'$, then the following condition (iii) is equivalent to (i).

(iii) *There exists an invertible $m \times m$-matrix R with entries in $\mathscr{E}_a$ such that $J(f') = RJ(f)$.*

To show Lemma 4.7, we recall the following fact in linear algebra.

Lemma 4.8. (cf. [22]) *Let A, B be $m \times m$-matrices with entries in $\mathbb{R}$. Then there exists an $m \times m$-matrices C with entries in $\mathbb{R}$ such that $C(E_m - BA) + A$ is invertible.*

Proof of Lemma 4.7:

The inclusion $\mathscr{J}_{f'} \subseteq \mathscr{J}_f$ is equivalent to that there exist $p_{ij} \in \mathscr{E}_a$ such that $df'_i = \sum_{j=1^m} p_{ij} df_j$, $(1 \leq i \leq m)$, namely that $J(f') = PJ(f)$ by setting $P = (p_{ij})$. Similarly, the inclusion $\mathscr{J}_f \subseteq \mathscr{J}_{f'}$ is equivalent to that there exist $q_{ij} \in \mathscr{E}_a$ such that $df_i = \sum_{j=1^m} q_{ij} df'_j$, $(1 \leq i \leq m)$, namely that $J(f) = QJ(f')$ by setting $Q = (q_{ij})$. Therefore the equivalence between (i) and (ii) is clear.

Suppose $m = m'$. By Lemma 4.8, there exists an $m \times m$-matrix C with entries in $\mathbb{R}$ such that

$$C(E_m - Q(a)P(a)) + P(a)$$

is invertible. Then $R := C(E_m - QP) + P$ is an invertible $m \times m$-matrix with entries in in $\mathscr{E}_a$. Then we have $(E_m - QP)J(f) = J(f) - QJ(f') = O$ and therefore $RJ(f) = C(E_m - QP)J(f) + PJ(f) = J(f')$. □

Remark 4.9. Related to Jacobi modules, we define the ramification module $\mathscr{R}_f \subseteq \mathscr{E}_a$ for a map-germ $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ by

$$\mathscr{R}_f := \{h \in \mathscr{E}_a \mid dh \in \mathscr{J}_f\},$$

using the Jacobi module $\mathscr{J}_f$. Then $\mathscr{R}_f = \mathscr{R}_{f'}$ if and only if $\mathscr{J}_f = \mathscr{J}_{f'}$. See, for details, the series of papers [6, 7, 8, 9, 10, 11].

Lemma 4.10. *Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$, $f' : (\mathbb{R}^n, a') \to (\mathbb{R}^{m'}, b')$ be map-germs. If $\mathscr{J}_f = \mathscr{J}_{f'}$, then*

$$J_f = J_{f'}, \quad \mathscr{N}_f = \mathscr{N}_{f'}.$$

Proof: The equality $J_f = J_{f'}$ follows from Lemma 4.7. For any $\eta \in \mathscr{V}_a$, the condition $\eta \in \mathscr{N}_f$ is equivalent to that $\omega(\eta) \in J_f = J_{f'}$ for any $\omega \in \mathscr{J}_f = \mathscr{J}_{f'}$, which is equivalent to that $\eta \in \mathscr{N}_{f'}$. Therefore we have $\mathscr{N}_f = \mathscr{N}_{f'}$. □

Lemma 4.11. *Let $f, f' : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ be proper frontals of corank 1. Then the conditions*

$$\lambda_f \cdot \mathscr{E}_a = \lambda_{f'} \cdot \mathscr{E}_a, \quad \mathscr{N}_f = \mathscr{N}_{f'},$$

imply that $\mathscr{J}_f = \mathscr{J}_{f'}$.

Proof: By the assumption we may take $\lambda_f = \lambda_{f'}$ and $\eta_f = \eta_{f'}$. and $\eta_f = \partial/\partial t_n$ for a system of coordinates $t_1, \dots, t_{n-1}, t_n$ of $(\mathbb{R}^n, a)$. Note that, by the assumption, the zero-locus of λ_f is nowhere dense. Then $f_*(\partial/\partial t_1), \dots, f_*(\partial/\partial t_{n-1}), (1/\lambda_f) f_*(\partial/\partial t_n)$ are linearly independent at a as elements of $\mathscr{E}_a^m$. Take additional $\xi_{n+1}, \dots, \xi_m$ to complete a basis of $\mathscr{E}_a^m$. Moreover by the assumption

$$f'_*(\partial/\partial t_1), \dots, f'_*(\partial/\partial t_{n-1}), (1/\lambda_f) f'_*(\partial/\partial t_n)$$

are linearly independent at a as elements of $\mathscr{E}_a^m$. Take additional $\xi'_{n+1}, \dots, \xi'_m$ to complete a basis of $\mathscr{E}_a^m$. Then define $R : (\mathbb{R}^n, a) \to \mathrm{GL}(m, \mathbb{R})$ by

$$Rf_*(\partial/\partial t_i) = f'_*(\partial/\partial t_i), 1 \leq i \leq n-1, \ R(1/\lambda_f) f_*(\partial/\partial t_n) = (1/\lambda_f) f'_*(\partial/\partial t_n), \ R\xi_j = \xi'_j, n+1 \leq j \leq m.$$

Then $Rf_*(\partial/\partial t_n) = f'_*(\partial/\partial t_n)$ and we have $RJ(f) = J(f')$. By Lemma 4.7, we have $\mathscr{J}_f = \mathscr{J}_{f'}$. □

We utilize the following in the next section:

Lemma 4.12. *Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ be an opening of $g : (\mathbb{R}^n, a) \to (\mathbb{R}^n, g(a))$ with respect to an adapted system of coordinates $(x_1, \dots, x_n, x_{n+1}, \dots, x_m)$. Then f and g are frontals and $\mathscr{J}_f = \mathscr{J}_g$. They have common Jacobian, same corank, and $\mathscr{N}_f = \mathscr{N}_g$. If they are of corank 1, then they have common kernel field.*

Proof: By Lemma 4.3, we have $\mathscr{J}_f = \mathscr{J}_g$. Then $J_f = J_g$, therefore $\lambda_f = \lambda_g$. Moreover, by Lemma 4.7, $\mathrm{Ker}(T_a f) = \mathrm{Ker}(T_a g) \subseteq T_a\mathbb{R}^n$. Therefore f and g have the same corank. Furthermore, for any $\eta \in \mathscr{V}_a$, the condition that $df_i(\eta) \in J_f, 1 \leq i \leq m$ is equivalent to that $dg_i(\eta) \in J_f = J_g, 1 \leq i \leq n$. Hence $\mathscr{N}_f = \mathscr{N}_g$. □

Definition 4.13. Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ and $f' : (\mathbb{R}^n, a') \to (\mathbb{R}^{m'}, b')$ be map-germs. Then f and f' are called $\mathscr{J}$*-equivalent* if there exists a diffeomorphism-germ $\sigma : (\mathbb{R}^n, a) \to (\mathbb{R}^n, a')$ such that $\mathscr{J}_{f'\circ\sigma} = \mathscr{J}_f$. Note that m and m' can be different.

By Lemma 4.6 and Lemma 4.11, we have

Corollary 4.14. *Let* $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ *and* $f' : (\mathbb{R}^n, a') \to (\mathbb{R}^{m'}, b')$ *be map-germs. If* f *and* f' *are* $\mathscr{A}$*-equivalent, then* f *and* f' *are* $\mathscr{J}$*-equivalent.*

Corollary 4.15. *Let* f, f' *be proper frontals. If* f *and* f' *are* $\mathscr{J}$*-equivalent, then* $(\lambda_f \cdot \mathscr{E}_a, \mathscr{N}_f)$ *is transformed to* $(\lambda_{f'} \cdot \mathscr{E}_{a'}, \mathscr{N}_{f'})$ *by a diffeomorphism-germ* $\sigma : (\mathbb{R}^n, a) \to (\mathbb{R}^n, a')$*. In particular* λ_f *and* $\lambda_{f'}$ *are* $\mathscr{K}$*-equivalent.*

Moreover if f *is of corank* 1 *and* $(\lambda_f \cdot \mathscr{E}_a, \mathscr{N}_f)$ *is transformed to* $(\lambda_{f'} \cdot \mathscr{E}_{a'}, \mathscr{N}_{f'})$ *by a diffeomorphism-germ* $\sigma : (\mathbb{R}^n, a) \to (\mathbb{R}^n, a')$*, then* f *and* f' *are* $\mathscr{J}$*-equivalent.*

On the vanishing order of a function for a vector field introduced in Definition 3.8, we have:

Lemma 4.16. *If* $\widetilde{h} = \rho h, \widetilde{\xi} = \nu\xi$ *for some* $\rho, \nu \in \mathscr{E}_a$ *with* $\rho(a) \neq 0, \xi(a) \neq 0$*, then* $\mathrm{ord}_a^{\widetilde{\xi}}(\widetilde{h}) = \mathrm{ord}_a^{\xi}(h)$*.*

If $\overline{h} = h \circ \sigma, \overline{\xi} = (T\sigma^{-1}) \circ \xi \circ \sigma$ *for some diffeomorphism-germ* $\sigma : (\mathbb{R}^n, a') \to (\mathbb{R}^n, a)$*, then* $\mathrm{ord}_{a'}^{\overline{\xi}}(\overline{h}) = \mathrm{ord}_a^{\xi}(h)$*.*

By Lemma 4.16 we have

Corollary 4.17. *Let* $f : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ *be a proper frontal of corank* 1*. Then* $\mathrm{ord}_a^{\eta}(\lambda)$ *is independent of the choices of the Jacobian* λ *and the kernel field* η *of* f*. If* $f' : (\mathbb{R}^n, a') \to (\mathbb{R}^{m'}, b')$ *is* $\mathscr{J}$*-equivalent to* f*, then* f' *is a proper frontal of corank* 1 *and* $\mathrm{ord}_a^{\eta'}(\lambda')$ *is equal to* $\mathrm{ord}_a^{\eta}(\lambda)$*, for any Jacobian* λ' *and any kernel filed* η' *of* f'*.*

5. Proofs of recognition theorems

In this section we give proofs of Theorems 3.9, 3.10, 3.11, 3.13, 3.14, 3.15, 3.17, 3.18, and 3.19.

Proof of Theorem 3.9: The equivalence of (1) and (1') is classically known (see [6]). The equivalence of (1') and (2) is proved in [21].

To study the condition, we set $g = (f_1, f_2)$. Then for the Jacobian λ and the kernel field η of g we also have $\eta\lambda(a) \neq 0$ (see Lemma4.12). By Theorem 3.4, g is $\mathscr{A}$-equivalent to the fold. Then the condition (3) means that f is a versal opening of the fold g. Since the cuspidal edge is characterized as the (mini)-versal opening of the fold map-germ, we have the equivalence of (3) and (1) by Theorem 4.5. □

Proof of Theorem 3.10: The equivalence of (1) and (1') is proved in Theorem 7.1 of [8]. The condition (3) means that f is a versal opening of the fold g. Since the embedded cuspidal edge is characterized as the versal opening of the fold map-germ, we have the equivalence of (3) and (1) by Theorem4.5. On the other hand, under the condition $\eta\lambda(a) \neq 0$, the condition $\mathrm{ord}_a^{\eta}(f_i) = 3$ for some $i, 3 \leq i \leq m$ is equivalent to that the Legendre lift $\widetilde{f}$ is an immersion i.e. f is a front. Therefore (3) and (2) are equivalent. □

Proof of Theorem 3.11. The equivalence of (1) and (1') is due to Cleave (see [8]).

Suppose the condition (2) is satisfied. Then f is $\mathscr{A}$-equivalent to the germ $g(t_1,t_2) = (t_1,t_2^2, f_3(t_1,t_2))$ at the origin with $\lambda = t_2, \eta = \partial/\partial t_2$, $(\eta^3 f_3)(0) = 0$ and $(d\lambda \wedge d(\eta^3 f_3))(0) \neq 0$. Since $df_3 \in \mathscr{J}_g$, in other word since $f_3 \in \mathscr{R}_g$ (Remark4.9), there exist functions A,B on $(\mathbb{R}^2,0)$ such that

$$f_3(t_1,t_2) = A(t_1,t_2^2) + B(t_1,t_2^2)t_2^3.$$

Then the condition $(\eta^3 f_3)(0) = 0$ is equivalent to $B(0,0) = 0$, and the condition

$$(d\lambda \wedge d(\eta^3 f_3))(0) \neq 0$$

is equivalent to $\frac{\partial B}{\partial t_1}(0,0) \neq 0$. Define diffeomorphism-germs $\sigma : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ by

$$\sigma(t_1,t_2) = (B(t_1,t_2^2),t_2)$$

and $\tau : (\mathbb{R}^3,0) \to (\mathbb{R}^3,0)$ by

$$\tau(x_1,x_2,x_3) = (B(x_1,x_2),x_2,x_3 - A(x_1,x_2)).$$

Then $(t_1,t_2^2,t_1t_2^3) \circ \sigma = \tau \circ (t_1,t_2^2, f_3)$ holds. Therefore f is $\mathscr{A}$-equivalent to folded umbrella. Hence we see that (2) implies (1). Conversely (1) implies (2) for some, so for any, adapted coordinates. □

Proof of Theorem 3.13: The $\mathscr{A}$-determinacy of tangent maps to curves of type $(1,2,4,5,\dots)$ is proved in Theorem 7.2 of [8]. Let $\gamma : (\mathbb{R},0) \to (\mathbb{R}^m,0)$ be the curve $t \mapsto (t,t^2,t^4,t^5,0,\dots)$. Then the tangent map $\mathrm{Tan}(\gamma) : (\mathbb{R}^2,0) \to (\mathbb{R}^m,0)$ is given by

$$\mathrm{Tan}(\gamma)(t,u) = (t+u,\ t^2 + 2ut,\ t^4 + 4ut^3,\ t^5 + 5ut^4,\ 0,\ \dots).$$

Then it is easy to see that $\mathrm{Tan}(\gamma)$ is $\mathscr{A}$-equivalent to $(t_1,t_2) \to (t_1,\ t_2^2,\ t_1t_2^3,\ t_2^5,\ 0,\dots,0)$. Hence we have the equivalence of (1) and (1').

Suppose f satisfies (2). Then f is an opening of (f_1, f_2), which is a fold by Theorem3.4. Therefore f is $\mathscr{A}$-equivalent to a frontal of form $(t_1,t_2^2, f_3, f_4,\dots)$ for an adapted coordinates. The Jacobian is given by $\lambda = t_2$ and the kernel field is given by $\eta = \partial/\partial t_2$. We write $f_i = A_i(t_1,t_2^2) + B_i(t_1,t_2^2)t_2^3$ for some A_i, B_i with $A_i(0,0) = 0, B_i(0,0) = 0$, $(3 \le i \le m)$. Then $f_i = \widetilde{A}_i(t_1,t_2^2)t_1t_2^3 + \widetilde{B}_i(t_1,t_2^2)t_2^5$. Then the condition (2) is equivalent to that, for some i, j with $3 \le i < j \le m$,

$$\begin{pmatrix} \widetilde{A}_i(0,0) & \widetilde{B}_i(0,0) \\ \widetilde{A}_j(0,0) & \widetilde{B}_j(0,0) \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R}).$$

Then f is $\mathscr{A}$-equivalent to $(t_1,\ t_2^2,\ t_1t_2^3,\ t_2^5,\ 0,\dots,0)$. Therefore (2) implies (1'). The converse is clear. □

Proof of Theorem 3.14: The equivalence of (1) and (1') is proved in Theorem 1 of [6]. The equivalence of (1') and (2) is proved in Proposition 1.3 of [21]. The condition that λ is $\mathscr{K}$-equivalent to t_1 and $\mathrm{ord}^\eta_a(\lambda) = 2$ is equivalent, by Theorem 3.4, to that f is an opening of Whitney's cusp

$$g(t_1,t_2) = (t_1,\ t_2^3 + t_1t_2).$$

The Jacobian is given by $\lambda = 3t_2^2 + t_1$ and the kernel field is given by $\eta = \partial/\partial t_2$. Set

$$U_1 = \frac{3}{4}t_2^4 + \frac{1}{2}t_1t_2^2, U_2 = \frac{3}{5}t_2^5 + \frac{1}{3}t_1t_2^3.$$

Then it is known that the ramification module $\mathscr{R}_g$ is generated by $1,U_1,U_2$ over g^* (see [6]). Since $f_3 \in \mathscr{R}_g$ is the third component for an adapted system of coordinates, f_3 is written as

$$f_3 = A \circ g + (B \circ g)U_1 + (C \circ g)U_2,$$

for some functions A,B,C with $A(0,0) = 0$, $\frac{\partial A}{\partial x_1}(0,0) = 0$, $\frac{\partial A}{\partial x_2}(0,0) = 0$. By the condition $\mathrm{ord}^\eta_a(f_3) = 4$, we have $B(0,0) \neq 0$. Then, by a change of adapted system of coordinates, We may suppose $f = (g, f_3)$ with $f_3 = U_1 + \Phi$, where $\Phi = (\widetilde{B} \circ g)U_1 + (D \circ g)U_2$ with $\widetilde{B}(0,0) = 0$. Then we set the family

$F_s = (g, U_1 + s\Phi)$. By the same infinitesimal method used in [6], we can show that the family F_s is trivialized by $\mathscr{A}$-equivalence. Hence $f = F_1$ is $\mathscr{A}$-equivalent to F_0, that is the normal form of (2). Therefore (3) implies (2). The converse is clear. □

Proof of Theorem 3.15: The equivalence of (1) and (1') is proved in [8]. The condition (2) implies, by Theorem 3.4, that f is an opening of Whitney's cusp. Using the same notations as in the proof of Theorem 3.14, we write f_k as $f_k = A_k \circ g + (B_k \circ g)U_1 + (C_k \circ g)U_2$, for some functions A_k, B_k, C_k with $A_k(0,0) = 0$, $\frac{\partial A_k}{\partial x_1}(0,0) = 0$, $\frac{\partial A_k}{\partial x_2}(0,0) = 0$. Then by the condition (2), we see that f is a versal opening (Definition 4.4) of g. On the other hand the map-germ of (1') is a versal opening of g ([8]). By Theorem 4.5, we see that (2) implies (1'). The converse implication (1') to (2) is clear. □

Proof of Theorem 3.17: The outline of the proof is similar to that of Theorem 3.14. The equivalence of (1) and (1') is proved in Theorem 1 of [6]. The equivalence of (1') and (2) is proved in [19]. The condition that λ is $\mathscr{K}$-equivalent to t_1t_2 and $\mathrm{ord}_a^\eta(\lambda) = 2$ is equivalent, by Theorem 3.4, to that f is an opening of bec à bec $g(t_1,t_2) = (t_1,\ t_2^3 + t_1t_2^2)$. The Jacobian is given by $\lambda = 3t_2^2 + 2t_1t_2$ and the kernel field is given by $\eta = \partial/\partial t_2$. Set $U_1 = \frac{3}{4}t_2^4 + \frac{2}{3}t_1t_2^3, U_2 = \frac{3}{5}t_2^5 + \frac{1}{2}t_1t_2^4$. Then it is known that the ramification module $\mathscr{R}_g$ is generated by $1, U_1, U_2$ over g^* (see [6]). Since $f_3 \in \mathscr{R}_g$ is the third component for an adapted system of coordinates, f_3 is written as $f_3 = A \circ g + (B \circ g)U_1 + (C \circ g)U_2$, for some functions A, B, C with $A(0,0) = 0$, $\frac{\partial A}{\partial x_1}(0,0) = 0$, $\frac{\partial A}{\partial x_2}(0,0) = 0$. By the condition $\mathrm{ord}_a^\eta(f_3) = 4$, we have $B(0,0) \neq 0$. Then, by a change of adapted system of coordinates, we may suppose $f = (g, f_3)$ with $f_3 = U_1 + \Phi$, where $\Phi = (\widetilde{B} \circ g)U_1 + (C \circ g)U_2$ with $\widetilde{B}(0,0) = 0$. Then, by the infinitesimal method used in [6], the family $F_s = (g, U_1 + s\Phi)$ is trivialized by $\mathscr{A}$-equivalence. Hence $f = F_1$ is $\mathscr{A}$-equivalent to F_0, that is the normal form of (2). Therefore (3) implies (2). The converse is clear. □

Proof of Theorem 3.18: Open Mond singularities are characterized as versal openings of bec à bec ([8]). Then Theorem3.18 is proved similarly as the proof of Theorem3.15. □

Proof of Theorem 3.19: The equivalence of (1) and (1') is proved in [6]. The condition (2) implies that f is an opening of bec à bec. Using the same notations in the proof of Theorem 3.17, we write f_3 as $f_3 = A \circ g + (B \circ g)U_1 + (C \circ g)U_2$, for some functions A, B, C with $A(0,0) = 0$, $\frac{\partial A}{\partial x_1}(0,0) = 0$, $\frac{\partial A}{\partial x_2}(0,0) = 0$. By the condition $\mathrm{ord}_a^\eta(f_3) = 5$, we have $B(0,0) = 0$ and $C(0,0) \neq 0$. Moreover, by the assumption, we may assume that $\mathrm{ord}_{(t_1,0)}^\eta f_3 \geq 4$ along the component $\{t_2 = 0\}$ of $S(f)$ and then $B(x_1, 0) = 0$. Then, by a change of adapted system of coordinates, we may suppose $f = (g, f_3)$ with $f_3 = U_2 + \Phi$, where $\Phi = (B \circ g)U_1 + (\widetilde{C} \circ g)U_2$ with $B(x_1,0) = 0, \widetilde{C}(0,0) = 0$. Then by the same infinitesimal method used in [6], the family $F_s = (g, U_2 + s\Phi)$ turns to be trivial under $\mathscr{A}$-equivalence. Hence $f = F_1$ is $\mathscr{A}$-equivalent to F_0, that is the normal form of (1'). Therefore (2) implies (1'). The converse is clear. □

6. An application to 3-dimensional Lorentzian geometry, and other topics

We announce the following result without explanations of notions. The details will be given in [16].

Theorem 6.1. ([2], [14, 16]) *Any null frontal surface in a Lorentzian* 3*-manifold turns to be a null tangent surface of a (directed) null curve, and any generic null frontal surface has only singularities, along the null curve, of type*

(I) *cuspidal edge* (CE), (II) *swallowtail* (SW)*, or* (III) *Shcherbak singularity* (SB).

Moreover the corresponding dual frontal in the space of null-geodesics has (I) *cuspidal edge* (CE)*,* (II) *Mond singularity* (MD)*, or* (III) *generic folded pleat* (GFP).

The same classification result holds not only for any Lorentzian metric but also for arbitrary non-degenerate (strictly convex) cone structure in any 3*-manifold.*

To show Theorem 6.1, we face the recognition problem on cuspidal edge, swallowtail, Scherbak singularity, Mond singularity, and "generic folded pleat". In fact we will use the recognition theorems introduced in the previous section and the following result on openings of Whitney's cusp. The following recognition result is proved by the same method of the above proof of Theorem3.14. The details will be given in [16].

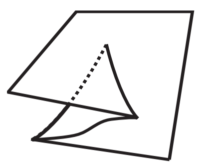
Theorem 6.2. (Recognition of folded pleat) *Let $f : (\mathbb{R}^2, a) \to (\mathbb{R}^3, b)$ be a frontal of corank* 1. *Then the following conditions are equivalent to each other:*

(1) *f is $\mathscr{A}$-equivalent to a folded pleat i.e. the singularity of tangent surface of a curve of type* $(2,3,5)$.

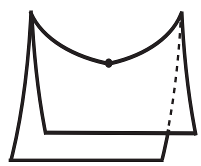
(1') *f is $\mathscr{A}$-equivalent to the germ $(t_1,\ t_2^3 + t_1t_2,\ \frac{3}{5}t_2^5 + \frac{1}{2}t_1t_2^3 + c(\frac{1}{2}t_2^6 + \frac{3}{4}t_1t_2^4))$ at the origin for some $c \in \mathbb{R}$.*

(2) *λ is $\mathscr{K}$-equivalent to the germ $(t_1,t_2) \mapsto t_1$ at the origin, $\mathrm{ord}_a^{\eta}(\lambda)(a) = 2$, f has an injective representative, and $\mathrm{ord}_p^{\eta}(f_3) = 5$.*

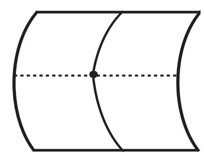
Note that a folded pleat singularity necessarily has an injective representative.



folded pleat



cuspidal swallowtail



cuspidal lips

Remark 6.3. Recall that the diffeomorphism classes (CE), (SW), (SB) and (MD) are exactly characterized as those of tangent surfaces in Euclidean space $\mathbb{R}^3$ of curves of type $(1,2,3)$, $(2,3,4)$, $(1,3,5)$, $(1,3,4)$ respectively. A map-germ $(\mathbb{R}^2, a) \to (\mathbb{R}^3, b)$ is called a *folded pleat* (FP) if it is diffeomorphic to the tangent surface of a curve of type $(2,3,5)$ in $\mathbb{R}^3$. The diffeomorphism classes of folded pleats fall into *two* classes, the generic folded pleat and the non-generic folded pleat. In the list of Theorem 6.1, it is claimed that only the generic folded pleat (GFP) appear. Theorem 6.2 do not solve the recognition of a singularity but a class of singularities, which consists of two singularities. Note that the parameter c in (1') of Theorem 6.2 is not a moduli, but provides just two $\mathscr{A}$-equivalence classes. To recognize the generic folded pleat, it is necessary an additional argument to distinguish generic and non-generic folded pleats.

In this occasion we introduce and prove the following two theorems of recognition:

Theorem 6.4. (Recognition of cuspidal swallowtail) *Let $(\mathbb{R}^2, a) \to (\mathbb{R}^3, b)$ be a frontal of corank* 1. *Then the following conditions are equivalent to each other:*

(1) *f is $\mathscr{A}$-equivalent to the cuspidal swallowtail i.e. the singularity of tangent surface of curves of type* $(3,4,5)$.

(1') *f is $\mathscr{A}$-equivalent to the germ $(t_1,t_2) \mapsto (t_1,\ t_2^4 + t_1t_2,\ \frac{4}{5}t_2^5 + \frac{1}{2}t_1t_2^2)$ at the origin.*

(2) *λ is $\mathscr{K}$-equivalent to the germ $(t_1,t_2) \mapsto t_1$ at the origin, $\mathrm{ord}_a^{\eta}(\lambda) = 3$ and $\mathrm{ord}_a^{\eta}(f_3) = 5$.*

Proof: In [8] it is proved that the condition (1) is equivalent to that f is $\mathscr{A}$-equivalent to the germ $(t,u) \mapsto (t^3 + 3u, t^4 + 4ut, t^5 + 5ut^2)$, which is $\mathscr{A}$-equivalent to the normal form of (1'). Therefore (1) and (1') are equivalent. In [24], the map-germ which is $\mathscr{A}$-equivalent to the germ $g : (t_1,t_2) \mapsto (t_1,\ t_2^4 + t_1t_2)$ at the origin is called a *swallowtail* and it is shown that a map-germ $g : (\mathbb{R}^2, a) \to (\mathbb{R}^2, g(a))$ is a swallowtail if and only if λ is $\mathscr{K}$-equivalent to the germ $(t_1,t_2) \mapsto t_1$ at the origin and $\mathrm{ord}_a^{\eta}(\lambda) = 3$. Suppose f satisfies

(2). Then f is an opening of swallowtail. Then f is $\mathscr{A}$-equivalent to a frontal of form $f=(g,f_3)$. We have the Jacobian $\lambda = 4t_2^3 + t_1$ and $\eta = \partial/\partial t_2$. We follow the method of [6]. Set

$$U = t_2^4 + t_1t_2,\ U_1 = \tfrac{4}{5}t_2^5 + \tfrac{1}{2}t_1t_2^2,\ U_2 = \tfrac{2}{3}t_2^6 + \tfrac{1}{3}t_1t_2^3,\ U_3 = \tfrac{4}{7}t_2^7 + \tfrac{1}{4}t_1t_2^4.$$

The third component f_3 is written as

$$f_3 = A\circ g + (B\circ g)U_1 + (C\circ g)U_2 + (D\circ g)U_3.$$

Then the condition $\mathrm{ord}_a^\eta(f_3) = 5$ implies that $B(0,0) \neq 0$. We may suppose $f=(g,f_3)$ with

$$f_3 = U_1 + \Phi, \Phi = (B\circ g)U_1 + (C\circ g)U_2 + (D\circ g)U_3, B(0,0) = 0.$$

Then the family $F_s = (g,\ U_1 + s\Phi)$ is trivialized by $\mathscr{A}$-equivalence. Thus $f = F_1$ is $\mathscr{A}$-equivalent to F_0 which is the normal form of (1'). Therefore (2) implies (1'). The converse is clear. Hence (1') and (2) are equivalent. $\square$

As for openings of the lips $(t_1,t_2) \to (t_1, t_2^3 + t_1^2t_2)$ (see [24]), we have

Theorem 6.5. (Recognition of cuspidal lips) *Let $(\mathbb{R}^2,a) \to (\mathbb{R}^3,b)$ be a frontal of corank 1. Then the following conditions are equivalent to each other:*

(1) *f is $\mathscr{A}$-equivalent to cuspidal lips i.e. $(t_1,t_2) \to (t_1, t_2^3 + t_1^2t_2, \frac{3}{4}t_2^4 + \frac{1}{2}t_1^2t_2^2)$.*

(2) *f is a front and λ is $\mathscr{K}$-equivalent to the germ $(t_1,t_2) \mapsto t_1^2 + t_2^2$ at the origin.*

(3) *λ is $\mathscr{K}$-equivalent to the germ $(t_1,t_2) \mapsto t_1^2 + t_2^2$ at the origin, and $\mathrm{ord}_a^\eta(f_3) = 4$.*

Proof: The equivalence of (1) and (2) is proved in [19]. Under the condition that λ is $\mathscr{K}$-equivalent to the germ $(t_1,t_2) \mapsto t_1^2 + t_2^2$ at the origin, the condition $\mathrm{ord}_a^\eta(f_3) = 4$ is equivalent to that the Legendre lift $\widetilde{f}$ is an immersion. Thus we have the equivalence of (2) and (3). $\square$

Remark 6.6. Cuspidal lips never appear as singularities of tangent surfaces.

We conclude the paper by presenting open questions:

Question 1. When does $\mathscr{J}$-equivalence imply $\mathscr{A}$-equivalence ?

Remark 6.7. For immersions, folds, cusps, lips, beaks, swallowtails : $(\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$, $\mathscr{J}$-equivalence of frontals of corank 1 implies $\mathscr{A}$-equivalence.

Example 6.8. ([23, 20]) Let $f, f' : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ be defined by $f(t_1,t_2) = (t_1, t_1t_2 + t_2^5 + t_2^7)$ (butterfly) and $f'(t_1,t_2) = (t_1, t_1t_2 + t_2^5)$ (elder butterfly). Then f is not $\mathscr{A}$-equivalent to f' and their recognition by Taylor coefficients is obtained by Kabata [20]. On the other hand we observe, by using the theory of implicit OED of first order, that f is $\mathscr{J}$-equivalent to f' in fact. Therefore we see that it is absolutely impossible to recognize them just in terms of kernel field η and Jacobian λ.

Question 2. When does $\mathscr{J}$-equivalence imply $\mathscr{K}$-equivalence ?

It can be shown, for map-germs of corank 1, that $\mathscr{J}$-equivalence implies $\mathscr{K}$-equivalence under a mild condition:

Lemma 6.9. *Let $f : (\mathbb{R}^n,a) \to (\mathbb{R}^m,b)$ and $f' : (\mathbb{R}^n,a') \to (\mathbb{R}^{m'},b')$ be map-germs of corank 1. If f and f' are $\mathscr{J}$-equivalent and f is $\mathscr{K}$-finite, then f and f' are $\mathscr{K}$-equivalent, i.e. $(f^*\mathfrak{m}_b)\mathscr{E}_a$ is transformed to $(f'^*\mathfrak{m}_{b'})\mathscr{E}_{a'}$ by a diffeomorphism-germ $\sigma : (\mathbb{R}^n,a) \to (\mathbb{R}^n,a')$. Here $\mathfrak{m}_b \subset \mathscr{E}_b$ is the maximal ideal. The condition that f is $\mathscr{K}$-finite means that $\dim_{\mathbb{R}}(\mathscr{E}_a/(f^*\mathfrak{m}_b)\mathscr{E}_a) < \infty$.*

Proof: By the assumption, f is $\mathscr{A}$-equivalent to $g : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$ of form

$$(t_1, \ldots, t_{n-1}, \varphi_n(t), \ldots, \varphi_m(t))$$

for some $\varphi_i \in \mathscr{E}_0, n \le i \le m$. Then $g^*(\mathfrak{m}_0)\mathscr{E}_0$ is generated by $t_1, \ldots, t_{n-1}, t_n^\ell$ for some ℓ and ℓ is uniquely determined by the minimum of orders of $\varphi_n(0, t_n), \ldots, \varphi_m(0, t_n)$ for t_n at 0. On the other hand, the Jacobi module $\mathscr{J}_g$ is generated by $dt_1, \ldots, dt_{n-1}, (\partial\varphi_n/\partial t_n)dt_n, \ldots, (\partial\varphi_n/\partial t_n)dt_n$, and the minimum of orders of $(\partial\varphi_n/\partial t_n)(0, t_n), \ldots, (\partial\varphi_m/\partial t_n)(0, t_n)$ for t_n at 0 is invariant under $\mathscr{J}$-equivalence. Therefore $\mathscr{K}$-equivalence class is also invariant under $\mathscr{J}$-equivalence. $\square$

REFERENCES

[1] V.I. Arnol'd, *Lagrangian manifold singularities, asymptotic rays and the open swallowtail,* Funct. Anal. Appl., **15** (1981). 235–246. DOI: 10.1007/bf01106152

[2] S. Chino, S. Izumiya, *Lightlike developables in Minkowski 3-space,* Demonstratio Mathematica **43–2** (2010), 387–399. DOI: 10.1515/dema-2013-0236

[3] S. Fujimori, K. Saji, M. Umehara, K. Yamada, *Singularities of maximal surfaces,* Math. Z. **259** (2008), 827–848. DOI: 10.1007/s00209-007-0250-0

[4] A.B. Givental, *Whitney singularities of solutions of partial differential equations,* J. Geom. Phys. **15–4** (1995), 353–368.

[5] G. Ishikawa, *Families of functions dominated by distributions of $\mathscr{C}$-classes of mappings,* Ann. Inst. Fourier, **33-2** (1983), 199–217.

[6] G. Ishikawa, *Developable of a curve and its determinacy relatively to the osculation-type*, Quarterly J. Math., **46** (1995), 437–451.

[7] G. Ishikawa, *Symplectic and Lagrange stabilities of open Whitney umbrellas*, Invent. math., **126–2** (1996), 215–234. DOI: 10.1007/s002220050095

[8] G. Ishikawa, *Singularities of tangent varieties to curves and surfaces,* Journal of Singularities, **6** (2012), 54–83. DOI: 10.5427/jsing.2012.6f

[9] G. Ishikawa, *Tangent varieties and openings of map-germs,* RIMS Kōkyūroku Bessatsu, **B38** (2013), 119–137.

[10] G. Ishikawa, *Openings of differentiable map-germs and unfoldings,* Topics on Real and Complex Singularities, Proc. of the 4th Japanese-Australian Workshop (JARCS4), Kobe 2011, World Scientific (2014), pp. 87–113. DOI: 10.1142/9789814596046_0007

[11] G. Ishikawa, *Singularities of frontals,* in "Singularities in Generic Geometry", Advanced Studies in Pure Mathematics **78**, Math. Soc. Japan., (2018), pp. 55–106. arχiv: 1609.00488 DOI: 10.2969/aspm/07810055

[12] G. Ishikawa, S. Janeczko, *Symplectic bifurcations of plane curves and isotropic liftings,* Quarterly J. of Math. Oxford, **54** (2003), 73–102. DOI: 10.1093/qjmath/54.1.73

[13] G. Ishikawa, Y. Machida, *Singularities of improper affine spheres and surfaces of constant Gaussian curvature,* International Journal of Mathematics, **17–3** (2006), 269–293. DOI: 10.1142/s0129167x06003485

[14] G. Ishikawa, Y. Machida, M. Takahashi, *Asymmetry in singularities of tangent surfaces in contact-cone Legendre-null duality,* Journal of Singularities, **3** (2011), 126–143.

[15] G. Ishikawa, Y. Machida, M. Takahashi, *Singularities of tangent surfaces in Cartan's split G_2-geometry,* Asian Journal of Mathematics, **20–2**, (2016), 353–382. DOI: 10.4310/ajm.2016.v20.n2.a6

[16] G. Ishikawa, Y. Machida, M. Takahashi, *Null frontal singular surfaces in Lorentzian 3-spaces,* in preparation.

[17] G. Ishikawa, T. Yamashita, *Singularities of tangent surfaces to generic space curves,* Journal of Geometry, **108** (2017), 301–318. DOI: 10.1007/s00022-016-0341-3

[18] G. Ishikawa, T. Yamashita, *Singularities of tangent surfaces to directed curves,* Topology and its Applications, **234** (2018), 198–208. DOI: 10.1016/j.topol.2017.11.018

[19] S. Izumiya, K. Saji, M. Takahashi, *Horospherical flat surfaces in Hyperbolic 3-space,* J. Math. Soc. Japan **62-3** (2010), 789–849. DOI: 10.2969/jmsj/06230789

[20] Y. Kabata, *Recognition of plane-to-plane map-germs,* Topology and its Applications, **202-1** (2016), 216–238. DOI: 10.1016/j.topol.2016.01.011

[21] M. Kokubu, W. Rossman, K. Saji, M. Umehara, K. Yamada, *Singularities of flat fronts in hyperbolic space,* Pacific J. of Math. **221-2** (2005), 303–351. DOI: 10.2140/pjm.2005.221.303

[22] J.N. Mather, *Stability of C^∞ mappings III: Finitely determined map-germs,* Publ. Math. I.H.E.S., **35** (1968), 279–308. DOI: 10.1007/bf02698926

[23] J.H. Rieger, *Families of maps from the plane to the plane,* J. London Math. Soc., **36** (1987), 351–369.

[24] K. Saji, *Criteria for singularities of smooth maps from the plane into the plane and their applications,* Hiroshima Math. J. **40** (2010), 229–239. DOI: 10.32917/hmj/1280754423

[25] K. Saji, M. Umehara, K. Yamada, *A_k singularities of wave fronts,* Math. Proc. Camb. Philos. Soc. **146-3** (2009), 731–746. DOI: 10.1017/s0305004108001977

[26] R.W. Sharp, *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program*, Graduate Texts in Mathematics **166**, Springer, (2000).

[27] H. Whitney, *On singularities of mappings of Euclidean spaces I, Mappings of the plane into the plane,* Ann. of Math. **62** (1955), 374–410. DOI: 10.1007/978-1-4612-2972-8_27

GOO ISHIKAWA, FACULTY OF SCIENCE, HOKKAIDO UNIVERSITY, SAPPORO 060-0810, JAPAN
Email address: ishikawa@math.sci.hokudai.ac.jp

Journal *of* Singularities
Volume 21 (2020), 167-186

Proc. of Geometric and Algebraic Singularity Theory, Będlewo, 2017
DOI: 10.5427/jsing.2020.21j

ON FAMILIES OF LAGRANGIAN SUBMANIFOLDS

SHYUICHI IZUMIYA AND MASATOMO TAKAHASHI

Dedicated to Professor Goo Ishikawa on the occasion of his 60th birthday

Abstract. Lagrangian equivalence among Lagrangian submanifolds and $S.P^+$-Legendrian equivalence among graph-like Legendrian unfoldings are equivalent. We investigate r-parameter families of Lagrangian submanifolds and r-parameter families of graph-like Legendrian unfoldings. Then we show that r-parameter families of Lagrangian equivalence and r-parameter families of $S.P^+$-Legendrian equivalence are equivalent. As an application, we give a generic classification of bifurcations of Lagrangian submanifold germs for lower dimensions.

1. Introduction

The study of singularities of caustics and wave fronts was the starting point of the theory of Lagrangian and Legendrian singularities developed by several mathematicians and physicists (cf. [1], [2, 5, 6, 7, 11, 18, 19, 29, 30]). The caustic is described as the set of critical values of the projection of a Lagrangian submanifold from the phase space onto the configuration space. Lagrangian equivalence among Lagrangian submanifold germs in the phase space was introduced for the study of oscillatory integrals on caustics (cf. [1, 4, 8]). By definition, Lagrangian equivalence implies caustic equivalence (i.e. diffeomorphic caustics). However, it has been known that caustic equivalence does not imply Lagrangian equivalence even generically. This is one of the main differences from the theory of Legendrian singularities. In the theory of Legendrian singularities, wave fronts equivalence (i.e. diffeomorphic wave fronts) implies Legendrian equivalence generically. This is the reason why people considered caustic equivalence instead of Lagrangian equivalence in many situations (cf. [1, 24, 30] etc).

On the other hand, the notion of graph-like Legendrian unfoldings was introduced in [9]. It belongs to a special class of the big Legendrian submanifolds which were introduced in [30]. In §2, we give brief reviews on the theories of Lagrangian singularities (cf. [1, 2, 6]), of big Legendrian submanifolds (cf. [20]) and of graph-like Legendrian unfoldings (cf. [21, 22]), respectively. One of the main results in the theory of graph-like Legendrian unfoldings is that Lagrangian equivalence among Lagrangian submanifolds and $S.P^+$-Legendrian equivalence (which was introduced in [10]) among graph-like Legendrian unfoldings are equivalent, see Theorem 2.8 (cf. [13]). It is known that two graph-like Legendrian unfoldings are $S.P^+$-Legendrian equivalent if and only if the corresponding graph-like wave front set germs are $S.P^+$-diffeomorphic generically [13, 14]. In this sense, $S.P^+$-Legendrian equivalence is geometric equivalence. It follows that the hidden relation between caustics and wave front propagations can be investigated and revealed. In

2010 *Mathematics Subject Classification.* 57R45, 53D12, 37J20, 58K40.

Key words and phrases. Lagrangian singularity, Legendrian singularity, graph-like Legendrian unfolding, bifurcation.

The authors would like to thank the referee for helpful comments to improve the original manuscript. The first author was partially supported by JSPS KAKENHI Grant Number JP 26287009 and the second author was partially supported by JSPS KAKENHI Grant Number JP 17K05238.

fact, we give several applications of Lagrangian singularity theory and graph-like Legendrian unfolding theory (cf. [13, 14, 15, 16, 21, 22, 23]).

On the other hand, if we consider r-parameter families of Lagrangian submanifold germs, the situation is not so simple. In [2, 30], V.I. Arnol'd and V.M. Zakalyukin gave a generic classification of bifurcations of caustics and wave fronts, and hence gave a generic classification of bifurcations of Legendrian submanifold germs by Legendrian equivalence. However, they only gave a generic classification of bifurcations of caustics by caustic equivalence. A generic classification of bifurcations of Lagrangian submanifold germs by Lagrangian equivalence has not been given in any contexts as far as the authors know. In this paper, we consider r-parameter families of Lagrangian submanifolds in §3 and r-parameter families of graph-like Legendrian unfoldings in §4, respectively. As a main result, we show that r-parameter Lagrangian equivalence among Lagrangian submanifolds families and r-parameter $S.P^+$-Legendrian equivalence among graph-like Legendrian unfoldings families are equivalent, see Theorem 5.1 in §5. Since $S.P^+$-Legendrian equivalence is geometric equivalence, it is much easier to investigate than Lagrangian equivalence. Therefore, as an application of Theorem 5.1, we give a generic classification of bifurcations of Lagrangian submanifolds by Lagrangian equivalence for lower dimensions, see Theorem 6.1 in §6. There appear functional moduli in the list of the classification even for lower dimensions.

All maps and manifolds considered here are differentiable of class C^∞.

2. Preliminaries

In order to fix the notations for describing the main results, we give brief reviews on the theories of Lagrangian singularities, of big Legendrian submanifolds and of graph-like Legendrian unfoldings, respectively. We also give a relation between the equivalence relations of Lagrangian submanifolds and graph-like Legendrian unfoldings (cf. [13, 16]).

2.1. **Lagrangian singularities.** We consider the cotangent bundle $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$ with the canonical symplectic structure $\omega = \sum_{i=1}^n dp_i \wedge dx_i$, where $(x,p) = (x_1,\dots,x_n,p_1,\dots,p_n)$ is the canonical coordinate on $T^*\mathbb{R}^n$. A submanifold $i : L \subset T^*\mathbb{R}^n$ is said to be a *Lagrangian submanifold* if $\dim L = n$ and $i^*\omega = 0$. The set of the critical values of $\pi \circ i$ is called the *caustic* of $i : L \subset T^*\mathbb{R}^n$, which is denoted by C_L. One of the main results in the theory of Lagrangian singularities is the description of Lagrangian submanifold germs by using families of function germs. For a function germ $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, we say that F is a *Morse family of functions* if the map germ

$$\Delta F = \left(\frac{\partial F}{\partial q_1},\dots,\frac{\partial F}{\partial q_k}\right) : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$$

is non-singular, where $(q,x) = (q_1,\dots,q_k,x_1,\dots,x_n) \in (\mathbb{R}^k \times \mathbb{R}^n, 0)$. In this case, we have a smooth n-dimensional submanifold germ $C(F) = (\Delta F)^{-1}(0) \subset (\mathbb{R}^k \times \mathbb{R}^n, 0)$ and a map germ $L(F) : (C(F), 0) \to T^*\mathbb{R}^n$ defined by

$$L(F)(q,x) = \left(x, \frac{\partial F}{\partial x_1}(q,x),\dots,\frac{\partial F}{\partial x_n}(q,x)\right).$$

We can show that $L(F)(C(F))$ is a Lagrangian submanifold germ. It is known that all Lagrangian submanifold germs in $T^*\mathbb{R}^n$ are constructed by the above method (cf. [2, page 300]).

A Morse family of functions $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ is called a *generating family* of $L(F)(C(F))$. Let $\pi_n : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be the canonical projection, then we can easily show that the critical value set of $\pi_n|_{C(F)}$ is the *bifurcation set* $\mathcal{B}_F$ of F, where

$$\mathcal{B}_F = \left\{x \in (\mathbb{R}^n, 0) \;\middle|\; \text{there exists } q \in (\mathbb{R}^k, 0) \text{ such that } (q,x) \in C(F),\ \operatorname{rank}\left(\frac{\partial^2 F}{\partial q_i \partial q_j}(q,x)\right) < k\right\},$$

so that we have $C_{L(F)(C(F))} = \mathcal{B}_F$.

We now define an equivalence relation among Lagrangian submanifold germs. Let

$$i : (L, x) \subset (T^*\mathbb{R}^n, p) \quad \text{and} \quad i' : (L', x') \subset (T^*\mathbb{R}^n, p')$$

be Lagrangian submanifold germs. Then we say that i and i' are *Lagrangian equivalent* if there exist a diffeomorphism germ $\sigma : (L, x) \to (L', x')$, a symplectic diffeomorphism germ $\hat{\tau} : (T^*\mathbb{R}^n, p) \to (T^*\mathbb{R}^n, p')$ and a diffeomorphism germ $\tau : (\mathbb{R}^n, \pi(p)) \to (\mathbb{R}^n, \pi(p'))$ such that $\hat{\tau} \circ i = i' \circ \sigma$ and $\pi \circ \hat{\tau} = \tau \circ \pi$. Then the caustic C_L is diffeomorphic to the caustic $C_{L'}$ by the diffeomorphism germ τ. However, it has been known that caustic equivalence does not imply Lagrangian equivalence even generically (cf. [2, 12, 16]).

A Lagrangian submanifold germ in $T^*\mathbb{R}^n$ at a point is said to be *Lagrange stable* if for every map with the given germ there is a neighbourhood in the space of Lagrangian submanifolds (in the Whitney C^∞-topology) and a neighbourhood of the original point such that each Lagrangian submanifold belonging to the first neighbourhood has in the second neighbourhood a point at which its germ is Lagrangian equivalent to the original germ.

We can interpret the Lagrangian equivalence by using the notion of generating families. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be function germs. We say that F and G are *P-$\mathcal{R}^+$-equivalent* if there exist a diffeomorphism germ

$$\Phi : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}^k \times \mathbb{R}^n, 0)$$

of the form $\Phi(q, x) = (\phi_1(q, x), \phi_2(x))$ and a function germ $\alpha : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ such that $G(q, x) = F(\Phi(q, x)) + \alpha(x)$. For any $F_1 : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ and $F_2 : (\mathbb{R}^{k'} \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, F_1 and F_2 are said to be *stably P-$\mathcal{R}^+$-equivalent* if they become P-$\mathcal{R}^+$-equivalent after the addition to the arguments q_i of new arguments q'_i and to the functions F_i of non-degenerate quadratic forms Q_i in the new arguments, that is, $F_1 + Q_1$ and $F_2 + Q_2$ are P-$\mathcal{R}^+$-equivalent. Then we have the following theorem (cf. [2, pages 304 and 325]):

Theorem 2.1. *Let $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ and $G : (\mathbb{R}^{k'} \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be Morse families of functions. Then $L(F)(C(F))$ and $L(G)(C(G))$ are Lagrangian equivalent if and only if F and G are stably P-$\mathcal{R}^+$-equivalent.*

2.2. **The theory of wave front propagations.** We consider one-parameter families of wave fronts and their bifurcations. The principal idea is that a one-parameter family of wave fronts is considered to be a wave front whose dimension is one dimension higher than each member of the family. This is called a *big wave front*. We distinguish space and time coordinates, so that we denote $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ and coordinates are denoted by $(x, t) = (x_1, \ldots, x_n, t) \in \mathbb{R}^n \times \mathbb{R}$. Then we consider the projective cotangent bundle $\overline{\pi} : PT^*(\mathbb{R}^n \times \mathbb{R}) \to \mathbb{R}^n \times \mathbb{R}$ over $\mathbb{R}^n \times \mathbb{R}$. Let $\overline{\Pi} : TPT^*(\mathbb{R}^n \times \mathbb{R}) \to PT^*(\mathbb{R}^n \times \mathbb{R})$ be the tangent bundle over $PT^*(\mathbb{R}^n \times \mathbb{R})$ and

$$d\overline{\pi} : TPT^*(\mathbb{R}^n \times \mathbb{R}) \to T(\mathbb{R}^n \times \mathbb{R})$$

the differential map of $\overline{\pi}$. For any $X \in TPT^*(\mathbb{R}^n \times \mathbb{R})$, there exists an element $\alpha \in T^*_{(x,t)}(\mathbb{R}^n \times \mathbb{R})$ such that $\overline{\Pi}(X) = [\alpha]$. For an element $V \in T_{(x,t)}(\mathbb{R}^n \times \mathbb{R})$, the property $\alpha(V) = 0$ does not depend on the choice of representative of the class $[\alpha]$. Thus we can define *the canonical contact structure* on $PT^*(\mathbb{R}^n \times \mathbb{R})$ by $K = \{X \in TPT^*(\mathbb{R}^n \times \mathbb{R}) \mid \overline{\Pi}(X)(d\overline{\pi}(X)) = 0\}$. Because of the trivialization $PT^*(\mathbb{R}^n \times \mathbb{R}) \cong (\mathbb{R}^n \times \mathbb{R}) \times P(\mathbb{R}^n \times \mathbb{R})^*$, we call

$$((x_1, \ldots, x_n, t), [\xi_1 : \cdots : \xi_n : \tau])$$

homogeneous coordinates, where $[\xi_1 : \cdots : \xi_n : \tau]$ are the homogeneous coordinates of the dual projective space $P(\mathbb{R}^n \times \mathbb{R})^*$. It is easy to show that $X \in K_{((x,t),[\xi:\tau])}$ if and only if

$$\sum_{i=1}^{n} \mu_i \xi_i + \lambda \tau = 0,$$

where $d\overline{\pi}(X) = \sum_{i=1}^{n} \mu_i(\partial/\partial x_i) + \lambda(\partial/\partial t)$. We remark that $PT^*(\mathbb{R}^n \times \mathbb{R})$ is a fiberwise compactification of the 1-jet space $J^1(\mathbb{R}^n, \mathbb{R})$ as follows: We consider an affine open subset

$$U_\tau = \{((x,t),[\xi : \tau]) | \tau \neq 0\}$$

of $PT^*(\mathbb{R}^n \times \mathbb{R})$. For any $((x,t),[\xi : \tau]) \in U_\tau$, we have

$$((x_1,\ldots,x_n,t),[\xi_1 : \cdots : \xi_n : \tau]) = ((x_1,\ldots,x_n,t),[-(\xi_1/\tau) : \cdots : -(\xi_n/\tau) : -1]),$$

so that we may adapt the corresponding *affine coordinates* $((x_1,\ldots,x_n,t),(p_1,\ldots,p_n))$, where $p_i = -\xi_i/\tau$. On U_τ we can easily show that $\theta^{-1}(0) = K|U_\tau$, where $\theta = dt - \sum_{i=1}^{n} p_i dx_i$. This means that U_τ may be identified with the 1-jet space $J^1(\mathbb{R}^n, \mathbb{R})$. We set

$$U_\tau = J^1_{GA}(\mathbb{R}^n, \mathbb{R}) \subset PT^*(\mathbb{R}^n \times \mathbb{R}).$$

We call the above coordinate system *a system of graph-like affine coordinates.* Throughout this paper, we use this identification.

A submanifold $i : \mathcal{L} \subset PT^*(\mathbb{R}^n \times \mathbb{R})$ is *a Legendrian submanifold* if $\dim \mathcal{L} = n$ and $di_p(T_p\mathcal{L}) \subset K_{i(p)}$ for any $p \in \mathcal{L}$. We say that a point $p \in \mathcal{L}$ is *a Legendrian singular point* if $\operatorname{rank} d(\overline{\pi} \circ i)_p < n$. For a Legendrian submanifold $i : \mathcal{L} \subset PT^*(\mathbb{R}^n \times \mathbb{R})$, $\overline{\pi} \circ i(\mathcal{L}) = W(\mathcal{L})$ is called *a big wave front.* We have a family of *small fronts*:

$$W_t(\mathcal{L}) = \pi_1(\pi_2^{-1}(t) \cap W(\mathcal{L})) \quad (t \in \mathbb{R}),$$

where $\pi_1 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $\pi_2 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ are the canonical projections defined by $\pi_1(x,t) = x$ and $\pi_2(x,t) = t$ respectively. In this sense, we call $\mathcal{L}$ a *big Legendrian submanifold.*

The *discriminant of the family* $\{W_t(\mathcal{L})\}_{t\in\mathbb{R}}$ is defined as the image of singular points of $\pi_1|_{W(\mathcal{L})}$. In the general case, the discriminant consists of three components: *the caustic* $C_L = \pi_1(\Sigma(W(\mathcal{L})))$, where $\Sigma(W(\mathcal{L}))$ is the set of singular points of $W(\mathcal{L})$ (i.e. the critical value set of the Legendrian mappings $\overline{\pi}|_{\mathcal{L}} = \overline{\pi} \circ i$); *the Maxwell stratified set* $M_{\mathcal{L}}$, the projection of the closure of the self intersection set of $W(\mathcal{L})$; and the critical value set $\Delta_{\mathcal{L}}$ of $\pi_1|_{W(\mathcal{L})\setminus\Sigma(W(\mathcal{L}))}$. In [20, 21, 31], it has been stated that $\Delta_{\mathcal{L}}$ is the *envelope of the family of momentary fronts.* However, we remark that $\Delta_{\mathcal{L}}$ is not necessarily the envelope of the family of the projection of smooth momentary fronts $\overline{\pi}(W_t(\mathcal{L}))$. It may happen that $\pi_2^{-1}(t) \cap W(\mathcal{L})$ is non-singular while $\pi_1|_{\pi_2^{-1}(t)\cap W(\mathcal{L})}$ has singularities, so that $\Delta_{\mathcal{L}}$ is the set of critical values of the family of mappings $\pi_1|_{\pi_2^{-1}(t)\cap W(\mathcal{L})}$ for smooth $\pi_2^{-1}(t) \cap W(\mathcal{L})$ (cf. [12]).

For any Legendrian submanifold germ $i : (\mathcal{L}, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$, there exists a generating family of i by the theory of Legendrian singularities [2]. Let $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be a function germ such that $(\mathcal{F}, d_2\mathcal{F}) : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R} \times \mathbb{R}^k, 0)$ is non-singular, where

$$d_2\mathcal{F}(q,x,t) = \left(\frac{\partial \mathcal{F}}{\partial q_1}(q,x,t), \ldots, \frac{\partial \mathcal{F}}{\partial q_k}(q,x,t) \right).$$

In this case, we call $\mathcal{F}$ a *big Morse family of hypersurfaces.* Then $\Sigma_*(\mathcal{F}) = (\mathcal{F}, d_2\mathcal{F})^{-1}(0)$ is a smooth n-dimensional submanifold germ. Define $\mathcal{L}_{\mathcal{F}} : (\Sigma_*(\mathcal{F}), 0) \to PT^*(\mathbb{R}^n \times \mathbb{R})$ by

$$\mathcal{L}_{\mathcal{F}}(q,x,t) = \left(x, t, \left[\frac{\partial \mathcal{F}}{\partial x}(q,x,t) : \frac{\partial \mathcal{F}}{\partial t}(q,x,t) \right] \right),$$

where
$$\left[\frac{\partial \mathcal{F}}{\partial x}(q,x,t):\frac{\partial \mathcal{F}}{\partial t}(q,x,t)\right]=\left[\frac{\partial \mathcal{F}}{\partial x_1}(q,x,t):\cdots:\frac{\partial \mathcal{F}}{\partial x_n}(q,x,t):\frac{\partial \mathcal{F}}{\partial t}(q,x,t)\right].$$
It is easy to show that $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ is a Legendrian submanifold germ. It is known that all big Legendrian submanifold germs are constructed by the above method (cf. [1,30]). We call $\mathcal{F}$ a *generating family of* $\mathcal{L}_{\mathcal{F}}$. The big wave front coincides with the *discriminant set* $D(\mathcal{F})$ of $\mathcal{F}$, where
$$D(\mathcal{F})=\Big\{(x,t)\in(\mathbb{R}^n\times\mathbb{R},0)\ \Big|\ \text{there exists } q\in(\mathbb{R}^k,0) \text{ such that } (q,x,t)\in\Sigma_*(\mathcal{F})\Big\},$$
so that we have $W(\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F})))=D(\mathcal{F})$.

We now consider an equivalence relation among big Legendrian submanifolds which preserves both the qualitative pictures of bifurcations and the discriminant of families of small fronts. Let $i:(\mathcal{L},p_0)\subset(PT^*(\mathbb{R}^n\times\mathbb{R}),p_0)$ and $i':(\mathcal{L}',p_0')\subset(PT^*(\mathbb{R}^n\times\mathbb{R}),p_0')$ be big Legendrian submanifold germs. We say that i and i' are *strictly parametrized*$^+$ *Legendrian equivalent* (or, briefly, $S.P^+$*-Legendrian equivalent*) if there exist diffeomorphism germs
$$\Phi:(\mathbb{R}^n\times\mathbb{R},\overline{\pi}(p_0))\to(\mathbb{R}^n\times\mathbb{R},\overline{\pi}(p_0'))$$
of the form $\Phi(x,t)=(\phi_1(x),t+\alpha(x))$ and $\Psi:(\mathcal{L},p_0)\to(\mathcal{L}',p_0')$ such that $\widehat{\Phi}\circ i=i\circ\Psi$, where $\widehat{\Phi}:(PT^*(\mathbb{R}^n\times\mathbb{R}),p_0)\to(PT^*(\mathbb{R}^n\times\mathbb{R}),p_0')$ is the unique contact lift of Φ. This equivalence relation was independently introduced in [10, 31] for the different purposes, respectively. We can define the notion of stability of big Legendrian submanifold germs with respect to $S.P^+$-Legendrian equivalence similar to the definition of Lagrangian stability in §2.1 (cf. [2, Part III]). However, we omit to give the definition here.

We study $S.P^+$-Legendrian equivalence by using the notion of generating families of Legendrian submanifold germs. Let $\mathcal{E}_{(q,x,t)}$ be the $\mathbb{R}$-algebra of function germs of (q,x,t)-variables. For function germs $\mathcal{F},\mathcal{G}:(\mathbb{R}^k\times(\mathbb{R}^n\times\mathbb{R}),0)\to(\mathbb{R},0)$, we say that $\mathcal{F}$ and $\mathcal{G}$ are *space-*$S.P^+$*-*$\mathcal{K}$*-equivalent* (or, briefly, s*-*$S.P^+$*-*$\mathcal{K}$*-equivalent*) if there exists a diffeomorphism germ
$$\Phi:(\mathbb{R}^k\times(\mathbb{R}^n\times\mathbb{R}),0)\to(\mathbb{R}^k\times(\mathbb{R}^n\times\mathbb{R}),0)$$
of the form $\Phi(q,x,t)=(\phi(q,x,t),\phi_1(x),t+\alpha(x))$ such that $\langle\mathcal{F}\circ\Phi\rangle_{\mathcal{E}_{(q,x,t)}}=\langle\mathcal{G}\rangle_{\mathcal{E}_{(q,x,t)}}$. The notion of $S.P^+$-$\mathcal{K}$-versal deformation plays an important role for our purpose. We define the extended tangent space of $\overline{f}:(\mathbb{R}^k\times\mathbb{R},0)\to(\mathbb{R},0)$ relative to $S.P^+$-$\mathcal{K}$ by
$$T_e(S.P^+\text{-}\mathcal{K})(\overline{f})=\left\langle\frac{\partial\overline{f}}{\partial q_1},\dots,\frac{\partial\overline{f}}{\partial q_k},\overline{f}\right\rangle_{\mathcal{E}_{(q,t)}}+\left\langle\frac{\partial\overline{f}}{\partial t}\right\rangle_{\mathbb{R}}.$$

We say that $\mathcal{F}$ is an $S.P^+$*-*$\mathcal{K}$*-versal deformation* of $\overline{f}=\mathcal{F}|_{\mathbb{R}^k\times\{0\}\times\mathbb{R}}$ if it satisfies
$$\mathcal{E}_{(q,t)}=T_e(S.P^+\text{-}\mathcal{K})(\overline{f})+\left\langle\frac{\partial\mathcal{F}}{\partial x_1}|_{\mathbb{R}^k\times\{0\}\times\mathbb{R}},\dots,\frac{\partial\mathcal{F}}{\partial x_n}|_{\mathbb{R}^k\times\{0\}\times\mathbb{R}}\right\rangle_{\mathbb{R}}.$$
Then we also have the following theorem.

Theorem 2.2. *Let* $\mathcal{F}:(\mathbb{R}^k\times(\mathbb{R}^n\times\mathbb{R}),0)\to(\mathbb{R},0)$ *and* $\mathcal{G}:(\mathbb{R}^{k'}\times(\mathbb{R}^n\times\mathbb{R}),0)\to(\mathbb{R},0)$ *be big Morse families of hypersurfaces.*
(1) $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ *and* $\mathcal{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G}))$ *are* $S.P^+$*-Legendrian equivalent if and only if* $\mathcal{F}$ *and* $\mathcal{G}$ *are stably* s*-*$S.P^+$*-*$\mathcal{K}$*-equivalent.*
(2) $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ *is* $S.P^+$*-Legendre stable if and only if* $\mathcal{F}$ *is an* $S.P^+$*-*$\mathcal{K}$*-versal deformation of* $\overline{f}=\mathcal{F}|_{\mathbb{R}^k\times\{0\}\times\mathbb{R}}$.

Since the big Legendrian submanifold germ $i : (\mathcal{L}, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$ is uniquely determined on the regular part of the big wave front $W(\mathcal{L})$, we have the following simple but significant property of Legendrian submanifold germs:

Proposition 2.3. *Let $i : (\mathcal{L}, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$ and $i' : (\mathcal{L}', p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$ be big Legendrian submanifold germs such that $\overline{\pi} \circ i, \overline{\pi} \circ i'$ are proper map germs and the regular sets of these map germs are dense respectively. Then $(\mathcal{L}, p_0) = (\mathcal{L}', p_0)$ if and only if*

$$(W(\mathcal{L}), \overline{\pi}(p_0)) = (W(\mathcal{L}'), \overline{\pi}(p_0)).$$

This result has been firstly pointed out by Zakalyukin [30]. Also see [25]. The assumption in the above proposition is a generic condition for i, i'. In particular, if i and i' are $S.P^+$-Legendre stable, then these satisfy the assumption.

Concerning the discriminant and the bifurcation of momentary fronts, we define the following equivalence relation among big wave front germs. Let $i : (\mathcal{L}, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$ and $i' : (\mathcal{L}', p_0') \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0')$ be big Legendrian submanifold germs. We say that $W(\mathcal{L})$ and $W(\mathcal{L}')$ are *$S.P^+$-diffeomorphic* if there exists a diffeomorphism germ

$$\Phi : (\mathbb{R}^n \times \mathbb{R}, \overline{\pi}(p_0)) \to (\mathbb{R}^n \times \mathbb{R}, \overline{\pi}(p_0'))$$

of the form $\Phi(x, t) = (\phi_1(x), t + \alpha(x))$ such that $\Phi(W(\mathcal{L})) = W(\mathcal{L}')$. Remark that the $S.P^+$-diffeomorphism among big wave front germs preserves the diffeomorphism types of discriminants [31]. By Proposition 2.3, we have the following proposition.

Proposition 2.4. *Let $i : (\mathcal{L}, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$ and $i' : (\mathcal{L}', p_0') \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0')$ be big Legendrian submanifold germs such that $\overline{\pi} \circ i, \overline{\pi} \circ i'$ are proper map germs and the regular sets of those map germs are dense respectively. Then i and i' are $S.P^+$-Legendrian equivalent if and only if $(W(\mathcal{L}), \overline{\pi}(p_0))$ and $(W(\mathcal{L}'), \overline{\pi}(p_0'))$ are $S.P^+$-diffeomorphic.*

2.3. **Graph-like Legendrian unfoldings.** In this subsection we explain the theory of graph-like Legendrian unfoldings. Graph-like Legendrian unfoldings belong to a special class of big Legendrian submanifolds. A big Legendrian submanifold $i : \mathcal{L} \subset PT^*(\mathbb{R}^n \times \mathbb{R})$ is said to be a *graph-like Legendrian unfolding* if $\mathcal{L} \subset J^1_{GA}(\mathbb{R}^n, \mathbb{R})$.
We call $W(\mathcal{L}) = \overline{\pi}(\mathcal{L})$ a *graph-like wave front* of $\mathcal{L}$, where $\overline{\pi} : J^1_{GA}(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}^n \times \mathbb{R}$ is the canonical projection. We define the mapping $\Pi : J^1_{GA}(\mathbb{R}^n, \mathbb{R}) \to T^*\mathbb{R}^n$ by $\Pi(x, t, p) = (x, p)$, where $(x, t, p) = (x_1, \dots, x_n, t, p_1, \dots, p_n)$ and the canonical contact form on $J^1_{GA}(\mathbb{R}^n, \mathbb{R})$ is given by $\theta = dt - \Sigma_{i=1}^n p_i dx_i$. Then we have the following proposition.

Proposition 2.5 ([12])**.** *For a graph-like Legendrian unfolding $\mathcal{L} \subset J^1_{GA}(\mathbb{R}^n, \mathbb{R})$, $z \in \mathcal{L}$ is a singular point of $\overline{\pi}|_{\mathcal{L}} : \mathcal{L} \to \mathbb{R}^n \times \mathbb{R}$ if and only if it is a singular point of $\pi_1 \circ \overline{\pi}|_{\mathcal{L}} : \mathcal{L} \to \mathbb{R}^n$. Moreover, $\Pi|_{\mathcal{L}} : \mathcal{L} \to T^*\mathbb{R}^n$ is immersive, so that $\Pi(\mathcal{L})$ is a Lagrangian submanifold in $T^*\mathbb{R}^n$.*

We have the following corollary of Proposition 2.5.

Corollary 2.6 ([12])**.** *For a graph-like Legendrian unfolding $\mathcal{L} \subset J^1_{GA}(\mathbb{R}^n, \mathbb{R})$, $\Delta_{\mathcal{L}}$ is the empty set so that the discriminant of the family of momentary fronts is $C_{\mathcal{L}} \cup M_{\mathcal{L}}$.*

Since $\mathcal{L}$ is a big Legendrian submanifold in $PT^*(\mathbb{R}^n \times \mathbb{R})$, it has a generating family

$$\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$$

at least locally. Since $\mathcal{L} \subset J^1_{GA}(\mathbb{R}^n, \mathbb{R}) = U_\tau \subset PT^*(\mathbb{R}^n \times \mathbb{R})$, it satisfies the condition $(\partial \mathcal{F}/\partial t)(0) \neq 0$. Let $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be a big Morse family of hypersurfaces. We say that $\mathcal{F}$ is a *graph-like Morse family of hypersurfaces* if $(\partial \mathcal{F}/\partial t)(0) \neq 0$. It is easy to show that the corresponding big Legendrian submanifold germ is a graph-like Legendrian unfolding. Of course, all graph-like Legendrian unfolding germs can be constructed by the above way. We also say that $\mathcal{F}$ is a *graph-like generating family* of $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$. We remark that the notion of graph-like Legendrian unfoldings and corresponding generating families have been introduced by the first named author in [9] to describe the perestroikas of wave fronts given as the solutions for general eikonal equations.

We can consider the following more restrictive class of graph-like generating families: Let $\mathcal{F}$ be a graph-like Morse family of hypersurfaces. By the implicit function theorem, there exists a function $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ such that $\langle \mathcal{F}(q,x,t)\rangle_{\mathcal{E}_{(q,x,t)}} = \langle F(q,x) - t\rangle_{\mathcal{E}_{(q,x,t)}}$. Then we have the following proposition.

Proposition 2.7 ([22]). *Let $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ and $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be function germs such that $\langle \mathcal{F}(q,x,t)\rangle_{\mathcal{E}_{(q,x,t)}} = \langle F(q,x) - t\rangle_{\mathcal{E}_{(q,x,t)}}$. Then $\mathcal{F}$ is a graph-like Morse family of hypersurfaces if and only if F is a Morse family of functions.*

We now consider the case $\mathcal{F}(q,x,t) = \lambda(q,x,t)(F(q,x) - t)$, for $\lambda(0) \neq 0$. In this case,

$$\Sigma_*(\mathcal{F}) = \{(q,x,F(q,x)) \in (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \mid (q,x) \in C(F)\},$$

where $C(F) = \Delta F^{-1}(0)$. Moreover, we have the Lagrangian submanifold germ

$$L(F)(C(F)) \subset T^*\mathbb{R}^n,$$

where $L(F)$ is defined by

$$L(F)(q,x) = \left(x, \frac{\partial F}{\partial x_1}(q,x), \ldots, \frac{\partial F}{\partial x_n}(q,x)\right).$$

Since $\mathcal{F}$ is a graph-like Morse family of hypersurfaces, we have a big Legendrian submanifold germ $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F})) \subset J^1_{GA}(\mathbb{R}^n, \mathbb{R})$, where $\mathcal{L}_{\mathcal{F}} : (\Sigma_*(\mathcal{F}), 0) \to J^1_{GA}(\mathbb{R}^n, \mathbb{R}) = T^*\mathbb{R}^n \times \mathbb{R}$ is defined by

$$\mathcal{L}_{\mathcal{F}}(q,x,t) = \left(x, t, -\frac{\frac{\partial \mathcal{F}}{\partial x_1}(q,x,t)}{\frac{\partial \mathcal{F}}{\partial t}(q,x,t)}, \ldots, -\frac{\frac{\partial \mathcal{F}}{\partial x_n}(q,x,t)}{\frac{\partial \mathcal{F}}{\partial t}(q,x,t)}\right).$$

We also define $\mathfrak{L}_F : (C(F), 0) \to J^1_{GA}(\mathbb{R}^n, \mathbb{R})$ by

$$\mathfrak{L}_F(q,x) = \left(x, F(q,x), \frac{\partial F}{\partial x_1}(q,x), \ldots, \frac{\partial F}{\partial x_n}(q,x)\right).$$

Since $\partial\mathcal{F}/\partial x_i = (\partial\lambda/\partial x_i)(F - t) + \lambda\partial F/\partial x_i$ and $\partial\mathcal{F}/\partial t = (\partial\lambda/\partial t)(F - t) - \lambda$, we have

$$(\partial\mathcal{F}/\partial x_i)(q,x,t) = \lambda(q,x,t)(\partial F/\partial x_i)(q,x,t)$$

and

$$(\partial\mathcal{F}/\partial t)(q,x,t) = -\lambda(q,x,t)$$

for $(q,x,t) \in \Sigma_*(\mathcal{F})$. It follows that $\mathfrak{L}_F(C(F)) = \mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$. By definition, we have

$$\Pi(\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))) = \Pi(\mathfrak{L}_F(C(F))) = L(F)(C(F)).$$

The graph-like wave front of $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F})) = \mathfrak{L}_F(C(F))$ is the graph of $F|_{C(F)}$. This is the reason why we call it a graph-like Legendrian unfolding.

For a graph-like Morse family of hypersurfaces $\mathcal{F}(q,x,t) = \lambda(q,x,t)(F(q,x)-t)$, $\mathcal{F}(q,x,t)$ and $\overline{F}(q,x,t) = F(q,x) - t$ are s-$S.P^+$-$\mathcal{K}$-equivalent, so that we consider $\overline{F}(q,x,t) = F(q,x) - t$ as a graph-like Morse family of hypersurfaces. Since $\overline{F}(q,x,t)$ is a big Morse family, we can use all the definitions of equivalence relations in §2.2. Moreover, we can translate the propositions and theorems into corresponding assertions in terms of graph-like Legendrian unfoldings. We can also consider the stability of graph-like Legendrian unfolding with respect to $S.P^+$-Legendrian equivalence which is analogous to the stability of Lagrangian submanifold germs with respect to Lagrangian equivalence in §2.1.

2.4. **Equivalence relations.** We consider a relation between the equivalence relations of Lagrangian submanifold germs and of graph-like Legendrian unfoldings (cf. [9, 10, 16, 20, 21, 31]).

Theorem 2.8 ([13]). *Let*

$$\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0) \quad \text{and} \quad \mathcal{G} : (\mathbb{R}^{k'} \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$$

be graph-like Morse families of hypersurfaces of the forms $\mathcal{F}(q, x, t) = \lambda(q, x, t)(F(q, x) - t)$ *and* $\mathcal{G}(q', x, t) = \mu(q', x, t)(G(q', x) - t)$. *Then Lagrangian submanifold germs* $L(F)(C(F))$ *and* $L(G)(C(G))$ *are Lagrangian equivalent if and only if the graph-like Legendrian unfoldings* $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ *and* $\mathcal{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G}))$ *are* $S.P^+$*-Legendrian equivalent.*

By definition, the set of Legendrian singular points of the graph-like Legendrian unfolding $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ coincides with the set of singular points of $\pi \circ L(F)$. Therefore the singularities of graph-like wave front of $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ lie on the caustic of $L(F)$. It follows that we can apply Proposition 2.4 to $S.P^+$-Legendrian equivalence. We have the following direct corollaries of Theorem 2.8.

Corollary 2.9. *With the same notations as those in Theorem 2.8, suppose that* $\overline{\pi} \circ \mathcal{L}_{\mathcal{F}}, \overline{\pi} \circ \mathcal{L}_{\mathcal{G}}$ *are proper map germs and the regular sets of these map germs are dense respectively. Then Lagrangian submanifold germs* $L(F)(C(F))$ *and* $L(G)(C(G))$ *are Lagrangian equivalent if and only if* $W(\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F})))$ *and* $W(\mathcal{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G})))$ *are* $S.P^+$*-diffeomorphic.*

Corollary 2.10. *Suppose that* $\mathcal{F}(q, x, t) = \lambda(q, x, t)(F(q, x) - t)$ *is a graph-like Morse family of hypersurfaces. Then* $L(F)(C(F))$ *is Lagrange stable if and only if* $\mathcal{L}(\Sigma_*(\mathcal{F}))$ *is* $S.P^+$*-Legendre stable.*

3. Families of Lagrangian submanifolds

We say that $i_r : L \times \mathbb{R}^r \subset T^*\mathbb{R}^n$ is an *r-parameter family of Lagrangian submanifolds* if $i|_{L \times \{s\}} : L \times \{s\} \subset T^*\mathbb{R}^n$ is a Lagrangian submanifold for each $s = (s_1, \dots, s_r) \in \mathbb{R}^r$. By the theory of Lagrangian singularity in §2.1, we have a Morse family of functions. Let

$$F : (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}, 0), (q, x, s) \to F(q, x, s)$$

be an r-parameter family of Morse families of functions, that is, for each fixed $s \in (\mathbb{R}^r, 0)$, $F_s(q, x) = F(q, x, s)$ is a Morse family of functions and it depends smoothly on s.

We consider the cotangent bundle $\pi_r : T^*(\mathbb{R}^n \times \mathbb{R}^r) \to \mathbb{R}^n \times \mathbb{R}^r$ over $\mathbb{R}^n \times \mathbb{R}^r$. Let

$$(x, s, p, u) = (x_i, s_j, p_i, u_j),\ i = 1, \dots, n, j = 1, \dots, r$$

be the canonical coordinates on $T^*(\mathbb{R}^n \times \mathbb{R}^r)$. Then the canonical symplectic structure on $T^*(\mathbb{R}^n \times \mathbb{R}^r)$ is given by the canonical 2-form $\omega_r = \sum_{i=1}^n dp_i \wedge dx_i + \sum_{j=1}^r du_j \wedge ds_j$. We denote the canonical projection by $\widetilde{\pi}_r : T^*(\mathbb{R}^n \times \mathbb{R}^r) \to T^*\mathbb{R}^n$.

Let $F : (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}, 0), (q, x, s) \mapsto F(q, x, s)$ be an r-parameter family of Morse families of functions. Then it is also a Morse family of functions as an $(n+r)$-parameter family of function germs. Therefore we have a Lagrangian submanifold germ $L(F)(C(F)) \subset T^*(\mathbb{R}^n \times \mathbb{R}^r)$, where $L(F) : (C(F), 0) \to T^*(\mathbb{R}^n \times \mathbb{R}^r)$ is defined in §2.1. Moreover, $\widetilde{\pi}_r \circ L(F)(C(F)) \subset T^*\mathbb{R}^n$ is an r-parameter family of Lagrangian submanifold germs. We call $L(F)(C(F))$ a *big Lagrangian submanifold germ.*

Let $i_r : (L \times \mathbb{R}^r, (x, 0)) \subset (T^*(\mathbb{R}^n \times \mathbb{R}^r), p)$ and $i'_r : (L' \times \mathbb{R}^r, (x', 0)) \subset (T^*(\mathbb{R}^n \times \mathbb{R}^r), p')$ be big Lagrangian submanifold germs. We say that i_r and i'_r are *r-parameter Lagrangian equivalent* (or, briefly, *r-Lagrangian equivalent*) if there exist a diffeomorphism germ

$$\sigma : (L \times \mathbb{R}^r, (x, 0)) \to (L' \times \mathbb{R}^r, (x', 0))$$

of the form $\sigma(u,s) = (\sigma_1(u,s), \varphi(s))$, a symplectic diffeomorphism germ

$$\hat{\tau} : (T^*(\mathbb{R}^n \times \mathbb{R}^r), p) \to (T^*(\mathbb{R}^n \times \mathbb{R}^r), p')$$

and a diffeomorphism germ $\tau : (\mathbb{R}^n \times \mathbb{R}^r, \pi(p)) \to (\mathbb{R}^n \times \mathbb{R}^r, \pi(p'))$ of the form

$$\tau(x,s) = (\tau_1(x,s), \varphi(s))$$

such that $\hat{\tau} \circ i_r = i'_r \circ \sigma$ and $\pi_r \circ \hat{\tau} = \tau \circ \pi_r$.

Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}, 0)$ be function germs. We say that F and G are *P-$\mathcal{R}^+$-equivalent as r-parameter families* (or, briefly, *r-P-$\mathcal{R}^+$-equivalent*) if there exist a diffeomorphism germ $\Phi : (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^r, 0)$ of the form $\Phi(q,x,s) = (\phi_1(q,x,s), \phi_2(x,s), \varphi(s))$ and a function germ $\alpha : (\mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}, 0)$ such that $G(q,x,s) = F(\Phi(q,x,s)) + \alpha(x,s)$. Then we also have the following theorem.

Theorem 3.1. *Let $F : (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}, 0)$ and $G : (\mathbb{R}^{k'} \times \mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}, 0)$ be r-parameter families of Morse families of functions. Then $L(F)(C(F))$ and $L(G)(C(G))$ are r-Lagrangian equivalent if and only if F and G are stably r-P-$\mathcal{R}^+$-equivalent.*

We also consider the stability of r-parameter families of Lagrangian submanifolds with respect to r-Lagrangian equivalence.

4. Families of graph-like Legendrian unfoldings

A big Legendrian submanifold

$$i : \mathcal{L} \times \mathbb{R}^r \subset PT^*(\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R})$$

is said to be an *r-parameter family of graph-like Legendrian unfoldings* if

$$\mathcal{L} \times \mathbb{R}^r \subset J^1_{GA}(\mathbb{R}^n \times \mathbb{R}^r, \mathbb{R}).$$

We call $W(\mathcal{L} \times \mathbb{R}^r) = \overline{\pi}_r(\mathcal{L} \times \mathbb{R}^r)$ an *r-parameter family of graph-like wave fronts* of $\mathcal{L} \times \mathbb{R}^r$, where $\overline{\pi}_r : J^1_{GA}(\mathbb{R}^n \times \mathbb{R}^r, \mathbb{R}) \to \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}$ is the canonical projection. By the theory of Legendrian singularity in §2.3, we have a graph-like Legendrian unfolding corresponding to the family of graph-like Legendrian unfoldings. Let

$$\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}), 0) \to (\mathbb{R}, 0), (q,x,s,t) \to \mathcal{F}(q,x,s,t)$$

be an r-parameter family of graph-like Morse families of hypersurfaces, that is, for each fixed $s \in (\mathbb{R}^r, 0)$, $\mathcal{F}_s(q,x,t) = \mathcal{F}(q,x,s,t)$ is a graph-like Morse family of hypersurfaces and it depends smoothly on s.

Let

$$i : (\mathcal{L} \times \mathbb{R}^r, (p,0)) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}), p_0) \quad \text{and} \quad i' : (\mathcal{L}' \times \mathbb{R}^r, (p',0)) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}), p'_0)$$

be Legendrian submanifold germs. We say that i and i' are *r-parameter $S.P^+$-Legendrian equivalent* (or, briefly *r-$S.P^+$-Legendrian equivalent*) if there exist diffeomorphism germs

$$\Phi : (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}, \overline{\pi}_r(p_0)) \to (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}, \overline{\pi}_r(p'_0))$$

of the form $\Phi(x,s,t) = (\phi_1(x,s), \varphi(s), t + \alpha(x,s))$ and $\Psi : (\mathcal{L} \times \mathbb{R}^r, p_0) \to (\mathcal{L}' \times \mathbb{R}^r, p'_0)$ of the form $\Psi(u,s) = (\psi_1(u,s), \varphi(s))$ such that $\widehat{\Phi} \circ i = i \circ \Psi$, where

$$\widehat{\Phi} : (PT^*(\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}), p_0) \to (PT^*(\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}), p'_0)$$

is the unique contact lift of Φ.

Let $\mathcal{F},\mathcal{G} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be function germs. We say that $\mathcal{F}$ and $\mathcal{G}$ are *r-parameter s-$S.P^+$-$\mathcal{K}$-equivalent* (or, briefly, *r-s-$S.P^+$-$\mathcal{K}$-equivalent*) if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}), 0) \to (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}), 0)$ of the form

$$\Phi(q, x, s, t) = (\phi(q, x, s, t), \phi_1(x, s), \varphi(s), t + \alpha(x, s))$$

such that $\langle \mathcal{F} \circ \Phi\rangle_{\mathcal{E}_{(q,x,s,t)}} = \langle \mathcal{G}\rangle_{\mathcal{E}_{(q,x,s,t)}}$. Then we also have the following theorem.

Theorem 4.1. *Let $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ and $\mathcal{G} : (\mathbb{R}^{k'} \times (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be r-parameter families of graph-like Legendrian unfoldings. Then $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ and $\mathcal{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G}))$ are r-$S.P^+$-Legendrian equivalent if and only if $\mathcal{F}$ and $\mathcal{G}$ are stably r-s-$S.P^+$-$\mathcal{K}$-equivalent.*

We also consider the stability of r-parameter families of graph-like Legendrian unfoldings with respect to r-$S.P^+$-Legendrian equivalence.

5. Relations between equivalence relations

We consider a relation of the r-parameter version of equivalence relations between r-parameter families of Lagrangian submanifolds and r-parameter families of graph-like Legendrian unfoldings. One of the main results in this paper is as follows:

Theorem 5.1. *Let $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ and $\mathcal{G} : (\mathbb{R}^{k'} \times (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be r-parameter families of graph-like Morse families of hypersurfaces of the forms*

$$\mathcal{F}(q, x, s, t) = \lambda(q, x, s, t)(F(q, x, s) - t) \quad \text{and} \quad \mathcal{G}(q', x, s, t) = \mu(q', x, s, t)(G(q', x, s) - t).$$

Then r-parameter families of Lagrangian submanifold germs $L(F)(C(F))$ and $L(G)(C(G))$ are r-Lagrangian equivalent if and only if the r-parameter families of graph-like Legendrian unfoldings $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ and $\mathcal{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G}))$ are r-$S.P^+$-Legendrian equivalent.

Proof. By Theorem 3.1, if $L(F)(C(F))$ and $L(G)(C(G))$ are r-Lagrangian equivalent, then F and G are stably r-P-$\mathcal{R}^+$-equivalent. In this case, we may assume that $k = k'$, F and G are r-P-$\mathcal{R}^+$-equivalent, so that there exist a diffeomorphism germ $\Phi : (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^r, 0)$ of the form $\Phi(q, x, s) = (\phi_1(q, x, s), \phi_2(x, s), \varphi(s))$ and a function germ $\alpha : (\mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}, 0)$ such that $G(q, x, s) = F(\Phi(q, x, s)) + \alpha(x, s)$. Then we define the diffeomorphism germ

$$\widetilde{\Phi} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}), 0) \to (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}), 0)$$

by $\widetilde{\Phi}(q, x, s, t) = (\phi_1(q, x, s), \phi_2(x, s), \varphi(s), t - \alpha(x, s))$. It follows that

$$\overline{G}(q, x, s, t) = G(q, x, s) - t = F \circ \Phi(q, x, s) - t + \alpha(x, s) = \overline{F} \circ \widetilde{\Phi}(q, x, s, t).$$

This means that $\mathcal{F}$ and $\mathcal{G}$ are r-s-$S.P^+$-$\mathcal{K}$-equivalent. By Theorem 4.1, $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ and $\mathcal{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G}))$ are r-$S.P^+$-Legendrian equivalent.

Conversely, we assume that $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ and $\mathcal{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G}))$ are r-$S.P^+$-Legendrian equivalent. Since $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F})) = \mathfrak{L}_F(C(F))$, $\mathcal{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G})) = \mathfrak{L}_G(C(G))$, it follows from the assumption that there exist diffeomorphism germs $\Phi : (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}, 0)$ of the form

$$\Phi(x, s, t) = (\phi_1(x, s), \varphi(s), t + \alpha(x, s))$$

and $\Psi : (C(F), 0) \to (C(G), 0)$ of the form $\Psi(u, s) = (\psi_1(u, s), \varphi(s))$ such that

$$\widehat{\Phi}(\mathfrak{L}_F(C(F))) = \mathfrak{L}_G(C(G) \circ \Psi).$$

Then we have $\Phi^{-1}(x, s, t) = (\phi_1^{-1}(x, s), \varphi^{-1}(s), t - \alpha(x, s))$, where $\phi_1^{-1} : (\mathbb{R}^n \times \mathbb{R}^s, 0) \to (\mathbb{R}^n, 0)$ satisfies the condition $\phi_1^{-1}(\phi_1(x, s), \varphi(s)) = x$ and $\alpha(x, s)$ means $\alpha(\phi_1^{-1}(x, s), \varphi^{-1}(s))$.

Therefore, the Jacobi matrix of Φ^{-1} at $\Phi(x,s,t)$ is given by

$$J_{\Phi(x,s,t)}\Phi^{-1} = \begin{pmatrix} \frac{\partial \phi_1^{-1}}{\partial x}(\phi_1(x,s),\varphi(s)) & \frac{\partial \phi_1^{-1}}{\partial s}(\phi_1(x,s),\varphi(s)) & 0 \\ 0 & \frac{\partial \varphi^{-1}}{\partial s}(\varphi(s)) & 0 \\ -\frac{\partial \alpha}{\partial x}(\phi_1(x,s),\varphi(s)) & -\frac{\partial \alpha}{\partial s}(\phi_1(x,s),\varphi(s)) & 1 \end{pmatrix}.$$

It follows that

$$\begin{aligned}\hat{\Phi}(x,s,t,[p:u:\tau]) = \Big(\Phi(x,s,t), \Big[p\cdot \frac{\partial \phi_1^{-1}}{\partial x}(\phi_1(x,s),\varphi(s)) - \tau\frac{\partial \alpha}{\partial x}(\phi_1(x,s),\varphi(s)) : \\ p\cdot \frac{\partial \phi_1^{-1}}{\partial s}(\phi_1(x,s),\varphi(s)) + u\cdot\frac{\partial \varphi^{-1}}{\partial s}(\varphi(s)) - \tau\frac{\partial \alpha}{\partial s}(\phi_1(x,s),\varphi(s)) : \tau\Big]\Big).\end{aligned}$$

Since $\tau \neq 0$, we have

$$\begin{aligned}&\Big[p\cdot \frac{\partial \phi_1^{-1}}{\partial x}(\phi_1(x,s),\varphi(s)) - \tau\frac{\partial \alpha}{\partial x}(\phi_1(x,s),\varphi(s)) : \\ &\quad p\cdot \frac{\partial \phi_1^{-1}}{\partial s}(\phi_1(x,s),\varphi(s)) + u\cdot\frac{\partial \varphi^{-1}}{\partial s}(\varphi(s)) - \tau\frac{\partial \alpha}{\partial s}(\phi_1(x,s),\varphi(s)) : \tau\Big] \\ &= \Big[-\frac{p}{\tau}\cdot \frac{\partial \phi_1^{-1}}{\partial x}(\phi_1(x,s),\varphi(s)) + \frac{\partial \alpha}{\partial x}(\phi_1(x,s),\varphi(s)) : \\ &\quad -\frac{p}{\tau}\cdot \frac{\partial \phi_1^{-1}}{\partial s}(\phi_1(x,s),\varphi(s)) - \frac{u}{\tau}\cdot\frac{\partial \varphi^{-1}}{\partial s}(\varphi(s)) + \frac{\partial \alpha}{\partial s}(\phi_1(x,s),\varphi(s)) : -1\Big].\end{aligned}$$

We consider the graph-like affine coordinates $(x,s,t,p,u) \in J^1_{GA}(\mathbb{R}^n\times\mathbb{R}^r,\mathbb{R})$, where we denote again $-p/\tau$ by p and $-u/\tau$ by u, respectively. By the form of $\hat{\Phi}$, we have

$$\hat{\Phi}(J^1_{GA}(\mathbb{R}^n\times\mathbb{R}^r,\mathbb{R})) = J^1_{GA}(\mathbb{R}^n\times\mathbb{R}^r,\mathbb{R}).$$

We define $\widetilde{\Phi} : T^*(\mathbb{R}^n\times\mathbb{R}^r) \to T^*(\mathbb{R}^n\times\mathbb{R}^r)$ by

$$\widetilde{\Phi}(x,s,p,u) = (\phi_1(x,s), \varphi(s), \phi_2(x,s,p), \phi_3(x,s,p,u)),$$

where

$$\begin{aligned}\phi_2(x,s,p) &= p\cdot \frac{\partial \phi_1^{-1}}{\partial x}(\phi_1(x,s),\varphi(s)) + \frac{\partial \alpha}{\partial x}(\phi_1(x,s),\varphi(s)), \\ \phi_3(x,s,p,u) &= p\cdot \frac{\partial \phi_1^{-1}}{\partial s}(\phi_1(x,s),\varphi(s)) + u\cdot\frac{\partial \varphi^{-1}}{\partial s}(\varphi(s)) + \frac{\partial \alpha}{\partial s}(\phi_1(x,s),\varphi(s)).\end{aligned}$$

Since $\hat{\Phi}$ is a contact diffeomorphism germ, there exists a non-zero function germ $\lambda : J^1_{GA}(\mathbb{R}^n\times\mathbb{R}^r,\mathbb{R}) \to \mathbb{R}$ such that $\hat{\Phi}^*\theta = \lambda\theta$, where $\theta = dt - \sum_{i=1}^n p_i dx_i - \sum_{j=1}^r u_j ds_j$. Therefore, we have

$$dt + d\alpha - \phi_2\cdot d\phi_1 - \phi_3\cdot d\varphi = \lambda(dt - p\cdot dx - u\cdot ds).$$

It follows that $\lambda = 1$ and

$$d\alpha - \phi_2\cdot d\phi_1 - \phi_3\cdot d\varphi = -p\cdot dx - u\cdot ds.$$

If we set $\overline{\theta} = -\sum_{i=1}^n p_i dx_i - \sum_{j=1}^r u_j ds_j$, then

$$\widetilde{\Phi}^*\omega = \widetilde{\Phi}^* d\overline{\theta} = d\widetilde{\Phi}^*\overline{\theta} = d(-d\alpha + \overline{\theta}) = -d(d\alpha) + d\overline{\theta} = \omega.$$

This means that $\widetilde{\Phi}$ is a symplectic diffeomorphism germ. Since

$$\Pi_r \circ \hat{\Phi}|_{J^1_{GA}(\mathbb{R}^n\times\mathbb{R}^r,\mathbb{R})} = \widetilde{\Phi}\circ\Pi_r|_{J^1_{GA}(\mathbb{R}^n\times\mathbb{R}^r,\mathbb{R})},$$

we have

$$\begin{aligned} L(G)(C(G)\circ\Psi) &= \Pi_r(\mathfrak{L}_G(C(G)\circ\Psi)) = \Pi_r\circ\hat{\Phi}(\mathfrak{L}_F(C(F))) \\ &= \widetilde{\Phi}\circ\Pi_r(\mathcal{L}_F(C(F))) = \widetilde{\Phi}\circ L(F)(C(F)), \end{aligned}$$

where

$$\Pi_r : J^1_{GA}(\mathbb{R}^n\times\mathbb{R}^r,\mathbb{R}) \to T^*(\mathbb{R}^n\times\mathbb{R}^r)$$

is the canonical projection $\Pi_r(x,s,t,p,u) = (x,s,p,u)$. It follows that $L(F)(C(F))$ and $L(G)(C(G))$ are r-Lagrangian equivalent. This completes the proof. □

Let $i : (\mathcal{L}\times\mathbb{R}^r, p_0) \subset J^1_{GA}(\mathbb{R}^n\times\mathbb{R}^r,\mathbb{R})$ and $i' : (\mathcal{L}'\times\mathbb{R}^r, p_0') \subset J^1_{GA}(\mathbb{R}^n\times\mathbb{R}^r,\mathbb{R})$ be r-parameter families of graph-like Legendrian unfoldings. We say that $W(\mathcal{L}\times\mathbb{R}^r)$ and $W(\mathcal{L}'\times\mathbb{R}^r)$ are *r-S.P$^+$-diffeomorphic* if there exists a diffeomorphism germ

$$\Phi : (\mathbb{R}^n\times\mathbb{R}^r\times\mathbb{R}, \overline{\pi}(p_0)) \to (\mathbb{R}^n\times\mathbb{R}^r\times\mathbb{R}, \overline{\pi}(p_0'))$$

of the form $\Phi(x,s,t) = (\phi_1(x,s),\varphi(s),t+\alpha(x,s))$ such that $\Phi(W(\mathcal{L}\times\mathbb{R}^r)) = W(\mathcal{L}'\times\mathbb{R}^r)$. We have the following direct corollaries of Theorem 5.1.

Corollary 5.2. *With the same notations as those in Theorem 5.1, suppose that*

$$\overline{\pi}_r\circ\mathcal{L}_{\mathcal{F}} \quad \text{and} \quad \overline{\pi}_r\circ\mathcal{L}_{\mathcal{G}}$$

are proper map germs and the regular sets of these map germs are dense respectively. Then r-parameter families of Lagrangian submanifold germs $L(F)(C(F))$ and $L(G)(C(G))$ are r-Lagrangian equivalent if and only if $W(\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F})))$ and $W(\mathcal{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G})))$ are r-S.P$^+$-diffeomorphic.

Corollary 5.3. *Suppose that $\mathcal{F}(q,x,s,t) = \lambda(q,x,s,t)(F(q,x,s)-t)$ is an r-parameter family of graph-like Morse families of hypersurfaces. Then $L(F)(C(F))$ is r-Lagrange stable if and only if $\mathcal{L}(\Sigma_*(\mathcal{F}))$ is r-S.P$^+$-Legendre stable.*

6. Classifications of bifurcations of Lagrangian submanifolds

We consider bifurcations of Lagrangian submanifold germs, that is, the case of $r=1$. As an application of Theorem 5.1, we give generic classifications of bifurcations of Lagrangian submanifold germs for lower dimensions by using one-parameter families of graph-like Legendrian unfoldings.

Theorem 6.1. *Let $1\le n\le 3$. A generic one-parameter family of Lagrangian submanifold germs $L(F)(C(F))$ of a one-parameter family of Morse families of functions*

$$F : (\mathbb{R}^k\times\mathbb{R}^n\times\mathbb{R},0)\to(\mathbb{R},0),$$

is one-parameter Lagrangian equivalent to the one-parameter family of Lagrangian submanifold germs of one of the following one-parameter families of Morse families of functions:

$n=1$;
(1) q_1,
(2) $\pm q_1^2 + x_1$,
(3) $q_1^3 + x_1q_1$,
(4) $\pm q_1^4 + \alpha(x_1,s)q_1^2 + x_1q_1$, $\partial\alpha/\partial s(0)\neq 0, \partial\alpha/\partial x_1(0) = 0$,

$n=2$;
(1) q_1,
(2) $\pm q_1^2 + x_1q_1$,
(3) $q_1^3 + x_1q_1 + x_2$,

$(4)_1$ $\pm q_1^4 + x_1 q_1^2 + x_2 q_2,$
$(4)_2$ $\pm q_1^4 + \alpha(x_1, x_2, s) q_1^2 + x_1 q_1 + x_2,\ \partial\alpha/\partial s(0) \neq 0, \partial\alpha/\partial x_1(0) = \partial\alpha/\partial x_2(0) = 0,$
$(5)_1$ $q_1^5 + \alpha(x_1, x_2, s) q_1^3 + x_1 q_1^2 + x_2 q_1,\ \partial\alpha/\partial s(0) \neq 0, \partial\alpha/\partial x_1(0) = \partial\alpha/\partial x_2(0) = 0,$
$(5)_2$ $q_1^5 + x_1 q_1^3 + \alpha(x_1, x_2, s) q_1^2 + x_2 q_1,\ \partial\alpha/\partial s(0) \neq 0, \partial\alpha/\partial x_1(0) = \partial\alpha/\partial x_2(0) = 0,$
(6) $q_1^3 \pm q_1 q_2^2 + \alpha(x_1, x_2, s) q_1^2 + x_1 q_1 + x_2 q_2,\ \partial\alpha/\partial s(0) \neq 0, \partial\alpha/\partial x_1(0) = \partial\alpha/\partial x_2(0) = 0,$

$n = 3$;
(1) $q_1,$
(2) $\pm q_1^2 + x_1 q_1,$
(3) $q_1^3 + x_1 q_1 + x_2,$
$(4)_1$ $\pm q_1^4 + x_1 q_1^2 + x_2 q_2 + x_3,$
$(4)_2$ $\pm q_1^4 + \alpha(x_1, x_2, x_3, s) q_1^2 + x_1 q_1 + x_2,\ \partial\alpha/\partial s(0) \neq 0, \partial\alpha/\partial x_i(0) = 0\ i = 1, 2, 3,$
$(5)_1$ $q_1^5 + x_1 q_1^3 + x_2 q_1^2 + x_3 q_1,$
$(5)_2$ $q_1^5 + \alpha(x_1, x_2, x_3, s) q_1^3 + x_1 q_1^2 + x_2 q_1 + x_3,\ \partial\alpha/\partial s(0) \neq 0, \partial\alpha/\partial x_i(0) = 0,$
$(5)_3$ $q_1^5 + x_1 q_1^3 + \alpha(x_1, x_2, x_3, s) q_1^2 + x_2 q_1 + x_3,\ \partial\alpha/\partial s(0) \neq 0, \partial\alpha/\partial x_i(0) = 0,$
$(6)_1$ $q_1^3 \pm q_1 q_2^2 + x_1 q_1^2 + x_2 q_1 + x_3 q_2,$
$(6)_2$ $q_1^3 \pm q_1 q_2^2 + \alpha(x_1, x_2, x_3, s) q_1^2 + x_1 q_1 + x_2 q_2 + x_3,\ \partial\alpha/\partial s(0) \neq 0, \partial\alpha/\partial x_i(0) = 0,$
$(7)_1$ $\pm q_1^6 + \alpha(x_1, x_2, x_3, s) q_1^4 + x_1 q_1^3 + x_2 q_1^2 + x_3 q_1,\ \partial\alpha/\partial s(0) \neq 0, \partial\alpha/\partial x_i(0) = 0,$
$(7)_2$ $\pm q_1^6 + x_1 q_1^4 + \alpha(x_1, x_2, x_3, s) q_1^3 + x_2 q_1^2 + x_3 q_1,\ \partial\alpha/\partial s(0) \neq 0, \partial\alpha/\partial x_i(0) = 0,$
$(7)_3$ $\pm q_1^6 + x_1 q_1^4 + x_1 q_1^3 + \alpha(x_1, x_2, x_3, s) q_1^2 + x_3 q_1,\ \partial\alpha/\partial s(0) \neq 0, \partial\alpha/\partial x_i(0) = 0,$
$(8)_1$ $\pm(q_1^2 q_2 + q_2^4) + \alpha(x_1, x_2, x_3, s) q_1^2 + x_1 q_2^2 + x_2 q_1 + x_3 q_2,\ \partial\alpha/\partial s(0) \neq 0, \partial\alpha/\partial x_i(0) = 0,$
$(8)_2$ $\pm(q_1^2 q_2 + q_2^4) + x_1 q_1^2 + \alpha(x_1, x_2, x_3, s) q_2^2 + x_2 q_1 + x_3 q_2,\ \partial\alpha/\partial s(0) \neq 0, \partial\alpha/\partial x_i(0) = 0,$
where $i = 1, 2, 3.$

The function germs α are called *functional moduli*. By definition of the one-parameter $S.P^+$-$\mathcal{K}$-equivalence relation, functional moduli must satisfy some extra conditions; however, we do not argue about such conditions here (cf. [17]).

In order to prove Theorem 6.1, we prepare some notations and results for the classification of function germs. We use a method for the classification of function germs in [26, 27, 28].

Let $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be a one-parameter family of graph-like Morse families of hypersurfaces of the form

$$\mathcal{F}(q, x, s, t) = \lambda(q, x, s, t)(F(q, x, s) - t).$$

We write $\overline{F}(q, x, s, t) = F(q, x, s) - t$. For an unfolding $\overline{F} : (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ of $\overline{f}(q, x, t) = f(q, x) - t$, $\overline{F}$ is a 1-$S.P^+$-$\mathcal{K}$-versal deformation of $\overline{f}$ if

$$\mathcal{E}_{(q,x,t)} = \left\langle \frac{\partial f}{\partial q}(q, x), f(q, x) - t \right\rangle_{\mathcal{E}_{(q,x,t)}} + \left\langle \frac{\partial f}{\partial x}(q, x), 1 \right\rangle_{\mathcal{E}_x} + \left\langle \frac{\partial F}{\partial s}|_{s=0} \right\rangle_{\mathbb{R}}.$$

It follows that if

$$\dim_{\mathbb{R}} \mathcal{E}_{(q,x,t)} / \left(\left\langle \frac{\partial f}{\partial q}(q, x), f(q, x) - t \right\rangle_{\mathcal{E}_{(q,x,t)}} + \left\langle \frac{\partial f}{\partial x}(q, x), 1 \right\rangle_{\mathcal{E}_x} \right) \leq 1,$$

then

$$\dim_{\mathbb{R}} \mathcal{E}_{(q,t)} / \left(\left\langle \frac{\partial f}{\partial q}(q), f(q) - t \right\rangle_{\mathcal{E}_{(q,t)}} + \langle 1 \rangle_{\mathbb{R}} \right) \leq n + 1.$$

However, the condition of 1-$S.P^+$-$\mathcal{K}$-versal deformations (that is, 1-$S.P^+$-Legendrian stability for corresponding Legendrian submanifold germs) is too strong for giving the classification. We assume that $\overline{F}(q, x, s, t)$ is an $S.P^+$-$\mathcal{K}$-versal deformation of $\overline{f}(q, t)$, namely,

$$\mathcal{E}_{(q,t)} = \left\langle \frac{\partial f}{\partial q}(q,x), f(q,x) - t \right\rangle_{\mathcal{E}_{(q,t)}} + \langle 1 \rangle_{\mathbb{R}} + \left\langle \frac{\partial F}{\partial x}|_{x=s=0}, \frac{\partial F}{\partial s}|_{x=s=0} \right\rangle_{\mathbb{R}}.$$

We give a quick review of the classification of $S.P^+$-$\mathcal{K}$-versal deformations with $S.P^+$-$\mathcal{K}$-cod ≤ 4. For details see [10]. Let F and $F' : (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be germs of unfoldings of f and $f' : (\mathbb{R}^k \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$, respectively. We say that F and F' are $S.P^+$-$\mathcal{K}$ (respectively, $S.P$-$\mathcal{K}$)-*equivalent* if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}, 0)$ of the form $\Phi(q,u,t) = (\phi_1(q,u,t), \phi_2(u), t + \alpha(u))$ (respectively, $\Phi(q,u,t) = (\phi_1(q,u,t), \phi_2(u), t))$ such that $\langle F \circ \Phi \rangle_{\mathcal{E}_{(q,u,t)}} = \langle F' \rangle_{\mathcal{E}_{(q,u,t)}}$. We also say that $F(q,u,t)$ is an $S.P^+$-$\mathcal{K}$ (respectively, $S.P$-$\mathcal{K}$)-*versal deformation* of $f = F|_{\mathbb{R}^k \times 0 \times \mathbb{R}}$ if

$$\mathcal{E}_{(q,t)} = \left\langle f, \frac{\partial f}{\partial q_1}, \dots, \frac{\partial g}{\partial q_k} \right\rangle_{\mathcal{E}_{(q,t)}} + \left\langle \frac{\partial f}{\partial t} \right\rangle_{\mathbb{R}} + \left\langle \frac{\partial F}{\partial u_1}|_{\mathbb{R}^k \times 0 \times \mathbb{R}}, \dots, \frac{\partial F}{\partial u_r}|_{\mathbb{R}^k \times 0 \times \mathbb{R}} \right\rangle_{\mathbb{R}}$$

$\Big($respectively, $\mathcal{E}_{(q,t)} = \left\langle f, \frac{\partial f}{\partial q_1}, \dots, \frac{\partial g}{\partial q_k} \right\rangle_{\mathcal{E}_{(q,t)}} + \left\langle \frac{\partial F}{\partial u_1}|_{\mathbb{R}^k \times 0 \times \mathbb{R}}, \dots, \frac{\partial F}{\partial u_r}|_{\mathbb{R}^k \times 0 \times \mathbb{R}} \right\rangle_{\mathbb{R}}\Big)$.

We say that f and f' are S-$\mathcal{K}$-*equivalent* if there exists a diffeomorphism germ

$$\Phi : (\mathbb{R}^k \times \mathbb{R}, 0) \to (\mathbb{R}^k \times \mathbb{R}, 0)$$

of the form $\Phi(q,t) = (\phi(q,t), t)$ such that $\langle f \circ \Phi \rangle_{\mathcal{E}_{(q,t)}} = \langle f' \rangle_{\mathcal{E}_{(q,t)}}$.

For each germ of a function $f : (\mathbb{R}^k \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$, we set

$$S.P\text{-}\mathcal{K}\text{-cod}(f) = \dim_{\mathbb{R}} \mathcal{E}_{(q,t)} \Big/ \left\langle f, \frac{\partial f}{\partial q_1}, \dots, \frac{\partial f}{\partial q_k} \right\rangle_{\mathcal{E}_{(q,t)}},$$

$$S.P^+\text{-}\mathcal{K}\text{-cod}(f) = \dim_{\mathbb{R}} \mathcal{E}_{(q,t)} \Big/ \left(\left\langle f, \frac{\partial f}{\partial q_1}, \dots, \frac{\partial f}{\partial q_k} \right\rangle_{\mathcal{E}_{(q,t)}} + \left\langle \frac{\partial f}{\partial t} \right\rangle_{\mathbb{R}} \right).$$

Then we have the following classifications:

Theorem 6.2 ([10, Theorem 4.2])**.** *Let $f : (\mathbb{R}^k \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a function germ with $S.P$-$\mathcal{K}$-cod $(f) \leq 5$. Then f is stably S-$\mathcal{K}$-equivalent to one of the germs in the following list:*

(1)	q_1,	$S.P\text{-}\mathcal{K}\text{-cod}(f) = 0$;	A_0,
(2)	$\pm t \pm q_1^2$,	$S.P\text{-}\mathcal{K}\text{-cod}(f) = 1$;	A_1,
(3)	$\pm t \pm q_1^3$,	$S.P\text{-}\mathcal{K}\text{-cod}(f) = 2$;	A_2,
(4)	$\pm t^2 \pm q_1^2$,	$S.P\text{-}\mathcal{K}\text{-cod}(f) = 2$;	B_2,
(5)	$\pm t \pm q_1^4$,	$S.P\text{-}\mathcal{K}\text{-cod}(f) = 3$;	A_3,
(6)	$\pm t^3 \pm q_1^2$,	$S.P\text{-}\mathcal{K}\text{-cod}(f) = 3$;	B_3,
(7)	$q_1^3 \pm tq_1$,	$S.P\text{-}\mathcal{K}\text{-cod}(f) = 3$;	C_3,
(8)	$\pm t + q_1^5$,	$S.P\text{-}\mathcal{K}\text{-cod}(f) = 4$;	A_4,
(9)	$\pm t + (q_1^3 \pm q_1 q_2^2)$,	$S.P\text{-}\mathcal{K}\text{-cod}(f) = 4$;	D_4,
(10)	$\pm t^2 + q_1^3$,	$S.P\text{-}\mathcal{K}\text{-cod}(f) = 4$;	F_4,
(11)	$\pm t^4 \pm q_1^2$,	$S.P\text{-}\mathcal{K}\text{-cod}(f) = 4$;	B_4,
(12)	$q_1^4 \pm tq_1$,	$S.P\text{-}\mathcal{K}\text{-cod}(f) = 4$;	C_4,
(13)	$\pm t + q_1^6$,	$S.P\text{-}\mathcal{K}\text{-cod}(f) = 5$;	A_5,
(14)	$\pm t \pm (q_1^4 + q_1 q_2^2)$,	$S.P\text{-}\mathcal{K}\text{-cod}(f) = 5$;	D_5,
(15)	$\pm t^5 \pm q_1^2$,	$S.P\text{-}\mathcal{K}\text{-cod}(f) = 5$;	B_5,
(16)	$q_1^5 \pm tq_1$,	$S.P\text{-}\mathcal{K}\text{-cod}(f) = 5$;	C_5.

We can construct an $S.P$-$\mathcal{K}$ (respectively, $S.P^+$-$\mathcal{K}$)-versal deformation for each normal form by the usual method (cf. [3]). Then the corresponding list is as follows:

$S.P$-$\mathcal{K}$-versal deformations:
(1) q_1,
(2) $\pm t \pm q_1^2 + u_1$,
(3) $\pm t \pm q_1^3 + u_1 q_1 + u_2$,
(4) $\pm t^2 \pm q_1^2 + u_1 t + u_2$,
(5) $\pm t \pm q_1^4 + u_1 q_1^2 + u_2 q_1 + u_3$,
(6) $\pm t^3 \pm q_1^2 + u_1 t^2 + u_2 t + u_3$,
(7) $q_1^3 \pm t q_1 + u_1 q_1^2 + u_2 q_1 + u_3$,
(8) $\pm t + q_1^5 + u_1 q_1^3 + u_2 q_1^2 + u_3 q_1^3 + u_4$,
(9) $\pm t + (q_1^3 \pm q_1 q_2^2) + u_1 q_1^2 + u_2 q_2 + u_3 q_1 + u_4$,
(10) $\pm t^2 + q_1^3 + u_1 t q_1 + u_2 q_1 + u_3 s + u_4$,
(11) $\pm t^4 \pm q_1^2 + u_1 t^3 + u_2 t^2 + u_3 t + u_4$,
(12) $q_1^4 \pm t q_1 + u_1 q_1^3 + u_2 q_1^2 + u_3 q_1 + u_4$,
(13) $\pm t \pm q_1^6 + u_1 q_1^4 + u_2 q_1^3 + u_3 q_1^2 + u_4 q_1 + u_5$,
(14) $\pm t \pm (q_1^4 + q_1 q_2^2) + u_1 q_1^2 + u_2 q_2^2 + u_3 q_1 + u_4 q_2 + u_5$,
(15) $\pm t^5 \pm q_1^2 + u_1 t^4 + u_2 t^3 + u_3 t^2 + u_4 t + u_5$,
(16) $q_1^5 \pm t q_1 + u_1 q_1^4 + u_2 q_1^3 + u_3 q_1^2 + u_4 q_1 + u_5$.
$S.P^+$-$\mathcal{K}$-versal deformations:

(1) q_1,
(2) $\pm t \pm q_1^2$,
(3) $\pm t \pm q_1^3 + v_1 q_1$,
(4) $\pm t^2 \pm q_1^2 + v_1$,
(5) $\pm t \pm q_1^4 + v_1 q_1^2 + v_2 q_1$,
(6) $\pm t^3 \pm q_1^2 + v_1 t + v_2$,
(7) $q_1^3 \pm t q_1 + v_1 q_1^2 + v_2$,
(8) $\pm t \pm q_1^5 + v_1 q_1^3 + v_2 q_1^2 + v_3 q_1^3$,
(9) $\pm t + (q_1^3 \pm q_1 q_2^2) + v_1 q_1^2 + v_2 q_2 + v_3 q_1$,
(10) $\pm t^2 + q_1^3 + v_1 t q_1 + v_2 q_1 + v_3$,
(11) $\pm t^4 \pm q_1^2 + v_1 t^2 + v_2 t + v_3$,
(12) $q_1^4 \pm t q_1 + v_1 q_1^3 + v_2 q_1^2 + v_3$,
(13) $\pm t \pm q_1^6 + v_1 q_1^4 + v_2 q_1^3 + v_3 q_1^2 + v_4 q_1$,
(14) $\pm t \pm (q_1^4 + q_1 q_2^2) + v_1 q_1^2 + v_2 q_2^2 + v_3 q_1 + v_4 q_2$,
(15) $\pm t^5 \pm q_1^2 + v_1 t^4 + v_2 t^3 + v_3 t^2 + v_4 t$,
(16) $q_1^5 \pm t q_1 + v_1 q_1^4 + v_2 q_1^3 + v_3 q_1^2 + v_4 q_1$.

We remark that the relation between $S.P^+$-$\mathcal{K}$-cod and $S.P$-$\mathcal{K}$-cod is given by

$$S.P^+\text{-}\mathcal{K} - \mathrm{cod}(f) \;=\; S.P\text{-}\mathcal{K} - \mathrm{cod}(f) + 1$$

by [10, Proposition 3.5].

The following theorem is useful and important for our purpose (cf. [3]).

Theorem 6.3. *Let F and $F' : (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be germs of functions which are $S.P^+$-$\mathcal{K}$ (respectively, $S.P$-$\mathcal{K}$)-versal deformations of $f = F|_{\mathbb{R}^k \times 0 \times \mathbb{R}}$ and $f' = F'|_{\mathbb{R}^k \times 0 \times \mathbb{R}}$ respectively. Then F and F' are $S.P^+$-$\mathcal{K}$ (respectively, $S.P$-$\mathcal{K}$)-equivalent if and only if f and f' are S-$\mathcal{K}$-equivalent.*

Proof of Theorem 6.1. Let $1 \leq n \leq 3$. We denote the set of one-parameter families of Lagrangian submanifolds by $L(U \times V, T^*(\mathbb{R}^n \times \mathbb{R}))$, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}$ are open domains around the origin. The set of Lagrangian stable one-parameter families of Lagrangian submanifolds is an open and dense subset in $L(U \times V, T^*(\mathbb{R}^n \times \mathbb{R}))$ (cf. [1, 2, 30]).

Therefore, by Corollary 2.10 and Theorem 5.1, we can give a classification of an $S.P^+$-$\mathcal{K}$-versal deformation of one-parameter graph-like Legendrian unfoldings under the one-parameter s-$S.P^+$-$\mathcal{K}$ equivalence.

We consider the case of $n = 3$. Since the classifications in the cases $n = 1$ and $n = 2$ are given by the similar method, we omit it. By Theorems 6.2, 6.3 and the form of

$$\overline{F}(q, x, s, t) = F(q, x, s) - t,$$

$\overline{F}$ is stably $S.P^+$-$\mathcal{K}$-equivalent to one of the germs in the following list:

$$\begin{array}{ll}
(1) & -t + q_1 + v_1 + v_2 + v_3 + v_4, \\
(2) & -t \pm q_1^2 + v_1 + v_2 + v_3 + v_4, \\
(3) & -t + q_1^3 + v_1 q_1 + v_2 + v_3 + v_4, \\
(4) & -t \pm q_1^4 + v_1 q_1^2 + v_2 q_1 + v_3 + v_4, \\
(5) & -t + q_1^5 + v_1 q_1^3 + v_2 q_1^2 + v_3 q_1 + v_4, \\
(6) & -t + (q_1^3 \pm q_1 q_2^2) + v_1 q_1^2 + v_2 q_2 + v_3 q_1 + v_4, \\
(7) & -t \pm q_1^6 + v_1 q_1^4 + v_2 q_1^3 + v_3 q_1^2 + v_4 q_1, \\
(8) & -t \pm (q_1^4 \pm q_1 q_2^2) + v_1 q_1^2 + v_2 q_2^2 + v_3 q_2 + v_4 q_1,
\end{array}$$

where $(v_1, v_2, v_3, v_4) \in (\mathbb{R}^4, 0)$. We would like to classify these germs by the one-parameter s-$S.P^+$-$\mathcal{K}$-equivalence. By the above normal forms, there exists a germ of a diffeomorphism $\phi : (\mathbb{R}^3 \times \mathbb{R}, 0) \to (\mathbb{R}^4, 0)$ such that $\overline{F}$ is stably one-parameter s-$S.P^+$-$\mathcal{K}$-equivalent to one of the germs in the following list:

$$\begin{array}{ll}
(1) & -t + q_1 + v_1(x,s) + v_2(x,s) + v_3(x,s) + v_4(x,s), \\
(2) & -t \pm q_1^2 + v_1(x,s) + v_2(x,s) + v_3(x,s) + v_4(x,s), \\
(3) & -t + q_1^3 + v_1(x,s) q_1 + v_2(x,s) + v_3(x,s) + v_4(x,s), \\
(4) & -t \pm q_1^4 + v_1(x,s) q_1^2 + v_2(x,s) q_1 + v_3(x,s) + v_4(x,s), \\
(5) & -t + q_1^5 + v_1(x,s) q_1^3 + v_2(x,s) q_1^2 + v_3(x,s) q_1 + v_4(x,s), \\
(6) & -t + (q_1^3 \pm q_1 q_2^2) + v_1(x,s) q_1^2 + v_2(x,s) q_2 + v_3(x,s) q_1 + v_4(x,s), \\
(7) & -t + \pm q_1^6 + v_1(x,s) q_1^4 + v_2(x,s) q_1^3 + v_3(x,s) q_1^2 + v_4(x,s) q_1, \\
(8) & -t \pm (q_1^4 \pm q_1 q_2^2) + v_1(x,s) q_1^2 + v_2(x,s) q_2^2 + v_3(x,s) q_2 + v_4(x,s) q_1,
\end{array}$$

where $x = (x_1, x_2, x_3) \in (\mathbb{R}^3, 0)$. Since $\overline{F}$ is a one-parameter family of graph-like Morse families of hypersurfaces, $\partial F / \partial q : (\mathbb{R}^k \times \mathbb{R}^3 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ is non-singular for each fixed $s \in (\mathbb{R}, 0)$, that is, we have a rank condition

$$\operatorname{rank} \left(\frac{\partial^2 F}{\partial q^2}, \frac{\partial^2 F}{\partial q \partial x} \right)(0) = k.$$

By the rank condition, (1), (2) and (3) are one-parameter s-$S.P$-$\mathcal{K}$-equivalent to

$$(1)\ -t+q_1,\quad (2)\ -t\pm q_1^2+x_1q_1,\quad (3)\ -t+q_1^3+x_1q_1+x_2,$$

respectively. In the case (4), we divide it into four cases: $(\partial v_1/\partial x_1)(0)\neq 0$, $(\partial v_1/\partial x_2)(0)\neq 0$, $(\partial v_1/\partial x_3)(0)\neq 0$ or $(\partial v_1/\partial s)(0)\neq 0$. In the first, second and third cases, $\overline{F}$ is one-parameter s-$S.P$-$\mathcal{K}$-equivalent to

$$(4)_1\ -t\pm q_1^4+x_1q_1^2+x_2q_1+x_3.$$

In the fourth case, $\overline{F}$ is one-parameter s-$S.P$-$\mathcal{K}$-equivalent to

$$(4)_2\ -t\pm q_1^4+\alpha(x,s)q_1^2+x_2q_1+x_3,$$

where $\alpha:(\mathbb{R}^3\times\mathbb{R},0)\to(\mathbb{R},0)$ is a smooth function with the conditions

$$(\partial\alpha/\partial s)(0)\neq 0, (\partial\alpha/\partial x_i)(0)=0, i=1,2,3.$$

In the case (5), $\overline{F}$ is one-parameter s-$S.P^+$-$\mathcal{K}$-equivalent to

$$\begin{aligned}
&(5)_1\quad -t+q_1^5+x_1q_1^3+x_2q_1^2+x_3q_1,\\
&(5)_2\quad -t+q_1^5+\alpha(x,s)q_1^3+x_1q_1^2+x_2q_1+x_3,\\
&(5)_3\quad -t+q_1^5+x_1q_1^3+\alpha(x,s)q_1^2+x_2q_1+x_3,
\end{aligned}$$

where $\alpha:(\mathbb{R}^3\times\mathbb{R},0)\to(\mathbb{R},0)$ is a smooth function with the conditions

$$(\partial\alpha/\partial s)(0)\neq 0, (\partial\alpha/\partial x_i)(0)=0, i=1,2,3.$$

In the cases (6) and (8), we can give the normal forms by the similar methods to those of the case (4). Moreover, in the case (7), we can also give the normal forms by the similar methods to those of the case (5). This completes the proof. □

Remark 6.4. In the generic classifications under one-parameter caustic equivalence in [1, 2, 30], the functional moduli have a special form. For instance, the functional moduli of the type $(7)_1$ in Theorem 6.1 are equivalent to the form $\alpha(x,s)=s$. Moreover, types $(7)_2$ and $(7)_3$ in Theorem 6.1 do not appear in the generic classifications under one-parameter caustic equivalence.

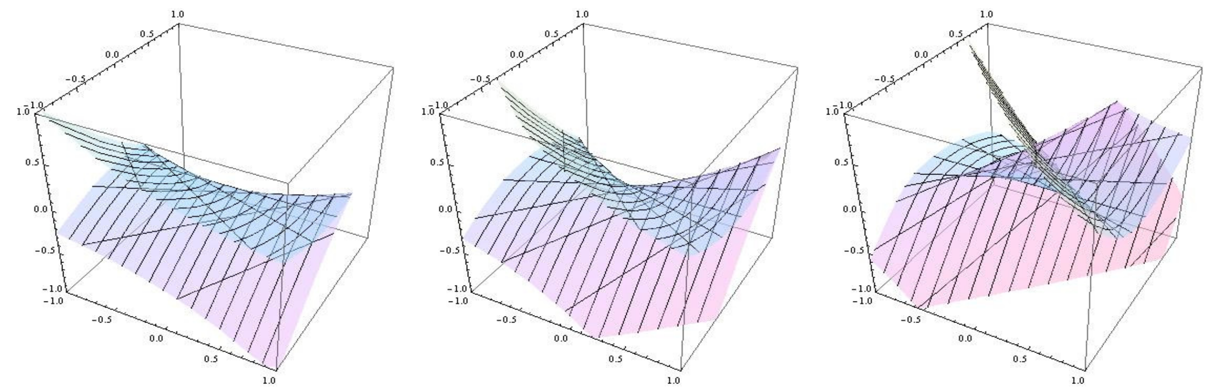
We give concrete examples of bifurcations of caustics for the types $(7)_1$ and $(7)_2$.

Example 6.5. Let $\overline{F}:(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R},0)\to(\mathbb{R},0)$ be given by

$$\overline{F}(q,x,s)=-t+q^6+\alpha(x_1,x_2,x_3,s)q^4+x_1q^3+x_2q^2+x_3q,$$

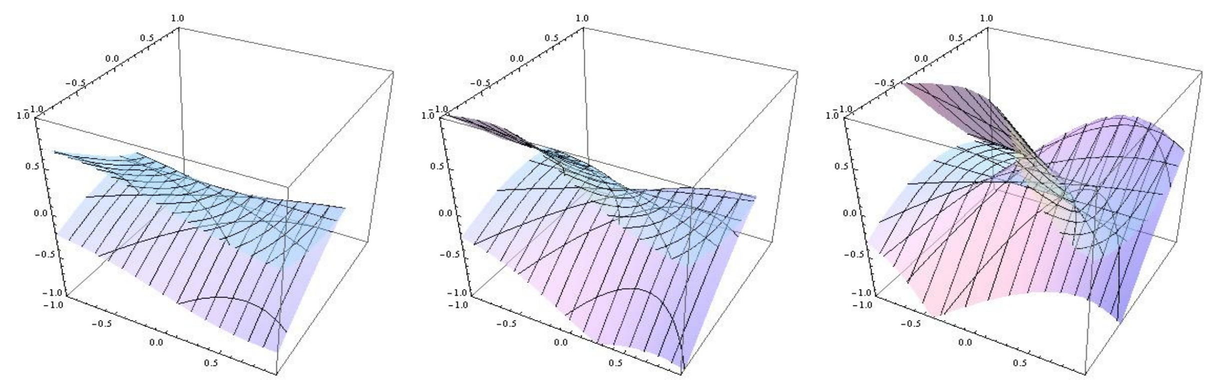
where $\partial\alpha/\partial s(0)\neq 0, \partial\alpha/\partial x_i(0)=0, i=1,2,3$. The one-parameter family of Lagrangian submanifold germs $L(F):(C(F),0)\to T^*\mathbb{R}^3$ is given by $L(F)(q,x,s)=(x,\partial F/\partial x(q,x,s))$.

If we take $\alpha(x,s) = s$, then the one-parameter family of caustics is given by the image of $(u,v,s) \mapsto (v, -15u^4 - 6su^2 - 3uv, 24u^5 + 8su^3 + 3vu^2)$; see Figure 1 (cf. [1, 2, 30]). If we take $\alpha(x,s) = s + x_1^2$, then the the one-parameter family of caustics is given by the image of $(u,v,s) \mapsto (v, -15u^4 - 6(s+v^2)u^2 - 3uv, 24u^5 + 8(s+v^2)u^3 + 3vu^2)$; see Figure 2.



$s > 0$ $s = 0$ $s < 0$

Figure 1. Type $(7)_1$ with $\alpha(x,s) = s$.



$s > 0$ $s = 0$ $s < 0$

Figure 2. Type $(7)_1$ with $\alpha(x,s) = s + x_1^2$.

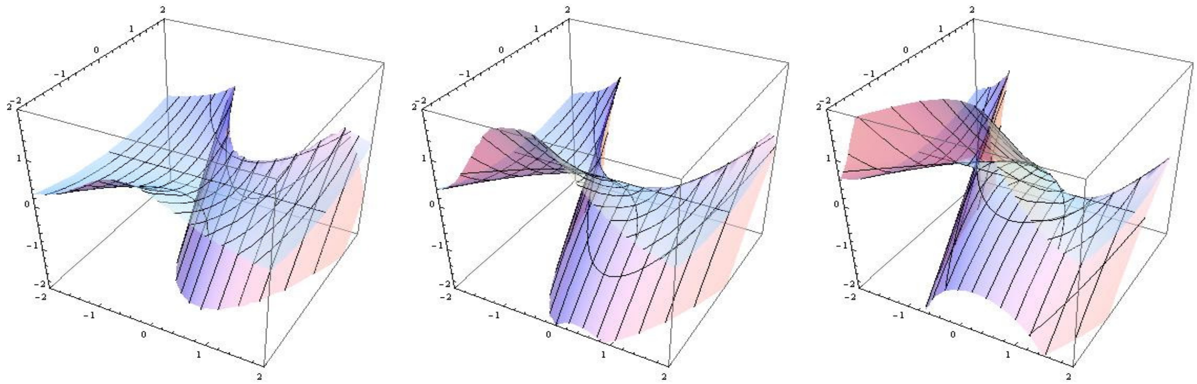
Example 6.6. Let $\overline{F} : (\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be given by

$$\overline{F}(q,x,s) = -t + q^6 + x_1 q^4 + \alpha(x_1,x_2,x_3,s) q^3 + x_2 q^2 + x_3 q,$$

where $\partial\alpha/\partial s(0) \neq 0, \partial\alpha/\partial x_i(0) = 0, i = 1,2,3$. If we take $\alpha(x,s) = s + x_1^2$, then the one-parameter family of caustics is given by the image of

$$(u,v,s) \mapsto (v, -15u^4 - 6vu^2 - 3(s+v^2)u, 24u^5 + 8vu^3 + 3(s+v^2)u^2);$$

see Figure 3.



$s > 0$ $s = 0$ $s < 0$

Figure 3. Type $(7)_2$ with $\alpha(x, s) = s + x_1^2$.

References

[1] V. I. Arnol'd, *Singularities of Caustics and Wave Fronts*. Mathematics and Its Applications. **62** Kluwer Academic Publishers (1990).

[2] V. I. Arnol'd, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of Differentiable Maps vol.* I. Birkhäuser (1986).

[3] J. Damon, *The unfolding and determinacy theorems for subgroups of $\mathcal{A}$ and $\mathcal{K}$*. Memoirs of Amer. Math. Soc. **50-306** (1984).

[4] J. J. Duistermaat, *Oscillatory Integrals, Lagrange Immersions and Unfolding of Singularities*. Communications of Pure and Applied Math. **XXVII** (1974), 207–281. DOI: 10.1002/cpa.3160270205

[5] J. Ehlers and E.T. Newman, *The theory of caustics and wave front singularities with physical applications*, J. Math. Physics. **41** (2000), 3344–3378. DOI: 10.1063/1.533316

[6] V. V. Goryunov and V. M. Zakalyukin, *Lagrangian and Legendrian singularities*, Real and complex singularities, Trends Math., Birkhauser, Basel, (2007), 169–185. DOI: 10.1007/978-3-7643-7776-2_12

[7] W. Hasse, M. Kriele and V. Perlick, *Caustics of wavefronts in general relativity*, Class. Quantum Grav. **13** (1996), 1161–1182. DOI: 10.1088/0264-9381/13/5/027

[8] L. Hörmander, *Fourier Integral Operators,I.* Acta. Math. **128** (1972), 79–183.

[9] S. Izumiya, *Perestroikas of optical wave fronts and graphlike Legendrian unfoldings*. J. Differential Geom. **38** (1993), 485–500. DOI: 10.4310/jdg/1214454479

[10] S. Izumiya, *Completely integrable holonomic systems of first-order differential equations*. Proc. Roy. Soc. Edinburgh. **125A** (1995), 567–586.

[11] S. Izumiya, *Differential Geometry from the viewpoint of Lagrangian or Legendrian singularity theory*. in Singularity Theory (ed., D. Chéniot et al), World Scientific (2007), 241–275. DOI: 10.1142/9789812707499_0008

[12] S. Izumiya, *The theory of graph-like Legendrian unfoldings and its applications*. J. Singl. **12** (2015), 53–79. DOI: 10.5427/jsing.2015.12d

[13] S. Izumiya, *Geometric interpretation of Lagrangian equivalence*. Canadian Math. Bull. (2016) 806–812. DOI: 10.4153/cmb-2016-056-2

[14] S. Izumiya, *Geometry of world sheets in Lorentz-Minkowski space*. RIMS Kôkyûroku Bessatsu, **B55** (2016), 89–109.

[15] S. Izumiya, *Caustics of world hyper-sheets in the Minkowski space-time*. Contemporary Mathematics, AMS, **675** (2016), 133–151. DOI: 10.1090/conm/675/13588

[16] S. Izumiya, *The theory of graph-like Legendrian unfoldings: Equivalence relations*. Advanced Studies in Pure Math., **78** (2018), 107–161. DOI: 10.2969/aspm/07810107

[17] S. Izumiya and Y. Kurokawa, *Holonomic systems of Clairaut type*. Differential Geom. Appl. **5** (1995), 219–235. DOI: 10.1016/0926-2245(95)92847-x

[18] S. Izumiya, D-H. Pei, T. Sano and E. Torii, *Evolutes of hyperbolic plane curves*. Acta Math. Sinica. **20**, (2004), 543–550. DOI: 10.1007/s10114-004-0301-y

[19] S. Izumiya, D-H. Pei and M. Takahashi, *Singularities of evolutes of hypersurfaces in hyperbolic space*. Proc. Edinburgh Math. Soc. **47** (2004), 131–153. DOI: 10.1017/s0013091503000312

[20] S. Izumiya and M. Takahashi, *Spacelike parallels and evolutes in Minkowski pseudo-spheres*. J. Geom. and Phys. **57** (2007), 1569–1600. DOI: 10.1016/j.geomphys.2007.01.008

[21] S. Izumiya and M. Takahashi, *Caustics and wave front propagations: Applications to differential geometry*. Banach Center Publications. Geometry and topology of caustics. **82** (2008), 125–142. DOI: 10.4064/bc82-0-9

[22] S. Izumiya and M. Takahashi, *On caustics of submanifolds and canal hypersurfaces in Euclidean space*. Topology and its appl. **159** (2012), 501–508. DOI: 10.1016/j.topol.2011.09.025

[23] S. Izumiya and M. Takahashi, *Pedal foliations and Gauss maps of hypersurfaces in Euclidean space*. J. Singlu. **6** (2012), 84–97. DOI: 10.5427/jsing.2012.6g

[24] S. Janeczko and M. Roberts, *Classification of symmetric caustics II. Caustic equivalence*. J. London Math. Soc. **48** (1993), 178–192. DOI: 10.1112/jlms/s2-48.1.178

[25] M. Kokubu, W. Rossman, K. Saji, M. Umehara and K. Yamada, *Singularities of flat fronts in hyperbolic space*, Pacific J. Math. **221** (2005), 303–351. DOI: 10.2140/pjm.2005.221.303

[26] M. Takahashi, *Bifurcations of ordinary differential equations of Clairaut type*. J. Diff. Equations. **190** (2003), 579–599. DOI: 10.1016/s0022-0396(02)00198-5

[27] M. Takahashi, *Bifurcations of holonomic systems of general Clairaut type*. Hokkaido Math. J. **35** (2006), 905–934. DOI: 10.14492/hokmj/1285766435

[28] M. Takahashi, *Bifurcations of completely integrable first order ordinary differential equations*. J. Math. Sci. (N. Y.) **144** (2007), 3854–3869. DOI: 10.1007/s10958-007-0239-6

[29] G. Wassermann, *Stability of Caustics*. Math. Ann. **216** (1975), 43–50.

[30] V. M. Zakalyukin, *Reconstructions of fronts and caustics depending on a parameter and versality of mappings*. J. Soviet Math. **27** (1983), 2713–2735. DOI: 10.1007/bf01084818

[31] V. M. Zakalyukin, *Envelope of Families of Wave Fronts and Control Theory*. Proc. Steklov Inst. Math. **209** (1995), 114–123.

Shyuichi Izumiya, Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan
Email address: `izumiya@math.sci.hokudai.ac.jp`

Masatomo Takahashi, Muroran Institute of Technology, Muroran 050-8585, Japan
Email address: `masatomo@mmm.muroran-it.ac.jp`

Journal *of* Singularities
Volume 21 (2020), 187-204

Proc. of Geometric and Algebraic Singularity Theory, Będlewo, 2017
DOI: 10.5427/jsing.2020.21k

SYMMETRIES OF SPECIAL 2-FLAGS

PIOTR MORMUL AND FERNAND PELLETIER

Abstract. This work is a continuation of authors' research interrupted in the year 2010. Derived are recursive relations describing for the first time all infinitesimal symmetries of special 2-flags (sometimes also misleadingly called 'Goursat 2-flags'). When algorithmized to the software level, they will give an answer filling in the gap in knowledge as of 2010: on one side the local finite classification of special 2-flags known in lengths not exceeding four, on the other side the existence of a continuous numerical modulus of that classification in length seven.

1. Introduction

The paper is devoted to 'special 2-flags', that is, strictly speaking, to rank 3 distributions generating special 2-flags. More particularly – to the symmetries of such distributions which are embeddable in flows. We exhibit, for the first time, recursive relations which describe all infinitesimal symmetries of special 2-flags. This is our main Theorem 2 in Section 7. The path leading to it is not short, for it includes, apart from the most basic definitions, also the recollection, in section 5.1, of the main bricks of the theory – the so-called *singularity classes* of special 2-flags. The initial data for those recurrences are triples of *free* smooth functions of three variables. Then, upon knowing the components of a symmetry up to certain flag's length, we derive closed form formulas for the pair of symmetry's components in the length augmented by one. In this way all infinitesimal symmetries are found, and, later, started to be used in the local classification issues for special 2-flags. As for this restricted class of objects, it is precisely defined below in Section 2.

Prior to that, however, we give, for the reader's orientation, some general information about the symmetries of some classes of subbundles (= geometric distributions) in the tangent bundles to manifolds. It appears that the size of a symmetry group may dramatically vary in function of a distribution.

There circulates a widely acknowledged folk theorem (cf. section 4 in [23] and p. 86 in [10]) saying that, outside the so-called stable range, distributions generic enough do not possess any nontrivial, even only local, symmetry. More to the point, in concrete classical classes of subbundles in the tangent bundle, like the '$3, 5$' or '$4, 7$' distributions, the (Lie) groups of symmetries are severely restricted in size: not bigger than 14-dimensional in the former (and maximal in the *flat* case, when the Cartan tensor – [3] – vanishes; [10], p. 88 and [2], p. 456), and not bigger than 21-dimensional in the latter (and maximal for the *instanton* distribution, [10], p. 90). It goes by itself that likewise restricted in size are the Lie algebras of vector fields – infinitesimal symmetries. (They always form a Lie algebra due to the Jacobi identity.)

It is quite to the contrary for the geometrical objects discussed in this work. Namely, by virtue of their rather stringent definition, the algebras of infinitesimal symetries are *infinite-dimensional*. Much like it is the case for the 1-flags, i. e., *Goursat flags* discussed here in considerable length, in the guise of 'forerunners', in – still introductory – Sections 3 and 4. (This discussion culminates in reproducing here a 1999 Theorem 1, for which a new, much more legible proof is now given. That new proof is instrumental for the main Section 7 of the present work. The infinitesimal symmetries for Goursat structures are parametrized by one free function of three variables – a so-called *contact hamiltonian*.)

2. Definition of special 2-flags

We start with basic motivations and consider first 2-flags of length 1. That is, rank 3 distributions $D \subset TM$, $\dim M = 5$ such that $D + [D, D] = TM$ (or, the same thing, $[D, D] = TM$, for $D \subset [D, D]$ whenever D is a distribution). In other words, the first order Lie brackets generate all the remaining tangent directions; distribution is 'two-step'. One thus enters the domain of the classical 'cinq variables' work [3]. It was shown there that every such two step D possessed *uniquely* determined corank 1 subdistribution F enjoying the property

$$[F,\, F] \subset D \tag{1}$$

(see equations (4) on p. 121 in [3]). Cartan calls such an accompanying subdistribution F *le système covariant* of the Pfaffian system D. Cartan firstly discerns a highly particular situation (a) when $[F, F] = F$ identically in the vicinity of a point. As a consequence, he infers that, in certain local coordinates t, x^0, y^0, x^1, y^1, D gets description

$$dx^0 - x^1 dt = 0 = dy^0 - y^1 dt.$$

In contemporary terminology, such D is, up to a local coordinate change, the classical *Cartan distribution*, or contact system, on the jet space $J^1(1, 2)$ of the 1-jets of functions $\mathbb{R}(t) \to \mathbb{R}^2(x,\, y)$, with $x^1 = \frac{dx^0}{dt}$ and $y^1 = \frac{dy^0}{dt}$. Its corank 1 covariant subdistribution F (reiterating, involutive in situation (a)!) is in these coordinates just $\mathrm{span}\big(\frac{\partial}{\partial x^1}\,,\ \frac{\partial}{\partial y^1}\big)$. In all what follows we will skip the symbol 'span' before a set of vector field generators.

By far more interesting is Cartan's situation (b) $[F, F] = D$ in the vicinity of a given point.[1] The covariant object F has then its 'curvature' and D is retrievable from F alone. We note that situation (b) is extremely rich geometrically and hides a *functional* modulus (one function of five variables) of the local classification of '3, 5' distributions with respect to the diffeomorphisms of base manifold.

We say that a general such D (with no extra information as to (a) or (b)) generates a 2-flag of length 1, while a D with its covariant system F involutive generates a *special* 2-flag of length 1. Therefore, the adjective 'special' in length 1 locally means nothing but 'jet-like'. How does it look like in bigger lengths/higher jets?

Let us analyze the contact system D on a concrete jet space $J^r(1, 2) =: M$ with $r \geq 1$. The main observation is that the sequence of modules of vector fields – consecutive Lie squares of D,

$$TM = D^0 \supset D^1 \supset D^2 \supset\ \cdots\ \supset D^{r-1} \supset D^r, \tag{2}$$

where $D^r = D$ and $[D^j,\, D^j] = D^{j-1}$ for $j = r,\, r-1, \ldots,\, 2,\, 1$, grows in ranks regularly by two: $3,\, 5,\, 7, \ldots,\, 2r+1,\, 2(r+1)+1 = \dim M$ independently of the underlying points in M. (Pay attention to the indexation, which starts with the biggest index r, following the notation put forward in [11].) The reason is that in passing from D^j to D^{j-1} one forgets about the j-th order derivatives, so that

$$D^{j-1} = \left(D^j,\ \frac{\partial}{\partial x^j},\, \frac{\partial}{\partial y^j}\right). \tag{3}$$

[1] Situations (a) and (b) do not exhaust all possibilities of the local behaviour of F; Elie Cartan used to be interested in clear situations only.

Therefore, all these modules of vector fields are actually *distributions* which together form a 2-flag of length r on M. Let us scrutinize the members of this flag. The natural coordinates in $J^r(1,2)$ are t, x^0, y^0, x^1, $y^1, \dots$, x^r, y^r, where $x^j = \frac{dx^{j-1}}{dt}$, $y^j = \frac{dy^{j-1}}{dt}$ for $j = 1, 2, \dots, r$. In these coordinates the member D^1 in (2) has a Pfaffian description $dx^0 - x^1 dt = 0 = dy^0 - y^1 dt$, hence it manifestly contains a corank 1 involutive subdistribution

$$F\colon = \left(\frac{\partial}{\partial x^j}, \, \frac{\partial}{\partial y^j} \, ; \; 1 \le j \le r \right).$$

Likewise, the next smaller member D^2 has description

$$dx^0 - x^1 dt = dy^0 - y^1 dt = 0 = dx^1 - x^2 dt = dy^1 - y^2 dt \,, \tag{4}$$

hence contains a corank 1 involutive subdistribution

$$\left(\frac{\partial}{\partial x^j}, \, \frac{\partial}{\partial y^j} \, ; \; 2 \le j \le r \right).$$

The key point is that the latter happens to be the Cauchy-characteristic module of D^1, denoted by $L(D^1)$ as in [11].[2] This pattern replicates itself all the way down the flag. The Pfaffian systems describing D^j gradually get larger sets of Pfaffian equations generators, while the Cauchy-characteristic modules get (with a shift in indices!) thinner. In fact, for $1 \le j < r$,

$$L(D^j) = \left(\frac{\partial}{\partial x^s}, \, \frac{\partial}{\partial y^s} \, ; \; j+1 \le s \le r \right)$$

sits inside D^{j+1} as a corank 1 subdistribution. For instance $L(D^{r-1})$ is a field of planes $\left(\frac{\partial}{\partial x^r}, \, \frac{\partial}{\partial y^r} \right)$ sitting inside a field of 3-spaces D^r, while $L(D^r) = (0)$. Moreover all these geometric objects nicely fit together into *Sandwich Diagram*, so called after a similar (if not identical) diagram assembled for Goursat distributions, or 1-flags, in [11]:

$$\begin{array}{ccccccccccccc}
TM = D^0 & \supset & D^1 & \supset & D^2 & \supset & \cdots & \supset & D^{r-1} & \supset & D^r & & \\
 & & \cup & & \cup & & & & \cup & & \cup & & \\
 & & F & \supset & L(D^1) & \supset & \cdots & \supset & L(D^{r-2}) & \supset & L(D^{r-1}) & \supset & L(D^r) = 0\,.
\end{array}$$

All vertical inclusions in the diagram are of codimension one, while all (drawn) horizontal inclusions are of codimension 2. The squares built by these inclusions can, indeed, be perceived as certain 'sandwiches'. For instance, in the leftmost sandwich F and D^2 are as if fillings, while D^1 and $L(D^1)$ constitute the covers (of dimensions differing by 3, one has to admit). At that, the sum $2 + 1$ of **co**dimensions, in D^1, of F and D^2 equals the dimension of the quotient space $D^1/L(D^1)$, so that it is natural to ask how the 2-dimensional plane $F/L(D^1)$ and the line $D^2/L(D^1)$ are mutually positioned in $D^1/L(D^1)$: do they intersect regularly, or else the plane subsumes the line?[3] Clearly, that question imposes by itself in further sandwiches 'indexed' by the upper right vertices D^3, $D^4, \dots, D^r$, as well.

This question has a trivial answer for the Cartan distribution $D = D^r$ analyzed above (all intersections *are* regular when $r \ge 2$). Yet a more pertinent question would be the following.

Assume the existence of Sandwich Diagram with all its above-listed dimensions, inclusions, involutivenesses and call such rank 3 distributions D^r *generating special* 2*-flags of length* r.

---

[2] For D – a distribution, $L(D)$ is, by definition, the module of *Cauchy-characteristic* vector fields with values in D infinitesimally preserving D. That module is automatically (the Jacobi identity) closed under the Lie bracket. It is noteworthy that for all the particular distributions D occurring in the present work, $L(D) \subset D$ is always not just a module included in D, but an involutive subdistribution of D of corank 2 (or 3, respectively) when $m = 1$ (or 2, respectively).

[3] The answer to this question suffices to geometrically tell the object (5) below from (4).

Are then those D^r locally 'jet-like', that is – locally equivalent to the Cartan contact distribution on $J^r(1,2)$?

For $r = 1$, we reiterate, yes ([3]), but for $r = 2$ already not. There suffices to seemingly slightly modify system (4) to

$$dx^0 - x^1dt = dy^0 - y^1dt = 0 = dt - x^2dx^1 = dy^1 - y^2dx^1\,. \tag{5}$$

This rank 3 distribution on $\mathbb{R}^7$ does generate a special 2-flag of length 2, yet is *not* locally equivalent to the 'jet-like' one around every point with $x^2 = 0$ (cf. [16], Prop. 1 (iii)). The argument there has been that the object (5) has at points $x^2 = 0$ the small growth vector[4] $(3, 5, 6, 7)$, while the contact system on $J^2(1,2)$ has everywhere the small growth vector $(3, 5, 7)$. Another, possibly even simpler argument is that at points $x^2 = 0$ there is no regular intersection in the only sandwich existing in that length: the line $D^2/L(D^1)$ collapses onto the plane $F/L(D^1)$, while the analogous line for (4) collapses nowhere.

Therefore it follows that the local theory of special multi-flags is not 'void' in the sense of boiling down to the contact systems on the jet spaces for curves. In fact, this theory is already fairly rich and still developing, including this work.

Let us reiterate the importance of 'special' for 2-flags to be tractable (and the same for multi-flags in general). Special, by the way of Sandwich Diagram, brings in so much stiffness as to result in the local models with **numerical moduli only**, no functional ones. While functional moduli, by simple and widely known dimension counts (cf., for instance, section 3 in [23]) are a commonplace in the local geometry of subbundles in tangent bundles. Even the already mentioned paper [3] about 2-flags of length 1 is not yet fully understood! On the other side, the initial departing models for us – contact systems on the jet spaces – are nowadays viewed as just the simplest 'baby' realizations of the special multi-flags.

Attention. This theory is even more neat in that it does not necessitate a definition via Sandwich Diagram as such. For it follows from the important works [1, 21] that, upon assuming only the properties of the upper row in Sandwich Diagram *and* the existence of a *whatever* corank one involutive subdistribution F in D^1, one **automatically** gets Sandwich Diagram in its entirety! In fact, (i) such an F is then unique, (ii) for $j = 1, 2, \ldots, r-1$ there holds

$$L(D^j) = D^{j+1} \cap F\,,$$

(iii) $L(D^r) = (0)$ and (iv) the $L(D^j)$'s are corank 1 subdistributions in D^{j+1}, so that Sandwich Diagram entirely holds.

Now that the focus is again on Sandwich Diagram, the ongoing question bears on the local geometry in the sandwiches 'indexed' by the upper right vertices $D^2, D^3, \ldots, D^r$. It naturally opens the way towards singularities. The first step in that direction is a, fairly rough, stratification of germs of special 2-flags into so-called sandwich classes – see the beginning of section 5.1. The second is further partitioning of sandwich classes into singularity classes, in the follow up of section 5.1.

3. Kumpera-Ruiz watching glasses for Goursat distributions

In order to gently introduce the reader to the main techniques of the paper, we present in this section a test case – derive the formulas for the infinitesimal symmetries of *Goursat* distributions which generate 1-flags. This will be instrumental during the presentation of similar things to-be-derived for special 2-flags in paper's subsequent sections.

---

[4] The small growth vector of a distribution D at a point p is the sequence of integers $\big(\dim V_j(p)\big)_{j\geq 1}$, where $V_1 = D,\ V_{j+1} = V_j + [D,\, V_j]$, which *ends* on the first biggest entry.

Recalling, a rank 2 distribution on a manifold M is Goursat when the tower of its consecutive Lie squares, understood as modules of vector fields, consist uniquely of regular distributions of ranks $3,\ 4,\ 5,\ \dots$ until $n = \dim M$.

With no loss in generality, Goursat distributions understood locally live on the stages of Goursat Monster Tower (GMT for short), by some authors called alternatively Semple Tower. The stages have been denoted in [12] by $\mathbb{P}^r\mathbb{R}^2$, $r \geq 2$. (On the stage $\mathbb{P}^r\mathbb{R}^2$ there lives a Goursat distribution of corank r.) The best glasses to watch Goursat distributions are Kumpera-Ruiz coordinates (KR for short), [9]. Those are semi-global sets of coordinates (their domain of definition is always dense in a given tower's stage) which critically depend on the strata of a most natural stratification of any given stage $\mathbb{P}^r\mathbb{R}^2$ – so-called *Kumpera-Ruiz classes*, KR-classes for short, see [11], p. 466. They exist in $\mathbb{P}^r\mathbb{R}^2$ in number 2^{r-2} and are univocally labelled by the words of length r over the alphabet $\{1,\, 2\}$, with two first letters always 1: $1.1.\,i_3.\,i_4.\dots.i_r$. (In [11] they were originally labelled by the subsets $I \subset \{3,\, 4, \dots,\, r\}$, a given I consisting of the indices j such that $i_j = 2$.) The KR classes are the main tool in the introductory part of our paper. Their generalizations for special 2-flags, so-called singularity classes, will play a similar role in the main part of the present contribution from Section 5 onwards.

To each KR-class attached are handy coordinates making that class *visible*. More precisely, due to the particular topology of the two lowest Monster's stages $\mathbb{P}^1\mathbb{R}^2$ and $\mathbb{P}^2\mathbb{R}^2$, they both are unions of pairs of open dense subsets, $\mathbb{P}^1\mathbb{R}^2 = U_1 \cup U_2$ and $\mathbb{P}^2\mathbb{R}^2 = V_1 \cup V_2$ such that, for each KR-class $\mathcal{C} = 1.1.\,i_3.\,i_4.\dots.i_r$ and indices $j,\, k \in \{1,\, 2\}$

$$\mathcal{C} \cap \pi_{r,1}^{-1}(U_j) \cap \pi_{r,2}^{-1}(V_k) \tag{6}$$

sits in the domain of Kumpera-Ruiz coordinates $x_1,\, x_2, \dots,\, x_{r+2}$ produced precisely for the data $\mathcal{C},\ j,\ k$.

Remark 1. The open dense sets U_j and V_k are related to the ways the Darboux theorem (in the contact 3D manifold $\mathbb{P}^1\mathbb{R}^2$) and Engel theorem (in the *Engel* 4D manifold $\mathbb{P}^2\mathbb{R}^2$) come into effect. In those coordinates

$$\Delta^r = \Big(Y[r],\, \partial_{r+2}\Big)\,, \tag{7}$$

where, in what follows, $\partial_j = \frac{\partial}{\partial x^j}$ and $Y[r]$ is a polynomial vector field defined recursively as follows.

Initially $Y[1] = \partial_1 + x^3\partial_2$ and $Y[2] = Y[1] + x^4\partial_3$. When, for $j \geq 3$, $Y[j-1]$ is already defined and $i_j = 1$, then $Y[j] = Y[j-1] + x^{j+2}\partial_{j+1}$. In the opposite case of $i_j = 2$ one puts $Y[j] = x^{j+2}Y[j-1] + \partial_{j+1}$. The eventual vector field $Y[r]$ in (7) is, therefore, polynomial of degree (1 + the # of letters 2 in the code of $\mathcal{C}$). That degree is maximal (and equal $r-1$) when the underlying KR-class is 1.1.2.2. . . 2 ($r-2$ letters 2 past the initial segment 1.1).

Remark 2. Whenever $i_j = 2$ in the code of $\mathcal{C}$, the variable x^{j+2} brought in at the j-th step of the above procedure *vanishes* at points of (6). This is a key property of the polynomial visualisations of Goursat distributions put forward in [9].

The KR-classes are invariant with respect to the local diffeomorphisms of Monster's relevant stages. They are only very rough approximations to local models (local normal forms). To really approach the orbits, one would need to know the (pseudo-)groups of infinitesimal symmetries of the structures Δ^r living on $\mathbb{P}^r\mathbb{R}^2$. Those groups are infinite-dimensional, for they consist of due prolongations of the contact vector fields which preserve the contact structure Δ^1. In order to see them, one puts on, no wonder, KR-glasses. That is, works and computes in chosen KR-coordinates.

4. Infinitesimal symmetries of Goursat flags

From now on we assume that KR-coordinates, pertinent for a fixed KR-class in length r, have been picked and frozen. In these coordinates, every concrete infinitesimal symmetry writes down as $\mathcal{Y}_f = \sum_{i=1}^r F^i\partial_i$, where the first three components are functions of one (smooth) generating function in three variables, say $f(x^1, x^2, x^3)$:

$$F^1 = -f_3\,, \qquad F^2 = f - x^3 f_3\,, \qquad F^3 = f_1 + x^3 f_2\,, \tag{8}$$

and the remaining components are other, more complicated functions of f depending on the KR-class in question, as will be recalled in what follows. Such one free function f is called a *contact hamiltonian*; the infinite dimensionality of the symmetry pseudogroup is visible.

When a vector field $\mathcal{Y}_f$ preserves infinitesimally the Goursat Δ^r, the *truncations* of $\mathcal{Y}_f$ do infinitesimally preseve all the earlier (older) Goursat structures showing up in the process of building up Δ^r. In fact, each component F^s, $s = 4, 5, \ldots, r+2$, depends **only** on the variables $x^1, x^2, \ldots, x^s$ and

$$\left[\sum_{i=1}^{j+2} F^i\partial_i\,,\ \Delta^j\right] \subset \Delta^j \tag{9}$$

for $j = 1, 2, \ldots, r$, where $\Delta^j = \Big(Y[j],\, \partial_{j+2}\Big)$, as in (7). This technically central statement is well-known in the theory of Goursat structures, compare for instance Proposition 1 in [14]. Besides, this triangle nature of the infinitesimal symmetries of Goursat structures will be clearly visible in the recurrences that are produced below. The first prolongation of an infinitesimal contactomorphism $\sum_{i=1}^3 F^i\partial_i$ is $\sum_{i=1}^4 F^i\partial_i$, and the new component is univocally determined by the previous ones,

$$F^4 = Y[2]F^3 - x^4\, Y[2]F^1, \tag{10}$$

compare p. 222 in [14]. Reiterating, the components F^1 and F^3 entering formula (10) depend on the first three variables, and the field $Y[2]$ differentiates them accordingly. In the outcome, the component F^4 depends on the first four variables, and so it goes further on. (This formula is, in fact, subsumed in the line of derivations that follow. It is given here prior to more involved relations that depend already on the KR-class underlying the KR coordinates in use.)

We work with a fixed class $\mathcal{C} = 1.1.\,i_3.\,i_4.\ldots i_r$ and with a fixed letter i_j in its code, $j \geq 3$. In order to word the recurrences governing the infinitesimal symmetries of $\mathcal{C}$, we need a

Definition of $s(j)$ for Goursat flags. There can, or cannot, be letters 2 before the letter i_j.

$$s(j) :\ = \begin{cases} 0\,, & \text{when there is no letter 2 in the code of } \mathcal{C} \text{ before } i_j\,, \\ s\,, & \text{the farthest position of a letter 2 before } i_j \text{ is } s, \text{ in the opposite case.} \end{cases}$$

Theorem 1 ([13]). *Suppose that the components $F^1, F^2, \ldots, F^{j+1}$, $j \geq 3$, of an infinitesimal symmetry $\mathcal{Y}_f$ of Δ^r in the vicinity of a KR-class $\mathcal{C} = 1.1.i_3.i_4.\ldots.i_r$ are already known. When $i_j = 1$, then*

$$F^{j+2} = \begin{cases} Y[j]F^{j+1} - x^{j+2}Y[2]F^1, & \text{when } s(j) = 0\,, \\ Y[j]F^{j+1} - x^{j+2}Y[s(j)]F^{s(j)+1}, & \text{when } s(j) \geq 3\,. \end{cases}$$

When $i_j = 2$, then

$$F^{j+2} = \begin{cases} x^{j+2}\Big(Y[2]F^1 - Y[j]F^{j+1}\Big), & \text{when } s(j) = 0\,, \\ x^{j+2}\Big(Y[s(j)]F^{s(j)+1} - Y[j]F^{j+1}\Big), & \text{when } s(j) \geq 3\,. \end{cases}$$

Note before the proof that, on the whole, there are 2^{j-2} versions of the formulas for the component function F^{j+2}, all of them encoded in this theorem. For that many KR-classes exist in length j. Those formulas are *polynomials* in the x variables, of growing degrees, with coefficients – partials (of growing orders) of a contact hamiltonian f.

The original proof of this theorem occupied full four pages in [13]. Now we are going to re-prove it in a much shorter manner. Then this new method will be generalized and applied to the 2-flags' case in the sections that follow.

To begin with, the truncation of the field $\mathcal{Y}_f$ to the Monster level j, $\sum_{i=1}^{j+2} F^i\partial_i$, preserves the Goursat structure Δ^j, as is noted already in (9). Implying, that

$$\left[\sum_{i=1}^{j+2} F^i\partial_i\,,\ Y[j]\right] = a_j Y[j] + b_j\partial_{j+2} \tag{11}$$

for certain unspecified functions a_j and b_j of variables $x^1, \ldots, x^{j+2}$.

Now we consider the situation $i_j = 1$. Remembering the construction of the field $Y[j]$ when the underlying KR-class is $\mathcal{C}$:

• when $s(j) = 0$, the first (∂_1) component on the LHS of (11) is $-Y[2]F^1$. And

•• when $s(j) \geq 3$, the $(s(j)+1)$-st component on the LHS of (11) is $-Y[s(j)]F^{s(j)+1}$. So

$$a_j = \begin{cases} -Y[2]F^1, & \text{when } s(j) = 0\,,\\ -Y[s(j)]F^{s(j)+1}, & \text{when } s(j) \geq 3\,. \end{cases} \tag{12}$$

One compares now the $(j+1)$-st components on the both sides of (11), obtaining

$$F^{j+2} - Y[j]F^{j+1} = a_j\, x^{j+2}\,.$$

Substituting on the RHS here the expressions (12) in due order, one gets closed form formulas for the ∂_{j+2} – component function F^{j+2}, as invoiced in the theorem. As for the coefficient function b_j in (11), it is – here and in what follows later – ascertained last, after finding out F^{j+2}.

In the situation $i_j = 2$ the arguments differ only technically. Now, regardless of the value of $s(j)$, the coefficient a_j can be extracted from (11) at the level ∂_{j+1}: on the LHS it is $-Y[j]F^{j+1}$, and it is a plain a_j on the RHS. Hence

$$a_j = -Y[j]F^{j+1}\,. \tag{13}$$

Then, no wonder, one compares the coefficients in (11) at: ∂_1, when $s(j) = 0$, or else at $\partial_{s(j)+1}$, when $s(j) \geq 3$. In the former case one fetches on the LHS the quantity $F^{j+2} - x^{j+2}\,Y[2]F^1$. In the latter, the quantity $F^{j+2} - x^{j+2}Y[s(j)]F^{s(j)+1}$.

At the same time one fetches $a_j\, x^{j+2}$ on the RHS, just irrelevantly of the case in question. That is, accounting for (13),

$$F^{j+2} - x^{j+2}\,Y[2]F^1 = -Y[j]F^{j+1}\,x^{j+2}$$

(when $s(j) = 0$), or else

$$F^{j+2} - x^{j+2}Y[s(j)]F^{s(j)+1} = -Y[j]F^{j+1}\,x^{j+2}$$

(when $s(j) \geq 3$). A closed form formula for F^{j+2}, invoiced earlier, follows immediately. Only then the b_j coefficient is got hold of. In order to conclude that the ascertained vector field actually *is* a symmetry of Δ^r one observes that, in each of the underlying 2^{r-2} situations,

$$\left[\sum_{i=1}^{r+2} F^i \partial_i\,,\ \partial_{r+2}\right] = \Big(-\,\partial_{r+2}F^{r+2}\Big)\,\partial_{r+2}\,,$$

because only its last component function F^{r+2} depends on the last variable x^{r+2}. Theorem 1 is now proved. □

5. Special 2-flags: a basic toolkit

Special 2-flags constitute a natural follow-up to Goursat flags. The latter compactify (in certain precise sense) the contact Cartan distributions on the jet spaces $J^r(1,1)$, while the former do the same with respect to the jet spaces $J^r(1,2)$.[5]

Sequences of Cartan prolongations of rank 3 distributions are the key players in producing (only locally, though) virtually all rank 3 distributions generating special 2-flags. There quickly emerges an immense tree of singularities of positive codimensions, all of them adjoining the unique open dense Cartan-like strata.

While the local classification problem is well advanced for the Goursat flags, most notably after the work [12], it is much less advanced for special 2-flags (or, more generally, for special multi-flags). It was first attacked in [8], then, in the chronological order, in: [15], [16], [22], [21], [17], [1], and [18]. After the year 2010 researchers were aiming at defining various invariant stratifications in the spaces of germs of special multi-flags: [19], [6], [5], [20]. The actual state of the art is reflected in a recent summarizing work [4]. The works [19] and [20] stand out due to a kinematical interpretation of the special 2-flags developed in them. Namely, a model of an articulated arm in the 3D space with an engine, or a spacecraft with attached string of satellites. The singularities related to various possible distributions of *right angles* between neighbouring segments are already well understood and encoded. However, the issue of constructing a kinematics-driven fine stratification analogous to Jean's one [7] of the car + trailers systems (modelling 1-flags) in terms of Jean's critical angles, is not yet solved. In particular, a faithful expression of the classes in the benchmark work [4], in the terms of an articulated arm in 3D space, seems to be out of reach. The issue mentioned above is, most likely, equivalent to that of computing all small growth vectors for distributions generating special 2-flags.

In the work [18] there was completed only the classification of special 2-flags in lengths not exceeding 4. At that time the machinery of infinitesimal symmetries for those objects was far from being assembled and the techniques in use were rather disparate. This notwithstanding, the precise number (34) of local equivalence classes of special 2-flags in length 4 was ascertained there (cf. the table below).

The driving force of the present work are the *singularity classes* (in the occurrence – of special 2-flags) known for 17 years already, [15]. They are technically most important for our purposes and results. We briefly recall their construction in the next section. For reader's convenience, here is the table of cardinalities of singularity classes, RV classes of Castro *et al* [4], and classes of the local equivalence of the special 2-flags, in function of flag's lengths not exceeding 7:

[5] Some researchers, e.g. in [5], use, instead of 'special multi-flags' a somehow misleading synonym 'Goursat multi-flags'.

length	# sing classes	# RV classes	# orbits
2	2	2	2
3	5	6	7
4	14	23	34
5	41	98	?
6	122	433	??
7	365	1935	∞

Question. How to partition a given singularity class of special 2-flags into (much finer!) RV classes of [4] ? And, all the more so, for special m-flags, $m > 2$? !

5.1. **Singularity classes of special 2-flags refining the sandwich classes.** We first divide all existing germs of special 2-flags of length r into 2^{r-1} pairwise disjoint *sandwich classes* in function of the geometry of the distinguished spaces in the sandwiches (at the reference point for a germ) in Sandwich Diagram on p. 3, and label those aggregates of germs by words of length r over the alphabet $\{1, \underline{2}\}$ starting (on the left) with 1, having the second cipher $\underline{2}$ iff $D^2(p) \subset F(p)$, and for $3 \le j \le r$ having the j-th cipher $\underline{2}$ iff $D^j(p) \subset L(D^{j-2})(p)$. More details about the sandwich classes are given in section 1.2 in [18].

This construction puts in relief possible non-transverse situations in the sandwiches. For instance, the second cipher is $\underline{2}$ iff the line $D^2(p)/L(D^1)(p)$ is not transverse, in the space $D^1(p)/L(D^1)$, to the codimension one subspace $F(p)/L(D^1)(p)$, and similarly in further sandwiches. This resembles very much the KR-classes of Goursat germs constructed in [11]. In length r the number of sandwiches has then been $r - 2$ (and so the # of KR classes 2^{r-2}). For 2-flags the number of sandwiches is $r - 1$ because the covariant distribution of D^1 comes into play and gives rise to one additional sandwich.

Passing to the main construction underlying our present contribution, we refine further the singularities of special 2-flags and recall from [15] how one passes from the sandwich classes to *singularity classes*. In fact, to any germ $\mathcal{F}$ of a special 2-flag associated is a word $\mathcal{W}(\mathcal{F})$ over the alphabet $\{1, 2, 3\}$, called the 'singularity class' of $\mathcal{F}$. It is a specification of the word 'sandwich class' for $\mathcal{F}$ (this last being over, reiterating, the alphabet $\{1, \underline{2}\}$) with the letters $\underline{2}$ replaced either by 2 or 3, in function of the geometry of $\mathcal{F}$.

In the definition that follows we keep fixed the germ of a rank-3 distribution D at $p \in M$, generating on M a special 2-flag $\mathcal{F}$ of length r.

Suppose that in the sandwich class $\mathcal{C}$ of D at p there appears somewhere, for the first time when reading from the left to right, the letter $\underline{2} = j_m$ (j_m is, as we know, not the first letter in $\mathcal{C}$) **and** that there are in $\mathcal{C}$ other letters $\underline{2} = j_s,\ m < s$, as well. We will specify each such j_s to one of the two: 2 or 3. (The specification of that first $j_m = \underline{2}$ will be made later and will be trivial.) Let the nearest $\underline{2}$ standing to the left to j_s be $\underline{2} = j_t,\ m \le t < s$. These two 'neighbouring' letters $\underline{2}$ are separated in $\mathcal{C}$ by $l = s - t - 1 \ge 0$ letters 1.

The gist of the construction consists in taking the *small flag* of precisely original flag's member D^s,

$$D^s = V_1 \subset V_2 \subset V_3 \subset V_4 \subset V_5 \subset \cdots ,$$

$V_{i+1} = V_i + [D^s,\ V_i]$, then focusing precisely on this new flag's member V_{2l+3}. Reiterating, in the t-th sandwich, there holds the inclusion: $F(p) \supset D^2(p)$ when $t = 2$, or else $L(D^{t-2})(p) \supset D^t(p)$ when $t > 2$. This serves as a preparation to our punch line (cf. [15, 17]).

Surprisingly perhaps, specifying j_s to 3 goes via replacing D^t by V_{2l+3} in the relevant sandwich inclusion at the reference point. That is to say, $j_s = \underline{2}$ is being specified to 3 if and only if $F(p) \supset V_{2l+3}(p)$ (when $t = 2$) or else $L(D^{t-2})(p) \supset V_{2l+3}(p)$ (when $t > 2$) holds.

In this way all non-first letters $\underline{2}$ in $\mathcal{C}$ are, one independently of another, specified to 2 or 3. Having that done, one simply replaces the first letter $\underline{2}$ by 2, and altogether obtains a word over $\{1, 2, 3\}$. It is the singularity class $\mathcal{W}(\mathcal{F})$ of $\mathcal{F}$ at p.

Example. In length 4 there exist the following fourteen singularity classes: 1.1.1.1, 1.1.1.2; 1.1.2.1, 1.1.2.2, 1.1.2.3; 1.2.1.1,[6] 1.2.1.2, 1.2.1.3, 1.2.2.1, 1.2.2.2, 1.2.2.3, 1.2.3.1, 1.2.3.2, 1.2.3.3. (cf. the table on p. 9).

(In length r the # of singularity classes is $\frac{1}{2}\big(3^{r-1}+1\big)$; the codimension of a class equals the # of 2's plus twice the # of 3's in the relevant code word.)

5.2. **New approach in the classification problem.** A new (2017) approach to the local classification of flags starts with the effective (recursive) computation of all infinitesimal symmetries of special 2-flags, extending the work done (in [13]) for 1-flags, reproduced with essential shortcuts in Section 4 above. The recursive patterns depend uniquely on the singularity classes of special 2-flags recapitulated above. Those classes are coarser, yes, but much fewer – see the table preceding section 5.1 – than the RV classes summarized (and so neatly systematized) in [4].

Polynomial visualisations of objects in the singularity classes, recalled in Section 6, are called EKR's (Extended Kumpera-Ruiz). They 'only' feature finite families of real parameters. Then the local classification problem is rephrased as a search for ultimate normalizations among such families of parameters. Having an explicit hold of the infinitesimal symmetries at each prolongation step, the freedom in varying those parameters will be ultimately reduced to solvability questions of (typically huge) systems of linear equations.

In fact, that linear algebra involves only partial derivatives, at the reference point, of the first three components of a given infinitesimal symmetry which are completely free functions of 3 variables (Lemma 1). Keeping the preceding part of a germ of a flag in question frozen imposes a sizeable set of linear conditions upon those derivatives up to certain order. Then some other linear combinations of them appear, or not, to be free – just in function of the local geometry of the prolonged distribution. This, in short, would determine the scope of possible normalizations in the new (emerging from prolongation) part of EKR's. See sections 8.1 and 8.2 below for more details.

6. EKR glasses for singularity classes of special 2-flags

According to section 5.1, the singularity classes of special 2-flags of length r are univocally encoded by words of length r over the alphabet $\{1, 2, 3\}$ such that: - the first letter is always 1, and - a letter 3, if any, must be preceded by a letter 2. That is to say, abusing notation a bit, for a singularity class $\mathcal{C} = 1.i_2.i_3 \dots i_r$ over $\{1,2,3\}$, a letter i_2 is either 1 or 2, and a letter 3 may show up not earlier than at the 3rd position, provided there is a letter 2 before it. (We call it, especially in the wider context of special m-flags with arbitrary m, 'the least upward jumps rule', cf. [16].)

For instance, $\mathcal{C} = 1.2.3$ is a legitimate singularity class of length 3 (and, in the occurrence, of codimension three in the pertinent Monster's stage No 3).

For each such $\mathcal{C}$ we are going to introduce coordinates, in the number of $2r+3$,

$$t,\, x^0,\, y^0,\, x^1,\, y^1, \dots,\, x^r,\, y^r\,, \tag{14}$$

in which the special rank 3 distribution – let us, from now on, call it Δ^r again – living on the Monster's r-th stage becomes visible. Those coordinates, we reiterate it, will sensitively depend

[6] See section 8.2 for more information about precisely this class.

on a class $\mathcal{C}$. In fact, skipping the geometric and also Lie-algebra-related arguments presented in detail in [17], within the domain of those coordinates (subsuming the class $\mathcal{C}$),

$$\Delta^r = \Big(Z[r],\, \partial_{x^r},\, \partial_{y^r}\Big)\,, \tag{15}$$

where the vector field $Z[r]$ is being defined recursively, shadowing step after step the code $1.i_2.i_3 \dots i_r$ of $\mathcal{C}$. The beginning of recurrence is $Z[1] = \partial_t + x^1\partial_{x^0} + y^1\partial_{y^0}$, and, quite simply, $\Delta^1 = \Big(Z[1],\, \partial_{x^1},\, \partial_{y^1}\Big)$ on $\mathbb{R}^5(t, x^0, y^0, x^1, y^1)$.

In the recurrence step one assumes description (15) known for $j-1$ in the place of r, where $1 \le j-1 \le r-1$, and puts

$$Z[j] \;=\; \begin{cases} Z[j-1] + x^j\partial_{x^{j-1}} + y^j\partial_{y^{j-1}}\,, & \text{when } i_j = 1\,,\\ x^jZ[j-1] + \partial_{x^{j-1}} + y^j\partial_{y^{j-1}}\,, & \text{when } i_j = 2\,,\\ x^jZ[j-1] + y^j\partial_{x^{j-1}} + \partial_{y^{j-1}}\,, & \text{when } i_j = 3\,. \end{cases} \tag{16}$$

In the end of this recurrence (for $j = r$) the description (15) *tout court* is arrived at, on $\mathbb{R}^{2r+3}$ in the variables (14). The final first vector field' generator $Z[r]$ is a, possibly deeply involved (in function of $\mathcal{C}$), polynomial vector field.

Our objective is to ascertain all infinitesimal symmetries $\mathcal{Y}$ of (15) in the vicinity of any particular class $\mathcal{C}$. They will, no wonder, sensitively depend on $\mathcal{C}$, too. Let us have such $\mathcal{Y}$ expanded in EKR coordinates chosen for $\mathcal{C}$:

$$\mathcal{Y} \;=\; A\,\partial_t + B\,\partial_{x^0} + C\,\partial_{y_0} + \sum_{s=1}^{r}\Big(F^s\,\partial_{x^s} + G^s\,\partial_{y^s}\Big)\,. \tag{17}$$

The first key property (needed later) is

Lemma 1. *The component functions A, B, C in (17) depend only on the variables t, x^0, y^0.*

Proof of Lemma 1. The reason is that, whatever the class $\mathcal{C}$, in the chosen EKR coordinates associated to $\mathcal{C}$ the bottom row in Sandwich Diagram has formally the same description as for the Cartan contact system on $J^r(1,2)$. In particular, because the relations (3) keep holding true in the vicinity of $\mathcal{C}$ in these coordinates, the covariant subdistribution F of D^1 is there invariably of the form

$$F \;=\; \big(\partial_{x^i},\; \partial_{y^i}\,;\; 1 \le i \le r\big)$$

The symmetry $\mathcal{Y}$, preserving $\Delta^r =: D$, preserves the derived flag $\big(D^j\big)_{j=r}^{0}$ of D, so preserves this F, too. Hence the first three components of $\mathcal{Y}$ can*not* depend on the variables x^i and y^i for $1 \le i \le r$, as stated in the lemma. □

Remark 3. Note, however, one essential difference with the 1-flags in that here are **three** free functions in the base of the theory, instead of just one contact hamiltonian there (in formulas (8)).

As before, one needs some additional information about the code of $\mathcal{C}$. So for $j = 2, 3, \dots, r$ we define

$$s(j) = \begin{cases} 0\,, & \text{when } i_2, \dots, i_{j-1} = 1\,,\\ \max\{s\colon\ 2 \le s < j\ \ \&\ \ i_s > 1\}\,, & \text{in the opposite case}\,. \end{cases}$$

Note that when $s(j) \ge 2$, then $i_{s(j)} = 2$ or else $i_{s(j)} = 3$. These two distinct (and disjoint) geometric situations account for bigger complexity of the recurrences to be produced. (The eventail of possible singularities of special 2-flags is much wider than for Goursat.)

7. Infinitesimal symmetries of special 2-flags brought under control

Our main theorem of the paper, Theorem 2 below, shows that every infinitesimal symmetry is uniquely determined by the singularity class under consideration together with symmetry's first three component functions, denoted traditionally A, B, C, **in an explicit, algorithmically computable manner**. Namely,

Theorem 2. *Let U be the domain of EKR coordinates (14) chosen for an arbitrarily fixed singularity class $1.i_2.i_3\ldots.i_r$. In those coordinates, all infinitesimal symmetries $\mathcal{Y}$ of Δ^r restricted to U are of a particular form (17), where A, B, C are free smooth functions of only t, x^0, y^0 and the F^s, G^s, $1 \le s \le r$, are univocally recursively determined by A, B, C and the class code, according to the formulae given in (20) and Lemmas 2, 3 and 4 below.*

PROOF. We are going to ascertain one by one (or rather two by two) the consecutive components of vector fields $\mathcal{Y}$ in (17) above, from F^1 and G^1 on, given the initial arbitrary function data A, B, C. To this end we will use the truncations $\mathcal{Y}[j]$ of $\mathcal{Y}$ to the spaces of coordinates of indices $\le j$, $j = 1, 2, \ldots, r$, on which the distributions Δ^j live:

$$\mathcal{Y}[j] \;=\; A\,\partial_t + B\,\partial_{x^0} + C\,\partial_{y_0} + \sum_{s=1}^{j}\Big(F^s\,\partial_{x^s} + G^s\,\partial_{y^s}\Big). \tag{18}$$

Attention. The formulas (20) right below and in Lemmas 2, 3 and 4 below are, in the first place, only *necessary* for $\mathcal{Y}$ to be a true symmetry of Δ^r. They became also *sufficient* in the last part of our (long) proof of Theorem 2.

To begin with, let us demonstrate the argument on the 'baby' components F^1 and G^1. The infinitesimal invariance condition

$$\big[\mathcal{Y}[1]\,,\ \Delta^1\big] \subset \Delta^1$$

clearly implies

$$[\mathcal{Y}[1]\,,\ Z[1]] = a_1\,Z[1] + b_1\partial_{x^1} + c_1\partial_{y^1}\,, \tag{19}$$

which in turn implies $a_1 = -Z[1]A$. At the same time $F^1 - ZB[1] = a_1\,x^1$ and $G^1 - Z[1]C = a_1\,y^1$. Putting all this together,

$$\begin{cases} F^1 \,=\, Z[1]B - x^1 Z[1]A\,,\\ G^1 \,=\, Z[1]C - y^1 Z[1]A\,. \end{cases} \tag{20}$$

So indeed the pair of new components in $\mathcal{Y}[1]$ is univocally determined by the base components A, B, C. As for the coefficients b_1 and c_1 in (19), they get ascertained only *after* F^1 and G^1 are found.

This inference is an instance of a general

Lemma 2. *Assuming that an infinitesimal symmetry $\mathcal{Y}[j-1]$ of Δ^{j-1} is already known for certain $2 \le j \le r$,* **in the situation** $i_j = 1$, *the ∂_{x^j} – and ∂_{y^j} – components of the prolongation $\mathcal{Y}[j]$ of $\mathcal{Y}[j-1]$ are as follows*

$$F^j \;=\; \begin{cases} Z[j]F^{j-1} -\; x^j Z[1]A\,, & \text{when } s(j) = 0\,,\\ Z[j]F^{j-1} -\; x^j Z[s(j)]F^{s(j)-1}\,, & \text{when } s(j) \ge 2,\ i_{s(j)} = 2\,,\\ Z[j]F^{j-1} -\; x^j Z[s(j)]G^{s(j)-1}\,, & \text{when } s(j) \ge 2,\ i_{s(j)} = 3\,. \end{cases}$$

$$G^j = \begin{cases} Z[j]G^{j-1} - y^j Z[1]A\,, & \text{when } s(j) = 0\,, \\ Z[j]G^{j-1} - y^j Z[s(j)]F^{s(j)-1}\,, & \text{when } s(j) \geq 2,\ i_{s(j)} = 2\,, \\ Z[j]G^{j-1} - y^j Z[s(j)]G^{s(j)-1}\,, & \text{when } s(j) \geq 2,\ i_{s(j)} = 3\,. \end{cases}$$

Proof of Lemma 2. The vector field $\mathcal{Y}[j]$ infinitesimally preserves the distribution Δ^j, whence

$$[\mathcal{Y}[j]\,,\ Z[j]] = a_j Z[j] + b_j\,\partial_{x^j} + c_j\,\partial_{y^j} \tag{21}$$

for certain unspecified functions a_j, b_j, c_j. The coefficient a_j is of central importance here. We typically work, here and in what will follow later, in the following order: - we firstly ascertain a_j, - secondly find (this is most important) F^j and G^j, - eventually ascertain the values of b_j and c_j.

The function a_j can be extracted from (21) by watching this vector equation on the level of such a component of $Z[j]$ which is identically 1. Inspecting the stepwise construction that leads from $Z[1]$ to $Z[j]$, there always *is* such a component! Namely, it is the ∂_t – component when $s(j) = 0$. When, on the contrary, $s(j) \geq 2$, it is either the $\partial_{x^{s(j)-1}}$ – component (when $i_{s(j)} = 2$), or else it is the $\partial_{y^{s(j)-1}}$ – component (when $i_{s(j)} = 3$). With thus specified information, it is a matter of course that

$$a_j = -\begin{cases} Z[1]A\,, & \text{when } s(j) = 0\,, \\ Z[s(j)]F^{s(j)-1}\,, & \text{when } s(j) \geq 2,\ i_{s(j)} = 2\,, \\ Z[s(j)]G^{s(j)-1}\,, & \text{when } s(j) \geq 2,\ i_{s(j)} = 3\,. \end{cases} \tag{22}$$

On the other hand, the same equation (21) watched on the level of $\partial_{x^{j-1}}$ reads

$$F^j - Z[j]F^{j-1} = a_j\,x^j\,,$$

and watched on the level of $\partial_{y^{j-1}}$ reads

$$G^j - Z[j]G^{j-1} = a_j\,y^j\,.$$

The needed expressions for F^j and G^j follow upon substituting the expression (22) of a_j into these two equations. □

Lemma 3. *Assuming that an infinitesimal symmetry $\mathcal{Y}[j-1]$ of Δ^{j-1} is already known for certain $2 \leq j \leq r$,* **in the situation** *$i_j = 2$, the ∂_{x^j} – and ∂_{y^j} – components of the prolongation $\mathcal{Y}[j]$ of $\mathcal{Y}[j-1]$ are as follows*

$$F^j = \begin{cases} x^j\left(Z[1]A - Z[j]F^{j-1}\right)\,, & \text{when } s(j) = 0\,, \\ x^j\left(Z[s(j)]F^{s(j)-1} - Z[j]F^{j-1}\right)\,, & \text{when } s(j) \geq 2,\ i_{s(j)} = 2\,, \\ x^j\left(Z[s(j)]G^{s(j)-1} - Z[j]F^{j-1}\right)\,, & \text{when } s(j) \geq 2,\ i_{s(j)} = 3\,. \end{cases}$$

$$G^j = Z[j]G^{j-1} - y^j Z[j]F^{j-1}\,.$$

Proof of Lemma 3. The vector equation (21) still holds true. Now the a_j coefficient can be (and easily) extracted from it at the level $\partial_{x^{j-1}}$, because the coefficient of the $\partial_{x^{j-1}}$ – component in $Z[j]$ is 1:

$$a_j = -Z[j]F^{j-1}. \tag{23}$$

At the same time writing down the equal sides of (21) at the level $\partial_{y^{j-1}}$,

$$G^j - Z[j]G^{j-1} = a_j\,y^j,$$

leads, by the way of (23), to the desired formula for G^j.

It is not that quick with the function F^j. It can be extracted from precisely one out of three levels of the ∂_t –, $\partial_{x^{s(j)-1}}$ –, or $\partial_{y^{s(j)-1}}$ – components. Because one, once again, looks for a component in $Z[j]$ with a coefficient 1, if 'enveloped' now in the factor x^j (because $i_j > 1$ in the proposition under proof).

In function of the position of that '1', equalling the relevant levels in (21), one gets precisely one relation out of the following three

$$\begin{cases} F^j - x^j Z[1]A = a_j\, x^j\,, & \text{when } s(j) = 0\,, \\ F^j - x^j Z[s(j)]F^{s(j)-1} = a_j\, x^j\,, & \text{when } s(j) \geq 2,\ i_{s(j)} = 2\,, \\ F^j - x^j Z[s(j)]G^{s(j)-1} = a_j\, x^j\,, & \text{when } s(j) \geq 2,\ i_{s(j)} = 3\,. \end{cases}$$

Then, accounting for (23), the desired formula for F^j follows. □

Lemma 4. *Assuming that an infinitesimal symmetry $\mathcal{Y}[j-1]$ of Δ^{j-1} is already known for certain $2 \leq j \leq r$,* **in the situation** $i_j = 3$*, the $\partial_{x^j}-$ and $\partial_{y^j}-$ components of the prolongation $\mathcal{Y}[j]$ of $\mathcal{Y}[j-1]$ are as follows*

$$F^j = \begin{cases} x^j \left(Z[1]A - Z[j]G^{j-1}\right)\,, & \text{when } s(j) = 0\,, \\ x^j \left(Z[s(j)]F^{s(j)-1} - Z[j]G^{j-1}\right)\,, & \text{when } s(j) \geq 2,\ i_{s(j)} = 2\,, \\ x^j \left(Z[s(j)]G^{s(j)-1} - Z[j]G^{j-1}\right)\,, & \text{when } s(j) \geq 2,\ i_{s(j)} = 3\,. \end{cases}$$

$$G^j = Z[j]F^{j-1} - y^j Z[j]G^{j-1}\,.$$

Proof of Lemma 4. Invariably, the vector equation (21) keeps holding true. The a_j coefficient on its right hand side can be extracted from it at the level $\partial_{y^{j-1}}$, because now the coefficient of the $\partial_{y^{j-1}}-$component in $Z[j]$ is 1:

$$a_j = -Z[j]G^{j-1}. \tag{24}$$

Then, writing simply down the equal sides of (21) at the level $\partial_{x^{j-1}}$,

$$G^j - Z[j]F^{j-1} = a_j\, y^j,$$

leads, by the way of (24), to the presently needed formula for G^j.

As for the function F^j, it can again be extracted from precisely one out of three levels of the ∂_t-, $\partial_{x^{s(j)-1}}-$, or $\partial_{y^{s(j)-1}}-$components. In function of the position of that key component '1' in the field $Z[j]$, equalling the sides of the relevant levels in (21), one gets precisely one relation out of the following three

$$\begin{cases} F^j - x^j Z[1]A = a_j\, x^j\,, & \text{when } s(j) = 0\,, \\ F^j - x^j Z[s(j)]F^{s(j)-1} = a_j\, x^j\,, & \text{when } s(j) \geq 2,\ i_{s(j)} = 2\,, \\ F^j - x^j Z[s(j)]G^{s(j)-1} = a_j\, x^j\,, & \text{when } s(j) \geq 2,\ i_{s(j)} = 3\,. \end{cases}$$

Upon accounting for (24), the expected formula for F^j follows. □

As already invoiced, the obtained recursive formulas – at this moment only necessary – are also sufficient for the produced vector field $\mathcal{Y}$ to actually *be* a symmetry of Δ^r. Indeed, knowing already that $\left[\mathcal{Y},\, Z[r]\right] \in \Delta^r$ (cf. the always holding true formulas (21) taken now for $j = r$), what only remains to be done is to take the remaining two generators of Δ^r and justify the vector fields' inclusions

$$\left[\mathcal{Y},\, \partial_{x^r}\right],\ \left[\mathcal{Y},\, \partial_{y^r}\right] \in \Delta^r.$$

To that end we note that Lemma 1 coupled with formulas (20) and all those listed in auxiliary Lemmas 2, 3 and 4 yield by simple induction that, for $j = 1, 2, \ldots, r$,

the components F^j and G^j of $\mathcal{Y}$ depend only on $t,\, x^0,\, y^0,\, x^1,\, y^1, \ldots, x^j,\, y^j$.

Using this information for $1 \le j \le r-1$ and again Lemma 1, one computes with ease

$$\big[\mathcal{Y},\, \partial_{x^r}\big] \;=\; -\big[\partial_{x^r},\, \mathcal{Y}\big] \;=\; \big(-\,\partial_{x^r}\, F^r\big)\, \partial_{x^r} + \big(-\,\partial_{x^r}\, G^r\big)\, \partial_{y^r}$$

and

$$\big[\mathcal{Y},\, \partial_{y^r}\big] \;=\; -\big[\partial_{y^r},\, \mathcal{Y}\big] \;=\; \big(-\,\partial_{y^r}\, F^r\big)\, \partial_{x^r} + \big(-\,\partial_{y^r}\, G^r\big)\, \partial_{y^r}\,.$$

Now, at long last, the proof of Theorem 2 is complete. □

8. Applications of recursively computable infinitesimal symmetries to the local classification problem

The main motivation underlying the present contribution has been to advance results in the local classification problem for special 2-flags – to propose a late follow-up to the work [18]. In fact, getting – recursively – hold of the infinitesimal symmetries of special 2-flags[7] opens a way to advance the local classification in lengths $r = 5$ (cf. in this respect, in particular, section 8.2) and $r = 6$ which have kept challenging the small monster community for the last 17 years (see the table preceding section 5.1).

8.1. **Continuous modulus in the class 1.2.1.2.1.2.1.** Reiterating already, the exact local classification of special 2-flags (and, all the more so, all special *multi*-flags) in lengths exceeding 4 is, in its generality, unknown. It is not excluded that a continuous modulus of the local classification hides itself already somewhere in length 6. Instead, we want to give an example in length 7 of the effectiveness of our formulae put forward in Section 7.

A possibly deepest fact communicated in [18] was

Theorem 3 ([18])**.** *In the singularity class $\mathcal{C} = 1.2.1.2.1.2.1$ of special 2-flags of length 7 there resides a continuous modulus of the local classification.*

This was originally proved (in the year 2003, as a matter of fact) by brute force, and here is how the infinitesimal symmetries may help.

PROOF. In the coordinates constructed for the class $\mathcal{C}$ we work with certain *germs* of the distribution Δ^7 which generates a locally universal special 2-flag of length 7. The reference points for those germs belong to $\mathcal{C}$. More precisely, these are the points, say P, with the coordinates

$$\begin{gathered} t = x^0 = y^0 = x^1 = y^1 = x^2 = y^2 = 0\,, \quad x^3 = 1\,, \\ y^3 = x^4 = y^4 = 0\,,\;\; x^5 = 1\,,\;\; y^5 = x^6 = y^6 = 0\,,\;\; x^7 = c\,,\;\; y^7 = 0\,. \end{gathered} \tag{25}$$

We *intend* to infinitesimally move such P only in the $\partial_{x^7}-$direction. (Compare, for instance, [11], where also only the farthest part of a flag – Goursat in that occurrence – was subject to possible movies.) That is, we look for an infinitesimal symmetry having at a point P of type (25) **all but the $\partial_{x^7}-$components vanishing**. Remembering about the triangle pattern of dependence of those component functions, this means the vanishing of A, B, C at $(0, 0, 0)$, the vanishing of $F^j\big(\pi_{7,j}(P)\big)$, $G^j\big(\pi_{7,j}(P)\big)$ for $j = 1, 2, \dots, 6$ and the vanishing of $G^7(P)$. The component $F^7(P)$ is not yet known and will be analyzed with care.

Initially we do not know how few/many such vector fields could exist. At any rate, any one of them is induced by certain functions A, B, C in the variables t, x, y. The recurrence formulae are known from Section 7. When, among other components of an infinitesimal symmetry, one

[7] As a matter of fact, our approach presented in this paper extends naturally to all special m-flags, $m \ge 2$ – this being the subject of another possible paper.

wants to express $F^7(P)$ via those basic unknown functions A, B, C, one goes *backwards* along the code of $\mathcal{C}$, and firstly applies Lemma 2 (because $i_7 = 1$), then Lemma 3 (because $i_6 = 2$), then again Lemma 2 (because $i_5 = 1$), and so on intermittently. Upon applying with care these lemmas due numbers of times, the above-listed vanishings mean in the terms of the functions in the base

$$0 = A(\mathbf{0}) = B(\mathbf{0}) = C(\mathbf{0}) = B_t(\mathbf{0}) = C_t(\mathbf{0}) = C_{x^0}(\mathbf{0}) = c\, C_{t\,x^0}(\mathbf{0}),$$

where the $\mathbf{0}$'s above stand for $(0,0,0)$, and – most important

$$0 = F^3\big(\pi_{7,3}(P)\big) = \big(3A_t - 2B_{x^0}\big)(0,0,0), \tag{26}$$

$$0 = F^5\big(\pi_{7,5}(P)\big) = \big(B_{x^0} - A_t\big)(0,0,0). \tag{27}$$

Now comes the punch line, because the outcome of the computations for F^7 is

$$F^7(P) = 3c\big(A_t - B_{x^0}\big)(0,0,0). \tag{28}$$

Relations (26) and (27) together imply $A_t(0,0,0) = B_{x^0}(0,0,0) = 0$. So $F^7(P) = 0$ by (28). That is, every infinitesimal symmetry of $\mathcal{C}$ must infinitesimally freeze at P the coordinate x^7, when it infinitesimally freezes all the remaining coordinates specified in (25). Theorem 3 is now (re-) proved. $\square$

Remark 4. In other terms, the germs of the structure Δ^7 at various points P as above (i. e., for different values of the parameter c) are pairwise non-equivalent. The local geometry of the distribution Δ^7 changes continuously within the discussed class $\mathcal{C}$.

8.2. **Towards the classification of the one step prolongations within singularity class 1.2.1.1.** We conclude the paper by excerpting from [18] the partition, into the orbits of the local classification, of the singularity class 1.2.1.1 (when the width $m = 2$, cf. Remark 5 on p. 37 there), and suggesting a line of possible continuation in the next length 5. This class is not chosen at random; it splits into maximal (6) number of orbits in that length 4, cf. Section 7 in [18]. The names of orbits are taken from that preprint. One means **the germs of** Δ^4, watched in the EKR coordinates constructed for 1.2.1.1, **at points**, say P, having

$$t = x^0 = y^0 = x^1 = y^1 = x^2 = y^2 = 0$$

and the four highest coordinates x^3, y^3, x^4, y^4 as in the following table

the orbit	$x^3\Big(\pi_{4,3}(P)\Big)$	$y^3\Big(\pi_{4,3}(P)\Big)$	$x^4(P)$	$y^4(P)$
$1.2.1_{-\mathrm{s,tra}}.1$	1	0	0	0
$1.2.1_{-\mathrm{s,tan}}.1_{-\mathrm{s,tra}}$	0	1	1	0
$1.2.1_{-\mathrm{s,tan}}.1_{-\mathrm{s,tan}}$	0	1	0	0
$1.2.1_{+\mathrm{s}}.1_{-\mathrm{s,tra}}$	0	0	1	0
$1.2.1_{+\mathrm{s}}.1_{-\mathrm{s,tan}}$	0	0	0	1
$1.2.1.1_{+\mathrm{s}}$	0	0	0	0

Upon prolonging Δ^4 to Δ^5 in the vicinity of points of 1.2.1.1, one is to work with points in the classes $1.2.1.1.i_5$, $i_5 \in \{1,\, 2,\, 3\}$. The classification result recalled in the table above applies now to the distribution $\big[\Delta^5,\, \Delta^5\big]$ and as such remains true, regardless of the value of i_5 (the Lie square of Δ^5 does *not* depend on new variables x^5, y^5). The same concerns the recursive formulae for the component functions F^j, G^j, $j = 1, 2, 3, 4$, of the infinitesimal symmetries of Δ^5. Yet, naturally, expressions for the components F^5, G^5 depend critically on the value of i_5. Sticking to the points P from the table, one is to analyze the expressions for $F^5(Q)$ and $G^5(Q)$, $Q \in 1.2.1.1.i_5$, $\pi_{5,4}(Q) = P$. They are linear in $x^5(Q)$, $y^5(Q)$, with coefficients depending on P and on certain partials at $(0, 0, 0)$ of the basic functions A, B, C. All the difficulty resides in the – unknown and hard to compute – coefficients standing next to those partials.

An instructive example is given in section 8.1. The coefficient standing next to $c = x^7(P)$ on the RHS of (28) has appeared *forced* to be zero by the earlier infinitesimal normalizations (26) and (27). Because of that phenomenon, even the outcome of the classification of singularity class 1.2.1.1.1 ($i_5 = 1$) is difficult to predict.

In general – in higher lengths – *systems* of coefficients in *growing* sets of partials of A, B, C would play decisive roles in freezing or not of the values of new incoming pairs of component functions of the infinitesimal symmetries. Linear algebra packages would eventually come in handy.

References

[1] J. Adachi; *Global stability of special multi-flags.* Israel J. Math. **179**, (2010), 29–56. DOI: 10.1007/s11856-010-0072-3

[2] R. L. Bryant, L. Hsu; *Rigidity of integral curves of rank 2 distributions.* Invent. math. **114**, 435–461. DOI: 10.1007/bf01232676

[3] E. Cartan; *Les systèmes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre.* Ann. Ecole Normale **27** (1910), 109–192. DOI: 10.24033/asens.618

[4] A. Castro, S. J. Colley, G. Kennedy, C. Shanbrom; *A coarse stratification of the monster tower.* Michigan Math. J. **66** (2017), 855–866. DOI: 10.1307/mmj/1508896892

[5] A. Castro, W. Howard; *A Monster tower approach to Goursat multi-flags.* Diff. Geom. Appl. **30** (2012), 405–427. DOI: 10.1016/j.difgeo.2012.06.005

[6] A. Castro, R. Montgomery; *Spatial curve singularities and the Monster/Semple tower.* Israel J. Math. **192** (2012), 381–427. DOI: 10.1007/s11856-012-0031-2

[7] F. Jean; *The car with n trailers: characterization of the singular configurations.* ESAIM: COCV **1** (1996), 241–266 (electronic). DOI: 10.1051/cocv:1996108

[8] A. Kumpera, J. L. Rubin; *Multi-flag systems and ordinary differential equations.* Nagoya Math. J. **166** (2002), 1–27. DOI: 10.1017/s0027763000008229

[9] A. Kumpera, C. Ruiz; *Sur l'équivalence locale des systèmes de Pfaff en drapeau.* In: "Monge-Ampère Equations and Related Topics", Rome 1982, 201–248.

[10] R. Montgomery, *A Tour of Subriemannian Geometries, their Geodesics, and Applications.* Math. Surveys Monographs **91**, AMS (2002). DOI: 10.1090/surv/091

[11] R. Montgomery, M. Zhitomirskii; *Geometric approach to Goursat flags.* Ann. Inst. H. Poincaré (AN) **18** (2001), 459–493. www.numdam.org/article/AIHPC__2001__18_4_459_0.pdf

[12] R. Montgomery, M. Zhitomirskii; *Points and Curves in the Monster Tower.* Memoirs AMS **956** (2010). DOI: 10.1090/s0065-9266-09-00598-5

[13] P. Mormul; *Goursat distributions with one singular hypersurface — constants important in their Kumpera-Ruiz pseudo-normal forms.* Preprint N° 185, Labo Topologie, Univ. de Bourgogne, Dijon (1999).

[14] P. Mormul; *Contact hamiltonians distinguishing locally certain Goursat systems.* Banach Center Publ. **51**, Warsaw 2000, 219–230. www.impan.pl/BC/BCP/51/mormu.pdf

[15] P. Mormul; *Geometric singularity classes for special k-flags, $k \geq 2$, of arbitrary length* (2003). www.mimuw.edu.pl/~mormul/Mor03.ps

[16] P. Mormul; *Multi-dimensional Cartan prolongation and special k-flags.* Banach Center Publ. **65**, Warsaw 2004, 157–178. DOI: 10.4064/bc65-0-12

[17] P. Mormul; *Singularity classes of special 2-flags.* SIGMA **5** (2009), 102 (electronic). www.elibm.org/ft/10008671000

[18] P. Mormul, F. Pelletier; *Special 2-flags in lengths not exceeding four: a study in strong nilpotency of distributions*, (2010). arχiv: 1011.1763

[19] F. Pelletier, M. Slayman; *Articulated arm and special multi-flags.* J. Math. Sci. Adv. Appl. **8** (2011), 9–41.

[20] F. Pelletier, M. Slayman; *Configurations of an articulated arm and singularities of special multi-flags.* SIGMA **10** (2014), 059 (electronic). DOI: 10.3842/sigma.2014.059

[21] K. Shibuya, K. Yamaguchi; *Drapeau theorem for differential systems.* Diff. Geom. Appl. **27** (2009), 793–808. DOI: 10.1016/j.difgeo.2009.03.017

[22] P. J. Vassiliou; *A constructive generalised Goursat normal form.* Diff. Geom. Appl. **24** (2006), 332–350. DOI: 10.1016/j.difgeo.2005.12.001

[23] M. Zhitomirskii; *Singularities and normal forms of smooth distributions.* Banach Center Publ. **32**, Warsaw 1993, 395–409. DOI: 10.4064/-32-1-395-409

Piotr Mormul, Institute of Mathematics, University of Warsaw, Banach str. 2, 02-097 Warsaw, Poland

Email address: `mormul@mimuw.edu.pl`

Fernand Pelletier, Université de Savoie Mont Blanc (LAMA),73370 Le Bourget-du-Lac, France

Email address: `fernand.pelletier@univ-smb.fr`

Journal *of* Singularities
Volume 21 (2020), 205-211

Proc. of Geometric and Algebraic Singularity Theory, Będlewo, 2017
DOI: 10.5427/jsing.2020.21l

KATO'S CHAOS CREATED BY QUADRATIC MAPPINGS ASSOCIATED WITH SPHERICAL ORTHOTOMIC CURVES

TAKASHI NISHIMURA

Abstract. In this paper, we first show that for a given generic spherical curve $\gamma : I \to S^n$ and a generic point $P \in S^n$, the spherical orthotomic curve relative to γ and P naturally yield a simple quadratic mapping $\Phi_P : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. Since S^n is compact and $\Phi_P|_{S^n} : S^n \to S^n$ is the spherical counterpart of the trivial expanding mapping $x \mapsto 2x$, it is natural to expect a chaotic behavior for the iteration of $\Phi_P|_{S^n}$. Accordingly, we show that $\Phi_P|_{S^n}$ (and incidentally $\Phi_P|_{D^{n+1}}$ as well) actually creates Kato's chaos. Therefore, by investigating spherical orthotomic curves, an example of singular quadratic mapping creating Kato's chaos is naturally obtained.

1. Introduction

Throughout this paper, let n be a non-negative integer. In addition, let S^n, D^{n+1} be the unit sphere and the unit closed disk of $\mathbb{R}^{n+1}$ respectively.

Let I be an interval. In [1], for a given plane unit-speed curve $\gamma : I \to \mathbb{R}^2$ and a given point $P \in \mathbb{R}^2$, the pedal curve $ped_{\gamma,P} : I \to \mathbb{R}^2$ and the orthotomic curve $ort_{\gamma,P} : I \to \mathbb{R}^2$ are defined as follows:

$$\begin{aligned} ped_{\gamma,P}(s) &= P + ((\gamma(s) - P) \cdot N(s))\, N(s), \\ ort_{\gamma,P}(s) &= P + 2\,((\gamma(s) - P) \cdot N(s))\, N(s). \end{aligned}$$

Here, $N(s)$ is the unit normal vector to γ at $\gamma(s)$. For instance, let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a parabola defined by $\gamma(t) = \left(t, t^2 - \frac{1}{4}\right)$ and let P be the origin $(0, 0)$. Let $\ell : \mathbb{R} \to \mathbb{R}$ be the arc-length of γ measured from $\gamma(0)$. Then, $ped_{\gamma\circ\ell^{-1},P}$ is just the affine tangent line to the parabola $\gamma \circ \ell^{-1}$ at $\gamma \circ \ell^{-1}(0)$ and $ort_{\gamma\circ\ell^{-1},P}$ is merely the directrix of the parabola with the focal point P. From this elementary example, in general, the orthotomic curve for a given unit-speed curve γ may be considered as a generalization of the directrix of a parabola in some sense. Moreover, as explained in pp. 175–177 in [1], orthotomic curves have a seismic application. This is a very interesting and very important practical application of orthotomic curves. Since pedal curves seem to be well-studied rather than orthotomic curves, we are interested in how to obtain the orthotomic curve from the pedal curve for a given unit-speed curve γ and a point P. By definition, it follows

$$\frac{ort_{\gamma,P}(s) + P}{2} = ped_{\gamma,P}(s)$$

and thus $ort_{\gamma,P}(s) = 2ped_{\gamma,P}(s) - P$. Therefore, by using the simple mapping $F_P : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F_P(x) = 2x - P,$$

we have the following:

$$ort_{\gamma,P}(s) = F_P \circ ped_{\gamma,P}(s).$$

2010 *Mathematics Subject Classification.* 37D45, 54H20, 26A18, 39B12 .
Key words and phrases. Kato's chaos, Sensitivity, Accessibility, Quadratic mapping, Spherical orthotomic curve.

Since F_P is nothing but the radial expansion with factor 2 with respect to the point P, the study of orthotomic curves may be completely reduced to the study of pedal curves in the plane curve case.

Similarly, in the case of S^n, by obtaining the orthotomic curve from the pedal curve for a given spherical unit-speed curve γ and a point P, we can get an expanding mapping $S^n \to S^n$ with similar properties as the above F_P. However, in this case, the space S^n is compact. Thus, this expanding mapping $S^n \to S^n$ is expected to have some kneading effect. This expectation leads us to study the iteration of this mapping. In order to get the expanding mapping $S^n \to S^n$, for a generic unit-speed curve $\gamma : I \to S^n$ and a generic point $P \in S^n$, the pedal curve $ped_{\gamma,P} : I \to S^n$ and the orthotomic curve $ort_{\gamma,P} : I \to S^n$ need to be defined reasonably. In [5, 6], a reasonable definition of spherical unit speed curve is given; and then for a spherical unit speed curve $\gamma : I \to S^n$ and a generic point $P \in S^n$, the spherical pedal curve $ped_{\gamma,P} : I \to S^n$ is defined reasonably. Notice that the well-definedness of $ped_{\gamma,P} : I \to S^n$ implies $P \cdot ped_{\gamma,P}(s) \neq 0$ for any $s \in I$ (see [5, 6]). Thus, by using the following relation which is reasonable in S^n,

$$\frac{ort_{\gamma,P}(s) + P}{2} = (P \cdot ped_{\gamma,P}(s))\, ped_{\gamma,P}(s),$$

the spherical orthotomic curve $ort_{\gamma,P} : I \to S^n$ is naturally defined as follows:

$$ort_{\gamma,P}(s) = 2\,(P \cdot ped_{\gamma,P}(s))\, ped_{\gamma,P}(s) - P.$$

Therefore, by using the mapping $\Phi_P : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ defined by

$$\Phi_P(x) = 2(P \cdot x)x - P,$$

the orthotomic curve is obtained from the pedal curve as follows:

$$ort_{\gamma,P}(s) = \Phi_P \circ ped_{\gamma,P}(s).$$

As in the following lemma, both $\Phi_P|_{S^n}$ and $\Phi_P|_{D^{n+1}}$ $(n \geq 0)$ are endomorphisms. Thus, $\Phi_P|_{S^n}$ $(n \geq 1)$ may be regarded as the spherical counterpart of the expansion F_P. By combining these facts and the compactness of S^n (resp., D^{n+1}), it is expected that not only $\Phi_P|_{S^n}$ but also $\Phi_P|_{D^{n+1}}$ may have a chaotic behavior of some kind.

Lemma 1. *For any $P \in S^n$, the following three hold:*

(1) $\Phi_P(S^n) \subset S^n$ *for any* $n \geq 0$.
(2) $\Phi_P(S^n) \supset S^n$ *for any* $n \geq 1$.
(3) $\Phi_P(D^{n+1}) = D^{n+1}$ *for any* $n \geq 0$.

For the proof of Lemma 1, see Section 2. The following two examples, too, show that for both $\Phi_P|_{S^n}$ and $\Phi_P|_{D^{n+1}}$, the chaotic behavior of their iteration deserves to be investigated.

Example 1. Suppose that $n = 1$ and $P = (1, 0)$. Then, $\Phi_P(x) = (2x_1^2 - 1, 2x_1x_2)$, where $x = (x_1, x_2)$. If x belongs to S^1, x may be written as $x = (\cos\theta, \sin\theta)$. Then,

$$\Phi_P|_{S^1}(\cos\theta, \sin\theta) = \left(2\cos^2\theta - 1, 2\cos\theta\sin\theta\right) = (\cos 2\theta, \sin 2\theta)\,.$$

Thus, the restricted mapping $\Phi_P|_{S^n}$ in this case is exactly the same mapping given in Chapter 1, Example 3.4 of Devaney's well-known book [2].

Example 2. Suppose that $n = 0$. Then, P is 1 or -1, and $\Phi_P(x) = 2x^2 - 1$ or $-2x^2 + 1$. Define the affine transformation $h_P : \mathbb{R} \to \mathbb{R}$ as follows:

$$h_P(x) = \begin{cases} -2x + 1 & (\text{if } \ P = 1), \\ \ \ 2x - 1 & (\text{if } \ P = -1). \end{cases}$$

Then, in each case, it is easily seen that $h_P^{-1} \circ \Phi_P \circ h_P(x) = 4x(1 - x)$. Therefore, in each case, $\Phi_P|_{D^1}$ has the same dynamics as Chapter 1, Example 8.9 of [2].

From Examples 1 and 2, it seems meaningful to study the chaotic behavior of iteration for $\Phi_P|_{S^n} : S^n \to S^n$ $(n \geq 1)$ or $\Phi_P|_{D^{n+1}} : D^{n+1} \to D^{n+1}$ $(n \geq 0)$, which is the main purpose of this paper.

Definition 1. Let (X, d) be a metric space with metric d and let $f : X \to X$ be a continuous mapping.

(1) The mapping f is said to be *sensitive* if there is a positive number $\lambda > 0$ such that for any $x \in X$ and any neighborhood U of x in X, there exists a point $y \in U$ and a non-negative integer $k \geq 0$ such that $d(f^k(x), f^k(y)) > \lambda$, where f^k stands for $\underbrace{f \circ \cdots \circ f}_{k\text{-tuples}}$.
(2) The mapping f is said to be *transitive* if for any non-empty open subsets $U, V \subset X$, there exists a positive integer $k > 0$ such that $f^k(U) \cap V \neq \emptyset$.
(3) The mapping f is said to be *accessible* if for any $\lambda > 0$ and any non-empty open subsets $U, V \subset X$, there exist two points $u \in U$, $v \in V$ and a positive integer $k > 0$ such that $d(f^k(u), f^k(v)) \leq \lambda$.
(4) The mapping f is said to be *topologically mixing* if for any non-empty open subsets $U, V \subset X$, there exists a positive integer $k > 0$ such that $f^m(U) \cap V \neq \emptyset$ for any $m \geq k$.
(5) The mapping f is said to be *chaotic in the sense of Devaney* ([2]) if f is sensitive, transitive and the set consisting of periodic points of f is dense in X.
(6) The mapping f is said to be *chaotic in the sense of Kato* ([3]) if f is sensitive and accessible.

By definition, it is clear that if a mapping $f : X \to X$ is topologically mixing, then it is transitive. Moreover, by [3], it is known that if a mapping $f : X \to X$ is topologically mixing, then it is chaotic in the sense of Kato. Although Kato's chaos has been well-investigated (for instance, see [3, 4, 7]), elementary examples which are singular and not transitive seem to have been desired. Theorem 1 gives such examples.

Theorem 1. (1) *Let P be a point of S^1.*

(1-1) *The endomorphism $\Phi_P|_{S^1} : S^1 \to S^1$ is chaotic in the sense of Devaney. Moreover, it is chaotic in the sense of Kato.*
(1-2) *The endomorphism $\Phi_P|_{D^2} : D^2 \to D^2$ is chaotic in the sense of Kato although it is not chaotic in the sense of Devaney.*

(2) *Let P be a point of S^0. Then, $\Phi_P|_{D^1} : D^1 \to D^1$ is chaotic in the sense of Devaney. Moreover, it is chaotic in the sense of Kato.*
(3) *Let m be an integer such that $m \geq 2$. Moreover, let P be a point of S^m. Then, both $\Phi_P|_{D^{m+1}} : D^{m+1} \to D^{m+1}$ and $\Phi_P|_{S^m} : S^m \to S^m$ are chaotic in the sense of Kato.*
(4) *Let m be an integer such that $m \geq 2$. Moreover, let P be a point of S^m. Then, neither $\Phi_P|_{D^{m+1}} : D^{m+1} \to D^{m+1}$ nor $\Phi_P|_{S^m} : S^m \to S^m$ is transitive. In particular, neither $\Phi_P|_{D^{m+1}} : D^{m+1} \to D^{m+1}$ nor $\Phi_P|_{S^m} : S^m \to S^m$ is chaotic in the sense of Devaney.*

This paper is organized as follows. In Section 2, the proof of Lemma 1 is given. Theorem 1 is proved in Section 3. Section 4 is an appendix where geometric properties of Φ_P are given though some of properties of Φ_P given in Section 4 already appear implicitly in Sections 2 and 3.

2. Proof of Lemma 1

2.1. **Proof of the assertion (1) of Lemma 1.** Let x be a point of S^n. Then, $x \cdot x = 1$ and we have the following:

$$\begin{aligned}\Phi_P(x) \cdot \Phi_P(x) &= (2(x \cdot P)x - P) \cdot (2(x \cdot P)x - P) \\ &= 4(x \cdot P)^2(x \cdot x) - 4(x \cdot P)^2 + (P \cdot P) \\ &= 4(x \cdot P)^2 - 4(x \cdot P)^2 + 1 = 1.\end{aligned}$$

This completes the proof of the assertion (1). □

2.2. **Proof of the assertion (2) of Lemma 1.** Let y be a point of S^n. Suppose that $y \neq -P$. Set

$$x = \frac{\frac{y+P}{2}}{||\frac{y+P}{2}||}.$$

Then, it follows

$$\begin{aligned}2(x \cdot P)x - P &= 2\left(\frac{\frac{y+P}{2}}{||\frac{y+P}{2}||} \cdot P\right)\frac{\frac{y+P}{2}}{||\frac{y+P}{2}||} - P \\ &= \frac{2}{||y+P||^2}\left((y \cdot P) + 1\right)(y+P) - P \\ &= \frac{1}{(1 + (y \cdot P))}\left((y \cdot P) + 1\right)(y+P) - P \\ &= (y+P) - P = y.\end{aligned}$$

Next, suppose that $y = -P$. Let x be a point of S^n such that $x \cdot P = 0$. Then,

$$2(x \cdot P)x - P = -P = y.$$

Therefore, we have the assertion (2). □

2.3. **Proof of the assertion (3) of Lemma 1.** Let x be a point of $\mathbb{R}^{n+1}$ such that $x \cdot x < 1$. Then, we have

$$\Phi_P(x) \cdot \Phi_P(x) < 4(x \cdot P)^2 - 4(x \cdot P)^2 + 1 = 1.$$

Conversely, let y be a point satisfying $y \cdot y < 1$. Notice that in this case $(y \cdot P) + 1 \geq -||y|| + 1 > 0$ and $1 + ||y||^2 + 2(y \cdot P) \geq 1 + ||y||^2 - 2||y|| = (1 - ||y||)^2 > 0$. Set

$$a = \sqrt{\frac{1 + ||y||^2 + 2(y \cdot P)}{2(y \cdot P) + 2}} \text{ and } x = a\frac{\frac{y+P}{2}}{||\frac{y+P}{2}||}.$$

Then,

$$\begin{aligned}2(x \cdot P)x - P &= 2\left(a\frac{\frac{y+P}{2}}{||\frac{y+P}{2}||} \cdot P\right)a\frac{\frac{y+P}{2}}{||\frac{y+P}{2}||} - P \\ &= \frac{2a^2}{||y+P||^2}\left((y \cdot P) + 1\right)(y+P) - P \\ &= \frac{2a^2}{(1 + ||y||^2 + 2(y \cdot P))}\left((y \cdot P) + 1\right)(y+P) - P \\ &= (y+P) - P = y.\end{aligned}$$

Therefore, the assertion (3) holds. □

3. Proof of Theorem 1

3.1. Proof of the assertion (1) of Theorem 1. We first show the assertion (1-1). Let x be a point of S^1. Set

$$P = (\cos\alpha, \sin\alpha) \text{ and } x = (\cos\theta, \sin\theta).$$

Then, it is easily seen that

$$\begin{aligned} & \Phi_P(\cos\theta, \sin\theta) \\ = \;& 2\left((\cos\alpha, \sin\alpha)\cdot(\cos\theta, \sin\theta)\right)(\cos\theta, \sin\theta) - (\cos\alpha, \sin\alpha) \\ = \;& (\cos(2\theta-\alpha), \sin(2\theta-\alpha)). \end{aligned}$$

It follows $\Phi_P^k(\cos(\theta+\alpha), \sin(\theta+\alpha)) = \left(\cos(2^k\theta+\alpha), \sin(2^k\theta+\alpha)\right)$ and therefore, by the same argument as in Example 8.6 of [2], $\Phi_P|_{S^1}$ is chaotic in the sense of Devaney. In order to show that $\Phi_P|_{S^1}$ is chaotic in the sense of Kato, it is sufficient to show that $\Phi_P|_{S^1}$ is accessible, which is easily seen by the above formula.

Next, we show the assertion (1-2). Since $\mathbb{R}^2$ may be regarded as $\mathbb{R}^2\times\{0\}\subset\mathbb{R}^3$, the given point $P\in S^1$ is naturally considered as a point of S^2. Then, $\Phi_P|_{S^2}$ and $\Phi_P|_{D^2}$ are semi-conjugate. Thus, the assertion (1-2) easily follows from the assertions (3) and (4) for $\Phi_P|_{S^2}$. □

3.2. Proof of the assertion (2) of Theorem 1. By Subsection 3.1 and Example 8.9 of [2], $\Phi_P|_{D^1}$ is chaotic in the sense of Devaney. Moreover, it is easily seen that the property of accessibility is preserved by semi-conjugacy. Thus, $\Phi_P|_{D^1}$ is chaotic in the sense of Kato as well. □

3.3. Proof of the assertion (3) of Theorem 1. Let Q be a point of $S^m - \{P, -P\}$. Set

$$P_Q^\perp = \frac{Q-(P\cdot Q)P}{||Q-(P\cdot Q)P||}.$$

Then, it follows $P_Q^\perp\in S^m$ and $P\cdot P_Q^\perp = 0$. Let x be a point of the circle $S^m\cap(\mathbb{R}P+\mathbb{R}P_Q^\perp)$. Then, x may be written as $x=\cos\theta\ P+\sin\theta\ P_Q^\perp$. Then, it is easily seen that

$$\Phi_P(\cos\theta\ P+\sin\theta\ P_Q^\perp) = \cos 2\theta\ P + \sin 2\theta\ P_Q^\perp.$$

Hence, for any non-empty open neighborhood U of Q in S^m there exists a positive integer i such that the circle $S^m\cap(\mathbb{R}P+\mathbb{R}P_Q^\perp)$ is contained in $\Phi_P^i(U)$. Therefore, $\Phi_P|_{S^m}$ is sensitive.

Next, take another point R. By the same argument as above, it is seen that for any non-empty open neighborhood V of R in S^m there exists a positive integer j such that the circle $S^m\cap(\mathbb{R}P+\mathbb{R}P_Q^\perp)$ is contained in $\Phi_P^j(V)$. Set $k=\max(i,j)$. Then, it follows

$$P\in\Phi_P^k(U)\cap\Phi_P^k(V).$$

Hence, $\Phi_P|_{S^m}$ is accessible.

Moreover, under the identification of S^m and $S^m\times\{0\}(\subset S^{m+1})$, the given point $P\in S^m$ is considered as a point of S^{m+1}. Then, $\Phi_P|_{S^{m+1}}$ and $\Phi_P|_{D^{m+1}}$ are semi-conjugate. Thus, $\Phi_P|_{D^{m+1}}$ is also sensitive and accessible. Therefore, both $\Phi_P|_{S^m}$ and $\Phi_P|_{D^{m+1}}$ are chaotic in the sense of Kato. □

3.4. Proof of the assertion (4) of Theorem 1. Let Q, R be points of S^m so that P, Q, R are linearly independent. Then, R does not belong to the circle $S^m\cap(\mathbb{R}P+\mathbb{R}P_Q^\perp)$ where $P_Q^\perp$ is the point constructed in Subsection 3.3. Thus, by the argument given in Subsection 3.3, there exist sufficiently small neighborhoods U (resp., V) of Q (resp., R) in S^m such that $\Phi_P^\ell(U)\cap V=\emptyset$ for any $\ell\geq 0$. Hence, $\Phi_P|_{S^m}$ is never transitive.

Again, under the identification of S^m and $S^m \times \{0\} (\subset S^{m+1})$, the given point $P \in S^m$ is considered as a point of S^{m+1}. Then, $\Phi_P|_{S^{m+1}}$ and $\Phi_P|_{D^{m+1}}$ are semi-conjugate. Thus, even $\Phi_P|_{D^{m+1}}$ is not transitive. □

4. Some properties of Φ_P

In this section, following the referee's suggestions, the geometric structure of Φ_P is studied.

Proposition 1. *Let P, $h : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be a point of $\mathbb{R}^{n+1}$ and an orthogonal linear mapping respectively. Set $\widetilde{P} = h(P)$. Then, the following equality holds:*

$$\Phi_{\widetilde{P}} \circ h = h \circ \Phi_P.$$

Proof. Let A be the orthogonal matrix corresponding to h. For any $x \in \mathbb{R}^{n+1}$, we have the following:

$$\begin{aligned} \Phi_{\widetilde{P}} \circ h(x) &= \Phi_{\widetilde{P}}(xA) \\ &= 2\left(\widetilde{P} \cdot xA\right) xA - \widetilde{P} \\ &= 2\left(PA \cdot xA\right) xA - PA \\ &= \left(2\left(P \cdot x\right) x - P\right) A \\ &= h \circ \Phi_P(x). \end{aligned}$$

□

Corollary 1. *Let P be a point of S^n and let $h : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be an orthogonal linear mapping such that $h(P) = (1, 0, \ldots, 0)$. Then, $h \circ \Phi_P \circ h^{-1}$ is the following mapping where $x = (x_1, x_2, \ldots, x_{n+1})$:*

$$h \circ \Phi_P \circ h^{-1}(x_1, x_2, \ldots, x_{n+1}) = (2x_1^2 - 1, 2x_1x_2, \ldots, 2x_1x_{n+1}).$$

Notice that if we understand that $x_2 \in \mathbb{R}^n$, then the form of Φ_P in Example 1 is exactly the same as the form of $h \circ \Phi_P \circ h^{-1}$ in Corollary 1. Moreover, the following holds.

Proposition 2. *Let P be a point of $\mathbb{R}^{n+1} - \{\mathbf{0}\}$. Then, the mapping Φ_P preserves any 2-dimensional linear subspace that contains P. Moreover, the restrictions of Φ_P to such linear subspaces are conjugated to each other.*

Proof. The proof of the first assertion of Proposition 2 is implicitly given in Subsection 3.3 although in Subsection 3.3 P is a point of S^n. Thus, it is omitted to give it here.

We show the second assertion of Proposition 2 by using the same symbols as in Subsection 3.3. Let $\widetilde{Q}$ be a point of $S^n - \left(\mathbb{R}P + \mathbb{R}P_Q^\perp\right)$ and let $h : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be an orthogonal linear mapping such that $h(P) = P$ and $h(Q) = \widetilde{Q}$. Then, it is trivially seen that h maps the 2-dimensional linear space $\left(\mathbb{R}P + \mathbb{R}P_Q^\perp\right)$ to $\left(\mathbb{R}P + \mathbb{R}P_{\widetilde{Q}}^\perp\right)$. Moreover, by Proposition 1, the following equality holds:

$$\Phi_{\widetilde{P}} \circ h = h \circ \Phi_P.$$

Therefore, the second assertion of Proposition 2 holds. □

Proposition 2 reduces the study of dynamical system of Φ_P to the 2-dimensional case, which is given in Example 1.

The final assertion is for the mapping Φ_P where $P = (1, 0, \ldots, 0)$.

Proposition 3. *Let $P = (1, 0, \ldots, 0) \in S^n$ and let $\Phi_P : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the mapping defined by*

$$\Phi_P(x_1, x_2, \ldots, x_{n+1}) = (2x_1^2 - 1, 2x_1x_2, \ldots, 2x_1x_{n+1}).$$

Let $(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$ be a point such that

$$\varphi(x_1, x_2, \ldots, x_{n+1}) = x_1^2 + \mu\left(x_2^2 + \cdots + x_{n+1}^2\right) = 1,$$

where μ is a positive real number. Then, $\varphi \circ \Phi_P(x_1, x_2, \ldots, x_{n+1}) = 1$. In other words, Φ_P preserves the level set $\varphi^{-1}(1)$.

Proof. Assume that $\varphi(x_1, x_2, \ldots, x_{n+1}) = 1$. Then,

$$\begin{aligned}
\varphi \circ \Phi_P(x_1, x_2, \ldots, x_{n+1}) &= \left(2x_1^2 - 1\right)^2 + \mu\left((2x_1x_2)^2 + \cdots + (2x_1x_{n+1})^2\right) \\
&= 4x_1^4 - 4x_1^2 + 1 + 4\mu\left(x_1^2x_2^2 + \cdots x_1^2x_{n+1}^2\right) \\
&= 4x_1^4 - 4x_1^2\left(1 - \mu\left(x_2^2 + \cdots + x_{n+1}^2\right)\right) + 1 \\
&= 4x_1^4 - 4x_1^4 + 1 \\
&= 1.
\end{aligned}$$

□

Notice that Φ_P does not necessarily preserve other level sets $\varphi^{-1}(c)$ $(c \neq 1)$. The case $\mu = 1$ of Proposition 3 suggests (1) of Lemma 1.

Acknowledgement

The author would like to express his sincere gratitude to the referee for careful reading of this paper and making invaluable suggestions. This work was partially supported by JSPS KAKENHI Grant Number JP17K05245.

References

[1] J. W. Bruce and P. J. Giblin, *Curves and Singularities (second edition)*, Cambridge University Press, 1992.
[2] R. Devaney, *An Introduction to Chaotic Dynamical Systems*, Addison-Wesley, Reading, MA, 2nd ed., 1989.
[3] H. Kato, *Everywhere chaotic homeomorphisms on manifolds and k-dimensional Menger manifolds*, Topology Appl., **72** (1996), 1–17. DOI: 10.1016/0166-8641(96)00008-9
[4] R. Li, H. Wang and Y. Zhao, *Kato's chaos in duopoly games*, Chaos, Solitons and Fractals, **84** (2016), 69–72. DOI: 10.1016/j.chaos.2016.01.006
[5] T. Nishimura, *Normal forms for singularities of pedal curves produced by non-singular dual curve germs in S^n*, Geom. Dedicata, **133** (2008), 59–66. DOI: 10.1007/s10711-008-9233-5
[6] T. Nishimura, *Singularities of pedal curves produced by singular dual curve germs in S^n*, Demonstratio Math., **43** (2010), 447–459. DOI: 10.1515/dema-2013-0240
[7] L. Wang, J. Liang and Z. Chu, *Weakly mixing property and chaos*, Arch. Math., **109** (2017), 83–89. DOI: 10.1007/s00013-017-1044-1

Takashi Nishimura: Research Institute of Environment and Information Sciences, Yokohama National University, Yokohama 240-8501, Japan
Email address: nishimura-takashi-yx@ynu.ac.jp

Journal *of* Singularities
Volume 21 (2020), 212-233

Proc. of Geometric and Algebraic Singularity Theory, Będlewo, 2017
DOI: 10.5427/jsing.2020.21m

A CLOSEDNESS THEOREM AND APPLICATIONS IN GEOMETRY OF RATIONAL POINTS OVER HENSELIAN VALUED FIELDS

KRZYSZTOF JAN NOWAK

Dedicated to Goo Ishikawa on the occasion of his 60th birthday

Abstract. We develop geometry of algebraic subvarieties of K^n over arbitrary Henselian valued fields K of equicharacteristic zero. This is a continuation of our previous article concerned with algebraic geometry over rank one valued fields. At the center of our approach is again the closedness theorem to the effect that the projections $K^n \times \mathbb{P}^m(K) \to K^n$ are definably closed maps. It enables, in particular, application of resolution of singularities in much the same way as over locally compact ground fields. As before, the proof of that theorem uses, among others, the local behavior of definable functions of one variable and fiber shrinking, being a relaxed version of curve selection. But now, to achieve the former result, we first examine functions given by algebraic power series. All our previous results will be established here in the general settings: several versions of curve selection (via resolution of singularities) and of the Łojasiewicz inequality (via two instances of quantifier elimination indicated below), extending continuous hereditarily rational functions as well as the theory of regulous functions, sets and sheaves, including Nullstellensatz and Cartan's theorems A and B. Two basic tools are quantifier elimination for Henselian valued fields due to Pas and relative quantifier elimination for ordered abelian groups (in a many-sorted language with imaginary auxiliary sorts) due to Cluckers–Halupczok. Other, new applications of the closedness theorem are piecewise continuity of definable functions, Hölder continuity of functions definable on closed bounded subsets of K^n, the existence of definable retractions onto closed definable subsets of K^n and a definable, non-Archimedean version of the Tietze–Urysohn extension theorem. In a recent paper, we established a version of the closedness theorem over Henselian valued fields with analytic structure along with several applications.

1. Introduction

Throughout the paper, K will be an arbitrary Henselian valued field of equicharacteristic zero with valuation v, value group Γ, valuation ring R and residue field $\Bbbk$. Examples of such fields are the quotient fields of the rings of formal power series and of Puiseux series with coefficients from a field $\Bbbk$ of characteristic zero as well as the fields of Hahn series (maximally complete valued fields also called Malcev–Neumann fields; cf. [27]):

$$\Bbbk((t^\Gamma)) := \left\{ f(t) = \sum_{\gamma\in\Gamma} a_\gamma t^\gamma : \ a_\gamma \in \Bbbk, \ \operatorname{supp} f(t) \text{ is well ordered} \right\}.$$

We consider the ground field K along with the three-sorted language $\mathcal{L}$ of Denef–Pas (cf. [53, 44]). The three sorts of $\mathcal{L}$ are: the valued field K-sort, the value group Γ-sort and the residue field $\Bbbk$-sort. The language of the K-sort is the language of rings; that of the Γ-sort is any augmentation

2000 *Mathematics Subject Classification.* Primary 12J25, 14B05, 14P10; Secondary 13J15, 14G27, 03C10.

Key words and phrases. valued fields, algebraic power series, closedness theorem, blowing up, descent property, quantifier elimination for Henselian valued fields, quantifier elimination for ordered abelian groups, fiber shrinking, curve selection, Łojasiewicz inequalities, hereditarily rational functions, regulous Nullstellensatz, regulous Cartan's theorems.

of the language of ordered abelian groups (and ∞); finally, that of the $\Bbbk$-sort is any augmentation of the language of rings. The only symbols of $\mathcal{L}$ connecting the sorts are two functions from the main K-sort to the auxiliary Γ-sort and $\Bbbk$-sort: the valuation map and an angular component map.

Every valued field K has a topology induced by its valuation v. Cartesian products K^n are equipped with the product topology, and their subsets inherit a topology, called the K-topology. This paper is a continuation of our paper [44] devoted to geometry over Henselian rank one valued fields, and includes our recent preprints [45, 46, 47]. The main aim is to prove (in Section 8) the closedness theorem stated below, and next to derive several results in the following Sections 9–14.

Theorem 1.1. *Let D be an $\mathcal{L}$-definable subset of K^n. Then the canonical projection*

$$\pi : D \times R^m \longrightarrow D$$

is definably closed in the K-topology, i.e. if $B \subset D \times R^m$ is an $\mathcal{L}$-definable closed subset, so is its image $\pi(B) \subset D$.

Remark 1.2. Not all valued fields K have an angular component map, but it exists if K has a cross section, which happens whenever K is $\aleph_1$-saturated (cf. [7, Chap. II]). Moreover, a valued field K has an angular component map whenever its residue field $\Bbbk$ is $\aleph_1$-saturated (cf. [54, Corollary 1.6]). In general, unlike for p-adic fields and their finite extensions, adding an angular component map does strengthen the family of definable sets. Since the K-topology is definable in the language of valued fields, the closedness theorem is a first order property. Therefore it is valid over arbitrary Henselian valued fields of equicharacteristic zero, because it can be proven using saturated elementary extensions, thus assuming that an angular component map exists.

Two basic tools applied in this paper are quantifier elimination for Henselian valued fields (along with preparation cell decomposition) due to Pas [53] and relative quantifier elimination for ordered abelian groups (in a many-sorted language with imaginary auxiliary sorts) due to Cluckers–Halupczok [8]. In the case where the ground field K is of rank one, Theorem 1.1 was established in our paper [44, Section 7], where instead we applied simply quantifier elimination for ordered abelian groups in the Presburger language. Of course, when K is a locally compact field, it holds by a routine topological argument.

As before, our approach relies on the local behavior of definable functions of one variable and the so-called fiber shrinking, being a relaxed version of curve selection. Over arbitrary Henselian valued fields, the former result will be established in Section 5, and the latter in Section 6. Now, however, in the proofs of fiber shrinking (Proposition 6.1) and the closedness theorem (Theorem 1.1), we also apply relative quantifier elimination for ordered abelian groups, due to Cluckers–Halupczok [8]. It will be recalled in Section 7.

Section 2 contains a version of the implicit function theorem (Proposition 2.5). In the next section, we provide a version of the Artin–Mazur theorem on algebraic power series (Proposition 3.3). Consequently, every algebraic power series over K determines a unique continuous function which is definable in the language of valued fields. Section 4 presents certain versions of the theorems of Abhyankar–Jung (Proposition 4.1) and Newton-Puiseux (Proposition 4.2) for Henselian subalgebras of formal power series which are closed under power substitution and division by a coordinate, given in our paper [43] (see also [52]). In Section 5, we use the foregoing results in analysis of functions of one variable, definable in the language of Denef–Pas, to establish a theorem on existence of the limit (Theorem 5.1).

The closedness theorem will allow us to establish several results as for instance: piecewise continuity of definable functions (Section 9), certain non-archimedean versions of curve selection (Section 10) and of the Łojasiewicz inequality with a direct consequence, Hölder continuity of definable functions on closed bounded subsets of K^n (Section 11) as well as extending hereditarily rational functions (Section 12) and the theory of regulous functions, sets and sheaves, including Nullstellensatz and Cartan's theorems A and B (Section 12). Over rank one valued fields, these results (except piecewise and Hölder continuity) were established in our paper [44]. The theory of hereditarily rational functions on the real and p-adic varieties was developed in the joint paper [30]. Yet another application of the closedness theorem is the existence of definable retractions onto closed definable subsets of K^n and a definable, non-Archimedean version of the Tietze–Urysohn extension theorem. These results are established for the algebraic case and for Henselian fields with analytic structure in our recent papers [49, 50, 51]. It is very plausible that they will also hold in the more general case of axiomatically based structures on Henselian valued fields.

The closedness theorem immediately yields five corollaries stated below. Corollaries 1.6 and 1.7, enable application of resolution of singularities and of transformation to a simple normal crossing by blowing up (cf. [28, Chap. III] for references and relatively short proofs) in much the same way as over locally compact ground fields.

Corollary 1.3. *Let D be an $\mathcal{L}$-definable subset of K^n and $\mathbb{P}^m(K)$ stand for the projective space of dimension m over K. Then the canonical projection $\pi : D \times \mathbb{P}^m(K) \longrightarrow D$ is definably closed.*

Corollary 1.4. *Let A be a closed $\mathcal{L}$-definable subset of $\mathbb{P}^m(K)$ or R^m. Then every continuous $\mathcal{L}$-definable map $f : A \to K^n$ is definably closed in the K-topology.*

Corollary 1.5. *Let ϕ_i, $i = 0, \ldots, m$, be regular functions on K^n, D be an $\mathcal{L}$-definable subset of K^n and $\sigma : Y \longrightarrow K\mathbb{A}^n$ the blow-up of the affine space $K\mathbb{A}^n$ with respect to the ideal $(\phi_0, \ldots, \phi_m)$. Then the restriction $\sigma : Y(K) \cap \sigma^{-1}(D) \longrightarrow D$ is a definably closed quotient map.*

Proof. Indeed, $Y(K)$ can be regarded as a closed algebraic subvariety of $K^n \times \mathbb{P}^m(K)$ and σ as the canonical projection. □

Corollary 1.6. *Let X be a smooth K-variety, D be an $\mathcal{L}$-definable subset of $X(K)$ and $\sigma : Y \longrightarrow X$ the blow-up along a smooth center. Then the restriction $\sigma : Y(K) \cap \sigma^{-1}(D) \longrightarrow D$ is a definably closed quotient map.*

Corollary 1.7. *(Descent property) Under the assumptions of the above corollary, every continuous $\mathcal{L}$-definable function $g : Y(K) \cap \sigma^{-1}(D) \longrightarrow K$ that is constant on the fibers of the blow-up σ descends to a (unique) continuous $\mathcal{L}$-definable function $f : D \longrightarrow K$.*

2. Some versions of the implicit function theorem

In this section, we give elementary proofs of some versions of the inverse mapping and implicit function theorems; cf. the versions established in the papers [55, Theorem 7.4], [22, Section 9], [36, Section 4] and [21, Proposition 3.1.4]. We begin with a simplest version (H) of Hensel's lemma in several variables, studied by Fisher [20]. Given an ideal $\mathfrak{m}$ of a ring R, let $\mathfrak{m}^{\times n}$ stand for the n-fold Cartesian product of $\mathfrak{m}$ and $R^\times$ for the set of units of R. The origin $(0, \ldots, 0) \in R^n$ is denoted by $\mathbf{0}$.

(H) *Assume that a ring R satisfies Hensel's conditions (i.e. it is linearly topologized, Hausdorff and complete) and that an ideal $\mathfrak{m}$ of R is closed. Let $f = (f_1, \dots, f_n)$ be an n-tuple of restricted power series $f_1, \dots, f_n \in R\{X\}$, $X = (X_1, \dots, X_n)$, J be its Jacobian determinant and $a \in R^n$. If $f(\mathbf{0}) \in \mathfrak{m}^{\times n}$ and $J(\mathbf{0}) \in R^\times$, then there is a unique $a \in \mathfrak{m}^{\times n}$ such that $f(a) = \mathbf{0}$.*

Proposition 2.1. *Under the above assumptions, f induces a bijection*

$$\mathfrak{m}^{\times n} \ni x \longrightarrow f(x) \in \mathfrak{m}^{\times n}$$

of $\mathfrak{m}^{\times n}$ onto itself.

Proof. For any $y \in \mathfrak{m}^{\times n}$, apply condition (H) to the restricted power series $f(X) - y$. □

If, moreover, the pair $(R, \mathfrak{m})$ satisfies Hensel's conditions (i.e. every element of $\mathfrak{m}$ is topologically nilpotent), then condition (H) holds by [5, Chap. III, §4.5].

Remark 2.2. Henselian local rings can be characterized both by the classical Hensel lemma and by condition (H): a local ring $(R, \mathfrak{m})$ is Henselian iff $(R, \mathfrak{m})$ with the discrete topology satisfies condition (H) (cf. [20, Proposition 2]).

Now consider a Henselian local ring $(R, \mathfrak{m})$. Let $f = (f_1, \dots, f_n)$ be an n-tuple of polynomials $f_1, \dots, f_n \in R[X]$, $X = (X_1, \dots, X_n)$ and J be its Jacobian determinant.

Corollary 2.3. *Suppose that $f(\mathbf{0}) \in \mathfrak{m}^{\times n}$ and $J(\mathbf{0}) \in R^\times$. Then f is a homeomorphism of $\mathfrak{m}^{\times n}$ onto itself in the $\mathfrak{m}$-adic topology. If, in addition, R is a Henselian valued ring with maximal ideal $\mathfrak{m}$, then f is a homeomorphism of $\mathfrak{m}^{\times n}$ onto itself in the valuation topology.*

Proof. Obviously, $J(a) \in R^\times$ for every $a \in \mathfrak{m}^{\times n}$. Let $\mathcal{M}$ be the jacobian matrix of f. Then

$$f(a + x) - f(a) = \mathcal{M}(a) \cdot x + g(x) = \mathcal{M}(a) \cdot (x + \mathcal{M}(a)^{-1} \cdot g(x))$$

for an n-tuple $g = (g_1, \dots, g_n)$ of polynomials $g_1, \dots, g_n \in (X)^2 R[X]$. Hence the assertion follows easily. □

The proposition below is a version of the inverse mapping theorem.

Proposition 2.4. *If $f(\mathbf{0}) = \mathbf{0}$ and $e := J(\mathbf{0}) \neq 0$, then f is an open embedding of $e \cdot \mathfrak{m}^{\times n}$ onto $e^2 \cdot \mathfrak{m}^{\times n}$.*

Proof. Let $\mathcal{N}$ be the adjugate of the matrix $\mathcal{M}(\mathbf{0})$ and $y = e^2 b$ with $b \in \mathfrak{m}^{\times n}$. Since

$$f(eX) = e \cdot \mathcal{M}(\mathbf{0}) \cdot X + e^2 g(X)$$

for an n-tuple $g = (g_1, \dots, g_n)$ of polynomials $g_1, \dots, g_n \in (X)^2 R[X]$, we get the equivalences

$$f(eX) = y \;\Leftrightarrow\; f(eX) - y = \mathbf{0} \;\Leftrightarrow\; e \cdot \mathcal{M}(\mathbf{0}) \cdot (X + \mathcal{N} g(X) - \mathcal{N} b) = \mathbf{0}.$$

Applying Corollary 2.3 to the map $h(X) := X + \mathcal{N} g(X)$, we get

$$f^{-1}(y) = ex \;\Leftrightarrow\; x = h^{-1}(\mathcal{N} b) \;\text{ and }\; f^{-1}(y) = e h^{-1}(\mathcal{N} \cdot y / e^2).$$

This finishes the proof. □

Further, let $0 \leq r < n$, $p = (p_{r+1}, \dots, p_n)$ be an $(n - r)$-tuple of polynomials

$$p_{r+1}, \dots, p_n \in R[X], \quad X = (X_1, \dots, X_n),$$

and

$$J := \frac{\partial(p_{r+1}, \dots, p_n)}{\partial(X_{r+1}, \dots, X_n)}, \quad e := J(\mathbf{0}).$$

Suppose that

$$\mathbf{0} \in V := \{x \in R^n : p_{r+1}(x) = \ldots = p_n(x) = 0\}.$$

In a similar fashion as above, we can establish the following version of the implicit function theorem.

Proposition 2.5. *If $e \neq 0$, then there exists a unique continuous map*

$$\phi : (e^2 \cdot \mathfrak{m})^{\times r} \longrightarrow (e \cdot \mathfrak{m})^{\times (n-r)}$$

which is definable in the language of valued fields and such that $\phi(0) = 0$ and the graph map

$$(e^2 \cdot \mathfrak{m})^{\times r} \ni u \longrightarrow (u, \phi(u)) \in (e^2 \cdot \mathfrak{m})^{\times r} \times (e \cdot \mathfrak{m})^{\times (n-r)}$$

is an open embedding into the zero locus V of the polynomials p and, more precisely, onto

$$V \cap \left[(e^2 \cdot \mathfrak{m})^{\times r} \times (e \cdot \mathfrak{m})^{\times (n-r)} \right].$$

Proof. Put $f(X) := (X_1, \ldots, X_r, p(X))$; of course, the jacobian determinant of f at $\mathbf{0} \in R^n$ is equal to e. Keep the notation from the proof of Proposition 2.4, take any $b \in e^2 \cdot \mathfrak{m}^{\times r}$ and put $y := (e^2 b, 0) \in R^n$. Then we have the equivalences

$$f(eX) = y \;\Leftrightarrow\; f(eX) - y = \mathbf{0} \;\Leftrightarrow\; e\mathcal{M}(\mathbf{0}) \cdot (X + \mathcal{N} g(X) - \mathcal{N} \cdot (b, 0)) = \mathbf{0}.$$

Applying Corollary 2.3 to the map $h(X) := X + \mathcal{N} g(X)$, we get

$$f^{-1}(y) = ex \;\Leftrightarrow\; x = h^{-1}(\mathcal{N} \cdot (b, 0)) \;\text{ and }\; f^{-1}(y) = eh^{-1}(\mathcal{N} \cdot y/e^2).$$

Therefore the function

$$\phi(u) := eh^{-1}(\mathcal{N} \cdot (u, 0)/e^2)$$

is the one we are looking for. □

3. Density property and a version of the Artin–Mazur theorem over Henselian valued fields

We say that a topological field K satisfies the *density property* (cf. [30, 44]) if the following equivalent conditions hold.

(1) If X is a smooth, irreducible K-variety and $\emptyset \neq U \subset X$ is a Zariski open subset, then $U(K)$ is dense in $X(K)$ in the K-topology.
(2) If C is a smooth, irreducible K-curve and $\emptyset \neq U$ is a Zariski open subset, then $U(K)$ is dense in $C(K)$ in the K-topology.
(3) If C is a smooth, irreducible K-curve, then $C(K)$ has no isolated points.

(This property is indispensable for ensuring reasonable topological and geometric properties of algebraic subsets of K^n; see [44] for the case where the ground field K is a Henselian rank one valued field.) The density property of Henselian non-trivially valued fields follows immediately from Proposition 2.5 and the Jacobian criterion for smoothness (see e.g. [17, Theorem 16.19]), recalled below for the reader's convenience.

Theorem 3.1. *Let $I = (p_1, \ldots, p_s) \subset K[X]$, $X = (X_1, \ldots, X_n)$ be an ideal, $A := K[X]/I$ and $V := \operatorname{Spec}(A)$. Suppose the origin $\mathbf{0} \in K^n$ lies in V (equivalently, $I \subset (X)K[X]$) and V is of dimension r at $\mathbf{0}$. Then the Jacobian matrix*

$$\mathcal{M} := \left[\frac{\partial p_i}{\partial X_j}(\mathbf{0}) : \; i = 1, \ldots, s, \; j = 1, \ldots, n \right]$$

has rank $\leq (n-r)$ and V is smooth at $\mathbf{0}$ iff $\mathcal{M}$ has exactly rank $(n-r)$. Furthermore, if V is smooth at $\mathbf{0}$ and

$$\mathcal{J} := \frac{\partial(p_{r+1}, \ldots, p_n)}{\partial(X_{r+1}, \ldots, X_n)}(\mathbf{0}) = \det \left[\frac{\partial p_i}{\partial X_j}(\mathbf{0}) : \; i, j = r+1, \ldots, n \right] \neq 0,$$

then $p_{r+1}, \dots, p_n$ *generate the localization* $I \cdot K[X]_{(X_1,\dots,X_n)}$ *of the ideal* I *with respect to the maximal ideal* $(X_1, \dots, X_n)$.

Remark 3.2. Under the above assumptions, consider the completion $\widehat{A} = K[[X]]/I \cdot K[[X]]$ of A in the (X)-adic topology. If $\mathcal{J} \neq 0$, it follows from the implicit function theorem for formal power series that there are unique power series

$$\phi_{r+1}, \dots, \phi_n \in (X_1, \dots, X_r) \cdot K[[X_1, \dots, X_r]]$$

such that

$$p_i(X_1, \dots, X_r, \phi_{r+1}(X_1, \dots, X_r), \dots, \phi_n(X_1, \dots, X_r)) = 0$$

for $i = r+1, \dots, n$. Therefore the homomorphism

$$\widehat{\alpha} : \widehat{A} \longrightarrow K[[X_1, \dots, X_r]], \quad X_j \mapsto X_j, \ X_k \mapsto \phi_k(X_1, \dots, X_r),$$

for $j = 1, \dots, r$ and $k = r+1, \dots, n$, is an isomorphism.

Conversely, suppose that $\widehat{\alpha}$ is an isomorphism; this means that the projection from V onto $\operatorname{Spec} K[X_1, \dots, X_r]$ is etale at $\mathbf{0}$. Then the local rings A and $\widehat{A}$ are regular and, moreover, it is easy to check that the determinant $\mathcal{J} \neq 0$ does not vanish after perhaps renumbering the polynomials $p_i(X)$.

We say that a formal power series $\phi \in K[[X]]$, $X = (X_1, \dots, X_n)$, is algebraic if it is algebraic over $K[X]$. The kernel of the homomorphism of K-algebras

$$\sigma : K[X, T] \longrightarrow K[[X]], \quad X_1 \mapsto X_1, \dots, X_n \mapsto X_n, \ T \mapsto \phi(X),$$

is, of course, a principal prime ideal: $\ker \sigma = (p) \subset K[X, T]$, where $p \in K[X, T]$ is a unique (up to a constant factor) irreducible polynomial, called an *irreducible polynomial* of ϕ.

We now state a version of the Artin–Mazur theorem (cf. [3, 4] for the classical versions).

Proposition 3.3. *Let* $\phi \in (X)K[[X]]$ *be an algebraic formal power series. Then there exist polynomials*

$$p_1, \dots, p_r \in K[X, Y], \quad Y = (Y_1, \dots, Y_r),$$

and formal power series $\phi_2, \dots, \phi_r \in K[[X]]$ *such that*

$$e := \frac{\partial(p_1, \dots, p_r)}{\partial(Y_1, \dots, Y_r)}(\mathbf{0}) = \det \left[\frac{\partial p_i}{\partial Y_j}(\mathbf{0}) : \ i, j = 1, \dots, r \right] \neq 0,$$

and

$$p_i(X_1, \dots, X_n, \phi_1(X), \dots, \phi_r(X)) = 0, \quad i = 1, \dots, r,$$

where $\phi_1 := \phi$.

Proof. Let $p_1(X, Y_1)$ be an irreducible polynomial of ϕ_1. Then the integral closure B of $A := K[X, Y_1]/(p_1)$ is a finite A-module and thus is of the form

$$B = K[X, Y]/(p_1, \dots, p_s), \quad Y = (Y_1, \dots, Y_r),$$

where $p_1, \dots, p_s \in K[X, Y]$. Obviously, A and B are of dimension n, and the induced embedding $\alpha : A \to K[[X]]$ extends to an embedding $\beta : B \to K[[X]]$. Put

$$\phi_k := \beta(Y_k) \in K[[X]], \quad k = 1, \dots, r.$$

Substituting $Y_k - \phi_k(0)$ for Y_k, we may assume that $\phi_k(0) = 0$ for all $k = 1, \dots, r$. Hence $p_i(\mathbf{0}) = 0$ for all $i = 1, \dots, s$.

The completion $\widehat{B}$ of B in the (X, Y)-adic topology is a local ring of dimension n, and the induced homomorphism

$$\widehat{\beta} : \widehat{B} = K[[X, Y]]/(p_1, \dots, p_s) \longrightarrow K[[X]]$$

is, of course, surjective. But, by the Zariski main theorem (cf. [59, Chap. VIII, § 13, Theorem 32]), $\widehat{B}$ is a normal domain. Comparison of dimensions shows that $\widehat{\beta}$ is an isomorphism. Now, it follows from Remark 3.2 that the determinant $e \neq 0$ does not vanish after perhaps renumbering the polynomials $p_i(X)$. This finishes the proof. □

Propositions 3.3 and 2.5 immediately yield the following

Corollary 3.4. *Let $\phi \in (X)K[[X]]$ be an algebraic power series with irreducible polynomial $p(X,T) \in K[X,T]$. Then there is an $a \in K$, $a \neq 0$, and a unique continuous function*

$$\widetilde{\phi} : a \cdot R^n \longrightarrow K$$

corresponding to ϕ, which is definable in the language of valued fields and such that $\widetilde{\phi}(0) = 0$ and $p(x, \widetilde{\phi}(x)) = 0$ for all $x \in a \cdot R^n$. □

For simplicity, we shall denote the induced continuous function by the same letter ϕ. This abuse of notation will not lead to confusion in general.

Remark 3.5. Clearly, the ring $K[[X]]_{alg}$ of algebraic power series is the henselization of the local ring $K[X]_{(X)}$ of regular functions. Therefore the implicit functions $\phi_{r+1}(u), \dots, \phi_n(u)$ from Proposition 2.5 correspond to unique algebraic power series

$$\phi_{r+1}(X_1, \dots, X_r), \dots, \phi_n(X_1, \dots, X_r)$$

without constant term. In fact, one can deduce by means of the classical version of the implicit function theorem for restricted power series (cf. [5, Chap. III, §4.5] or [20]) that $\phi_{r+1}, \dots, \phi_n$ are of the form

$$\phi_k(X_1, \dots, X_r) = e \cdot \omega_k(X_1/e^2, \dots, X_r/e^2), \quad k = r+1, \dots, n,$$

where $\omega_k(X_1, \dots, X_r) \in R[[X_1, \dots, X_r]]$ and $e \in R$.

4. The Newton–Puiseux and Abhyankar–Jung Theorems

Here we are going to provide a version of the Newton–Puiseux theorem, which will be used in analysis of definable functions of one variable in the next section.

We call a polynomial

$$f(X;T) = T^s + a_{s-1}(X)T^{n-1} + \dots + a_0(X) \in K[[X]][T],$$

$X = (X_1, \dots, X_s)$, quasiordinary if its discriminant $D(X)$ is a normal crossing:

$$D(X) = X^\alpha \cdot u(X) \quad \text{ with } \quad \alpha \in \mathbb{N}^s, \; u(X) \in k[[X]], \; u(0) \neq 0.$$

Let K be an algebraically closed field of characteristic zero. Consider a henselian $K[X]$-subalgebra $K\langle X \rangle$ of the formal power series ring $K[[X]]$ which is closed under reciprocal (whence it is a local ring), power substitution and division by a coordinate. For positive integers $r_1, \dots, r_n$ put

$$K\langle X_1^{1/r_1}, \dots, X_n^{1/r_n} \rangle := \left\{ a(X_1^{1/r_1}, \dots, X_n^{1/r_n}) : a(X) \in K\langle X \rangle \right\};$$

when $r_1 = \ldots = r_m = r$, we denote the above algebra by $K\langle X^{1/r} \rangle$.

In our paper [43] (see also [52]), we established a version of the Abhyankar–Jung theorem recalled below. This axiomatic approach to that theorem was given for the first time in our preprint [42].

Proposition 4.1. *Under the above assumptions, every quasiordinary polynomial*

$$f(X;T) = T^s + a_{s-1}(X)T^{s-1} + \dots + a_0(X) \in K\langle X \rangle[T]$$

has all its roots in $K\langle X^{1/r} \rangle$ for some $r \in \mathbb{N}$; actually, one can take $r = s!$.

A particular case is the following version of the Newton-Puiseux theorem.

Corollary 4.2. *Let X denote one variable. Every polynomial*

$$f(X;T) = T^s + a_{s-1}(X)T^{s-1} + \cdots + a_0(X) \in K\langle X\rangle[T]$$

has all its roots in $K\langle X^{1/r}\rangle$ for some $r \in \mathbb{N}$; one can take $r = s!$. Equivalently, the polynomial $f(X^r, T)$ splits into T-linear factors. If $f(X,T)$ is irreducible, then $r = s$ will do and

$$f(X^s, T) = \prod_{i=1}^{s} (T - \phi(\epsilon^i X)),$$

where $\phi(X) \in K\langle X\rangle$ and ϵ is a primitive root of unity.

Remark 4.3. Since the proof of these theorems is of finitary character, it is easy to check that if the ground field K of characteristic zero is not algebraically closed, they remain valid for the Henselian subalgebra $\overline{K} \otimes_K K\langle X\rangle$ of $\overline{K}[[X]]$, where $\overline{K}$ denotes the algebraic closure of K.

The ring $K[[X]]_{alg}$ of algebraic power series is a local Henselian ring closed under power substitutions and division by a coordinate. Thus the above results apply to the algebra

$$K\langle X\rangle = K[[X]]_{alg}.$$

5. Definable functions of one variable

At this stage, we can readily to proceed with analysis of definable functions of one variable over arbitrary Henselian valued fields of equicharacteristic zero. We wish to establish a general version of the theorem on existence of the limit stated below. It was proven in [44, Proposition 5.2] over rank one valued fields. Now the language $\mathcal{L}$ under consideration is the three-sorted language of Denef–Pas.

Theorem 5.1. *(Existence of the limit) Let $f : A \to K$ be an $\mathcal{L}$-definable function on a subset A of K and suppose 0 is an accumulation point of A. Then there is a finite partition of A into $\mathcal{L}$-definable sets $A_1, \ldots, A_r$ and points $w_1 \ldots, w_r \in \mathbb{P}^1(K)$ such that*

$$\lim_{x\to 0} f|A_i\,(x) = w_i \quad \textit{for} \;\; i = 1, \ldots, r.$$

Moreover, there is a neighborhood U of 0 such that each definable set

$$\{(v(x), v(f(x))) :\; x \in (A_i \cap U) \setminus \{0\}\} \subset \Gamma \times (\Gamma \cup \;\{\infty\}),\;\; i = 1, \ldots, r,$$

is contained in an affine line with rational slope $q \cdot l = p_i \cdot k + \beta_i, \quad i = 1, \ldots, r,$ with $p_i, q \in \mathbb{Z}$, $q > 0$, $\beta_i \in \Gamma$, or in $\Gamma \times \{\infty\}$.

Proof. Having the Newton–Puiseux theorem for algebraic power series at hand, we can repeat mutatis mutandis the proof from loc. cit. as briefly outlined below. In that paper, the field L is the completion of the algebraic closure $\overline{K}$ of the ground field K. Here, in view of Corollary 4.3, the K-algebras $L\{X\}$ and $\widehat{K}\{X\}$ should be just replaced with $\overline{K} \otimes_K K[[X]]_{alg}$ and $K[[X]]_{alg}$, respectively. Then the reasonings follow almost verbatim. Note also that Lemma 5.1 (to the effect that K is a closed subspace of $\overline{K}$) holds true for arbitrary Henselian valued fields of equicharacteristic zero. This follows directly from that the field K is algebraically maximal (as it is Henselian and finitely ramified; see e.g. [18, Chap. 4]). □

We conclude with the following comment. The above proposition along with the technique of fiber shrinking from [44, Section 6] were two basic tools in the proof of the closedness theorem [44, Theorem 3.1] over Henselian rank one valued fields, which plays an important role in Henselian geometry.

6. Fiber shrinking

Consider a Henselian valued field K of equicharacteristic zero along with the three-sorted language $\mathcal{L}$ of Denef–Pas. In this section, we remind the reader the concept of fiber shrinking introduced in our paper [44, Section 6].

Let A be an $\mathcal{L}$-definable subset of K^n with accumulation point $a = (a_1, \dots, a_n) \in K^n$ and E an $\mathcal{L}$-definable subset of K with accumulation point a_1. We call an $\mathcal{L}$-definable family of sets $\Phi = \bigcup_{t \in E} \{t\} \times \Phi_t \subset A$ an $\mathcal{L}$-definable x_1-fiber shrinking for the set A at a if

$$\lim_{t \to a_1} \Phi_t = (a_2, \dots, a_n),$$

i.e. for any neighborhood U of $(a_2, \dots, a_n) \in K^{n-1}$, there is a neighborhood V of $a_1 \in K$ such that $\emptyset \neq \Phi_t \subset U$ for every $t \in V \cap E$, $t \neq a_1$. When $n = 1$, A is itself a fiber shrinking for the subset A of K at an accumulation point $a \in K$.

Proposition 6.1. *(Fiber shrinking) Every $\mathcal{L}$-definable subset A of K^n with accumulation point $a \in K^n$ has, after a permutation of the coordinates, an $\mathcal{L}$-definable x_1-fiber shrinking at a.*

In the case where the ground field K is of rank one, the proof of Proposition 6.1 was given in [44, Section 6]. In the general case, it can be repeated verbatim once we demonstrate the following result on definable subsets in the value group sort Γ.

Lemma 6.2. *Let Γ be an ordered abelian group and P be a definable subset of Γ^n. Suppose that $(\infty, \dots, \infty)$ is an accumulation point of P, i.e. for any $\delta \in \Gamma$ the set*

$$\{x \in P : x_1 > \delta, \dots, x_n > \delta\} \neq \emptyset$$

is non-empty. Then there is an affine semi-line

$$L = \{(r_1 k + \gamma_1, \dots, r_n k + \gamma_n) : \ k \in \Gamma, \ k \geq 0\} \quad \text{with} \ \ r_1, \dots, r_n \in \mathbb{N},$$

passing through a point $\gamma = (\gamma_1, \dots, \gamma_n) \in P$ and such that $(\infty, \dots, \infty)$ is an accumulation point of the intersection $P \cap L$ too.

In [44, Section 6], Lemma 6.2 was established for archimedean groups by means of quantifier elimination in the Presburger language. Now, in the general case, it follows in a similar fashion by means of relative quantifier elimination for ordered abelian groups in the language $\mathcal{L}_{qe}$ due to Cluckers–Halupczok [8], outlined in the next section. Indeed, applying Theorem 7.1 along with Remarks 7.2 and 7.3), it is not difficult to see that the parametrized congruence conditions which occur in the description of the set P are not an essential obstacle to finding the line L we are looking for. Therefore the lemma reduces, likewise as it was in [44, Section 6], to a problem of semi-linear geometry.

7. Quantifier elimination for ordered abelian groups

It is well known that archimedean ordered abelian groups admit quantifier elimination in the Presburger language. Much more complicated are quantifier elimination results for non-archimedean groups (especially those with infinite rank), going back as far as Gurevich [24]. He established a transfer of sentences from ordered abelian groups to so-called coloured chains (i.e. linearly ordered sets with additional unary predicates), enhanced later to allow arbitrary formulas. This was done in his doctoral dissertation "The decision problem for some algebraic theories" (Sverdlovsk, 1968), and by Schmitt in his habilitation dissertation "Model theory of ordered abelian groups" (Heidelberg, 1982); see also the paper [56]. Such a transfer is a kind of relative quantifier elimination, which allows Gurevich–Schmitt [25], in their study of the NIP property, to lift model theoretic properties from ordered sets to ordered abelian groups or, in other words, to transform statements on ordered abelian groups into those on coloured chains.

Instead Cluckers–Halupczok [8] introduce a suitable many-sorted language $\mathcal{L}_{qe}$ with main group sort Γ and auxiliary imaginary sorts (with canonical parameters for some definable families of convex subgroups) which carry the structure of a linearly ordered set with some additional unary predicates. They provide quantifier elimination relative to the auxiliary sorts, where each definable set in the group sort is a union of a family of quantifier free definable sets with parameter running a definable (with quantifiers) set of the auxiliary sorts.

Fortunately, sometimes it is possible to directly deduce information about ordered abelian groups without any deeper knowledge of the auxiliary sorts. For instance, this may be illustrated by their theorem on piecewise linearity of definable functions [8, Corollary 1.10] as well as by Proposition 6.2 and application of quantifier elimination in the proof of the closedness theorem in Section 4.

Now we briefly recall the language $\mathcal{L}_{qe}$ taking care of points essential for our applications. The main group sort Γ is with the constant 0, the binary function $+$ and the unary function $-$. The collection $\mathcal{A}$ of auxiliary sorts consists of certain imaginary sorts:

$$\mathcal{A} := \{\mathcal{S}_p, \mathcal{T}_p, \mathcal{T}_p^+ : p \in \mathbb{P}\};$$

here $\mathbb{P}$ stands for the set of prime numbers. By abuse of notation, $\mathcal{A}$ will also denote the union of the auxiliary sorts. In this section, we denote Γ-sort variables by $x, y, z, \ldots$ and auxiliary sorts variables by $\eta, \theta, \zeta, \ldots$.

Further, the language $\mathcal{L}_{qe}$ consists of some unary predicates on $\mathcal{S}_p$, $p \in \mathbb{P}$, some binary order relations on $\mathcal{A}$, a ternary relation

$$x \equiv_{m,\alpha}^{m'} y \ \text{ on } \ \Gamma \times \Gamma \times \mathcal{S}_p \ \text{ for each } \ p \in \mathbb{P}, \ m, m' \in \mathbb{N},$$

and finally predicates for the ternary relations $x \diamond_\alpha y + k_\alpha$ on $\Gamma \times \Gamma \times \mathcal{A}$, where $\diamond \in \{=, <, \equiv_m\}$, $m \in \mathbb{N}$, $k \in \mathbb{Z}$ and α is the third operand running any of the auxiliary sorts $\mathcal{A}$.

We now explain the meaning of the above ternary relations, which are defined by means of certain definable convex subgroups Γ_α and $\Gamma_\alpha^{m'}$ of Γ with $\alpha \in \mathcal{A}$ and $m' \in \mathbb{N}$. Namely we write

$$x \equiv_{m,\alpha}^{m'} y \ \text{ iff } \ x - y \in \Gamma_\alpha^{m'} + m\Gamma.$$

Further, let 1_α denote the minimal positive element of Γ/Γ_α if Γ/Γ_α is discrete and $1_\alpha := 0$ otherwise, and set $k_\alpha := k \cdot 1_\alpha$ for all $k \in \mathbb{Z}$. By definition we write

$$x \diamond_\alpha y + k_\alpha \quad \text{iff} \quad x \,(\mathrm{mod}\, \Gamma_\alpha) \diamond y \ (\mathrm{mod}\, \Gamma_\alpha) + k_\alpha.$$

(Thus the language $\mathcal{L}_{qe}$ incorporates the Presburger language on all quotients Γ/Γ_α.) Note also that the ordinary predicates $<$ and $\equiv_m$ on Γ are Γ-quantifier-free definable in the language $\mathcal{L}_{qe}$.

Now we can readily formulate quantifier elimination relative to the auxiliary sorts ([8, Theorem 1.8]).

Theorem 7.1. *In the theory T of ordered abelian groups, each $\mathcal{L}_{qe}$-formula $\phi(\bar{x}, \bar{\eta})$ is equivalent to an $\mathcal{L}_{qe}$-formula $\psi(\bar{x}, \bar{\eta})$ in family union form, i.e.*

$$\psi(\bar{x}, \bar{\eta}) = \bigvee_{i=1}^{k} \exists \bar{\theta} \ \left[\chi_i(\bar{\eta}, \bar{\theta}) \wedge \omega_i(\bar{x}, \bar{\theta})\right],$$

where $\bar{\theta}$ are $\mathcal{A}$-variables, the formulas $\chi_i(\bar{\eta}, \bar{\theta})$ live purely in the auxiliary sorts $\mathcal{A}$, each $\omega_i(\bar{x}, \bar{\theta})$ is a conjunction of literals (i.e. atomic or negated atomic formulas) and T implies that the

$\mathcal{L}_{qe}(\mathcal{A})$-formulas

$$\{\chi_i(\bar{\eta},\bar{\alpha}) \wedge \omega_i(\bar{x},\bar{\alpha}) : \ i = 1,\dots,k, \ \bar{\alpha} \in \mathcal{A}\}$$

are pairwise inconsistent.

Remark 7.2. The sets definable (or, definable with parameters) in the main group sort Γ resemble to some extent the sets which are definable in the Presburger language. Indeed, the atomic formulas involved in the formulas $\omega_i(\bar{x},\bar{\theta})$ are of the form $t(\bar{x}) \diamond_{\theta_j} k_{\theta_j}$, where $t(\bar{x})$ is a $\mathbb{Z}$-linear combination (respectively, a $\mathbb{Z}$-linear combination plus an element of Γ), the predicates

$$\diamond \in \{=, <, \equiv_m, \equiv_m^{m'}\} \ \text{ with some } \ m, m' \in \mathbb{N},$$

θ_j is one of the entries of $\bar{\theta}$ and $k \in \mathbb{Z}$; here $k = 0$ if $\diamond$ is $\equiv_m^{m'}$. Clearly, while linear equalities and inequalities define polyhedra, congruence conditions define sets which consist of entire cosets of $m\Gamma$ for finitely many $m \in \mathbb{N}$.

Remark 7.3. Note also that the sets given by atomic formulas $t(\bar{x}) \diamond_{\theta_j} k_{\theta_j}$ consist of entire cosets of the subgroups Γ_{θ_j}. Therefore, the union of those subgroups Γ_{θ_j} which essentially occur in a formula in family union form, describing a proper subset of Γ^n, is not cofinal with Γ. This observation is often useful as, for instance, in the proofs of fiber shrinking and Theorem 1.1.

8. Proof of the closedness theorem

In the proof of Theorem 1.1, we shall generally follow the ideas from our previous paper [44, Section 7]. We must show that if B is an $\mathcal{L}$-definable subset of $D \times (K^{\circ})^n$ and a point a lies in the closure of $A := \pi(B)$, then there is a point b in the closure of B such that $\pi(b) = a$. Again, the proof reduces easily to the case $m = 1$ and next, by means of fiber shrinking (Proposition 6.1), to the case $n = 1$. We may obviously assume that $a = 0 \notin A$.

Whereas in the paper [44] preparation cell decomposition (due to Pas; see [53, Theorem 3.2] and [44, Theorem 2.4]) was combined with quantifier elimination in the Γ sort in the Presburger language, here it is combined with relative quantifier elimination in the language $\mathcal{L}_{qe}$ considered in Section 7. In a similar manner as in [44], we can now assume that B is a subset F of a cell C of the form presented below. Let $a(x,\xi), b(x,\xi), c(x,\xi) : D \longrightarrow K$ be three $\mathcal{L}$-definable functions on an $\mathcal{L}$-definable subset D of $K^2 \times \Bbbk^m$ and let $\nu \in \mathbb{N}$ is a positive integer. For each $\xi \in \Bbbk^m$ set

$$C(\xi) := \Big\{ (x,y) \in K_x^n \times K_y : \ (x,\xi) \in D,$$
$$v(a(x,\xi)) \lhd_1 v((y - c(x,\xi))^{\nu}) \lhd_2 v(b(x,\xi)), \ \overline{ac}(y - c(x,\xi)) = \xi_1 \Big\},$$

where $\lhd_1, \lhd_2$ stand for $<, \leq$ or no condition in any occurrence. A cell C is by definition a disjoint union of the fibres $C(\xi)$. The subset F of C is a union of fibers $F(\xi)$ of the form

$$F(\xi) := \Big\{ (x,y) \in C(\xi) : \ \exists \ \bar{\theta} \ \chi(\bar{\theta}) \ \wedge$$
$$\bigwedge_{i \in I_a} v(a_i(x,\xi)) \lhd_{1,\theta_{j_i}} v((y - c(x,\xi))^{\nu_i}), \ \bigwedge_{i \in I_b} v((y - c(x,\xi))^{\nu_i}) \lhd_{2,\theta_{j_i}} v(b_i(x,\xi))$$
$$\wedge \ \bigwedge_{i \in I_f} v((y - c(x,\xi))^{\nu_i}) \diamond_{\theta_{j_i}} v(f_i(x,\xi)) \Big\},$$

where I_a, I_b, I_f are finite (possibly empty) sets of indices, a_i, b_i, f_i are $\mathcal{L}$-definable functions, $\nu_i, M \in \mathbb{N}$ are positive integers, $\lhd_1$, $\lhd_2$ stand for $<$ or $\leq$, the predicates

$$\diamond \in \{\equiv_M, \neg \equiv_M, \equiv_M^{m'}, \neg \equiv_M^{m'}\} \text{ with some } m' \in \mathbb{N},$$

and θ_{j_i} is one of the entries of $\bar{\theta}$.

As before, since every $\mathcal{L}$-definable subset in the Cartesian product $\Gamma^n \times \Bbbk^m$ of auxiliary sorts is a finite union of the Cartesian products of definable subsets in Γ^n and in $\Bbbk^m$, we can assume that B is one fiber $F(\xi')$ for a parameter $\xi' \in \Bbbk^m$. For simplicity, we abbreviate

$$c(x,\xi'), a(x,\xi'), b(x,\xi'), a_i(x,\xi'), b_i(x,\xi'), f_i(x,\xi')$$

to

$$c(x), a(x), b(x), a_i(x), b_i(x), f_i(x)$$

with $i \in I_a$, $i \in I_b$ and $i \in I_f$. Denote by $E \subset K$ the common domain of these functions; then 0 is an accumulation point of E.

By the theorem on existence of the limit (Theorem 5.1), we can assume that the limits

$$c(0), a(0), b(0), a_i(0), b_i(0), f_i(0)$$

of the functions

$$c(x), a(x), b(x), a_i(x), b_i(x), f_i(x)$$

when $x \to 0$ exist in R. Moreover, there is a neighborhood U of 0 such that, each definable set

$$\{(v(x), v(f_i(x))) : \ x \in (E \cap U) \setminus \{0\}\} \subset \Gamma \times (\Gamma \cup \ \{\infty\}), \quad i \in I_f,$$

is contained in an affine line with rational slope

$$q \cdot l = p_i \cdot k + \beta_i, \quad i \in I_f, \tag{8.1}$$

with $p_i, q \in \mathbb{Z}$, $q > 0$, $\beta_i \in \Gamma$, or in $\Gamma \times \{\infty\}$.

The role of the center $c(x)$ is, of course, immaterial. We may assume, without loss of generality, that it vanishes, $c(x) \equiv 0$, for if a point $b = (0, w) \in K^2$ lies in the closure of the cell with zero center, the point $(0, w + c(0))$ lies in the closure of the cell with center $c(x)$.

Observe now that If $\lhd_1$ occurs and $a(0) = 0$, the set $F(\xi')$ is itself an x-fiber shrinking at $(0,0)$ and the point $b = (0,0)$ is an accumulation point of B lying over $a = 0$, as desired. And so is the point $b = (0,0)$ if $\lhd_{1,\theta_{j_i}}$ occurs and $a_i(0) = 0$ for some $i \in I_a$, because then the set $F(\xi')$ contains the x-fiber shrinking

$$F(\xi') \cap \{(x,y) \in E \times K : \ v(a_i(x)) \lhd_1 v(y^{\nu_i})\}.$$

So suppose that either only $\lhd_2$ occur or $\lhd_1$ occur and, moreover, $a(0) \neq 0$ and $a_i(0) \neq 0$ for all $i \in I_a$. By elimination of K-quantifiers, the set $v(E)$ is a definable subset of Γ. Further, it is easy to check, applying Theorem 7.1 ff. likewise as it was in Lemma 6.2, that the set $v(E)$ is given near infinity only by finitely many parametrized congruence conditions of the form

$$v(E) = \left\{ k \in \Gamma : \ k > \beta \ \wedge \ \exists \, \bar{\theta} \ \omega(\bar{\theta}) \ \wedge \ \bigwedge_{i=1}^{s} \ m_i k \diamond_{N,\theta_{j_i}} \gamma_i \right\}. \tag{8.2}$$

where $\beta, \gamma_i \in \Gamma$, $m_i, N \in \mathbb{N}$ for $i = 1, \ldots, s$, the predicates

$$\diamond \in \{\equiv_N, \neg \equiv_N, \equiv_N^{m'}, \neg \equiv_N^{m'}\} \text{ with some } m' \in \mathbb{N},$$

and θ_{j_i} is one of the entries of $\bar{\theta}$. Obviously, after perhaps shrinking the neighborhood of zero, we may assume that

$$v(a(x)) = v(a(0)) \ \text{ and } \ v(a_i(x)) = v(a_i(0))$$

for all $i \in I_a$ and $x \in E \setminus \{0\}$, $v(x) > \beta$.

Now, take an element $(u, w) \in F(\xi')$ with $u \in E \setminus \{0\}$, $v(u) > \beta$. In order to complete the proof, it suffices to show that $(0, w)$ is an accumulation point of $F(\xi')$. To this end, observe that, by equality 8.2, there is a point $x \in E$ arbitrarily close to 0 such that

$$v(x) \in v(u) + qMN \cdot \Gamma.$$

By equality 8.1, we get $v(f_i(x)) \in v(f_i(u)) + p_i MN \cdot \Gamma,\ i \in I_f$, and hence

$$v\left(f_i(x)\right) \equiv_M v(f_i(u)), \quad i \in I_f. \tag{8.3}$$

Clearly, in the vicinity of zero we have

$$v(y^{\nu}) \lhd_2 v(b(x, \xi)) \quad \text{and} \quad \bigwedge_{i \in I_b} v(y^{\nu_i}) \lhd_{2,\theta_{j_i}} v(b_i(x, \xi)).$$

Therefore equality 8.3 along with the definition of the fibre $F(\xi')$ yield $(x, w) \in F(\xi')$, concluding the proof of the closedness theorem.

9. Piecewise continuity of definable functions

Further, let $\mathcal{L}$ be the three-sorted language $\mathcal{L}$ of Denef–Pas. The main purpose of this section is to prove the following

Theorem 9.1. *Let $A \subset K^n$ and $f : A \to \mathbb{P}^1(K)$ be an $\mathcal{L}$-definable function. Then f is piecewise continuous, i.e. there is a finite partition of A into $\mathcal{L}$-definable locally closed subsets $A_1, \ldots, A_s$ of K^n such that the restriction of f to each A_i is continuous.*

We immediately obtain

Corollary 9.2. *The conclusion of the above theorem holds for any $\mathcal{L}$-definable function $f : A \to K$.*

The proof of Theorem 9.1 relies on two basic ingredients. The first one is concerned with a theory of algebraic dimension and decomposition of definable sets into a finite union of locally closed definable subsets we begin with. It was established by van den Dries [13] for certain expansions of rings (and Henselian valued fields, in particular) which admit quantifier elimination and are equipped with a topological system. The second one is the closedness theorem (Theorem 1.1).

Consider an infinite integral domain D with quotient field K. One of the fundamental concepts introduced by van den Dries [13] is that of a *topological system* on a given expansion $\mathcal{D}$ of a domain D in a language $\widetilde{\mathcal{L}}$. That concept incorporates both Zariski-type and definable topologies. We remind the reader that it consists of a topology τ_n on each set D^n, $n \in \mathbb{N}$, such that:

1) For any n-ary $\widetilde{\mathcal{L}}_D$-terms $t_1, \ldots, t_s$, $n, s \in \mathbb{N}$, the induced map

$$D^n \ni a \longrightarrow (t_1(a), \ldots, t_s(a)) \in D^s$$

is continuous.

2) Every singleton $\{a\}$, $a \in D$, is a closed subset of D.

3) For any n-ary relation symbol R of the language $\widetilde{\mathcal{L}}$ and any sequence $1 \le i_1 < \ldots < i_k \le n$, $1 \le k \le n$, the two sets

$$\{(a_{i_1}, \ldots, a_{i_k}) \in D^k :\ \mathcal{D} \models R((a_{i_1}, \ldots, a_{i_k})^{\&}),\ a_{i_1} \neq 0, \ldots, a_{i_k} \neq 0\},$$
$$\{(a_{i_1}, \ldots, a_{i_k}) \in D^k :\ \mathcal{D} \models \neg R((a_{i_1}, \ldots, a_{i_k})^{\&}),\ a_{i_1} \neq 0, \ldots, a_{i_k} \neq 0\}$$

are open in D^k; here $(a_{i_1}, \ldots, a_{i_k})^{\&}$ denotes the element of D^n whose i_j-th coordinate is a_{i_j}, $j = 1, \ldots, k$, and whose remaining coordinates are zero.

Finite intersections of closed sets of the form $\{a \in D^n : t(a) = 0\}$, where t is an n-ary $\widetilde{\mathcal{L}}_D$-term, will be called *special closed subsets* of D^n. Finite intersections of open sets of the form

$$\{a \in D^n : t(a) \neq 0\},$$

$$\{a \in D^n : \mathcal{D} \models R((t_{i_1}(a), \dots, t_{i_k}(a))^{\&}),\, t_{i_1}(a) \neq 0, \dots, t_{i_k}(a) \neq 0\}$$

or

$$\{a \in D^n : \mathcal{D} \models \neg R((t_{i_1}(a), \dots, t_{i_k}(a))^{\&}),\, t_{i_1}(a) \neq 0, \dots, t_{i_k}(a) \neq 0\},$$

where t, t_{i_1}, t_{i_k} are $\widetilde{\mathcal{L}}_D$-terms, will be called *special open subsets* of D^n. Finally, an intersection of a special open and a special closed subsets of D^n will be called a *special locally closed* subset of D^n. Every quantifier-free $\widetilde{\mathcal{L}}$-definable set is a finite union of special locally closed sets.

Suppose now that the language $\widetilde{\mathcal{L}}$ extends the language of rings and has no extra function symbols of arity > 0 and that an $\widetilde{\mathcal{L}}$-expansion $\mathcal{D}$ of the domain D under study admits quantifier elimination and is equipped with a topological system such that every non-empty special open subset of D is infinite. These conditions ensure that $\mathcal{D}$ is algebraically bounded and algebraic dimension is a dimension function on $\mathcal{D}$ ([13, Proposition 2.15 and 2.7]). Algebraic dimension is the only dimension function on $\mathcal{D}$ whenever, in addition, D is a non-trivially valued field and the topology τ_1 is induced by its valuation. Then, for simplicity, the algebraic dimension of an $\widetilde{\mathcal{L}}$-definable set E will be denoted by $\dim E$.

Now we recall the following two basic results from the paper [13, Propositions 2.17 and 2.23]:

Proposition 9.3. *Every $\widetilde{\mathcal{L}}$-definable subset of D^n is a finite union of intersections of Zariski closed with special open subsets of D^n and, a fortiori, a finite union of locally closed $\widetilde{\mathcal{L}}$-definable subsets of D^n.*

Proposition 9.4. *Let E be an $\widetilde{\mathcal{L}}$-definable subset of D^n, and let $\overline{E}$ stand for its closure and $\partial E := \overline{E} \setminus E$ for its frontier. Then*

$$\text{alg.dim}\,(\partial E) < \text{alg.dim}\,(E).$$

It is not difficult to strengthen the former proposition as follows.

Corollary 9.5. *Every $\widetilde{\mathcal{L}}$-definable set is a finite disjoint union of locally closed sets.*

Quantifier elimination due to Pas [53, Theorem 4.1] (more precisely, elimination of K-quantifiers) enables translation of the language $\mathcal{L}$ of Denef–Pas on K into a language $\widetilde{\mathcal{L}}$ described above, which is equipped with the topological system wherein τ_n is the K-topology on K^n, $n \in \mathbb{N}$. Indeed, we must augment the language of rings by adding extra relation symbols for the inverse images under the valuation and angular component map of relations on the value group and residue field, respectively. More precisely, we must add the names of sets of the form

$$\{a \in K^n : (v(a_1), \dots, v(a_n)) \in P\} \quad \text{and} \quad \{a \in K^n : (\overline{ac}\, a_1, \dots, \overline{ac}\, a_n) \in Q\},$$

where P and Q are definable subsets of Γ^n and $\Bbbk^n$ (as the auxiliary sorts of the language $\mathcal{L}$), respectively.

Summing up, the foregoing results apply in the case of Henselian non-trivially valued fields with the three-sorted language $\mathcal{L}$ of Denef–Pas. Now we can readily prove Theorem 9.1.

Proof. Consider an $\mathcal{L}$-definable function $f : A \to \mathbb{P}^1(K)$ and its graph

$$E := \{(x, f(x)) : x \in A\} \subset K^n \times \mathbb{P}^1(K).$$

We shall proceed with induction with respect to the dimension $d = \dim A = \dim E$ of the source and graph of f. By Corollary 9.5, we can assume that the graph E is a locally closed subset of $K^n \times \mathbb{P}^1(K)$ of dimension d and that the conclusion of the theorem holds for functions with source and graph of dimension $< d$.

Let F be the closure of E in $K^n \times \mathbb{P}^1(K)$ and $\partial E := F \setminus E$ be the frontier of E. Since E is locally closed, the frontier ∂E is a closed subset of $K^n \times \mathbb{P}^1(K)$ as well. Let

$$\pi : K^n \times \mathbb{P}^1(K) \longrightarrow K^n$$

be the canonical projection. Then, by virtue of the closedness theorem, the images $\pi(F)$ and $\pi(\partial E)$ are closed subsets of K^n. Further,

$$\dim F = \dim \pi(F) = d \quad \text{and} \quad \dim \pi(\partial E) \leq \dim \partial E < d;$$

the last inequality holds by Proposition 9.4. Putting $B := \pi(F) \setminus \pi(\partial E) \subset \pi(E) = A$, we thus get $\dim B = d$ and $\dim (A \setminus B) < d$. Clearly, the set

$$E_0 := E \cap (B \times \mathbb{P}^1(K)) = F \cap (B \times \mathbb{P}^1(K))$$

is a closed subset of $B \times \mathbb{P}^1(K)$ and is the graph of the restriction $f_0 : B \longrightarrow \mathbb{P}^1(K)$ of f to B. Again, it follows immediately from the closedness theorem that the restriction $\pi_0 : E_0 \longrightarrow B$ of the projection π to E_0 is a definably closed map. Therefore f_0 is a continuous function. But, by the induction hypothesis, the restriction of f to $A \setminus B$ satisfies the conclusion of the theorem, whence so does the function f. This completes the proof. □

10. Curve selection

We now pass to curve selection over non-locally compact ground fields under study. While the real version of curve selection goes back to the papers [6, 58] (see also [40, 41, 4]), the p-adic one was achieved in the papers [57, 12].

In this section we give two versions of curve selection which are counterparts of the ones from our paper [44, Proposition 8.1 and 8.2] over rank one valued fields. The first one is concerned with valuative semialgebraic sets and we can repeat verbatim its proof which relies on transformation to a normal crossing by blowing up and the closedness theorem.

By a valuative semialgebraic subset of K^n we mean a (finite) Boolean combination of elementary valuative semialgebraic subsets, i.e. sets of the form $\{x \in K^n : \; v(f(x)) \leq v(g(x))\}$, where f and g are regular functions on K^n. We call a map φ semialgebraic if its graph is a valuative semialgebraic set.

Proposition 10.1. *Let A be a valuative semialgebraic subset of K^n. If a point $a \in K^n$ lies in the closure (in the K-topology) of $A \setminus \{a\}$, then there is a semialgebraic map $\varphi : R \longrightarrow K^n$ given by algebraic power series such that*

$$\varphi(0) = a \quad \textit{and} \quad \varphi(R \setminus \{0\}) \subset A \setminus \{a\}.$$

We now turn to the general version of curve selection for $\mathcal{L}$-definable sets. Under the circumstances, we apply relative quantifier elimination in a many-sorted language due to Cluckers–Halupczok rather than simply quantifier elimination in the Presburger language for rank one valued fields. The passage between the two corresponding reasonings for curve selection is similar to that for fiber shrinking. Nevertheless we provide a detailed proof for more clarity and the reader's convenience. Note that both fiber shrinking and curve selection apply Lemma 6.2.

Proposition 10.2. *Let A be an $\mathcal{L}$-definable subset of K^n. If a point $a \in K^n$ lies in the closure (in the K-topology) of $A \setminus \{a\}$, then there exist a semialgebraic map $\varphi : R \longrightarrow K^n$ given by algebraic power series and an $\mathcal{L}$-definable subset E of R with accumulation point 0 such that*

$$\varphi(0) = a \quad \textit{and} \quad \varphi(E \setminus \{0\}) \subset A \setminus \{a\}.$$

Proof. As before, we proceed with induction with respect to the dimension of the ambient space n. The case $n=1$ being evident, suppose $n>1$. By elimination of K-quantifiers, the set $A \setminus \{a\}$ is a finite union of sets defined by conditions of the form

$$(v(f_1(x)),\dots,v(f_r(x))) \in P, \quad (\overline{ac}\, g_1(x),\dots,\overline{ac}\, g_s(x)) \in Q,$$

where $f_i, g_j \in K[x]$ are polynomials, and P and Q are definable subsets of Γ^r and $\Bbbk^s$, respectively. Without loss of generality, we may assume that A is such a set and $a=0$.

Take a finite composite $\sigma : Y \longrightarrow K\mathbb{A}^n$ of blow-ups along smooth centers such that the pull-backs $f_1^\sigma,\dots,f_r^\sigma$ and $g_1^\sigma,\dots,g_s^\sigma$ are normal crossing divisors unless they vanish. Since the restriction $\sigma : Y(K) \longrightarrow K^n$ is definably closed (Corollary 1.6), there is a point $b \in Y(K) \cap \sigma^{-1}(a)$ which lies in the closure of the set $B := Y(K) \cap \sigma^{-1}(A \setminus \{a\})$. Take local coordinates $y_1 \dots, y_n$ near b in which $b=0$ and every pull-back above is a normal crossing. We shall first select a semialgebraic map $\psi : R \longrightarrow Y(K)$ given by restricted power series and an $\mathcal{L}$-definable subset E of R with accumulation point 0 such that $\psi(0)=b$ and $\psi(E \setminus \{0\}) \subset B$.

Since the valuation map and the angular component map composed with a continuous function are locally constant near any point at which this function does not vanish, the conditions which describe the set B near b are of the form

$$(v(y_1),\dots,v(y_n)) \in \widetilde{P}, \quad (\overline{ac}\, y_1,\dots,\overline{ac}\, y_n) \in \widetilde{Q},$$

where $\widetilde{P}$ and $\widetilde{Q}$ are definable subsets of Γ^n and $\Bbbk^n$, respectively.

The set B_0 determined by the conditions

$$(v(y_1),\dots,v(y_n)) \in \widetilde{P},$$

$$(\overline{ac}\, y_1,\dots,\overline{ac}\, y_n) \in \widetilde{Q} \cap \bigcup_{i=1}^{n} \{\xi_i = 0\},$$

is contained near b in the union of hyperplanes $\{y_i=0\}$, $i=1,\dots,n$. If b is an accumulation point of the set B_0, then the desired map ψ exists by the induction hypothesis. Otherwise b is an accumulation point of the set $B_1 := B \setminus B_0$.

Now we are going to apply relative quantifier elimination in the value group sort Γ. Similarly, as in the proof of Lemma 6.2, the parametrized congruence conditions which occur in the description of the definable subset $\widetilde{P}$ of Γ^n, achieved via quantifier elimination, are not an essential obstacle to finding the desired map ψ, but affect only the definable subset E of R. Neither are the conditions

$$\widetilde{Q} \setminus \bigcup_{i=1}^{n} \{\xi_i = 0\}$$

imposed on the angular components of the coordinates $y_1,\dots,y_n$, because none of them vanishes here. Therefore, in order to select the map ψ, we must first of all analyze the linear conditions (equalities and inequalities) which occur in the description of the set $\widetilde{P}$.

The set $\widetilde{P}$ has an accumulation point $(\infty,\dots,\infty)$ as $b=0$ is an accumulation point of B. By Lemma 6.2, there is an affine semi-line

$$L = \{(r_1 t + \gamma_1,\dots,r_n t + \gamma_n) : t \in \Gamma,\ t \geq 0\} \quad \text{with} \quad r_1,\dots,r_n \in \mathbb{N},$$

passing through a point $\gamma = (\gamma_1,\dots,\gamma_n) \in P$ and such that $(\infty,\dots,\infty)$ is an accumulation point of the intersection $P \cap L$ too.

Now, take some elements

$$(\xi_1, \ldots, \xi_n) \in \widetilde{Q} \setminus \bigcup_{i=1}^{n} \{\xi_i = 0\}$$

and next some elements $w_1, \ldots, w_n \in K$ for which

$$v(w_1) = \gamma_1, \ldots, v(w_n) = \gamma_n \quad \text{and} \quad \overline{ac}\, w_1 = \xi_1, \ldots, \overline{ac}\, w_n = \xi_n.$$

It is not difficult to check that there exists an $\mathcal{L}$-definable subset E of R which is determined by a finite number of parametrized congruence conditions (in the many-sorted language $\mathcal{L}_{qe}$ described in Section 7) imposed on $v(t)$ and the conditions $\overline{ac}\, t = 1$ such that the subset

$$F := \{(w_1 \cdot t^{r_1}, \ldots, w_n \cdot t^{r_n}) : \ t \in E\}$$

of the arc

$$\psi : R \to Y, \quad \psi(t) = (w_1 \cdot t^{r_1}, \ldots, w_n \cdot t^{r_n})$$

is contained in B_1. Then $\varphi := \sigma \circ \psi$ is the map we are looking for. This completes the proof. □

11. The Łojasiewicz inequalities

In this section we provide certain two versions of the Łojasiewicz inequality which generalize the ones from [44, Propositions 9.1 and 9.2] to the case of arbitrary Henselian valued fields. Moreover, the first one is now formulated for several functions $g_1, \ldots, g_m$. For its proof we still need the following easy consequence of the closedness theorem.

Proposition 11.1. *Let $f : A \to K$ be a continuous $\mathcal{L}$-definable function on a closed bounded subset $A \subset K^n$. Then f is a bounded function, i.e. there is an $\omega \in \Gamma$ such that $v(f(x)) \geq \omega$ for all $x \in A$.*

We adopt the following notation:

$$v(x) = v(x_1, \ldots, x_n) := \min \{v(x_1), \ldots, v(x_n)\}$$

for $x = (x_1, \ldots, x_n) \in K^n$.

Theorem 11.2. *Let $f, g_1, \ldots, g_m : A \to K$ be continuous $\mathcal{L}$-definable functions on a closed (in the K-topology) bounded subset A of K^m. If*

$$\{x \in A : g_1(x) = \ldots = g_m(x) = 0\} \subset \{x \in A : f(x) = 0\},$$

then there exist a positive integer s and a constant $\beta \in \Gamma$ such that

$$s \cdot v(f(x)) + \beta \geq v((g_1(x), \ldots, g_m(x)))$$

for all $x \in A$.

Proof. Put $g = (g_1, \ldots, g_m)$. It is easy to check that the set $A_\gamma := \{x \in A : \ v(f(x)) = \gamma\}$ is a closed $\mathcal{L}$-definable subset of A for every $\gamma \in \Gamma$. By the hypothesis and the closedness theorem, the set $g(A_\gamma)$ is a closed $\mathcal{L}$-definable subset of $K^m \setminus \{0\}$, $\gamma \in \Gamma$. The set $v(g(A_\gamma))$ is thus bounded from above, i.e. $v(g(A_\gamma)) \leq \alpha(\gamma)$ for some $\alpha(\gamma) \in \Gamma$. By elimination of K-quantifiers, the set

$$\Lambda := \{(v(f(x)), v(g(x))) \in \Gamma^2 : \ x \in A, \ f(x) \neq 0\} \ \subset \ \{(\gamma, \delta) \in \Gamma^2 : \ \delta \leq \alpha(\gamma)\}$$

is a definable subset of Γ^2 in the many-sorted language $\mathcal{L}_{qe}$ from Section 7. Applying Theorem 7.1 ff., we see that this set is described by a finite number of parametrized linear equalities and inequalities, and of parametrized congruence conditions. Hence

$$\Lambda \cap \{(\gamma, \delta) \in \Gamma^2 : \ \gamma > \gamma_0\} \subset \{(\gamma, \delta) \in \Gamma^2 : \ \delta \leq s \cdot \gamma\}$$

for a positive integer s and some $\gamma_0 \in \Gamma$. We thus get

$$v(g(x)) \leq s \cdot v(f(x)) \ \text{ if } \ x \in A,\, v(f(x)) > \gamma_0.$$

Again, by the hypothesis, we have $g(\{x \in A : \ v(f(x)) \leq \gamma_0\}) \subset K^m \setminus \{0\}$. Therefore it follows from the closedness theorem that the set $\{v(g(x)) \in \Gamma : \ v(f(x)) \leq \gamma_0\}$ is bounded from above, say, by a $\theta \in \Gamma$. Taking an $\omega \in \Gamma$ as in Proposition 11.1 and putting $\beta := \max\{0, \theta - s \cdot \omega\}$, we get

$$s \cdot v(f(x)) - v(g(x)) + \beta \geq 0, \quad \text{for all} \ \ x \in A,$$

as desired. □

A direct consequence of Theorem 11.2 is the following result on Hölder continuity of definable functions.

Proposition 11.3. *Let $f : A \to K$ be a continuous $\mathcal{L}$-definable function on a closed bounded subset $A \subset K^n$. Then f is Hölder continuous with a positive integer s and a constant $\beta \in \Gamma$, i.e.*

$$s \cdot v(f(x) - f(z)) + \beta \geq v(x - z)$$

for all $x, z \in A$.

Proof. Apply Theorem 11.2 to the functions

$$f(x) - f(y) \ \text{ and } \ g_i(x, y) = x_i - y_i, \ i = 1, \ldots, n.$$

□

We immediately obtain

Corollary 11.4. *Every continuous $\mathcal{L}$-definable function $f : A \to K$ on a closed bounded subset $A \subset K^n$ is uniformly continuous.*

Now we state a version of the Łojasiewicz inequality for continuous definable functions of a locally closed subset of K^n.

Theorem 11.5. *Let $f, g : A \to K$ be two continuous $\mathcal{L}$-definable functions on a locally closed subset A of K^n. If*

$$\{x \in A : g(x) = 0\} \subset \{x \in A : f(x) = 0\},$$

then there exist a positive integer s and a continuous $\mathcal{L}$-definable function h on A such that $f^s(x) = h(x) \cdot g(x)$ for all $x \in A$.

Proof. It is easy to check that the set A is of the form $A := U \cap F$, where U and F are two $\mathcal{L}$-definable subsets of K^n, U is open and F is closed in the K-topology.

We shall adapt the foregoing arguments. Since the set U is open, its complement $V := K^n \setminus U$ is closed in K^n and A is the following union of open and closed subsets of K^n and of $\mathbb{P}^n(K)$:

$$X_\beta := \{x \in K^n : \ v(x_1), \ldots, v(x_n) \geq -\beta, \ \ v(x - y) \leq \beta \quad \text{for all} \ \ y \in V\},$$

where $\beta \in \Gamma$, $\beta \geq 0$. As before, we see that the sets

$$A_{\beta,\gamma} := \{x \in X_\beta : \ v(f(x)) = \gamma\} \ \text{ with } \beta, \gamma \in \Gamma$$

are closed $\mathcal{L}$-definable subsets of $\mathbb{P}^n(K)$, and next that the sets $g(A_{\beta,\gamma})$ are closed $\mathcal{L}$-definable subsets of $K \setminus \{0\}$ for all $\beta, \gamma \in \Gamma$. Likewise, we get

$$\Lambda := \{(\beta, v(f(x)), v(g(x))) \in \Gamma^3 : \ x \in X_\beta, \ f(x) \neq 0\} \ \subset \ \{(\beta, \gamma, \delta) \in \Gamma^3 : \ \delta < \alpha(\beta, \gamma)\}$$

for some $\alpha(\beta, \gamma) \in \Gamma$.

Λ is a definable subset of Γ^3 in the many-sorted language $\mathcal{L}_{qe}$, and thus is described by a finite number of parametrized linear equalities and inequalities, and of parametrized congruence conditions. Again, the above inclusion reduces to an analysis of those linear equalities and inequalities. Consequently, there exist a positive integer $s \in \mathbb{N}$ and elements $\gamma_0(\beta) \in \Gamma$ such that

$$\Lambda \cap \{(\beta, \gamma, \delta) \in \Gamma^3 : \ \gamma > \gamma_0(\beta)\} \subset \{(\beta, \gamma, \delta) \in \Gamma^3 : \ \delta < s \cdot \gamma\}.$$

Since A is the union of the sets X_β, it is not difficult to check that the quotient f^s/g extends by zero through the zero set of the denominator to a (unique) continuous $\mathcal{L}$-definable function on A, which is the desired result. $\square$

We conclude this section with a theorem which is much stronger than its counterpart, [44, Proposition 12.1], concerning continuous rational functions. The proof we give now resembles the above one, without applying transformation to a normal crossing. Put

$$\mathcal{D}(f) := \{x \in A : f(x) \neq 0\} \ \text{ and } \ \mathcal{Z}(f) := \{x \in A : f(x) = 0\}.$$

Theorem 11.6. *Let $f : A \to K$ be a continuous $\mathcal{L}$-definable function on a locally closed subset A of K^n and $g : \mathcal{D}(f) \to K$ a continuous $\mathcal{L}$-definable function. Then $f^s \cdot g$ extends, for $s \gg 0$, by zero through the set $\mathcal{Z}(f)$ to a (unique) continuous $\mathcal{L}$-definable function on A.*

Proof. As in the proof of Theorem 11.5, let $A = U \cap F$ and consider the same sets $X_\beta \subset K^n$, $\beta \in \Gamma$, and $\Lambda \subset \Gamma^3$. Under the assumptions, we get

$$\Lambda \subset \{(\beta, \gamma, \delta) \in \Gamma^3 : \ \delta > \alpha(\beta, \gamma)\}$$

for some $\alpha(\beta, \gamma) \in \Gamma$. Now, in a similar fashion as before, we can find an integer $r \in \mathbb{Z}$ and elements $\gamma_0(\beta) \in \Gamma$ such that

$$\Lambda \cap \{(\beta, \gamma, \delta) \in \Gamma^3 : \ \gamma > \gamma_0(\beta)\} \subset \{(\beta, \gamma, \delta) \in \Gamma^3 : \ \delta > r \cdot \gamma\}.$$

Take a positive integer $s \in \mathbb{N}$ such that $s + r > 0$. Then, as in the proof of Theorem 11.5, it is not difficult to check that the function $f^s \cdot g$ extends by zero through the zero set of f to a (unique) continuous $\mathcal{L}$-definable function on A, which is the desired result. $\square$

Remark 11.7. Note that Theorem 11.6 is, in fact, a strengthening of Theorem 11.5, and has many important applications. In particular, it plays a crucial role in the proof of the Nullstellensatz for regulous (i.e. continuous and rational) functions on K^n.

12. Continuous hereditarily rational functions and regulous functions and sheaves

Continuous rational functions on singular real algebraic varieties, unlike those on non-singular real algebraic varieties, often behave quite unusually. This is illustrated by many examples from the paper [30, Section 1], and gives rise to the concept of hereditarily rational functions. We shall assume that the ground field K is not algebraically closed. Otherwise, the notion of a continuous rational function on a normal variety coincides with that of a regular function and, in general, the study of continuous rational functions leads to the concept of *seminormality* and *seminormalization*; cf. [1, 2] or [29, Section 10.2] for a recent treatment. Let K be topological field with the density property. For a K-variety Z, let $Z(K)$ denote the set of all K-points on Z. We say that a continuous function $f : Z(K) \longrightarrow K$ is *hereditarily rational* if for every irreducible subvariety $Y \subset Z$ there exists a Zariski dense open subvariety $Y^0 \subset Y$ such that $f|_{Y^0(K)}$ is regular. Below we recall an extension theorem, which plays a crucial role in the theory of continuous rational functions. It says roughly that continuous rational extendability to the non-singular ambient space is ensured by (and in fact equivalent to) the intrinsic property to be continuous hereditarily rational. This theorem was first proven for real and p-adic varieties in [30], and next over Henselian rank one valued fields in [44, Section 10]. The proof of the latter result relied on the closedness theorem (Theorem 1.1), the descent property (Corollary 1.7) and the Łojasiewicz inequality (Theorem 11.5), and can now be repeated verbatim for the case where K is an arbitrary Henselian valued field K of equicharacteristic zero.

Theorem 12.1. *Let X be a non-singular K-variety and $W \subset Z \subset X$ closed subvarieties. Let f be a continuous hereditarily rational function on $Z(K)$ that is regular at all K-points of $Z(K) \setminus W(K)$. Then f extends to a continuous hereditarily rational function F on $X(K)$ that is regular at all K-points of $X(K) \setminus W(K)$.*

The corresponding theorem for hereditarily rational functions of class $\mathcal{C}^k$, $k \in \mathbb{N}$, remains an open problem as yet. This leads to the concept of k-regulous functions, $k \in \mathbb{N}$, on a subvariety $Z(K)$ of a non-singular K-variety $X(K)$, i.e. those functions on $Z(K)$ which are the restrictions to $Z(K)$ of rational functions of class $\mathcal{C}^k$ on $X(K)$.

In real algebraic geometry, the theory of regulous functions, varieties and sheaves was developed by Fichou–Huisman–Mangolte–Monnier [19]. Regulous geometry over Henselian rank one valued fields was studied in our paper [44, Sections 11, 12, 13]. The basic tools we applied are the closedness theorem, descent property, the Łojasiewicz inequalities and transformation to a normal crossing by blowing up. We should emphasize that all those our results, including the Nullstellensatz and Cartan's theorems A and B for regulous quasi-coherent sheaves, remain true over arbitrary Henselian valued fields (of equicharacteristic zero) with almost the same proofs.

We conclude this paper with the following comment.

Remark 12.2. In our recent paper [48], we established a definable, non-Archimedean version of the closedness theorem over Henselian valued fields (of equicharacteristic zero) with analytic structure along with several applications. Let us mention, finally, that the theory of analytic structures goes back to the work of many mathematicians (see e.g. [12, 14, 37, 16, 15, 38, 39, 9, 10, 11]).

References

[1] A. Andreotti, E. Bombieri, *Sugli omeomorfismi delle varietà algebriche*, Ann. Scuola Norm. Sup Pisa **23** (3) (1969), 431–450.

[2] A. Andreotti, F. Norguet, *La convexité holomorphe dans l'espace analytique des cycles d'une variété algébrique*, Ann. Scuola Norm. Sup. Pisa **21** (3) (1967), 31–82.

[3] M. Artin, B. Mazur, *On periodic points*, Ann. Math. **81** (1965), 82–99.

[4] J. Bochnak, M. Coste, M.-F. Roy, *Real Algebraic Geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 36, Springer-Verlag, Berlin, 1998.

[5] N. Bourbaki, *Algèbre Commutative*, Hermann, Paris, 1962.

[6] F. Bruhat, H. Cartan, *Sur la structure des sous-ensembles analytiques réels*, C. R. Acad. Sci. **244** (1957), 988–990.

[7] G. Cherlin, *Model Theoretic Algebra, Selected Topics*, Lect. Notes Math. **521**, Springer-Verlag, Berlin, 1976.

[8] R. Cluckers, E. Halupczok, *Quantifier elimination in ordered abelian groups*, Confluentes Math. **3** (2011), 587–615. DOI: 10.1142/s1793744211000473

[9] R. Cluckers, L. Lipshitz, Z. Robinson, *Analytic cell decomposition and analytic motivic integration*, Ann. Sci. École Norm. Sup. (4) **39** (2006), 535–568. DOI: 10.1016/j.ansens.2006.03.001

[10] R. Cluckers, L. Lipshitz, *Fields with analytic structure*, J. Eur. Math. Soc. **13** (2011), 1147–1223. DOI: 10.4171/jems/278

[11] R. Cluckers, L. Lipshitz, *Strictly convergent analytic structures*, J. Eur. Math. Soc. **19** (2017), 107–149. DOI: 10.4171/jems/662

[12] J. Denef, L. van den Dries, *p-adic and real subanalytic sets*, Ann. Math. **128** (1988), 79–138. DOI: 10.2307/1971463

[13] L. van den Dries, *Dimension of definable sets, algebraic boundedness and Henselian fields*, Ann. Pure Appl. Logic **45** (1989), 189–209. DOI: 10.1016/0168-0072(89)90061-4
[14] L. van den Dries, *Analytic Ax–Kochen–Ershov theorems*, Contemporary Mathematics **131** (1992), 379–392.
[15] L. van den Dries, D. Haskell, D. Macpherson, *One dimensional p-adic subanalytic sets*, J. London Math. Soc. **56** (1999), 1–20. DOI: 10.1112/s0024610798006917
[16] L. van den Dries, A. Macintyre, D. Marker, *The elementary theory of restricted analytic fields with exponentiation*, Ann. Math. **140** (1994), 183–205. DOI: 10.2307/2118545
[17] D. Eisenbud, *Commutative Algebra with a View Towards Algebraic Geometry*, Graduate Texts in Math. **150**, Springer-Verlag, New York, 1994.
[18] A.J. Engler, A. Prestel, *Valued Fields*, Springer-Verlag, Berlin, 2005.
[19] G. Fichou, J. Huisman, F. Mangolte, J.-P. Monnier, *Fonctions régulues*, J. Reine Angew. Math. **718** (2016), 103–151. DOI: 10.1515/crelle-2014-0034
[20] B. Fisher, *A note on Hensel's lemma in several variables*, Proc. Amer. Math. Soc. **125** (11) (1997), 3185–3189.
[21] O. Gabber, P. Gille, L. Moret-Bailly, *Fibrés principaux sur les corps valués henséliens*, Algebraic Geometry **1** (2014), 573–612. DOI: 10.14231/ag-2014-025
[22] B. Green, F. Pop, P. Roquette, *On Rumely's local global principle*, Jber. d. Dt. Math.-Verein. **97** (1995), 43–74.
[23] A. Grothendieck, *Éléments de Géométrie Algébrique. III. Étude cohomologique des faisceaux cohérents*, Publ. Math. IHES **11** (1961) and **17** (1963).
[24] Y. Gurevich, *Elementary properties of ordered abelian groups*, Algebra i Logika Seminar, **3** (1964), 5–39 (in Russian); Amer. Math. Soc. Transl., II Ser. **46** (1965), 165–192 (in English).
[25] Y. Gurevich, P.H. Schmitt, *The theory of ordered abelian groups does not have the independence property*, Trans. Amer. Math. Soc. **284** (1984), 171-Â–182. DOI: 10.1090/s0002-9947-1984-0742419-0
[26] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics 52, Springer-Verlag, 1977.
[27] I. Kaplansky, *Maximal fields with valuations I and II*, Duke Math. J. **9** (1942), 303–321 and **12** (1945), 243–248. DOI: 10.1215/s0012-7094-42-00922-0
[28] J. Kollár, Lectures on resolution of singularities, Ann, Math. Studies, Vol. 166, Princeton Univ. Press, Princeton, New Jersey, 2007.
[29] J. Kollár, *Singularities of the minimal model program*, (With a collaboration of S. Kovács) Cambridge Tracts in Mathematics, vol. 200. Cambridge Univ. Press, Cambridge, 2013.
[30] J. Kollár, K. Nowak, *Continuous rational functions on real and p-adic varieties*, Math. Zeitschrift **279** (2015), 85–97.
[31] W. Kucharz, *Approximation by continuous rational maps into spheres*, J. Eur. Math. Soc. **16** (2014), 1555–1569. DOI: 10.4171/jems/469
[32] W. Kucharz, *Continuous rational maps into spheres*, Math. Zeitschrift **283** (2016), 1201-1215 DOI: 10.1007/s00209-016-1639-4
[33] W. Kucharz, *Piecewise-regular maps*, Math. Ann. **372** (2018), 1545-1574 DOI: 10.1007/s00208-017-1607-2
[34] W. Kucharz, K. Kurdyka, *Stratified-algebraic vector bundles*, J. Reine Angew. Math., published online 2016; DOI: 10.1515/crelle-2015-0105
[35] W. Kucharz, M. Zieliński, *Regulous vector bundles*, arχiv: 1703.05566
[36] F.-V. Kuhlmann, *Maps on ultrametric spaces, Hensel's lemma and differential equations over valued fields*, Comm. Algebra **39** (2011), 1730–1776. DOI: 10.1080/00927871003789157
[37] L. Lipshitz, *Rigid subanalytic sets*, Amer. J. Math. **115** (1993), 77-108.
[38] L. Lipshitz, Z. Robinson, *Rings of Separated Power Series and Quasi-Affinoid Geometry*, Astérisque **264** (2000).
[39] L. Lipshitz, Z. Robinson, *Uniform properties of rigid subanalytic sets*, Trans. Amer. Math. Soc. 357 (11) (2005), 4349–4377.
[40] S. Łojasiewicz, *Ensembles Semi-analytiques*, I.H.E.S., Bures-sur-Yvette, 1965.
[41] J. Milnor, *Singular points of complex hypersurfaces*, Princeton Univ. Press, Princeton, New Jersey, 1968.
[42] K.J. Nowak, *On the Abhyankar–Jung theorem for Henselian $k[x]$-algebras of formal power series*, IMUJ Preprint **2009/02** (2009); available online imuj2009/pr0902.pdf
[43] K.J. Nowak, *Supplement to the paper "Quasianalytic perturbation of multiparameter hyperbolic polynomials and symmetric matrices" (Ann. Polon. Math. 101 (2011), 275–291)*, Ann. Polon. Math. **103** (2012), 101-107. DOI: 10.4064/ap103-1-8
[44] K.J. Nowak, *Some results of algebraic geometry over Henselian rank one valued fields*, Sel. Math. New Ser. **23** (2017), 455–495. DOI: 10.1007/s00029-016-0245-y

[45] K.J. Nowak, *Piecewise continuity of functions definable over Henselian rank one valued fields*, arχiv: 1702.07849
[46] K.J. Nowak, *On functions given by algebraic power series over Henselian valued fields*, arχiv: 1703.08203
[47] K.J. Nowak, *The closedness theorem over Henselian valued fields*, arχiv: 1704.01093
[48] K.J. Nowak, *Some results of geometry over Henselian fields with analytic structure*, (2018), arχiv: 1808.02481
[49] K.J. Nowak, *Definable retractions and a non-Archimedean Tietze–Urysohn theorem over Henselian valued fields*, (2018), arχiv: 1808.09782
[50] K.J. Nowak, *Definable retractions over complete fields with separated power series*, (2019), arχiv: 1901.00162
[51] K.J. Nowak, *Definable transformation to normal crossings over Henselian fields with separated analytic structure*, Symmetry **11** (7) (2019), 934. DOI: 10.3390/sym11070934
[52] A. Parusiński, G. Rond, *The Abhyankar–Jung theorem*, J. Algebra **365** (2012), 29-41.
[53] J. Pas, *Uniform p-adic cell decomposition and local zeta functions*, J. Reine Angew. Math. **399** (1989), 137–172. DOI: 10.1515/crll.1989.399.137
[54] J. Pas, *On the angular component map modulo p*, J. Symbolic Logic **55** (1990), 1125–1129. DOI: 10.2307/2274477
[55] A. Prestel, M. Ziegler, *Model theoretic methods in the theory of topological fields*, J. Reine Angew. Math. **299–300** (1978), 318–341. DOI: 10.1515/crll.1978.299-300.318
[56] P.H. Schmitt, *Model and substructure complete theories of ordered abelian groups*; In: *Models and Sets* (Proceedings of Logic Colloquium '83), Lect. Notes Math. **1103**, Springer-Verlag, Berlin, 1984, 389–418. DOI: 10.1007/bfb0099396
[57] P. Scowcroft, L. van den Dries, *On the structure of semi-algebraic sets over p-adic fields*, J. Symbolic Logic **53** (4) (1988), 1138–1164. DOI: 10.1017/s0022481200027973
[58] A.H. Wallace, *Algebraic approximation of curves*, Canadian J. Math. **10** (1958), 242–278.
[59] O. Zariski, P. Samuel, *Commutative Algebra*, Vol. II, Van Nostrand, Princeton, 1960.

Krzysztof Jan Nowak, Institute of Mathematics, Faculty of Mathematics and Computer Science, Jagiellonian University, ul. Profesora Łojasiewicza 6, 30-348 Kraków, Poland
Email address: nowak@im.uj.edu.pl

Journal *of* Singularities
Volume 21 (2020), 234-248

Proc. of Geometric and Algebraic Singularity Theory, Będlewo, 2017
DOI: 10.5427/jsing.2020.21n

LINKING BETWEEN SINGULAR LOCUS AND REGULAR FIBERS

OSAMU SAEKI

Dedicated to Professor Goo Ishikawa on the occasion of his 60th birthday

Abstract. Given a null-cobordant oriented framed link L in a closed oriented 3–manifold M, we determine those links in $M \setminus L$ which can be realized as the singular point set of a generic map $M \to \mathbf{R}^2$ that has L as an oriented framed regular fiber. Then, we study the linking behavior between the singular point set and regular fibers for generic maps of M into $\mathbf{R}^2$.

1. Introduction

Topology of generic C^∞ maps of manifolds of dimension ≥ 2 into the plane $\mathbf{R}^2$ has been extensively studied as a natural generalization of Morse theory, which studies generic maps into the real line $\mathbf{R}$. For a Morse function, singular points, or critical points, are isolated and their positions in the source manifold are not interesting except for their cardinalities or indices. On the other hand, for a generic map into the plane, the singular point set is a smooth submanifold of dimension one in the source manifold and its position may be non-trivial. In [14], the author studied the position of the singular point set and characterized those smooth 1–dimensional submanifolds which arise as the singular point set of a generic map.

On the other hand, each regular fiber of such a generic map into $\mathbf{R}^2$ is of codimension two and is disjoint from the singular point set. Therefore, the singular point set and regular fibers may be non-trivially linked.

In September 2018[1] Professor David Chillingworth asked the author the following question: *for a generic map $f : \mathbf{R}^3 \to \mathbf{R}^2$, must every component of a regular fiber be linked by at least one component of the singular point set ?*[2]

In this paper, we concentrate on generic maps of *closed* (i.e. compact and boundaryless) 3–dimensional manifolds, instead of $\mathbf{R}^3$, and study the linking behavior between the singular point set and regular fibers in the source 3–manifold. More precisely, let M be a closed oriented 3–manifold and $f : M \to \mathbf{R}^2$ a generic C^∞ map. Generic maps that we consider in this paper are called *excellent maps*, as defined in §2, and have fold and cusp singularities. In our 3–dimensional case, both the singular point set and regular fibers have dimension one, and they constitute disjoint links in M. We study their relative positions in the 3–manifold M.

For example, let us consider the unit sphere $S^3 \subset \mathbf{R}^4$ and let $\pi : \mathbf{R}^4 \to \mathbf{R}^2$ be the standard projection defined by $\pi(x_1, x_2, x_3, x_4) = (x_1, x_2)$ for $(x_1, x_2, x_3, x_4) \in \mathbf{R}^4$. Then,

$$f_0 = \pi|_{S^3} : S^3 \to \mathbf{R}^2$$

2000 *Mathematics Subject Classification.* Primary 57R45; Secondary 58K30, 57M25, 57R20, 57R70.
Key words and phrases. Excellent map, 3–manifold, singular point set, regular fiber, relative Stiefel–Whitney class, framing.

[1] In the conference "Geometric and Algebraic Singularity Theory" held in honor of the 60th birthday of Goo Ishikawa, in Będlewo, Poland.

[2] This question originates from a physical study of phase singularities, nodal lines, or optical polarization knots. For details, the reader is referred to [1, 2, 4, 8].

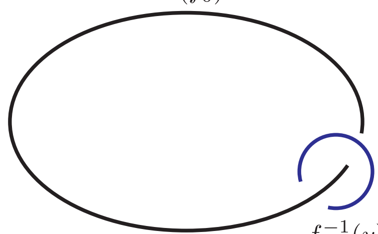


FIGURE 1. Singular point set and a regular fiber for a specific map $f_0 : S^3 \to \mathbf{R}^2$

is an excellent map whose singular point set $S(f_0) = \{(x_1, x_2, 0, 0) \in S^3\}$ consists only of definite fold singularities and is a trivial knot in S^3. Furthermore, for $y = (y_1, y_2)$ with $y_1^2 + y_2^2 < 1$, the regular fiber $f_0^{-1}(y) = \{(y_1, y_2, x_3, x_4) \in S^3\}$ is an unknotted circle linked with $S(f_0)$ (see Fig. 1). So, in this example, the answer to the above question is positive.

The present paper is organized as follows. In §2, we will first see that regular fibers are naturally oriented and framed; i.e. they have natural normal framings induced by the generic map $f : M \to \mathbf{R}^2$. Furthermore, they bound compact oriented normally framed surfaces embedded in M. Conversely, in [13], it has been shown that if an oriented normally framed link in M bounds a compact oriented normally framed surface, then it is realized as a regular fiber of a generic map of M into $\mathbf{R}^2$. Then, in Theorem 2.3, given such a framed link L in M, we characterize those unoriented links in $M \smallsetminus L$ that arise as the singular point set of a generic map $f : M \to \mathbf{R}^2$ such that L coincides with a framed regular fiber of f. The characterization is given in terms of a relative characteristic class (see [7]) which is the obstruction to extending a certain trivialization of the tangent bundle of M on a neighborhood of L to the whole M.

In §3, we will study the relative characteristic class which arises as the obstruction as above. As a consequence, we will show that if a regular fiber has an odd number of components, then it necessarily links with the singular point set (see Remark 3.5). We will also give a result which enables us to identify the obstruction for local links that are embedded inside an open 3–disk.

In §4, by utilizing the results obtained in §3, we show that there exist generic maps $S^3 \to \mathbf{R}^2$ such that a regular fiber, which is a 2–component link, and the singular point set are split; i.e. they lie inside disjoint 3–disks. We also see that there exists such an example for every closed oriented 3–manifold M. We also give two explicit examples of generic maps $S^3 \to \mathbf{R}^2$ which exhibit non-linking phenomena between regular fibers and the singular point set.

Finally in §5, we address the original question concerning generic maps of $\mathbf{R}^3$ into the plane. By utilizing results obtained in [6] on regular fibers of submersions $\mathbf{R}^3 \to \mathbf{R}^2$, we answer to the question negatively, by constructing counter examples.

Throughout the paper, manifolds and maps are differentiable of class C^∞ unless otherwise indicated. All (co)homology groups are with $\mathbf{Z}_2$–coefficients unless otherwise indicated. The symbol "$\cong$" means an appropriate isomorphism between algebraic objects.

2. MAIN THEOREM

Let M be a closed oriented 3–dimensional manifold. We say that a map $f : M \to \mathbf{R}^2$ is *excellent* if its singularities consist only of fold and cusp singularities, where a *fold singularity* (or a *cusp singularity*) is modeled on the map germ

$$(x, y, z) \mapsto (x, y^2 \pm z^2) \quad (\text{resp. } (x, y, z) \mapsto (x, y^3 + xy - z^2))$$

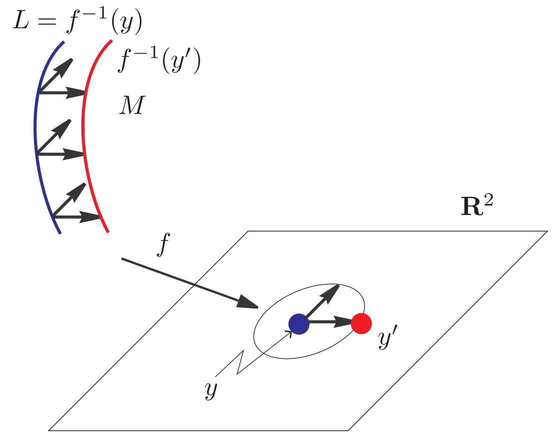


FIGURE 2. Framing for a regular fiber

at the origin. We say that a fold singularity is *definite* (resp. *indefinite*) if it is modeled on the map germ $(x, y, z) \mapsto (x, y^2 + z^2)$ (resp. $(x, y, z) \mapsto (x, y^2 - z^2)$).

It is known that the set of excellent maps is always open and dense in the mapping space $C^\infty(M, \mathbf{R}^2)$ endowed with the Whitney C^∞ topology (for example, see [5, 18]).

In the following, for a map $f : M \to \mathbf{R}^2$, we denote by $S(f)$ the set of singular points of f. If f is an excellent map, then we see easily that $S(f)$ is a link in M, i.e. a disjoint union of finitely many smoothly embedded circles. For a regular value $y \in \mathbf{R}^2$, if $L = f^{-1}(y)$ is non-empty, then we call it a *regular fiber*, which is also a link in M and is disjoint from $S(f)$. We fix an orientation of $\mathbf{R}^2$ once and for all, and then a regular fiber is naturally oriented, since M is oriented. Furthermore, L is naturally *framed*: its framing is given as the pull-back of the trivial normal framing of the point y in $\mathbf{R}^2$ (see Fig. 2). In other words, taking a small disk neighborhood of y in $\mathbf{R}^2$ consisting entirely of regular values, let y' be a point in its boundary, then $f^{-1}(y')$ represents the framed longitude of the framed link L.

LEMMA 2.1. *A framed regular fiber L of an excellent map $f : M \to \mathbf{R}^2$ over a regular point $y \in \mathbf{R}^2$ is always framed null-cobordant. In other words, there exists a compact oriented surface V embedded in M whose boundary coincides with L and which is consistent with the framed longitude.*

Proof. Let ℓ be a half line in $\mathbf{R}^2$ emanating from y. We may assume that it is transverse to the map f. Then, $V = f^{-1}(\ell)$ gives the desired surface (see Fig. 3). □

In [13, Proposition 5.1], it has been shown that every null-cobordant oriented framed link L in M can be realized as an oriented framed regular fiber of an excellent map $f : M \to \mathbf{R}^2$. In this case, the singular point set $S(f)$ is a link disjoint from L. Then, it is natural to ask which links in $M \smallsetminus L$ appear as the singular point set of such an excellent map.

In order to state our first theorem, let us prepare some notations and terminologies. For a (unoriented) link J in $M \smallsetminus L$, we denote by $[J]_2 \in H_1(M \smallsetminus L)$ the $\mathbf{Z}_2$–homology class represented by J. Let $N(L)$ be a small tubular neighborhood of L in M disjoint from J. Since L is a framed link, we have a natural 3–framing of M over $\partial N(L)$, i.e. a trivialization of $TM|_{\partial N(L)}$. The obstruction to extending this framing over $M \smallsetminus \operatorname{Int} N(L)$ is the relative Stiefel–Whitney class (see [7]), denoted by $w_2(M, L)$, which is an element of the $\mathbf{Z}_2$–cohomology group

$$H^2(M \smallsetminus \operatorname{Int} N(L), \partial N(L)) \cong H^2(M, N(L)) \cong H^2(M, L),$$

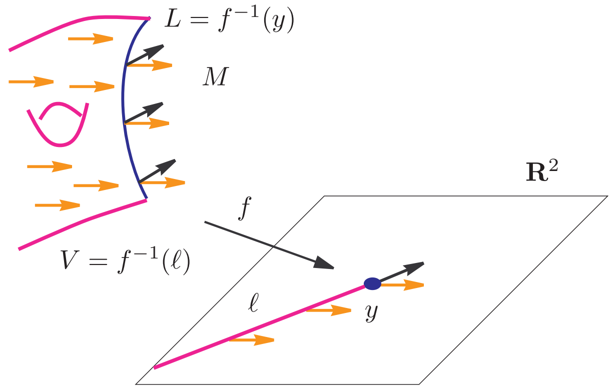


FIGURE 3. Constructing a framed null-cobordism

where the first isomorphism is given by excision and the second one is given by the natural homotopy equivalence $(M, L) \to (M, N(L))$. Note that by Poincaré–Lefschetz duality, we have

$$H^2(M \setminus \operatorname{Int} N(L), \partial N(L)) \cong H_1(M \setminus \operatorname{Int} N(L)) \cong H_1(M \setminus L).$$

REMARK 2.2. Let $j : (M, \emptyset) \to (M, L)$ be the inclusion. Then the induced homomorphism $j^* : H^2(M, L) \to H^2(M)$ maps $w_2(M, L)$ to the second Stiefel–Whitney class $w_2(M)$ of M, which vanishes. By the cohomology exact sequence

$$H^1(L) \xrightarrow{\delta} H^2(M, L) \xrightarrow{j^*} H^2(M),$$

we see that $w_2(M, L) = \delta(\alpha)$ for some $\alpha \in H^1(L)$.

Now, one of the main theorems of this paper is the following.

THEOREM 2.3. *Let L be an oriented null-cobordant framed link in a closed oriented 3–manifold M, and J be an unoriented link in M disjoint from L. Then, there exist an excellent map $f : M \to \mathbf{R}^2$ and a regular value $y \in \mathbf{R}^2$ such that $f^{-1}(y)$ coincides with L as an oriented framed link and that $S(f) = J$ if and only if $[J]_2 \in H_1(M \setminus L)$ is Poincaré dual to $w_2(M, L) \in H^2(M, L)$.*

Proof. Suppose that $f : M \to \mathbf{R}^2$ is an excellent map such that L coincides with $f^{-1}(y)$ as a framed link for a regular value $y \in \mathbf{R}^2$ and that $J = S(f)$. Then, we have the following, which is originally due to Thom [16].

LEMMA 2.4. *If $f : M \to \mathbf{R}^2$ is an excellent map and $y \in \mathbf{R}^2$ is a regular value, then for $L = f^{-1}(y)$, $[S(f)]_2 \in H_1(M \setminus L)$ is Poincaré dual to $w_2(M, L) \in H^2(M, L)$.*

For the sake of completeness, we include a short proof here.

Proof of Lemma 2.4. Since f is a submersion outside of $S(f)$, we can extend the framing on $N(L)$ to $M \setminus S(f)$. Then, we see easily that $S(f)$ is exactly the obstruction locus and by definition of the relative Stiefel–Whitney class, we have the desired conclusion. $\square$

Conversely, suppose that $[J]_2 \in H_1(M \setminus L)$ is Poincaré dual to $w_2(M, L)$. Let $g : M \to \mathbf{R}^2$ be an arbitrary excellent map for which there exists a regular value $y \in \mathbf{R}^2$ such that $g^{-1}(y)$ coincides with L as a framed link. Such an excellent map always exists by [13]. Then, we see that $[S(g)]_2 \in H_1(M \setminus L)$ is Poincaré dual to $w_2(M, L)$ by Lemma 2.4. By our assumption, this implies that J and $S(g)$ are $\mathbf{Z}_2$–homologous in $M \setminus L$. Set $S(g) = J_0$.

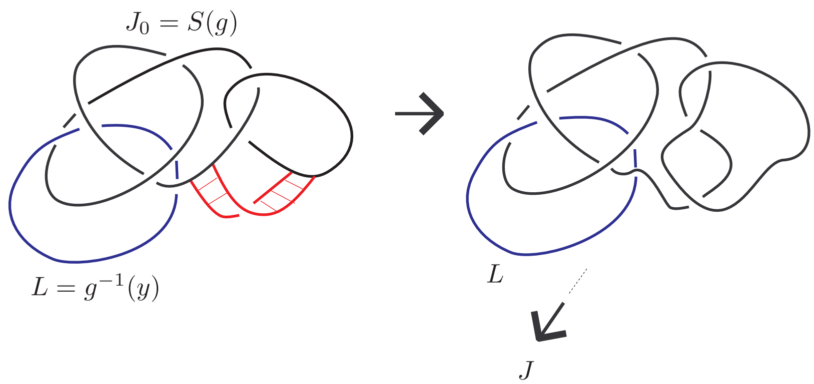


FIGURE 4. Starting from J_0, we get J up to isotopy by a finite iteration of band operations inside $M \smallsetminus L$.

LEMMA 2.5 ([14]). *If $[J_0]_2 = [J]_2 \in H_1(M \smallsetminus L)$, then by modifying J_0 by a finite iteration of band operations inside $M \smallsetminus L$, we can get J, up to isotopy.*

Here, a band operation on J_0 is defined as follows. Set $I_1 = I_2 = [-1, 1]$, and let

$$\varphi : I_1 \times I_2 \to M \smallsetminus L$$

be an embedding of a band such that $\varphi(I_1 \times I_2) \cap J_0 = \varphi(\{-1, 1\} \times I_2)$. Then a *band operation* applied to J_0 transforms it to $(J_0 \smallsetminus \varphi(\{-1,1\} \times I_2)) \cup \varphi(I_1 \times \{-1,1\})$ with the corners smoothed. Lemma 2.5 states that repeating this procedure finitely many times, we get a link isotopic to J in $M \smallsetminus L$, starting from J_0 (see Fig. 4).

Proof of Lemma 2.5. First, we may assume that both J_0 and J are connected, by using band operations. Here, note that the reverse of a band operation is again a band operation.

Now, we orient J_0 and J arbitrarily. Since $[J]_2 = [J_0]_2$ in $H_1(M \smallsetminus L)$, we have $[J] = [J_0] + 2\gamma$ for some $\gamma \in H_1(M \smallsetminus L; \mathbf{Z})$, where $[J]$ and $[J_0] \in H_1(M \smallsetminus L; \mathbf{Z})$ are the $\mathbf{Z}$–homology classes represented by J and J_0, respectively. Using a band whose center curve corresponds to γ, we may assume $[J_0] = [J]$ in $H_1(M \smallsetminus L; \mathbf{Z})$ (see the left hand side picture of Fig. 5).

Recall that $H_1(M \smallsetminus L; \mathbf{Z})$ is the abelianization of $\pi_1(M \smallsetminus L)$. By realizing commutators in $\pi_1(M \smallsetminus L)$ by band operations, we may assume J_0 and J are freely homotopic (see the right hand side picture of Fig. 5).

Then, for dimensional reasons, J_0 is regularly homotopic to J. This implies that J_0 is transformed to J by a finite iteration of "crossing changes" in knot theory, up to isotopy.

Finally, we can realize each "crossing change" by two band operations as depicted in Fig. 6. This completes the proof of Lemma 2.5. (For more details, the reader is referred to [14].) □

LEMMA 2.6 ([14]). *Each band operation applied to $S(g)$ can be realized by a generic deformation of $g : M \to \mathbf{R}^2$ which does not modify $g^{-1}(N(y))$ for a small disk neighborhood $N(y)$ of y in $\mathbf{R}^2$. In other words, for a link J_1 obtained by a band operation to $S(g)$ in $M \smallsetminus g^{-1}(y)$, there exists a generic 1–parameter deformation from g to g_1 in such a way that $g_1 : M \to \mathbf{R}^2$ is an excellent map with $S(g_1) = J_1$, $g_1^{-1}(N(y)) = g^{-1}(N(y))$ and $g_1|_{g_1^{-1}(N(y))} = g|_{g^{-1}(N(y))}$.*

The above lemma can be proved by using Levine's cusp elimination techniques [9] (see Fig. 7). For details, the reader is referred to [14].

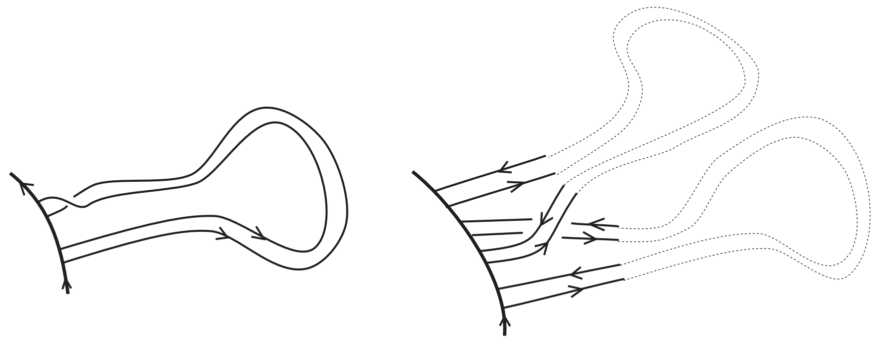


FIGURE 5. Modifying J_0 appropriately

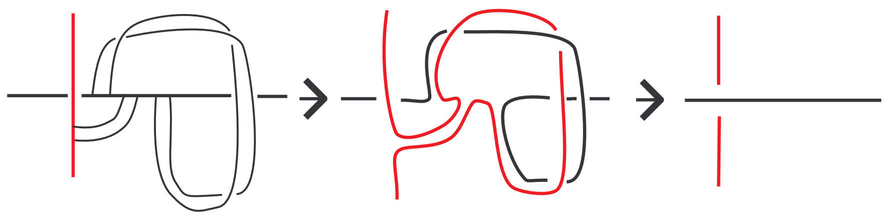


FIGURE 6. Realizing a crossing change by two band operations

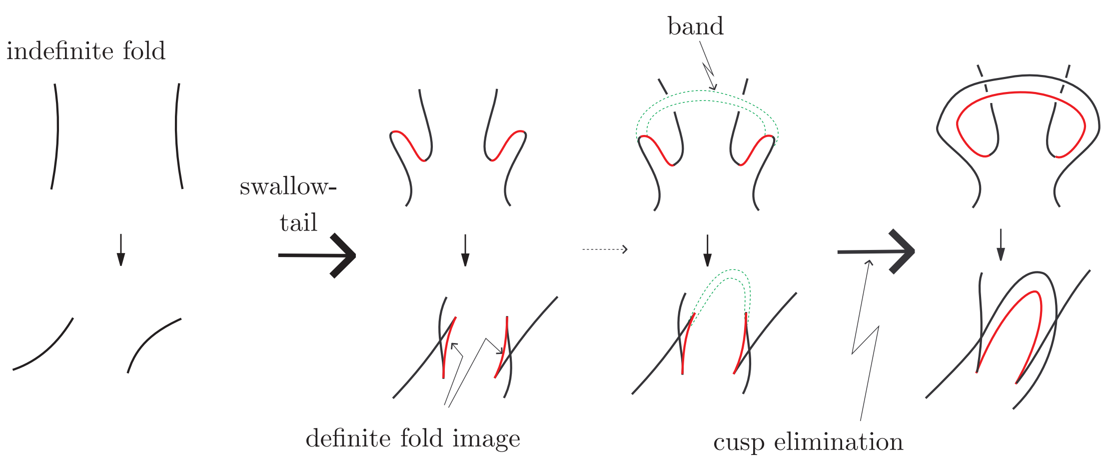


FIGURE 7. An example of a cusp elimination along a curve corresponding to a band operation. The upper row depicts a change of the singular point set in the source 3–manifold M, while the lower row depicts the corresponding change of the singular point set image in $\mathbf{R}^2$.

Now let us go back to the proof of Theorem 2.3. Combining Lemmas 2.5 and 2.6, we can deform g with $S(g) = J_0$ to an excellent map $f : M \to \mathbf{R}^2$ with $S(f) = J$, keeping the condition $g^{-1}(y) = f^{-1}(y) = L$. This completes the proof. $\square$

Remark 2.7. As in [14], suppose J is decomposed as a disjoint union

$$J = F_0 \cup F_1 \cup C,$$

where F_0 and F_1 are finite disjoint unions of open arcs and circles, C is a finite set of points, and each point of C is adjacent to both F_0 and F_1. If both F_0 and F_1 are non-empty, then in Theorem 2.3, we can find an excellent map f such that $S(f) = J$, F_0 is the set of definite fold singularities, F_1 is the set of indefinite fold singularities, and C is the set of cusp singularities.

Remark 2.8. Let $g : M \to \mathbf{R}^2$ be an excellent map for which there exists a regular value y such that $g^{-1}(y)$ coincides with L as a framed link. In the situation of Theorem 2.3, we see that $[J]_2 \in H_1(M)$ is Poincaré dual to $w_2(M)$, which vanishes, by Remark 2.2. Then, we can apply the modification techniques developed in [14] without touching L to obtain an excellent map $h : M \to \mathbf{R}^2$ homotopic to g such that $S(h)$ is isotopic to J in M. However, in order to obtain an excellent map h' such that $S(h')$ coincides with J, we need to further modify h. In such a modification process, the regular fiber over y may change, since in the course of the isotopy, the link may cross L. In §3, we will see that not every $\mathbf{Z}_2$ null-homologous link J in M can be realized as above, depending on its position relative to L.

Generalizing our Theorem 2.3, we can also obtain the following, which can be proved by the same argument. Details are left to the reader.

Theorem 2.9. *Let M be a closed oriented 3–manifold and $L_1, L_2, \ldots, L_\ell$, and J be disjoint links in M. Suppose that $L_1, L_2, \ldots, L_\ell$ are oriented and null-cobordant framed links, and that they bound disjoint compact oriented framed surfaces. Furthermore, J is an unoriented link. Then, there exist an excellent map $f : M \to \mathbf{R}^2$ and distinct regular values $y_1, y_2, \ldots, y_\ell \in \mathbf{R}^2$ of f such that $f^{-1}(y_i) = L_i$ as framed links for $i = 1, 2, \ldots, \ell$, and $J = S(f)$ if and only if $[J]_2 \in H_1(M \smallsetminus L)$ is Poincaré dual to $w_2(M, L)$, where $L = L_1 \cup L_2 \cup \cdots \cup L_\ell$.*

For maps into S^2, we have a similar result as follows. Recall that, for a closed oriented 3–dimensional manifold M, the homotopy classes of M into S^2 are in one-to-one correspondence with the framed cobordism classes of closed oriented framed 1–dimensional submanifolds in M by the Pontrjagin–Thom construction. For the classification of the homotopy set $[M, S^2]$ for a closed oriented 3–manifold M, the reader is referred to [3].

Theorem 2.10. *Let M be a closed oriented 3–manifold and fix a homotopy class of a map $g : M \to S^2$. Let L be an oriented framed link in M which corresponds to the homotopy class of g. Then, for an unoriented link J in $M \smallsetminus L$, there exist an excellent map $f : M \to S^2$ homotopic to g and a regular value $y \in S^2$ of f such that $f^{-1}(y)$ coincides with L as a framed link and $J = S(f)$ if and only if $[J]_2 \in H_1(M \smallsetminus L)$ is Poincaré dual to $w_2(M, L)$.*

The proof of Theorem 2.10 is similar to that of Theorem 2.3 and is left to the reader. Note that Theorem 2.3 corresponds to the case of a null-homotopic map g in Theorem 2.10 in a certain sense.

3. Obstruction

In order to apply Theorem 2.3 in practical situations, let us study the obstruction class $w_2(M, L)$ more in detail, where M is a closed oriented 3–manifold and L is a framed link in M.

As we saw in Remark 2.2, there exists an $\alpha \in H^1(L)$ such that $\delta(\alpha) = w_2(M, L)$, although such a cohomology class may not be unique. In fact, such an α can be explicitly given as follows. Set $L = L_1 \cup L_2 \cup \cdots \cup L_t$, where L_s are the components of L, $s = 1, 2, \ldots, t$. It is known that a closed oriented 3–manifold M is always parallelizable, i.e. its tangent bundle is trivial. Let us fix a framing τ of M, where τ can be identified with a trivialization of the tangent bundle

TM. Once such a framing τ is fixed, we can compare it with the specific framing given on each component L_s of the framed link L. This defines a well-defined element a_s in $\pi_1(SO(3)) \cong \mathbf{Z}_2$. Then, we have the following.

LEMMA 3.1. *Let $\alpha \in H^1(L)$ be the unique cohomology class such that the Kronecker product $\langle \alpha, [L_s]_2 \rangle \in \mathbf{Z}_2$ coincides with a_s for each component L_s of L. Then, we have $\delta(\alpha) = w_2(M, L)$.*

Proof. For each component L_s, let K_s be the boundary of a small meridian disk D_s^2 of L_s. We may assume that K_s is contained in $M \smallsetminus N(L)$. Then, by using τ, we can extend the framing over $\partial N(L)$ given by the framed link L to

$$(M \smallsetminus \operatorname{Int} N(L)) \smallsetminus \left(\cup_{s=1}^{t} K_s\right).$$

If $a_s = 0$, then this framing further extends across K_s: otherwise, it does not. Therefore, $w_2(M, L)$ is Poincaré dual to the sum of those $[K_s]_2$ such that $a_s \neq 0$.

Let us consider the commutative diagram

$$\begin{array}{ccc} H^1(L) & \xrightarrow{\delta} & H^2(M, L) \\ {\scriptstyle p}\downarrow & & {\scriptstyle q}\downarrow \\ H_2(M, M \smallsetminus L) & \xrightarrow{\partial} & H_1(M \smallsetminus L), \end{array}$$

where the first (or the second) row is a part of the cohomology (resp. homology) exact sequence for the pair (M, L) (resp. $(M, M \smallsetminus L)$), and the vertical maps are the duality isomorphisms. By the construction of α, we see that $p(\alpha)$ is represented by the sum of those $[D_s^2, \partial D_s^2]_2$ such that $a_s \neq 0$, where $[D_s^2, \partial D_s^2]_2 \in H_2(M, M \smallsetminus L)$ is the $\mathbf{Z}_2$–homology class represented by the pair $(D_s^2, \partial D_s^2)$. Since $\partial [D_s^2, \partial D_s^2]_2 = [K_s]_2 \in H_1(M \smallsetminus L)$, we have the desired conclusion by the commutativity of the diagram. $\square$

For example, if the framing on L coincides with τ up to homotopy, then $\alpha = 0$ and consequently we have $w_2(M, L) = 0$.

Note that the framing τ may not be unique. The set of homotopy classes of such framings is in one-to-one correspondence with the homotopy set $[M, SO(3)]$. If we consider the set of homotopy classes of framings on the 2–skeleton of M, then each such framing up to homotopy defines a *spin structure* on M, and the set of spin structures is in one-to-one correspondence with $H^1(M)$ (see [11]).

By the cohomology exact sequence,

$$H^1(M) \xrightarrow{i^*} H^1(L) \xrightarrow{\delta} H^2(M, L) \xrightarrow{j^*} H^2(M),$$

we see that for every element $\beta \in \operatorname{Im} i^*$, we could choose $\alpha + \beta$ instead of α, where $i : L \to M$ is the inclusion map. The observation in the previous paragraph shows that this corresponds to choosing another framing, say τ', which is "twisted along β".

REMARK 3.2. As we saw in Remark 2.2, $w_2(M, L)$ is in the kernel of

$$j^* : H^2(M, L) \longrightarrow H^2(M),$$

which coincides with $\operatorname{Im} \delta \cong H^1(L) / \operatorname{Im} i^*$. Note that if L is framed null-cobordant, then this latter group is non-trivial, since L bounds a compact surface in M and hence $[L]_2 = 0$ in $H_1(M)$.

If we change the framing of a component L_s of L, then $w_2(M, L)$ changes in general. The difference is described by $\delta[L_s]_2^*$, where $[L_s]_2^*$ is the dual to the homology class $[L_s]_2 \in H_1(L)$ represented by L_s with respect to the basis of $H_1(L)$ consisting of the homology classes represented by the components of L. This follows from the observation described in [7, pp. 520–521].

(However, we need to be careful, since if we change the framing of L_s, then the resulting framed link may not be framed null-cobordant any more.)

REMARK 3.3. Let L be an oriented link in a closed oriented 3–manifold M. Then, we can easily show that it bounds a compact oriented surface in M if and only if L represents zero in $H_1(M;\mathbf{Z})$.

In order to apply Theorem 2.3 in practical situations, we have the following proposition which helps to identify the obstruction $w_2(M,L)$.

PROPOSITION 3.4. *Let L be an oriented framed link which bounds a compact oriented surface V consistent with the framing. Let $\alpha \in H^1(L)$ be an element such that $\delta(\alpha) = w_2(M,L)$. Then, we have*

$$\begin{aligned}\langle w_2(M,L), [V,\partial V]_2\rangle &= \langle \delta(\alpha), [V,\partial V]_2\rangle \\ &= \langle \alpha, [L]_2\rangle \\ &\equiv \chi(V) \pmod 2 \\ &\equiv \sharp L \pmod 2,\end{aligned}$$

where $\langle\cdot\,,\,\cdot\rangle$ is the Kronecker product, $[V,\partial V]_2 \in H_2(M,L)$ is the fundamental class of V in $\mathbf{Z}_2$–coefficients, $\chi(V)$ denotes the Euler characteristic of V, and $\sharp L$ denotes the number of components of L.

The above proposition is similar to the Poincaré–Hopf theorem for vector fields. It can be proved by decomposing V into simplices, and by computing the contribution of each simplex. We omit the details.

The above proposition can also be proved as follows. First, we construct an excellent map $f : M \to \mathbf{R}^2$ such that for a regular value y, $f^{-1}(y)$ coincides with L as a framed link and that for a half line ℓ emanating from y in $\mathbf{R}^2$ transverse to f, we have $f^{-1}(\ell) = V$. Such an excellent map is constructed in [13]. Then, the map $f|_V : V \to \ell$ is a Morse function and its number of critical points coincides with the number of intersection points of V and $S(f)$. As $[S(f)]_2$ is Poincaré dual to $w_2(M,L)$, we see that this number modulo 2 coincides with $\langle w_2(M,L), [V,\partial V]_2\rangle$. Since the number of critical points of the Morse function is congruent modulo 2 to $\chi(V)$, we get the result. The congruence $\chi(V) \equiv \sharp L \pmod 2$ is obvious, since V is a compact orientable surface and $\partial V = L$.

REMARK 3.5. The above proposition shows the following. If $f : M \to \mathbf{R}^2$ is an excellent map and $y \in \mathbf{R}^2$ is a regular value such that $L = f^{-1}(y)$ has an odd number of components, then every compact oriented surface V in M bounded by L compatible with the framing of L intersects with $S(f)$. If $H_1(M) = 0$, then this implies that the $\mathbf{Z}_2$ linking number of L and $S(f)$ in M does not vanish. Thus, in this case, the regular fiber L necessarily links with $S(f)$ (see Fig. 8). In particular, if a regular fiber is connected, then it is necessarily linked with at least one component of $S(f)$.

Let us now consider the case of a local knot component. Suppose that the oriented framed link L contains a component K that lies in the interior of a closed 3–disk D embedded in M. Set $U = \operatorname{Int} D$, which is an open set of M diffeomorphic to $\mathbf{R}^3$. In the following, let us identify U with $\mathbf{R}^3$. In this case, up to homotopy, we may assume that the framing τ for M over U is given by the standard framing of $\mathbf{R}^3$.

Let $\pi : \mathbf{R}^3 \to H$ be the orthogonal projection onto a generic hyperplane $H \cong \mathbf{R}^2$ in the sense that $\pi|_K$ is an immersion with normal crossings. On the other hand, we may assume that the first vector field defining the framing τ over K is tangent to K consistent with the orientation.

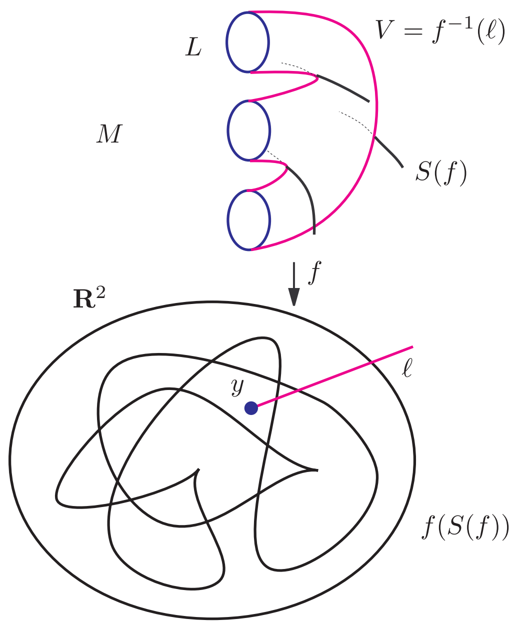


FIGURE 8. Regular fiber with an odd number of components links with the singular point set.

Since $\pi|_K$ is an immersion, we may assume that at each point x of K the remaining two vector fields give a basis for a 2–plane $N_x \subset T_x\mathbf{R}^3$ transverse to T_xK containing the direction $H^\perp$ perpendicular to H. Then, we count the number of times modulo 2 the 2–framing rotates in N_x with respect to a fixed positive direction of $H^\perp$ while $x \in K$ goes around K once. This number is denoted by $t_v(K)$, which is an element in $\mathbf{Z}_2$. Then, we have the following.

LEMMA 3.6. *Let $\alpha \in H^1(L)$ be an arbitrary element such that $\delta(\alpha) = w_2(M, L)$. Then, we have*

$$\langle \alpha, [K]_2 \rangle \equiv t_v(K) + c(K) + 1 \pmod 2,$$

where $c(K)$ denotes the number of crossings of the immersion $\pi|_K : K \to H$ with normal crossings.

Proof. Since the framing τ is standard on $U = \mathbf{R}^3$, in order for the obstruction to vanish on K, we need to have that the winding number of $\pi(K)$ on H is even as long as $t_v(K) = 0$. On the other hand, by [17], we have that the winding number has the same parity as $c(K)+1$. Thus, by the observation in [7, pp. 520–521], we have the conclusion. □

4. EXAMPLES

In this section, we give some explicit examples which imply that the answer to the problem posed in §1 for closed oriented 3–manifolds is negative in general.

EXAMPLE 4.1. Let L be a 2–component framed link $h^{-1}(\{y_1, y_2\})$ in S^3 that consists of two framed fibers of the positive Hopf fibration $h : S^3 \to S^2$, for $y_1 \neq y_2$ in S^2, where we reverse the orientation of one of the components and the framings are induced by h. By taking the inverse image $h^{-1}(a)$ of an embedded arc a in S^2 connecting y_1 and y_2, we see that L is framed null-cobordant (see Fig. 9). By Lemma 3.6, we have that $w_2(S^3, L)$ vanishes. This can also be proved as follows. Let us take two distinct points $p_1, p_2 \in S^2 \setminus \{y_1, y_2\}$. Since $S^2 \setminus \{p_1, p_2\}$ is

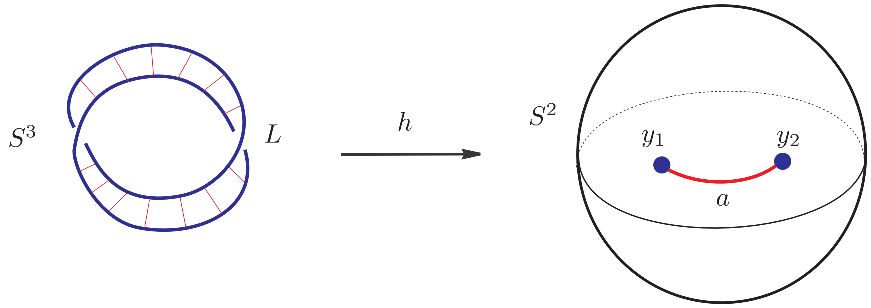


FIGURE 9. Framed Hopf link which is null-cobordant

diffeomorphic to an open annulus $S^1 \times (-1, 1)$, it has a 2–framing. By pulling back this 2–framing by the Hopf fibration h, we see that the framing of $TS^3|_L$ naturally extends to $S^3 \smallsetminus h^{-1}(\{p_1, p_2\})$. This means that $w_2(S^3, L)$ is Poincaré dual to $h^{-1}(\{p_1, p_2\})$. Since

$$[h^{-1}(p_1)]_2 = [h^{-1}(p_2)]_2 \in H_1(S^3 \smallsetminus L),$$

we see that $w_2(S^3, L)$ vanishes.

Therefore, by Theorem 2.3, an arbitrary link J split from L can be realized as the singular point set of an excellent map $S^3 \to \mathbf{R}^2$ with L a framed regular fiber, since $[J]_2 = 0$ is Poincaré dual to $w_2(S^3, L) = 0$. In this example, the components of the regular fiber L do not link with the singular point set!

Note that L has an even number of components. This is consistent with the observation given in Remark 3.5.

Let M be an arbitrary closed oriented 3–manifold. By considering the above 2–component link L as embedded in $\mathbf{R}^3 \subset S^3$ and by embedding it to M, we get the same result for M as well. This gives counter examples to the question presented in §1 for closed oriented 3–manifolds.

We will give two explicit examples of excellent maps on S^3 which give counter examples.

EXAMPLE 4.2. Let $h : S^3 \to S^2$ be the (positive) Hopf fibration. Let $p_N = (0, 0, 1)$ and $p_S = (0, 0, -1)$ be the north and the south poles of S^2, respectively, where we identify S^2 with the unit sphere in $\mathbf{R}^3$. We decompose S^2 as $S^2 = D_N \cup D_S \cup A$, where D_N (or D_S) is a small 2–disk neighborhood of p_N (resp. p_S) in S^2 with $D_N \cap D_S = \emptyset$, and A is the annulus obtained as the closure of $S^2 \smallsetminus (D_N \cup D_S)$.

Note that the fibration h is trivial on each of D_N, D_S and A. Let us fix a trivialization

$$h^{-1}(A) = S^1 \times A = S^1 \times ([-1, 1] \times S^1) = (S^1 \times [-1, 1]) \times S^1, \tag{4.1}$$

where we identify A with $[-1, 1] \times S^1$ so that $\{1\} \times S^1$ (or $\{-1\} \times S^1$) coincides with ∂D_N (resp. ∂D_S). We take the trivialization of $h^{-1}(A)$ in such a way that it extends to a trivialization of h over $D_N \cup A$. Note that in (4.1), the first S^1–factor corresponds to the fibers of h and the last S^1–factor corresponds to the equatorial direction of S^2 in the target.

Let $k : S^1 \times [-1, 1] \to [1, \infty)$ be a Morse function such that

(1) $k^{-1}(1) = S^1 \times \{-1, 1\}$,
(2) k has no critical point in a small neighborhood of $S^1 \times \{-1, 1\}$,
(3) k has exactly two critical points in such a way that one of them has index 1 and the other has index 2.

Using the above ingredients, let us now construct an excellent map $f : S^3 \to \mathbf{R}^2$ as follows. On $h^{-1}(D_N)$ (or on $h^{-1}(D_S)$), we define $f = i_N \circ h$ (resp. $f = i_S \circ h$), where $i_N : D_N \to \mathbf{R}^2$

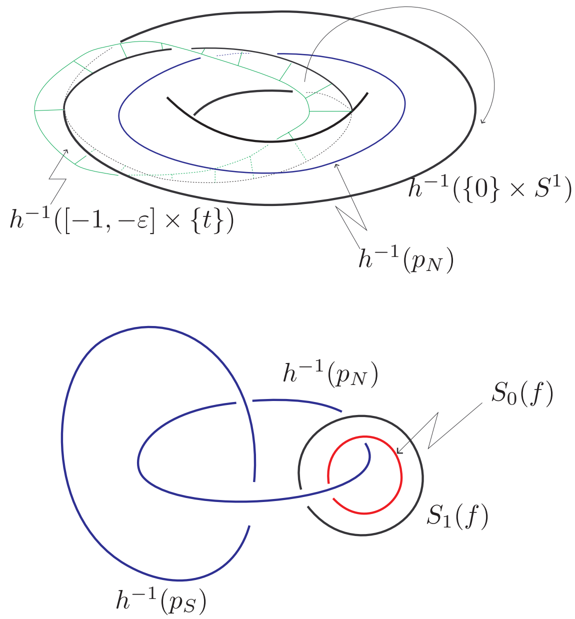


FIGURE 10. Framed regular fiber and the singular point set of the excellent map $f : S^3 \to \mathbf{R}^2$ in Example 4.2

(resp. $i_S : D_S \to \mathbf{R}^2$) is an orientation preserving (resp. reversing) embedding onto the unit disk in $\mathbf{R}^2$ such that $i_N(p_N) = i_S(p_S)$ coincides with the origin $\mathbf{0}$. Furthermore, we choose i_N and i_S such that for each $t \in S^1$, $i_N(1,t) = i_S(-1,t)$ holds for $(1,t)$ and $(-1,t) \in [-1,1]\times S^1 = A$. On $h^{-1}(A) = (S^1 \times [-1,1]) \times S^1$, we define f by $f(x,t) = \eta(k(x),t)$ for $x \in S^1 \times [-1,1]$ and $t \in S^1$, where $\eta : [1,\infty)\times S^1 \to \mathbf{R}^2$ is an embedding such that its image is the complement of the open unit disk in $\mathbf{R}^2$ and that $\eta(\{1\}\times S^1)$ coincides with the unit circle in $\mathbf{R}^2$. We choose η consistently with i_N and i_S, i.e. we require the condition that $\eta(1,t) = i_N(1,t) = i_S(-1,t)$ for every $t \in S^1$. Then, the map $f : S^3 \to \mathbf{R}^2$ thus constructed is well-defined.

By modifying f near the attached tori $h^{-1}(\partial D_N \cup \partial D_S)$ appropriately, we may assume that f is a smooth excellent map. Furthermore, the origin $\mathbf{0}$ of $\mathbf{R}^2$ is a regular value and $f^{-1}(\mathbf{0})$ is a framed regular fiber as in Example 4.1. Note that $S(f)$ has two components: one consists of definite fold singularities and the other of indefinite fold singularities.

The situation is as depicted in Fig. 10. The torus in the top figure represents $h^{-1}(\{0\}\times S^1)$ for $\{0\}\times S^1 \subset [-1,1]\times S^1 = A$, and it separates the regular fiber components $h^{-1}(p_N)$ and $h^{-1}(p_S)$ of f. The annulus depicts $h^{-1}([-1,-\varepsilon]\times\{t\})$ for some small $\varepsilon > 0$ and for some $t \in S^1$. We may assume that the critical points of k on $h^{-1}([-1,1]\times\{t\})$ are contained in $h^{-1}([-1,-\varepsilon]\times\{t\})$. As t varies in S^1 in the positive direction, the annulus rotates as depicted in that figure. Therefore, the critical points of k on the annulus sweep out a 2–component link $S(f) = S_0(f) \cup S_1(f)$ as depicted in the bottom figure, where $S_0(f)$ (or $S_1(f)$) is the set of definite (resp. indefinite) fold singularities of f.

In this example, the regular fiber component $h^{-1}(p_S)$ of f does not link with $S(f)$.

EXAMPLE 4.3. We have yet another example $g : S^3 \to \mathbf{R}^2$ constructed as follows. In the following, we use the same notations as in Example 4.2. We define g on $h^{-1}(D_N \cup D_S)$ in exactly the same way as f. On the other hand, we replace f on $h^{-1}(A)$ with the map F defined

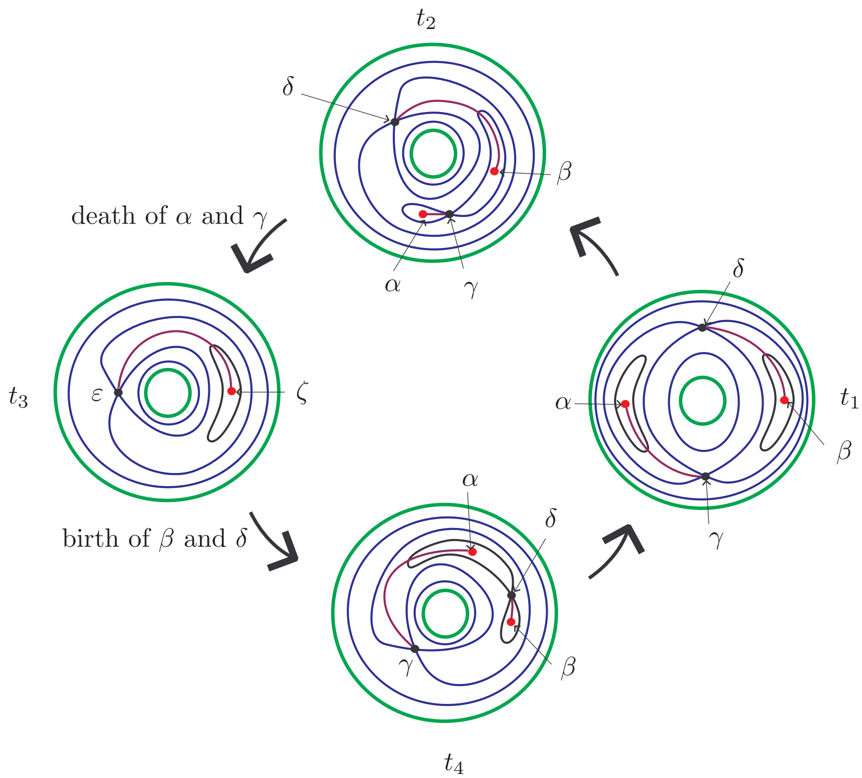


FIGURE 11. Level sets of $k_t : S^1 \times [-1, 1] \to [1, \infty)$ for $t = t_1, t_2, t_3$ and $t_4 \in S^1$, which correspond to those in Fig. 12.

by $F(x, t) = \eta(k_t(x), t)$ for $x \in S^1 \times [-1, 1]$ and $t \in S^1$, where $\eta : [1, \infty) \times S^1 \to \mathbf{R}^2$ is the embedding as in the above example, and $k_t : S^1 \times [-1, 1] \to [1, \infty)$, $t \in S^1$, is a generic 1–parameter family of functions on the annulus whose level sets are as depicted in Fig. 11, where the green circles depict the boundary components of the annulus and correspond to the level set $k_t^{-1}(1)$. Note that for $t \in S^1$, k_t is a Morse function, except for two values where a birth or a death of a pair of critical points occurs. In the figure, the red points depict critical points of index 2 and the black ones of index 1. The singular value set of F is as depicted in Fig. 12, and the critical points in Fig. 11 correspond to the curves $\alpha, \beta, \gamma, \delta, \varepsilon$ and ζ in Fig. 12.

In this way, we get an excellent map $g : S^3 \to \mathbf{R}^2$ with exactly two cusp singularities such that $S(g)$ consists of a circle. Furthermore, we see that $S(g)$ bounds a 2–disk in S^3 disjoint from the regular fiber $g^{-1}(\mathbf{0})$. Such a disk can be found by tracing the brown curves in Fig. 11. Therefore, $S(g)$ is an unknotted circle in S^3 and is split from the regular fiber over the origin $\mathbf{0}$. This again gives a desired counter example.

REMARK 4.4. The above examples show that the answer to the following question (see §1) is, in general, negative for excellent maps of S^3 into $\mathbf{R}^2$: *must every component of a regular fiber be linked by at least one component of the singular point set ?*

REMARK 4.5. Let $f : M \to \mathbf{R}^2$ be an excellent map of a closed oriented 3–manifold M. We assume that f is C^∞ stable, i.e. $f|_{S(f)}$ satisfies certain transversality conditions (for details, see [5, 10]). Such a C^∞ stable map f is *simple* if it has no cusp singularities and for every

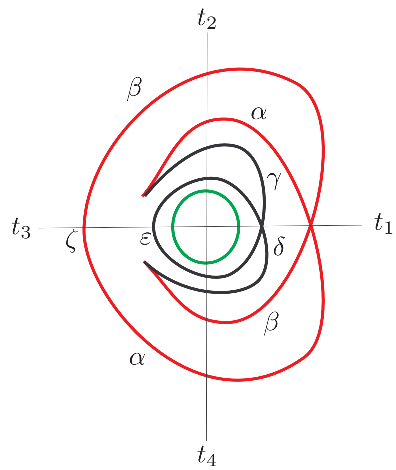


FIGURE 12. Singular value set of F, where the green circle in the center corresponds to the image of $\eta(\{1\} \times S^1)$, the red curve corresponds to the image of the definite fold singularities, and the black one to the image of the indefinite fold singularities. The values t_1, t_2, t_3 and $t_4 \in S^1$ correspond to those in Fig. 11.

$y \in f(S(f))$, each component of $f^{-1}(y)$ contains at most one singular point. In this case, by [15], regular fibers, the singular point set, or their unions are all graph links: i.e. their exteriors are unions of circle bundles over surfaces attached along their torus boundaries. The realization problem of graph links as regular fibers or the singular point set has been addressed in [15]. See also [12].

5. Maps of $\mathbf{R}^3$ into $\mathbf{R}^2$

Let us consider the following problem (see §1 and Remark 4.4).

PROBLEM 5.1. For a generic map $f : \mathbf{R}^3 \to \mathbf{R}^2$, must every component of a regular fiber be linked by at least one component of the singular point set $S(f)$?

In order to answer negatively to the above problem, we use the following theorem which is due to Hector and Peralta-Salas [6].

THEOREM 5.2 (Hector and Peralta-Salas, 2012). *Let $L = L_1 \cup L_2 \cup \cdots \cup L_\mu \subset \mathbf{R}^3$ be an oriented link. Then, there exist a submersion $f : \mathbf{R}^3 \to \mathbf{R}^2$ and a regular value $y \in \mathbf{R}^2$ such that $f^{-1}(y) = L$ if and only if for all i with $1 \le i \le \mu$, we have*

$$\sum_{j \neq i} \mathrm{lk}(L_i, L_j) \equiv 1 \pmod 2,$$

where lk *denotes the linking number.*

Now, let L be a link that satisfies the condition as described in Theorem 5.2 (for example, a Hopf link). Then, there exist a submersion $f : \mathbf{R}^3 \to \mathbf{R}^2$ and a regular value $y \in \mathbf{R}^2$ with $L = f^{-1}(y)$.

Take a point $p \in \mathbf{R}^3 \setminus L$ and its small 3–disk neighborhood $N(p) \subset \mathbf{R}^3 \setminus L$. Then, we can deform f in $N(p)$ so that the resulting map $g : \mathbf{R}^3 \to \mathbf{R}^2$ is excellent and $S(g)$ is an unknotted circle in $N(p)$ (use the move called "lip" or "birth". See [14, Lemma 3.1]). Then, no component of $L = g^{-1}(y)$ links with $S(g)$.

This gives a negative answer to Problem 5.1.

We finish this paper by posing some open problems.

PROBLEM 5.3. Can we generalize Theorem 2.3 for generic maps $f : M \to \mathbf{R}^2$ for closed non-orientable 3–manifolds? How about generic maps of general closed n–dimensional manifolds into $\mathbf{R}^p$ with $n > p > 1$?

Acknowledgment

The author is most thankful to Professor David Chillingworth for posing the question as in Problem 5.1, which motivated the study developed in this paper, and for indicating the references [1, 2, 4, 8] in physics. He would also like to thank Kazuto Takao for helpful discussions and for the anonymous referee for various invaluable comments which improved the presentation of the paper. The author has been supported in part by JSPS KAKENHI Grant Numbers JP15K13438, JP16K13754, JP16H03936, JP17H06128.

References

[1] M.V. Berry and M.R. Dennis, *Knotted and linked phase singularities in monochromatic waves*, Proc. R. Soc. Lond. A **457** (2001), 2251–2263. DOI: 10.1098/rspa.2001.0826

[2] M.V Berry and M.R. Dennis, *Topological events on wave dislocation lines: birth and death of loops, and reconnection*, J. Physics A: Mathematical and Theoretical **40** (2007), 65–74. DOI: 10.1088/1751-8113/40/1/004

[3] M. Cencelj, D. Repovš and M.B. Skopenkov, *Classification of framed links in 3–manifolds*, Proc. Indian Acad. Sci. (Math. Sci.) **117** (2007), 301–306. DOI: 10.1007/s12044-007-0025-x

[4] M.R. Dennis, *Local phase structure of wave dislocation lines: twist and twirl*, J. Opt. A: Pure Appl. Opt. **6** (2004), S202–S208. DOI: 10.1088/1464-4258/6/5/011

[5] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Grad. Texts in Math., Vol. 14, Springer–Verlag, New York, Heidelberg, Berlin, 1973.

[6] G. Hector and D. Peralta-Salas, *Integrable embeddings and foliations*, Amer. J. Math. **134** (2012), 773–825. DOI: 10.1353/ajm.2012.0018

[7] M.A. Kervaire, *Relative characteristic classes*, Amer. J. Math. **79** (1957), 517–558.

[8] H. Larocque, D. Sugic, D. Mortimer, A.J. Taylor, R. Fickler, R.W. Boyd, M.R. Dennis, and E. Karimi , *Reconstructing the topology of optical polarization knots*, Nature Physics (2018), https://doi.org/10.1038/s41567-018-0229-2.

[9] H.I. Levine, *Elimination of cusps*, Topology **3**, Suppl. 2 (1965), 263–296. DOI: 10.1007/bfb0075066

[10] H. Levine, *Classifying immersions into $\mathbf{R}^4$ over stable maps of 3-manifolds into $\mathbf{R}^2$*, Lecture Notes in Math., Vol. 1157, Springer–Verlag, Berlin, 1985.

[11] J. Milnor, *Spin structures on manifolds*, Enseignement Math. (2) **9** (1963), 198–203.

[12] O. Saeki, *Simple stable maps of 3–manifolds into surfaces II*, J. Fac. Sci. Univ. Tokyo **40** (1993), 73–124.

[13] O. Saeki, *Stable maps and links in 3–manifolds*, Kodai Math. J. **17** (1994), 518–529. DOI: 10.2996/kmj/1138040047

[14] O. Saeki, *Constructing generic smooth maps of a manifold into a surface with prescribed singular loci*, Ann. Inst. Fourier (Grenoble) **45** (1995), 1135–1162. DOI: 10.5802/aif.1489

[15] O. Saeki, *Simple stable maps of 3-manifolds into surfaces*, Topology **35** (1996), 671–698. DOI: 10.1016/0040-9383(95)00034-8

[16] R. Thom, *Les singularités des applications différentiables*, Ann. Inst. Fourier, Grenoble **6** (1955–1956), 43–87.

[17] H. Whitney, *On regular closed curves in the plane*, Compo. Math. **4** (1937), 276–284.

[18] H. Whitney, *On singularities of mappings of Euclidean spaces. I. Mappings of the plane into the plane*, Ann. of Math. (2) **62** (1955), 374–410. DOI: 10.1007/978-1-4612-2972-8_27

Osamu Saeki, Institute of Mathematics for Industry, Kyushu University, Motooka 744, Nishi-ku, Fukuoka 819-0395, Japan

Email address: saeki@imi.kyushu-u.ac.jp

Journal *of* Singularities
Volume 21 (2020), 249-267

Proc. of Geometric and Algebraic Singularity Theory, Będlewo, 2017
DOI: 10.5427/jsing.2020.21o

GEOMETRIC ALGEBRA AND SINGULARITIES OF RULED AND DEVELOPABLE SURFACES

JUNKI TANAKA AND TORU OHMOTO

Dedicated to Professor Goo Ishikawa on the occasion of his 60th birthday.

Abstract. Any ruled surface in $\mathbb{R}^3$ is described as a curve of unit dual vectors in the algebra of dual quaternions (=the even Clifford algebra $C\ell^+(0,3,1)$). Combining this classical framework and $\mathcal{A}$-classification theory of C^∞ map-germs $(\mathbb{R}^2,0) \to (\mathbb{R}^3,0)$, we characterize local diffeomorphic types of singular ruled surfaces in terms of geometric invariants. In particular, using a theorem of G. Ishikawa, we show that local topological type of singular developable surfaces is completely determined by vanishing order of the *dual torsion* $\check{\tau}$, that generalizes an old result of D. Mond for tangent developables of non-singular space curves. This work suggests that Geometric Algebra would be useful for studying singularities of geometric objects in classical Klein geometries.

1. Introduction

A *ruled surface* in Euclidean space $\mathbb{R}^3$ is a surface formed by a 1-parameter family of straight lines, called *rulings*; at least partly, it admits a parametrization of the form $F(s,t) = \boldsymbol{r}(s) + t\boldsymbol{e}(s)$ with $|\boldsymbol{e}(s)| = 1$, $s \in I$, $t \in \mathbb{R}$, where I is an open interval. A *developable surface* is a ruled surface which is locally planar (i.e. the Gaussian curvature is constantly zero). The parametrization map $F : I \times \mathbb{R} \to \mathbb{R}^3$ may be singular at some point (s_0, t_0), that is, the differential $dF(s_0, t_0)$ may have rank one, and then the surface (= the image of F) has a particularly singular shape around that point. In this paper, we study local diffeomorphic types of the singular surface and its bifurcations (see Fig.1). All maps and manifolds are assumed to be of class C^∞ throughout.

The main feature of this paper is to combine classical line geometry using *dual quaternions* [2, 3, 17, 21] and $\mathcal{A}$-classification theory of singularities of (frontal) maps $\mathbb{R}^2 \to \mathbb{R}^3$ [15, 5, 9, 8]. Here $\mathcal{A}$ denotes a natural equivalence relation in singularity theory of C^∞ maps; two map-germs $f, g : (\mathbb{R}^2,0) \to (\mathbb{R}^3,0)$ are *$\mathcal{A}$-equivalent* if there exist diffeomorphism-germs $\sigma : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ and $\varphi : (\mathbb{R}^3,0) \to (\mathbb{R}^3,0)$ such that $g = \varphi \circ f \circ \sigma^{-1}$. We simply say the *$\mathcal{A}$-type* of a map-germ to mean its $\mathcal{A}$-equivalence class. As a weaker notion, *topological $\mathcal{A}$-equivalence* is defined by taking σ and φ to be homeomorphism-germs. We also use the $\mathcal{A}$-equivalence with the target changes being rotations $\varphi \in SO(3)$, which is called *rigid equivalence* throughout the present paper. Our aim is to classify germs of parametrization maps F of ruled surfaces in $\mathbb{R}^3$ up to $\mathcal{A}$-equivalence and rigid equivalence.

1.1. **Ruled surfaces.** Geometric Algebra is a neat tool for studying motions in classical geometry; in case of Euclidean 3-space, it is the algebra of dual quaternions (e.g. Selig [21]). As an application, any ruled surface in $\mathbb{R}^3$ is described as a curve of *unit dual vectors*

$$\check{\boldsymbol{v}} : I \to \check{\mathbb{U}} \subset \mathbb{D}^3, \quad \check{\boldsymbol{v}}(s) = \boldsymbol{v}_0(s) + \varepsilon \boldsymbol{v}_1(s).$$

2010 *Mathematics Subject Classification.* 53A25, 53A05, 15A66, 57R45, 58K40.
Key words and phrases. Differential line geometry, Clifford algebra, Ruled surfaces, Developable surfaces, Singularities of smooth maps.

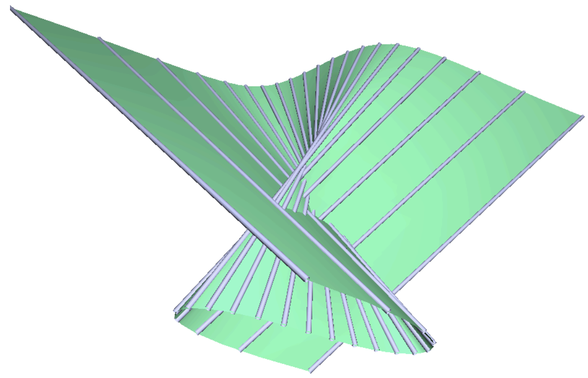


FIGURE 1. Deforming Mond's H_2-singularity via a family of ruled surfaces: the surface has two crosscaps and one triple point.

Here $\mathbb{D} = \mathbb{R} \oplus \varepsilon\mathbb{R}$ with $\varepsilon^2 = 0$ is the $\mathbb{R}$-algebra of *dual numbers*, and $\mathbb{D}^3 = \mathbb{R}^3 \oplus \varepsilon\mathbb{R}^3$ is the space of *dual vectors*, and especially, the space of unit dual vectors is given by

$$\check{\mathbb{U}} := \{\check{\boldsymbol{v}} = \boldsymbol{v}_0 + \varepsilon \boldsymbol{v}_1 \in \mathbb{D}^3,\ |\boldsymbol{v}_0| = 1,\ \boldsymbol{v}_0 \cdot \boldsymbol{v}_1 = 0\},$$

which is a 4-dimensional submanifold in the 6-dimensional space $\mathbb{D}^3$. Obviously, $\check{\mathbb{U}}$ is diffeomorphic to the total space of the (co)tangent bundle TS^2. It is naturally identified with the space of oriented lines in $\mathbb{R}^3$, by assigning to a unit dual vector $\check{\boldsymbol{v}}$ an oriented line $\boldsymbol{v}_0 \times \boldsymbol{v}_1 + t\boldsymbol{v}_0$ $(t \in \mathbb{R})$, see §2.1 for the detail. In our context, as the space of ruled surfaces in $\mathbb{R}^3$, we consider the space $C^\infty(I, \check{\mathbb{U}})$ of all smooth curves in $\check{\mathbb{U}}$ endowed with the Whintey C^∞-topology.

Assume that our ruled surface is *non-cylindrical*, i.e., $\boldsymbol{v}_0'(s) \neq 0$ for any $s \in I$, then the curve $\check{\boldsymbol{v}}$ admits the Frenet formula in $\mathbb{D}^3$ with *complete* differential invariants, the *dual curvature* and the *dual torsion*

$$\check{\kappa}(s) = \kappa_0(s) + \varepsilon\kappa_1(s), \quad \check{\tau}(s) = \tau_0(s) + \varepsilon\tau_1(s) \quad \in \mathbb{D}.$$

Here we may take s to be the arclength of the spherical curve $\boldsymbol{v}_0(s)$, that is equivalent to $\kappa_0(s) \equiv 1$, thus three real functions κ_1, τ_0, τ_1 are essential. In particular, $\kappa_1(s_0) = 0$ if and only if F is singular at (s_0, t_0) for some t_0; such t_0 is unique (Lemma 2.3).

We determine which $\mathcal{A}$-types of singular germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ appear in generic families of ruled surfaces. Assume that F is singular at $(s_0, t_0) = (0, 0)$ and $F(0, 0) = 0$, after taking parallel translations if needed. From the *dual Bouquet formula* of $\check{\boldsymbol{v}}$ at $s = 0$ in $\mathbb{D}^3$, we derive a *canonical* Taylor expansion of parameterization map F (§3.2), where $o(n)$ denotes Landau's notation of function-germs of order greater than n:

$$\begin{cases} x &= t - \frac{1}{2}ts^2 + \frac{\tau_1(0)}{2}s^3 + o(3), \\ y &= ts - \frac{\tau_1(0)}{2}s^2 - \frac{2\tau_0(0)\kappa_1'(0) + \tau_1'(0)}{6}s^3 + o(3), \\ z &= \frac{\kappa_1'(0)}{2}s^2 + \frac{\tau_0(0)}{2}ts^2 + \frac{\kappa_1''(0) - 2\tau_0(0)\tau_1(0)}{6}s^3 + o(3). \end{cases}$$

Then we apply to the jet of F the criteria for detecting $\mathcal{A}$-types of map-germs in Mond [14, 15].

Theorem 1.1. *The $\mathcal{A}$-classification of singularities of F arising in generic at most 3-parameter families of non-cylindrical ruled surfaces is given as in Table 1; in particular, for each $\mathcal{A}$-type in that table, the canonical expansion with the described condition is regarded as a normal form of the jet of ruled surface-germ under rigid equivalence.*

	normal form	ℓ	cond. at $s = s_0$ (with $\kappa_1(s_0) = 0$)
S_0	(x, y^2, xy)	2	$\kappa_1' \neq 0$
$S_1^\pm$	$(x, y^2, y^3 \pm x^2 y)$	3	$\kappa_1' = 0,\ \ \tau_1 \neq 0,\ \ \kappa_1''(\kappa_1'' - 2\tau_0\tau_1) \gtrless 0$
S_2	$(x, y^2, y^3 + x^3 y)$	4	$\kappa_1' = \kappa_1'' = 0,\ \kappa_1^{(3)}\tau_0\tau_1 \neq 0$
$B_2^\pm$	$(x, y^2, x^2 y \pm y^5)$		$\kappa_1' = 0,\ \kappa_1'' = 2\tau_0\tau_1 \neq 0,\ b_2 \gtrless 0$
H_2	$(x, xy + y^5, y^3)$		$\kappa_1' = \tau_1 = 0,\ \kappa_1'' \neq 0,\ h_2 \neq 0$
$S_3^\pm$	$(x, y^2, y^3 \pm x^4 y)$	5	$\kappa_1' = \kappa_1'' = \kappa_1^{(3)} = 0,\ \kappa_1^{(4)}\tau_0\tau_1 \gtrless 0$
$C_3^\pm$	$(x, y^2, xy^3 \pm x^3 y)$		$\kappa_1' = \kappa_1'' = \tau_0 = 0,\ \tau_1 \neq 0,\ \kappa_1^{(3)}(\kappa_1^{(3)} - 2\tau_0'\tau_1) \gtrless 0$
$B_3^\pm$	$(x, y^2, x^2 y \pm y^7)$		$\kappa_1' = 0,\ \kappa_1'' = 2\tau_0\tau_1 \neq 0,\ b_2 = 0,\ b_3 \gtrless 0$
H_3	$(x, xy + y^7, y^3)$		$\kappa_1' = \tau_1 = 0,\ \kappa_1'' \neq 0,\ h_2 = 0,\ h_3 \neq 0$
P_3	$(x, xy + y^3, xy^2 + p_4 y^4)$		$\kappa_1' = \kappa_1'' = \tau_1 = 0,\ \tau_0\tau_1' \neq 0,\ p_4 \neq 0, 1, \frac{1}{2}, \frac{3}{2}.$

TABLE 1. $\mathcal{A}$-types of singularities of ruled surfaces. Assume that $\kappa_1(s_0) = 0$, then F is singular at a unique point lying on the ruling corresponding to s_0. This table characterizes the $\mathcal{A}$-type of the germ of F at that point. Here, $\kappa_1', \kappa_1'', \cdots$ denote derivatives at $s = s_0$ for short, e.g. κ_1' means $\frac{d}{ds}\kappa_1(s_0)$, and b_2, b_3, h_2, h_3, p_4 are some polynomials of those derivatives (see §3.2). The letters $\lesseqgtr, \gtreqless, \pm$ are in the same order. In the second column, ℓ means $\mathcal{A}$-codimension of the map-germ.

Precisely saying, via a variant of Thom's transversality theorem (§3.3), we show that there exists a dense subset $\mathcal{O}$ in the mapping space $\mathcal{R}_W$ consisting of families of non-cylindrical $\check{\boldsymbol{v}} : I \times W \to \check{\mathbb{U}}$ with parameter space W of dimension ≤ 3 so that for any family belonging to $\mathcal{O}$ and for any $\lambda \in W$, the germ of the corresponding paramatrization map $F(-, \lambda) : I \times \mathbb{R} \to \mathbb{R}^3$ at every point (s_0, t_0) is $\mathcal{A}$-equivalent to either an immersion-germ or one of the singular germs in Table 1.

Obviously, normal forms under rigid equivalence have functional moduli: those are nothing but $\kappa_1(s)$, $\tau_0(s)$ and $\tau_1(s)$ satisfying the prescribed condition on derivatives at $s = s_0$.

Remark 1.2. (**Realization**) Izumiya-Takeuchi [10] firstly proved in a rigorous way that a generic singularity of ruled surfaces is only of type crosscap S_0, and Martins and Nuño-Ballesteros [13] showed that any $\mathcal{A}$-simple map-germ $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is $\mathcal{A}$-equivalent to a germ of ruled surface. By our theorem, $\mathcal{A}$-types which are not realized by ruled surfaces must have $\mathcal{A}$-codimension ≥ 6. This is sharp: for example, the 3-jet $(x, y^3, x^2 y)$, over which there are $\mathcal{A}$-orbits of codimension 6, is never $\mathcal{A}^3$-equivalent to 3-jets of any non-cylindrical nor cylindrical ruled surfaces (Remark 3.3). The realizability of versal families of $\mathcal{A}$-types via families of ruled surfaces can also be verified: for each germ in Table 1, an $\mathcal{A}_e$-versal deformation is obtained via deforming three invariants κ_1, τ_0, τ_1 appropriately (Remark 3.4).

Remark 1.3. (**Conformal GA**) Our approach would be applicable to other Clifford algebras and corresponding geometries. For instance, Izumiya-Saji-Takahashi [9] classified local singularities of horospherical flat surfaces in Lorentzian space (conformal spherical geometry); a horospherical surface is described by a curve in the Lie algebra $\mathfrak{so}(3, 1)$. Conformal Geometric Algebra may fit with this setting as well and our approach should work.

Remark 1.4. (**Framed curves**) Take the space of dual vectors $\mathbb{D}^3$ instead of $\check{\mathbb{U}}$. A curve $I \to \mathbb{D}^3$ corresponds to a *framed curve*, which describes a 1-parameter family of Euclidean motions of $\mathbb{R}^3$;

	normal form	ℓ	cond. at $s = s_0$
cE	(x, y^2, y^3)	1	$\tau_0 \neq 0, \ \ \tau_1 \neq 0$
cS_0	(x, y^2, xy^3)	2	$\tau_1 \neq 0, \ \ \tau_0 = 0, \ \ \tau_0' \neq 0$
cS_1^+	$(x, y^2, y^3(x^2 + y^2))$	3	$\tau_1 \neq 0, \ \ \tau_0 = \tau_0' = 0, \ \ \tau_0'' \neq 0$
cC_3^+	$(x, y^2, y^3(x^3 + xy^2))$	4	$\tau_1 \neq 0, \ \ \tau_0 = \tau_0' = \tau_0'' = 0, \ \ \tau_0''' \neq 0$
Sw	$(x, xy + 2y^3, xy^2 + 3y^4)$	2	$\tau_0 \neq 0, \ \ \tau_1 = 0, \ \ \tau_1' \neq 0$
cA_4	$(x, xy + \frac{5}{2}y^4, xy^2 + 4y^5)$	3	$\tau_0 \neq 0, \ \ \tau_1 = \tau_1' = 0, \ \ \tau_1'' \neq 0$
cA_5	$(x, xy + 3y^5, xy^2 + 5y^6)^\dagger$	4	$\tau_0 \neq 0, \ \ \tau_1 = \tau_1' = \tau_1'' = 0, \ \ \tau_1''' \neq 0$
T_1	$(x, xy + y^3, 0) + o(3)$	3	$\tau_0 = \tau_1 = 0, \ \ \tau_1' \neq 0$
T_2	$(x, xy, 0) + o(3)$	4	$\tau_0 = \tau_1 = \tau_1' = 0$

TABLE 2. $\mathcal{A}$-types of singularities of developable surfaces. An exception is the type cA_5; the condition implies that the germ is *topologically* $\mathcal{A}$-equivalent to the normal form † (in this case, the striction curve σ is topologically determinative in the sense of Ishikawa [5]).

various geometric aspects of framed curves have recently been studied by e.g. Honda-Takahashi [4]. Since the dual Frenet formula is available for regular framed curves, we may rebuild the theory by using dual quaternions. That would be useful for singularity analysis in several topics of applied mathematics such as 3D-interpolation via ruled/developable surfaces, 1-parameter motions of axes in robotics, and so on (cf. [17, 21]).

1.2. **Developable surfaces.** For a non-cylindrical ruled surface, it is developable (the Gaussian curvature is constantly zero) if and only if $\kappa_1 = 0$ identically, see §2. Thus two real functions τ_0, τ_1 are complete invariants of such developables. Izumiya-Takeuchi [10] classified generic singularities of developable surfaces rigorously, and Kurokawa [12] treated a similar task for 1-parameter families of developables. We generalize their results systematically using the complete invariants.

Theorem 1.5. *The $\mathcal{A}$-classification of singularities of F arising in generic at most 2-parameter families of non-cylindrical developable surfaces is given as in Table 2; in particular, for each $\mathcal{A}$-type in that table, the canonical expansion with the described condition is regarded as a normal form of the jet of developable-germ under rigid equivalence.*

Remark 1.6. (**Realization**) In our classification process §4.1, we see that non-cylindrical developables do not admit $\mathcal{A}$-types

$$cS_1^- : (x, y^2, y^3(x^2 - y^2)) \ \text{ nor } \ cC_3^- : (x, y^2, y^3(x^3 - xy^2))$$

(for the former, it was shown in [12]), while cS_1^+ and cC_3^+ appear. Furthermore, $\tau_1 \neq 0$ and $\tau_0 = \tau_0' = \tau_0'' = 0$ if and only if the 5-jet of F is equivalent to $(x, y^2, 0)$, and thus, for instance, we see that frontal singularities of cuspidal S and B-types

$$cS_* : (x, y^2, y^3(y^2 + h(x, y^2))), \quad cB_* : (x, y^2, y^3(x^2 + h(x, y^2)))$$

$(h(x, y^2) = o(2))$ never appear in our developable surfaces. Similarly, since $\tau_1 = 0$ if and only if the 2-jet is reduced to $(x, xy, 0)$, wavefronts of cuspidal beaks/lips type $A_3^\pm$ and purse/pyramid types D_k never appear. Indeed, their 2-jets are equivalent to $(x, 0, 0)$ and $(x^2 \pm y^2, xy, 0)$ respectively (it is obvious to see no appearance of D_k, for the corank of our maps F is at most one).

A non-cylindrical developable surface, which is not a cone, is re-parametrized as the tangent developable of the *striction curve* $\sigma(s)$ (Lemma 2.4). Here $\sigma(s)$ may be singular; recall that for a possibly singular space curve, its tangent developable is defined by the closure of the union of tangent lines at smooth points; indeed, it is a frontal surface, see §2.4 (cf. Ishikawa [6]). A space curve-germ is said to be of type $(m, m+\ell, m+\ell+r)$ if it is $\mathcal{A}$-equivalent to the germ

$$x = s^m + o(m), \;\; y = s^{m+\ell} + o(m+\ell), \;\; z = s^{m+\ell+r} + o(m+\ell+r)$$

(the curve is said to be of *finite type* if $m, n, \ell < \infty$). A type of curve-germ is called *smoothly determinative* (resp. *topologically determinative*) if it determines the $\mathcal{A}$-type (resp. topological $\mathcal{A}$-type) of the tangent developable. Ishikawa [5, 6] gave the following complete characterization (Mond [16] for the case of $m = 1$, i.e. smooth curves):

(i) smoothly determinative types are only $(1, 2, 2+r)$, $(2, 3, 4)$, $(1, 3, 4)$, $(3, 4, 5)$ and $(1, 3, 5)$;
(ii) $(m, m+\ell, m+\ell+r)$ is topologically determinative if and only if ℓ or r is odd, or $m = 1$ and ℓ, r are both even.

Using this result, we obtain a complete topological $\mathcal{A}$-classification of singularities of non-cylindrical developable surfaces:

Theorem 1.7. (Topological classification) *For a non-cylindrical developable surface, the germ of its striction curve $\sigma(s)$ at $s = s_0$ has the type*

$$(m, m+1, m+1+r),$$

where $m-1$ and $r-1$ are orders of τ_1 and τ_0 at $s = s_0$, respectively, i.e.,

$$\tau_1 = \tau_1' = \cdots = \tau_1^{(m-2)} = \tau_0 = \tau_0' = \cdots = \tau_0^{(r-2)} = 0,$$
$$\tau_1^{(m-1)}\tau_0^{(r-1)} \neq 0.$$

In particular, topological $\mathcal{A}$-types of the germ of F at singular points are completely determined by orders of the dual torsion $\check{\tau} = \tau_0 + \varepsilon\tau_1$.

Remark 1.8. Theorem 1.7 is regarded as the *dual version* of a result of Mond [16] and Ishikawa [5]: $\mathcal{A}$-type of the tangent developable of a *non-singular* space curve σ with non-zero curvature is determined by the vanishing order of its torsion function. This is the case that σ is of type $(1, 2, 2+r)$, and then the torsion of σ has the same order of τ_0 (Lemma 2.4). Note that in our theorem above, $\sigma(s)$ can be *singular* (i.e., $m \geq 2$) and the non-zero curvature condition is replaced by the non-cylindrical condition.

Remark 1.9. Table 2 is separated into three parts. One is the case of $\tau_1(s_0) \neq 0$; they are the tangent developables of non-singular curves of type $(1, 2, 2+r)$, which are *frontal singularities* as mentioned in Remark 1.8. The second is the case of $\tau_0(s_0) \neq 0$; they are the tangent developables of singular curves of types $(2, 3, 4)$, $(3, 4, 5)$ and $(4, 5, 6)$, which are *wavefronts* – the former two types are smoothly determinative, while the third one is topologically determinative, by Ishikawa's characterization. In the remaining part, types T_0 and T_1 are tangent developable of curves of type $(2, 3, 4+r)$ $(r \geq 1)$. Tangent developables of curves of other types (e.g., $(1, 3, 3+r), (2, 4, 4+r)$) are cylindrical at $s = s_0$.

Remark 1.10. Not only striction curves but also several other kind of characteristic curves on a ruled surface can be discussed. For instance, flecnodal curves are important in projective differential geometry of surfaces [11, 20].

The rest of this paper is organized as follows. In §2, we briefly review two main ingredients for non-experts in each subject – the first is the algebra of dual quaternions, which is the most basic Geometric Algebra, and the second is about useful criteria for detecting $\mathcal{A}$-types in singularity theory of maps. In §3, we apply the $\mathcal{A}$-criteria to the canonical Taylor expansion of F at singular points and prove Theorem 1.1. In §4, we proceed to the case of developable surfaces and prove Theorems 1.5 and 1.7.

This paper is based on the first author's master thesis [22]. This work was supported by JSPS KAKENHI Grant Numbers JP15K13452, JP17H0612818 and JP18K18714.

2. Preliminaries

Geometric Algebra is a new look at Clifford algebras, which is nowadays recognized as a very neat tool for describing motions in Klein geometries in the context of a variety of applications to physics, mechanics and computer vision. In §§2.1 and 2.2, we give a very quick summary on the geometric algebra for 3-dimensional Euclidean motions and its application to the geometry of ruled surfaces. A good compact reference is the nineth chapter of Selig's textbook [21] (also see [2, 10, 7, 17]).

In §§2.3 and 2.4, we briefly describe some basic notions in Singularity Theory, which will be used in §§3 and 4. We deal with two classes of C^∞ maps from a surface into $\mathbb{R}^3$; ordinary smooth maps of corank at most one, i.e. $\dim\ker df \le 1$ (Mond [15]) and frontal maps (Ishikawa [5], Izumiya-Saji [8]).

2.1. **Dual quaternions.** Let $\mathbb{H}$ denote the field of quaternions: $q = a + bi + cj + dk$. The conjugate of q is $\bar{q} = a - bi - cj - dk$ and the norm is given by $|q| = \sqrt{q\bar{q}}$. Decompose $\mathbb{H}$ into the real and the imaginary parts, $\mathbb{H} = \mathbb{R} \oplus \operatorname{Im}\mathbb{H}$, where one identifies $bi + cj + dk \in \operatorname{Im}\mathbb{H}$ with $\boldsymbol{v} = (b, c, d)^T \in \mathbb{R}^3$ equipped with the standard inner and exterior products. We write $q = a + \boldsymbol{v}$, then the multiplication of $\mathbb{H}$ is written as

$$(a + \boldsymbol{v})(b + \boldsymbol{u}) = (ab - \boldsymbol{v}\cdot\boldsymbol{u}) + (a\boldsymbol{u} + b\boldsymbol{v} + \boldsymbol{v}\times\boldsymbol{u}).$$

The quaternionic unitary group

$$\mathbb{H}_1 = Sp(1) = \{q \in \mathbb{H}, |q| = 1\}$$

is naturally isomorphic to $SU(2)$, that doubly covers $SO(3)$; indeed, $\pm q \in \mathbb{H}_1$ defines the rotation $\boldsymbol{x} \mapsto q\boldsymbol{x}\bar{q}$. The Lie algebra of $\mathbb{H}_1$ is just $\operatorname{Im}\mathbb{H} = \mathbb{R}^3$.

Put $\mathbb{D} = \mathbb{R}[\varepsilon]/\langle\varepsilon^2\rangle$, and call it the algebra of *dual numbers*. A dual number $a + \varepsilon b$ is invertible if $a \neq 0$, and it has a square root if $a > 0$. The $\mathbb{R}$-algebra of *dual quaternions* is defined by

$$\check{\mathbb{H}} := \mathbb{D}^4 = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{D} = \{\ \check{q} = q_0 + \varepsilon q_1 \mid q_0, q_1 \in \mathbb{H}\ \}.$$

That is identified with the even Clifford algebra $C\ell^+(0,3,1)$ [21, §9.3]. The conjugate of $\check{q}$ is defined by $\check{q}^* := \bar{q}_0 + \varepsilon\bar{q}_1$, and then $\check{q}\check{q}^* = |q_0|^2 + \varepsilon\mathrm{Re}[q_1\bar{q}_0]$. The Lie group of *unit dual quaternions* is defined by

$$\check{\mathbb{H}}_1 := \{\ \check{q} \in \check{\mathbb{H}} \mid \check{q}\check{q}^* = 1\ \}.$$

This group is isomorphic to the semi-direct product $\mathbb{H}_1 \ltimes \operatorname{Im}\mathbb{H} = Sp(1) \ltimes \mathbb{R}^3$ via the correspondance $\check{q} \leftrightarrow (q_0, q_1\bar{q}_0)$. Then, $\check{\mathbb{H}}_1$ doubly covers $SE(3) = SO(3) \ltimes \mathbb{R}^3$, the group of Euclidean motions of $\mathbb{R}^3$; the action $\check{\Theta}$ of $\check{\mathbb{H}}_1$ on $\boldsymbol{x} \in \mathbb{R}^3$ is given by

$$1 + \varepsilon\check{\Theta}(\check{q})\boldsymbol{x} := \check{q}(1 + \varepsilon\boldsymbol{x})\check{q}^* = 1 + \varepsilon(q_0\boldsymbol{x}\bar{q}_0 + 2q_1\bar{q}_0).$$

That is, q_0 and $2q_1\bar{q}_0$ express a rotation and a parallel translation, respectively. The Lie algebra of $\check{\mathbb{H}}_1$ is canonically identified with the space of *dual vectors*

$$\mathbb{D}^3 = \operatorname{Im}\mathbb{H} \otimes_{\mathbb{R}} \mathbb{D}, \quad \check{\boldsymbol{v}} = \boldsymbol{v}_0 + \varepsilon\boldsymbol{v}_1 \quad (\boldsymbol{v}_0, \boldsymbol{v}_1 \in \operatorname{Im}\mathbb{H} = \mathbb{R}^3),$$

which is a $\mathbb{D}$-submodule of $\check{\mathbb{H}} = \mathbb{D}^4$. The standard inner and exterior products of $\mathbb{R}^3$ are extended to $\mathbb{D}$-bilinear operations on $\mathbb{D}^3$;

$$\check{\boldsymbol{u}}\cdot\check{\boldsymbol{v}} := -\tfrac{1}{2}(\check{\boldsymbol{u}}\check{\boldsymbol{v}} + \check{\boldsymbol{v}}\check{\boldsymbol{u}}) \in \mathbb{D}, \qquad \check{\boldsymbol{u}}\times\check{\boldsymbol{v}} := \tfrac{1}{2}(\check{\boldsymbol{u}}\check{\boldsymbol{v}} - \check{\boldsymbol{v}}\check{\boldsymbol{u}}) \in \mathbb{D}^3.$$

A *unit dual vector* means a dual vector $\check{\boldsymbol{v}} \in \mathbb{D}^3$ with $\check{\boldsymbol{v}} \cdot \check{\boldsymbol{v}} = 1$, i.e., $|\boldsymbol{v}_0| = 1$ and $\boldsymbol{v}_0 \cdot \boldsymbol{v}_1 = 0$ (it is also called a 2-*blade* in the Clifford algebra $C\ell(0,3,1)$ [21, §10.1]). Denote the set of unit dual vectors by $\check{\mathbb{U}}$, which is identified with the space of oriented lines in $\mathbb{R}^3$ in the following way:

$$\text{oriented lines} : \boldsymbol{v}_0 \times \boldsymbol{v}_1 + t\boldsymbol{v}_0 \quad \overset{1:1}{\longleftrightarrow} \quad \text{unit dual vectors} : \check{\boldsymbol{v}} = \boldsymbol{v}_0 + \varepsilon \boldsymbol{v}_1.$$

This expression is very useful [21, §9.3]: for instance,

(i) a point $\boldsymbol{a} \in \mathbb{R}^3$ lies on the line corresponding to a unit dual vector $\boldsymbol{v}_0 + \varepsilon\boldsymbol{v}_1$ if and only if $\boldsymbol{a} \times \boldsymbol{v}_0 = \boldsymbol{v}_1$;

(ii) two lines intersect perpendicularly if and only if the corresponding unit dual vectors $\check{\boldsymbol{u}}$ and $\check{\boldsymbol{v}}$ satisfy that $\check{\boldsymbol{u}} \cdot \check{\boldsymbol{v}} = 0$.

2.2. **Ruled and developable surfaces.** Using the identification just mentioned above, a ruled surface is exactly described as a curve of unit dual vectors:

$$\check{\boldsymbol{v}} : I \to \check{\mathbb{U}} \subset \mathbb{D}^3, \quad \check{\boldsymbol{v}}(s) = \boldsymbol{v}_0(s) + \varepsilon \boldsymbol{v}_1(s)$$

(I an open interval) with $|\boldsymbol{v}_0(s)| = 1$ and $\boldsymbol{v}_0(s) \cdot \boldsymbol{v}_1(s) = 0$ ($s \in I$). Interpreting it as an object in $\mathbb{R}^3$, we have a parametrization

$$F(s,t) = \boldsymbol{r}(s) + t\boldsymbol{e}(s) \quad (\boldsymbol{r} = \boldsymbol{v}_0 \times \boldsymbol{v}_1, \quad \boldsymbol{e} = \boldsymbol{v}_0).$$

Note that $|\boldsymbol{e}(s)| = 1$ and $\boldsymbol{r} \cdot \boldsymbol{e} = 0$. Let R_s denote the ruling defined by $\check{\boldsymbol{v}}(s)$ and put

$$R = R(\check{\boldsymbol{v}}) := \bigcup_{s \in I} R_s \ \subset \mathbb{R}^3.$$

Formally, $\check{\boldsymbol{v}}(s)$ looks like a $\mathbb{D}$-version of the velocity vector of a space curve. That leads us to define the *curvature* $\check{\kappa}(s)$ of $\check{\boldsymbol{v}}$ by

$$\check{\kappa}(s) = \kappa_0(s) + \varepsilon\kappa_1(s) := \sqrt{\check{\boldsymbol{v}}'(s) \cdot \check{\boldsymbol{v}}'(s)} = |\boldsymbol{v}_0'| + \varepsilon \frac{\boldsymbol{v}_0' \cdot \boldsymbol{v}_1'}{|\boldsymbol{v}_0'|} \ \in \mathbb{D},$$

provided $\check{\boldsymbol{v}}$ is non-cylindrical, i.e., $\boldsymbol{v}_0'(s) \neq 0$ ($s \in I$). Here $(\ )'$ means $\frac{d}{ds}$. From now on, we assume that

$$|\boldsymbol{v}_0'(s)| = 1$$

by taking s to be the arc-length of $\boldsymbol{v}_0$. Then, $\check{\kappa} = 1 + \varepsilon \boldsymbol{v}_0' \cdot \boldsymbol{v}_1'$ and thus $\check{\kappa}^{-1} = 1 - \varepsilon \boldsymbol{v}_0' \cdot \boldsymbol{v}_1'$. Put

$$\check{\boldsymbol{n}}(s) = \boldsymbol{n}_0(s) + \varepsilon \boldsymbol{n}_1(s) := \check{\kappa}^{-1} \check{\boldsymbol{v}}'(s),$$

and

$$\check{\boldsymbol{t}}(s) = \boldsymbol{t}_0(s) + \varepsilon \boldsymbol{t}_1(s) := \check{\boldsymbol{v}}(s) \times \check{\boldsymbol{n}}(s).$$

Then for every $s \in I$, three dual vectors $\check{\boldsymbol{v}}(s)$, $\check{\boldsymbol{n}}(s)$ and $\check{\boldsymbol{t}}(s)$ form a basis of the $\mathbb{D}$-module $\operatorname{Im} \check{\mathbb{H}} = \mathbb{D}^3$ satisfying

$$\check{\boldsymbol{v}} \times \check{\boldsymbol{n}} = \check{\boldsymbol{t}}, \quad \check{\boldsymbol{t}} \times \check{\boldsymbol{v}} = \check{\boldsymbol{n}}, \quad \check{\boldsymbol{n}} \times \check{\boldsymbol{t}} = \check{\boldsymbol{v}},$$
$$\check{\boldsymbol{v}} \cdot \check{\boldsymbol{n}} = \check{\boldsymbol{n}} \cdot \check{\boldsymbol{t}} = \check{\boldsymbol{t}} \cdot \check{\boldsymbol{v}} = 0, \quad \check{\boldsymbol{v}} \cdot \check{\boldsymbol{v}} = \check{\boldsymbol{n}} \cdot \check{\boldsymbol{n}} = \check{\boldsymbol{t}} \cdot \check{\boldsymbol{t}} = 1.$$

From these relations and the property (ii) of unit dual vectors mentioned before, we see that three lines corresponding to unit dual vectors $\check{\boldsymbol{v}}, \check{\boldsymbol{n}}, \check{\boldsymbol{t}}$ meet at one point and are mutually perpendicular; in particular, $\boldsymbol{v}_0, \boldsymbol{n}_0, \boldsymbol{t}_0$ forms an orthonormal basis of $\mathbb{R}^3$.

We define the *torsion* $\check{\tau}(s)$ of $\check{\boldsymbol{v}}$ by

$$\check{\tau}(s) = \tau_0(s) + \varepsilon\tau_1(s) := \check{\boldsymbol{n}}'(s) \cdot \check{\boldsymbol{t}}(s) \quad \in \mathbb{D}.$$

The following theorem is classical:

Theorem 2.1. (cf. Guggenheimmer [2, §8.2], Selig [21, §9.4]) *Assume that s is the arc-length of $\boldsymbol{v}_0$, i.e. $\kappa_0(s) = |\boldsymbol{v}_0'(s)| = 1$.*

(1) (Frenet formula) *It holds that*

$$\frac{d}{ds}\begin{bmatrix} \check{\boldsymbol{v}}(s) \\ \check{\boldsymbol{n}}(s) \\ \check{\boldsymbol{t}}(s) \end{bmatrix} = \begin{bmatrix} 0 & \check{\kappa}(s) & 0 \\ -\check{\kappa}(s) & 0 & \check{\tau}(s) \\ 0 & -\check{\tau}(s) & 0 \end{bmatrix}\begin{bmatrix} \check{\boldsymbol{v}}(s) \\ \check{\boldsymbol{n}}(s) \\ \check{\boldsymbol{t}}(s) \end{bmatrix}.$$

(2) *The dual curvature $\check{\kappa}(s)$ and the dual torsion $\check{\tau}(s)$ are complete invariants of the ruled surface R up to Euclidean motions. That is, for two curves $\check{\boldsymbol{v}}_1$ and $\check{\boldsymbol{v}}_2$, they have the same invariants $\check{\kappa}$ and $\check{\tau}$ if and only if ruled surfaces $R(\check{\boldsymbol{v}}_1)$ and $R(\check{\boldsymbol{v}}_2)$ in $\mathbb{R}^3$ are transformed to each other by some Euclidean motion.*

(3) *$R(\check{\boldsymbol{v}})$ is a developable surface (including a cone) if and only if $\kappa_1 = 0$ identically. In particular, τ_0, τ_1 are complete invariants of the developable surface.*

The *striction curve* of a ruled surface R is the curve having minimal length which meets all the rulings of R. Let $F(s,t) = \boldsymbol{r}(s) + t\boldsymbol{e}(s)$ be a canonical parametrization

$$(\boldsymbol{r}\cdot\boldsymbol{e} = 0, \quad |\boldsymbol{e}| = |\boldsymbol{e}'| = 1),$$

then the striction curve $\sigma(s)$ is characterized by the equation $\sigma'\cdot\boldsymbol{e}' = 0$ (cf. [21, p.218], [10, Lemma 2.1], [17, §5.3]). We then have the following:

Lemma 2.2. *For a non-cylindrical ruled surface, it holds that*

(1) $\sigma(s) = \boldsymbol{r}(s) - (\boldsymbol{r}'(s)\cdot\boldsymbol{e}'(s))\boldsymbol{e}(s)$,
(2) $\sigma\times\boldsymbol{v}_0 = \boldsymbol{v}_1$, $\sigma\times\boldsymbol{n}_0 = \boldsymbol{n}_1$ *and* $\sigma\times\boldsymbol{t}_0 = \boldsymbol{t}_1$,
(3) $\sigma'(s) = \tau_1(s)\boldsymbol{v}_0(s) + \kappa_1(s)\boldsymbol{t}_0(s)$,
(4) $\kappa_1 = \det(\boldsymbol{e},\boldsymbol{e}',\boldsymbol{r}')$, $\tau_0 = \det(\boldsymbol{e},\boldsymbol{e}',\boldsymbol{e}'')$, $\tau_1 = \sigma'\cdot\boldsymbol{e}$.

From (2) and the property (i) of unit dual vectors in §2.1, it follows that $\sigma(s)$ lies on each of three lines corresponding to unit dual vectors $\check{\boldsymbol{v}}(s), \check{\boldsymbol{n}}(s), \check{\boldsymbol{t}}(s)$, that is, $\sigma(s)$ is the locus of the center of moving orthogonal frames. For completeness we prove the lemma, although it is elementary.

Proof: It is easy to see (1) by differentiating $\sigma(s) = \boldsymbol{r}(s) + t(s)\boldsymbol{e}(s)$. We show (2). First, by $\check{\boldsymbol{n}}\cdot\check{\boldsymbol{v}} = 0$, we see that $\boldsymbol{n}_1\cdot\boldsymbol{v}_0 = -\boldsymbol{v}_1\cdot\boldsymbol{n}_0$, and similarly $\boldsymbol{n}_1\cdot\boldsymbol{t}_0 = -\boldsymbol{t}_1\cdot\boldsymbol{n}_0$. By the Frenet formula, $\boldsymbol{v}_0' = \boldsymbol{n}_0$, $\boldsymbol{t}_0' = -\tau_0\boldsymbol{n}_0$, $\boldsymbol{v}_1' = \kappa_1\boldsymbol{n}_0 + \boldsymbol{n}_1$ and $\boldsymbol{t}_1' = -\tau_0\boldsymbol{n}_1 - \tau_1\boldsymbol{n}_0$. Since $\boldsymbol{r} = \boldsymbol{v}_0\times\boldsymbol{v}_1$ and $\boldsymbol{e} = \boldsymbol{v}_0$, it follows from (1) that

$$\sigma = -(\boldsymbol{t}_1\cdot\boldsymbol{n}_0)\boldsymbol{v}_0 - (\boldsymbol{v}_1\cdot\boldsymbol{t}_0)\boldsymbol{n}_0 - (\boldsymbol{n}_1\cdot\boldsymbol{v}_0)\boldsymbol{t}_0.$$

Thus $\sigma\times\boldsymbol{v}_0 = -(\boldsymbol{v}_1\cdot\boldsymbol{t}_0)\boldsymbol{n}_0\times\boldsymbol{v}_0 - (\boldsymbol{n}_1\cdot\boldsymbol{v}_0)\boldsymbol{t}_0\times\boldsymbol{v}_0 = (\boldsymbol{v}_1\cdot\boldsymbol{t}_0)\boldsymbol{t}_0 + (\boldsymbol{v}_1\cdot\boldsymbol{n}_0)\boldsymbol{n}_0 = \boldsymbol{v}_1$, for $\boldsymbol{v}_1\cdot\boldsymbol{v}_0 = 0$. That yields (2). Differentiating the first one of (2),

$$0 = (\sigma\times\boldsymbol{v}_0)' - \boldsymbol{v}_1' = (\sigma'\times\boldsymbol{v}_0 + \sigma\times\boldsymbol{n}_0) - (\kappa_1\boldsymbol{n}_0 + \boldsymbol{n}_1) = \sigma'\times\boldsymbol{v}_0 - \kappa_1\boldsymbol{n}_0$$

and similarly $\sigma'\times\boldsymbol{t}_0 + \tau_1\boldsymbol{n}_0 = 0$. Substitute $\sigma' = a\boldsymbol{v}_0 + b\boldsymbol{n}_0 + c\boldsymbol{t}_0$ for those equalities, we obtain $a = \tau_1$, $b = 0$, $c = \kappa_1$, that is (3). Finally, (4) is easy, e.g., $\kappa_1 = \boldsymbol{v}_0'\cdot\boldsymbol{v}_1' = \boldsymbol{e}'\cdot(\boldsymbol{r}'\times\boldsymbol{e}) = \det(\boldsymbol{e},\boldsymbol{e}',\boldsymbol{r}')$. □

Lemma 2.3. (cf. Izumiya et al [10, Lemma 2.2], [7, §1]) *For a non-cylindrical ruled surface, F is singular at (s_0,t_0) if and only if $\kappa_1(s_0) = 0$ and $t_0 = -\boldsymbol{r}'(s_0)\cdot\boldsymbol{e}'(s_0)$. The singular value $F(s_0,t_0)$ is the point $\sigma(s_0)$ where the curve $\sigma(s)$ is tangent to the ruling R_{s_0} or $\sigma'(s_0) = 0$.*

Proof: $\frac{\partial F}{\partial s}(s_0)\times\frac{\partial F}{\partial t}(s_0) = (\boldsymbol{r}'(s_0) + t_0\boldsymbol{e}'(s_0))\times\boldsymbol{e}(s_0) = 0 \Leftrightarrow \boldsymbol{r}'(s_0) = \alpha\boldsymbol{e}(s_0) - t_0\boldsymbol{e}'(s_0)$ for some $\alpha\neq 0 \Leftrightarrow \det(\boldsymbol{e}(s_0),\boldsymbol{e}'(s_0),\boldsymbol{r}'(s_0)) = 0$ and $t_0 = -\boldsymbol{r}'(s_0)\cdot\boldsymbol{e}'(s_0)$. The second claim follows from (3) in Lemma 2.2. □

In case of $\kappa_1 = 0$ identically, Lemmas 2.2 and 2.3 imply that singular points of F form a non-singular curve $s \mapsto (s, -\boldsymbol{r}'(s)\cdot \boldsymbol{e}'(s)) \in I \times \mathbb{R}$ and the image of this curve is just the striction curve $\sigma(s)$. Note that $\sigma(s)$ is a non-singular space curve, if $\tau_1 \neq 0$; especially, F is written by $\sigma(s) + \tilde{t}\sigma'(s)$ with $\tilde{t} = (t + \boldsymbol{r}'(s)\cdot \boldsymbol{e}'(s))/\tau_1$.

Lemma 2.4. (Izumiya et al [7, §1]) *A non-cylindrical developable surface, which is not a cone, is re-parametrized as the tangent developable of the striction curve $\sigma(s)$. The curve σ is non-singular whenever $\tau_1 \neq 0$, and then the curvature κ_σ and the torsion τ_σ of σ are given respectively by*

$$\kappa_\sigma = \frac{|\sigma' \times \sigma''|}{|\sigma'|^3} = \frac{1}{\tau_1}, \qquad \tau_\sigma = \frac{\det(\sigma', \sigma'', \sigma''')}{|\sigma' \times \sigma''|^2} = \frac{\tau_0}{\tau_1}.$$

2.3. **$\mathcal{A}$-classification of map-germs.** A *singular* point of $f : M \to N$ between manifolds means a point $p \in M$ where df_p is neither injective nor surjective (then $f(p) \in N$ is called a *singular value* of f); we denote by $S(f) \subset M$ the set of singular points of f. Two maps $\tilde{f} : U \to N$ and $\tilde{g} : V \to N$ on neighborhoods U and V of $p \in M$ define the same *map-germ at p* if there is a neighborbood $W \subset U \cap V$ of p so that $\tilde{f}|_W \equiv \tilde{g}|_W$; a *map-germ at p* is an equivalence class of maps under this relation, denoted by $f : (M, p) \to (N, f(p))$. Two map-germs at p have the same *k-jet* if they have the same Taylor polynomials at p of order k in some local coordinates; a k-jet is such an equivalence class of map-germs, denoted by $j^k f(p)$. Two germs $f : (M, p) \to (N, q)$ and $g : (M', p') \to (N', q')$ are *$\mathcal{A}$-equivalent* if they commute each other via diffeomorphism-germs σ and τ:

$$\begin{array}{ccc} (M,p) & \xrightarrow{f} & (N,q) \\ {\scriptstyle\sigma}\downarrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\downarrow{\scriptstyle\tau} \\ (M',p') & \xrightarrow[g]{} & (N',q') \end{array}$$

For simplicity, we consider map-germs $(\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$ and the $\mathcal{A}$-equivalence by the action of diffeomorphisms σ and τ preserving the origins. At the k-jet level, *$\mathcal{A}^k$-equivalence* is defined. A germ $f : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$ is said to be *k-$\mathcal{A}$-determined* if any germs $g : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$ with $j^k g(0) = j^k f(0)$ is $\mathcal{A}$-equivalent to f; such germs are collectively referred to as *finitely $\mathcal{A}$-determined* germs. For instance, the germ (x, y^2, xy) is 2-determined. Let $J^k(m, n)$ be the *jet space* consisting of all k jets of $(\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$, which is identified with the affine space of Taylor coefficients of order r $(1 \leq r \leq k)$ in a fixed system of local coordinates. The codimension of the $\mathcal{A}$-orbit of a germ f in the space of all map-germs $(\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$ is called the *$\mathcal{A}$-codimension* of f; the $\mathcal{A}$-codimension of f is finite if and only if f is finitely $\mathcal{A}$-determined (see e.g. [1]).

Thanks to finite determinacy, the process of $\mathcal{A}$-classification is reduced to a finite dimensional problem: we stratify $J^k(m, n)$ invariantly under the $\mathcal{A}^k$-equivalence step by step from low order k and low codimension. For instance, using several determinacy criteria, $\mathcal{A}$-classification of map-germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ up to certain codimension has been established in Mond [14, 15]. In §3, we will follow Mond's classification process.

Furthermore, in Mond [14, 16], a special class of map-germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is considered. A map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is of class *CE* (i.e. cuspidal edge), if rank $df(0) = 1$ and the singular point set $S(f)$ is non-singular. A germ f in CE is *k-$\mathcal{A}$-determined in CE* if any germ g in CE with the same k-jet as $j^k f(0)$ is $\mathcal{A}$-equivalent to f. In §4, we will use the following criteria of determinacy in CE [16, Lem.1.1, Prop.1.2].

Proposition 2.5 (Mond [16]). *It holds that*
i) If $f \in CE$ and $j^2f(0) = (x, y^2, 0)$, then f is $\mathcal{A}$-equivalent to the germ

$$g(x, y) = (x, y^2, y^3 p(x, y^2))$$

for some smooth function $p(u, v)$;
ii) $f(x, y) = (x, y^2, y^3)$ is 3-determined in CE;
iii) $f(x, y) = (x, y^2, yp(x, y^2))$ and $g(x, y) = (x, y^2, yq(x, y^2))$ are $\mathcal{A}$-equivalent if and only if $\tilde{f}(x, y) = (x, y^2, y^3 p(x, y^2))$ and $\tilde{g}(x, y) = (x, y^2, y^3 q(x, y^2))$ are $\mathcal{A}$-equivalent. In particular, f is $(k-2)$-determined if and only if $\tilde{f}$ is k-determined in CE.

2.4. **Singularities of frontal surfaces.** There is a special class of surfaces, called *frontal surfaces*. Let $ST^*\mathbb{R}^3$ be the spherical cotangent bundle with respect to the standard metric of $\mathbb{R}^3$ equipped with the standard contact structure. Let U be an open set of $\mathbb{R}^2$. A map $\iota : U \to ST^*\mathbb{R}^3$ is called *isotropic* if it satisfies that the image $d\iota(T_pU)$ is contained in the contact plane $K_{\iota(p)}$ for any $p \in U$. A *frontal map* is the composed map $f = \pi \circ \iota : U \to \mathbb{R}^3$ of an isotropic map ι and the projection $\pi : ST^*\mathbb{R}^3 \to \mathbb{R}^3$. The image (possibly singular) surface is called to be *frontal*. An isotropic immersion ι is usually called a *Lagrange immersion*, and $\pi \circ \iota$ and its image are called a *Lagrange map* and a *wavefront*, respectively. Let $f : U \to \mathbb{R}^3$ be a frontal map with $\nu : U \to S^2$ so that $\iota = (f, \nu) : U \to ST^*\mathbb{R}^3 = \mathbb{R}^3 \times S^2$ is an isotropic map. We identifies $T\mathbb{R}^3 \simeq T^*\mathbb{R}^3$ using the standard metric, then the unit vector ν is always orthogonal to the subspace $df(T_pU)$ at any $p \in U$. Let x, y be coordinates of U and put $\lambda(x, y) = \det\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \nu\right](x, y)$; then the singular point set $S(f)$ is defined by $\lambda(x, y) = 0$. If $d\lambda(p) \neq 0$, then p is called a *non-degenerate* singular point. In particular, if p is non-degenerate and $\operatorname{rank} df_p = 1$, the germ f at p is of class CE.

For a developable surface with $\boldsymbol{e} \times \boldsymbol{e}' \neq 0$, set $f : U \to \mathbb{R}^3$ to be $f(s, t) := \boldsymbol{r}(s) + t\boldsymbol{e}(s)$. Then f is a frontal map; in fact, it suffices to put $\nu = \boldsymbol{e} \times \boldsymbol{e}'/|\boldsymbol{e} \times \boldsymbol{e}'|$ (then $\frac{\partial f}{\partial t} \cdot \nu = \boldsymbol{e} \cdot \nu = 0$ and $\frac{\partial f}{\partial s} \cdot \nu = (\boldsymbol{r}' + t\boldsymbol{e}') \cdot \nu = \det(\boldsymbol{r}', \boldsymbol{e}, \boldsymbol{e}') = 0$). Note that any singularities of f are non-degenerate and have corank one (see the comment before Lemma 2.4). There are two cases:

If $\iota = (f, \nu)$ is singular, then it is easy to see that the 2-jet of f is $\mathcal{A}^2$-equivalent to $(x, y^2, 0)$, and hence Mond's criteria for map-germs of class CE (Proposition 2.5) can be applied.

If ι is non-singular, i.e. ι is a Legendre immersion, then the 2-jet is equivalent to $(x, xy, 0)$, and thus Proposition 2.5 is useless. In this case, we employ the Legendre singularity theory. There are known useful criteria of [8] (precisely saying, the topological type cA_5 is not dealt in [8] but the same argument as in Appendix of [8] works as well):

Proposition 2.6. (Izumiya-Saji [8, Theorem 8.1]) *Let $f : U \to \mathbb{R}^3$ be a Legendre map, and p a non-degenerate singular point with $\operatorname{rank} df_p = 1$. Let η be an arbitrary vector field around p so that $\eta(q)$ spans $\ker df_q$ at any $q \in S(f)$. Then f is $\mathcal{A}$-equivalent to cE, Sw, cA_4 or cA_5 if the following condition holds:*

$$\begin{array}{l|l}
cE & \eta\lambda(p) \neq 0, \\
Sw & \eta\lambda(p) = 0,\ \eta\eta\lambda(p) \neq 0, \\
cA_4 & \eta\lambda(p) = \eta\eta\lambda(p) = 0,\ \eta\eta\eta\lambda(p) \neq 0, \\
cA_5 & \eta\lambda(p) = \eta\eta\lambda(p) = \eta\eta\eta\lambda(p) = 0,\ \eta\eta\eta\eta\lambda(p) \neq 0.
\end{array}$$

Through the theory of frontal maps and generating functions, Ishikawa [5, 6] showed that the tangent developable of a curve of type

$$(m, m + \ell, m + \ell + r)$$

has a parametrization $F : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ defined by

$$\begin{array}{rcl} x & = & t, \\ y & = & s^{m+\ell} + s^{m+\ell+1}\varphi(s) + t(s^\ell + s^{\ell+1}\phi(s)), \\ z & = & (\ell+r)(m+\ell+r)\displaystyle\int_0^s u^r \frac{\partial y(u,t)}{\partial u} du \\ & = & (\ell+r)(m+\ell)s^{m+\ell+r} + \cdots + t(\ell(m+\ell+r)s^{\ell+r} + \cdots) \end{array}$$

with some C^∞ functions $\varphi(s)$ and $\phi(s)$. These two function must be related to invariants τ_0 and τ_1. It is also shown [5, Thm 2.1] that the topological type of the tangent developable of a space curve is determined by type $(m, m+\ell, m+\ell+r)$ of the curve, unless both ℓ, r are even, as mentioned in Introduction.

3. Singularities of ruled surfaces

In this section, we prove Theorem 1.1 (2); first we give a certain stratification of the jet space of triples of functions $(\kappa_1, \tau_0, \tau_1)$, and then discuss a variant of Thom's transversality theorem.

3.1. **Dual Bouquet formula.** Consider a curve $\check{\boldsymbol{v}} : I \to \mathbb{D}^3$, $\check{\boldsymbol{v}}(s) = \boldsymbol{v}_0(s) + \varepsilon \boldsymbol{v}_1(s)$, with $\check{\boldsymbol{v}} \cdot \check{\boldsymbol{v}} = 1$ and $|\boldsymbol{v}_0'(s)| = 1$ as in §2.2. We are concerned with the germ of $\check{\boldsymbol{v}}$ at the origin ($s_0 = 0$). Throughout this section, let $\check{\kappa}, \check{\tau}, \check{\kappa}', \check{\tau}', \cdots$ denote their values at $s = 0$ for short, e.g. $\check{\kappa}' = \check{\kappa}'(0)$, unless specifically mentioned.

By iterated uses of the Frenet formula (Theorem 2.1 (1)), we obtain the "Bouquet formula" of the curve in $\mathbb{D}^3$ at $s = 0$;

$$\check{\boldsymbol{v}}(s) = \sum_{n=0}^{r} \frac{\check{\boldsymbol{v}}^{(n)}(0)}{n!} s^n + o(r) \quad \in \mathbb{D}^3$$

with

$$\begin{array}{rcl} \check{\boldsymbol{v}}'(0) & = & \check{\kappa}\,\check{\boldsymbol{n}}(0), \\ \check{\boldsymbol{v}}''(0) & = & -\check{\kappa}^2\,\check{\boldsymbol{v}}(0) + \check{\kappa}'\,\check{\boldsymbol{n}}(0) + \check{\kappa}\check{\tau}\,\check{\boldsymbol{t}}(0), \\ \check{\boldsymbol{v}}^{(3)}(0) & = & -3\check{\kappa}\check{\kappa}'\,\check{\boldsymbol{v}}(0) + (\check{\kappa}'' - \check{\kappa}^3 - \check{\kappa}\check{\tau}^2)\,\check{\boldsymbol{n}}(0) + (2\check{\kappa}'\check{\tau} + \check{\kappa}\check{\tau}')\,\check{\boldsymbol{t}}(0), \\ \check{\boldsymbol{v}}^{(4)}(0) & = & (\check{\kappa}^4 + \check{\kappa}^2\check{\tau}^2 - 4\check{\kappa}\check{\kappa}'')\check{\boldsymbol{v}}(0) + (\check{\kappa}^{(3)} - 6\check{\kappa}^2\check{\kappa}' - 3\check{\kappa}'\check{\tau}^2 - 3\check{\kappa}\check{\tau}\check{\tau}')\check{\boldsymbol{n}}(0) \\ & & +(3\check{\kappa}''\check{\tau} + 3\check{\kappa}'\check{\tau}' - \check{\kappa}^3\check{\tau} + \check{\kappa}\check{\tau}'' - \check{\kappa}\check{\tau}^3)\check{\boldsymbol{t}}(0), \\ \check{\boldsymbol{v}}^{(5)}(0) & = & (10\check{\kappa}^3\check{\kappa}' + 5\check{\kappa}\check{\kappa}'\check{\tau}^2 + 5\check{\kappa}^2\check{\tau}\check{\tau}' - 5\check{\kappa}\check{\kappa}^{(3)})\check{\boldsymbol{v}}(0) + (\check{\kappa}^{(4)} - 6\check{\kappa}^2\check{\kappa}'' - 6\check{\kappa}''\check{\tau}^2 \\ & & -12\check{\kappa}'\check{\tau}\check{\tau}' - 3\check{\kappa}(\check{\tau}')^2 - 4\check{\kappa}\check{\tau}\check{\tau}'' + \check{\kappa}^3\check{\tau}^2 + \check{\kappa}\check{\tau}^4)\check{\boldsymbol{n}}(0) + (4\check{\kappa}^{(3)}\check{\tau} + 6\check{\kappa}''\check{\tau}' \\ & & +3\check{\kappa}'\check{\tau}'' - 9\check{\kappa}^2\check{\kappa}'\check{\tau} - \check{\kappa}^3\check{\tau}' + \check{\kappa}'\check{\tau}'' + \check{\kappa}\check{\tau}^{(3)} - 4\check{\kappa}'\check{\tau}^3 - 6\check{\kappa}\check{\tau}^2\check{\tau}')\check{\boldsymbol{t}}(0), \end{array}$$

and so on. A similar but more naïve expansion written by Plücker coordinates, instead of dual quaternions, can be found in a classical book of Hlavatý [3].

Since dual vectors $\{\check{\boldsymbol{v}}(0), \check{\boldsymbol{n}}(0), \check{\boldsymbol{t}}(0)\}$ form a $\mathbb{D}$-basis of $\operatorname{Im}\check{\mathbb{H}} = \mathbb{D}^3$, we may write

$$\check{\boldsymbol{v}}(s) = [\,\check{\boldsymbol{v}}(0), \check{\boldsymbol{n}}(0), \check{\boldsymbol{t}}(0)\,]\;\check{\boldsymbol{w}}(s),$$

and by the above derivatives $\check{\boldsymbol{v}}^{(k)}(0)$, one computes

$$\check{\boldsymbol{w}}(s) = [1 - \tfrac{1}{2}\check{\kappa}^2 s + \cdots, \check{\kappa}s + \tfrac{1}{2}\check{\kappa}' s + \cdots, \tfrac{1}{2}\check{\kappa}\check{\tau}s^2 + \cdots]^T \in \mathbb{D}^3.$$

Recall that three oriented lines in $\mathbb{R}^3$ determined by unit dual vectors $\check{\boldsymbol{v}}(0), \check{\boldsymbol{n}}(0), \check{\boldsymbol{t}}(0)$ meet at one point, which is nothing but the striction point $\sigma(0)$, as mentioned just after Lemma 2.2. By an Euclidean motion, the triple of lines can be transformed to standard coordinate axises of $\mathbb{R}^3$, i.e., $\boldsymbol{v}_0(0), \boldsymbol{n}_0(0), \boldsymbol{t}_0(0)$ are sent to the standard basis i, j, k of $\operatorname{Im}\mathbb{H} = \mathbb{R}^3$, respectively, and

$\boldsymbol{v}_1(0) = \boldsymbol{n}_1(0) = \boldsymbol{t}_1(0) = 0 \in \mathbb{R}^3$. Namely, we may assume that the 3×3 matrix (with entries in $\mathbb{D}$) $[\,\check{\boldsymbol{v}}(0), \check{\boldsymbol{n}}(0), \check{\boldsymbol{t}}(0)\,]$ is the identity matrix, so $\check{\boldsymbol{v}}(s) = \check{\boldsymbol{w}}(s)$. Then

$$\check{\boldsymbol{v}}(s) = \boldsymbol{v}_0(s) + \varepsilon \boldsymbol{v}_1(s) = \begin{bmatrix} 1 \\ s \\ 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 \\ \kappa_1 s \\ 0 \end{bmatrix} + o(1).$$

At a point $(0, t_0) \in I \times \mathbb{R}$, the Taylor expansion of $F(s,t) = \boldsymbol{v}_0(s) \times \boldsymbol{v}_1(s) + t\boldsymbol{v}_0(s)$ is immediately obtained; in particular, $F(0,t_0) = [t_0, 0, 0]^T$ and

$$dF(0,t_0) = \begin{bmatrix} 0 & 1 \\ t_0 & 0 \\ \kappa_1 & 0 \end{bmatrix}.$$

This gives an alternative proof of Lemma 2.3: F is singular at $(0,t_0)$ if and only if $\kappa_1(0) = t_0 = 0$ ($t_0 = 0$ means that the point is just the striction point $\sigma(0)$ lying on the ruling). Assume that F is singular at the origin. Then we obtain a *canonical Taylor expansion* of F:

$$\begin{aligned} &F(s,t) = \\ &\left(t - \tfrac{1}{2}ts^2 + \tfrac{\tau_1}{2}s^3,\ ts - \tfrac{\tau_1}{2}s^2 - \tfrac{2\tau_0\kappa_1' + \tau_1'}{6}s^3,\ \tfrac{\kappa_1'}{2}s^2 + \tfrac{\tau_0}{2}ts^2 + \tfrac{\kappa_1'' - 2\tau_0\tau_1}{6}s^3\right) + o(3). \end{aligned} \tag{1}$$

Remark 3.1. (Truncated polynomial maps) Let $F(s,t)$ be as in (1), and set

$$\bar{F}(s,t) = (\bar{\boldsymbol{v}}_0(s) \times \bar{\boldsymbol{v}}_1(s)) + t\bar{\boldsymbol{v}}_0(s)$$

to be a polynomial map of order k with $j^k\bar{F}(0) = j^kF(0)$. Denote by $\bar{s}$ the arc-length of the curve $\bar{\boldsymbol{v}}_0(s)$, then $\bar{s} := s + o(k)$, and thus k-jets at 0 of the dual curvature and the dual torsion do not change from those of F. That gives examples of polynomial ruled surfaces with prescribed k-jets of $\check{\kappa}$ and $\check{\tau}$ at a point.

3.2. **Recognition of singularity types.** Now our task is to find appropriate diffeomorphism-germs of the source and the target for reducing jets of $F(s,t)$ to normal forms in $\mathcal{A}$-classification step by step; for such computations, we have used the software *Mathematica*.

Let (X,Y,Z) be the coordinates of the target $\mathbb{R}^3$. Below, $\kappa_1, \kappa_1', \cdots$ denote their values at $s = 0$ unless specifically mentioned. From now on, assume that $\kappa_1 (= \kappa_1(0)) = 0$. Put $y = s$ and $x = t - \frac{1}{2}ts^2 + \frac{\tau_1}{2}s^3 + \cdots$ which is the first component of F in the form (1) above. With this new coordinates (x,y) of the source $\mathbb{R}^2$, we set

$$\begin{aligned} f(x,y) &:= F(y, t(x,y)) = (x, f_2(x,y), f_3(x,y)) \\ &= \left(x,\ xy - \tfrac{1}{2}\tau_1 y^2 - \tfrac{1}{6}\tau_1' y^3,\ \tfrac{1}{2}\kappa_1' y^2 + \tfrac{1}{2}\tau_0 xy^2 + \tfrac{1}{6}(\kappa_1'' - 2\tau_0\tau_1)y^3\right) + o(3). \end{aligned} \tag{2}$$

Note that $f(x,y)$ is still of the form $\tilde{\boldsymbol{r}}(y) + x\tilde{\boldsymbol{e}}(y)$. Now, we apply to this germ $f(x,y)$ the recognition trees in Mond's classification [15, Figs.1, 2]. Below, $S_k^\pm$, $B_k^\pm$, $C_k^\pm$, H_k and F_4 denote Mond's notations of $\mathcal{A}$-simple germs [15].

- **2-jet**: Crosscap S_0 is 2-determined, thus it follows from (2) that

$$f \sim_{\mathcal{A}} S_0 : (x, xy, y^2) \iff \kappa_1 = 0, \quad \kappa_1' \neq 0.$$

Let $\kappa_1' = 0$. Then the 2-jet is equivalent to either of $(x, xy, 0)$ or $(x, y^2, 0)$, according to whether $\tau_1 = 0$ or not. We compute the second and third component of f as

$$\begin{aligned}
f_2 &= xy - \tfrac{1}{2}\tau_1 y^2 - \tfrac{1}{6}\tau_1' y^3 \\
&\quad + \tfrac{1}{24}((8 - 4\tau_0^2)xy^3 + (-5\tau_1 + 3\tau_0^2\tau_1 - 3\tau_0\kappa_1'' - \tau_1'')y^4) \\
&\quad + \tfrac{1}{120}(-15\tau_0\tau_0' xy^4 + (12\tau_0\tau_0'\tau_1 - 9\tau_1' + 6\tau_0^2\tau_1' - 6\tau_0'\kappa_1'' - 4\tau_0\kappa_1^{(3)} - \tau_1^{(3)})y^5) \\
&\quad + o(5), \\
f_3 &= \tfrac{1}{6}(3\tau_0 xy^2 + (\kappa_1'' - 2\tau_0\tau_1)y^3) \\
&\quad + \tfrac{1}{24}(4\tau_0' xy^3 + (-3\tau_1\tau_0' - 3\tau_0\tau_1' + \kappa_1^{(3)})y^4) \\
&\quad + \tfrac{1}{120}((25\tau_0 - 5\tau_0^3 + 5\tau_0'')xy^4 + (-16\tau_0\tau_1 + 4\tau_0^3\tau_1 - 6\tau_0' tau_1' - 6\tau_0^2\kappa_1'' \\
&\quad - 4y^5\tau_1\tau_0'' - 4y^5\tau_0\tau_1'' + \kappa_1^{(4)})y^5) + o(5).
\end{aligned}$$

• **3-jet**: Let $\kappa_1 = \kappa_1' = 0$ and $\tau_1 \neq 0$. First, let us remove the term xy from f_2; take $\bar{x} = x$ and $\bar{y} = y - \frac{1}{\tau_1}x$, then we see that

$$j^3 f(0) \sim (x, y^2 + \tfrac{\tau_1'}{\tau_1^3}x^2 y + \tfrac{\tau_1'}{3\tau_1}y^3, \kappa_1'' x^2 y + \tfrac{1}{3}\tau_1^2(\kappa_1'' - 2\tau_0\tau_1)y^3). \tag{3}$$

The first two components can be transformed to (x, y^2) by a coordinate change of (x, y) with identical linear part and by a target coordinate change of (X, Y), since the plane-to-plane germ (x, y^2) is 2-determined (stable germ). Hence $j^3 f(0)$ is equivalent to one of the following:

$$\left\{\begin{array}{lll}
(x, y^2, y^3 \pm x^2 y) & \kappa_1''(\kappa_1'' - 2\tau_0\tau_1) \gtreqless 0,\ \tau_1 \neq 0 & \cdots S_1^{\pm}, \\
(x, y^2, y^3) & \kappa_1'' = 0,\ \tau_0\tau_1 \neq 0 & \cdots S, \\
(x, y^2, x^2 y) & \kappa_1'' = 2\tau_0\tau_1 \neq 0 & \cdots B, \\
(x, y^2, 0) & \kappa_1'' = \tau_0 = 0,\ \tau_1 \neq 0 & \cdots C.
\end{array}\right. \tag{4}$$

Note that $S_1^{\pm}$ is 3-determined, thus this case is clarified.

Let $\tau_1 = 0$. Then from (2), we have

$$j^3 f(0) \sim \left(x,\ xy - \tfrac{1}{6}\tau_1' y^3, \tfrac{1}{2}\tau_0 xy^2 + \tfrac{1}{6}\kappa_1'' y^3\right).$$

In the same way as above, $j^3 f(0)$ is reduced to one of the following:

$$\left\{\begin{array}{lll}
(x, xy, y^3) & \kappa_1'' \neq 0,\ \tau_1 = 0 & \cdots H, \\
(x, xy + y^3, xy^2) & \kappa_1'' = \tau_1 = 0,\ \tau_0\tau_1' \neq 0 & \cdots P, \\
(x, xy, xy^2) & \kappa_1'' = \tau_1 = \tau_1' = 0,\ \tau_0 \neq 0, & \\
(x, xy + y^3, 0) & \kappa_1'' = \tau_0 = \tau_1 = 0,\ \tau_1' \neq 0, & \\
(x, xy, 0) & \kappa_1'' = \tau_0 = \tau_1 = \tau_1' = 0. &
\end{array}\right. \tag{5}$$

Each of last three types has codimension ≥ 6, so we omit them here. Below, for types $S, B, \cdots, P$ in (4) and (5), we detect $\mathcal{A}$-types with codimension ≤ 5 by checking higher jets and the determinacy.

• **S-type**: Let $\kappa_1 = \kappa_1' = 0$ and $\tau_0\tau_1 \neq 0$. Then a computation shows that

$$\begin{aligned}
\kappa_1'' = 0 \quad &\Longrightarrow \quad j^4 f(0) \sim \left(x, y^2, y^3 - \tfrac{\kappa_1^{(3)}}{2\tau_0\tau_1^4}x^3 y\right), \\
\kappa_1'' = \kappa_1^{(3)} = 0 \quad &\Longrightarrow \quad j^5 f(0) \sim \left(x, y^2, y^3 - \tfrac{\kappa_1^{(4)}}{8\tau_0\tau_1^5}x^4 y\right), \\
\kappa_1'' = \kappa_1^{(3)} = \kappa_1^{(4)} = 0 \quad &\Longrightarrow \quad j^6 f(0) \sim \left(x, y^2, y^3 - \tfrac{\kappa_1^{(5)}}{40\tau_0\tau_1^6}x^5 y\right).
\end{aligned}$$

Note that S_k is $(k+2)$-determined (its codimension is $k+2$), thus f is of type $S_k^\pm$ $(k=2,3,4)$ if and only if $\kappa_1=\kappa_1'=\cdots=\kappa_1^{(k)}=0$ and $\kappa_1^{(k+1)}\tau_0\tau_1 \lessgtr 0$ (seemingly, it is so for any k).

• B**-type**: Let $\kappa_1=\kappa_1'=\kappa_1''-2\tau_0\tau_1=0$ and $\kappa_1''\neq 0$. Then it would be $\mathcal{A}$-equivalent to B_k-type [15, 4.1:17, Table 3]. For instance,

$$j^5f(0)\sim(x,y^2,x^2y+b_2y^5)$$

with

$$\begin{aligned} b_2 = \; & 48\tau_0^2\tau_1^2(\tau_0^2-2)-20(\tau_0^2(\tau_1')^2+\tau_1^2(\tau_0')^2)-56\tau_0\tau_1\tau_0'\tau_1' \\ & -24\tau_0\tau_1(\tau_0\tau_1''+\tau_1\tau_0'')+20\kappa_1^{(3)}(\tau_0\tau_1'+\tau_1\tau_0')-5(\kappa_1^{(3)})^2+6\kappa_1^{(4)}\tau_0\tau_1. \end{aligned}$$

Since B_2 is 5-determined,

$$f\sim_{\mathcal{A}} B_2^\pm:(x,y^2,x^2y\pm y^5) \iff b_2\gtrless 0.$$

Let b_3 be the coefficient of y^7 in the last component of $j^7f(0)$, which is written as a polynomial in derivatives of invariants at $s=0$, then $B_3^\pm:(x,y^2,x^2y\pm y^7)$ is detected by the condition that $b_2=0$ and $b_3\neq 0$. Here B_3 is of codimension 5.

• C**-type**: Let $\kappa_1=\kappa_1'=\kappa_1''=\tau_0=0,\ \tau_1\neq 0$. Through

$$\psi(X,Y,Z)=\left(\tfrac{1}{\tau_1}X,\ Y,\ \tfrac{1}{\tau_1}\left(Z-aY^2-bX^2Y\right)\right)$$

with $a=\frac{1}{4}(\kappa_1^{(3)}-3\tau_1\tau_0')$, $b=\frac{3}{2\tau_1^2}(\kappa_1^{(3)}-\tau_1\tau_0')$, we see that

$$j^4f(0)\sim\left(x,\ y^2,\ \kappa_1^{(3)}x^3y+(\kappa_1^{(3)}-2\tau_1\tau_0')xy^3\right).$$

Since C_3 is 4-determined (of codimension 5),

$$f\sim_{\mathcal{A}} C_3^\pm:(x,y^2,xy^3\pm x^3y) \iff \kappa_1^{(3)}(\kappa_1^{(3)}-2\tau_1\tau_0')\gtrless 0.$$

• H**-type**: Let $\kappa_1=\kappa_1'=\tau_1=0$ and $\kappa_1''\neq 0$. Then it would be $\mathcal{A}$-equivalent to H_k-type [15, 4.2.1:2]. A lengthy computation shows that

$$j^5f(0)\sim\left(x,\ xy+h_2y^5,\ y^3\right)$$

with

$$\begin{aligned} h_2 = \; & -15\tau_0^2(\tau_1')^3-24\tau_0'(\tau_1')^2\kappa_1''-36\tau_1'(\kappa_1'')^2-15\tau_0^2\tau_1'(\kappa_1'')^2-24\tau_0'(\kappa_1'')^3 \\ & -21\tau_0\tau_1'\kappa_1''\tau_1''+20\tau_0(\tau_1')^2\kappa_1^{(3)}-\tau_0(\kappa_1'')^2\kappa_1^{(3)}+5\kappa_1''\tau_1''\kappa_1^{(3)}-5\tau_1'(\kappa_1^{(3)})^2 \\ & -4(\kappa_1'')^2\tau_1^{(3)}+4\tau_1'\kappa_1''\kappa_1^{(4)}. \end{aligned}$$

Since H_2 is 5-determined,

$$f\sim_{\mathcal{A}} H_2^\pm:(x,xy\pm y^5,y^3) \iff h_2\gtrless 0.$$

Let h_3 be the coefficient of y^8 in the middle component of $j^8f(0)$, then $H_3:(x,xy+y^8,y^3)$ is detected by $h_2=0$ and $h_3\neq 0$ (H_3 is of codimension 5).

• P**-type**: Let $\kappa_1=\kappa_1'=\kappa_1''=\tau_1=0$ and $\tau_0\tau_1'\neq 0$. Then we see that there is a polynomial p_4 in derivatives of κ_1,τ_0,τ_1 so that

$$f\sim_{\mathcal{A}} P_3:(x,\ xy+y^3,\ xy^2+p_4y^4)$$

for $p_4\neq 0,\frac{1}{2},1,\frac{3}{2}$ [15, §4.2].

Remark 3.2. (Characterization of C_k and F_4) Among $\mathcal{A}$-simple germs obtained in Mond [15], we have just discussed germs of type $S_k^{\pm}$, $B_k^{\pm}$ and H_k. So there remain C_k ($k \geq 4$) and F_4, which are the next to C_3-type above. Suppose that $\kappa_1^{(3)}(\kappa_1^{(3)} - 2\tau_1\tau_0') = 0$. Then we have the following condition for each of them.

- If $\kappa_1^{(3)} = 0$ and $\tau_1\tau_0' \neq 0$, then $j^4 f(0) \sim (x, y^2, xy^3)$ and

$$\kappa_1^{(3)} = 0 \qquad \Longrightarrow \qquad j^5 f(0) \sim \left(x, y^2, xy^3 - \frac{\kappa_1^{(4)}}{8\tau_0'\tau_1^4} x^4 y\right),$$
$$\kappa_1^{(3)} = \kappa_1^{(4)} = 0 \qquad \Longrightarrow \qquad j^6 f(0) \sim \left(x, y^2, xy^3 - \frac{\kappa_1^{(5)}}{40\tau_0'\tau_1^5} x^5 y\right).$$

Since $C_k^{\pm} : (x, y^2, xy^3 \pm x^k y)$ is $(k+1)$-determined, we see that f is of type $C_k^{\pm}$ ($k = 4, 5$) if and only if $\tau_0 = \kappa_1 = \kappa_1' = \cdots = \kappa_1^{(k-1)} = 0$ and $\kappa_1^{(k)}\tau_0'\tau_1 \lessgtr 0$ (seemingly, it is so for any k).

- If $\kappa_1^{(3)} = 2\tau_1\tau_0' \neq 0$, we have $j^4 f(0) \sim (x, y^2, x^3 y)$ and

$$f \sim_{\mathcal{A}} F_4 : (x, y^2, x^3 y + y^5) \Longleftrightarrow 3\kappa_1^{(4)} - 8\tau_0'\tau_1' - 12\tau_1\tau_0'' \neq 0.$$

Remark 3.3. (Non-realizable jets) Let us continue the argument in Remark 3.2. If $\kappa_1^{(3)} = \tau_0' = 0$, then f should be of codimension ≥ 7 and a computation shows that

$$j^5 f(0) \sim \left(x, y^2, \kappa_1^{(4)} x^4 y + (\kappa_1^{(4)} - 4\tau_1\tau_0'')y^5 + 2\sqrt{5}(\kappa_1^{(4)} - 2\tau_1\tau_0'')x^2 y^3\right).$$

In particular, if two of three coefficients $\kappa_1^{(4)}$, $\kappa_1^{(4)} - 4\tau_1\tau_0''$, $\kappa_1^{(4)} - 2\tau_1\tau_0''$ are zero, then all are zero. Thus, for instance, the following 5-jets are not equivalent to jets of any non-cylindrical ruled surface:

$$(x, y^2, x^4 y), \quad (x, y^2, x^2 y^3), \quad (x, y^2, y^5).$$

The 5-jet (x, y^2, y^5) is obviously realizable by a cylinder, while the 5-jets

$$(x, y^2, x^4 y) \quad \text{and} \quad (x, y^2, x^2 y^3)$$

are not equivalent to jets of *any* ruled surfaces, even if we drop the condition $\boldsymbol{e}'(0) \neq 0$. In fact, put $F = \boldsymbol{r}(s) + t\boldsymbol{e}(s)$ with $\boldsymbol{r}(s)\cdot\boldsymbol{e}(s) = 0$ and $\boldsymbol{e}(s) = (1, 0, 0) + o(s)$. If F is singular at $(s, t) = (0, 0)$ and $\boldsymbol{r}(0) = 0$, then $\boldsymbol{r}(s) = o(s)$. It is easy to see that $F \sim_{\mathcal{A}} f = (x, y^2 h(x, y), y^3 g(x, y))$ with some functions h, g of the form $p(y) + xq(y)$, and thus the 5-jet of F is never equivalent to those two jets mentioned above. By the same reason, the $\mathcal{A}^3$-orbit of the 3-jet $(x, y^3, x^2 y)$ is not realized by jets of any ruled surfaces (the 2-jet $(x, 0, 0)$ never appears in non-cylindrical ruled surfaces as seen before, and the 3-jet is not realizable by ruled surfaces with $\boldsymbol{e}'(0) = 0$, that is shown in the same way as above).

3.3. **Transversality.** To precisely state *genericity* of ruled surfaces, we need an appropriate mapping space (moduli space) equipped with a certain topology. By the definition, a *residual* subset of a mapping space is a union of countably many open dense subsets. When maps having a prescribed condition form a residual subset, we often say that such a map is *generic*, abusing words. Let I be an open interval containing $0 \in \mathbb{R}$ and let u denote the coordinate of I. As the mapping space of non-cylindrical ruled surfaces, we take

$$\mathcal{R} := \{\, \check{\boldsymbol{v}} = \boldsymbol{v}_0 + \varepsilon\boldsymbol{v}_1 \in C^\infty(I, \check{\mathbb{U}}) \mid \boldsymbol{v}_0'(u) \neq 0 \,(u \in I) \,\}$$

equipped with Whitney C^∞ topology. As a remark, Izumiya and Takeuchi [10] and Martins and Nuño-Ballesteros [13] took the space $C^\infty(I, \mathbb{R}^3 \times S^2)$ instead of $C^\infty(I, \check{\mathbb{U}})$, but the difference does not affect the matter of genericity arguments – given a pair $(\boldsymbol{r}, \boldsymbol{e})$ of base and director curves, we simply assign a curve $\check{\boldsymbol{v}} : I \to \check{\mathbb{U}}$ with $\boldsymbol{v}_0 = \boldsymbol{e}$ and $\boldsymbol{v}_1 = \boldsymbol{r} \times \boldsymbol{e}$.

Also we put

$$\mathcal{M} := C^\infty(I, \mathbb{R}_{>0} \times \mathbb{R}^3)$$

of quadruples $(\kappa_0, \kappa_1, \tau_0, \tau_1)$ of real-valued functions with $\kappa_0(u) > 0$ equipped with Whitney C^∞ topology. Any curve $\check{\boldsymbol{v}}(u)$ in $\mathcal{R}$ defines $\mathbb{D}$-valued functions, $\check{\kappa}(u)$ and $\check{\tau}(u)$ (parameterized by a general parameter $u \in I$), that produces a continuous map $\Phi : \mathcal{R} \to \mathcal{M}$. Obviously, Φ is surjective. In fact, given a quadruple of functions $(\kappa_0(u), \kappa_1(u), \tau_0(u), \tau_1(u)) \in \mathcal{M}$, put a new parameter $s := s(u) = \int_0^u \kappa_0(u)du$ and define $\kappa_1(s) := \kappa_1(u(s))$, etc. Then, three functions $\kappa_1(s), \tau_0(s), \tau_1(s)$ determines, up to Euclidean motions, the curve $\check{\boldsymbol{v}}(s) = \boldsymbol{v}_0(s) + \varepsilon \boldsymbol{v}_1(s)$ by solving the ordinary differential equation determined by the Frenet formula. The ambiguity is fixed by the initial values $\check{\boldsymbol{v}}(0), \check{\boldsymbol{n}}(0), \check{\boldsymbol{t}}(0)$, which corresponds to the initial orthogonal axes in $\mathbb{R}^3$ at $u = 0$. Put $\check{\boldsymbol{v}}(u) := \check{\boldsymbol{v}}(s(u)) \in \mathcal{R}$; the set of such cruves is exactly the preimage via Φ of the given quadruple of functions. That implies that for a dense subset $O \subset \mathcal{M}$, the preimage $\Phi^{-1}(O)$ is also dense in $\mathcal{R}$.

The above construction is extended for a parametric version. Let W be an open subset of $\mathbb{R}^p$ $(0 \le p \le 3)$, and consider the subspace $\mathcal{R}_W$ of $C^\infty(I \times W, \check{\mathbb{U}})$ which consists of maps $\check{\boldsymbol{v}}(u, \lambda) = \boldsymbol{v}_0(u, \lambda) + \varepsilon \boldsymbol{v}_1(u, \lambda)$ with parameter $\lambda \in W$ satisfying $\partial \boldsymbol{v}_0/\partial u \neq 0$ at any (u, λ). Put $\mathcal{M}_W$ to be the mapping space of $I \times W \to \mathbb{R}_{>0} \times \mathbb{R}^3$, and then a surjective continuous map $\Phi : \mathcal{R}_W \to \mathcal{M}_W$ is defined in entirely the same way as above. For a dense subset $O \subset \mathcal{M}_W$, the preimage $\Phi^{-1}(O)$ is also in $\mathcal{R}_W$.

As seen in the previous section, we have obtained a semi-algebraic stratification of the jet space $J^r := \mathbb{R}^3 \times J^r(1,3)$ up to codimension 4 (r sufficiently large). In fact, any strata are defined by the conditions in Table 1 of (in)equalities in Taylor coefficients $\{\kappa_1^{(k)}, \tau_0^{(k)}, \tau_1^{(k)}\}_{0 \le k \le r}$, which form a system of coordinates of the affine space J^r. Notice that these Taylor coefficients are with respect to the arclength parameter s. For each quadruple $(\kappa_0, \kappa_1, \tau_0, \tau_1) \in \mathcal{M}_W$, we put

$$s = s(u, \lambda) := \int_0^u \kappa_0(u, \lambda) du, \qquad \varphi(u, \lambda) = (\kappa_1(u, \lambda), \tau_0(u, \lambda), \tau_1(u, \lambda)).$$

By the assumption that $\partial s/\partial u = \kappa_0 > 0$, let $\bar{\varphi}(s, \lambda) := \varphi(u(s, \lambda), \lambda)$. Then we define

$$\Psi : I \times W \times \mathcal{M}_W \to J^r, \qquad \Psi(u, \lambda, (\kappa_0, \varphi)) := j^r_s \bar{\varphi}(s(u, \lambda), \lambda),$$

where $j^r_s \bar{\varphi}$ means the r-jet respect to the parameter s. By a version of Thom's transversality theorem (Lemma 4.6 in [1]), there is a dense subset O of $\mathcal{M}_W$ so that for any $\varphi \in O$, the jet extension $\Psi_{\kappa_0, \varphi} : I \times W \to J^r$ is transverse to every stratum of our stratification of J^r. Hence, $\Phi^{-1}(O)$ is dense in $\mathcal{R}_W$, and for any element of $\Phi^{-1}(O)$, only $\mathcal{A}$-singularity types listed in Table 1 appears. This completes the proof of (2) in Theorem 1.1. $\square$

Remark 3.4. ($\mathcal{A}_e$-versal deformations) For each type in Table 1, an $\mathcal{A}_e$-versal deformation of the germ is realized by a generic family of non-cylindrical ruled surfaces. This is directly checked by computations. For instance, as in Table 1, the $S_1^\pm$-singularity of ruled surface at $s = 0$ is characterized by $\kappa_1(0) = \kappa_1'(0) = 0$, $\kappa_1''(0) \neq 0, 2\tau_0(0)\tau_1(0)$ and $\tau_1(0) \neq 0$. Suppose that $\varphi = (\kappa_1(s), \tau_0(s), \tau_1(s)) : I \to \mathbb{R}^3$ satisfies this condition. Define a 1-parameter family $I \times \mathbb{R} \to \mathbb{R}^3$ by $\varphi(s, \lambda) := \varphi(s) + (\lambda, 0, 0)$, then obviously, its 1-jet extension $j^1_s \varphi$ is transverse at $(0, 0)$ to the stratum defined by $\kappa_1 = \kappa_1' = 0$ in $J^1 = \mathbb{R}^3 \times J^1(1, 3)$. This family yields a 1-parameter family $F(s, t, \lambda) = (t, ts - \frac{\tau_1}{2}s^2, \lambda s) + o(2)$ of ruled surfaces. By using a coordinate change of $x = t + \cdots$ (= first component of $F(s, t, \lambda)$) and $y = s$ and some target changes, we see that the germ of $F(s, t, \lambda)$ is equivalent to $(x, y^2, y^3 \pm x^2 y + \lambda y)$, which is an $\mathcal{A}_e$-miniversal deformation of $S_1^\pm$-singularity.

4. Singularities of developable surfaces

4.1. **Recognition of singularity types.** For non-cylindrical developable surfaces, $\kappa_1(s) \equiv 0$ identically. Hence the Taylor expansion of f is (2) with $\kappa_1^{(k)} = 0$ for all k:

$$f(x,y) := F(y, t(x,y)) = (x, xy - \tfrac{1}{2}\tau_1 y^2 - \tfrac{1}{6}\tau_1' y^3, \tfrac{1}{2}\tau_0 xy^2 + \tfrac{1}{3}\tau_0\tau_1 y^3) + o(3).$$

Using the $\mathcal{A}$-criteria mentioned in §2, we classify singularities arising in generic families of developable surfaces. Notice that there are two different aspects; singularities of frontal surfaces correspond to the case of $\tau_1 \neq 0$, while singularities of wavefronts correspond to the case of $\tau_1 = 0$. Below we prove Theorem 1.5.

- **Case of** $\tau_1 \neq 0$: By $s = y + \tau_1^{-1}x$ and some linear change of the target, we have

$$f = (x, y^2 + o(2), f_3(x,y)) \quad \text{with} \quad f_3 = \tau_0 y^3 + o(3).$$

Note that (x, y^2) is 2-determined and that each term $x^k y^{2l}$ in f_3 can be removed by a coordinate change of the target $(X,Y,Z) \mapsto (X,Y,Z - X^kY^l)$. Use Proposition 2.5 in §2 ([16]) for determinacy in CE.

(i) If $\tau_0 \neq 0$, then $f \sim_{\mathcal{A}} (x, y^2, y^3)$, since it is 3-determined in CE.

(ii) Let $\tau_0 = 0$. Computing the 4-jet, we see

$$f_3 = \tau_0'(6x^2y^2 + 8\tau_1 xy^3 + 3\tau_1^2 y^4) + o(4).$$

If $\tau_0' \neq 0$, then $f \sim_{\mathcal{A}} (x, y^2, xy^3)$, for the germ is 4-determined in CE. Hence f is of type cuspidal crosscap.

(iii) Let $\tau_0 = \tau_0' = 0$. Computing the 5-jet, we see

$$f_3 = \tau_0''(10x^3y^2 + 20\tau_1 x^2y^3 + 15\tau_1^2 xy^4 + 4\tau_1^3 y^5) + o(5).$$

If $\tau_0'' \neq 0$, by target changes using $X = x$ and $Y = y^2$, terms x^3y^2 and xy^4 can be removed from $Z = f_3$, thus we see that $f \sim_{\mathcal{A}} (x, y^2, y^3(x^2 + y^2))$, for this germ is 5-determined in CE. That is cuspidal S_1^+-type. Note that cuspidal S_1^- never appears.

(iv) Let $\tau_0 = \tau_0' = \tau_0'' = 0$. Computing the 6-jet, we see

$$f_3 = \tau_0'''(15x^4y^2 + 40\tau_1 x^3y^3 + 45\tau_1^2 x^2y^4 + 24\tau_1^3 xy^5 + 5\tau_1^4 y^6) + o(6).$$

If $\tau_0''' \neq 0$, then $f \sim_{\mathcal{A}} (x, y^2, y^3(x^3 + xy^2))$, for the germ is 6-determined in CE. That is cuspidal C_3^+-type, while cuspidal C_3^- does not appear. Note that $\tau_0 = \tau_0' = \tau_0'' = 0$ if and only if the 5-jet of f is equivalent to $(x, y^2, 0)$, thus cuspidal S and B-types never appear, as mentioned in Remark 1.6.

- **Case of** $\tau_1 = 0$: Then $f = (x, xy - \frac{1}{6}\tau_1' y^3, \frac{1}{2}\tau_0 xy^2) + o(3)$. Note that $j^2 f(0) \sim (x, xy, 0)$, thus types $A_3^{\pm}$ and D_k never appear (Remark 1.6).

If $\tau_0 = 0$, $j^3 f(0)$ is equivalent to either $(x, xy + y^3, 0)$ or $(x, xy, 0)$, that is of type T_1 or T_2 (codimension $3, 4$) in Table 2. Now assume that $\tau_0 \neq 0$. Write

$$f = (x, f_2(x,y), f_3(x,y)) = (x, xy - \tfrac{1}{6}\tau_1' y^3, xy^2) + o(3).$$

The singular point set $S(F)$ is defined by $(f_2)_y = (f_3)_y = 0$, and through a computation, it is simplified as $\lambda = 0$ with

$$\lambda = x - \tfrac{1}{2}\tau_1' y^2 - \tfrac{1}{6}\tau_1'' y^3 - \tfrac{1}{24}(\tau_1''' - 3\tau_1')y^4 + o(4).$$

We may take $\eta = \partial/\partial y$ as a vector field which generates $\ker dF$ along $S(F)$. Then, $\eta\lambda(0) = 0$, $\eta\eta\lambda(0) = -\tau_1'$, $\eta\eta\eta\lambda(0) = -\tau_1''$ and $\eta\eta\eta\eta\lambda(0) = -(\tau_1''' - 3\tau_1')$. Hence, by Izumiya-Saji's criteria in §2.5, we have the conditions for detecting Sw, cA_4 and cA_5.

4.2. **Topological classification.** We prove Theorem 1.7. Let $\sigma(s)$ be the striction curve of a non-cylindrical developable surface. Assume that $\sigma(0) = 0 \in \mathbb{R}^3$, and consider the germ $\sigma : (\mathbb{R}, 0) \to (\mathbb{R}^3, 0)$. Since $\{\boldsymbol{v}_0(s), \boldsymbol{n}_0(s), \boldsymbol{t}_0(s)\}$ form a basis of $\mathbb{R}^3$ for each s, we denote the k-th derivative by

$$\sigma^{(k)}(s) = A_k(s)\boldsymbol{v}_0(s) + B_k(s)\boldsymbol{n}_0(s) + C_k(s)\boldsymbol{t}_0(s) \quad (k \geq 1)$$

where $A_k(s), B_k(s), C_k(s)$ are some functions. Then, with respect to the basis $\{\boldsymbol{v}_0(0), \boldsymbol{n}_0(0), \boldsymbol{t}_0(0)\}$, the expansion of σ at $s = 0$ is given by

$$\sigma(s) = (A_1(0)s + \tfrac{1}{2}A_2(0)s^2 + \cdots, B_1(0)s + \tfrac{1}{2}B_2(0)s^2 + \cdots, C_1(0)s + \tfrac{1}{2}C_2(0)s^2 + \cdots).$$

Now assume that σ is of type (m, n_1, n_2), i.e.,

$$\begin{cases} A_1(0) = \cdots = A_{m-1}(0) = 0,\ A_m(0) \neq 0, \\ B_1(0) = \cdots = B_{n_1-1}(0) = 0,\ B_{n_1}(0) \neq 0, \\ C_1(0) = \cdots = C_{n_2-1}(0) = 0,\ C_{n_2}(0) \neq 0. \end{cases}$$

Since $\sigma'(s) = \tau_1(s)\boldsymbol{v}_0(s)$ for a developable surface (Lemma 2.2 (iii)), we see that $A_1(s) = \tau_1(s)$ and $B_1(s) \equiv C_1(s) \equiv 0$. By the Frenet formula (Theorem 2.1 (1)),

$$\begin{aligned} \sigma^{(k+1)} &= (\sigma^{(k)})' = \{A_k\boldsymbol{v}_0 + B_k\boldsymbol{n}_0 + C_k\boldsymbol{t}_0\}' \\ &= (A_k' - B_k)\boldsymbol{v}_0 + (B_k' + A_k - C_k\tau_0)\boldsymbol{n}_0 + (C_k' + B_k\tau_0)\boldsymbol{t}_0 \\ &= A_{k+1}\boldsymbol{v}_0 + B_{k+1}\boldsymbol{n}_0 + C_{k+1}\boldsymbol{t}_0. \end{aligned}$$

Thus for $k = 1$, we have $A_2(s) = \tau_1'(s)$, $B_2(s) = \tau_1(s)$, $C_2(s) \equiv 0$, and for $k = 2$, $A_3(s) = \tau_1''(s) - \tau_1(s)$, $B_3(s) = 2\tau_1'(s)$ and $C_3(s) = \tau_0(s)\tau_1(s)$. For $k \geq 3$, there are some smooth functions $a_{k,*}(s)$, $b_{k,*}(s)$, $c_{k,*,*}(s)$ and positive numbers $\beta_k, \gamma_{k,0}, \cdots, \gamma_{k,k-3} > 0$ such that

$$\begin{aligned} A_k(s) &= a_{k,0}(s)\tau_1(s) + \cdots + a_{k,k-2}(s)\tau_1^{(k-2)}(s) + \tau_1^{(k-1)}(s), \\ B_k(s) &= b_{k,0}(s)\tau_1(s) + \cdots + b_{k,k-3}(s)\tau_1^{(k-3)}(s) + \beta_k\tau_1^{(k-2)}(s), \\ C_k(s) &= \{c_{k,0,0}(s)\tau_0(s) + \cdots + \gamma_{k,0}\tau_0^{(k-4)}(s)\}\tau_1(s) \\ &\quad + \{c_{k,1,0}(s)\tau_0(s) + \cdots + \gamma_{k,1}\tau_0^{(k-5)}(s)\}\tau_1'(s) + \cdots \\ &\quad + \{c_{k,k-4,0}(s)\tau_0(s) + \gamma_{k,k-4}\tau_0'(s)\}\tau_1^{(k-4)}(s) + \gamma_{k,k-3}\tau_0(s)\tau_1^{(k-3)}(s). \end{aligned}$$

Hence, by the assumption on $A_k(0)$, we have

$$\tau_1(0) = \cdots = \tau_1^{(m-2)}(0) = 0, \quad \tau_1^{(m-1)}(0) \neq 0,$$

and thus

$$B_1(0) = \cdots = B_m(0) = 0,\ \ B_{m+1}(0) \neq 0, \quad C_1(0) = \cdots = C_{m+2}(0) = 0.$$

In particular,

$$n_1 = m + 1,\ \ n_2 = m + 1 + r \quad (r \geq 1).$$

By the above formula of $C_k(s)$ with $k = m + 1 + r$, we see

$$\tau_0(0) = \cdots = \tau_0^{(r-2)}(0) = 0, \quad \tau_0^{(r-1)}(0) \neq 0.$$

Conversely, if the order of τ_0 and τ_1 are r and $m - 1$, respectively, then the type of σ is

$$(m, m + 1, m + 1 + r).$$

This completes the proof. □

References

[1] M. Golubitsky and V. Guillemin, *Stable Mappings and Their Singularities*, GTM 14, Springer-Verlag (1973).

[2] H. W. Guggenheimer, *Differential Geometry*, McGraw-Hill (1963).

[3] V. Hlavarý, *Differential line geometry*, Noordhoff, Groningen-London (1953).

[4] S. Honda and M. Takahashi, Framed curves in the Euclidean space, Adv. Geom. **16** (3) (2016), 265–276. DOI: 10.1112/s0024610700001095

[5] G. Ishikawa, Topological classification of the tangent developables of space curves, J. London Math. Soc. (2) **62** (1999), 583–598.

[6] G. Ishikawa, Singularities of developable surfaces, Singularity Theory, Proc. European Sing. Conf. (Liverpool, 1996), ed. W. Bruce and D. Mond, Cambridge Univ. Press (1999), 403–418. DOI: 10.1017/cbo9780511569265.025

[7] S. Izumiya, M. C. Romero Fuster, M. A. S. Ruas and F. Tari, *Differential Geometry from a Singularity Theory Viewpoint*, World Scientific (2016). DOI: 10.1142/9108

[8] S. Izumiya and K. Saji, The mandala of Legendrian dualities for pseudo-spheres in Lorentz-Minkowski space and "flat" spacelike surfaces, J. Singularities **2** (2010), 92–127. DOI: 10.5427/jsing.2010.2g

[9] S. Izumiya, K. Saji and M. Takahashi, Horospherical flat surfaces in Hyperbolic 3-space, J. Math. Soc. Japan **62** (2010), 789–849. DOI: 10.2969/jmsj/06230789

[10] S. Izumiya and N. Takeuchi, Singularities of ruled surfaces in $\mathbb{R}^3$, Math. Proc. Camb. Phil. Soc. **130** (2001), 1–11. DOI: 10.1017/s0305004100004643

[11] Y. Kabata, Recognition of plane-to-plane map-germs, Topology and its Appl. **202** (2016), 216–238. DOI: 10.1016/j.topol.2016.01.011

[12] H. Kurokawa, On generic singularities of 1-parameter families of developable surfaces (in Japanese), Master Thesis, Hokkaido University (2013).

[13] R. Martins and J. J. Nuño-Ballesteros, Finitely determined singularities of ruled surfaces in $\mathbb{R}^3$, Math. Proc. Camb. Phil. Soc. **147** (2009), 701–733. DOI: 10.1017/s0305004109002618

[14] D. Mond, *Classification of certain singularities and applications to differential geometry*, Ph.D. thesis, The University of Liverpool (1982).

[15] D. Mond, On the classification of germs of maps from $\mathbb{R}^2$ to $\mathbb{R}^3$, Proc. London Math. Soc. (3) **50** (1985), 333–369. DOI: 10.1112/plms/s3-50.2.333

[16] D. Mond, Singularities of the tangent developable surface of a space curve, Quart. J. Math. Oxford Ser. (2), **40** (1989), 79–91. DOI: 10.1093/qmath/40.1.79

[17] H. Pottmann and J. Wallner, *Computational Line Geometry*, Mathematics and Visualization, Springer (2001).

[18] K. Saji, Criteria for S_k singularities and their applications, Jour. Gökova Geom. Top. **4** (2010), 67–81.

[19] G. Salmon, *A treatise on the analytic geometry of three dimensions*, 4th edition, Dublin (1882).

[20] H. Sano, Y. Kabata, J. L. Deolindo Silva and T. Ohmoto, Classification of jets of surfaces in 3-space via central projection, Bull. Brasilian Math. Soc. New Series **48** (2017), 623–639. DOI: 10.1007/s00574-017-0036-x

[21] J. M. Selig, *Geometric Fundamentals of Robotics, 2nd edition*, Monographs in Computer Science, Springer (2005).

[22] J. Tanaka, Clifford algebra and singularities of ruled surfaces (in Japanese), Master Thesis, Hokkaido University (2016).

J. Tanaka, Kobo Co, LTD, Kobe, Japan
Email address: junki.tanaka@khobho.co.jp

T. Ohmoto, Department of Mathematics, Faculty of Science, Global Station of Bigdata and cybersecurity (GSB), Hokkaido University, Sapporo 060-0810, Japan
Email address: ohmoto@math.sci.hokudai.ac.jp

Journal *of* Singularities
Volume 21 (2020), 268-281

Proc. of Geometric and Algebraic Singularity Theory, Będlewo, 2017
DOI: 10.5427/jsing.2020.21p

APPLICATION OF SINGULARITY THEORY TO BIFURCATION OF BAND STRUCTURES IN CRYSTALS

H. TERAMOTO, A. TSUCHIDA, K. KONDO, S. IZUMIYA, M. TODA, T. KOMATSUZAKI

Abstract. Starting from the mean-field Hamiltonian of an electron in a crystal, we briefly review some known facts about its spectral structures and how singularities come into play in such spectral structures, and then provide our future perspective. We also estimate lower bounds of codimensions for the case where more than two bands to cross at a point.

1. Introduction

As in the song by Prof. Goo Ishikawa [10], singularity is everywhere. In this paper, we provide one example of such singularities appearing in solid-state physics [25]. Let

$$\hat{H} = -\frac{1}{2}\Delta + V(x) \tag{1}$$

be a Schrödinger operator on $L^2(\mathbb{R}^d)$, where $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian on $\mathbb{R}^d$, and $V\colon \mathbb{R}^d \to \mathbb{R}$. We assume there is a basis

$$\{\gamma_1, \ldots, \gamma_d\} \tag{2}$$

in $\mathbb{R}^d$ such that $V(x + \gamma_i) = V(x)$ holds for all $x \in \mathbb{R}^d$ and $i \in \{1, \ldots, d\}$. This Schrödinger operator appears in the following situation: an electron moving in a periodic potential in the bulk of a crystal ($d = 3$) or on the surface of a crystal ($d = 2$). A crystal consists of atoms and electrons interacting with each other. This Schrödinger operator is simplified to study the behavior of one of the electrons in the crystal; the effect of all the other electrons and atoms on the electron at $x \in \mathbb{R}^d$ is approximated by an averaged potential $V(x)$. One can also add a spin degree of freedom as in [15]. Some of mathematical justifications of this can be found in [13, 6, 5].

In Sec. 2, we briefly review what is known about spectral structures of the operator Eq. (1). There, band structures arise in the spectral structures as a consequence of the periodicity of the potential. Some of the topological features of the bands may be characterized by twisted-equivariant K-theory. Explaining the theory is beyond the scope of this paper but one of the established facts is that the bands cannot change their topology unless some of their band gaps close. Recently, it has become possible to manipulate band structures by changing the material properties of crystals and let some of the bands collide with each other [9]. To understand how such collisions trigger their topological changes, it is important to understand band geometries in neighborhoods of band crossings and their unfoldings. Having this goal in mind, in Sec. 3, we review our recent results on classification of band geometries in neighborhoods of band crossings in terms of the theory of singularities [25]. In Sec. 4, we discuss our future perspective along this direction.

2. Brief Review of Schrödinger operators with periodic potentials

In this section, we briefly review spectral properties of Schrödinger operators with periodic potentials by following [20, 16, 17]. For the definitions of terms in this section, see [21, 19, 20]. In this context, the basis in Eq. (2) is determined by the geometric structure of the crystal [2]. The lattice defined by

$$\Gamma = \left\{ \gamma \in \mathbb{R}^d \middle| \gamma = \sum_{j=1}^{d} n_j \gamma_j, (n_1, \cdots, n_d) \in \mathbb{Z}^d \right\} \tag{3}$$

is denoted as the Bravais lattice and its dual lattice

$$\Gamma^* = \left\{ k \in \mathbb{R}^d \,|k \cdot \gamma \in 2\pi\mathbb{Z}, \text{for all } \gamma \in \Gamma \right\} \tag{4}$$

is denoted as the inverse Bravais lattice. To fix the notation, we denote the centered fundamental domain of Γ by

$$Y = \left\{ x \in \mathbb{R}^d \middle| x = \sum_{j=1}^{d} \alpha_j \gamma_j, \text{for } \alpha_j \in \left[-\frac{1}{2}, \frac{1}{2} \right] \right\}, \tag{5}$$

and the centered fundamental domain of Γ^* by

$$Y^* = \left\{ k \in \mathbb{R}^d \middle| k = \sum_{j=1}^{d} \alpha_j \gamma_j^*, \text{for } \alpha_j \in \left[-\frac{1}{2}, \frac{1}{2} \right] \right\}, \tag{6}$$

where $\left\{\gamma_j^*\right\}_{j\in\{1,\cdots,d\}}$ is the dual basis to $\{\gamma_j\}_{j\in\{1,\cdots,d\}}$ such that $\gamma_i^* \cdot \gamma_j = 2\pi\delta_{i,j}$ holds for all $i, j \in \{1, \cdots, d\}$.

To investigate the spectral structure of the Schrödinger operator Eq. (1) on $L^2\left(\mathbb{R}^d\right)$, we show the operator is unitary equivalent to one decomposable by the direct integral decomposition. To do that, we introduce the following notation.

2.1. **Constant Fiber Direct Integral and Direct Integral Decomposition.**

Let $\mathcal{H}' = L^2\left(\mathbb{T}^d\right)$ be a Hilbert space on the torus $\mathbb{T}^d = \mathbb{R}^d/\Gamma$ with the inner product $(\cdot,\cdot)_{\mathcal{H}'}$, and let $L^2\left(Y^*, dk; \mathcal{H}'\right)$ be the set of measurable functions f on Y^* with values in $\mathcal{H}'$ which satisfy $\int_{Y^*} \|f(k)\|_{\mathcal{H}'}^2 dk < \infty$, where $\|\cdot\|_{\mathcal{H}'}$ is the norm induced from the inner product $(\cdot,\cdot)_{\mathcal{H}'}$. We call $\mathcal{H} = L^2\left(Y^*, dk; \mathcal{H}'\right)$ a constant fiber direct integral by following [20] and write

$$\mathcal{H} = \int_{Y^*}^{\oplus} \mathcal{H}' dk. \tag{7}$$

Note that $\mathcal{H}$ is a Hilbert space equipped with an inner product

$$(f, g)_{\mathcal{H}} = \int_{Y^*} \left(f(k), g(k)\right)_{\mathcal{H}'} dk \tag{8}$$

for $f, g \in \mathcal{H}$.

Next we would like to introduce the direct integral decomposition of an operator associated with a constant fiber direct integral. Suppose $A(\cdot)$ is a function from Y^* to the set of self-adjoint operators on a Hilbert space $\mathcal{H}'$. The function is measurable if and only if the function $(A(\cdot) + i)^{-1}$ is measurable, where i is the operator multiplied by the imaginary number i. Note that the spectrum of a self-adjoint operator is on the real line and thus $-i$ is in the resolvent set of the operator. Therefore, the function $(A(\cdot) + i)^{-1}$ is a well-defined function from Y^* to the set of bounded operators on $\mathcal{H}'$, $\mathcal{L}(\mathcal{H}')$. Such a function is called measurable if for each $\phi, \psi \in \mathcal{H}'$, $\left(\phi, (A(\cdot)+i)^{-1}\psi\right)_{\mathcal{H}'}$ is measurable.

Let $A(\cdot)$ be a measurable function from Y^* with the Lebesgue measure to the set of self-adjoint operators on a Hilbert space $\mathcal{H}'$. We define an operator A on $\mathcal{H} = \int_{Y^*}^{\oplus} \mathcal{H}'\, dk$ having $A(\cdot)$ as direct sum components with domain

$$D(A) = \left\{ \psi \in \mathcal{H} \,\middle|\, \psi(k) \in D(A(k)) \text{ a.e. } k \in Y^*;\ \int_{Y^*} \|A(k)\psi(k)\|_{\mathcal{H}'}^2\, dk < \infty \right\} \tag{9}$$

by $(A\psi)(k) = A(k)\psi(k)$ for all $k \in Y^*$ and for $\psi \in D(A)$, where $D(A(k)) \subset \mathcal{H}'$ is the domain of the operator $A(k)$ for $k \in Y^*$. If an operator A on $\mathcal{H} = \int_{Y^*}^{\oplus} \mathcal{H}'\, dk$ can be decomposed in this form, we say that the operator A admits direct integral decomposition and write

$$A = \int_{Y^*}^{\oplus} A(k)\ dk. \tag{10}$$

Next, let us introduce the modified Bloch-Floquet transformation [28]. By using the transformation, the operator in Eq. (1) is shown to be unitary equivalent to one that admits direct integral decomposition.

2.2. **Modified Bloch-Floquet Transformation.** Let $\mathcal{S}(\mathbb{R}^d)$ be the set of rapid decreasing functions on $\mathbb{R}^d$, i.e.,

$$\mathcal{S}(\mathbb{R}^d) = \left\{ \psi \in C^\infty(\mathbb{R}^d) \,\middle|\, \|\psi\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^d} \left|x^\alpha D^\beta \psi(x)\right| < \infty, \text{for all } \alpha, \beta \in I_+^d \right\}, \tag{11}$$

where I_+^d is the set of all d-tuples of nonnegative integers, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$ and

$$D^\beta \phi(x) = \frac{\partial^{|\beta|}\phi(x)}{\partial x_1^{\beta_1} \cdots \partial x_d^{\beta_d}} \quad \left(|\beta| = \sum_{j=1}^{d} \beta_d\right) \tag{12}$$

for $\alpha, \beta \in I_+^d$, and $|Y^*|$ is the volume of Y^*. Let $L^2_{\text{loc}}(\mathbb{R}^d)$ be the set of locally square-integrable functions on $\mathbb{R}^d$, i.e.,

$$L^2_{\text{loc}}(\mathbb{R}^d) = \left\{ \psi\colon \mathbb{R}^d \to \mathbb{C} \,\middle|\, \int_K |\psi|^2\ dx < \infty, \text{for any compact set } K \subset \mathbb{R}^d \right\}. \tag{13}$$

For $\psi \in \mathcal{S}(\mathbb{R}^d)$, we define the modified Bloch-Floquet transform

$$\tilde{\mathcal{U}}_{\text{BF}}\colon \mathcal{S}(\mathbb{R}^d) \to L^2_{\text{loc}}\left(\mathbb{R}^d, dk; L^2_{\text{loc}}(\mathbb{R}^d)\right) \tag{14}$$

as

$$\left(\tilde{\mathcal{U}}_{\text{BF}}\psi\right)(k,x) = \frac{1}{|Y^*|^{1/2}} \sum_{\gamma\in\Gamma} e^{-ik\cdot(x+\gamma)}\psi(x+\gamma) \tag{15}$$

for $x \in \mathbb{R}^d$ and $k \in \mathbb{R}^d$, where $|Y^*|$ is the volume of Y^*. In what follows, we construct $\mathcal{U}_{\text{BF}}\colon L^2(\mathbb{R}^d) \to \mathcal{H}$ from $\tilde{\mathcal{U}}_{\text{BF}}\colon \mathcal{S}(\mathbb{R}^d) \to L^2_{\text{loc}}\left(\mathbb{R}^d, dk; L^2_{\text{loc}}(\mathbb{R}^d)\right)$ by following [17].

First note that

$$\left(\tilde{\mathcal{U}}_{\text{BF}}\psi\right)(k, x+\gamma') = \left(\tilde{\mathcal{U}}_{\text{BF}}\psi\right)(k,x) \tag{16}$$

$$\left(\tilde{\mathcal{U}}_{\text{BF}}\psi\right)(k+\gamma^*, x) = e^{-i\gamma^*\cdot x}\left(\tilde{\mathcal{U}}_{\text{BF}}\psi\right)(k,x) \tag{17}$$

holds for all $\gamma' \in \Gamma$ and $\gamma^* \in \Gamma^*$ and the function is periodic in $x \in \mathbb{R}^d$, and thus $\left(\tilde{\mathcal{U}}_{\text{BF}}\phi\right)(k,\cdot)$ can be regarded as an element of $\mathcal{H}'$ for each $k \in \mathbb{R}^d$.

Next, by introducing a unitary representation of the group Γ^*, $\tau\colon \Gamma^* \to \mathcal{U}(\mathcal{H}')$, as $(\tau(\gamma^*)\phi)(x) = e^{i\gamma^*\cdot x}\phi(x)$ for $\phi \in \mathcal{H}'$, $x \in \mathbb{R}^d$, and $\gamma^* \in \Gamma^*$, the function

$$\left(\tilde{\mathcal{U}}_{\mathrm{BF}}\psi\right) \in L^2_{\mathrm{loc}}\left(\mathbb{R}^d, dk; \mathcal{H}'\right)$$

can be regarded as an element of the Hilbert space

(18)
$$\mathcal{H}_\tau = \left\{\psi \in L^2_{\mathrm{loc}}\left(\mathbb{R}^d, dk; \mathcal{H}'\right) \middle| \psi(k-\gamma^*, \cdot) = \tau(\gamma^*)\psi(k,\cdot), \text{for all } \gamma^* \in \Gamma^* \text{ for a.e. } k \in \mathbb{R}^d\right\}.$$

Since there is a natural isomorphism between $\mathcal{H}_\tau$ and $L^2(Y^*, dk; \mathcal{H}')$ given by restriction from $\mathbb{R}^d$ to Y^*, we get $\mathcal{H}_\tau \simeq \mathcal{H} = \int_{Y^*}^{\oplus} \mathcal{H}'\, dk$.

In addition, $\left(\tilde{\mathcal{U}}_{\mathrm{BF}}\psi_1, \tilde{\mathcal{U}}_{\mathrm{BF}}\psi_2\right)_{\mathcal{H}} = (\psi_1, \psi_2)_{L^2(\mathbb{R}^d)}$ holds for $\psi_1, \psi_2 \in \mathcal{S}\left(\mathbb{R}^d\right)$. This can be shown as follows: First, note that

$$\begin{aligned}\left(\tilde{\mathcal{U}}_{\mathrm{BF}}\psi_1, \tilde{\mathcal{U}}_{\mathrm{BF}}\psi_2\right)_{\mathcal{H}} &= \int_{Y^*}\left(\tilde{\mathcal{U}}_{\mathrm{BF}}\psi_1(k,\cdot), \tilde{\mathcal{U}}_{\mathrm{BF}}\psi_2(k,\cdot)\right)_{\mathcal{H}'} dk \\ &= \frac{1}{|Y^*|}\int_{Y^*}\int_{\mathbb{T}^d}\sum_{\gamma',\gamma\in\Gamma} e^{ik\cdot\left(x+\gamma'\right)-ik\cdot(x+\gamma)}\overline{\psi}_1(x+\gamma')\,\psi_2(x+\gamma)\ dxdk\end{aligned}$$

holds where $\bar{\cdot}$ is the complex conjugate of an operand. Since the sum in the integrand converges uniformly for all $x \in \mathbb{T}^d$ and $k \in Y^*$ and the domains of the integrations are compact, the sums and integrals can be interchanged to get

(19)
$$\frac{1}{|Y^*|}\sum_{\gamma',\gamma\in\Gamma}\int_{Y^*}\int_{\mathbb{T}^d} e^{ik\cdot\left(\gamma'-\gamma\right)}\overline{\psi}_1(x+\gamma')\,\psi_2(x+\gamma)\ dxdk.$$

By integrating it with respect to k and using

$$\frac{1}{|Y^*|}\int_{Y^*} e^{ik\cdot\left(\gamma'-\gamma\right)}dk = \delta_{\gamma',\gamma},$$

where $\delta_{\gamma',\gamma} = \begin{cases} 1 & (\gamma' = \gamma) \\ 0 & (\gamma' \neq \gamma)\end{cases}$, we get

(20)
$$\sum_{\gamma\in\Gamma}\int_{\mathbb{T}^d}\overline{\psi}_1(x+\gamma)\,\psi_2(x+\gamma)\ dx.$$

This is equal to

(21)
$$(\psi_1, \psi_2)_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d}\overline{\psi}_1(x)\,\psi_2(x)\ dx$$

and thus proves the claim. Since $\mathcal{S}\left(\mathbb{R}^d\right)$ is dense in $L^2\left(\mathbb{R}^d\right)$, the modified Bloch-Floquet operator can be extended to be a unitary operator $\mathcal{U}_{\mathrm{BF}}\colon L^2\left(\mathbb{R}^d\right) \to \mathcal{H}$ with inverse given by

(22)
$$\left(\mathcal{U}_{\mathrm{BF}}^{-1}\psi\right)(x) = \frac{1}{|Y|^{1/2}}\int_{Y^*}\psi(k,[x])\,e^{ik\cdot x}\ dk,$$

where $[\cdot]$ refers to the decomposition $x = \gamma_x + [x]$ with $\gamma_x \in \Gamma$ and $[x] \in Y$.

2.3. **Direct Integral Decomposition of Eq.** (1)**.** Suppose $d = 1, 2, 3$ and $V: \mathbb{R}^d \to \mathbb{R}$ is Γ-periodic and $V \in L^2_{\mathrm{loc}}\left(\mathbb{R}^d\right)$. Then,

$$\hat{H}_{\mathrm{BF}} = \mathcal{U}_{\mathrm{BF}} \hat{H} \mathcal{U}_{\mathrm{BF}}^{-1} = \int_{Y^*}^{\oplus} \hat{H}\left(k\right)\ dk \tag{23}$$

holds with fiber operator

$$\hat{H}\left(k\right) = \frac{1}{2}\left(-i\nabla_x + k\right)^2 + V\left(x\right) \tag{24}$$

for $k \in Y^*$ acting on the k-independent domain $D_0 = W^{2,2}\left(\mathbb{T}^d\right) \subset \mathcal{H}'$, where

$$W^{2,2}\left(\mathbb{T}^d\right) = \left\{\psi \in \mathcal{H}' \left| D^\alpha \psi \in \mathcal{H}', \text{for all } |\alpha| \le 2\right.\right\} \tag{25}$$

is the Sobolev space where $D^\alpha \psi$ is the differential of ψ in the weak sense, i.e., one satisfies

$$\int_{\mathbb{T}^d} \left(D^\alpha \psi\right)\left(x\right) \phi\left(x\right)\ dx = \left(-1\right)^{|\alpha|} \int_{\mathbb{T}^d} \psi\left(x\right)\left(D^\alpha \phi\right)\left(x\right)\ dx \tag{26}$$

for all $\phi \in C^\infty\left(\mathbb{T}^d\right)$.

To prove the claim in Eq. (23), let us show the following:

$$\mathcal{U}_{\mathrm{BF}}\left(-\Delta\right)\mathcal{U}_{\mathrm{BF}}^{-1} = \int_{Y^*}^{\oplus} \left(-i\nabla_x + k\right)^2 dk. \tag{27}$$

Let A be the operator on the right hand side of Eq. (27). The operator $A\left(k\right) = \left(-i\nabla_x + k\right)^2$ is self-adjoint for $k \in Y^*$ acting on the k-independent domain $D_0 = W^{2,2}\left(\mathbb{T}^d\right) \subset \mathcal{H}'$ and is measurable, therefore, **Theorem XIII.85 (a)** in [20] guarantees that the operator A is self-adjoint as well. We shall show that if $\psi \in \mathcal{S}\left(\mathbb{R}^d\right)$, then, $\mathcal{U}_{\mathrm{BF}}\psi \in D\left(A\right)$ and

$$\mathcal{U}_{\mathrm{BF}}\left(-\Delta\psi\right) = A\left(\mathcal{U}_{\mathrm{BF}}\psi\right).$$

Since $-\Delta$ is essentially self-adjoint on $\mathcal{S}\left(\mathbb{R}^d\right)$ and A is self-adjoint, Eq. (27) follows because this means that $-\Delta$ has the unique self-adjoint extension that should coincide with the self-adjoint operator $\mathcal{U}_{\mathrm{BF}}^{-1} A \mathcal{U}_{\mathrm{BF}}$. Take an arbitrary $\psi \in \mathcal{S}\left(\mathbb{R}^d\right)$. Then,

$$\mathcal{U}_{\mathrm{BF}}\left(-\Delta\psi\right)\left(k, x\right) = \frac{1}{\left|Y^*\right|^{1/2}} \sum_{\gamma\in\Gamma} e^{-ik\cdot(x+\gamma)}\left(-\Delta\psi\right)\left(x+\gamma\right) \tag{28}$$

$$= \frac{1}{\left|Y^*\right|^{1/2}} \sum_{\gamma\in\Gamma} \left(-i\nabla_x + k\right) e^{-ik\cdot(x+\gamma)}\left(-i\nabla_x\psi\right)\left(x+\gamma\right) \tag{29}$$

$$= \frac{1}{\left|Y^*\right|^{1/2}} \sum_{\gamma\in\Gamma} A\left(k\right) e^{-ik\cdot(x+\gamma)}\psi\left(x+\gamma\right) \tag{30}$$

holds. Since the sum converges uniformly for $x \in Y$, the sum and differential can be interchanged and we get

$$A\left(k\right) \frac{1}{\left|Y^*\right|^{1/2}} \sum_{\gamma\in\Gamma} e^{-ik\cdot(x+\gamma)}\psi\left(x+\gamma\right) \tag{31}$$

and this equals to $\left(A\left(\mathcal{U}_{\mathrm{BF}}\psi\right)\right)\left(k, x\right)$. For each $k \in Y^*$,

$$\left(A\left(\mathcal{U}_{\mathrm{BF}}\psi\right)\right)\left(k, x+\gamma\right) = \left(A\left(\mathcal{U}_{\mathrm{BF}}\psi\right)\right)\left(k, x\right) \tag{32}$$

for all $\gamma \in \Gamma$, thus $\left(A\left(\mathcal{U}_{\mathrm{BF}}\psi\right)\right)\left(k, \cdot\right) \in \mathcal{H}'$. This proves $\mathcal{U}_{\mathrm{BF}}\psi \in D\left(A\right)$. In the same manner, we can prove

$$\mathcal{U}_{\mathrm{BF}} V\left(x\right) \mathcal{U}_{\mathrm{BF}}^{-1} = \int_{Y^*}^{\oplus} V\left(x\right)\ dk. \tag{33}$$

Let B be the operator on the right hand side of Eq. (33). By noting that

(34) $$|(\psi, V\psi)_{\mathcal{H}'}| \leq \|V\|_{\mathcal{H}'} (\psi, \psi)_{\mathcal{H}'} = 0 \times (\psi, A(k)\psi) + \beta (\psi, \psi)_{\mathcal{H}'}$$

holds for all $k \in Y^*$ and $\psi \in W^{2,2}(\mathbb{T}^d)$ where $\beta = \|V\|_{\mathcal{H}'}$ and using **Theorem XIII.85 (g)** in [20], we conclude that $\hat{H}_{\mathrm{BF}} = \int_{Y^*}^{\oplus} \hat{H}(k)\ dk$ is self-adjoint on $W^{2,2}(\mathbb{T}^d)$ as well and this proves the claim in Eq. (23).

Note that $\lambda \in \sigma\left(\hat{H}_{\mathrm{BF}}\right)$ if and only if

(35) $$\left|\left\{k \middle| \sigma\left(\hat{H}(k)\right) \cap (\lambda - \epsilon, \lambda + \epsilon) \neq \emptyset\right\}\right| > 0$$

holds for all $\epsilon > 0$, where $|\cdot|$ is the Lebesgue measure on Y^* by **Theorem XIII.85 (d)** in [20]. By using this fact, we can restore the spectrum of $\hat{H}_{\mathrm{BF}}$ from the spectrum of $\hat{H}(k)$ for each $k \in Y^*$.

2.4. **Spectral Structures of $\hat{H}(k)$.** Suppose $k \in Y^*$. We investigate the spectral structures of the operator $\hat{H}(k)$ on $\mathcal{H}'$. To do that, let us investigate the spectral structures of the unperturbed operator $\hat{H}_0(k) = \frac{1}{2}(-i\nabla_x + k)^2$ on $\mathcal{H}'$. This operator is self-adjoint on $W^{2,2}(\mathbb{T}^d)$ bounded from below, and has the complete set of eigenvectors $\phi_n(x) = \frac{1}{|Y^*|^{1/2}} e^{i\sum_{j=1}^d n_j \gamma_j^* \cdot x}$ with the eigenvalues $\frac{1}{2}\left(\sum_{j=1}^d n_j \gamma_j^* + k\right)^2$ for $n \in \mathbb{Z}^d$. From this information, we deduce the spectral structures of $\hat{H}(k)$ in what follows. First, note that $\hat{H}_0(k)$ has a compact resolvent, which can be shown by using **Theorem XIII.64** in [20]. Second, note that V is in $L^2(\mathbb{T}^d)$ and is symmetric and satisfies Eq. (34) for all $k \in Y^*$ and $\psi \in W^{2,2}(\mathbb{T}^d)$. Then, $\hat{H}(k) = \hat{H}_0(k) + V$ is self-adjoint and bounded from below as well and has a compact resolvent, which can be shown by using **Theorem XIII.68** in [20]. Then, by using **Theorem XIII.64** in [20], we conclude that $\hat{H}(k)$ has a complete set of eigenvectors with eigenvalues $E_0(k) \leq E_1(k) \leq \cdots$ where $E_j(k) \to \infty$ as $j \to \infty$. Since

(36) $$H(k + \gamma^*) = \tau(\gamma^*)^{-1} H(k) \tau(\gamma^*)$$

holds for all $\gamma^* \in \Gamma^*$, $E_j(k)$ is Γ^*-periodic function of k for $j \in \mathbb{N} \cup \{0\}$. In the context of band theory in solid-state physics, the eigenvalues $E_j(k)$ parametrized by $k \in Y^*$ for each $j \in \mathbb{N} \cup \{0\}$ are called a band and we denote a band as a set of the eigenvalues parametrized by $k \in Y^*$ having a common index $j \in \mathbb{N} \cup \{0\}$.

3. Singularities in the spectral structures of the Schrödinger operator

In this section, we review our recent progress on classification of geometric structures of bands in a neighborhood of a band crossing in the bulk of a crystal ($d = 3$), under the condition that either time-reversal symmetry or space-inversion symmetry is broken [25]. Under this condition, band crossings, i.e, $E_j(k) = E_l(k)$ for $j \neq l$, occur only at a finite number of points $k \in Y^*$ in general. Among these band crossings, two-band crossings occur most generically and thus we first focus on a two-band crossing. Such band crossings are important because the band cannot change its topology unless its band gaps close.

Without loss of generality, we can assume a two-band crossing occurs at the origin $k = 0 \in \mathbb{R}^3$ in order to analyze the local geometry, and let $E_\pm(k)$ $(E_-(k) \leq E_+(k))$ be two bands involved in the crossing. Let $\sigma(k) = \{E_\pm(k)\}$ be the set of the eigenvalues. In addition, we assume that there exists an open neighborhood of the origin U $(\subset \mathbb{R}^d)$ in which the gap condition

(37) $$\inf_{k \in U} d\left(\sigma(k), \sigma\left(\hat{H}(k)\right) \setminus \sigma(k)\right) > 0$$

holds, where $d(\cdot,\cdot)$ is the Euclidean distance between the two sets. Under the gap condition, the projection operator $P(k) : \mathcal{H}' \to \mathcal{H}'$ can be defined by using the Dunford integral such as

$$P(k) = -\frac{1}{2\pi i}\int_C \left(\hat{H}(k) - z\right)^{-1} dz, \tag{38}$$

where the integration path C on $\mathbb{C}$ is chosen so that it encloses $\sigma(k)$ $(k \in U)$ counterclockwise. Under this setting, by using **Proposition 2.1.** in [17], the map $k \mapsto P(k)$ is of class C^∞ from $\mathbb{R}^d$ to $\mathcal{L}(\mathcal{H}')$ equipped with the operator norm. This implies that there exists an open neighborhood $U_0 \subset U$ in which $\|P(k) - P(0)\| < 1$ holds. In the open neighborhood, we can use Nagy's formula [12]

$$W(k) = \left(1 - (P(k) - P(0))^2\right)^{-1/2} (P(k)P(0) + (1 - P(k))(1 - P(0))) \tag{39}$$

to get a smooth orthogonal frame $\chi_j(k) = W(k)\chi_j(0)$ $(j = 1, 2)$ for

$$\text{Ran } P(k) = \{\psi \in \mathcal{H}' \,|\text{There exists } \psi' \in \mathcal{H}' \text{ such that } \psi = P(k)\psi' \text{ holds.}\} \ \ (k \in U_0) \tag{40}$$

where $\chi_j(0)$ $(j = 1, 2)$ is an orthogonal basis spanning Ran $P(0)$. By defining

$$H_{jl}(k) = \left(\chi_j(k), \hat{H}(k)\chi_l(k)\right)$$

for $j, l = 1, 2$, the map

$$H : k \mapsto \begin{pmatrix} H_{11}(k) & H_{12}(k) \\ H_{21}(k) & H_{22}(k) \end{pmatrix} \tag{41}$$

is a C^∞ map from U_0 to the set of 2×2 Hermite matrices and the two eigenvalues $E_\pm(k)$ can be written as

$$E_\pm(k) = \frac{H_{11}(k) + H_{22}(k) \pm \sqrt{(H_{11}(k) - H_{22}(k))^2 + \overline{H_{12}(k)}H_{21}(k)}}{2}. \tag{42}$$

If we consider the relative difference between the two eigenvalues, the trace part of the matrix

$$\frac{H_{11}(k) + H_{22}(k)}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{43}$$

is irrelevant and thus we subtract the trace part so that the target image of the map H is in the set of 2×2-traceless Hermite matrix $\text{Herm}_0(2)$ for $k \in U_0$. Since we assume $E_+(0) = E_-(0)$, the map should satisfy $H_{11}(0) = H_{22}(0)$ and $\overline{H_{12}(0)}H_{21}(0) = |H_{21}(0)|^2 = 0$. In conjunction with Trace $H(0) = H_{11}(0) + H_{22}(0) = 0$, we get $H(0) = O_2$ where O_2 is the 2×2 zero matrix. Under this setting, the map H can be written as $H\colon \left(\mathbb{R}^3, 0\right) \to (\text{Herm}_0(2), O_2)$. Having this setting in mind, we introduce our framework [25, 11] to classify Hamiltonians in a neighborhood of a multi-band crossing in the next section.

3.1. **Settings.** Let $M_m(\mathbb{C})$ be the set of $m \times m$ complex matrices, $\text{Herm}_0(m)$ be the set of $m \times m$ trace-less Hermite matrices

$$\text{Herm}_0(m) = \left\{X \in M_m(\mathbb{C}) \,\middle|\, X^\dagger = X, \text{Trace } X = 0\right\}, \tag{44}$$

and $SU(m)$ be the set of $m \times m$ special unitary matrices

$$SU(m) = \left\{X \in M_m(\mathbb{C}) \,\middle|\, X^\dagger X = XX^\dagger = I_m, \det X = 1\right\}, \tag{45}$$

where I_m is the $m \times m$ unit matrix. Let $H, H'\colon (\mathbb{R}^n, 0) \to (\text{Herm}_0(m), O_m)$ be C^∞ map-germs where $n \in \mathbb{N}$ and O_m is the $m \times m$ zero matrix.

Definition 3.1. *We say that H and H' are $\mathcal{SU}(m)$-equivalent if there exists a map-germ $U\colon (\mathbb{R}^n, 0) \to (SU(m), U(0))$ and a diffeomorphism-germ $s\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $H \circ s(k) = U(k) H'(k) U^\dagger(k)$ holds for all $k \in \mathbb{R}^n$.*

For example, the case where $m = 2$ and $n = 3$ corresponds to the geometric classification of Hamiltonians in the bulk of a crystal in a neighborhood of a two-band crossing. In this case, a map-germ $H\colon \left(\mathbb{R}^3, 0\right) \to (\mathrm{Herm}_0(2), O_2)$ can be written as

$$H\colon k \mapsto \begin{pmatrix} \delta(k) & \beta(k) - i\gamma(k) \\ \beta(k) + i\gamma(k) & -\delta(k) \end{pmatrix} \tag{46}$$

$$= \beta(k)\sigma_1 + \gamma(k)\sigma_2 + \delta(k)\sigma_3 \tag{47}$$

$$= (\beta(k), \gamma(k), \delta(k)) \cdot \sigma, \tag{48}$$

where $\beta, \gamma, \delta\colon \left(\mathbb{R}^3, 0\right) \to (\mathbb{R}, 0)$ are map-germs, $k = (k_1, k_2, k_3) \in \mathbb{R}^3$ a Bloch wavenumber,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \text{ and } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{49}$$

are three Pauli matrices, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, and $(\beta(k), \gamma(k), \delta(k)) \cdot \sigma$ is an inner product between the two vectors $(\beta(k), \gamma(k), \delta(k))$ and σ. If one considers a map-germ

$$H'\colon k \mapsto U(k) H(k) U^\dagger(k)$$

where $U\colon (\mathbb{R}^n, 0) \to (SU(m), U(0))$, the image of the map-germ H' is also in $(\mathrm{Herm}_0(2), O_2)$ and $H'(k)$ and $H(k)$ are unitary equivalent for $k \in \mathbb{R}^3$. Therefore, it is natural to consider the two map-germs H' and H as equivalent in their geometric classification of bands. Contrastingly, the role that the diffeomorphism-germ $s\colon \left(\mathbb{R}^3, 0\right) \to \left(\mathbb{R}^3, 0\right)$ plays in the definition may be strange in this context because the source space $\mathbb{R}^3$ is spanned by a Bloch wavenumber k and introducing arbitrary nonlinear transformations to that space is not at all natural. Depending on which geometrical features one wants to preserve, one can have several other choices:

1. Restrict a class of $s\colon \left(\mathbb{R}^3, 0\right) \to \left(\mathbb{R}^3, 0\right)$ to the set of orthogonal transformations.
2. Relax a class of $s\colon \left(\mathbb{R}^3, 0\right) \to \left(\mathbb{R}^3, 0\right)$ to the set of homeomorphisms.

In case of 1, surely all the details of the graph of the eigenvalues against k are preserved. To understand a phenomenon such as in [7], in which the star-like shape of the Fermi surface is essential, it is important not to miss the details. However, if you restrict a class of

$$s\colon \left(\mathbb{R}^3, 0\right) \to \left(\mathbb{R}^3, 0\right)$$

to the set of orthogonal transformations, you will end up with infinitely many classes as many as all the possible graphs of the eigenvalues against k and this classification may be too fine to be useful. Contrastingly, if you relax a class of $s\colon \left(\mathbb{R}^3, 0\right) \to \left(\mathbb{R}^3, 0\right)$ to the set of homeomorphisms, you may end up with a finite number of classes up to a certain codimension but you will miss important information like multiplicity, which tells the maximum possible number of generic band crossings that can appear if you perturb the Hamiltonians smoothly [25]. Here, we set a class of $s\colon \left(\mathbb{R}^3, 0\right) \to \left(\mathbb{R}^3, 0\right)$ to the set of diffeomorphisms as in the definition so that we can get a finite number of classes up to a certain codimension and at the same time we do not miss important quantities like multiplicity and Chern number.

Let $\mathcal{E}_n = \{f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}, f(0))\}$ be the ring of function-germs with the maximal ideal $\mathcal{M}_n$. Let $\mathcal{E}_{n,m} = \{H\colon (\mathbb{R}^n, 0) \to (\mathrm{Herm}_0(m), H(0))\}$ and

$$\mathcal{M}_n\mathcal{E}_{n,m} = \mathcal{M}_n\mathcal{E}_{n,m} = \{H\colon (\mathbb{R}^n, 0) \to (\mathrm{Herm}_0(m), O_m)\}.$$

For $\mathcal{SU}(m)$-equivalence, we define its tangent space $T\mathcal{SU}(m)$ at $H \in \mathcal{M}_n\mathcal{E}_{n,m}$ as the set of infinitesimal actions of map-germs U and s as

$$T\mathcal{SU}(m)(H) = \left\{ \left.\frac{\partial H_\epsilon(k)}{\partial \epsilon}\right|_{\epsilon=0} \middle| \begin{array}{c} H_\epsilon(k) = U_\epsilon(k)\, H \circ s_\epsilon(k)\, U_\epsilon^\dagger(k), \\ U_\epsilon\colon (\mathbb{R}^n, 0) \to (SU(m), U_\epsilon(0)), \\ s_\epsilon\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0), \\ U_{\epsilon=0} = I_m, s_{\epsilon=0} = \mathrm{id}_n \end{array} \right\} (\subset \mathcal{E}_{n,m}), \tag{50}$$

where $\mathrm{id}_n\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is the identity. In a similar manner, we define its extended tangent space as the set of infinitesimal actions of U and s that may map the origin to a point different from the origin as

$$T_e\mathcal{SU}(m)(H) = \left\{ \left.\frac{\partial H_\epsilon(k)}{\partial \epsilon}\right|_{\epsilon=0} \middle| \begin{array}{c} H_\epsilon(k) = U_\epsilon(k)\, H \circ s_\epsilon(k)\, U_\epsilon^\dagger(k), \\ U_\epsilon\colon (\mathbb{R}^n, 0) \to (SU(m), U_\epsilon(0)), \\ s_\epsilon\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, s_\epsilon(0)), \\ U_{\epsilon=0} = I_m, s_{\epsilon=0} = \mathrm{id}_n \end{array} \right\} (\subset \mathcal{E}_{n,m}). \tag{51}$$

Note that the tangent space $T\mathcal{SU}(m)(H)$ and the extended tangent space $T_e\mathcal{SU}(m)(H)$ are modules over $\mathcal{E}_n$. We define the codimension of $H \in \mathcal{M}_n\mathcal{E}_{n,m}$ as $\dim_{\mathbb{R}} \dfrac{\mathcal{E}_{n,m}}{T_e\mathcal{SU}(m)(H)}$, which is the dimension of the quotient module $\dfrac{\mathcal{E}_{n,m}}{T_e\mathcal{SU}(m)(H)}$ regarded as a vector space over $\mathbb{R}$. For example, if $n = 3, m = 2$ as in the example above and $H(k) = \left(k_1, k_2, k_3^\ell\right) \cdot \sigma$ where $\ell \in \mathbb{N}$, its tangent space, extended tangent space, quotient module, and codimension are:

$$\begin{aligned} T\mathcal{SU}(m)(H) = {} & \langle (-k_2, k_1, 0) \cdot \sigma, \left(k_3^\ell, 0, -k_1\right) \cdot \sigma, \left(0, -k_3^\ell, k_2\right) \cdot \sigma \rangle_{\mathcal{E}_n} \\ & + \mathcal{M}_n \langle (1, 0, 0) \cdot \sigma, (0, 1, 0) \cdot \sigma, \left(0, 0, \ell k_3^{\ell-1}\right) \cdot \sigma \rangle_{\mathcal{E}_n}, \end{aligned} \tag{52}$$

$$\begin{aligned} T_e\mathcal{SU}(m)(H) = {} & \langle (-k_2, k_1, 0) \cdot \sigma, \left(k_3^\ell, 0, -k_1\right) \cdot \sigma, \left(0, -k_3^\ell, k_2\right) \cdot \sigma, \\ & (1, 0, 0) \cdot \sigma, (0, 1, 0) \cdot \sigma, \left(0, 0, \ell k_3^{\ell-1}\right) \cdot \sigma \rangle_{\mathcal{E}_n}, \end{aligned} \tag{53}$$

$$\frac{\mathcal{E}_{n,m}}{T_e\mathcal{SU}(m)(H)} = \langle (0, 0, 1) \cdot \sigma, (0, 0, k_3) \cdot \sigma, \cdots, \left(0, 0, k_3^{\ell-2}\right) \cdot \sigma \rangle_{\mathbb{R}}, \tag{54}$$

and

$$\dim_{\mathbb{R}} \frac{\mathcal{E}_{n,m}}{T_e\mathcal{SU}(m)(H)} = \ell - 1,$$

respectively, where $\langle \cdots \rangle_A$ is the A-module generated by the elements in the bracket. Under this setting, we get the following classification of $\mathcal{M}_3\mathcal{E}_{3,2}$ under $\mathcal{SU}(m)$-equivalence [25]. In [25], the classes represented by the map-germs

$$\left(k_1, k_2 k_3, k_2^2 \pm r k_3^{\ell+2}\right) \cdot \sigma \ \ (r > 0, \ell = 3, 4, 5) \tag{55}$$

are missing and we correct the result by adding the representatives to Table 1. The detail of the correction is reported in [26].

3.2. Classification of $\mathcal{M}_3\mathcal{E}_{3,2}$ under $\mathcal{SU}(2)$-equivalence.

Theorem 3.1 ([25])**.** *If the codimension of a map-germ in $\mathcal{M}_3\mathcal{E}_{2,3}$ is less than 8, the map-germ is $\mathcal{SU}(2)$-equivalent to one and only one of the map-germs listed in Table. 1.*

$\hat{H}(k)$	*ranges*	*mult*	*Ch*$_\pm$	*codim*
$(k_1, k_2, k_3) \cdot \sigma$		1	∓ 1	0
$\left(k_1, k_2, k_3^\ell\right) \cdot \sigma$	$\ell = 2, \cdots, 8$	ℓ	$\begin{cases} \mp 1 & (\ell\text{:}odd) \\ 0 & (\ell\text{:}even) \end{cases}$	$\ell - 1$
$\left(k_1, k_2^2, k_3^2 + rk_2^2\right) \cdot \sigma$	$r \in [0, \infty)$	4	0	5
$\left(k_1, k_2k_3, \frac{r}{2}\left(k_2^2 - k_3^2\right)\right) \cdot \sigma$	$r \in (0, 1)$	*4*	± 2	5
$\left(k_1, k_2k_3, k_2^2 + rk_3^{\ell+2}\right) \cdot \sigma$	$r \in (0, \infty), \ell = 1, 3$	$\ell + 4$	± 1	$\ell + 4$
$\left(k_1, k_2k_3, k_2^2 + rk_3^4\right) \cdot \sigma$	$r \in (0, \infty)$	6	0	6
$\left(k_1, k_2k_3, k_2^2 - rk_3^4\right) \cdot \sigma$	$r \in (0, \infty)$	6	± 2	6
$\left(k_1, k_2^2 - k_3^2 + rk_3^3, 2k_2k_3\right) \cdot \sigma$	$r \in (0, \infty)$	4	± 2	7
$\left(k_1, k_2^2 \pm k_3^2, rk_3^3\right) \cdot \sigma$	$r \in (0, \infty)$	6	0	7

TABLE 1. List of map-germs in each class of codimension less than 8 where "ranges" are possible ranges for the parameters r and ℓ, "mult" multiplicity, "Ch$_\pm$" Chern numbers of the upper and lower energy levels, and "codim" $\mathcal{SU}(2)_e$-codimension.

Here we define the multiplicity [14] and Chern number [27, 3, 23] as follows: Let

$$\hat{H}(k) = (\beta(k), \gamma(k), \delta(k)) \cdot \sigma \tag{56}$$

be a map-germ. Let $\langle \beta, \gamma, \delta \rangle_{\mathcal{E}_3}$ be the ideal in $\mathcal{E}_3$ generated by the matrix elements of the map-germ. We define the multiplicity of the map-germ as

$$\dim_{\mathbb{R}} \mathcal{E}_3 / \langle \beta, \gamma, \delta \rangle_{\mathcal{E}_3}, \tag{57}$$

i.e., the dimension of the quotient ring $\mathcal{E}_3 / \langle \beta, \gamma, \delta \rangle_{\mathcal{E}_3}$ regarded as a vector space over $\mathbb{R}$. Next, we define the Chern number. Here, we assume $\hat{H}(k) \neq (0,0,0) \cdot \sigma$ except for the origin $k = \mathbf{0}$. In this case, two eigenfunctions of the matrix, $\psi^{(\pm)}(k)$, can be chosen so that they depend smoothly on the Bloch wavenumber k $\left(\in \mathbb{R}^3\right)$ except for the origin $k = \mathbf{0}$. Let their corresponding eigenvalues be

$$E^{(\pm)}(k) \;\; \left(E^{(+)}(k) \geq E^{(-)}(k)\right). \tag{58}$$

Note that $\hat{H}(k)\psi^{(\pm)}(k) = E^{(\pm)}(k)\psi^{(\pm)}(k)$ holds. In terms of the two eigenfunctions, Berry curvatures are defined as

$$B^{(\pm)}(k) = i \sum_{j,j'=1}^{3} \frac{\partial}{\partial k_j} \left(\psi^{(\pm)}(k)^* \cdot \frac{\partial \psi^{(\pm)}(k)}{\partial k_{j'}}\right) dk_j \wedge dk_{j'}, \tag{59}$$

for $k \neq 0$. Note that the Berry curvature is well-defined except for the origin $k = \mathbf{0}$. Let S be an arbitrary 2-dimensional sphere enclosing the origin $k = \mathbf{0}$. Then, the Chern number is defined as

$$\mathrm{Ch}_\pm = \frac{1}{2\pi} \int_S B^{(\pm)}(k). \tag{60}$$

This number does not depend on how we choose the sphere S as long as the sphere encloses the origin. For the calculation of the multiplicity and Chern number, see [25].

In this classification, the class of codimension 0 is the most generic class and its normal form has a Weyl point at the origin $k = 0$. Other classes of higher codimension appear on verges of bifurcations. Bands cannot change their topology without colliding with others and these classes are expected to provide invaluable information on which types of geometric changes happen if two bands collide with each other.

When we presented this result in front of Prof. Goo Ishikawa in a workshop of differential geometry and singularity theory and their applications in Morioka, Japan, 2017,
Prof. Goo Ishikawa pointed out that band crossings among three or higher number of bands might be relevant for such a high codimension as 8. To answer Prof. Goo Ishikawa's question, we would like to show a list of lower bounds of codimensions of map-germs in $\mathcal{E}_{n,m}$ under $\mathcal{SU}(m)$-equivalence. The codimension of a map-germ in $\mathcal{E}_{n,m}$ having an m-fold degeneracy at the origin should be larger than this lower bound.

3.3. Lower bound of codimension of map-germs in $\mathcal{M}_n\mathcal{E}_{n,m}$ under $\mathcal{SU}(m)$-equivalence.

Let $\gamma_j \in \mathrm{Herm}_0(m)$ $\left(j = 1, \cdots, m^2-1\right)$ be bases of $\mathrm{Herm}_0(m)$.

Theorem 3.2. *Codimension of a C^∞ map-germ $H \in \mathcal{M}_n\mathcal{E}_{n,m}$ is equal to or greater than*

$$\max_{d\in\mathbb{N}\cup\{0\}} \left\{ \left(m^2-1\right) \frac{(n+d-1)!}{(n-1)!d!} - n \sum_{d'=0}^{d} \frac{(n+d'-1)!}{(n-1)!d'!} \right\}. \tag{61}$$

Proof. Take an arbitrary $H \in \mathcal{M}_n\mathcal{E}_{n,m}$ and $d \in \mathbb{N}\cup\{0\}$. We estimate the lower bound of its codimension. First note that

$$\dim_{\mathbb{R}} \frac{\mathcal{E}_{n,m}}{T_e\mathcal{SU}(m)(H)} \geq \dim_{\mathbb{R}} \frac{\mathcal{E}_{n,m}}{T_e\mathcal{SU}(m)(H) + \mathcal{M}_n^{d+1}\mathcal{E}_{n,m}} \tag{62}$$

holds.

Second note that $\dfrac{\mathcal{E}_{n,m}}{T_e\mathcal{SU}(m)(H) + \mathcal{M}_n^{d+1}\mathcal{E}_{n,m}}$ is isomorphic to

$$\frac{\mathcal{E}_{n,m}/\mathcal{M}_n^{d+1}\mathcal{E}_{n,m}}{\left(T_e\mathcal{SU}(m)(H) + \mathcal{M}_n^{d+1}\mathcal{E}_{n,m}\right)/\mathcal{M}_n^{d+1}\mathcal{E}_{n,m}} \tag{63}$$

by using **(2.6) Theorem. (Third isomorphism theorem)** in [1]. $\mathcal{E}_{n,m}/\mathcal{M}_n^{d+1}\mathcal{E}_{n,m}$ is an $\left(m^2-1\right)\left(\sum_{d'=0}^{d} \dfrac{(n+d'-1)!}{(n-1)!d'!}\right)$-dimensional vector space over $\mathbb{R}$.

$$\left(T_e\mathcal{SU}(m)(H) + \mathcal{M}_n^{d+1}\mathcal{E}_{n,m}\right)/\mathcal{M}_n^{d+1}\mathcal{E}_{n,m} \tag{64}$$

is a vector space over $\mathbb{R}$ spanned by

$$k_1^{d_1}k_2^{d_2}\cdots k_n^{d_n}\frac{\partial H(k)}{\partial k_j} + \mathcal{M}_n^{d+1}\mathcal{E}_{n,m}$$

for $j = 1, \cdots, n$ and $d_1 + d_2 + \cdots + d_n \leq d$ and $k_1^{d_1}k_2^{d_2}\cdots k_n^{d_n}\left[\gamma_j, H(k)\right] + \mathcal{M}_n^{d+1}\mathcal{E}_{n,m}$ for $j = 1, \cdots, n$ and $d_1 + d_2 + \cdots + d_n \leq d-1$ where $[A, B] = AB - BA$ for $A, B \in \mathrm{Herm}_0(m)$, which is a vector space in $\mathcal{E}_{n,m}/\mathcal{M}_n^{d+1}\mathcal{E}_{n,m}$ of dimension at most

$$\left(\left(m^2-1\right)\left(\sum_{d'=0}^{\max\{d-1,0\}} \frac{(n+d'-1)!}{(n-1)!d'!}\right) + n\sum_{d'=0}^{d} \frac{(n+d'-1)!}{(n-1)!d'!}\right).$$

Therefore,

$$\begin{aligned}(65)\quad \dim_{\mathbb{R}} \frac{\mathcal{E}_{n,m}/\mathcal{M}_n^{d+1}\mathcal{E}_{n,m}}{\left(T_e\mathcal{SU}\left(m\right)\left(H\right)+\mathcal{M}_n^{d+1}\mathcal{E}_{n,m}\right)/\mathcal{M}_n\mathcal{E}_{n,m}} &\geq \left(m^2-1\right)\left(\sum_{d'=0}^{d}\frac{(n+d'-1)!}{(n-1)!d'!}\right)\\ &-\left(\left(m^2-1\right)\left(\sum_{d'=0}^{\max\{d-1,0\}}\frac{(n+d'-1)!}{(n-1)!d'!}\right)+n\sum_{d'=0}^{d}\frac{(n+d'-1)!}{(n-1)!d'!}\right),\\ &=\left(m^2-1\right)\frac{(n+d-1)!}{(n-1)!d!}-n\sum_{d'=0}^{d}\frac{(n+d'-1)!}{(n-1)!d'!}\end{aligned}$$

holds. Since $d\in\mathbb{N}\cup\{0\}$ is arbitrary, this proves the theorem. $\square$

If we set $n=3$, we get lower bounds of codimensions for $m=2,3,4,5,6$ in Table. 2. The results in Table 2 imply multiple band crossings may be less generic compared to two band crossings.

m	2	3	4	5	6
codimension ($\geq$)	0	20	180	840	2783

TABLE 2. Lower bounds of codimensions relative to $\mathcal{SU}\left(m\right)$-equivalence for $n=3$ estimated by Eq. (61).

However, if we consider the codimension of a moduli family of map-germs, it can still have a small codimension. To investigate it, we need to classify $\mathcal{E}_{n,m}$ not based on the codimension of the extended tangent space in [25] but that substracted by the number of moduli parameters.

4. Future Perspectives

So far we have classified local geometric structures of bands in a neighborhood of a two-band crossing by classifying underlying Hamiltonians in the bulk of a crystal when either time-reversal symmetry or spacial inversion symmetry is broken. We have also estimated lower bounds of codimension for multi-band crossings. This should be the first step to understand global geometric structures of bands and their bifurcations. Steps further along this line of research are: Classification of local geometric structures of bands in a neighborhood of a multi-band crossing

(1) relative to a Fermi level.
(2) on a surface.
(3) in the bulk under time-reversal and spacial-inversion symmetries.

Point 1 is important to study the geometry of the Fermi surface, i.e., the intersection between bands and a Fermi level. For example, the geometry of a Fermi surface determines a type of semimetails [24]. This requires studying not only relative differences between bands, but also their differences relative to a Fermi level. We also need to take the trace part of Eq. (43) into account to classify geometric structures of Fermi surfaces.

Point 2 is important to understand geometric structures of bands on a surface, such as a Dirac-cone [7, 29, 4] and is also important for studying its engineering [9]. The geometric structures depend strongly on a crystallographic symmetry and the presence or absence of time-reversal symmetry and thus it is important to take these symmetries into account. This can be done if we extend our framework [25, 11] to an equivariant framework.

If the effective Hamiltonian of a crystal has a spin degree of freedom and symmetry as in Point 3, every band has two-fold degeneracy such as $E_0(k) = E_1(k) \leq E_2(k) = E_3(k) \leq \cdots$ for $k \in Y^*$ and it is important to take the degeneracy along with symmetries into account. Under this condition, band crossings that occur most generically are crossings of two pairs of bands. To classify geometric structures of such crossings, we need to classify 4×4 Hamiltonians instead of 2×2 ones because four bands are involved in the crossings. Such crossings appearing at time reversal invariant momentum (TRIM) points play a major role for topological properties of global band structures [8, 18]. Bifurcations occurring at TRIMs are shown to trigger topological changes in a lattice model in Chapter 3 in [22]. To study such bifurcations in our framework, we need to extend our framework [25, 11] to a framework in a multi-germ setting.

5. Acknowledgment

H.T. thanks Prof. Goo Ishikawa for his valuable comments on this study. This paper is an extended version of lecture notes by H.T. in the Singularity Workshop in Biwako in 2017 and H.T. thanks all of the organizers and participants of the workshop. This paper was finalized during his visit to UAM on the occasion of the TraX International Conference 2018 and H.T. thanks all the organizers especially Prof. Thomas Bartsch, Prof. Florentino Borondo Rodríguez, Prof. Rosa María Benito Zafrilla and Prof. Fabio Revuelta for their hospitality during his stay in Madrid. H.T. was supported by the Grant-in-Aid for Young Scientists (B) (No. 16K1770) and JST PRESTO Grant Number JPMJPR16E8, Japan. K.K. was partially supported by the Grant-in-Aid for Scientific Research (C) from JSPS (No. 16K04872) and by the Center for Spintronics Research Network(CSRN), Tohoku University. S.I. is supported by the Grant-in-Aid for Scientific Research (B) from JSPS (No. 26287009) and partially supported by the Grant-in-Aid for Scientific Research (A) from JSPS (No. 26247006). M.T. is supported by the Cooperative Research Program of the Network Joint Research Center for Materials and Devices, the Dynamic Alliance for Open Innovation Bridging Human, Environment and Materials, Grants-in-Aid for Challenging Exploratory Research (No. 22654047 and No. 25610105 to M.T.) from Japan Society for the Promotion of Science (JSPS), Grants-in-Aid for Scientific Research (C) (No. 24540394, No. 26400421, and No. 19K03653 to M.T.) from Japan Society for the Promotion of Science (JSPS). T.K. is supported by the Grant-in-Aid for Scientific Research (B) (No. 15KT0055).

References

1. W. A. Adkins and S. H. Weintrab, *Algebra, An Approach via Module Theory*, 6 ed., vol. A, Springer-Verlag, 1992.
2. M. I. Aroyo (ed.), *International Table for Crystallography*, 6 ed., vol. A, Wiley, 2016.
3. M. V. Berry, *Quantal Phase Factors Accompanying Adiabatic Changes*, Proc. R. Soc. Lond. **A392** (1983), 45–57.
4. C. X. Liu and X. L. Qi and H. Zang and X. Dai and Z. Fang and S. C. Zhang, *Model Hamiltonian for topological insulators*, Phys. Rev. B **82** (2010), 045122-1–045122-19.
5. E. Cancés, A. Deleurence, and M. Lewin, *A new approach to the modeling of local defects in crystals: The reduced hartree-fock case*, Commun. Math. Phys. **281** (2008), 129–177.
6. I. Catto, C. L. Bris, and P. L. Lions, *On the thermodynamic limit for hartree-fock type models*, Ann. I. H. Poincaré **6** (2001), 687–760.
7. L. Fu, *Hexagonal Warping Effects in the Surface States of the Topological Insulator Bi_2Te_3*, Phys. Rev. Lett. **103** (2009), 266801-1–266801-4.
8. M. Z. Hasan and C. L. Kane, *Colloquium: Topological insulators*, Rev. Mod. Phys. **82** (2010), 3045–3067. DOI: 10.1103/revmodphys.82.3045
9. K. Honma, T. Sato, S. Souma, K. Sugawara, Y. Tanaka, and T. Takahashi, *Switching of Dirac-Fermion Mass at the Interface of Ultrathin Ferromagnet and Rashba Metal*, Phys. Rev. Lett. **115** (2015), 266401-1–266401-5.
10. G. Ishikawa, *Singularity Song*, `http://www.math.sci.hokudai.ac.jp/~ishikawa/ondo.html`, 2003.

11. S. Izumiya, M. Takahashi, and H. Teramoto, *Geometric equivalence among smooth section-germs of vector bundles with respect to structure groups*, in preparation.
12. T. Kato, *Perturbation theory for linear operators*, Classics in mathematics, Springer, 1976.
13. E. H. Lieb and B. Simon, *The thomas-fermi theory of atoms, molecules and solids*, Adv. Math. **23** (1977), 22–116.
14. J. N. Mather, *Stability of C^∞ mappings, IV: Classification of stable germs by $\mathbb{R}$ algebras*, Publ. Math. I. H. E. S. **37** (1969), 223–248.
15. D. Monaco and G. Panati, *Symmetry and Localization in Periodic Crystals: Triviality of Bloch Bundles with a Fermionic Time-Reversal Symmetry*, Acta Appl. Math. **137** (2015), 185–203. DOI: 10.1007/s10440-014-9995-8
16. G. Panati, *Triviality of Bloch and Bloch-Dirac Bundles*, Ann. Henri Poincaré **8** (2007), 995–1011. DOI: 10.1007/s00023-007-0326-8
17. G. Panati and A. Pisante, *Bloch Bundles, Marzari-Vanderbilt Functional and Maximally Localized Wannier Functions*, Comm. Math. Phys. **322** (2013), 835–875. DOI: 10.1007/s00220-013-1741-y
18. X.-L. Qi and S.-C. Zhang, *Topological insulators and superconductors*, Rev. Mod. Phys. **83** (2011), 1057–1110. DOI: 10.1103/revmodphys.83.1057
19. M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, vol. II, Academic Press, 1975.
20. ———, *Methods of Modern Mathematical Physics*, vol. IV, Academic Press, 1978.
21. ———, *Methods of Modern Mathematical Physics*, vol. I, Academic Press, 1980.
22. S.-Q. Shen, *Topological Insulators, Dirac Equation in Condensed Matters*, 1 ed., Springer, 2012.
23. B. Simon, *Holonomy, the Quantum Adiabatic Theorem, and Berry's Phase*, Phys. Rev. Lett. **51** (1983), 2167–2170. DOI: 10.1103/physrevlett.51.2167
24. A. A. Soluyanov, D. Gresch, Z. Wang, Q. Wu, M. Troyer, X. Dai, and B. A. Bernevig, *Type-II Weyl semimetals*, Nature **527** (2015), 495–498. DOI: 10.1038/nature15768
25. H. Teramoto, K. Kondo, S. Izumiya, M. Toda, and T. Komatsuzaki, *Classification of Hamiltonians in neighborhoods of band crossings in terms of the theory of singularities*, J. Math. Phys. **58** (2017), 073502-1–073502-39.
26. ———, *Erratum: "Classification of Hamiltonians in neighborhoods of band crossings in terms of the theory of singularities" [J. Math. Phys. 58, 073502 (2017)]*, J. Math. Phys. **60** (2019), 129901-1–129901-2.
27. D. J. Thouless, M. Kohmoto, P. Nightingale, and M. den Nijs, *Quantized Hall Conductance in a Two-Dimensional Periodic Potential*, Phys. Rev. Lett. **49** (1982), 405–408.
28. J. Zak, *Dynamics of electrons in solids in external fields*, Phys. Rev. **168** (1968), 686–695. DOI: 10.1103/physrev.168.686
29. H. Zhang, C.-X. Liu, X.-L. Gi, X. Dai, Z. Fang, and S.-C. Zhang, *Topological insulators in Bi_2Se_3, Bi_2Te_3 and Sb_2Te_3 with a single Dirac cone on the surface*, Nat. Phys. **5** (2009), 438–442.

Journal *of* Singularities
Volume 21 (2020), 282-300

Proc. of Geometric and Algebraic Singularity Theory, Będlewo, 2017
DOI: 10.5427/jsing.2020.21q

A DESCRIPTION OF A RESULT OF DELIGNE BY LOG HIGHER ALBANESE MAP

SAMPEI USUI

Dedicated to Goo Ishikawa on his sixtieth birthday

Abstract. In a joint work with Kazuya Kato and Chikara Nakayama, log higher Albanese manifolds were constructed as an application of log mixed Hodge theory with group action. In this framework, we describe a work of Deligne on some nilpotent quotients of the fundamental group of the projective line minus three points, where polylogarithms appear. As a result, we have q-expansions of higher Albanese maps at boundary points, i.e., log higher Albanese maps over the boundary.

0. Introduction

We review the results of [11]: - General theory of log mixed Hodge structures with polarizable graded quotients endowed with group actions. - Description of the functors represented by higher Albanese manifolds in terms of tensor functors. - Toroidal partial compactifications of higher Albanese manifolds to get log higher Albanese manifolds, and describe the functors represented by them.

We describe a result of Deligne in [3] about polylogarithms, which were appeared in higher Albanese maps, in terms of the log higher Albanese maps. The advantage of our formulation is that log higher Albanese maps have q expansions at the boundary points over which we observe directly $\zeta(n)$ $(n \geq 2)$ as values of polylogarithms.

For readers' convenience, we add as Appendix a summary of the related result of Deligne in [3].

Actually, for the present description in Section 3, it is enough to use the formulation of spaces of nilpotent orbits in [10] Part III. The formulation of them in [11] is reviewed in Sections 1 and 2 for further study in the case of higher Albanese manifolds with non-trivial Hodge structures.

1. Log mixed Hodge structures with group action

We review general formulations and results of log mixed Hodge structures with group action in [11] and [10] Parts III, IV, in a minimal size for the later use of this paper. The full version will be appeared in [10] Part V.

1.1. A *log structure* on a ringed space $(S, \mathcal{O}_S)$ consists of a sheaf of monoids M on S and a homomorphism $\alpha : M \to \mathcal{O}_S$ such that $\alpha^{-1}(\mathcal{O}_S^\times) \xrightarrow{\sim} \mathcal{O}_S^\times$.

1.2. Example. Let $S = \mathbf{C}$ and $\{0\}$ a divisor. The associated log structure is

$$M = \{f \in \mathcal{O}_S \mid f \text{ is invertible on } S \smallsetminus \{0\}\}.$$

2010 *Mathematics Subject Classification.* Primary 14C30; Secondary 14D07, 32G20.

Key words and phrases. Hodge theory, log Hodge structure, log higher Albanese map, polylogarithm, zeta value.

$S^{\log}$ is defined to be the set of all pairs (s, h) consisting of a point $s \in S$ and an argument function h which is a homomorphism $M_s \to \mathbf{S}^1$ of monoids whose restriction to $\mathcal{O}_{S,s}^{\times}$ is $u \mapsto u(s)/|u(s)|$.

In this case, the ringed space $(S^{\log}, \mathcal{O}_S^{\log})$ is explained as follows. Let

$$\tilde{S}^{\log} := \mathbf{C} \cup (\mathbf{R} \times i\infty) = \mathbf{R} \times i(\mathbf{R} \cup \infty)$$

endowed with coordinate function $z = x + iy$ $(-\infty < y \le \infty)$. Let $S^{\log} := (\mathbf{C} \cup (\mathbf{R} \times i\infty))/\mathbf{Z}$, and consider maps $\tilde{S}^{\log} \to S^{\log} \to S : z = x + iy \mapsto (e^{-2\pi y}, e^{2\pi i x}) \mapsto q := e^{2\pi i z}$. Note that $(e^{-2\pi y}, e^{2\pi i x})$ is a polar coordinate extended over $-\infty < y \le \infty$, and $S^{\log} \to S$ is a real oriented blowing-up at $\{0\}$, which is proper. $h : M_0 \to \mathbf{S}^1$ in $t := (0, h) \in S^{\log}$ sends q to $e^{2\pi i x}$. Since z is considered as a branch of $(2\pi i)^{-1} \log(q)$, we have $\mathcal{O}_{S,t}^{\log} = \mathcal{O}_{S,0}[z]$ which is isomorphic to a polynomial algebra $\mathcal{O}_{S,0}[T]$ of one indeterminate T over $\mathcal{O}_{S,0}$ under $z \leftrightarrow T$ ([12] 2.2.5).

For more general and finer treatment, see [9], [12] 2.2.

1.3. Let G be a linear algebraic group over $\mathbf{Q}$. Let G_u be the unipotent radical of G and let $G_{\mathrm{red}} = G/G_u$. Let $\mathrm{Rep}(G)$ be the category of finite-dimensional linear representations of G over $\mathbf{Q}$.

1.4. Let $k_0 : \mathbf{G}_m \to G_{\mathrm{red}}$ be a $\mathbf{Q}$-rational and central homomorphism. Assume that, for one (hence all) lifting $\mathbf{G}_{m,\mathbf{R}} \to G_{\mathbf{R}}$ of k_0, the adjoint action of $\mathbf{G}_{m,\mathbf{R}}$ on $\mathrm{Lie}\,(G_u)_{\mathbf{R}} = \mathbf{R} \otimes_{\mathbf{Q}} \mathrm{Lie}\,(G_u)$ is of weight ≤ -1.

Then, for any $V \in \mathrm{Rep}(G)$, the action of $\mathbf{G}_m$ on V via a lifting $\mathbf{G}_m \to G$ of k_0 defines an increasing filtration W on V over $\mathbf{Q}$, called *weight filtration*, which is independent of the lifting.

1.5. Assume that we are given a homomorphism $k_0 : \mathbf{G}_m \to G_{\mathrm{red}}$ as in 1.4. A *G-mixed Hodge structure* (*G-MHS*, for short) *of type* k_0 is an exact $\otimes$-functor ([4] 2.7) from $\mathrm{Rep}(G)$ to the category of $\mathbf{Q}$-mixed Hodge structures keeping the underlying vector spaces with weight filtrations.

1.6. As in [2], let $S_{\mathbf{C}/\mathbf{R}}$ be the Weil restriction of scalars of $\mathbf{G}_m$ from $\mathbf{C}$ to $\mathbf{R}$. It represents the functor $A \mapsto (\mathbf{C} \otimes_{\mathbf{R}} A)^{\times}$ for commutative rings A over $\mathbf{R}$. In particular, $S_{\mathbf{C}/\mathbf{R}}(\mathbf{R}) = \mathbf{C}^{\times}$, which is understood as $\mathbf{C}^{\times}$ regarded as an algebraic group over $\mathbf{R}$.

Let $w : \mathbf{G}_m \to S_{\mathbf{C}/\mathbf{R}}$ be the homomorphism induced from the natural map $A^{\times} \to (\mathbf{C} \otimes_{\mathbf{R}} A)^{\times}$.

1.7. The following (1) and (2) are equivalent:

(1) A finite-dimensional linear representation of $S_{\mathbf{C}/\mathbf{R}}$ over $\mathbf{R}$.

(2) A finite-dimensional $\mathbf{R}$-vector space V with a decomposition

$$V_{\mathbf{C}} := \mathbf{C} \otimes_{\mathbf{R}} V = \bigoplus_{p,q \in \mathbf{Z}} V_{\mathbf{C}}^{p,q}$$

such that, for any p, q, $V_{\mathbf{C}}^{q,p}$ is complex conjugate of $V_{\mathbf{C}}^{p,q}$ (Hodge decomposition).

For a finite-dimensional linear representation V of $S_{\mathbf{C}/\mathbf{R}}$, the corresponding decomposition is defined by

$$V_{\mathbf{C}}^{p,q} := \{v \in V_{\mathbf{C}} \mid [z]v = z^p \bar{z}^q v \text{ for } z \in \mathbf{C}^{\times}\}.$$

Here $[z]$ denotes $z \in \mathbf{C}^{\times}$ regarded as an element of $S_{\mathbf{C}/\mathbf{R}}(\mathbf{R})$.

1.8. Let H be a G-MHS of type k_0 (1.5). By 1.7 and Tannaka duality (cf. [4] 1.12 Théorème), the Hodge decompositions of gr^W of $H(V)$ for $V \in \mathrm{Rep}(G)$ give a homomorphism $S_{\mathbf{C}/\mathbf{R}} \to (G_{\mathrm{red}})_{\mathbf{R}}$ such that the composite $\mathbf{G}_m \overset{w}{\to} S_{\mathbf{C}/\mathbf{R}} \to (G_{\mathrm{red}})_{\mathbf{R}}$ coincides with k_0. We call this

$$S_{\mathbf{C}/\mathbf{R}} \to (G_{\mathrm{red}})_{\mathbf{R}}$$

the *associated homomorphism with* H.

1.9. Let $k_0 : \mathbf{G}_m \to G_{\mathrm{red}}$ be as in 1.4. Fix a homomorphism $h_0 : S_{\mathbf{C}/\mathbf{R}} \to (G_{\mathrm{red}})_{\mathbf{R}}$ such that $h_0 \circ w = k_0$.

G-mixed Hodge structure of type h_0 is a G-mixed Hodge structure of type k_0 (1.5) whose associated homomorphism $S_{\mathbf{C}/\mathbf{R}} \to (G_{\mathrm{red}})_{\mathbf{R}}$ (1.8) is $G_{\mathrm{red}}(\mathbf{R})$-conjugate to h_0.

The *period domain* $D = D(G, h_0)$ associated to (G, h_0) is defined to be the set of isomorphism classes of G-mixed Hodge structures of type h_0.

1.10. Usual period domains of Griffiths [5] and their generalization for mixed Hodge structures [13] are special cases of the present period domains.

Let $\Lambda = (H_0, W, (\langle\ ,\ \rangle_w)_w, (h^{p,q})_{p,q})$ be the Hodge data as usual as in [10] Part III. Let G be the subgroup of $\mathrm{Aut}(H_{0,\mathbf{Q}}, W)$ consisting of elements which induce *similitudes for* $\langle\ ,\ \rangle_w$ for each w. That is, $G := \{g \in \mathrm{Aut}(H_{0,\mathbf{Q}}, W) \mid$ for any w, there is a $t_w \in \mathbf{G}_m$ such that $\langle gx, gy\rangle_w = t_w\langle x, y\rangle_w$ for any $x, y \in \mathrm{gr}^W_w\}$. Let $G_1 := \mathrm{Aut}(H_{0,\mathbf{Q}}, W, (\langle\ ,\ \rangle_w)_w) \subset G$.

Let $D(\Lambda)$ be the period domain of [13]. Then $D(\Lambda)$ is identified with an open and closed part of D in this paper as follows.

Assume that $D(\Lambda)$ is not empty and fix an $\mathbf{r} \in D(\Lambda)$. Then the Hodge decomposition of $\mathrm{gr}^W\mathbf{r}$ induces $h_0 : S_{\mathbf{C}/\mathbf{R}} \to (G_{\mathrm{red}})_{\mathbf{R}}$. (We have $\langle [z]x, [z]y\rangle_w = |z|^{2w}\langle x, y\rangle_w$ for $z \in \mathbf{C}^\times$ (see 1.6 for $[z]$).) Consider the associated period domain D (1.9). Then D is a finite disjoint union of $G_1(\mathbf{R})G_u(\mathbf{C})$-orbits which are open and closed in D. Let $\mathcal{D}$ be the $G_1(\mathbf{R})G_u(\mathbf{C})$-orbit in D consisting of points whose associated homomorphisms $S_{\mathbf{C}/\mathbf{R}} \to (D_{\mathrm{red}})_{\mathbf{R}}$ are $(G_1/G_u)(\mathbf{R})$-conjugate to h_0. Then the map $H \mapsto H(H_{0,\mathbf{Q}})$ gives a $G_1(\mathbf{R})G_u(\mathbf{C})$-equivariant isomorphism $\mathcal{D} \simeq D(\Lambda)$.

1.11. Fix a homomorphism $h_0 : S_{\mathbf{C}/\mathbf{R}} \to (G_{\mathrm{red}})_{\mathbf{R}}$ as in 1.9.

Let $\mathcal{C}$ be the category of triples (V, W, F) consisting of a finite-dimensional $\mathbf{Q}$-vector space V, an increasing filtration W on V (called the weight filtration), and a decreasing filtration F on $V_{\mathbf{C}}$ (called the Hodge filtration).

Let Y be the set of all isomorphism classes of exact $\otimes$-functors from $\mathrm{Rep}(G)$ to $\mathcal{C}$ preserving the underlying vector spaces with weight filtrations.

Then $G(\mathbf{C})$ acts on Y by changing the Hodge filtration F. We have $D \subset Y$ and D is a $G(\mathbf{R})G_u(\mathbf{C})$-orbit in Y (cf. [11] Proposition 3.2.5). We define $\check{D} := G(\mathbf{C})D$ in Y. Thus

$$D \subset \check{D} = G(\mathbf{C})D \subset Y.$$

$\check{D}$ is a $G(\mathbf{C})$-homogeneous space and D is an open subspace. Hence $\check{D}$ and D are complex analytic manifolds.

1.12. Let $h_0 : S_{\mathbf{C}/\mathbf{R}} \to (G_{\mathrm{red}})_{\mathbf{R}}$ be as in 1.9. Let C be the image of $i \in \mathbf{C}^\times = S_{\mathbf{C}/\mathbf{R}}(\mathbf{R})$ by h_0 in $(G_{\mathrm{red}})(\mathbf{R})$ (Weil operator). We say that h_0 is $\mathbf{R}$-*polarizable* if $\{a \in (G_{\mathrm{red}})'(\mathbf{R}) \mid Ca = aC\}$ is a maximal compact subgroup of $(G_{\mathrm{red}})'(\mathbf{R})$. Here $(G_{\mathrm{red}})'$ denotes the commutator subgroup of G_{red}.

1.13. Let $h_0 : S_{\mathbf{C}/\mathbf{R}} \to (G_{\mathrm{red}})_{\mathbf{R}}$ be $\mathbf{R}$-polarizable (1.12).

Let Γ be a subgroup of $G(\mathbf{Q})$ for which there is a faithful $V \in \mathrm{Rep}(G)$ and a $\mathbf{Z}$-lattice L in V such that L is stable under the action of Γ.

Then the following holds ([11] Proposition 3.3.4):
(1) The action of Γ on D is proper, and the quotient space $\Gamma \setminus D$ is Hausdorff.
(2) If Γ is torsion-free and if $\gamma p = p$ with $\gamma \in \Gamma$ for some $p \in D$, then $\gamma = 1$.
(3) If Γ is torsion-free, then the projection $D \to \Gamma \setminus D$ is a local homeomorphism.

1.14. Let (G, h_0) be as above.

A *nilpotent cone* is a cone σ over $\mathbf{R}_{\geq 0}$ in $\mathrm{Lie}\,(G)_{\mathbf{R}}$ generated by a finite number of mutually commuting elements such that, for any $V \in \mathrm{Rep}(G)$, the image of σ under the induced map $\mathrm{Lie}\,(G)_{\mathbf{R}} \to \mathrm{End}_{\mathbf{R}}(V)$ consists of nilpotent operators.

For $F \in \check{D}$ and a nilpotent cone σ, $(\sigma, \exp(\sigma_{\mathbf{C}})F)$ is a *nilpotent orbit* if it satisfies the following conditions: Take a generators $N_1, \dots, N_n \in \mathrm{Lie}\,(G)_{\mathbf{R}}$ of the cone σ.

(1) (admissibility) There is a faithful $V \in \mathrm{Rep}(G)$ such that the relative monodromy weight filtrations $M(N_j, W)$ on V exist for all $1 \leq j \leq n$.

(2) (Griffiths transversality) $N_j F^p \subset F^{p-1}$ for any $1 \leq j \leq n$, $p \in \mathbf{Z}$.

(3) (limit mixed Hodge property) $\exp(\sum_{j=1}^n iy_j N_j)F \in D$ if $y_j \in \mathbf{R}_{>0}$ are sufficiently large.

This is well-defined, i.e., independent of choices of generators $N_1, \dots, N_n$.

Note that, for admissibility, the above condition (1) is enough under the assumption of $\mathbf{R}$-polarizability (cf. [7], [10] III Proposition 1.3.4, Remark in 2.2.2, [8] Proposition 2.1.10).

1.15. A *weak fan* Σ *in* $\mathrm{Lie}\,(G)$ is a nonempty set of sharp rational nilpotent cones satisfying the conditions that it is closed under taking faces and that any $\sigma, \sigma' \in \Sigma$ coincide if they have a common interior point and if there is an $F \in \check{D}$ such that both $(\sigma, \exp(\sigma_{\mathbf{C}})F)$ and $(\sigma', \exp(\sigma'_{\mathbf{C}})F)$ are nilpotent orbits.

For a weak fan Σ in $\mathrm{Lie}\,(G)$, let D_Σ be the set of all nilpotent orbits $(\sigma, \exp(\sigma_{\mathbf{C}})F)$ with $\sigma \in \Sigma$ and $F \in \check{D}$.

1.16. Let Γ be a subgroup of $G(\mathbf{Q})$ as in 1.13.

A weak fan Σ in 1.15 is said to be *strongly compatible with* Γ if Σ is stable under the adjoint action of Γ and each cone $\sigma \in \Sigma$ is generated over $\mathbf{R}_{\geq 0}$ by log of $\exp(\sigma) \cap \Gamma$.

1.17. $\mathcal{B}$ denotes the category of locally ringed spaces with a covering by open sets each of which has the strong topology in an analytic space. $\mathcal{B}(\log)$ denotes the category of objects of $\mathcal{B}$ endowed with an fs log structure. For precise definitions of these, see [12] 3.2.4, [10] Part III 1.1.

1.18. Let $S \in \mathcal{B}(\log)$. A $\mathbf{Q}$-*log mixed Hodge structure* ($\mathbf{Q}$-*LMH*, for short) *with* $\mathbf{R}$-*polarizable graded quotients on* S is $(H_{\mathbf{Q}}, W, H_{\mathcal{O}}, F)$ consisting of locally constant sheaf $H_{\mathbf{Q}}$ with an increasing filtration W of $H_{\mathbf{Q}}$ on $(S^{\log}, \mathcal{O}_S^{\log})$, locally free sheaf $H_{\mathcal{O}}$ with a decreasing filtration F of $H_{\mathcal{O}}$ on $(S, \mathcal{O}_S)$ such that gr_F^p is locally free for all p, and a specified isomorphism $\mathcal{O}_S^{\log} \otimes_{\mathbf{Q}} H_{\mathbf{Q}} \simeq \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} H_{\mathcal{O}}$, whose pullbacks to each fs log point $s \in S$ satisfy the following conditions (1)–(3).

(1) (admissibility) Let $N_1, \dots, N_n$ be a generator of the local monodromy cone

$$C(s) := \mathrm{Hom}\,(M_s/\mathcal{O}_s^\times, \mathbf{R}_{\geq 0}) \subset \pi_1(s^{\log}).$$

The relative monodromy weight filtrations $M(N_j, W)$ exists for all $1 \leq j \leq n$.

(2) (Griffiths transversality) $\nabla F^p \subset \omega_s^{1,\log} \otimes_{\mathcal{O}_s} F^{p-1}$ for all $p \in \mathbf{Z}$.

(3) ($\mathbf{R}$-polarizability on graded quotients) For each $w \in \mathbf{Z}$, there is a non-degenerate $(-1)^w$-symmetric bilinear form $\langle\ ,\ \rangle_w : H(\mathrm{gr}_w^W)_{\mathbf{R}} \times H(\mathrm{gr}_w^W)_{\mathbf{R}} \to \mathbf{R}$ over $\mathbf{R}$ such that the quadruple $(H(\mathrm{gr}_w^W), \langle\ ,\ \rangle_w, H(\mathrm{gr}_w^W)_{\mathcal{O}}, F(\mathrm{gr}_w^W))$ is an $\mathbf{R}$-polarized log Hodge structure of weight w on s. The last part means as follows. Let $q_1, \dots, q_r \in M_s \smallsetminus \mathcal{O}_s^\times$ whose classes generate the monoid $M_s/\mathcal{O}_s^\times$. For $t \in s^{\log}$ and $a \in \mathrm{sp}(t)$ with $\exp(a(\log(q_j)))$ sufficiently small for all

$1 \le j \le r$, $(H(\mathrm{gr}^W_w), \langle\ ,\ \rangle_w, H(\mathrm{gr}^W_w)_{\mathcal{O}}, F(\mathrm{gr}^W_w)(a))$ is an $\mathbf{R}$-polarized Hodge structure. Here we use $H(\mathrm{gr}^W_w)_{\mathbf{R}} := \mathbf{R} \otimes_{\mathbf{Q}} H(\mathrm{gr}^W_w)$, $H(\mathrm{gr}^W_w)_{\mathcal{O}} := \mathcal{O}_s \otimes_{\mathbf{Q}} H(\mathrm{gr}^W_w)$, $F(\mathrm{gr}^W_w) := F(H(\mathrm{gr}^W_w)_{\mathcal{O}})$.

Note that, in [12] Definition 2.4.7, [10] Part III 1.3.2, rational polarizations on graded quotients were used. But, in the present paper, we use $\mathbf{R}$-polarizability on graded quotients. Even under this latter condition, the proof of [10] Part III Proposition 1.3.4 works.

1.19. Definition. Given (G, h_0) and Γ as in 1.13. Let $S \in \mathcal{B}(\log)$.

A *G-log mixed Hodge structure with a Γ-level structure on* S is (H, μ) consisting of an exact $\otimes$-functor $H : \mathrm{Rep}(G) \to \mathrm{LMH}(S); (V, W) \mapsto (V, W, F)$ and a global section μ of the quotient sheaf $\Gamma\backslash\mathcal{I}$.

Here $\mathcal{I}$ is the following sheaf on $S^{\log}$. For an open set U of $S^{\log}$, $\mathcal{I}(U)$ is the set of all isomorphisms $H_{\mathbf{Q}}|_U \xrightarrow{\sim} \mathrm{id}$ of $\otimes$-functors from $\mathrm{Rep}(G)$ to the category of local systems of $\mathbf{Q}$-modules V on U preserving the weight filtration W.

1.20. Let (G, h_0) be as in 1.13 and let Γ and Σ be as in 1.16.

A G-LMH H on S with a Γ-level structure μ is said to be *of type* (h_0, Σ) if the following (i) and (ii) are satisfied for any $s \in S$ and any $t \in s^{\log}$. Take a $\otimes$-isomorphism $\tilde{\mu}_t : H_{\mathbf{Q},t} \xrightarrow{\sim} \mathrm{id}$ which belongs to μ_t.

(i) There is a cone $\sigma \in \Sigma$ such that the logarithm of the action of the cone $\mathrm{Hom}\,((M_S/\mathcal{O}_S^\times)_s, \mathbf{N}) \subset \pi_1(s^{\log})$ on $H_{\mathbf{Q},t}$ is contained, via $\tilde{\mu}_t$, in $\sigma \subset \mathrm{Lie}\,(G)_{\mathbf{R}}$.

(ii) Let $\sigma \in \Sigma$ be the smallest cone satisfying (i). Let $a : \mathcal{O}^{\log}_{S,t} \to \mathbf{C}$ be a ring homomorphism which induces the evaluation $\mathcal{O}_{S,s} \to \mathbf{C}$ at s and consider the Hodge filtration F of the functor $V \mapsto \tilde{\mu}_t a(H(V))$ in Y. Then this functor belongs to $\check{D}$ and (σ, F) generates a nilpotent orbit.

If (H, μ) is of type (h_0, Σ), we have a map $S \to \Gamma \setminus D_\Sigma$, called the *period map* associated to (H, μ), which sends $s \in S$ to the class of the nilpotent orbit $(\sigma, Z) \in D_\Sigma$ which is obtained in (ii).

1.21. Let (G, h_0) be as in 1.13 and let Γ and Σ be as in 1.16.

Introduce on $\Gamma \setminus D_\Sigma$ the *strong topology*, that is the strongest topology for which the period map $S \to \Gamma \setminus D_\Sigma$ is continuous for all (S, H, μ), and introduce a sheaf of holomorphic functions $\mathcal{O}$ and a log structure M.

Theorem 1.22. *Let (G, h_0, Γ, Σ) be as in 1.21. Assume that h_0 is $\mathbf{R}$-polarizable (1.12). Then*

(1) *$\Gamma \setminus D_\Sigma$ is Hausdorff.*

From hereafter, assume that Γ is neat.

(2) *$\Gamma \setminus D_\Sigma$ is a log manifold (*[10] Part III 1.1.5*). In particular, $\Gamma \setminus D_\Sigma$ belongs to $\mathcal{B}(\log)$.*

(3) *$\Gamma \setminus D_\Sigma$ represents the contravariant functor from $\mathcal{B}(\log)$ to* (Set):

$S \mapsto \{$*isomorphism class of G-LMH over S with a Γ-level structure of type (h_0, Σ)* $\}$.

(4) *Let S be a connected, log smooth, fs log analytic space, and let U be the open subspace of S consisting of all points of S at which the log structure of S is trivial. Assume that $S \smallsetminus U$ is a smooth divisor.*

Let (H, μ) be a G-MHS over U of type h_0 (1.9) endowed with a Γ-level structure (1.19). Let $\varphi : U \to \Gamma \setminus D$ be the associated period map. Assume that (H, μ) extends to a G-LMH over S with a Γ-level structure of type (h_0, Σ).

Then, φ extends to a morphism $\overline{\varphi}$ in $\mathcal{B}(\log)$ in the following commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{\overline{\varphi}} & \Gamma \setminus D_\Sigma \\ \bigcup & & \bigcup \\ U & \xrightarrow{\varphi} & \Gamma \setminus D. \end{array}$$

2. Log higher Albanese manifolds

We review here formulations and results of higher Albanese manifolds in [6] and of log higher Albanese manifolds in [11].

2.1. Let X be a connected smooth algebraic variety over $\mathbf{C}$. Fix $b \in X$. Let Γ be a quotient group of $\pi_1(X, b)$ which is torsion-free and nilpotent.

Let $\mathcal{G} = \mathcal{G}_\Gamma$ be the unipotent algebraic group over $\mathbf{Q}$ whose Lie algebra is defined as follows: Let I be the augmentation ideal $\mathrm{Ker}(\mathbf{Q}[\Gamma] \to \mathbf{Q})$ of $\mathbf{Q}[\Gamma]$. Then $\mathrm{Lie}\,(\mathcal{G})$ is the $\mathbf{Q}$-subspace of $\mathbf{Q}[\Gamma]^\wedge := \varprojlim_n \mathbf{Q}[\Gamma]/I^n$ generated by all $\log(\gamma)$ $(\gamma \in \Gamma)$. The Lie product of $\mathrm{Lie}\,(\mathcal{G})$ is defined by $[x, y] = xy - yx$. We have $\Gamma \subset \mathcal{G}(\mathbf{Q})$.

2.2. Let $\pi_1 = \pi_1(X, b)$. Let J be the augmentation ideal $\mathrm{Ker}(\mathbf{Q}[\pi_1] \to \mathbf{Q})$ of $\mathbf{Q}[\pi_1]$. For a positive integer n, let Γ_n be the image of $\pi_1 \to \mathbf{Q}[\pi_1]/J^n$. Then $\mathrm{Lie}\,(\mathcal{G}_{\Gamma_n})$ has a mixed Hodge structure induced from de Rham theory on the path spaces over X by Chen's iterated integral.

For a given Γ as in 2.1, there exists $n \geq 1$ such that Γ is a quotient of Γ_n. Hereafter we assume that $\mathrm{Lie}\,(\mathcal{G}_\Gamma)$ has a quotient mixed Hodge structure of the one on $\mathrm{Lie}\,(\mathcal{G}_{\Gamma_n})$. Note that this mixed Hodge structure on $\mathrm{Lie}\,(\mathcal{G}_\Gamma)$ is independent of the choice of n.

We note that there is an insufficient statement on mixed Hodge structure on $\mathrm{Lie}\,(\mathcal{G}_\Gamma)$ in [11] 6.1.2. The authors of [11] agreed to correct this part, so as to assume the existence of this mixed Hodge structure on $\mathrm{Lie}\,(\mathcal{G}_\Gamma)$ as above in the present paper.

Let $\mathcal{G} = \mathcal{G}_\Gamma$. Let $F^0\mathrm{Lie}\,(\mathcal{G})_\mathbf{C}$ be the 0-th Hodge filter on $\mathrm{Lie}\,(\mathcal{G})_\mathbf{C}$ and let $F^0\mathcal{G}(\mathbf{C})$ be the corresponding subgroup of $\mathcal{G}(\mathbf{C})$. The *higher Albanese manifold* associated to (X, Γ) is defined in [6] as

$$A_{X,\Gamma} := \Gamma \setminus \mathcal{G}(\mathbf{C})/F^0\mathcal{G}(\mathbf{C}).$$

2.3. Take a $\mathbf{Q}$-MHS V_0 with polarizable gr^W having the $\mathbf{Q}$-MHS on $\mathrm{Lie}\,(\mathcal{G})_\mathbf{Q}$ with $\mathcal{G} = \mathcal{G}_\Gamma$ in 2.2 as a direct summand.

Let $Q \subset \mathrm{Aut}(V_{0,\mathbf{Q}})$ be the *Mumford-Tate group* associated to V_0, i.e., the Tannaka group of the Tannaka category $\langle V_0 \rangle$ generated by V_0: $\langle V_0 \rangle \xrightarrow{\sim} \mathrm{Rep}(Q)$. Explicitly, it is the smallest $\mathbf{Q}$-subgroup Q of $\mathrm{Aut}(V_{0,\mathbf{Q}})$ such that $Q_\mathbf{R}$ contains the image of the homomorphism $h : S_{\mathbf{C}/\mathbf{R}} \to \mathrm{Aut}(V_{0,\mathbf{R}})$ and such that $\mathrm{Lie}\,(Q)_\mathbf{R}$ contains δ. Here h and δ are determined by the canonical splitting of the $\mathbf{Q}$-MHS V_0 ([1], [10] Part II 1.2).

The action of Q on $\mathrm{Lie}\,(\mathcal{G})$ induces an action of Q on $\mathcal{G}$. By this, define a semi-direct product G of Q and $\mathcal{G}$:

$$1 \to \mathcal{G} \to G \to Q \to 1.$$

We have $\mathcal{G} \subset G_u$. We have $h_0 : S_{\mathbf{C}/\mathbf{R}} \to (Q_{\mathrm{red}})_\mathbf{R} = (G_{\mathrm{red}})_\mathbf{R}$ by using the Hodge decomposition of $\mathrm{gr}^W V_0$.

2.4. Let D_G (resp. D_Q) be the period domain D for G (resp. Q) and h_0 in 2.3. We have a canonical map $\Gamma \setminus D_G \to D_Q$ induced by $G \to Q$.

Let $b_Q \in D_Q$ be the isomorphism class of the evident functor $\mathrm{Rep}(Q) \to \mathbf{Q}$-MHS by definition in 2.3, and let $b_G \in D_G$ be the isomorphism class of $\mathrm{Rep}(G) \to \mathrm{Rep}(Q) \overset{b_Q}{\to} \mathbf{Q}$-MHS via the section $Q \hookrightarrow G$.

Let $\mathcal{D}$ be the fiber of the map $D_G \to D_Q$ over b_Q.

Theorem 2.5. *The map $G_u(\mathbf{C}) \to D_G$; $g \mapsto g \cdot b_G$ induces an isomorphism*

$$A_{X,\Gamma} = \Gamma \setminus \mathcal{G}(\mathbf{C})/F^0\mathcal{G}(\mathbf{C}) \simeq \Gamma \setminus \mathcal{D}$$

of analytic manifolds.

2.6. Let $\mathcal{C}_{X,\Gamma}$ be the category of variations of $\mathbf{Q}$-MHS $\mathcal{H}$ on X satisfying the following three conditions:

(1) For any $w \in \mathbf{Z}$, $\mathrm{gr}^W_w \mathcal{H}$ is a constant polarizable Hodge structure.

(2) $\mathcal{H}$ is *good at infinity* in the sense of [6] (1.5), i.e., there exists a smooth compactification $\overline{X}$ of X with normal crossing boundary divisor $\overline{X} \smallsetminus X$ such that the Hodge filtration bundles extend to sub-bundles of the canonical extension of $\mathcal{O}$-module of $\mathcal{H}$ which induce the corresponding thing for each $\mathrm{gr}^W_w \mathcal{H}$, and that, for the nilpotent logarithm N_j of a local monodromy transformation about a component of $\overline{X} \smallsetminus X$, the relative monodromy weight filtration $M(N_j, W)$ exists.

(3) The monodromy action of $\pi_1(X, b)$ factors through Γ.

Hain and Zucker showed

Theorem 2.7. ([6] (1.6) Theorem). *The category $\mathcal{C}_{X,\Gamma}$ is equivalent to the category of $\mathbf{Q}$-MHS V with polarizable $\mathrm{gr}^W V$ endowed with an action of* $\mathrm{Lie}\,(\mathcal{G})$ *such that* $\mathrm{Lie}\,(\mathcal{G}) \otimes V \to V$ *is a homomorphism of MHS.*

2.8. Define a contravariant functor $\mathcal{F}_\Gamma : \mathcal{B}(\log) \to \mathrm{Sets}$ as follows: For $S \in \mathcal{B}(\log)$, $\mathcal{F}_\Gamma(S)$ is the set of isomorphism classes of pairs (H, μ) of an exact $\otimes$-functor $H : \mathcal{C}_{X,\Gamma} \to \mathrm{MHS}(S)$ and a Γ-level structure μ satisfying the following condition (i). Here a Γ-*level structure* means a global section of the sheaf $\Gamma \setminus \mathcal{I}$, where $\mathcal{I}$ is the sheaf of functorial $\otimes$-isomorphisms $H(\mathcal{H})_{\mathbf{Q}} \overset{\sim}{\to} \mathcal{H}(b)_{\mathbf{Q}}$ of $\mathbf{Q}$-local systems preserving weight filtrations.

(i) For any $\mathbf{Q}$-MHS h, we have a functorial $\otimes$-isomorphism $H(h_X) \cong h_S$ such that the induced isomorphism of local systems $H(h_X)_{\mathbf{Q}} \cong h_{\mathbf{Q}} = h_X(b)_{\mathbf{Q}}$ belongs to μ. Here h_X (resp. h_S) denotes the constant variation (resp. family) of $\mathbf{Q}$-MHS over X (resp. S) associated to h.

Theorem 2.9. *Let the notation be as in 2.8. The functor $\mathcal{F}_\Gamma$ is represented by $A_{X,\Gamma} \simeq \Gamma \setminus \mathcal{D}$.*

This follows from Theorem 2.5 and Theorem 2.7.

Let $\varphi : X \to A_{X,\Gamma}$ be the higher Albanese map.

2.10. Let Σ be a weak fan in $\mathrm{Lie}\,(G)$ such that $\sigma \subset \mathrm{Lie}\,(\mathcal{G})_{\mathbf{R}}$ for any $\sigma \in \Sigma$. Assume that Σ and Γ are strongly compatible. Let $\Gamma \setminus D_{G,\Sigma} \to D_Q$ be a canonical morphism induced by $G \to Q$. Define

$$A_{X,\Gamma,\Sigma} := (\text{the fiber of } \Gamma \setminus D_{G,\Sigma} \to D_Q \text{ over } b_Q) \in \mathcal{B}(\log)$$

Define a contravariant functor $\mathcal{F}_{\Gamma,\Sigma} : \mathcal{B}(\log) \to \mathrm{Sets}$ as follows: For $S \in \mathcal{B}(\log)$, $\mathcal{F}_{\Gamma,\Sigma}(S)$ is the set of isomorphism classes of pairs (H, μ) consisting of an exact $\otimes$-functor $H : \mathcal{C}_{X,\Gamma} \to \mathrm{LMH}(S)$ and a Γ-level structure μ satisfying the condition (i) in 2.8 and also the following condition (ii).

(ii) The following (ii-1) and (ii-2) are satisfied for any $s \in S$ and any $t \in s^{\log}$. Let

$$\tilde{\mu}_t : H(\mathcal{H})_{\mathbf{Q},t} \cong \mathcal{H}(b)_{\mathbf{Q}}$$

be a functorial $\otimes$-isomorphism which belongs to μ_t.

(ii-1) There is a $\sigma \in \Sigma$ such that the logarithm of the action of the local monodromy cone $\mathrm{Hom}\,((M_S/\mathcal{O}_S^\times)_s, \mathbf{N}) \subset \pi_1(s^{\log})$ on $H_{\mathbf{Q},t}$ is contained, via $\tilde{\mu}_t$, in $\sigma \subset \mathrm{Lie}\,(\mathcal{G})_{\mathbf{R}}$.

(ii-2) Let $\sigma \in \Sigma$ be the smallest cone which satisfies (ii-1) and let $a : \mathcal{O}^{\log}_{S,t} \to \mathbf{C}$ be a ring homomorphism which induces the evaluation $\mathcal{O}_{S,s} \to \mathbf{C}$ at s. Then, for each $\mathcal{H} \in \mathcal{C}_{X,\Gamma}$, $(\sigma, \tilde{\mu}_t(a(H(\mathcal{H}))))$ generates a nilpotent orbit in the sense of [10] Part III, 2.2.2.

Theorem 2.11. *Let the notation be as in 2.9 and 2.10.*

(1) *The functor $\mathcal{F}_{\Gamma,\Sigma}$ is represented by $A_{X,\Gamma,\Sigma}$.*

(2) *Let $\overline{X}$ be a smooth algebraic variety over $\mathbf{C}$ which contains X as a dense open subset such that the complement $\overline{X} \smallsetminus X$ is a smooth divisor. Endow $\overline{X}$ with the log structure associated to this divisor. Assume that Σ is the fan consisting of all rational nilpotent cones in $\mathrm{Lie}\,(\mathcal{G})_{\mathbf{R}}$ of rank ≤ 1 (denoted by Ξ in [11] 6.2.5). Then, the* higher Albanese map *$\varphi : X \to A_{X,\Gamma}$ extends uniquely to a morphism $\overline{\varphi} : \overline{X} \to A_{X,\Gamma,\Sigma}$ of log manifolds.*

Since an object of $\mathcal{C}_{X,\Gamma}$ is good at infinity (2.6), it extends to an LMH over $\overline{X}$. Hence (2) follows from (1) and the general theorem 1.22 (4).

3. Description of a result of Deligne by log higher Albanese map

For a group $\Gamma^{(n)}$ in 3.3 below, Deligne [3] showed that polylogarithms appear in the higher Albanese map $X \to A_{X,\Gamma^{(n)}}$ (cf. Section A below). Here we describe them in our framework in [11] (Section 2 in the present paper).

3.1. Let $X := \mathbf{P}^1(\mathbf{C}) \smallsetminus \{0,1,\infty\} \subset \overline{X} := \mathbf{P}^1(\mathbf{C})$ with affine coordinate x. Let $b := (0,1)$ the "tangential base point" over $0 \in \overline{X}$ with tangent $v_0 \in T_0(\overline{X}) = \mathrm{Hom}_{\mathbf{C}}(m_0/m_0^2, \mathbf{C})$ defined by $v_0(x) = 1$ in [3] Section 15. This is understood in log geometry in the following way. Let $y = (0,h) \in \overline{X}^{\log}$ be the point lying over $0 \in X$, where $h : M^{\mathrm{gp}}_{\overline{X},0} = \mathcal{O}^{\times}_{\overline{X},0} \times x^{\mathbf{Z}} \to \mathbf{S}^1$ is the argument function which is a group homomorphism sending $f \in \mathcal{O}^{\times}_{\overline{X},0}$ to $f(0)/|f(0)|$ and x to $v_0(x)/|v_0(x)| = 1$ ([11] 6.3.7). Let $u_0 \in \mathcal{O}^{\log}_{\overline{X},y}$ be the branch of $\log(x)$ having real value on $\mathbf{R}_{>0}$. (This u_0 is the branch denoted by $f \in \mathcal{O}^{\log}_{\overline{X},y}$ in [11] 6.3.7 (ii), and u_0 can be also regarded as the function $2\pi i z$ on $\tilde{S}^{\log}$ in 1.1.1.) Then the corresponding base point in the boundary in our sense is $b = (y,a)$, where $a : \mathcal{O}^{\log}_{\overline{X},y} = \mathbf{C}\{x\}[u_0] \to \mathbf{C}$ is the specialization which is a ring homomorphism sending x to 0 and u_0 to $a(u_0) = \log(v_0(x)) = \log(1) = 0$ ([11] 6.3.7 (ii)).

See [11] 6.3.6, 6.3.7 for more general description of the above correspondence of boundary points.

3.2. The inclusion $X \subset \mathbf{G}_m(\mathbf{C}) = \mathbf{C}^{\times}$ induces $\pi_1(X,b) \to \pi_1(\mathbf{G}_m(\mathbf{C}),b) = \mathbf{Z}(1)$. Let K be its kernel, and let $\Gamma := \pi_1(X,b)/[K,K]$ and $\Gamma_1 := K/[K,K]$. Then

$$1 \to \Gamma_1 \to \Gamma \to \mathbf{Z}(1) \to 1.$$

3.3. Let $Z^n\Gamma$ be the descending central series of Γ defined by $Z^{n+1}\Gamma := [Z^n\Gamma, \Gamma]$ starting with $Z^1\Gamma = \Gamma$.

Let $\Gamma^{(n)} := \Gamma/Z^{n+1}(\Gamma)$ and $\Gamma_1^{(n)} := \mathrm{Image}\,(\Gamma_1 \to \Gamma^{(n)})$. Let $\gamma_0, \gamma_1 \in \Gamma^{(n)}$ be the classes of small loops anticlockwise around 0 and clockwise around 1, respectively. Then, we have

$$\Gamma^{(n)} = \langle \gamma_0, \gamma_1 \rangle,\ (\mathrm{ad}\,\gamma_0)^{k-1}\gamma_1\ (1 \le k \le n) \text{ are commutative, } \Gamma_1^{(n)} = \textstyle\sum_{k=1}^n \mathbf{Z}(\mathrm{ad}\,\gamma_0)^{k-1}\gamma_1.$$

3.4. Let $\Lambda = (V, W, (\langle\ ,\ \rangle_w)_{w\in\mathbf{Z}}, (h^{p,q})_{p,q\in\mathbf{Z}})$ be as follows. V is a free $\mathbf{Z}$-module with basis $e_1, e_2, e_3, \ldots, e_{n+1}$. W is a weight filtration on $V_{\mathbf{Q}}$ defined by

$$W_{-2n-1} = 0 \subset W_{-2n} = W_{-2n+1} = \mathbf{Q}e_1 \subset W_{-2n+2} = W_{-2n+3} = W_{-2n+1} + \mathbf{Q}e_2$$
$$\subset \cdots \subset W_0 = W_{-1} + \mathbf{Q}e_{n+1} = V_{\mathbf{Q}}.$$

$\langle\ ,\ \rangle_w : \mathrm{gr}^W_w(V_{\mathbf{Q}}) \times \mathrm{gr}^W_w(V_{\mathbf{Q}}) \to \mathbf{Q}$ ($w \in \mathbf{Z}$) are the $\mathbf{Q}$-bilinear forms characterized by

$$\langle e_{n+1+k}, e_{n+1+k}\rangle_{2k} = 1$$

for $k = 0, -1, \ldots, -n$. $h^{k,k} = 1$ for $k = 0, -1, \ldots, -n$, and $h^{p,q} = 0$ for the other (p, q).

Let $D(\Lambda)$ be the period domain in [10] Part III with universal Hodge filtration F:

$$F^1 = 0 \subset F^0 = \mathbf{C}(e_{n+1} + \sum_{n\ge j\ge 1} a_{j,n+1}e_j) \subset F^{-1} = F^0 + \mathbf{C}(e_n + \sum_{n-1\ge j\ge 1} a_{j,n}e_j)$$

$$\subset \cdots \subset F^{-n} = F^{-n+1} + \mathbf{C}e_1 = V_{\mathbf{C}}.$$

3.5. Let $\mathcal{G}$ be the unipotent group $\mathcal{G}$ in 2.1 for $\Gamma^{(n)}$. Define an action of $\mathrm{Lie}(\mathcal{G})$ on $V_{\mathbf{Q}}$ by $N_0 = \log(\gamma_0)$, $N_1 = \log(\gamma_1)$:

$$N_0e_j = e_{j-1}\ (j = 2, \ldots, n), \quad N_0e_j = 0\ (j = 1, n+1),$$

$$N_1e_{n+1} = -e_n, \quad N_1e_j = 0\ (j = 1, 2, \ldots, n).$$

Then

$$(-N_0 + N_1)^j = (-N_0)^j + (-\mathrm{Ad}N_0)^{j-1}N_1 \quad (1 \le j \le n+1).$$

3.6. Let X be as in 3.1 and $\Gamma^{(n)}$ be as in 3.3. We consider the higher Albanese manifold $A_{X,\Gamma^{(n)}}$ of X by using the base point b in 3.1.

The $\mathbf{Q}$-MHS on $\mathrm{Lie}(\mathcal{G})$ is as follows: N_0 and N_1 are of Hodge type $(-1, -1)$ and compatible with bracket and hence $F^0\mathcal{G}(\mathbf{C}) = \{1\}$. Thus the higher Albanese manifold is

$$A_{X,\Gamma^{(n)}} = \Gamma^{(n)} \setminus \mathcal{G}(\mathbf{C}).$$

Lemma 3.7. *Let F and N_j $(j = 0, 1)$ be as in 3.4 and in 3.5.*

(i) *We have the following.*

(1) (N_0, F) *satisfies the Griffiths transversality if and only if*

$$a_{k,n+1} = 0 \quad (2 \le k \le n); \quad a_{1,k} = a_{l-k+1,l} \quad (2 \le k < l \le n).$$

(2) (N_1, F) *satisfies the Griffiths transversality if and only if*

$$a_{k,n} = 0 \quad (1 \le k \le n-1).$$

(3) $(-N_0 + N_1, F)$ *satisfies the Griffiths transversality if and only if*

$$a_{1,k} = a_{l-k+1,l} \quad (2 \le k < l \le n+1).$$

(ii) *The following three conditions are equivalent.*

(1) *The Lie action* $\mathrm{Lie}(\mathcal{G}) \otimes V \to V$ *in 3.5 is a homomorphism of MHS with respect to the MHS on* $\mathrm{Lie}(\mathcal{G})$ *in 3.6 and the MHS* (V, W, F) *in 3.4.*

(2) *For $j = 0$ and 1, (N_j, F) satisfies the Griffiths transversality.*

(3) $a_{j,k} = 0$ *unless* $(j, k) = (1, n+1)$.

The assertions are easily verified by direct computation.

3.8. For any fixed $a \in \mathbf{C}$, denote by $F(a)$ the Hodge filtration in 3.7 (ii) (3) with $a_{1,n+1} = a$. By the action in 3.5, we define

$$\mathcal{D} := \exp\left(\mathbf{C}N_0 + \sum_{k=1}^{n} \mathbf{C}(\mathrm{Ad}N_0)^{k-1}N_1\right) F(a) \subset D(\Lambda).$$

Then, this $\mathcal{D}$ coincides with $\mathcal{D}$ in 2.4. Hence $\mathcal{G}(\mathbf{C}) \simeq \mathcal{D}$ and $A_{X,\Gamma^{(n)}} \simeq \Gamma^{(n)} \setminus \mathcal{D}$ as complex analytic manifolds.

3.9. Let $\varphi : X \to A_{X,\Gamma^{(n)}} \simeq \Gamma^{(n)} \setminus \mathcal{D}$ be the composite of higher Albanese map and the isomorphism in 3.8. Let $F(x)$ be the pullback by φ of the universal Hodge filtration on $\Gamma^{(n)} \setminus \mathcal{D}$.

Since $F(x)$ is rigid by Theorem 2.7, we consider a connection equation:

$$dF(x) = \omega F(x), \quad \omega := (2\pi i)^{-1}\frac{dx}{x}N_0 + (2\pi i)^{-1}\frac{dx}{1-x}N_1.$$

That is,

$$da_{k-1,k}(x) = (2\pi i)^{-1}\frac{dx}{x} \quad (2 \le k \le n),$$

$$da_{n,n+1}(x) = -(2\pi i)^{-1}\frac{dx}{1-x},$$

$$da_{j,k}(x) = (2\pi i)^{-1}a_{j+1,k}(x)\frac{dx}{x} \quad (3 \le k \le n+1,\ 1 \le j \le k-2).$$

3.10. This system is solved by iterated integrals. The solutions are

$$a_{j,k}(x) = \frac{1}{(k-j)!}((2\pi i)^{-1}\log(x))^{k-j} \quad (2 \le k \le n,\ 1 \le j \le k-1),$$

$$a_{j,n+1}(x) = -(2\pi i)^{-n-1+j}l_{n+1-j}(x) \quad (1 \le j \le n).$$

Here the $l_j(x)$ are polylogarithms, in particular $l_1(x) = -\log(1-x)$.

3.11. Table of solutions:

$$\begin{pmatrix} 1 & a_{1,2} & \dots & a_{1,n} & a_{1,n+1} \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & 0 & \ddots & a_{n-1,n} & a_{n-1,n+1} \\ \vdots & \vdots & \ddots & 1 & a_{n,n+1} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (2\pi i)^{-1}\log(x) & \dots & \frac{((2\pi i)^{-1}\log(x))^{n-1}}{(n-1)!} & -(2\pi i)^{-n}l_n(x) \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & 0 & \ddots & (2\pi i)^{-1}\log(x) & -(2\pi i)^{-2}l_2(x) \\ \vdots & \vdots & \ddots & 1 & -(2\pi i)^{-1}l_1(x) \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Note that, for $1 \le j \le n$,

$$\exp((2\pi i)^{-1}\log(x)N_0)e_j = e_j + (2\pi i)^{-1}\log(x)e_{j-1} + \dots + \frac{1}{(j-1)!}((2\pi i)^{-1}\log(x))^{j-1}e_1,$$

for $j = n+1$,

$$\exp\left(-\sum_{n \ge k \ge 1}(2\pi i)^{-k}l_k(x)(\mathrm{Ad}N_0)^{k-1}N_1\right)e_{n+1} = e_{n+1} - \left(\sum_{n \ge k \ge 1}(2\pi i)^{-k}l_k(x)e_{n+1-k}\right).$$

3.12. For $\alpha, \beta, \lambda_2, \dots, \lambda_n \in \mathbf{C}$, let $F = F(\alpha, \beta, \lambda_2, \dots, \lambda_n)$ be a Hodge filtration:

$$F^1 = 0 \subset F^0 = \mathbf{C}(e_{n+1} + \beta e_n + \lambda_2 e_{n-1} + \dots + \lambda_n e_1)$$

$$\subset F^{-1} = F^0 + \mathbf{C}\left(e_n + \alpha e_{n-1} + \frac{\alpha^2}{2!}e_{n-2} + \dots + \frac{\alpha^{n-1}}{(n-1)!}e_1\right) \subset \cdots$$

$$\subset F^{-n+1} = F^{-n+2} + \mathbf{C}(e_2 + \alpha e_1) \subset F^{-n} = F^{-n+1} + \mathbf{C}e_1 = V_{\mathbf{C}}.$$

3.13. Let $\varphi : X \to A_{X,\Gamma^{(n)}} \simeq \Gamma^{(n)} \setminus \mathcal{D}$ be the higher Albanese map in 3.9. We have a commutative diagram

$$\begin{array}{ccccc} & & \tilde{\varphi}(X) & \subset & \mathcal{D} \\ & \tilde{\varphi} \nearrow & \downarrow & & \downarrow \\ X & \xrightarrow[\varphi]{\sim} & \varphi(X) & \subset & \Gamma^{(n)} \setminus \mathcal{D} \end{array}$$

where $\tilde{\varphi} : X \to \mathcal{D}$ is a multi-valued map corresponding to the Hodge filtration

$$x \mapsto F((2\pi i)^{-1} \log(x), -(2\pi i)^{-1} l_1(x), \dots, -(2\pi i)^{-n} l_n(x))$$

in the notation in 3.12. $\tilde{\varphi}(X) \to X$ and $\tilde{\varphi}(X) \to \varphi(X)$ are $\Gamma^{(n)}$-torsors and $\varphi : X \overset{\sim}{\to} \varphi(X)$ is an isomorphism.

3.14. Let Σ be the set of all cones of the form $\mathbf{R}_{\geq 0} N$ with $N \in \mathrm{Lie}\,(\mathcal{G})$. We consider the extended period domain $D(\Lambda)_\Sigma$ in [10] Part III. This is only a set. By using the strong topology ([12] Section 3.1), the quotient $\Gamma^{(n)} \setminus D(\Lambda)_\Sigma$ has a structure of a log manifold. Define $\Gamma^{(n)} \setminus \mathcal{D}_\Sigma$ to be the closure of $\Gamma^{(n)} \setminus \mathcal{D}$ in $\Gamma^{(n)} \setminus D(\Lambda)_\Sigma$. This inherits a structure of log manifold. We have $A_{X,\Gamma^{(n)},\Sigma} \simeq \Gamma^{(n)} \setminus \mathcal{D}_\Sigma$ in the category $\mathcal{B}(\log)$.

Let $N \in \mathrm{Lie}\,(\mathcal{G})$ and $\sigma := \mathbf{R}_{\geq 0} N$. Let Γ_σ be the group generated by the monoid $\Gamma^{(n)} \cap \exp(\sigma)$. If we use as Σ the fan consisting of the cone σ and 0, also denoted by σ by abuse of notation, we have $A_{X,\Gamma_\sigma,\sigma} \simeq \Gamma_\sigma \setminus \mathcal{D}_\sigma$ in the category $\mathcal{B}(\log)$.

3.15. Let N_0 be as in 3.5 and set $\sigma_0 = \mathbf{R}_{\geq 0} N_0$. Let $F = F(\alpha, \beta, \lambda_2, \dots, \lambda_n)$ be as in 3.12. By Lemma 3.7 (i) (1), (N_0, F) satisfies the Griffiths transversality if and only if

$$\beta = \lambda_2 = \cdots = \lambda_{n-1} = 0.$$

If this is the case, (N_0, F) generates a σ_0-nilpotent orbit, since admissibility and $\mathbf{R}$-polarizability on gr^W trivially hold. We describe the local structure of $\Gamma_{\sigma_0} \setminus \mathcal{D}_{\sigma_0}$ near the image p_0 of this nilpotent orbit.

Let $Y := \{(q, \beta, \lambda_2, \dots, \lambda_n) \in \mathbf{C}^{n+1} \mid \beta = \lambda_2 = \cdots = \lambda_{n-1} = 0 \text{ if } q = 0\}$ be the log manifold with the strong topology, with the structure sheaf of rings which is the inverse image of the sheaf of holomorphic functions on $\mathbf{C}^{n+1}$, and with the log structure generated by q. Then there is an open neighborhood U of $(0, 0, \dots, 0, \lambda_n)$ in $\mathbf{C}^{n+1}$ and an open immersion

$$Y \cap U \hookrightarrow \Gamma_{\sigma_0} \setminus \mathcal{D}_{\sigma_0}$$

of log manifolds which sends

$$(q, \beta, \lambda_2, \dots, \lambda_n) \in Y \cap U$$

with $q \neq 0$ to the class of $F(\alpha, \beta, \lambda_2, \dots, \lambda_n)$, where $\alpha \in \mathbf{C}$ is such that $q = e^{2\pi i \alpha}$, and which sends $(0, 0, \dots, 0, \lambda_n)$ to p_0.

3.16. Near $x = 0$, a nilpotent orbit in naive sense is

$$\exp((2\pi i)^{-1} \log(x) N_0) F(0, 0, \dots, 0, \lambda_n^0) = F((2\pi i)^{-1} \log(x), 0, \dots, 0, \lambda_n^0), \tag{1}$$

where $\lambda_n^0 = -(2\pi i)^{-n} l_n(0)$. The corresponding "higher Albanese map" (i.e., local version about 0 of $\tilde{\varphi}$ in 3.13) is

$$F((2\pi i)^{-1} \log(x), -(2\pi i)^{-1} l_1(x), \dots, -(2\pi i)^{-n} l_n(x)) \tag{2}$$

under the condition $l_j(0) = 0$ $(1 \leq j \leq n-1)$. These two are asymptotic when x goes to the boundary point $b = (y, a)$ with $y = (0, h) \in \overline{X}^{\log}$ and a being the specialization at y in 3.1.

3.17. As above, let u_0 be the branch of $\log(x)$ in 3.1 and T an indeterminate over $\mathcal{O}_{\overline{X},0}$. Then, by 1.1.1, we have an isomorphism $\mathcal{O}^{\log}_{\overline{X},y} = \mathcal{O}_{\overline{X},0}[u_0] \simeq \mathcal{O}_{\overline{X},0}[T]$ of $\mathcal{O}_{\overline{X},0}$-algebras under $(2\pi i)^{-1}u_0 \leftrightarrow T$. Consider an $\mathcal{O}_{\overline{X},0}$-algebra homomorphism $\mathcal{O}_{\overline{X},0}[T] \to \mathcal{O}_{\overline{X},0}$, $T \mapsto x$.

Under the initial condition in 3.16 given by the base point b in 3.1, we have

$$l_j(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^j} \quad (1 \le j \le n-1), \quad l_n(x) = c + \sum_{k=1}^{\infty} \frac{x^k}{k^n}$$

on a simply connected neighborhood $\overline{X}_0$ of $0 \in \overline{X}$, where $c := -(2\pi i)^n \lambda_n^0$.

Let $\alpha = (2\pi i)^{-1}\log(x)$. Then, as

$$x \to 0,\ \exp(-\alpha N_0)(F \text{ in } 3.16\ (2))$$

converges to $F(0,0,\dots,0,\lambda_n^0)$ in $\mathcal{D}$ (3.8), and hence the class of (F in 3.16 (2)) converges to the class p_0 (3.15) of the nilpotent orbit $(\sigma_0, \exp(\sigma_{0,\mathbf{C}})F(0,0,\dots,0,\lambda_n^0))$ in $\Gamma_{\sigma_0} \setminus \mathcal{D}_{\sigma_0}$. We thus have an extension of the higher Albanese map over $\overline{X}_0$ (Theorem 2.11 (2)):

$$\overline{\varphi}_0 : \overline{X}_0 \to \Gamma_{\sigma_0} \setminus \mathcal{D}_{\sigma_0}.$$

This is a morphism in the category $\mathcal{B}(\log)$. The log structure on the source (resp. the target) is given by x (resp. q). The pullback of the universal log mixed Hodge structure on the target coincides with the log mixed Hodge structure on the source.

3.18. By using log mixed Hodge theory, 3.16 is described as follows.

Taking the images of the nilpotent orbit in naive sense 3.16 (1) and the "higher Albanese map" 3.16 (2), we have their real analytic extensions with boundary

$$\overline{\nu}_0^{\log},\ \overline{\varphi}_0^{\log} : \overline{X}_0^{\log} \to (\Gamma_{\sigma_0} \setminus \mathcal{D}_{\sigma_0})^{\log}.$$

Here, $\overline{X}_0^{\log}$ is like Example 1.1.1, and $(\Gamma_{\sigma_0} \setminus \mathcal{D}_{\sigma_0})^{\log}$ coincides with the moduli of nilpotent i-orbits $\Gamma_{\sigma_0} \setminus \mathcal{D}^\sharp_{\sigma_0}$ in the present situation ([10] III Theorem 2.5.6).

Let $\tilde{\overline{X}}_0^{\log}$ be the universal covering of $\overline{X}_0^{\log}$. The above maps are still lifted to

$$\tilde{\overline{\nu}}_0^{\log},\ \tilde{\overline{\varphi}}_0^{\log} : \tilde{\overline{X}}_0^{\log} \to \mathcal{D}^\sharp_{\sigma_0}.$$

The boundary point b in 3.16 can be understood as the point

$$b = (z = 0 + i\infty) = (u_0 = -\infty + i0) \in \tilde{\overline{X}}_0^{\log}.$$

We have $(\exp(-(2\pi i)^{-1}\log(x)N_0)(3.16\ (2)))(b) = F(0,0,\dots,0,\lambda_n^0)$, and

$$\tilde{\overline{\nu}}_0^{\log}(b) = \tilde{\overline{\varphi}}_0^{\log}(b) = (\text{nilpotent } i\text{-orbit generated by } (N_0, F(0,0,\dots,0,\lambda_n^0))) \in \mathcal{D}^\sharp_{\sigma_0}.$$

3.19. Let now $\sigma_1 = \mathbf{R}_{\ge 0}N_1$ for N_1 in 3.5. Let $F = F(\alpha,\beta,\lambda_2,\dots,\lambda_n)$ be as in 3.12. By Lemma 3.7 (i) (2), (N_1, F) satisfies the Griffiths transversality if and only if $\alpha = 0$. If this is the case, (N_1, F) generates a σ_1-nilpotent orbit, since admissibility and $\mathbf{R}$-polarizability on gr^W trivially hold. We have a similar description of the local structure of $\Gamma_{\sigma_1} \setminus \mathcal{D}_{\sigma_1}$ near the image p_1 of this nilpotent orbit.

Let Y be the log manifold $\{(\alpha, q, \lambda_2, \dots, \lambda_n) \in \mathbf{C}^{n+1} \mid \alpha = 0 \text{ if } q = 0\}$ with the strong topology, the structure sheaf and the log structure defined by q. Then there is an open neighborhood U of $(0,0,\lambda_2,\dots,\lambda_n)$ in $\mathbf{C}^{n+1}$ and an open immersion

$$Y \cap U \hookrightarrow \Gamma_{\sigma_1} \setminus \mathcal{D}_{\sigma_1}$$

of log manifolds which sends

$$(\alpha, q, \lambda_2, \dots, \lambda_n) \in Y \cap U$$

with $q \neq 0$ to the class of $F(\alpha, \beta, \lambda_2, \ldots, \lambda_n)$, where $\beta \in \mathbf{C}$ is such that $q = e^{2\pi i\beta}$, and which sends $(0, 0, \lambda_2, \ldots, \lambda_n)$ to p_1.

3.20. We assume the initial condition in 3.16. Near $x = 1$, a nilpotent orbit in naive sense is

$$\begin{aligned}(1) \qquad &\exp((2\pi i)^{-1}\log(1-x)N_1) \cdot F(0, 0, -(2\pi i)^{-2}\zeta(2), \ldots, -(2\pi i)^{-n}(c+\zeta(n))) \\ &= F(0, -(2\pi i)^{-1}l_1(x), -(2\pi i)^{-2}\zeta(2), \ldots, -(2\pi i)^{-n}(c+\zeta(n))).\end{aligned}$$

The corresponding "higher Albanese map" (i.e., local version about 1 of $\tilde{\varphi}$ in 3.13) is

$$(2) \qquad F((2\pi i)^{-1}\log(x), -(2\pi i)^{-1}l_1(x), \ldots, -(2\pi i)^{-n}l_n(x)).$$

These two are asymptotic when x goes to the tangential boundary point $\tilde{p}_1 := (1, -1)$ with tangent $v_1 \in T_1(\overline{X}) = \mathrm{Hom}_{\mathbf{C}}(m_1/m_1^2, \mathbf{C})$ defined by $v_1(1-x) = -1$. This is the boundary point in our sense described as follows. Let u_1 be the branch of $\log(1-x)$ having real value on $\mathbf{R}_{<1}$. Then the corresponding point in the boundary in our sense is $\tilde{p}_1 = (y, a)$ with $y = (1, h) \in \overline{X}^{\log}$ such that the argument function $h : M^{\mathrm{gp}}_{\overline{X},1} = \mathcal{O}^{\times}_{\overline{X},1} \times (1-x)^{\mathbf{Z}} \to \mathbf{S}^1$ is a group homomorphism sending $f \in \mathcal{O}^{\times}_{\overline{X},1}$ to $f(1)/|f(1)|$ and $1-x$ to $v_1(1-x)/|v_1(1-x)| = -1$, and the specialization $a : \mathcal{O}^{\log}_{\overline{X},y} = \mathbf{C}\{1-x\}[u_1] \to \mathbf{C}$ is a ring homomorphism sending $1-x$ to 0 and u_1 to

$$a(u_1) = -a(-u_1) = \log(v_1(-(1-x))) = \log(1) = 0$$

(cf. [11] 6.3.7 (ii)).

3.21. As above, let u_1 be the branch of $\log(1-x)$ and T an indeterminate over $\mathcal{O}_{\overline{X},1}$. Then, by 1.1.1, we have an isomorphism

$$\mathcal{O}^{\log}_{\overline{X},y} = \mathcal{O}_{\overline{X},1}[u_1] \simeq \mathcal{O}_{\overline{X},1}[T]$$

of $\mathcal{O}_{\overline{X},1}$-algebras under $(2\pi i)^{-1}u_1 \leftrightarrow T$. Consider an $\mathcal{O}_{\overline{X},1}$-algebra homomorphism

$$\mathcal{O}_{\overline{X},1}[T] \to \mathcal{O}_{\overline{X},1}, \ T \mapsto 1-x.$$

Let $\beta = (2\pi i)^{-1}\log(1-x)$. Then, as $x \to 1$ in $\overline{X}$ along the real axis starting from b over 0 to 1, $\exp(-\beta N_1)(F$ in 3.20 (2)) converges to $F(0, 0, -(2\pi i)^{-2}\zeta(2), \ldots, -(2\pi i)^{-n}(c+\zeta(n)))$ in $\mathcal{D}$ (3.8), and hence the class of (F in 3.20 (2)) converges to the class p_1 (3.19) of the nilpotent orbit

$$(\sigma_1, \exp(\sigma_{1,\mathbf{C}})F(0, 0, -(2\pi i)^{-2}\zeta(2), \ldots, -(2\pi i)^{-n}(c+\zeta(n))))$$

in $\Gamma_{\sigma_1} \setminus \mathcal{D}_{\sigma_1}$. We thus have an extension of the higher Albanese map over a simply connected neighborhood $\overline{X}_1$ of 1 in $\overline{X}$ (Theorem 2.11 (2)):

$$\overline{\varphi}_1 : \overline{X}_1 \to \Gamma_{\sigma_1} \setminus \mathcal{D}_{\sigma_1}.$$

This is a morphism in the category $\mathcal{B}(\log)$. The log structure on the source (resp. the target) is given by $1-x$ (resp. q). The pullback of the universal log mixed Hodge structure on the target coincides with the log mixed Hodge structure on the source.

3.22. By using log mixed Hodge theory, 3.20 is described as follows.

Taking the images of the nilpotent orbit in naive sense 3.20 (1) and the "higher Albanese map" 3.20 (2), we have their real analytic extensions with boundary

$$\overline{\nu}_1^{\log}, \ \overline{\varphi}_1^{\log} : \overline{X}_1^{\log} \to (\Gamma_{\sigma_1} \setminus \mathcal{D}_{\sigma_1})^{\log}.$$

Here, $\overline{X}_1^{\log}$ is similar to Example 1.1.1 over $x = 1$, and $(\Gamma_{\sigma_1} \setminus \mathcal{D}_{\sigma_1})^{\log}$ coincides with the moduli of nilpotent i-orbits $\Gamma_{\sigma_1} \setminus \mathcal{D}^{\sharp}_{\sigma_1}$ in the present situation ([10] III Theorem 2.5.6).

Let $\tilde{\overline{X}}_1^{\log}$ be the universal covering of $\overline{X}_1^{\log}$. The above maps are still lifted to

$$\tilde{\overline{\nu}}_1^{\log},\ \tilde{\overline{\varphi}}_1^{\log} : \tilde{\overline{X}}_1^{\log} \to \mathcal{D}^{\sharp}_{\sigma_1}.$$

The boundary point $\tilde{p}_1$ in 3.20 can be understood as the point

$$\tilde{p}_1 = (z_1 = 0 + i\infty) = (u_1 = -\infty + i0) \in \tilde{\overline{X}}_1^{\log},$$

(where $2\pi i z_1 := u_1$). We have

$$(\exp(-(2\pi i)^{-1}\log(1-x)N_1)(3.20\,(2)))(\tilde{p}_1) = F(0, 0, -(2\pi i)^{-2}\zeta(2), \dots, -(2\pi i)^{-n}(c + \zeta(n))),$$

and $\tilde{\overline{\nu}}_1^{\log}(\tilde{p}_1) = \tilde{\overline{\varphi}}_1^{\log}(\tilde{p}_1) \in \mathcal{D}^{\sharp}_{\sigma_1}$ is the nilpotent i-orbit generated by

$$(N_1, F(0, 0, -(2\pi i)^{-2}\zeta(2), \dots, -(2\pi i)^{-n}(c + \zeta(n)))).$$

3.23. In order to describe the local structure near $x = \infty$, we take a local coordinate $\xi := x^{-1}$.

By abuse of notation, let $F(\xi)$ be the pullback of the universal Hodge filtration by the composite $\varphi : X \to A_{X,\Gamma^{(n)}} \simeq \Gamma^{(n)} \setminus \mathcal{D}$ of higher Albanese map and the isomorphism in 3.8.

Since $d\log(x) = -d\log(\xi)$ and $-d\log(x-1) = d\log(\xi) - d\log(1-\xi)$, a connection equation in 3.9 now is

$$dF(\xi) = \omega F(\xi), \quad \omega := (2\pi i)^{-1}\frac{d\xi}{\xi}(-N_0 + N_1) + (2\pi i)^{-1}\frac{d\xi}{1-\xi}N_1.$$

That is,

$$da_{k-1,k}(\xi) = -(2\pi i)^{-1}\frac{d\xi}{\xi} \quad (2 \le k \le n),$$

$$da_{n,n+1}(\xi) = -(2\pi i)^{-1}\frac{d\xi}{\xi} - (2\pi i)^{-1}\frac{d\xi}{1-\xi},$$

$$da_{j,k}(\xi) = -(2\pi i)^{-1}a_{j+1,k}(\xi)\frac{d\xi}{\xi} \quad (3 \le k \le n+1,\ 1 \le j \le k-2).$$

3.24. This system is solved by iterated integrals as before, and the solutions are

$$a_{j,k}(\xi) = \frac{1}{(k-j)!}(-(2\pi i)^{-1}\log(\xi))^{k-j} \quad (2 \le k \le n,\ 1 \le j \le k-1),$$

$$a_{j,n+1}(\xi) = \frac{1}{(n+1-j)!}(-(2\pi i)^{-1}\log(\xi))^{n+1-j} + (-(2\pi i)^{-1})^{n+1-j} l_{n+1-j}(\xi) \quad (1 \le j \le n).$$

3.25. Table of solutions:

$$\begin{pmatrix} 1 & a_{1,2} & \dots & a_{1,n} & a_{1,n+1} \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & 0 & \ddots & a_{n-1,n} & a_{n-1,n+1} \\ \vdots & \vdots & \ddots & 1 & a_{n,n+1} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -(2\pi i)^{-1}\log(\xi) & \dots & \frac{(-(2\pi i)^{-1}\log(\xi))^{n-1}}{(n-1)!} & \frac{(-(2\pi i)^{-1}\log(\xi))^{n}}{n!} + (-(2\pi i)^{-1})^n l_n(\xi) \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & 0 & \ddots & -(2\pi i)^{-1}\log(\xi) & \frac{(-(2\pi i)^{-1}\log(\xi))^{2}}{2!} + (-(2\pi i)^{-1})^2 l_2(\xi) \\ \vdots & \vdots & \ddots & 1 & -(2\pi i)^{-1}\log(\xi) - (2\pi i)^{-1} l_1(\xi) \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

3.26. Let now $\sigma_\infty = \mathbf{R}_{\geq 0} N_\infty$ with $N_\infty := -N_0 + N_1$ for N_0, N_1 in 3.5. Let

$$F = F(-\alpha', \beta', \lambda'_2, \dots, \lambda'_n)$$

be as in 3.12. By Lemma 3.7 (i) (3), (N_∞, F) satisfies the Griffiths transversality if and only if $\beta' = -\alpha', \lambda'_2 = \frac{(-\alpha')^2}{2!}, \dots, \lambda'_{n-1} = \frac{(-\alpha')^{n-1}}{(n-1)!}$. If this is the case, (N_∞, F) generates a σ_∞-nilpotent orbit, since admissibility and $\mathbf{R}$-polarizability on gr^W trivially hold. We describe the local structure of $\Gamma_{\sigma_\infty} \setminus \mathcal{D}_{\sigma_\infty}$ near the image p_∞ of this nilpotent orbit.

Let

$$Y := \left\{ (q', \beta', \lambda'_2, \dots, \lambda'_n) \in \mathbf{C}^{n+1} \,|\, \beta' = -\alpha', \lambda'_2 = \frac{(-\alpha')^2}{2!}, \dots, \lambda'_{n-1} = \frac{(-\alpha')^{n-1}}{(n-1)!} \text{ if } q' = 0 \right\}$$

be the log manifold with the strong topology, with the structure sheaf of rings which is the inverse image of the sheaf of holomorphic functions on $\mathbf{C}^{n+1}$, and with the log structure generated by q'. Then there is an open neighborhood U of $(0, 0, \dots, 0, \lambda'_n)$ in $\mathbf{C}^{n+1}$ and an open immersion

$$Y \cap U \hookrightarrow \Gamma_{\sigma_\infty} \setminus \mathcal{D}_{\sigma_\infty}$$

of log manifolds which sends $(q', \beta', \lambda'_2, \dots, \lambda'_n) \in Y \cap U$ with $q' \neq 0$ to the class of

$$F(-\alpha', \beta', \lambda'_2, \dots, \lambda'_n),$$

where $\alpha' \in \mathbf{C}$ is such that $q' = e^{2\pi i \alpha'}$, and which sends $(0, 0, \dots, 0, \lambda'_n)$ to p_∞.

3.27. Near $x = \infty$, i.e., $\xi = 0$, a nilpotent orbit in naive sense is

$$(1) \qquad \exp((2\pi i)^{-1}\log(\xi) N_\infty) F(0, 0, \dots, 0, \lambda'^{0}_n)$$

$$= F\Big(-(2\pi i)^{-1}\log(\xi), -(2\pi i)^{-1}\log(\xi), \frac{(-(2\pi i)^{-1}\log(\xi))^2}{2!}, \dots, \frac{(-(2\pi i)^{-1}\log(\xi))^n}{n!} + \lambda'^{0}_n\Big),$$

where $\lambda'^{0}_n = (-(2\pi i)^{-1})^n l_n(0)$. The corresponding "higher Albanese map" (i.e., local version about ∞ of $\tilde{\varphi}$ in 3.13) is

$$(2) \qquad F\Big(-(2\pi i)^{-1}\log(\xi), -(2\pi i)^{-1}\log(\xi) - (2\pi i)^{-1} l_1(\xi), \dots, \frac{(-(2\pi i)^{-1}\log(\xi))^n}{n!} + (-(2\pi i)^{-1})^n l_n(\xi)\Big)$$

under the condition $l_j(0) = 0$ $(1 \leq j \leq n-1)$. These two are asymptotic when ξ goes to the boundary point b' described as follows.

Changing ∞ and ξ into 0 and x, respectively, $b' = (\infty, 1)$ corresponds to the tangential boundary point $(0, 1)$ of Deligne, i.e., b' is the tangential base point over $\infty \in \overline{X}$ with tangent $v' \in T_\infty(\overline{X}) = \mathrm{Hom}_{\mathbf{C}}(m_\infty / m_\infty^2, \mathbf{C})$ defined by $v'(\xi) = 1$.

This corresponds to our boundary point $b' = (y', a')$ with $y' = (\infty, h') \in \overline{X}^{\log}$ described as follows. Let u' be the branch of $\log(\xi)$ having real value on $\mathbf{R}_{>0}$. The argument function

$$h' : M^{\mathrm{gp}}_{\overline{X},\infty} = \mathcal{O}^{\times}_{\overline{X},\infty} \times \xi^{\mathbf{Z}} \to \mathbf{S}^1$$

is a group homomorphism sending $f \in \mathcal{O}^{\times}_{\overline{X},\infty}$ to $f(\xi = 0)/|f(\xi = 0)|$ and ξ to $v'(\xi)/|v'(\xi)| = 1$, and the specialization $a' : \mathcal{O}^{\log}_{\overline{X},y'} = \mathbf{C}\{\xi\}[u'] \to \mathbf{C}$ is a ring homomorphism sending ξ to 0 and u' to $a'(u') = \log(v'(\xi)) = \log(1) = 0$.

3.28. As above, let u' be the branch of $\log(\xi)$ and T an indeterminate over $\mathcal{O}_{\overline{X},\infty}$. Then, by 1.1.1, we have an isomorphism $\mathcal{O}^{\log}_{\overline{X},y'} = \mathcal{O}_{\overline{X},\infty}[u'] \simeq \mathcal{O}_{\overline{X},\infty}[T]$ of $\mathcal{O}_{\overline{X},\infty}$-algebras under $(2\pi i)^{-1}u' \leftrightarrow T$. Consider an $\mathcal{O}_{\overline{X},\infty}$-algebra homomorphism $\mathcal{O}_{\overline{X},\infty}[T] \to \mathcal{O}_{\overline{X},\infty}$, $T \mapsto \xi$.

Let $\alpha' = (2\pi i)^{-1}\log(\xi)$. Then, as $\xi \to 0$, $\exp(-\alpha' N_\infty)(F$ in 3.27 (2)) converges to

$$F(0, 0, \dots, 0, \lambda'^{0}_{n})$$

in $\mathcal{D}$ (3.8), and hence the class of (F in 3.27 (2)) converges to the class p_∞ (3.26) of the nilpotent orbit $(\sigma_\infty, \exp(\sigma_{\infty,\mathbf{C}})F(0, 0, \dots, 0, \lambda'^{0}_{n}))$ in $\Gamma_{\sigma_\infty} \setminus \mathcal{D}_{\sigma_\infty}$. We thus have an extension of the higher Albanese map over $\overline{X}_\infty$ (Theorem 2.11 (2)):

$$\overline{\varphi}_\infty : \overline{X}_\infty \to \Gamma_{\sigma_\infty} \setminus \mathcal{D}_{\sigma_\infty}.$$

This is a morphism in the category $\mathcal{B}(\log)$. The log structure on the source (resp. the target) is given by ξ (resp. q). The pullback of the universal log mixed Hodge structure on the target coincides with the log mixed Hodge structure on the source.

3.29. By using log mixed Hodge theory, 3.27 is described as follows.

Taking the images of the nilpotent orbit in naive sense 3.27 (1) and the "higher Albanese map" 3.27 (2), we have their real analytic extensions with boundary

$$\overline{\nu}^{\log}_\infty,\ \overline{\varphi}^{\log}_\infty : \overline{X}^{\log}_\infty \to (\Gamma_{\sigma_\infty} \setminus \mathcal{D}_{\sigma_\infty})^{\log}.$$

Here, $\overline{X}^{\log}_\infty$ is like Example 1.1.1, and $(\Gamma_{\sigma_\infty} \setminus \mathcal{D}_{\sigma_\infty})^{\log}$ coincides with the moduli of nilpotent i-orbits $\Gamma_{\sigma_\infty} \setminus \mathcal{D}^{\sharp}_{\sigma_\infty}$ in the present situation ([10] III Theorem 2.5.6).

Let $\tilde{\overline{X}}^{\log}_\infty$ be the universal covering of $\overline{X}^{\log}_\infty$. The above maps are still lifted to

$$\tilde{\overline{\nu}}^{\log}_\infty,\ \tilde{\overline{\varphi}}^{\log}_\infty : \tilde{\overline{X}}^{\log}_\infty \to \mathcal{D}^{\sharp}_{\sigma_\infty}.$$

The boundary point b' in 3.27 can be understood as the point

$$b' = (z' = 0 + i\infty) = (u' = -\infty + i0) \in \tilde{\overline{X}}^{\log}_\infty$$

(where $2\pi i z' := u'$). We have $(\exp(-(2\pi i)^{-1}\log(\xi)N_\infty)(3.27\,(2)))(b') = F(0, 0, \dots, 0, \lambda'^{0}_{n})$, and

$$\tilde{\overline{\nu}}^{\log}_\infty(b') = \tilde{\overline{\varphi}}^{\log}_\infty(b') = (\text{nilpotent } i\text{-orbit generated by } (N_\infty, F(0, 0, \dots, 0, \lambda'^{0}_{n}))) \in \mathcal{D}^{\sharp}_{\sigma_\infty}.$$

3.30. For any $\sigma \in \Sigma$, $\Gamma_\sigma \setminus \mathcal{D}_\sigma \to \Gamma^{(n)} \setminus \mathcal{D}_\Sigma$ is a local homeomorphism. This is analogously proved as [12] Theorem A (iv).

Summing-up, we have a global extension over $\overline{X}$ of the higher Albanese map which is an isomorphism over its image:

$$\overline{\varphi} : \overline{X} \xrightarrow{\sim} \overline{\varphi}(\overline{X}) \subset A_{X,\Gamma^{(n)},\Sigma} \simeq \Gamma^{(n)} \setminus \mathcal{D}_\Sigma.$$

3.31. To study analytic continuations and extensions of polylogarithms in the spaces of nilpotent i-orbits $D^\sharp_\Sigma$, in the spaces of SL(2)-orbits $D_{\mathrm{SL}(2)}$, and in spaces of Borel-Serre orbits D_{BS} is an interesting problem. See [10] for these extended period domains and their relations which are described as a fundamental diagram.

A. Summary of a result of Deligne in [3]

We add here a summary of a result of Deligne in [3] for readers' convenience.

A.1. Just as 3.1–3.2, consider the situation $X := \mathbf{P}^1(\mathbf{C}) \smallsetminus \{0,1,\infty\} \subset \overline{X} := \mathbf{P}^1(\mathbf{C})$. Let $b := (0,1)$ the "tangential base point" over $0 \in \overline{X}$ with tangent 1.

Consider the quotient group Γ of $\pi_1(X,b)$ as in [3] 16.14 (cf. 3.2): The inclusion

$$X \subset \mathbf{G}_m(\mathbf{C}) = \mathbf{C}^\times$$

induces $\pi_1(X,b) \to \pi_1(\mathbf{G}_m(\mathbf{C}),b) = \mathbf{Z}(1)_B$ (suffix B means Betti, cf. [3]). Let

$$K := \mathrm{Ker}\,(\pi_1(X,b) \to \mathbf{Z}(1)_B).$$

Let $\Gamma := \pi_1(X,b)/[K,K]$ and $\Gamma_1 := K/[K,K]$. Then, we have an exact sequence

$$1 \to \Gamma_1 \to \Gamma \to \mathbf{Z}(1)_B \to 1.$$

A.2. ([3] 16.15). Let $\mu_0, \mu_1 : \mathbf{Z}(1)_B \to \Gamma$ be the monodromies around $0,1$, respectively. Take a generator u of $\mathbf{Z}(1)_B$ (e.g. $u = 2\pi i$), put $a_j = \mu_j(u)$ $(j = 0,1)$. Then, $\Gamma = \langle a_0, a_1 \rangle$ with relation: conjugates of a_1 are commutative.

Γ_1 is a representation of $\mathbf{Z}(1)_B$ with basis (conjugates of a_1) under the action

$$\gamma \mapsto \mu_0(t)\gamma\mu_0(t)^{-1} \ (\gamma \in \Gamma_1, t \in \mathbf{Z}(1)_B),$$

i.e., $\Gamma_1 = \mathbf{Z}[\mathbf{Z}(1)_B] \cdot a_1$, where $\sum_k c_k(a_0^k a_1 a_0^{-k}) = \sum_k c_k \cdot (2\pi i \cdot k) \cdot a_1$.

These are described as

$$\Gamma_1 = \mathbf{Z}[\mathbf{Z}(1)_B] \cdot a_1 \simeq \mathbf{Z}[u,u^{-1}] \cdot \frac{du}{u}, \quad \Gamma = \mathbf{Z}(1)_B \ltimes \Gamma_1,$$

$$\sum_k c_k(a_0^k a_1 a_0^{-k}) = \sum_k c_k \cdot (2\pi i \cdot k) \cdot a_1 \simeq \sum_k c_k u^k \frac{du}{u}$$

([3] 16.16). Action of $\mathbf{Z}(1)_B$ on Γ_1 is given by multiplication in $\mathbf{Z}[\mathbf{Z}(1)_B] = \mathbf{Z}[u,u^{-1}]$.

A.3. The descending central series of Γ induces a filtration on Γ_1:

$$Z^N(\Gamma) \cap \Gamma_1 = ((u-1)^{N-1}) \cdot \frac{du}{u} \quad (N \geq 1)$$

Let $\Gamma^{(N)} := \Gamma/Z^{N+1}(\Gamma)$ and $\Gamma_1^{(N)} := \mathrm{Image}(\Gamma_1 \to \Gamma^{(N)})$. Then

$$\Gamma_1^{(N)} = \mathbf{Z}[u,u^{-1}]/(u-1)^N \cdot \frac{du}{u}.$$

Put $u = e^v$ and hence $v = \log u$. Then

$$\mathbf{Q} \otimes \Gamma_1^{(N)} = \mathbf{Q}[u,u^{-1}]/(u-1)^N \cdot \frac{du}{u} = \mathbf{Q}[v]/(v^N) \cdot dv$$

and we have

$$\Gamma_1^{(N)} = \left\{ \sum_{k=0}^{N-1} c_k \exp(kv) dv \,\middle|\, c_k \in \mathbf{Z} \right\}.$$

A.4. As groups, identify

$$\varphi : \mathbf{Q}[v]/(v^N) \cdot dv \xrightarrow{\sim} \prod_1^N \mathbf{Q}(n)_B : \quad v^{n-1}dv = \frac{1}{n}dv^n \mapsto u^{\otimes n}.$$

Then

$$\sum_{k=0}^{N-1} c_k \exp(kv)dv \quad \overset{\varphi}{\mapsto} \quad \sum_{k=0}^{N-1} c_k \left(\sum_{n=0}^{N-1} \frac{1}{n!} k^n u^{\otimes n} \right) \otimes u = \sum_{n=1}^{N} \left(\sum_{k=0}^{N-1} c_k \frac{k^{n-1}}{(n-1)!} \right) u^{\otimes n}.$$

Hence

Proposition A.4.1. ([3] 16.17). $\quad (n-1)! \cdot \mathrm{pr}_n \circ \varphi(\Gamma_1^{(N)}) = \mathbf{Z}(n)_B.$

A.5. ([3] 16.12). Define a Lie algebra action of $\mathbf{Q}(1)$ on $\prod_1^N \mathbf{Q}(n)$ by

$$a * (b_1, b_2, \ldots, b_N) = (0, ab_1, \ldots, ab_{N-1}),$$

and $\mathbf{Q}(1) \ltimes \prod_1^N \mathbf{Q}(n)$ the associated semi-direct product of Lie algebra.

Let $\mu_0, \mu_1 : \mathbf{Q}(1) \to \mathbf{Q}(1) \ltimes \prod_1^N \mathbf{Q}(n)$ be morphisms of Lie algebras such that μ_0 is the identity onto the first factor $\mathbf{Q}(1)$ and μ_1 is the identity onto the factor $\mathbf{Q}(1)$ in the product $\prod_1^N \mathbf{Q}(n)$.

By abuse of notation, let $\mu_0, \mu_1 : \mathbf{Q}(1) \to \mathbf{Q} \otimes \mathrm{Lie}\,\Gamma^{(N)}$. Then there exists a unique Lie algebra isomorphism respecting each μ_0, μ_1:

$$\mathbf{Q}(1) \ltimes \prod_1^N \mathbf{Q}(n) \xrightarrow{\sim} \mathbf{Q} \otimes \mathrm{Lie}\,\Gamma^{(N)} = \mathbf{Q}(1) \ltimes (\mathbf{Q} \otimes \mathrm{Lie}\,\Gamma_1^{(N)})$$

which is given by μ_0 and $\nu_n := (\mathrm{ad}\,\mu_0)^{n-1}(\mu_1)$ $(1 \le n \le N)$.

A.6. Let $\mathrm{Lie}\,U_{\mathrm{DR}}^{(N)}$ be the de Rham realization of iterated Tate motive in [3] 16.13. Let

$$e_\alpha := \mu_\alpha(1) \in \mathrm{Lie}\,U_{\mathrm{DR}}^{(N)} \quad \big(1 = \exp(2\pi i) \in \mathbf{Q}(1)_{\mathrm{DR}},\ \alpha = 0, 1\big).$$

Take coordinates $(u, (v_n)_{1 \le n \le N})$ of $U_{\mathrm{DR}}^{(N)}$ as follows:

$$(u, (v_n)_n) \mapsto \exp(ue_0) \exp\left(\sum_{n=1}^{N} v_n (\mathrm{Ad} e_0)^{n-1}(e_1) \right).$$

Lemma A.6.1. ([3] 19.3.1). *Let $z \in \mathbf{C}^\times \smallsetminus \mathbf{R}_{\ge 1}$. The end point of the image in $U_{DR}^{(N)}(\mathbf{C})$ of the line segment from $(0, z)$ to z has coordinates $u = 0$, $v_n = -l_n(z)$.*

Proof. Let $z_1, z_2 \in \mathbf{C}^\times \smallsetminus \mathbf{R}_{\ge 1}$. Take a path from z_1 to z_2, and take an iterated integral $I_{z_1}^{z_2}$ of

$$dI(t) = \left(\frac{dt}{t} e_0 + \frac{dt}{t-1} e_1 \right) \cdot I(t)$$

for $I(t) = 1 + ue_0 + \sum_n v_n (\mathrm{Ad} e_0)^{n-1}(e_1)$. Note

$$e_0 * e_0 = e_0,\ e_0 * (\mathrm{Ad} e_0)^{n-1}(e_1) = (\mathrm{Ad} e_0)^n(e_1) \quad (1 \le n \le N),$$

$$e_1 * e_0 = 0,\ e_1 * e_1 = e_1,\ e_1 * (\mathrm{Ad} e_0)^{n-1}(e_1) = 0 \quad (2 \le n \le N).$$

The corresponding differential equation is

$$du = \frac{dt}{t}, \quad dv_1 = \frac{dt}{t-1}, \quad dv_n = v_{n-1} \frac{dt}{t}.$$

Take $I(z_1) = \text{identity} \in U_{\mathrm{DR}}^{(N)}(\mathbf{C})$ as an initial condition and consider z_2 as a variable.

If z_1 is a tangential base point $(0,\tau)$ ([3] Section 15), replace the initial condition by

$$I(t)\exp\left(-\log\left(\frac{t}{\tau}\right)\right)\to \text{identity}\quad \text{as } t\to 0.$$

For the line segment from $(0,z)$ to z, we have

$$u=\log\left(\frac{t}{z}\right),\quad v_n=-l_n(t).\quad \square$$

Acknowlegements. The author thanks Kazuya Kato and Chikara Nakayama for joint research. This note grew up in the preparation for and during the workshop "Hodge theory and algebraic geometry" held at the end of August, 2017, Tokyo Denki University. The author thanks Tetsushi Ito, Satoshi Minabe, Taro Fujisawa, and Atsushi Ikeda for the good occasion and helpful discussions. The author thanks the referee for careful reading and valuable comments.

S. Usui was partially supported by JSPS Grants-in-Aid for Scientific Research (C) 17K05200.

References

[1] E. Cattani, A. Kaplan and W. Schmid, *Degeneration of Hodge structures*, Ann. of Math. **123** (1986), 457–535.

[2] P. Deligne, *Travaux de Shimura*, Séminaire Bourbaki (1970/71), Exp. 389, Lecture Notes in Math., **244**, Springer (1971), 123–165. DOI: 10.1007/bfb0058700

[3] P. Deligne, *Groupe fondamental de la droite projective moins trois points*, in "Galois group over **Q**", MSRI Publ. 16 (1989), 79–297. DOI: 10.1007/978-1-4613-9649-9_3

[4] P. Deligne, *Catégories tannakiennes*, in Grothendieck Festschrift, Springer (1990), 111–195.

[5] P. A. Griffiths, *Periods of integrals on algebraic manifolds. I. Construction and properties of modular varieties*, Amer. J. Math. **90** (1968), 568–626.

[6] R. M. Hain and S. Zucker, *Unipotent variation of mixed Hodge structure*, Invent. Math. **88** (1987), 83–124. DOI: 10.1007/bf01405093

[7] M. Kashiwara, *A study of variation of mixed Hodge structure*, Publ. RIMS, Kyoto Univ. **22** (1986), 991–1024.

[8] K. Kato, *On* SL(2)*-orbit theorems*, Kyoto J. Math. **54** (2014), 841–861. DOI: 10.1215/21562261-2801840

[9] K. Kato and C. Nakayama, *Log Betti cohomology, log étale cohomology, and log de Rham cohomology of log schemes over* **C**, Kodai Math. J. **22** (1999), 161–186. DOI: 10.2996/kmj/1138044041

[10] K. Kato, C. Nakayama and S. Usui, *Classifying spaces of degenerating mixed Hodge structures, I.*, Adv. Stud. Pure Math. **54** (2009), 187–222, *II.*, Kyoto J. Math. **51** (2011), 149–261, *III.*, J. Algebraic Geometry **22** (2013), 671–772, *IV.*, Kyoto J. Math. **58** (2018), 289-426, *V.*, in preparation.

[11] K. Kato, C. Nakayama and S. Usui, *Extended period domains, Algebraic groups, and higher Albanese manifolds*, in "Hodge theory and L^2-analysis", ALM 39 (2017), 451–473.

[12] K. Kato and S. Usui, *Classifying spaces of degenerating polarized Hodge structures*, Ann. Math. Studies **169**, Princeton University Press (2009). DOI: 10.1515/9781400837113

[13] S. Usui, *Variation of mixed Hodge structure arising from family of logarithmic deformations. II. Classifying space*, Duke Math. J. **51-4** (1984), 851–875. DOI: 10.1215/s0012-7094-85-05227-5

Sampei Usui, Graduate School of Science, Osaka University, Toyonaka, Osaka, 560-0043, Japan
Email address: usui@math.sci.osaka-u.ac.jp

Proceedings of the international conference
"Geometric and Algebraic Singularity Theory"
in honor of Goo Ishikawa
on the occasion of his 60th Birthday,
10-16 September 2017, Będlewo, Poland

www.journalofsing.org
Worldwide Center of Mathematics, LLC