Journal *of* Singularities
Volume 21 (2020), 249-267

Proc. of Geometric and Algebraic Singularity Theory, Będlewo, 2017
DOI: 10.5427/jsing.2020.21o

GEOMETRIC ALGEBRA AND SINGULARITIES OF RULED AND DEVELOPABLE SURFACES

JUNKI TANAKA AND TORU OHMOTO

Dedicated to Professor Goo Ishikawa on the occasion of his 60th birthday.

Abstract. Any ruled surface in $\mathbb{R}^3$ is described as a curve of unit dual vectors in the algebra of dual quaternions (=the even Clifford algebra $C\ell^+(0,3,1)$). Combining this classical framework and $\mathcal{A}$-classification theory of C^∞ map-germs $(\mathbb{R}^2,0) \to (\mathbb{R}^3,0)$, we characterize local diffeomorphic types of singular ruled surfaces in terms of geometric invariants. In particular, using a theorem of G. Ishikawa, we show that local topological type of singular developable surfaces is completely determined by vanishing order of the *dual torsion* $\check{\tau}$, that generalizes an old result of D. Mond for tangent developables of non-singular space curves. This work suggests that Geometric Algebra would be useful for studying singularities of geometric objects in classical Klein geometries.

1. Introduction

A *ruled surface* in Euclidean space $\mathbb{R}^3$ is a surface formed by a 1-parameter family of straight lines, called *rulings*; at least partly, it admits a parametrization of the form $F(s,t) = \boldsymbol{r}(s) + t\boldsymbol{e}(s)$ with $|\boldsymbol{e}(s)| = 1$, $s \in I$, $t \in \mathbb{R}$, where I is an open interval. A *developable surface* is a ruled surface which is locally planar (i.e. the Gaussian curvature is constantly zero). The parametrization map $F : I \times \mathbb{R} \to \mathbb{R}^3$ may be singular at some point (s_0, t_0), that is, the differential $dF(s_0,t_0)$ may have rank one, and then the surface (= the image of F) has a particularly singular shape around that point. In this paper, we study local diffeomorphic types of the singular surface and its bifurcations (see Fig.1). All maps and manifolds are assumed to be of class C^∞ throughout.

The main feature of this paper is to combine classical line geometry using *dual quaternions* [2, 3, 17, 21] and $\mathcal{A}$-classification theory of singularities of (frontal) maps $\mathbb{R}^2 \to \mathbb{R}^3$ [15, 5, 9, 8]. Here $\mathcal{A}$ denotes a natural equivalence relation in singularity theory of C^∞ maps; two map-germs $f, g : (\mathbb{R}^2,0) \to (\mathbb{R}^3,0)$ are *$\mathcal{A}$-equivalent* if there exist diffeomorphism-germs $\sigma : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ and $\varphi : (\mathbb{R}^3,0) \to (\mathbb{R}^3,0)$ such that $g = \varphi \circ f \circ \sigma^{-1}$. We simply say the *$\mathcal{A}$-type* of a map-germ to mean its $\mathcal{A}$-equivalence class. As a weaker notion, *topological $\mathcal{A}$-equivalence* is defined by taking σ and φ to be homeomorphism-germs. We also use the $\mathcal{A}$-equivalence with the target changes being rotations $\varphi \in SO(3)$, which is called *rigid equivalence* throughout the present paper. Our aim is to classify germs of parametrization maps F of ruled surfaces in $\mathbb{R}^3$ up to $\mathcal{A}$-equivalence and rigid equivalence.

1.1. **Ruled surfaces.** Geometric Algebra is a neat tool for studying motions in classical geometry; in case of Euclidean 3-space, it is the algebra of dual quaternions (e.g. Selig [21]). As an application, any ruled surface in $\mathbb{R}^3$ is described as a curve of *unit dual vectors*

$$\check{\boldsymbol{v}} : I \to \check{\mathbb{U}} \subset \mathbb{D}^3, \quad \check{\boldsymbol{v}}(s) = \boldsymbol{v}_0(s) + \varepsilon \boldsymbol{v}_1(s).$$

2010 *Mathematics Subject Classification.* 53A25, 53A05, 15A66, 57R45, 58K40.

Key words and phrases. Differential line geometry, Clifford algebra, Ruled surfaces, Developable surfaces, Singularities of smooth maps.

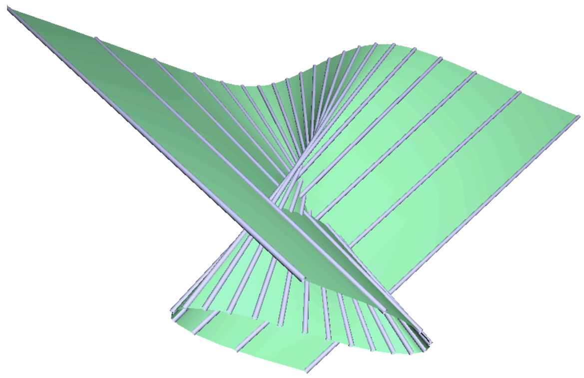

FIGURE 1. Deforming Mond's H_2-singularity via a family of ruled surfaces: the surface has two crosscaps and one triple point.

Here $\mathbb{D} = \mathbb{R} \oplus \varepsilon\mathbb{R}$ with $\varepsilon^2 = 0$ is the $\mathbb{R}$-algebra of *dual numbers*, and $\mathbb{D}^3 = \mathbb{R}^3 \oplus \varepsilon\mathbb{R}^3$ is the space of *dual vectors*, and especially, the space of unit dual vectors is given by

$$\check{\mathbb{U}} := \{\check{\boldsymbol{v}} = \boldsymbol{v}_0 + \varepsilon \boldsymbol{v}_1 \in \mathbb{D}^3,\ |\boldsymbol{v}_0| = 1,\ \boldsymbol{v}_0 \cdot \boldsymbol{v}_1 = 0\},$$

which is a 4-dimensional submanifold in the 6-dimensional space $\mathbb{D}^3$. Obviously, $\check{\mathbb{U}}$ is diffeomorphic to the total space of the (co)tangent bundle TS^2. It is naturally identified with the space of oriented lines in $\mathbb{R}^3$, by assigning to a unit dual vector $\check{\boldsymbol{v}}$ an oriented line $\boldsymbol{v}_0 \times \boldsymbol{v}_1 + t\boldsymbol{v}_0$ ($t \in \mathbb{R}$), see §2.1 for the detail. In our context, as the space of ruled surfaces in $\mathbb{R}^3$, we consider the space $C^\infty(I, \check{\mathbb{U}})$ of all smooth curves in $\check{\mathbb{U}}$ endowed with the Whintey C^∞-topology.

Assume that our ruled surface is *non-cylindrical*, i.e., $\boldsymbol{v}_0'(s) \neq 0$ for any $s \in I$, then the curve $\check{\boldsymbol{v}}$ admits the Frenet formula in $\mathbb{D}^3$ with *complete* differential invariants, the *dual curvature* and the *dual torsion*

$$\check{\kappa}(s) = \kappa_0(s) + \varepsilon\kappa_1(s), \quad \check{\tau}(s) = \tau_0(s) + \varepsilon\tau_1(s) \quad \in \mathbb{D}.$$

Here we may take s to be the arclength of the spherical curve $\boldsymbol{v}_0(s)$, that is equivalent to $\kappa_0(s) \equiv 1$, thus three real functions κ_1, τ_0, τ_1 are essential. In particular, $\kappa_1(s_0) = 0$ if and only if F is singular at (s_0, t_0) for some t_0; such t_0 is unique (Lemma 2.3).

We determine which $\mathcal{A}$-types of singular germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ appear in generic families of ruled surfaces. Assume that F is singular at $(s_0, t_0) = (0, 0)$ and $F(0, 0) = 0$, after taking parallel translations if needed. From the *dual Bouquet formula* of $\check{\boldsymbol{v}}$ at $s = 0$ in $\mathbb{D}^3$, we derive a *canonical* Taylor expansion of parameterization map F (§3.2), where $o(n)$ denotes Landau's notation of function-germs of order greater than n:

$$\begin{cases} x &= t - \frac{1}{2}ts^2 + \frac{\tau_1(0)}{2}s^3 + o(3), \\ y &= ts - \frac{\tau_1(0)}{2}s^2 - \frac{2\tau_0(0)\kappa_1'(0) + \tau_1'(0)}{6}s^3 + o(3), \\ z &= \frac{\kappa_1'(0)}{2}s^2 + \frac{\tau_0(0)}{2}ts^2 + \frac{\kappa_1''(0) - 2\tau_0(0)\tau_1(0)}{6}s^3 + o(3). \end{cases}$$

Then we apply to the jet of F the criteria for detecting $\mathcal{A}$-types of map-germs in Mond [14, 15].

Theorem 1.1. *The $\mathcal{A}$-classification of singularities of F arising in generic at most 3-parameter families of non-cylindrical ruled surfaces is given as in Table 1; in particular, for each $\mathcal{A}$-type in that table, the canonical expansion with the described condition is regarded as a normal form of the jet of ruled surface-germ under rigid equivalence.*

	normal form	ℓ	cond. at $s = s_0$ (with $\kappa_1(s_0) = 0$)
S_0	(x, y^2, xy)	2	$\kappa_1' \neq 0$
$S_1^{\pm}$	$(x, y^2, y^3 \pm x^2 y)$	3	$\kappa_1' = 0,\ \ \tau_1 \neq 0,\ \ \kappa_1''(\kappa_1'' - 2\tau_0\tau_1) \gtrless 0$
S_2	$(x, y^2, y^3 + x^3 y)$	4	$\kappa_1' = \kappa_1'' = 0,\ \kappa_1^{(3)}\tau_0\tau_1 \neq 0$
$B_2^{\pm}$	$(x, y^2, x^2 y \pm y^5)$		$\kappa_1' = 0,\ \kappa_1'' = 2\tau_0\tau_1 \neq 0,\ b_2 \gtrless 0$
H_2	$(x, xy + y^5, y^3)$		$\kappa_1' = \tau_1 = 0,\ \kappa_1'' \neq 0,\ h_2 \neq 0$
$S_3^{\pm}$	$(x, y^2, y^3 \pm x^4 y)$	5	$\kappa_1' = \kappa_1'' = \kappa_1^{(3)} = 0,\ \kappa_1^{(4)}\tau_0\tau_1 \gtrless 0$
$C_3^{\pm}$	$(x, y^2, xy^3 \pm x^3 y)$		$\kappa_1' = \kappa_1'' = \tau_0 = 0,\ \tau_1 \neq 0,\ \kappa_1^{(3)}(\kappa_1^{(3)} - 2\tau_0'\tau_1) \gtrless 0$
$B_3^{\pm}$	$(x, y^2, x^2 y \pm y^7)$		$\kappa_1' = 0,\ \kappa_1'' = 2\tau_0\tau_1 \neq 0,\ b_2 = 0,\ b_3 \gtrless 0$
H_3	$(x, xy + y^7, y^3)$		$\kappa_1' = \tau_1 = 0,\ \kappa_1'' \neq 0,\ h_2 = 0,\ h_3 \neq 0$
P_3	$(x, xy + y^3,$ $xy^2 + p_4 y^4)$		$\kappa_1' = \kappa_1'' = \tau_1 = 0,\ \tau_0\tau_1' \neq 0,\ p_4 \neq 0, 1, \frac{1}{2}, \frac{3}{2}.$

TABLE 1. $\mathcal{A}$-types of singularities of ruled surfaces. Assume that $\kappa_1(s_0) = 0$, then F is singular at a unique point lying on the ruling corresponding to s_0. This table characterizes the $\mathcal{A}$-type of the germ of F at that point. Here, $\kappa_1', \kappa_1'', \cdots$ denote derivatives at $s = s_0$ for short, e.g. κ_1' means $\frac{d}{ds}\kappa_1(s_0)$, and b_2, b_3, h_2, h_3, p_4 are some polynomials of those derivatives (see §3.2). The letters $\lesseqgtr, \gtreqless, \pm$ are in the same order. In the second column, ℓ means $\mathcal{A}$-codimension of the map-germ.

Precisely saying, via a variant of Thom's transversality theorem (§3.3), we show that there exists a dense subset $\mathcal{O}$ in the mapping space $\mathcal{R}_W$ consisting of families of non-cylindrical $\check{\boldsymbol{v}} : I \times W \to \check{\mathbb{U}}$ with parameter space W of dimension ≤ 3 so that for any family belonging to $\mathcal{O}$ and for any $\lambda \in W$, the germ of the corresponding paramatrization map $F(-, \lambda) : I \times \mathbb{R} \to \mathbb{R}^3$ at every point (s_0, t_0) is $\mathcal{A}$-equivalent to either an immersion-germ or one of the singular germs in Table 1.

Obviously, normal forms under rigid equivalence have functional moduli: those are nothing but $\kappa_1(s)$, $\tau_0(s)$ and $\tau_1(s)$ satisfying the prescribed condition on derivatives at $s = s_0$.

Remark 1.2. (**Realization**) Izumiya-Takeuchi [10] firstly proved in a rigorous way that a generic singularity of ruled surfaces is only of type crosscap S_0, and Martins and Nuño-Ballesteros [13] showed that any $\mathcal{A}$-simple map-germ $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is $\mathcal{A}$-equivalent to a germ of ruled surface. By our theorem, $\mathcal{A}$-types which are not realized by ruled surfaces must have $\mathcal{A}$-codimension ≥ 6. This is sharp: for example, the 3-jet $(x, y^3, x^2 y)$, over which there are $\mathcal{A}$-orbits of codimension 6, is never $\mathcal{A}^3$-equivalent to 3-jets of any non-cylindrical nor cylindrical ruled surfaces (Remark 3.3). The realizability of versal families of $\mathcal{A}$-types via families of ruled surfaces can also be verified: for each germ in Table 1, an $\mathcal{A}_e$-versal deformation is obtained via deforming three invariants κ_1, τ_0, τ_1 appropriately (Remark 3.4).

Remark 1.3. (**Conformal GA**) Our approach would be applicable to other Clifford algebras and corresponding geometries. For instance, Izumiya-Saji-Takahashi [9] classified local singularities of horospherical flat surfaces in Lorentzian space (conformal spherical geometry); a horospherical surface is described by a curve in the Lie algebra $\mathfrak{so}(3, 1)$. Conformal Geometric Algebra may fit with this setting as well and our approach should work.

Remark 1.4. (**Framed curves**) Take the space of dual vectors $\mathbb{D}^3$ instead of $\check{\mathbb{U}}$. A curve $I \to \mathbb{D}^3$ corresponds to a *framed curve*, which describes a 1-parameter family of Euclidean motions of $\mathbb{R}^3$;

	normal form	ℓ	cond. at $s = s_0$
cE	(x, y^2, y^3)	1	$\tau_0 \neq 0,\ \ \tau_1 \neq 0$
cS_0	(x, y^2, xy^3)	2	$\tau_1 \neq 0,\ \ \tau_0 = 0,\ \ \tau_0' \neq 0$
cS_1^+	$(x, y^2, y^3(x^2 + y^2))$	3	$\tau_1 \neq 0,\ \ \tau_0 = \tau_0' = 0,\ \ \tau_0'' \neq 0$
cC_3^+	$(x, y^2, y^3(x^3 + xy^2))$	4	$\tau_1 \neq 0,\ \ \tau_0 = \tau_0' = \tau_0'' = 0,\ \ \tau_0''' \neq 0$
Sw	$(x, xy + 2y^3, xy^2 + 3y^4)$	2	$\tau_0 \neq 0,\ \ \tau_1 = 0,\ \ \tau_1' \neq 0$
cA_4	$(x, xy + \frac{5}{2}y^4, xy^2 + 4y^5)$	3	$\tau_0 \neq 0,\ \ \tau_1 = \tau_1' = 0,\ \ \tau_1'' \neq 0$
cA_5	$(x, xy + 3y^5, xy^2 + 5y^6)^\dagger$	4	$\tau_0 \neq 0,\ \ \tau_1 = \tau_1' = \tau_1'' = 0,\ \ \tau_1''' \neq 0$
T_1	$(x, xy + y^3, 0) + o(3)$	3	$\tau_0 = \tau_1 = 0,\ \ \tau_1' \neq 0$
T_2	$(x, xy, 0) + o(3)$	4	$\tau_0 = \tau_1 = \tau_1' = 0$

TABLE 2. $\mathcal{A}$-types of singularities of developable surfaces. An exception is the type cA_5; the condition implies that the germ is *topologically* $\mathcal{A}$-equivalent to the normal form † (in this case, the striction curve σ is topologically determinative in the sense of Ishikawa [5]).

various geometric aspects of framed curves have recently been studied by e.g. Honda-Takahashi [4]. Since the dual Frenet formula is available for regular framed curves, we may rebuild the theory by using dual quaternions. That would be useful for singularity analysis in several topics of applied mathematics such as 3D-interpolation via ruled/developable surfaces, 1-parameter motions of axes in robotics, and so on (cf. [17, 21]).

1.2. **Developable surfaces.** For a non-cylindrical ruled surface, it is developable (the Gaussian curvature is constantly zero) if and only if $\kappa_1 = 0$ identically, see §2. Thus two real functions τ_0, τ_1 are complete invariants of such developables. Izumiya-Takeuchi [10] classified generic singularities of developable surfaces rigorously, and Kurokawa [12] treated a similar task for 1-parameter families of developables. We generalize their results systematically using the complete invariants.

Theorem 1.5. *The $\mathcal{A}$-classification of singularities of F arising in generic at most 2-parameter families of non-cylindrical developable surfaces is given as in Table 2; in particular, for each $\mathcal{A}$-type in that table, the canonical expansion with the described condition is regarded as a normal form of the jet of developable-germ under rigid equivalence.*

Remark 1.6. (**Realization**) In our classification process §4.1, we see that non-cylindrical developables do not admit $\mathcal{A}$-types

$$cS_1^- : (x, y^2, y^3(x^2 - y^2)) \ \text{ nor } \ cC_3^- : (x, y^2, y^3(x^3 - xy^2))$$

(for the former, it was shown in [12]), while cS_1^+ and cC_3^+ appear. Furthermore, $\tau_1 \neq 0$ and $\tau_0 = \tau_0' = \tau_0'' = 0$ if and only if the 5-jet of F is equivalent to $(x, y^2, 0)$, and thus, for instance, we see that frontal singularities of cuspidal S and B-types

$$cS_* : (x, y^2, y^3(y^2 + h(x, y^2))), \quad cB_* : (x, y^2, y^3(x^2 + h(x, y^2)))$$

$(h(x, y^2) = o(2))$ never appear in our developable surfaces. Similarly, since $\tau_1 = 0$ if and only if the 2-jet is reduced to $(x, xy, 0)$, wavefronts of cuspidal beaks/lips type $A_3^\pm$ and purse/pyramid types D_k never appear. Indeed, their 2-jets are equivalent to $(x, 0, 0)$ and $(x^2 \pm y^2, xy, 0)$ respectively (it is obvious to see no appearance of D_k, for the corank of our maps F is at most one).

A non-cylindrical developable surface, which is not a cone, is re-parametrized as the tangent developable of the *striction curve* $\sigma(s)$ (Lemma 2.4). Here $\sigma(s)$ may be singular; recall that for a possibly singular space curve, its tangent developable is defined by the closure of the union of tangent lines at smooth points; indeed, it is a frontal surface, see §2.4 (cf. Ishikawa [6]). A space curve-germ is said to be of type $(m, m+\ell, m+\ell+r)$ if it is $\mathcal{A}$-equivalent to the germ

$$x = s^m + o(m), \;\; y = s^{m+\ell} + o(m+\ell), \;\; z = s^{m+\ell+r} + o(m+\ell+r)$$

(the curve is said to be of *finite type* if $m, n, \ell < \infty$). A type of curve-germ is called *smoothly determinative* (resp. *topologically determinative*) if it determines the $\mathcal{A}$-type (resp. topological $\mathcal{A}$-type) of the tangent developable. Ishikawa [5, 6] gave the following complete characterization (Mond [16] for the case of $m = 1$, i.e. smooth curves):

(i) smoothly determinative types are only $(1, 2, 2+r)$, $(2, 3, 4)$, $(1, 3, 4)$, $(3, 4, 5)$ and $(1, 3, 5)$;
(ii) $(m, m+\ell, m+\ell+r)$ is topologically determinative if and only if ℓ or r is odd, or $m = 1$ and ℓ, r are both even.

Using this result, we obtain a complete topological $\mathcal{A}$-classification of singularities of non-cylindrical developable surfaces:

Theorem 1.7. (Topological classification) *For a non-cylindrical developable surface, the germ of its striction curve $\sigma(s)$ at $s = s_0$ has the type*

$$(m, m+1, m+1+r),$$

where $m-1$ and $r-1$ are orders of τ_1 and τ_0 at $s = s_0$, respectively, i.e.,

$$\tau_1 = \tau_1' = \cdots = \tau_1^{(m-2)} = \tau_0 = \tau_0' = \cdots = \tau_0^{(r-2)} = 0,$$
$$\tau_1^{(m-1)}\tau_0^{(r-1)} \neq 0.$$

In particular, topological $\mathcal{A}$-types of the germ of F at singular points are completely determined by orders of the dual torsion $\check{\tau} = \tau_0 + \varepsilon\tau_1$.

Remark 1.8. Theorem 1.7 is regarded as the *dual version* of a result of Mond [16] and Ishikawa [5]: $\mathcal{A}$-type of the tangent developable of a *non-singular* space curve σ with non-zero curvature is determined by the vanishing order of its torsion function. This is the case that σ is of type $(1, 2, 2+r)$, and then the torsion of σ has the same order of τ_0 (Lemma 2.4). Note that in our theorem above, $\sigma(s)$ can be *singular* (i.e., $m \geq 2$) and the non-zero curvature condition is replaced by the non-cylindrical condition.

Remark 1.9. Table 2 is separated into three parts. One is the case of $\tau_1(s_0) \neq 0$; they are the tangent developables of non-singular curves of type $(1, 2, 2+r)$, which are *frontal singularities* as mentioned in Remark 1.8. The second is the case of $\tau_0(s_0) \neq 0$; they are the tangent developables of singular curves of types $(2, 3, 4)$, $(3, 4, 5)$ and $(4, 5, 6)$, which are *wavefronts* – the former two types are smoothly determinative, while the third one is topologically determinative, by Ishikawa's characterization. In the remaining part, types T_0 and T_1 are tangent developable of curves of type $(2, 3, 4+r)$ $(r \geq 1)$. Tangent developables of curves of other types (e.g., $(1, 3, 3+r), (2, 4, 4+r)$) are cylindrical at $s = s_0$.

Remark 1.10. Not only striction curves but also several other kind of characteristic curves on a ruled surface can be discussed. For instance, flecnodal curves are important in projective differential geometry of surfaces [11, 20].

The rest of this paper is organized as follows. In §2, we briefly review two main ingredients for non-experts in each subject – the first is the algebra of dual quaternions, which is the most basic Geometric Algebra, and the second is about useful criteria for detecting $\mathcal{A}$-types in singularity theory of maps. In §3, we apply the $\mathcal{A}$-criteria to the canonical Taylor expansion of F at singular points and prove Theorem 1.1. In §4, we proceed to the case of developable surfaces and prove Theorems 1.5 and 1.7.

This paper is based on the first author's master thesis [22]. This work was supported by JSPS KAKENHI Grant Numbers JP15K13452, JP17H0612818 and JP18K18714.

2. Preliminaries

Geometric Algebra is a new look at Clifford algebras, which is nowadays recognized as a very neat tool for describing motions in Klein geometries in the context of a variety of applications to physics, mechanics and computer vision. In §§2.1 and 2.2, we give a very quick summary on the geometric algebra for 3-dimensional Euclidean motions and its application to the geometry of ruled surfaces. A good compact reference is the nineth chapter of Selig's textbook [21] (also see [2, 10, 7, 17]).

In §§2.3 and 2.4, we briefly describe some basic notions in Singularity Theory, which will be used in §§3 and 4. We deal with two classes of C^∞ maps from a surface into $\mathbb{R}^3$; ordinary smooth maps of corank at most one, i.e. $\dim\ker df \le 1$ (Mond [15]) and frontal maps (Ishikawa [5], Izumiya-Saji [8]).

2.1. **Dual quaternions.** Let $\mathbb{H}$ denote the field of quaternions: $q = a + bi + cj + dk$. The conjugate of q is $\bar{q} = a - bi - cj - dk$ and the norm is given by $|q| = \sqrt{q\bar{q}}$. Decompose $\mathbb{H}$ into the real and the imaginary parts, $\mathbb{H} = \mathbb{R} \oplus \operatorname{Im}\mathbb{H}$, where one identifies $bi + cj + dk \in \operatorname{Im}\mathbb{H}$ with $\boldsymbol{v} = (b, c, d)^T \in \mathbb{R}^3$ equipped with the standard inner and exterior products. We write $q = a + \boldsymbol{v}$, then the multiplication of $\mathbb{H}$ is written as

$$(a + \boldsymbol{v})(b + \boldsymbol{u}) = (ab - \boldsymbol{v}\cdot\boldsymbol{u}) + (a\boldsymbol{u} + b\boldsymbol{v} + \boldsymbol{v}\times\boldsymbol{u}).$$

The quaternionic unitary group

$$\mathbb{H}_1 = Sp(1) = \{q \in \mathbb{H}, |q| = 1\}$$

is naturally isomorphic to $SU(2)$, that doubly covers $SO(3)$; indeed, $\pm q \in \mathbb{H}_1$ defines the rotation $\boldsymbol{x} \mapsto q\boldsymbol{x}\bar{q}$. The Lie algebra of $\mathbb{H}_1$ is just $\operatorname{Im}\mathbb{H} = \mathbb{R}^3$.

Put $\mathbb{D} = \mathbb{R}[\varepsilon]/\langle\varepsilon^2\rangle$, and call it the algebra of *dual numbers*. A dual number $a + \varepsilon b$ is invertible if $a \neq 0$, and it has a square root if $a > 0$. The $\mathbb{R}$-algebra of *dual quaternions* is defined by

$$\check{\mathbb{H}} := \mathbb{D}^4 = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{D} = \{\ \check{q} = q_0 + \varepsilon q_1 \mid q_0, q_1 \in \mathbb{H}\ \}.$$

That is identified with the even Clifford algebra $C\ell^+(0,3,1)$ [21, §9.3]. The conjugate of $\check{q}$ is defined by $\check{q}^* := \bar{q}_0 + \varepsilon\bar{q}_1$, and then $\check{q}\check{q}^* = |q_0|^2 + \varepsilon\mathrm{Re}[q_1\bar{q}_0]$. The Lie group of *unit dual quaternions* is defined by

$$\check{\mathbb{H}}_1 := \{\ \check{q} \in \check{\mathbb{H}} \mid \check{q}\check{q}^* = 1\ \}.$$

This group is isomorphic to the semi-direct product $\mathbb{H}_1 \ltimes \operatorname{Im}\mathbb{H} = Sp(1) \ltimes \mathbb{R}^3$ via the correspondance $\check{q} \leftrightarrow (q_0, q_1\bar{q}_0)$. Then, $\check{\mathbb{H}}_1$ doubly covers $SE(3) = SO(3) \ltimes \mathbb{R}^3$, the group of Euclidean motions of $\mathbb{R}^3$; the action $\check{\Theta}$ of $\check{\mathbb{H}}_1$ on $\boldsymbol{x} \in \mathbb{R}^3$ is given by

$$1 + \varepsilon\check{\Theta}(\check{q})\boldsymbol{x} := \check{q}(1 + \varepsilon\boldsymbol{x})\check{q}^* = 1 + \varepsilon(q_0\boldsymbol{x}\bar{q}_0 + 2q_1\bar{q}_0).$$

That is, q_0 and $2q_1\bar{q}_0$ express a rotation and a parallel translation, respectively. The Lie algebra of $\check{\mathbb{H}}_1$ is canonically identified with the space of *dual vectors*

$$\mathbb{D}^3 = \operatorname{Im}\mathbb{H} \otimes_{\mathbb{R}} \mathbb{D}, \quad \check{\boldsymbol{v}} = \boldsymbol{v}_0 + \varepsilon\boldsymbol{v}_1 \quad (\boldsymbol{v}_0, \boldsymbol{v}_1 \in \operatorname{Im}\mathbb{H} = \mathbb{R}^3),$$

which is a $\mathbb{D}$-submodule of $\check{\mathbb{H}} = \mathbb{D}^4$. The standard inner and exterior products of $\mathbb{R}^3$ are extended to $\mathbb{D}$-bilinear operations on $\mathbb{D}^3$;

$$\check{\boldsymbol{u}}\cdot\check{\boldsymbol{v}} := -\tfrac{1}{2}(\check{\boldsymbol{u}}\check{\boldsymbol{v}} + \check{\boldsymbol{v}}\check{\boldsymbol{u}}) \in \mathbb{D}, \qquad \check{\boldsymbol{u}}\times\check{\boldsymbol{v}} := \tfrac{1}{2}(\check{\boldsymbol{u}}\check{\boldsymbol{v}} - \check{\boldsymbol{v}}\check{\boldsymbol{u}}) \in \mathbb{D}^3.$$

A *unit dual vector* means a dual vector $\check{\boldsymbol{v}} \in \mathbb{D}^3$ with $\check{\boldsymbol{v}} \cdot \check{\boldsymbol{v}} = 1$, i.e., $|\boldsymbol{v}_0| = 1$ and $\boldsymbol{v}_0 \cdot \boldsymbol{v}_1 = 0$ (it is also called a 2-*blade* in the Clifford algebra $C\ell(0,3,1)$ [21, §10.1]). Denote the set of unit dual vectors by $\check{\mathbb{U}}$, which is identified with the space of oriented lines in $\mathbb{R}^3$ in the following way:

$$\text{oriented lines} : \boldsymbol{v}_0 \times \boldsymbol{v}_1 + t\boldsymbol{v}_0 \quad \overset{1:1}{\longleftrightarrow} \quad \text{unit dual vectors} : \check{\boldsymbol{v}} = \boldsymbol{v}_0 + \varepsilon \boldsymbol{v}_1.$$

This expression is very useful [21, §9.3]: for instance,

(i) a point $\boldsymbol{a} \in \mathbb{R}^3$ lies on the line corresponding to a unit dual vector $\boldsymbol{v}_0 + \varepsilon\boldsymbol{v}_1$ if and only if $\boldsymbol{a} \times \boldsymbol{v}_0 = \boldsymbol{v}_1$;

(ii) two lines intersect perpendicularly if and only if the corresponding unit dual vectors $\check{\boldsymbol{u}}$ and $\check{\boldsymbol{v}}$ satisfy that $\check{\boldsymbol{u}} \cdot \check{\boldsymbol{v}} = 0$.

2.2. **Ruled and developable surfaces.** Using the identification just mentioned above, a ruled surface is exactly described as a curve of unit dual vectors:

$$\check{\boldsymbol{v}} : I \to \check{\mathbb{U}} \subset \mathbb{D}^3, \quad \check{\boldsymbol{v}}(s) = \boldsymbol{v}_0(s) + \varepsilon \boldsymbol{v}_1(s)$$

(I an open interval) with $|\boldsymbol{v}_0(s)| = 1$ and $\boldsymbol{v}_0(s) \cdot \boldsymbol{v}_1(s) = 0$ $(s \in I)$. Interpreting it as an object in $\mathbb{R}^3$, we have a parametrization

$$F(s,t) = \boldsymbol{r}(s) + t\boldsymbol{e}(s) \quad (\boldsymbol{r} = \boldsymbol{v}_0 \times \boldsymbol{v}_1, \quad \boldsymbol{e} = \boldsymbol{v}_0).$$

Note that $|\boldsymbol{e}(s)| = 1$ and $\boldsymbol{r} \cdot \boldsymbol{e} = 0$. Let R_s denote the ruling defined by $\check{\boldsymbol{v}}(s)$ and put

$$R = R(\check{\boldsymbol{v}}) := \bigcup_{s \in I} R_s \ \subset \mathbb{R}^3.$$

Formally, $\check{\boldsymbol{v}}(s)$ looks like a $\mathbb{D}$-version of the velocity vector of a space curve. That leads us to define the *curvature* $\check{\kappa}(s)$ of $\check{\boldsymbol{v}}$ by

$$\check{\kappa}(s) = \kappa_0(s) + \varepsilon\kappa_1(s) := \sqrt{\check{\boldsymbol{v}}'(s) \cdot \check{\boldsymbol{v}}'(s)} = |\boldsymbol{v}_0'| + \varepsilon \frac{\boldsymbol{v}_0' \cdot \boldsymbol{v}_1'}{|\boldsymbol{v}_0'|} \ \in \mathbb{D},$$

provided $\check{\boldsymbol{v}}$ is non-cylindrical, i.e., $\boldsymbol{v}_0'(s) \neq 0$ $(s \in I)$. Here $(\ )'$ means $\frac{d}{ds}$. From now on, we assume that

$$|\boldsymbol{v}_0'(s)| = 1$$

by taking s to be the arc-length of $\boldsymbol{v}_0$. Then, $\check{\kappa} = 1 + \varepsilon \boldsymbol{v}_0' \cdot \boldsymbol{v}_1'$ and thus $\check{\kappa}^{-1} = 1 - \varepsilon \boldsymbol{v}_0' \cdot \boldsymbol{v}_1'$. Put

$$\check{\boldsymbol{n}}(s) = \boldsymbol{n}_0(s) + \varepsilon \boldsymbol{n}_1(s) := \check{\kappa}^{-1} \check{\boldsymbol{v}}'(s),$$

and

$$\check{\boldsymbol{t}}(s) = \boldsymbol{t}_0(s) + \varepsilon \boldsymbol{t}_1(s) := \check{\boldsymbol{v}}(s) \times \check{\boldsymbol{n}}(s).$$

Then for every $s \in I$, three dual vectors $\check{\boldsymbol{v}}(s)$, $\check{\boldsymbol{n}}(s)$ and $\check{\boldsymbol{t}}(s)$ form a basis of the $\mathbb{D}$-module $\operatorname{Im} \check{\mathbb{H}} = \mathbb{D}^3$ satisfying

$$\check{\boldsymbol{v}} \times \check{\boldsymbol{n}} = \check{\boldsymbol{t}}, \quad \check{\boldsymbol{t}} \times \check{\boldsymbol{v}} = \check{\boldsymbol{n}}, \quad \check{\boldsymbol{n}} \times \check{\boldsymbol{t}} = \check{\boldsymbol{v}},$$
$$\check{\boldsymbol{v}} \cdot \check{\boldsymbol{n}} = \check{\boldsymbol{n}} \cdot \check{\boldsymbol{t}} = \check{\boldsymbol{t}} \cdot \check{\boldsymbol{v}} = 0, \quad \check{\boldsymbol{v}} \cdot \check{\boldsymbol{v}} = \check{\boldsymbol{n}} \cdot \check{\boldsymbol{n}} = \check{\boldsymbol{t}} \cdot \check{\boldsymbol{t}} = 1.$$

From these relations and the property (ii) of unit dual vectors mentioned before, we see that three lines corresponding to unit dual vectors $\check{\boldsymbol{v}}, \check{\boldsymbol{n}}, \check{\boldsymbol{t}}$ meet at one point and are mutually perpendicular; in particular, $\boldsymbol{v}_0, \boldsymbol{n}_0, \boldsymbol{t}_0$ forms an orthonormal basis of $\mathbb{R}^3$.

We define the *torsion* $\check{\tau}(s)$ of $\check{\boldsymbol{v}}$ by

$$\check{\tau}(s) = \tau_0(s) + \varepsilon\tau_1(s) := \check{\boldsymbol{n}}'(s) \cdot \check{\boldsymbol{t}}(s) \quad \in \mathbb{D}.$$

The following theorem is classical:

Theorem 2.1. (cf. Guggenheimmer [2, §8.2], Selig [21, §9.4]) *Assume that s is the arc-length of $\boldsymbol{v}_0$, i.e. $\kappa_0(s) = |\boldsymbol{v}_0'(s)| = 1$.*

(1) (Frenet formula) *It holds that*

$$\frac{d}{ds}\begin{bmatrix}\check{\boldsymbol{v}}(s)\\ \check{\boldsymbol{n}}(s)\\ \check{\boldsymbol{t}}(s)\end{bmatrix}=\begin{bmatrix}0 & \check{\kappa}(s) & 0\\ -\check{\kappa}(s) & 0 & \check{\tau}(s)\\ 0 & -\check{\tau}(s) & 0\end{bmatrix}\begin{bmatrix}\check{\boldsymbol{v}}(s)\\ \check{\boldsymbol{n}}(s)\\ \check{\boldsymbol{t}}(s)\end{bmatrix}.$$

(2) *The dual curvature $\check{\kappa}(s)$ and the dual torsion $\check{\tau}(s)$ are complete invariants of the ruled surface R up to Euclidean motions. That is, for two curves $\check{\boldsymbol{v}}_1$ and $\check{\boldsymbol{v}}_2$, they have the same invariants $\check{\kappa}$ and $\check{\tau}$ if and only if ruled surfaces $R(\check{\boldsymbol{v}}_1)$ and $R(\check{\boldsymbol{v}}_2)$ in $\mathbb{R}^3$ are transformed to each other by some Euclidean motion.*

(3) *$R(\check{\boldsymbol{v}})$ is a developable surface (including a cone) if and only if $\kappa_1 = 0$ identically. In particular, τ_0, τ_1 are complete invariants of the developable surface.*

The *striction curve* of a ruled surface R is the curve having minimal length which meets all the rulings of R. Let $F(s,t) = \boldsymbol{r}(s) + t\boldsymbol{e}(s)$ be a canonical parametrization

$$(\boldsymbol{r}\cdot\boldsymbol{e} = 0, \quad |\boldsymbol{e}| = |\boldsymbol{e}'| = 1),$$

then the striction curve $\sigma(s)$ is characterized by the equation $\sigma'\cdot\boldsymbol{e}' = 0$ (cf. [21, p.218], [10, Lemma 2.1], [17, §5.3]). We then have the following:

Lemma 2.2. *For a non-cylindrical ruled surface, it holds that*

(1) $\sigma(s) = \boldsymbol{r}(s) - (\boldsymbol{r}'(s)\cdot\boldsymbol{e}'(s))\boldsymbol{e}(s)$,

(2) $\sigma\times\boldsymbol{v}_0 = \boldsymbol{v}_1$, $\sigma\times\boldsymbol{n}_0 = \boldsymbol{n}_1$ *and* $\sigma\times\boldsymbol{t}_0 = \boldsymbol{t}_1$,

(3) $\sigma'(s) = \tau_1(s)\boldsymbol{v}_0(s) + \kappa_1(s)\boldsymbol{t}_0(s)$,

(4) $\kappa_1 = \det(\boldsymbol{e},\boldsymbol{e}',\boldsymbol{r}')$, $\tau_0 = \det(\boldsymbol{e},\boldsymbol{e}',\boldsymbol{e}'')$, $\tau_1 = \sigma'\cdot\boldsymbol{e}$.

From (2) and the property (i) of unit dual vectors in §2.1, it follows that $\sigma(s)$ lies on each of three lines corresponding to unit dual vectors $\check{\boldsymbol{v}}(s), \check{\boldsymbol{n}}(s), \check{\boldsymbol{t}}(s)$, that is, $\sigma(s)$ is the locus of the center of moving orthogonal frames. For completeness we prove the lemma, although it is elementary.

Proof : It is easy to see (1) by differentiating $\sigma(s) = \boldsymbol{r}(s) + t(s)\boldsymbol{e}(s)$. We show (2). First, by $\check{\boldsymbol{n}}\cdot\check{\boldsymbol{v}} = 0$, we see that $\boldsymbol{n}_1\cdot\boldsymbol{v}_0 = -\boldsymbol{v}_1\cdot\boldsymbol{n}_0$, and similarly $\boldsymbol{n}_1\cdot\boldsymbol{t}_0 = -\boldsymbol{t}_1\cdot\boldsymbol{n}_0$. By the Frenet formula, $\boldsymbol{v}_0' = \boldsymbol{n}_0$, $\boldsymbol{t}_0' = -\tau_0\boldsymbol{n}_0$, $\boldsymbol{v}_1' = \kappa_1\boldsymbol{n}_0 + \boldsymbol{n}_1$ and $\boldsymbol{t}_1' = -\tau_0\boldsymbol{n}_1 - \tau_1\boldsymbol{n}_0$. Since $\boldsymbol{r} = \boldsymbol{v}_0\times\boldsymbol{v}_1$ and $\boldsymbol{e} = \boldsymbol{v}_0$, it follows from (1) that

$$\sigma = -(\boldsymbol{t}_1\cdot\boldsymbol{n}_0)\boldsymbol{v}_0 - (\boldsymbol{v}_1\cdot\boldsymbol{t}_0)\boldsymbol{n}_0 - (\boldsymbol{n}_1\cdot\boldsymbol{v}_0)\boldsymbol{t}_0.$$

Thus $\sigma\times\boldsymbol{v}_0 = -(\boldsymbol{v}_1\cdot\boldsymbol{t}_0)\boldsymbol{n}_0\times\boldsymbol{v}_0 - (\boldsymbol{n}_1\cdot\boldsymbol{v}_0)\boldsymbol{t}_0\times\boldsymbol{v}_0 = (\boldsymbol{v}_1\cdot\boldsymbol{t}_0)\boldsymbol{t}_0 + (\boldsymbol{v}_1\cdot\boldsymbol{n}_0)\boldsymbol{n}_0 = \boldsymbol{v}_1$, for $\boldsymbol{v}_1\cdot\boldsymbol{v}_0 = 0$. That yields (2). Differentiating the first one of (2),

$$0 = (\sigma\times\boldsymbol{v}_0)' - \boldsymbol{v}_1' = (\sigma'\times\boldsymbol{v}_0 + \sigma\times\boldsymbol{n}_0) - (\kappa_1\boldsymbol{n}_0 + \boldsymbol{n}_1) = \sigma'\times\boldsymbol{v}_0 - \kappa_1\boldsymbol{n}_0$$

and similarly $\sigma'\times\boldsymbol{t}_0 + \tau_1\boldsymbol{n}_0 = 0$. Substitute $\sigma' = a\boldsymbol{v}_0 + b\boldsymbol{n}_0 + c\boldsymbol{t}_0$ for those equalities, we obtain $a = \tau_1$, $b = 0$, $c = \kappa_1$, that is (3). Finally, (4) is easy, e.g., $\kappa_1 = \boldsymbol{v}_0'\cdot\boldsymbol{v}_1' = \boldsymbol{e}'\cdot(\boldsymbol{r}'\times\boldsymbol{e}) = \det(\boldsymbol{e},\boldsymbol{e}',\boldsymbol{r}')$. □

Lemma 2.3. (cf. Izumiya et al [10, Lemma 2.2], [7, §1]) *For a non-cylindrical ruled surface, F is singular at (s_0, t_0) if and only if $\kappa_1(s_0) = 0$ and $t_0 = -\boldsymbol{r}'(s_0)\cdot\boldsymbol{e}'(s_0)$. The singular value $F(s_0, t_0)$ is the point $\sigma(s_0)$ where the curve $\sigma(s)$ is tangent to the ruling R_{s_0} or $\sigma'(s_0) = 0$.*

Proof : $\frac{\partial F}{\partial s}(s_0)\times\frac{\partial F}{\partial t}(s_0) = (\boldsymbol{r}'(s_0) + t_0\boldsymbol{e}'(s_0))\times\boldsymbol{e}(s_0) = 0 \Leftrightarrow \boldsymbol{r}'(s_0) = \alpha\boldsymbol{e}(s_0) - t_0\boldsymbol{e}'(s_0)$ for some $\alpha\neq 0 \Leftrightarrow \det(\boldsymbol{e}(s_0),\boldsymbol{e}'(s_0),\boldsymbol{r}'(s_0)) = 0$ and $t_0 = -\boldsymbol{r}'(s_0)\cdot\boldsymbol{e}'(s_0)$. The second claim follows from (3) in Lemma 2.2. □

In case of $\kappa_1 = 0$ identically, Lemmas 2.2 and 2.3 imply that singular points of F form a non-singular curve $s \mapsto (s, -\boldsymbol{r}'(s)\cdot\boldsymbol{e}'(s)) \in I \times \mathbb{R}$ and the image of this curve is just the striction curve $\sigma(s)$. Note that $\sigma(s)$ is a non-singular space curve, if $\tau_1 \neq 0$; especially, F is written by $\sigma(s) + \tilde{t}\sigma'(s)$ with $\tilde{t} = (t + \boldsymbol{r}'(s)\cdot\boldsymbol{e}'(s))/\tau_1$.

Lemma 2.4. (Izumiya et al [7, §1]) *A non-cylindrical developable surface, which is not a cone, is re-parametrized as the tangent developable of the striction curve $\sigma(s)$. The curve σ is non-singular whenever $\tau_1 \neq 0$, and then the curvature κ_σ and the torsion τ_σ of σ are given respectively by*

$$\kappa_\sigma = \frac{|\sigma' \times \sigma''|}{|\sigma'|^3} = \frac{1}{\tau_1}, \qquad \tau_\sigma = \frac{\det(\sigma', \sigma'', \sigma''')}{|\sigma' \times \sigma''|^2} = \frac{\tau_0}{\tau_1}.$$

2.3. **$\mathcal{A}$-classification of map-germs.** A *singular* point of $f : M \to N$ between manifolds means a point $p \in M$ where df_p is neither injective nor surjective (then $f(p) \in N$ is called a *singular value* of f); we denote by $S(f) \subset M$ the set of singular points of f. Two maps $\tilde{f} : U \to N$ and $\tilde{g} : V \to N$ on neighborhoods U and V of $p \in M$ define the same *map-germ at p* if there is a neighborbood $W \subset U \cap V$ of p so that $\tilde{f}|_W \equiv \tilde{g}|_W$; a *map-germ at p* is an equivalence class of maps under this relation, denoted by $f : (M, p) \to (N, f(p))$. Two map-germs at p have the same *k-jet* if they have the same Taylor polynomials at p of order k in some local coordinates; a k-jet is such an equivalence class of map-germs, denoted by $j^k f(p)$. Two germs $f : (M, p) \to (N, q)$ and $g : (M', p') \to (N', q')$ are *$\mathcal{A}$-equivalent* if they commute each other via diffeomorphism-germs σ and τ:

$$\begin{array}{ccc} (M,p) & \xrightarrow{f} & (N,q) \\ {\scriptstyle\sigma}\downarrow\simeq & & \simeq\downarrow{\scriptstyle\tau} \\ (M',p') & \xrightarrow[g]{} & (N',q') \end{array}$$

For simplicity, we consider map-germs $(\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$ and the $\mathcal{A}$-equivalence by the action of diffeomorphisms σ and τ preserving the origins. At the k-jet level, *$\mathcal{A}^k$-equivalence* is defined. A germ $f : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$ is said to be *k-$\mathcal{A}$-determined* if any germs $g : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$ with $j^k g(0) = j^k f(0)$ is $\mathcal{A}$-equivalent to f; such germs are collectively referred to as *finitely $\mathcal{A}$-determined* germs. For instance, the germ (x, y^2, xy) is 2-determined. Let $J^k(m, n)$ be the *jet space* consisting of all k jets of $(\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$, which is identified with the affine space of Taylor coefficients of order r ($1 \le r \le k$) in a fixed system of local coordinates. The codimension of the $\mathcal{A}$-orbit of a germ f in the space of all map-germs $(\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$ is called the *$\mathcal{A}$-codimension* of f; the $\mathcal{A}$-codimension of f is finite if and only if f is finitely $\mathcal{A}$-determined (see e.g. [1]).

Thanks to finite determinacy, the process of $\mathcal{A}$-classification is reduced to a finite dimensional problem: we stratify $J^k(m, n)$ invariantly under the $\mathcal{A}^k$-equivalence step by step from low order k and low codimension. For instance, using several determinacy criteria, $\mathcal{A}$-classification of map-germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ up to certain codimension has been established in Mond [14, 15]. In §3, we will follow Mond's classification process.

Furthermore, in Mond [14, 16], a special class of map-germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is considered. A map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is of class *CE* (i.e. cuspidal edge), if $\operatorname{rank} df(0) = 1$ and the singular point set $S(f)$ is non-singular. A germ f in CE is *k-$\mathcal{A}$-determined in CE* if any germ g in CE with the same k-jet as $j^k f(0)$ is $\mathcal{A}$-equivalent to f. In §4, we will use the following criteria of determinacy in CE [16, Lem.1.1, Prop.1.2].

Proposition 2.5 (Mond [16])**.** *It holds that*
i) If $f \in CE$ and $j^2 f(0) = (x, y^2, 0)$, then f is $\mathcal{A}$-equivalent to the germ

$$g(x, y) = (x, y^2, y^3 p(x, y^2))$$

for some smooth function $p(u, v)$;
ii) $f(x, y) = (x, y^2, y^3)$ is 3-determined in CE;
iii) $f(x, y) = (x, y^2, yp(x, y^2))$ and $g(x, y) = (x, y^2, yq(x, y^2))$ are $\mathcal{A}$-equivalent if and only if $\tilde{f}(x, y) = (x, y^2, y^3 p(x, y^2))$ and $\tilde{g}(x, y) = (x, y^2, y^3 q(x, y^2))$ are $\mathcal{A}$-equivalent. In particular, f is $(k-2)$-determined if and only if $\tilde{f}$ is k-determined in CE.

2.4. **Singularities of frontal surfaces.** There is a special class of surfaces, called *frontal surfaces*. Let $ST^*\mathbb{R}^3$ be the spherical cotangent bundle with respect to the standard metric of $\mathbb{R}^3$ equipped with the standard contact structure. Let U be an open set of $\mathbb{R}^2$. A map $\iota : U \to ST^*\mathbb{R}^3$ is called *isotropic* if it satisfies that the image $d\iota(T_pU)$ is contained in the contact plane $K_{\iota(p)}$ for any $p \in U$. A *frontal map* is the composed map $f = \pi \circ \iota : U \to \mathbb{R}^3$ of an isotropic map ι and the projection $\pi : ST^*\mathbb{R}^3 \to \mathbb{R}^3$. The image (possibly singular) surface is called to be *frontal*. An isotropic immersion ι is usually called a *Lagrange immersion*, and $\pi \circ \iota$ and its image are called a *Lagrange map* and a *wavefront*, respectively. Let $f : U \to \mathbb{R}^3$ be a frontal map with $\nu : U \to S^2$ so that $\iota = (f, \nu) : U \to ST^*\mathbb{R}^3 = \mathbb{R}^3 \times S^2$ is an isotropic map. We identifies $T\mathbb{R}^3 \simeq T^*\mathbb{R}^3$ using the standard metric, then the unit vector ν is always orthogonal to the subspace $df(T_pU)$ at any $p \in U$. Let x, y be coordinates of U and put $\lambda(x, y) = \det\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \nu\right](x, y)$; then the singular point set $S(f)$ is defined by $\lambda(x, y) = 0$. If $d\lambda(p) \neq 0$, then p is called a *non-degenerate* singular point. In particular, if p is non-degenerate and $\operatorname{rank} df_p = 1$, the germ f at p is of class CE.

For a developable surface with $\boldsymbol{e} \times \boldsymbol{e}' \neq 0$, set $f : U \to \mathbb{R}^3$ to be $f(s, t) := \boldsymbol{r}(s) + t\boldsymbol{e}(s)$. Then f is a frontal map; in fact, it suffices to put $\nu = \boldsymbol{e} \times \boldsymbol{e}'/|\boldsymbol{e} \times \boldsymbol{e}'|$ (then $\frac{\partial f}{\partial t} \cdot \nu = \boldsymbol{e} \cdot \nu = 0$ and $\frac{\partial f}{\partial s} \cdot \nu = (\boldsymbol{r}' + t\boldsymbol{e}') \cdot \nu = \det(\boldsymbol{r}', \boldsymbol{e}, \boldsymbol{e}') = 0$). Note that any singularities of f are non-degenerate and have corank one (see the comment before Lemma 2.4). There are two cases:

If $\iota = (f, \nu)$ is singular, then it is easy to see that the 2-jet of f is $\mathcal{A}^2$-equivalent to $(x, y^2, 0)$, and hence Mond's criteria for map-germs of class CE (Proposition 2.5) can be applied.

If ι is non-singular, i.e. ι is a Legendre immersion, then the 2-jet is equivalent to $(x, xy, 0)$, and thus Proposition 2.5 is useless. In this case, we employ the Legendre singularity theory. There are known useful criteria of [8] (precisely saying, the topological type cA_5 is not dealt in [8] but the same argument as in Appendix of [8] works as well):

Proposition 2.6. (Izumiya-Saji [8, Theorem 8.1]) *Let $f : U \to \mathbb{R}^3$ be a Legendre map, and p a non-degenerate singular point with rank $df_p = 1$. Let η be an arbitrary vector field around p so that $\eta(q)$ spans $\ker df_q$ at any $q \in S(f)$. Then f is $\mathcal{A}$-equivalent to cE, Sw, cA_4 or cA_5 if the following condition holds:*

$$\begin{array}{l|l} cE & \eta\lambda(p) \neq 0, \\ Sw & \eta\lambda(p) = 0,\ \eta\eta\lambda(p) \neq 0, \\ cA_4 & \eta\lambda(p) = \eta\eta\lambda(p) = 0,\ \eta\eta\eta\lambda(p) \neq 0, \\ cA_5 & \eta\lambda(p) = \eta\eta\lambda(p) = \eta\eta\eta\lambda(p) = 0,\ \eta\eta\eta\eta\lambda(p) \neq 0. \end{array}$$

Through the theory of frontal maps and generating functions, Ishikawa [5, 6] showed that the tangent developable of a curve of type

$$(m, m + \ell, m + \ell + r)$$

has a parametrization $F : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ defined by

$$\begin{aligned} x &= t, \\ y &= s^{m+\ell} + s^{m+\ell+1}\varphi(s) + t(s^\ell + s^{\ell+1}\phi(s)), \\ z &= (\ell+r)(m+\ell+r)\int_0^s u^r \frac{\partial y(u,t)}{\partial u} du \\ &= (\ell+r)(m+\ell)s^{m+\ell+r} + \cdots + t(\ell(m+\ell+r)s^{\ell+r} + \cdots) \end{aligned}$$

with some C^∞ functions $\varphi(s)$ and $\phi(s)$. These two function must be related to invariants τ_0 and τ_1. It is also shown [5, Thm 2.1] that the topological type of the tangent developable of a space curve is determined by type $(m, m+\ell, m+\ell+r)$ of the curve, unless both ℓ, r are even, as mentioned in Introduction.

3. Singularities of ruled surfaces

In this section, we prove Theorem 1.1 (2); first we give a certain stratification of the jet space of triples of functions $(\kappa_1, \tau_0, \tau_1)$, and then discuss a variant of Thom's transversality theorem.

3.1. **Dual Bouquet formula.** Consider a curve $\check{\boldsymbol{v}} : I \to \mathbb{D}^3$, $\check{\boldsymbol{v}}(s) = \boldsymbol{v}_0(s) + \varepsilon \boldsymbol{v}_1(s)$, with $\check{\boldsymbol{v}} \cdot \check{\boldsymbol{v}} = 1$ and $|\boldsymbol{v}_0'(s)| = 1$ as in §2.2. We are concerned with the germ of $\check{\boldsymbol{v}}$ at the origin ($s_0 = 0$). Throughout this section, let $\check\kappa, \check\tau, \check\kappa', \check\tau', \cdots$ denote their values at $s = 0$ for short, e.g. $\check\kappa' = \check\kappa'(0)$, unless specifically mentioned.

By iterated uses of the Frenet formula (Theorem 2.1 (1)), we obtain the "Bouquet formula" of the curve in $\mathbb{D}^3$ at $s = 0$;

$$\check{\boldsymbol{v}}(s) = \sum_{n=0}^{r} \frac{\check{\boldsymbol{v}}^{(n)}(0)}{n!} s^n + o(r) \quad \in \mathbb{D}^3$$

with

$$\begin{aligned} \check{\boldsymbol{v}}'(0) &= \check\kappa\, \check{\boldsymbol{n}}(0), \\ \check{\boldsymbol{v}}''(0) &= -\check\kappa^2\, \check{\boldsymbol{v}}(0) + \check\kappa'\, \check{\boldsymbol{n}}(0) + \check\kappa\check\tau\, \check{\boldsymbol{t}}(0), \\ \check{\boldsymbol{v}}^{(3)}(0) &= -3\check\kappa\check\kappa'\, \check{\boldsymbol{v}}(0) + (\check\kappa'' - \check\kappa^3 - \check\kappa\check\tau^2)\, \check{\boldsymbol{n}}(0) + (2\check\kappa'\check\tau + \check\kappa\check\tau')\, \check{\boldsymbol{t}}(0), \\ \check{\boldsymbol{v}}^{(4)}(0) &= (\check\kappa^4 + \check\kappa^2\check\tau^2 - 4\check\kappa\check\kappa'')\check{\boldsymbol{v}}(0) + (\check\kappa^{(3)} - 6\check\kappa^2\check\kappa' - 3\check\kappa'\check\tau^2 - 3\check\kappa\check\tau\check\tau')\check{\boldsymbol{n}}(0) \\ &\quad + (3\check\kappa''\check\tau + 3\check\kappa'\check\tau' - \check\kappa^3\check\tau + \check\kappa\check\tau'' - \check\kappa\check\tau^3)\check{\boldsymbol{t}}(0), \\ \check{\boldsymbol{v}}^{(5)}(0) &= (10\check\kappa^3\check\kappa' + 5\check\kappa\check\kappa'\check\tau^2 + 5\check\kappa^2\check\tau\check\tau' - 5\check\kappa\check\kappa^{(3)})\check{\boldsymbol{v}}(0) + (\check\kappa^{(4)} - 6\check\kappa^2\check\kappa'' - 6\check\kappa''\check\tau^2 \\ &\quad - 12\check\kappa'\check\tau\check\tau' - 3\check\kappa(\check\tau')^2 - 4\check\kappa\check\tau\check\tau'' + \check\kappa^3\check\tau^2 + \check\kappa\check\tau^4)\check{\boldsymbol{n}}(0) + (4\check\kappa^{(3)}\check\tau + 6\check\kappa''\check\tau' \\ &\quad + 3\check\kappa'\check\tau'' - 9\check\kappa^2\check\kappa'\check\tau - \check\kappa^3\check\tau' + \check\kappa'\check\tau'' + \check\kappa\check\tau^{(3)} - 4\check\kappa'\check\tau^3 - 6\check\kappa\check\tau^2\check\tau')\check{\boldsymbol{t}}(0), \end{aligned}$$

and so on. A similar but more naïve expansion written by Plücker coordinates, instead of dual quaternions, can be found in a classical book of Hlavatý [3].

Since dual vectors $\{\check{\boldsymbol{v}}(0), \check{\boldsymbol{n}}(0), \check{\boldsymbol{t}}(0)\}$ form a $\mathbb{D}$-basis of $\operatorname{Im}\check{\mathbb{H}} = \mathbb{D}^3$, we may write

$$\check{\boldsymbol{v}}(s) = [\,\check{\boldsymbol{v}}(0), \check{\boldsymbol{n}}(0), \check{\boldsymbol{t}}(0)\,]\, \check{\boldsymbol{w}}(s),$$

and by the above derivatives $\check{\boldsymbol{v}}^{(k)}(0)$, one computes

$$\check{\boldsymbol{w}}(s) = [1 - \tfrac{1}{2}\check\kappa^2 s + \cdots, \check\kappa s + \tfrac{1}{2}\check\kappa' s + \cdots, \tfrac{1}{2}\check\kappa\check\tau s^2 + \cdots]^T \in \mathbb{D}^3.$$

Recall that three oriented lines in $\mathbb{R}^3$ determined by unit dual vectors $\check{\boldsymbol{v}}(0), \check{\boldsymbol{n}}(0), \check{\boldsymbol{t}}(0)$ meet at one point, which is nothing but the striction point $\sigma(0)$, as mentioned just after Lemma 2.2. By an Euclidean motion, the triple of lines can be transformed to standard coordinate axises of $\mathbb{R}^3$, i.e., $\boldsymbol{v}_0(0), \boldsymbol{n}_0(0), \boldsymbol{t}_0(0)$ are sent to the standard basis i, j, k of $\operatorname{Im}\mathbb{H} = \mathbb{R}^3$, respectively, and

$\boldsymbol{v}_1(0) = \boldsymbol{n}_1(0) = \boldsymbol{t}_1(0) = 0 \in \mathbb{R}^3$. Namely, we may assume that the 3×3 matrix (with entries in $\mathbb{D}$) $[\,\check{\boldsymbol{v}}(0), \check{\boldsymbol{n}}(0), \check{\boldsymbol{t}}(0)\,]$ is the identity matrix, so $\check{\boldsymbol{v}}(s) = \check{\boldsymbol{w}}(s)$. Then

$$\check{\boldsymbol{v}}(s) = \boldsymbol{v}_0(s) + \varepsilon \boldsymbol{v}_1(s) = \begin{bmatrix} 1 \\ s \\ 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 \\ \kappa_1 s \\ 0 \end{bmatrix} + o(1).$$

At a point $(0, t_0) \in I \times \mathbb{R}$, the Taylor expansion of $F(s,t) = \boldsymbol{v}_0(s) \times \boldsymbol{v}_1(s) + t\boldsymbol{v}_0(s)$ is immediately obtained; in particular, $F(0,t_0) = [t_0, 0, 0]^T$ and

$$dF(0,t_0) = \begin{bmatrix} 0 & 1 \\ t_0 & 0 \\ \kappa_1 & 0 \end{bmatrix}.$$

This gives an alternative proof of Lemma 2.3: F is singular at $(0,t_0)$ if and only if $\kappa_1(0) = t_0 = 0$ ($t_0 = 0$ means that the point is just the striction point $\sigma(0)$ lying on the ruling). Assume that F is singular at the origin. Then we obtain a *canonical Taylor expansion* of F:

$$(1) \qquad \begin{aligned} &F(s,t) = \\ &\left(t - \tfrac{1}{2}ts^2 + \tfrac{\tau_1}{2}s^3,\ ts - \tfrac{\tau_1}{2}s^2 - \tfrac{2\tau_0\kappa_1' + \tau_1'}{6}s^3,\ \tfrac{\kappa_1'}{2}s^2 + \tfrac{\tau_0}{2}ts^2 + \tfrac{\kappa_1'' - 2\tau_0\tau_1}{6}s^3\right) + o(3). \end{aligned}$$

Remark 3.1. (Truncated polynomial maps) Let $F(s,t)$ be as in (1), and set

$$\bar{F}(s,t) = (\bar{\boldsymbol{v}}_0(s) \times \bar{\boldsymbol{v}}_1(s)) + t\bar{\boldsymbol{v}}_0(s)$$

to be a polynomial map of order k with $j^k\bar{F}(0) = j^kF(0)$. Denote by $\bar{s}$ the arc-length of the curve $\bar{\boldsymbol{v}}_0(s)$, then $\bar{s} := s + o(k)$, and thus k-jets at 0 of the dual curvature and the dual torsion do not change from those of F. That gives examples of polynomial ruled surfaces with prescribed k-jets of $\check{\kappa}$ and $\check{\tau}$ at a point.

3.2. **Recognition of singularity types.** Now our task is to find appropriate diffeomorphism-germs of the source and the target for reducing jets of $F(s,t)$ to normal forms in $\mathcal{A}$-classification step by step; for such computations, we have used the software *Mathematica*.

Let (X, Y, Z) be the coordinates of the target $\mathbb{R}^3$. Below, $\kappa_1, \kappa_1', \cdots$ denote their values at $s = 0$ unless specifically mentioned. From now on, assume that $\kappa_1 (= \kappa_1(0)) = 0$. Put $y = s$ and $x = t - \frac{1}{2}ts^2 + \frac{\tau_1}{2}s^3 + \cdots$ which is the first component of F in the form (1) above. With this new coordinates (x,y) of the source $\mathbb{R}^2$, we set

$$(2) \qquad \begin{aligned} f(x,y) &:= F(y, t(x,y)) = (x, f_2(x,y), f_3(x,y)) \\ &= \left(x,\ xy - \tfrac{1}{2}\tau_1 y^2 - \tfrac{1}{6}\tau_1' y^3,\ \tfrac{1}{2}\kappa_1' y^2 + \tfrac{1}{2}\tau_0 xy^2 + \tfrac{1}{6}(\kappa_1'' - 2\tau_0\tau_1)y^3\right) + o(3). \end{aligned}$$

Note that $f(x,y)$ is still of the form $\tilde{\boldsymbol{r}}(y) + x\tilde{\boldsymbol{e}}(y)$. Now, we apply to this germ $f(x,y)$ the recognition trees in Mond's classification [15, Figs.1, 2]. Below, $S_k^\pm$, $B_k^\pm$, $C_k^\pm$, H_k and F_4 denote Mond's notations of $\mathcal{A}$-simple germs [15].

- **2-jet**: Crosscap S_0 is 2-determined, thus it follows from (2) that

$$f \sim_{\mathcal{A}} S_0 : (x, xy, y^2) \iff \kappa_1 = 0, \quad \kappa_1' \neq 0.$$

Let $\kappa_1'=0$. Then the 2-jet is equivalent to either of $(x,xy,0)$ or $(x,y^2,0)$, according to whether $\tau_1=0$ or not. We compute the second and third component of f as

$$\begin{aligned}
f_2 &= xy-\tfrac{1}{2}\tau_1y^2-\tfrac{1}{6}\tau_1'y^3\\
&\quad+\tfrac{1}{24}((8-4\tau_0^2)xy^3+(-5\tau_1+3\tau_0^2\tau_1-3\tau_0\kappa_1''-\tau_1'')y^4)\\
&\quad+\tfrac{1}{120}(-15\tau_0\tau_0'xy^4+(12\tau_0\tau_0'\tau_1-9\tau_1'+6\tau_0^2\tau_1'-6\tau_0'\kappa_1''-4\tau_0\kappa_1^{(3)}-\tau_1^{(3)})y^5)\\
&\quad+o(5),\\
f_3 &= \tfrac{1}{6}(3\tau_0xy^2+(\kappa_1''-2\tau_0\tau_1)y^3)\\
&\quad+\tfrac{1}{24}(4\tau_0'xy^3+(-3\tau_1\tau_0'-3\tau_0\tau_1'+\kappa_1^{(3)})y^4)\\
&\quad+\tfrac{1}{120}((25\tau_0-5\tau_0^3+5\tau_0'')xy^4+(-16\tau_0\tau_1+4\tau_0^3\tau_1-6\tau_0'tau_1'-6\tau_0^2\kappa_1''\\
&\quad-4y^5\tau_1\tau_0''-4y^5\tau_0\tau_1''+\kappa_1^{(4)})y^5)+o(5).
\end{aligned}$$

• **3-jet**: Let $\kappa_1=\kappa_1'=0$ and $\tau_1\neq0$. First, let us remove the term xy from f_2; take $\bar{x}=x$ and $\bar{y}=y-\frac{1}{\tau_1}x$, then we see that

$$j^3f(0)\sim(x,y^2+\tfrac{\tau_1'}{\tau_1^3}x^2y+\tfrac{\tau_1'}{3\tau_1}y^3,\kappa_1''x^2y+\tfrac{1}{3}\tau_1^2(\kappa_1''-2\tau_0\tau_1)y^3). \tag{3}$$

The first two components can be transformed to (x,y^2) by a coordinate change of (x,y) with identical linear part and by a target coordinate change of (X,Y), since the plane-to-plane germ (x,y^2) is 2-determined (stable germ). Hence $j^3f(0)$ is equivalent to one of the following:

$$\begin{cases}
(x,y^2,y^3\pm x^2y) & \kappa_1''(\kappa_1''-2\tau_0\tau_1)\gtrless0,\ \tau_1\neq0 & \cdots S_1^{\pm},\\
(x,y^2,y^3) & \kappa_1''=0,\ \tau_0\tau_1\neq0 & \cdots S,\\
(x,y^2,x^2y) & \kappa_1''=2\tau_0\tau_1\neq0 & \cdots B,\\
(x,y^2,0) & \kappa_1''=\tau_0=0,\ \tau_1\neq0 & \cdots C.
\end{cases} \tag{4}$$

Note that $S_1^{\pm}$ is 3-determined, thus this case is clarified.

Let $\tau_1=0$. Then from (2), we have

$$j^3f(0)\sim\left(x,\ xy-\tfrac{1}{6}\tau_1'y^3,\tfrac{1}{2}\tau_0xy^2+\tfrac{1}{6}\kappa_1''y^3\right).$$

In the same way as above, $j^3f(0)$ is reduced to one of the following:

$$\begin{cases}
(x,xy,y^3) & \kappa_1''\neq0,\ \tau_1=0 & \cdots H,\\
(x,xy+y^3,xy^2) & \kappa_1''=\tau_1=0,\ \tau_0\tau_1'\neq0 & \cdots P,\\
(x,xy,xy^2) & \kappa_1''=\tau_1=\tau_1'=0,\ \tau_0\neq0, & \\
(x,xy+y^3,0) & \kappa_1''=\tau_0=\tau_1=0,\ \tau_1'\neq0, & \\
(x,xy,0) & \kappa_1''=\tau_0=\tau_1=\tau_1'=0. &
\end{cases} \tag{5}$$

Each of last three types has codimension ≥ 6, so we omit them here. Below, for types $S,B,\cdots,P$ in (4) and (5), we detect $\mathcal{A}$-types with codimension ≤ 5 by checking higher jets and the determinacy.

• **S-type**: Let $\kappa_1=\kappa_1'=0$ and $\tau_0\tau_1\neq0$. Then a computation shows that

$$\begin{aligned}
\kappa_1''=0 \quad&\Longrightarrow\quad j^4f(0)\sim\left(x,y^2,y^3-\tfrac{\kappa_1^{(3)}}{2\tau_0\tau_1^4}x^3y\right),\\
\kappa_1''=\kappa_1^{(3)}=0 \quad&\Longrightarrow\quad j^5f(0)\sim\left(x,y^2,y^3-\tfrac{\kappa_1^{(4)}}{8\tau_0\tau_1^5}x^4y\right),\\
\kappa_1''=\kappa_1^{(3)}=\kappa_1^{(4)}=0 \quad&\Longrightarrow\quad j^6f(0)\sim\left(x,y^2,y^3-\tfrac{\kappa_1^{(5)}}{40\tau_0\tau_1^6}x^5y\right).
\end{aligned}$$

Note that S_k is $(k+2)$-determined (its codimension is $k+2$), thus f is of type $S_k^{\pm}$ $(k=2,3,4)$ if and only if $\kappa_1=\kappa_1'=\cdots=\kappa_1^{(k)}=0$ and $\kappa_1^{(k+1)}\tau_0\tau_1 \lessgtr 0$ (seemingly, it is so for any k).

• B**-type**: Let $\kappa_1=\kappa_1'=\kappa_1''-2\tau_0\tau_1=0$ and $\kappa_1''\neq 0$. Then it would be $\mathcal{A}$-equivalent to B_k-type [15, 4.1:17, Table 3]. For instance,

$$j^5f(0)\sim(x,y^2,x^2y+b_2y^5)$$

with

$$\begin{aligned} b_2= \; & 48\tau_0^2\tau_1^2(\tau_0^2-2)-20(\tau_0^2(\tau_1')^2+\tau_1^2(\tau_0')^2)-56\tau_0\tau_1\tau_0'\tau_1' \\ & -24\tau_0\tau_1(\tau_0\tau_1''+\tau_1\tau_0'')+20\kappa_1^{(3)}(\tau_0\tau_1'+\tau_1\tau_0')-5(\kappa_1^{(3)})^2+6\kappa_1^{(4)}\tau_0\tau_1. \end{aligned}$$

Since B_2 is 5-determined,

$$f\sim_{\mathcal{A}} B_2^{\pm}:(x,y^2,x^2y\pm y^5) \iff b_2\gtrless 0.$$

Let b_3 be the coefficient of y^7 in the last component of $j^7f(0)$, which is written as a polynomial in derivatives of invariants at $s=0$, then $B_3^{\pm}:(x,y^2,x^2y\pm y^7)$ is detected by the condition that $b_2=0$ and $b_3\neq 0$. Here B_3 is of codimension 5.

• C**-type**: Let $\kappa_1=\kappa_1'=\kappa_1''=\tau_0=0,\ \tau_1\neq 0$. Through

$$\psi(X,Y,Z)=\left(\tfrac{1}{\tau_1}X,\ Y,\ \tfrac{1}{\tau_1}\left(Z-aY^2-bX^2Y\right)\right)$$

with $a=\frac{1}{4}(\kappa_1^{(3)}-3\tau_1\tau_0')$, $b=\frac{3}{2\tau_1^2}(\kappa_1^{(3)}-\tau_1\tau_0')$, we see that

$$j^4f(0)\sim\left(x,\ y^2,\ \kappa_1^{(3)}x^3y+(\kappa_1^{(3)}-2\tau_1\tau_0')xy^3\right).$$

Since C_3 is 4-determined (of codimension 5),

$$f\sim_{\mathcal{A}} C_3^{\pm}:(x,y^2,xy^3\pm x^3y) \iff \kappa_1^{(3)}(\kappa_1^{(3)}-2\tau_1\tau_0')\gtrless 0.$$

• H**-type**: Let $\kappa_1=\kappa_1'=\tau_1=0$ and $\kappa_1''\neq 0$. Then it would be $\mathcal{A}$-equivalent to H_k-type [15, 4.2.1:2]. A lengthy computation shows that

$$j^5f(0)\sim\left(x,\ xy+h_2y^5,\ y^3\right)$$

with

$$\begin{aligned} h_2= \; & -15\tau_0^2(\tau_1')^3-24\tau_0'(\tau_1')^2\kappa_1''-36\tau_1'(\kappa_1'')^2-15\tau_0^2\tau_1'(\kappa_1'')^2-24\tau_0'(\kappa_1'')^3 \\ & -21\tau_0\tau_1'\kappa_1''\tau_1''+20\tau_0(\tau_1')^2\kappa_1^{(3)}-\tau_0(\kappa_1'')^2\kappa_1^{(3)}+5\kappa_1''\tau_1''\kappa_1^{(3)}-5\tau_1'(\kappa_1^{(3)})^2 \\ & -4(\kappa_1'')^2\tau_1^{(3)}+4\tau_1'\kappa_1''\kappa_1^{(4)}. \end{aligned}$$

Since H_2 is 5-determined,

$$f\sim_{\mathcal{A}} H_2^{\pm}:(x,xy\pm y^5,y^3) \iff h_2\gtrless 0.$$

Let h_3 be the coefficient of y^8 in the middle component of $j^8f(0)$, then $H_3:(x,xy+y^8,y^3)$ is detected by $h_2=0$ and $h_3\neq 0$ (H_3 is of codimension 5).

• P**-type**: Let $\kappa_1=\kappa_1'=\kappa_1''=\tau_1=0$ and $\tau_0\tau_1'\neq 0$. Then we see that there is a polynomial p_4 in derivatives of κ_1,τ_0,τ_1 so that

$$f\sim_{\mathcal{A}} P_3:(x,\ xy+y^3,\ xy^2+p_4y^4)$$

for $p_4\neq 0,\frac{1}{2},1,\frac{3}{2}$ [15, §4.2].

Remark 3.2. (Characterization of C_k and F_4) Among $\mathcal{A}$-simple germs obtained in Mond [15], we have just discussed germs of type $S_k^\pm$, $B_k^\pm$ and H_k. So there remain C_k ($k \geq 4$) and F_4, which are the next to C_3-type above. Suppose that $\kappa_1^{(3)}(\kappa_1^{(3)} - 2\tau_1\tau_0') = 0$. Then we have the following condition for each of them.

- If $\kappa_1^{(3)} = 0$ and $\tau_1\tau_0' \neq 0$, then $j^4 f(0) \sim (x, y^2, xy^3)$ and

$$\kappa_1^{(3)} = 0 \qquad \Longrightarrow \qquad j^5 f(0) \sim \left(x, y^2, xy^3 - \frac{\kappa_1^{(4)}}{8\tau_0'\tau_1^4} x^4 y\right),$$

$$\kappa_1^{(3)} = \kappa_1^{(4)} = 0 \qquad \Longrightarrow \qquad j^6 f(0) \sim \left(x, y^2, xy^3 - \frac{\kappa_1^{(5)}}{40\tau_0'\tau_1^5} x^5 y\right).$$

Since $C_k^\pm : (x, y^2, xy^3 \pm x^k y)$ is $(k+1)$-determined, we see that f is of type $C_k^\pm$ ($k = 4, 5$) if and only if $\tau_0 = \kappa_1 = \kappa_1' = \cdots = \kappa_1^{(k-1)} = 0$ and $\kappa_1^{(k)}\tau_0'\tau_1 \lessgtr 0$ (seemingly, it is so for any k).

- If $\kappa_1^{(3)} = 2\tau_1\tau_0' \neq 0$, we have $j^4 f(0) \sim (x, y^2, x^3 y)$ and

$$f \sim_{\mathcal{A}} F_4 : (x, y^2, x^3 y + y^5) \Longleftrightarrow 3\kappa_1^{(4)} - 8\tau_0'\tau_1' - 12\tau_1\tau_0'' \neq 0.$$

Remark 3.3. (Non-realizable jets) Let us continue the argument in Remark 3.2. If $\kappa_1^{(3)} = \tau_0' = 0$, then f should be of codimension ≥ 7 and a computation shows that

$$j^5 f(0) \sim \left(x, y^2, \kappa_1^{(4)} x^4 y + (\kappa_1^{(4)} - 4\tau_1\tau_0'')y^5 + 2\sqrt{5}(\kappa_1^{(4)} - 2\tau_1\tau_0'')x^2 y^3\right).$$

In particular, if two of three coefficients $\kappa_1^{(4)}$, $\kappa_1^{(4)} - 4\tau_1\tau_0''$, $\kappa_1^{(4)} - 2\tau_1\tau_0''$ are zero, then all are zero. Thus, for instance, the following 5-jets are not equivalent to jets of any non-cylindrical ruled surface:

$$(x, y^2, x^4 y), \quad (x, y^2, x^2 y^3), \quad (x, y^2, y^5).$$

The 5-jet (x, y^2, y^5) is obviously realizable by a cylinder, while the 5-jets

$$(x, y^2, x^4 y) \quad \text{and} \quad (x, y^2, x^2 y^3)$$

are not equivalent to jets of *any* ruled surfaces, even if we drop the condition $\boldsymbol{e}'(0) \neq 0$. In fact, put $F = \boldsymbol{r}(s) + t\boldsymbol{e}(s)$ with $\boldsymbol{r}(s) \cdot \boldsymbol{e}(s) = 0$ and $\boldsymbol{e}(s) = (1, 0, 0) + o(s)$. If F is singular at $(s, t) = (0, 0)$ and $\boldsymbol{r}(0) = 0$, then $\boldsymbol{r}(s) = o(s)$. It is easy to see that $F \sim_{\mathcal{A}} f = (x, y^2 h(x, y), y^3 g(x, y))$ with some functions h, g of the form $p(y) + xq(y)$, and thus the 5-jet of F is never equivalent to those two jets mentioned above. By the same reason, the $\mathcal{A}^3$-orbit of the 3-jet $(x, y^3, x^2 y)$ is not realized by jets of any ruled surfaces (the 2-jet $(x, 0, 0)$ never appears in non-cylindrical ruled surfaces as seen before, and the 3-jet is not realizable by ruled surfaces with $\boldsymbol{e}'(0) = 0$, that is shown in the same way as above).

3.3. **Transversality.** To precisely state *genericity* of ruled surfaces, we need an appropriate mapping space (moduli space) equipped with a certain topology. By the definition, a *residual* subset of a mapping space is a union of countably many open dense subsets. When maps having a prescribed condition form a residual subset, we often say that such a map is *generic*, abusing words. Let I be an open interval containing $0 \in \mathbb{R}$ and let u denote the coordinate of I. As the mapping space of non-cylindrical ruled surfaces, we take

$$\mathcal{R} := \{\, \check{\boldsymbol{v}} = \boldsymbol{v}_0 + \varepsilon \boldsymbol{v}_1 \in C^\infty(I, \check{\mathbb{U}}) \,|\, \boldsymbol{v}_0'(u) \neq 0 \,(u \in I)\,\}$$

equipped with Whitney C^∞ topology. As a remark, Izumiya and Takeuchi [10] and Martins and Nuño-Ballesteros [13] took the space $C^\infty(I, \mathbb{R}^3 \times S^2)$ instead of $C^\infty(I, \check{\mathbb{U}})$, but the difference does not affect the matter of genericity arguments – given a pair $(\boldsymbol{r}, \boldsymbol{e})$ of base and director curves, we simply assign a curve $\check{\boldsymbol{v}} : I \to \check{\mathbb{U}}$ with $\boldsymbol{v}_0 = \boldsymbol{e}$ and $\boldsymbol{v}_1 = \boldsymbol{r} \times \boldsymbol{e}$.

Also we put

$$\mathcal{M} := C^\infty(I, \mathbb{R}_{>0} \times \mathbb{R}^3)$$

of quadruples $(\kappa_0, \kappa_1, \tau_0, \tau_1)$ of real-valued functions with $\kappa_0(u) > 0$ equipped with Whitney C^∞ topology. Any curve $\check{\boldsymbol{v}}(u)$ in $\mathcal{R}$ defines $\mathbb{D}$-valued functions, $\check{\kappa}(u)$ and $\check{\tau}(u)$ (parameterized by a general parameter $u \in I$), that produces a continuous map $\Phi : \mathcal{R} \to \mathcal{M}$. Obviously, Φ is surjective. In fact, given a quadruple of functions $(\kappa_0(u), \kappa_1(u), \tau_0(u), \tau_1(u)) \in \mathcal{M}$, put a new parameter $s := s(u) = \int_0^u \kappa_0(u)du$ and define $\kappa_1(s) := \kappa_1(u(s))$, etc. Then, three functions $\kappa_1(s), \tau_0(s), \tau_1(s)$ determines, up to Euclidean motions, the curve $\check{\boldsymbol{v}}(s) = \boldsymbol{v}_0(s) + \varepsilon \boldsymbol{v}_1(s)$ by solving the ordinary differential equation determined by the Frenet formula. The ambiguity is fixed by the initial values $\check{\boldsymbol{v}}(0), \check{\boldsymbol{n}}(0), \check{\boldsymbol{t}}(0)$, which corresponds to the initial orthogonal axes in $\mathbb{R}^3$ at $u = 0$. Put $\check{\boldsymbol{v}}(u) := \check{\boldsymbol{v}}(s(u)) \in \mathcal{R}$; the set of such cruves is exactly the preimage via Φ of the given quadruple of functions. That implies that for a dense subset $O \subset \mathcal{M}$, the preimage $\Phi^{-1}(O)$ is also dense in $\mathcal{R}$.

The above construction is extended for a parametric version. Let W be an open subset of $\mathbb{R}^p$ $(0 \le p \le 3)$, and consider the subspace $\mathcal{R}_W$ of $C^\infty(I \times W, \check{\mathbb{U}})$ which consists of maps $\check{\boldsymbol{v}}(u, \lambda) = \boldsymbol{v}_0(u, \lambda) + \varepsilon \boldsymbol{v}_1(u, \lambda)$ with parameter $\lambda \in W$ satisfying $\partial \boldsymbol{v}_0/\partial u \neq 0$ at any (u, λ). Put $\mathcal{M}_W$ to be the mapping space of $I \times W \to \mathbb{R}_{>0} \times \mathbb{R}^3$, and then a surjective continuous map $\Phi : \mathcal{R}_W \to \mathcal{M}_W$ is defined in entirely the same way as above. For a dense subset $O \subset \mathcal{M}_W$, the preimage $\Phi^{-1}(O)$ is also in $\mathcal{R}_W$.

As seen in the previous section, we have obtained a semi-algebraic stratification of the jet space $J^r := \mathbb{R}^3 \times J^r(1, 3)$ up to codimension 4 (r sufficiently large). In fact, any strata are defined by the conditions in Table 1 of (in)equalities in Taylor coefficients $\{\kappa_1^{(k)}, \tau_0^{(k)}, \tau_1^{(k)}\}_{0 \le k \le r}$, which form a system of coordinates of the affine space J^r. Notice that these Taylor coefficients are with respect to the arclength parameter s. For each quadruple $(\kappa_0, \kappa_1, \tau_0, \tau_1) \in \mathcal{M}_W$, we put

$$s = s(u, \lambda) := \int_0^u \kappa_0(u, \lambda)du, \qquad \varphi(u, \lambda) = (\kappa_1(u, \lambda), \tau_0(u, \lambda), \tau_1(u, \lambda)).$$

By the assumption that $\partial s/\partial u = \kappa_0 > 0$, let $\bar{\varphi}(s, \lambda) := \varphi(u(s, \lambda), \lambda)$. Then we define

$$\Psi : I \times W \times \mathcal{M}_W \to J^r, \qquad \Psi(u, \lambda, (\kappa_0, \varphi)) := j^r_s \bar{\varphi}(s(u, \lambda), \lambda),$$

where $j^r_s \bar{\varphi}$ means the r-jet respect to the parameter s. By a version of Thom's transversality theorem (Lemma 4.6 in [1]), there is a dense subset O of $\mathcal{M}_W$ so that for any $\varphi \in O$, the jet extension $\Psi_{\kappa_0, \varphi} : I \times W \to J^r$ is transverse to every stratum of our stratification of J^r. Hence, $\Phi^{-1}(O)$ is dense in $\mathcal{R}_W$, and for any element of $\Phi^{-1}(O)$, only $\mathcal{A}$-singularity types listed in Table 1 appears. This completes the proof of (2) in Theorem 1.1. □

Remark 3.4. ($\mathcal{A}_e$-versal deformations) For each type in Table 1, an $\mathcal{A}_e$-versal deformation of the germ is realized by a generic family of non-cylindrical ruled surfaces. This is directly checked by computations. For instance, as in Table 1, the $S_1^\pm$-singularity of ruled surface at $s = 0$ is characterized by $\kappa_1(0) = \kappa_1'(0) = 0$, $\kappa_1''(0) \neq 0, 2\tau_0(0)\tau_1(0)$ and $\tau_1(0) \neq 0$. Suppose that $\varphi = (\kappa_1(s), \tau_0(s), \tau_1(s)) : I \to \mathbb{R}^3$ satisfies this condition. Define a 1-parameter family $I \times \mathbb{R} \to \mathbb{R}^3$ by $\varphi(s, \lambda) := \varphi(s) + (\lambda, 0, 0)$, then obviously, its 1-jet extension $j^1_s \varphi$ is transverse at $(0, 0)$ to the stratum defined by $\kappa_1 = \kappa_1' = 0$ in $J^1 = \mathbb{R}^3 \times J^1(1, 3)$. This family yields a 1-parameter family $F(s, t, \lambda) = (t, ts - \frac{\tau_1}{2}s^2, \lambda s) + o(2)$ of ruled surfaces. By using a coordinate change of $x = t + \cdots$ (= first component of $F(s, t, \lambda)$) and $y = s$ and some target changes, we see that the germ of $F(s, t, \lambda)$ is equivalent to $(x, y^2, y^3 \pm x^2 y + \lambda y)$, which is an $\mathcal{A}_e$-miniversal deformation of $S_1^\pm$-singularity.

4. Singularities of developable surfaces

4.1. **Recognition of singularity types.** For non-cylindrical developable surfaces, $\kappa_1(s) \equiv 0$ identically. Hence the Taylor expansion of f is (2) with $\kappa_1^{(k)} = 0$ for all k:

$$f(x,y) := F(y, t(x,y)) = (x, xy - \tfrac{1}{2}\tau_1 y^2 - \tfrac{1}{6}\tau_1' y^3, \tfrac{1}{2}\tau_0 xy^2 + \tfrac{1}{3}\tau_0\tau_1 y^3) + o(3).$$

Using the $\mathcal{A}$-criteria mentioned in §2, we classify singularities arising in generic families of developable surfaces. Notice that there are two different aspects; singularities of frontal surfaces correspond to the case of $\tau_1 \neq 0$, while singularities of wavefronts correspond to the case of $\tau_1 = 0$. Below we prove Theorem 1.5.

- **Case of** $\tau_1 \neq 0$: By $s = y + \tau_1^{-1}x$ and some linear change of the target, we have

$$f = (x, y^2 + o(2), f_3(x,y)) \quad \text{with} \quad f_3 = \tau_0 y^3 + o(3).$$

Note that (x, y^2) is 2-determined and that each term $x^k y^{2l}$ in f_3 can be removed by a coordinate change of the target $(X, Y, Z) \mapsto (X, Y, Z - X^k Y^l)$. Use Proposition 2.5 in §2 ([16]) for determinacy in CE.

(i) If $\tau_0 \neq 0$, then $f \sim_{\mathcal{A}} (x, y^2, y^3)$, since it is 3-determined in CE.

(ii) Let $\tau_0 = 0$. Computing the 4-jet, we see

$$f_3 = \tau_0'(6x^2y^2 + 8\tau_1 xy^3 + 3\tau_1^2 y^4) + o(4).$$

If $\tau_0' \neq 0$, then $f \sim_{\mathcal{A}} (x, y^2, xy^3)$, for the germ is 4-determined in CE. Hence f is of type cuspidal crosscap.

(iii) Let $\tau_0 = \tau_0' = 0$. Computing the 5-jet, we see

$$f_3 = \tau_0''(10x^3y^2 + 20\tau_1 x^2y^3 + 15\tau_1^2 xy^4 + 4\tau_1^3 y^5) + o(5).$$

If $\tau_0'' \neq 0$, by target changes using $X = x$ and $Y = y^2$, terms x^3y^2 and xy^4 can be removed from $Z = f_3$, thus we see that $f \sim_{\mathcal{A}} (x, y^2, y^3(x^2 + y^2))$, for this germ is 5-determined in CE. That is cuspidal S_1^+-type. Note that cuspidal S_1^- never appears.

(iv) Let $\tau_0 = \tau_0' = \tau_0'' = 0$. Computing the 6-jet, we see

$$f_3 = \tau_0'''(15x^4y^2 + 40\tau_1 x^3y^3 + 45\tau_1^2 x^2y^4 + 24\tau_1^3 xy^5 + 5\tau_1^4 y^6) + o(6).$$

If $\tau_0''' \neq 0$, then $f \sim_{\mathcal{A}} (x, y^2, y^3(x^3 + xy^2))$, for the germ is 6-determined in CE. That is cuspidal C_3^+-type, while cuspidal C_3^- does not appear. Note that $\tau_0 = \tau_0' = \tau_0'' = 0$ if and only if the 5-jet of f is equivalent to $(x, y^2, 0)$, thus cuspidal S and B-types never appear, as mentioned in Remark 1.6.

- **Case of** $\tau_1 = 0$: Then $f = (x, xy - \frac{1}{6}\tau_1' y^3, \frac{1}{2}\tau_0 xy^2) + o(3)$. Note that $j^2 f(0) \sim (x, xy, 0)$, thus types $A_3^{\pm}$ and D_k never appear (Remark 1.6).

If $\tau_0 = 0$, $j^3 f(0)$ is equivalent to either $(x, xy + y^3, 0)$ or $(x, xy, 0)$, that is of type T_1 or T_2 (codimension $3, 4$) in Table 2. Now assume that $\tau_0 \neq 0$. Write

$$f = (x, f_2(x,y), f_3(x,y)) = (x, xy - \tfrac{1}{6}\tau_1' y^3, xy^2) + o(3).$$

The singular point set $S(F)$ is defined by $(f_2)_y = (f_3)_y = 0$, and through a computation, it is simplified as $\lambda = 0$ with

$$\lambda = x - \tfrac{1}{2}\tau_1' y^2 - \tfrac{1}{6}\tau_1'' y^3 - \tfrac{1}{24}(\tau_1''' - 3\tau_1')y^4 + o(4).$$

We may take $\eta = \partial/\partial y$ as a vector field which generates $\ker dF$ along $S(F)$. Then, $\eta\lambda(0) = 0$, $\eta\eta\lambda(0) = -\tau_1'$, $\eta\eta\eta\lambda(0) = -\tau_1''$ and $\eta\eta\eta\eta\lambda(0) = -(\tau_1''' - 3\tau_1')$. Hence, by Izumiya-Saji's criteria in §2.5, we have the conditions for detecting Sw, cA_4 and cA_5.

4.2. **Topological classification.** We prove Theorem 1.7. Let $\sigma(s)$ be the striction curve of a non-cylindrical developable surface. Assume that $\sigma(0) = 0 \in \mathbb{R}^3$, and consider the germ $\sigma : (\mathbb{R}, 0) \to (\mathbb{R}^3, 0)$. Since $\{\boldsymbol{v}_0(s), \boldsymbol{n}_0(s), \boldsymbol{t}_0(s)\}$ form a basis of $\mathbb{R}^3$ for each s, we denote the k-th derivative by

$$\sigma^{(k)}(s) = A_k(s)\boldsymbol{v}_0(s) + B_k(s)\boldsymbol{n}_0(s) + C_k(s)\boldsymbol{t}_0(s) \quad (k \geq 1)$$

where $A_k(s), B_k(s), C_k(s)$ are some functions. Then, with respect to the basis $\{\boldsymbol{v}_0(0), \boldsymbol{n}_0(0), \boldsymbol{t}_0(0)\}$, the expansion of σ at $s = 0$ is given by

$$\sigma(s) = (A_1(0)s + \tfrac{1}{2}A_2(0)s^2 + \cdots, B_1(0)s + \tfrac{1}{2}B_2(0)s^2 + \cdots, C_1(0)s + \tfrac{1}{2}C_2(0)s^2 + \cdots).$$

Now assume that σ is of type (m, n_1, n_2), i.e.,

$$\begin{cases} A_1(0) = \cdots = A_{m-1}(0) = 0, \ A_m(0) \neq 0, \\ B_1(0) = \cdots = B_{n_1-1}(0) = 0, \ B_{n_1}(0) \neq 0, \\ C_1(0) = \cdots = C_{n_2-1}(0) = 0, \ C_{n_2}(0) \neq 0. \end{cases}$$

Since $\sigma'(s) = \tau_1(s)\boldsymbol{v}_0(s)$ for a developable surface (Lemma 2.2 (iii)), we see that $A_1(s) = \tau_1(s)$ and $B_1(s) \equiv C_1(s) \equiv 0$. By the Frenet formula (Theorem 2.1 (1)),

$$\begin{aligned} \sigma^{(k+1)} &= (\sigma^{(k)})' = \{A_k\boldsymbol{v}_0 + B_k\boldsymbol{n}_0 + C_k\boldsymbol{t}_0\}' \\ &= (A_k' - B_k)\boldsymbol{v}_0 + (B_k' + A_k - C_k\tau_0)\boldsymbol{n}_0 + (C_k' + B_k\tau_0)\boldsymbol{t}_0 \\ &= A_{k+1}\boldsymbol{v}_0 + B_{k+1}\boldsymbol{n}_0 + C_{k+1}\boldsymbol{t}_0. \end{aligned}$$

Thus for $k = 1$, we have $A_2(s) = \tau_1'(s)$, $B_2(s) = \tau_1(s)$, $C_2(s) \equiv 0$, and for $k = 2$, $A_3(s) = \tau_1''(s) - \tau_1(s)$, $B_3(s) = 2\tau_1'(s)$ and $C_3(s) = \tau_0(s)\tau_1(s)$. For $k \geq 3$, there are some smooth functions $a_{k,*}(s)$, $b_{k,*}(s)$, $c_{k,*,*}(s)$ and positive numbers $\beta_k, \gamma_{k,0}, \cdots, \gamma_{k,k-3} > 0$ such that

$$\begin{aligned} A_k(s) &= a_{k,0}(s)\tau_1(s) + \cdots + a_{k,k-2}(s)\tau_1^{(k-2)}(s) + \tau_1^{(k-1)}(s), \\ B_k(s) &= b_{k,0}(s)\tau_1(s) + \cdots + b_{k,k-3}(s)\tau_1^{(k-3)}(s) + \beta_k\tau_1^{(k-2)}(s), \\ C_k(s) &= \{c_{k,0,0}(s)\tau_0(s) + \cdots + \gamma_{k,0}\tau_0^{(k-4)}(s)\}\tau_1(s) \\ &\quad + \{c_{k,1,0}(s)\tau_0(s) + \cdots + \gamma_{k,1}\tau_0^{(k-5)}(s)\}\tau_1'(s) + \cdots \\ &\quad + \{c_{k,k-4,0}(s)\tau_0(s) + \gamma_{k,k-4}\tau_0'(s)\}\tau_1^{(k-4)}(s) + \gamma_{k,k-3}\tau_0(s)\tau_1^{(k-3)}(s). \end{aligned}$$

Hence, by the assumption on $A_k(0)$, we have

$$\tau_1(0) = \cdots = \tau_1^{(m-2)}(0) = 0, \quad \tau_1^{(m-1)}(0) \neq 0,$$

and thus

$$B_1(0) = \cdots = B_m(0) = 0, \ \ B_{m+1}(0) \neq 0, \quad C_1(0) = \cdots = C_{m+2}(0) = 0.$$

In particular,

$$n_1 = m + 1, \ \ n_2 = m + 1 + r \quad (r \geq 1).$$

By the above formula of $C_k(s)$ with $k = m + 1 + r$, we see

$$\tau_0(0) = \cdots = \tau_0^{(r-2)}(0) = 0, \quad \tau_0^{(r-1)}(0) \neq 0.$$

Conversely, if the order of τ_0 and τ_1 are r and $m - 1$, respectively, then the type of σ is

$$(m, m + 1, m + 1 + r).$$

This completes the proof. ∎

References

[1] M. Golubitsky and V. Guillemin, *Stable Mappings and Their Singularities*, GTM 14, Springer-Verlag (1973).

[2] H. W. Guggenheimer, *Differential Geometry*, McGraw-Hill (1963).

[3] V. Hlavarý, *Differential line geometry*, Noordhoff, Groningen-London (1953).

[4] S. Honda and M. Takahashi, Framed curves in the Euclidean space, Adv. Geom. **16** (3) (2016), 265–276. DOI: 10.1112/s0024610700001095

[5] G. Ishikawa, Topological classification of the tangent developables of space curves, J. London Math. Soc. (2) **62** (1999), 583–598.

[6] G. Ishikawa, Singularities of developable surfaces, Singularity Theory, Proc. European Sing. Conf. (Liverpool, 1996), ed. W. Bruce and D. Mond, Cambridge Univ. Press (1999), 403–418. DOI: 10.1017/cbo9780511569265.025

[7] S. Izumiya, M. C. Romero Fuster, M. A. S. Ruas and F. Tari, *Differential Geometry from a Singularity Theory Viewpoint*, World Scientific (2016). DOI: 10.1142/9108

[8] S. Izumiya and K. Saji, The mandala of Legendrian dualities for pseudo-spheres in Lorentz-Minkowski space and "flat" spacelike surfaces, J. Singularities **2** (2010), 92–127. DOI: 10.5427/jsing.2010.2g

[9] S. Izumiya, K. Saji and M. Takahashi, Horospherical flat surfaces in Hyperbolic 3-space, J. Math. Soc. Japan **62** (2010), 789–849. DOI: 10.2969/jmsj/06230789

[10] S. Izumiya and N. Takeuchi, Singularities of ruled surfaces in $\mathbb{R}^3$, Math. Proc. Camb. Phil. Soc. **130** (2001), 1–11. DOI: 10.1017/s0305004100004643

[11] Y. Kabata, Recognition of plane-to-plane map-germs, Topology and its Appl. **202** (2016), 216–238. DOI: 10.1016/j.topol.2016.01.011

[12] H. Kurokawa, On generic singularities of 1-parameter families of developable surfaces (in Japanese), Master Thesis, Hokkaido University (2013).

[13] R. Martins and J. J. Nuño-Ballesteros, Finitely determined singularities of ruled surfaces in $\mathbb{R}^3$, Math. Proc. Camb. Phil. Soc. **147** (2009), 701–733. DOI: 10.1017/s0305004109002618

[14] D. Mond, *Classification of certain singularities and applications to differential geometry*, Ph.D. thesis, The University of Liverpool (1982).

[15] D. Mond, On the classification of germs of maps from $\mathbb{R}^2$ to $\mathbb{R}^3$, Proc. London Math. Soc. (3) **50** (1985), 333–369. DOI: 10.1112/plms/s3-50.2.333

[16] D. Mond, Singularities of the tangent developable surface of a space curve, Quart. J. Math. Oxford Ser. (2), **40** (1989), 79–91. DOI: 10.1093/qmath/40.1.79

[17] H. Pottmann and J. Wallner, *Computational Line Geometry*, Mathematics and Visualization, Springer (2001).

[18] K. Saji, Criteria for S_k singularities and their applications, Jour. Gökova Geom. Top. **4** (2010), 67–81.

[19] G. Salmon, *A treatise on the analytic geometry of three dimensions*, 4th edition, Dublin (1882).

[20] H. Sano, Y. Kabata, J. L. Deolindo Silva and T. Ohmoto, Classification of jets of surfaces in 3-space via central projection, Bull. Brasilian Math. Soc. New Series **48** (2017), 623–639. DOI: 10.1007/s00574-017-0036-x

[21] J. M. Selig, *Geometric Fundamentals of Robotics, 2nd edition*, Monographs in Computer Science, Springer (2005).

[22] J. Tanaka, Clifford algebra and singularities of ruled surfaces (in Japanese), Master Thesis, Hokkaido University (2016).

J. Tanaka, Kobo Co, LTD, Kobe, Japan
Email address: junki.tanaka@khobho.co.jp

T. Ohmoto, Department of Mathematics, Faculty of Science, Global Station of Bigdata and cybersecurity (GSB), Hokkaido University, Sapporo 060-0810, Japan
Email address: ohmoto@math.sci.hokudai.ac.jp