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LINKING BETWEEN SINGULAR LOCUS AND REGULAR FIBERS

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Dedicated to Professor Goo Ishikawa on the occasion of his 60th birthday

Abstract. Given a null-cobordant oriented framed link L in a closed oriented 3–manifold M, we determine those links in $M \smallsetminus L$ which can be realized as the singular point set of a generic map $M \to \mathbf{R}^2$ that has L as an oriented framed regular fiber. Then, we study the linking behavior between the singular point set and regular fibers for generic maps of M into $\mathbf{R}^2$.

1. Introduction

Topology of generic C^∞ maps of manifolds of dimension ≥ 2 into the plane $\mathbf{R}^2$ has been extensively studied as a natural generalization of Morse theory, which studies generic maps into the real line $\mathbf{R}$. For a Morse function, singular points, or critical points, are isolated and their positions in the source manifold are not interesting except for their cardinalities or indices. On the other hand, for a generic map into the plane, the singular point set is a smooth submanifold of dimension one in the source manifold and its position may be non-trivial. In [14], the author studied the position of the singular point set and characterized those smooth 1–dimensional submanifolds which arise as the singular point set of a generic map.

On the other hand, each regular fiber of such a generic map into $\mathbf{R}^2$ is of codimension two and is disjoint from the singular point set. Therefore, the singular point set and regular fibers may be non-trivially linked.

In September 2018[1] Professor David Chillingworth asked the author the following question: *for a generic map $f : \mathbf{R}^3 \to \mathbf{R}^2$, must every component of a regular fiber be linked by at least one component of the singular point set ?*[2]

In this paper, we concentrate on generic maps of *closed* (i.e. compact and boundaryless) 3–dimensional manifolds, instead of $\mathbf{R}^3$, and study the linking behavior between the singular point set and regular fibers in the source 3–manifold. More precisely, let M be a closed oriented 3–manifold and $f : M \to \mathbf{R}^2$ a generic C^∞ map. Generic maps that we consider in this paper are called *excellent maps*, as defined in §2, and have fold and cusp singularities. In our 3–dimensional case, both the singular point set and regular fibers have dimension one, and they constitute disjoint links in M. We study their relative positions in the 3–manifold M.

For example, let us consider the unit sphere $S^3 \subset \mathbf{R}^4$ and let $\pi : \mathbf{R}^4 \to \mathbf{R}^2$ be the standard projection defined by $\pi(x_1, x_2, x_3, x_4) = (x_1, x_2)$ for $(x_1, x_2, x_3, x_4) \in \mathbf{R}^4$. Then,

$$f_0 = \pi|_{S^3} : S^3 \to \mathbf{R}^2$$

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[1] In the conference "Geometric and Algebraic Singularity Theory" held in honor of the 60th birthday of Goo Ishikawa, in Będlewo, Poland.

[2] This question originates from a physical study of phase singularities, nodal lines, or optical polarization knots. For details, the reader is referred to [1, 2, 4, 8].

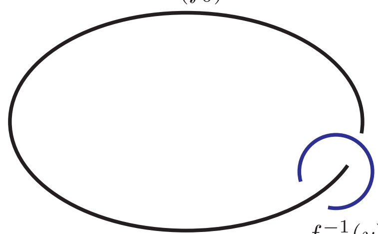

FIGURE 1. Singular point set and a regular fiber for a specific map $f_0 : S^3 \to \mathbf{R}^2$

is an excellent map whose singular point set $S(f_0) = \{(x_1, x_2, 0, 0) \in S^3\}$ consists only of definite fold singularities and is a trivial knot in S^3. Furthermore, for $y = (y_1, y_2)$ with $y_1^2 + y_2^2 < 1$, the regular fiber $f_0^{-1}(y) = \{(y_1, y_2, x_3, x_4) \in S^3\}$ is an unknotted circle linked with $S(f_0)$ (see Fig. 1). So, in this example, the answer to the above question is positive.

The present paper is organized as follows. In §2, we will first see that regular fibers are naturally oriented and framed; i.e. they have natural normal framings induced by the generic map $f : M \to \mathbf{R}^2$. Furthermore, they bound compact oriented normally framed surfaces embedded in M. Conversely, in [13], it has been shown that if an oriented normally framed link in M bounds a compact oriented normally framed surface, then it is realized as a regular fiber of a generic map of M into $\mathbf{R}^2$. Then, in Theorem 2.3, given such a framed link L in M, we characterize those unoriented links in $M \smallsetminus L$ that arise as the singular point set of a generic map $f : M \to \mathbf{R}^2$ such that L coincides with a framed regular fiber of f. The characterization is given in terms of a relative characteristic class (see [7]) which is the obstruction to extending a certain trivialization of the tangent bundle of M on a neighborhood of L to the whole M.

In §3, we will study the relative characteristic class which arises as the obstruction as above. As a consequence, we will show that if a regular fiber has an odd number of components, then it necessarily links with the singular point set (see Remark 3.5). We will also give a result which enables us to identify the obstruction for local links that are embedded inside an open 3–disk.

In §4, by utilizing the results obtained in §3, we show that there exist generic maps $S^3 \to \mathbf{R}^2$ such that a regular fiber, which is a 2–component link, and the singular point set are split; i.e. they lie inside disjoint 3–disks. We also see that there exists such an example for every closed oriented 3–manifold M. We also give two explicit examples of generic maps $S^3 \to \mathbf{R}^2$ which exhibit non-linking phenomena between regular fibers and the singular point set.

Finally in §5, we address the original question concerning generic maps of $\mathbf{R}^3$ into the plane. By utilizing results obtained in [6] on regular fibers of submersions $\mathbf{R}^3 \to \mathbf{R}^2$, we answer to the question negatively, by constructing counter examples.

Throughout the paper, manifolds and maps are differentiable of class C^∞ unless otherwise indicated. All (co)homology groups are with $\mathbf{Z}_2$–coefficients unless otherwise indicated. The symbol "$\cong$" means an appropriate isomorphism between algebraic objects.

2. Main theorem

Let M be a closed oriented 3–dimensional manifold. We say that a map $f : M \to \mathbf{R}^2$ is *excellent* if its singularities consist only of fold and cusp singularities, where a *fold singularity* (or a *cusp singularity*) is modeled on the map germ

$$(x, y, z) \mapsto (x, y^2 \pm z^2) \quad \text{(resp. } (x, y, z) \mapsto (x, y^3 + xy - z^2))$$

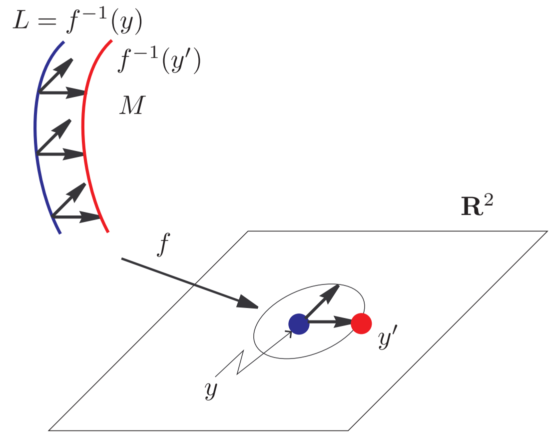

FIGURE 2. Framing for a regular fiber

at the origin. We say that a fold singularity is *definite* (resp. *indefinite*) if it is modeled on the map germ $(x, y, z) \mapsto (x, y^2 + z^2)$ (resp. $(x, y, z) \mapsto (x, y^2 - z^2)$).

It is known that the set of excellent maps is always open and dense in the mapping space $C^\infty(M, \mathbf{R}^2)$ endowed with the Whitney C^∞ topology (for example, see [5, 18]).

In the following, for a map $f : M \to \mathbf{R}^2$, we denote by $S(f)$ the set of singular points of f. If f is an excellent map, then we see easily that $S(f)$ is a link in M, i.e. a disjoint union of finitely many smoothly embedded circles. For a regular value $y \in \mathbf{R}^2$, if $L = f^{-1}(y)$ is nonempty, then we call it a *regular fiber*, which is also a link in M and is disjoint from $S(f)$. We fix an orientation of $\mathbf{R}^2$ once and for all, and then a regular fiber is naturally oriented, since M is oriented. Furthermore, L is naturally *framed*: its framing is given as the pull-back of the trivial normal framing of the point y in $\mathbf{R}^2$ (see Fig. 2). In other words, taking a small disk neighborhood of y in $\mathbf{R}^2$ consisting entirely of regular values, let y' be a point in its boundary, then $f^{-1}(y')$ represents the framed longitude of the framed link L.

LEMMA 2.1. *A framed regular fiber L of an excellent map $f : M \to \mathbf{R}^2$ over a regular point $y \in \mathbf{R}^2$ is always framed null-cobordant. In other words, there exists a compact oriented surface V embedded in M whose boundary coincides with L and which is consistent with the framed longitude.*

Proof. Let ℓ be a half line in $\mathbf{R}^2$ emanating from y. We may assume that it is transverse to the map f. Then, $V = f^{-1}(\ell)$ gives the desired surface (see Fig. 3). □

In [13, Proposition 5.1], it has been shown that every null-cobordant oriented framed link L in M can be realized as an oriented framed regular fiber of an excellent map $f : M \to \mathbf{R}^2$. In this case, the singular point set $S(f)$ is a link disjoint from L. Then, it is natural to ask which links in $M \setminus L$ appear as the singular point set of such an excellent map.

In order to state our first theorem, let us prepare some notations and terminologies. For a (unoriented) link J in $M \setminus L$, we denote by $[J]_2 \in H_1(M \setminus L)$ the $\mathbf{Z}_2$–homology class represented by J. Let $N(L)$ be a small tubular neighborhood of L in M disjoint from J. Since L is a framed link, we have a natural 3–framing of M over $\partial N(L)$, i.e. a trivialization of $TM|_{\partial N(L)}$. The obstruction to extending this framing over $M \setminus \operatorname{Int} N(L)$ is the relative Stiefel–Whitney class (see [7]), denoted by $w_2(M, L)$, which is an element of the $\mathbf{Z}_2$–cohomology group

$$H^2(M \setminus \operatorname{Int} N(L), \partial N(L)) \cong H^2(M, N(L)) \cong H^2(M, L),$$

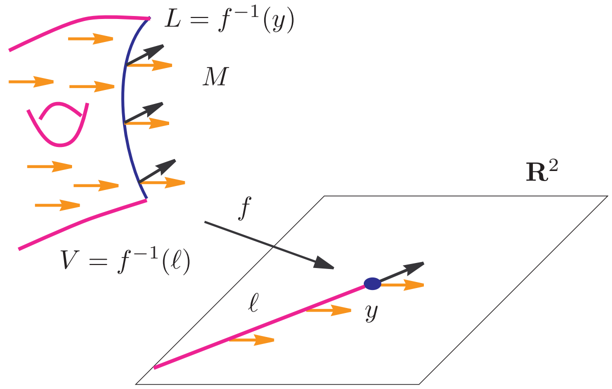

FIGURE 3. Constructing a framed null-cobordism

where the first isomorphism is given by excision and the second one is given by the natural homotopy equivalence $(M, L) \to (M, N(L))$. Note that by Poincaré–Lefschetz duality, we have

$$H^2(M \setminus \operatorname{Int} N(L), \partial N(L)) \cong H_1(M \setminus \operatorname{Int} N(L)) \cong H_1(M \setminus L).$$

REMARK 2.2. Let $j : (M, \emptyset) \to (M, L)$ be the inclusion. Then the induced homomorphism $j^* : H^2(M, L) \to H^2(M)$ maps $w_2(M, L)$ to the second Stiefel–Whitney class $w_2(M)$ of M, which vanishes. By the cohomology exact sequence

$$H^1(L) \xrightarrow{\delta} H^2(M, L) \xrightarrow{j^*} H^2(M),$$

we see that $w_2(M, L) = \delta(\alpha)$ for some $\alpha \in H^1(L)$.

Now, one of the main theorems of this paper is the following.

THEOREM 2.3. *Let L be an oriented null-cobordant framed link in a closed oriented 3–manifold M, and J be an unoriented link in M disjoint from L. Then, there exist an excellent map $f : M \to \mathbf{R}^2$ and a regular value $y \in \mathbf{R}^2$ such that $f^{-1}(y)$ coincides with L as an oriented framed link and that $S(f) = J$ if and only if $[J]_2 \in H_1(M \setminus L)$ is Poincaré dual to $w_2(M, L) \in H^2(M, L)$.*

Proof. Suppose that $f : M \to \mathbf{R}^2$ is an excellent map such that L coincides with $f^{-1}(y)$ as a framed link for a regular value $y \in \mathbf{R}^2$ and that $J = S(f)$. Then, we have the following, which is originally due to Thom [16].

LEMMA 2.4. *If $f : M \to \mathbf{R}^2$ is an excellent map and $y \in \mathbf{R}^2$ is a regular value, then for $L = f^{-1}(y)$, $[S(f)]_2 \in H_1(M \setminus L)$ is Poincaré dual to $w_2(M, L) \in H^2(M, L)$.*

For the sake of completeness, we include a short proof here.

Proof of Lemma 2.4. Since f is a submersion outside of $S(f)$, we can extend the framing on $N(L)$ to $M \setminus S(f)$. Then, we see easily that $S(f)$ is exactly the obstruction locus and by definition of the relative Stiefel–Whitney class, we have the desired conclusion. □

Conversely, suppose that $[J]_2 \in H_1(M \setminus L)$ is Poincaré dual to $w_2(M, L)$. Let $g : M \to \mathbf{R}^2$ be an arbitrary excellent map for which there exists a regular value $y \in \mathbf{R}^2$ such that $g^{-1}(y)$ coincides with L as a framed link. Such an excellent map always exists by [13]. Then, we see that $[S(g)]_2 \in H_1(M \setminus L)$ is Poincaré dual to $w_2(M, L)$ by Lemma 2.4. By our assumption, this implies that J and $S(g)$ are $\mathbf{Z}_2$–homologous in $M \setminus L$. Set $S(g) = J_0$.

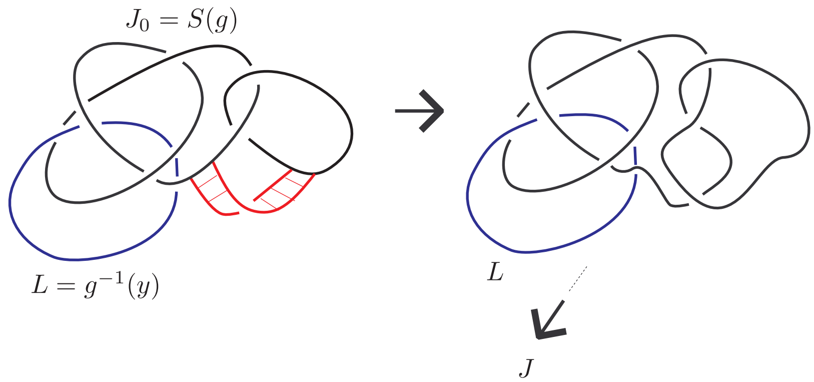

FIGURE 4. Starting from J_0, we get J up to isotopy by a finite iteration of band operations inside $M \setminus L$.

LEMMA 2.5 ([14]). *If $[J_0]_2 = [J]_2 \in H_1(M \setminus L)$, then by modifying J_0 by a finite iteration of band operations inside $M \setminus L$, we can get J, up to isotopy.*

Here, a band operation on J_0 is defined as follows. Set $I_1 = I_2 = [-1, 1]$, and let

$$\varphi : I_1 \times I_2 \to M \setminus L$$

be an embedding of a band such that $\varphi(I_1 \times I_2) \cap J_0 = \varphi(\{-1, 1\} \times I_2)$. Then a *band operation* applied to J_0 transforms it to $(J_0 \setminus \varphi(\{-1, 1\} \times I_2)) \cup \varphi(I_1 \times \{-1, 1\})$ with the corners smoothed. Lemma 2.5 states that repeating this procedure finitely many times, we get a link isotopic to J in $M \setminus L$, starting from J_0 (see Fig. 4).

Proof of Lemma 2.5. First, we may assume that both J_0 and J are connected, by using band operations. Here, note that the reverse of a band operation is again a band operation.

Now, we orient J_0 and J arbitrarily. Since $[J]_2 = [J_0]_2$ in $H_1(M \setminus L)$, we have $[J] = [J_0] + 2\gamma$ for some $\gamma \in H_1(M \setminus L; \mathbf{Z})$, where $[J]$ and $[J_0] \in H_1(M \setminus L; \mathbf{Z})$ are the $\mathbf{Z}$–homology classes represented by J and J_0, respectively. Using a band whose center curve corresponds to γ, we may assume $[J_0] = [J]$ in $H_1(M \setminus L; \mathbf{Z})$ (see the left hand side picture of Fig. 5).

Recall that $H_1(M \setminus L; \mathbf{Z})$ is the abelianization of $\pi_1(M \setminus L)$. By realizing commutators in $\pi_1(M \setminus L)$ by band operations, we may assume J_0 and J are freely homotopic (see the right hand side picture of Fig. 5).

Then, for dimensional reasons, J_0 is regularly homotopic to J. This implies that J_0 is transformed to J by a finite iteration of "crossing changes" in knot theory, up to isotopy.

Finally, we can realize each "crossing change" by two band operations as depicted in Fig. 6. This completes the proof of Lemma 2.5. (For more details, the reader is referred to [14].) $\square$

LEMMA 2.6 ([14]). *Each band operation applied to $S(g)$ can be realized by a generic deformation of $g : M \to \mathbf{R}^2$ which does not modify $g^{-1}(N(y))$ for a small disk neighborhood $N(y)$ of y in $\mathbf{R}^2$. In other words, for a link J_1 obtained by a band operation to $S(g)$ in $M \setminus g^{-1}(y)$, there exists a generic 1–parameter deformation from g to g_1 in such a way that $g_1 : M \to \mathbf{R}^2$ is an excellent map with $S(g_1) = J_1$, $g_1^{-1}(N(y)) = g^{-1}(N(y))$ and $g_1|_{g_1^{-1}(N(y))} = g|_{g^{-1}(N(y))}$.*

The above lemma can be proved by using Levine's cusp elimination techniques [9] (see Fig. 7). For details, the reader is referred to [14].

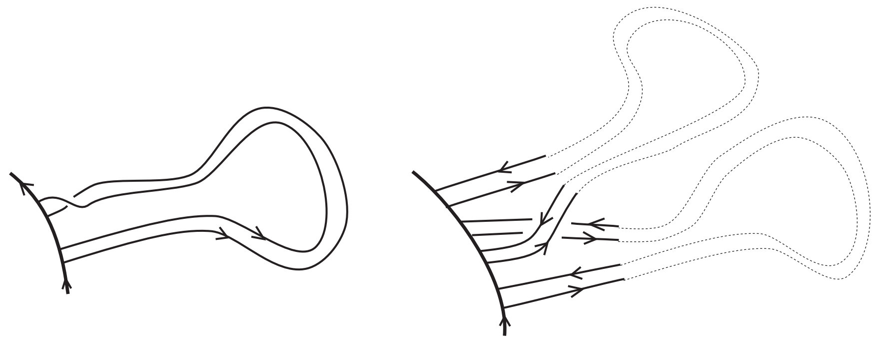

FIGURE 5. Modifying J_0 appropriately

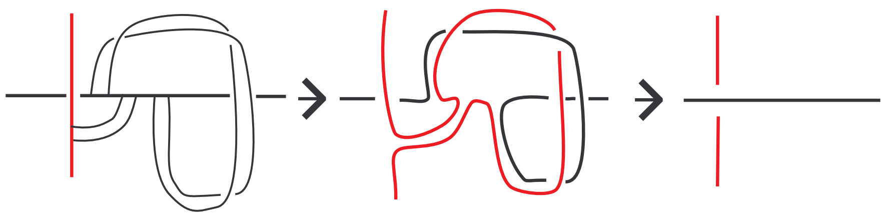

FIGURE 6. Realizing a crossing change by two band operations

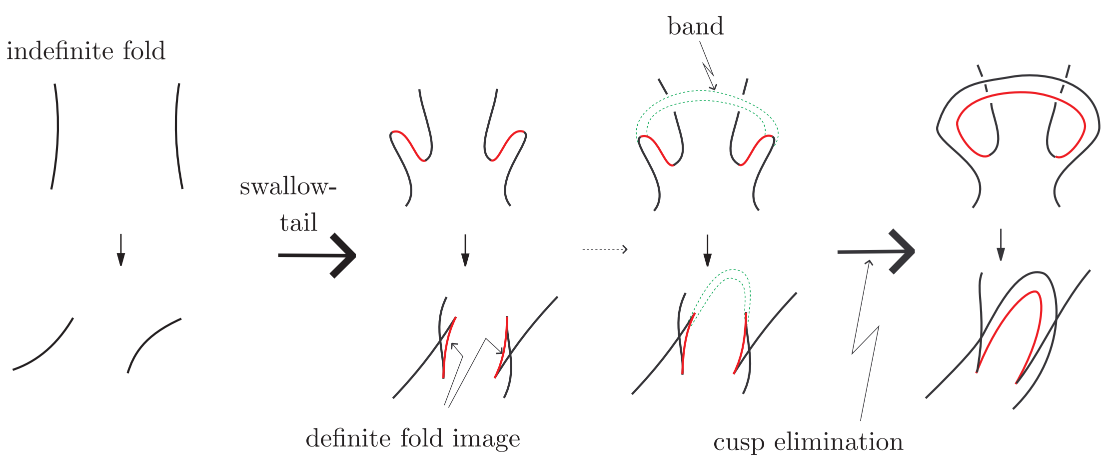

FIGURE 7. An example of a cusp elimination along a curve corresponding to a band operation. The upper row depicts a change of the singular point set in the source 3–manifold M, while the lower row depicts the corresponding change of the singular point set image in $\mathbf{R}^2$.

Now let us go back to the proof of Theorem 2.3. Combining Lemmas 2.5 and 2.6, we can deform g with $S(g) = J_0$ to an excellent map $f : M \to \mathbf{R}^2$ with $S(f) = J$, keeping the condition $g^{-1}(y) = f^{-1}(y) = L$. This completes the proof. $\square$

Remark 2.7. As in [14], suppose J is decomposed as a disjoint union

$$J = F_0 \cup F_1 \cup C,$$

where F_0 and F_1 are finite disjoint unions of open arcs and circles, C is a finite set of points, and each point of C is adjacent to both F_0 and F_1. If both F_0 and F_1 are non-empty, then in Theorem 2.3, we can find an excellent map f such that $S(f) = J$, F_0 is the set of definite fold singularities, F_1 is the set of indefinite fold singularities, and C is the set of cusp singularities.

Remark 2.8. Let $g : M \to \mathbf{R}^2$ be an excellent map for which there exists a regular value y such that $g^{-1}(y)$ coincides with L as a framed link. In the situation of Theorem 2.3, we see that $[J]_2 \in H_1(M)$ is Poincaré dual to $w_2(M)$, which vanishes, by Remark 2.2. Then, we can apply the modification techniques developed in [14] without touching L to obtain an excellent map $h : M \to \mathbf{R}^2$ homotopic to g such that $S(h)$ is isotopic to J in M. However, in order to obtain an excellent map h' such that $S(h')$ coincides with J, we need to further modify h. In such a modification process, the regular fiber over y may change, since in the course of the isotopy, the link may cross L. In §3, we will see that not every $\mathbf{Z}_2$ null-homologous link J in M can be realized as above, depending on its position relative to L.

Generalizing our Theorem 2.3, we can also obtain the following, which can be proved by the same argument. Details are left to the reader.

Theorem 2.9. *Let M be a closed oriented 3–manifold and $L_1, L_2, \ldots, L_\ell$, and J be disjoint links in M. Suppose that $L_1, L_2, \ldots, L_\ell$ are oriented and null-cobordant framed links, and that they bound disjoint compact oriented framed surfaces. Furthermore, J is an unoriented link. Then, there exist an excellent map $f : M \to \mathbf{R}^2$ and distinct regular values $y_1, y_2, \ldots, y_\ell \in \mathbf{R}^2$ of f such that $f^{-1}(y_i) = L_i$ as framed links for $i = 1, 2, \ldots, \ell$, and $J = S(f)$ if and only if $[J]_2 \in H_1(M \setminus L)$ is Poincaré dual to $w_2(M, L)$, where $L = L_1 \cup L_2 \cup \cdots \cup L_\ell$.*

For maps into S^2, we have a similar result as follows. Recall that, for a closed oriented 3–dimensional manifold M, the homotopy classes of M into S^2 are in one-to-one correspondence with the framed cobordism classes of closed oriented framed 1–dimensional submanifolds in M by the Pontrjagin–Thom construction. For the classification of the homotopy set $[M, S^2]$ for a closed oriented 3–manifold M, the reader is referred to [3].

Theorem 2.10. *Let M be a closed oriented 3–manifold and fix a homotopy class of a map $g : M \to S^2$. Let L be an oriented framed link in M which corresponds to the homotopy class of g. Then, for an unoriented link J in $M \setminus L$, there exist an excellent map $f : M \to S^2$ homotopic to g and a regular value $y \in S^2$ of f such that $f^{-1}(y)$ coincides with L as a framed link and $J = S(f)$ if and only if $[J]_2 \in H_1(M \setminus L)$ is Poincaré dual to $w_2(M, L)$.*

The proof of Theorem 2.10 is similar to that of Theorem 2.3 and is left to the reader. Note that Theorem 2.3 corresponds to the case of a null-homotopic map g in Theorem 2.10 in a certain sense.

3. Obstruction

In order to apply Theorem 2.3 in practical situations, let us study the obstruction class $w_2(M, L)$ more in detail, where M is a closed oriented 3–manifold and L is a framed link in M.

As we saw in Remark 2.2, there exists an $\alpha \in H^1(L)$ such that $\delta(\alpha) = w_2(M, L)$, although such a cohomology class may not be unique. In fact, such an α can be explicitly given as follows. Set $L = L_1 \cup L_2 \cup \cdots \cup L_t$, where L_s are the components of L, $s = 1, 2, \ldots, t$. It is known that a closed oriented 3–manifold M is always parallelizable, i.e. its tangent bundle is trivial. Let us fix a framing τ of M, where τ can be identified with a trivialization of the tangent bundle

TM. Once such a framing τ is fixed, we can compare it with the specific framing given on each component L_s of the framed link L. This defines a well-defined element a_s in $\pi_1(SO(3)) \cong \mathbf{Z}_2$. Then, we have the following.

LEMMA 3.1. *Let $\alpha \in H^1(L)$ be the unique cohomology class such that the Kronecker product $\langle \alpha, [L_s]_2 \rangle \in \mathbf{Z}_2$ coincides with a_s for each component L_s of L. Then, we have $\delta(\alpha) = w_2(M, L)$.*

Proof. For each component L_s, let K_s be the boundary of a small meridian disk D_s^2 of L_s. We may assume that K_s is contained in $M \setminus N(L)$. Then, by using τ, we can extend the framing over $\partial N(L)$ given by the framed link L to

$$(M \setminus \operatorname{Int} N(L)) \setminus \left(\cup_{s=1}^{t} K_s\right).$$

If $a_s = 0$, then this framing further extends across K_s: otherwise, it does not. Therefore, $w_2(M, L)$ is Poincaré dual to the sum of those $[K_s]_2$ such that $a_s \neq 0$.

Let us consider the commutative diagram

$$\begin{array}{ccc} H^1(L) & \xrightarrow{\delta} & H^2(M, L) \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle q}\big\downarrow \\ H_2(M, M \setminus L) & \xrightarrow{\partial} & H_1(M \setminus L), \end{array}$$

where the first (or the second) row is a part of the cohomology (resp. homology) exact sequence for the pair (M, L) (resp. $(M, M \setminus L)$), and the vertical maps are the duality isomorphisms. By the construction of α, we see that $p(\alpha)$ is represented by the sum of those $[D_s^2, \partial D_s^2]_2$ such that $a_s \neq 0$, where $[D_s^2, \partial D_s^2]_2 \in H_2(M, M \setminus L)$ is the $\mathbf{Z}_2$–homology class represented by the pair $(D_s^2, \partial D_s^2)$. Since $\partial[D_s^2, \partial D_s^2]_2 = [K_s]_2 \in H_1(M \setminus L)$, we have the desired conclusion by the commutativity of the diagram. □

For example, if the framing on L coincides with τ up to homotopy, then $\alpha = 0$ and consequently we have $w_2(M, L) = 0$.

Note that the framing τ may not be unique. The set of homotopy classes of such framings is in one-to-one correspondence with the homotopy set $[M, SO(3)]$. If we consider the set of homotopy classes of framings on the 2–skeleton of M, then each such framing up to homotopy defines a *spin structure* on M, and the set of spin structures is in one-to-one correspondence with $H^1(M)$ (see [11]).

By the cohomology exact sequence,

$$H^1(M) \xrightarrow{i^*} H^1(L) \xrightarrow{\delta} H^2(M, L) \xrightarrow{j^*} H^2(M),$$

we see that for every element $\beta \in \operatorname{Im} i^*$, we could choose $\alpha + \beta$ instead of α, where $i : L \to M$ is the inclusion map. The observation in the previous paragraph shows that this corresponds to choosing another framing, say τ', which is "twisted along β".

REMARK 3.2. As we saw in Remark 2.2, $w_2(M, L)$ is in the kernel of

$$j^* : H^2(M, L) \longrightarrow H^2(M),$$

which coincides with $\operatorname{Im} \delta \cong H^1(L)/\operatorname{Im} i^*$. Note that if L is framed null-cobordant, then this latter group is non-trivial, since L bounds a compact surface in M and hence $[L]_2 = 0$ in $H_1(M)$.

If we change the framing of a component L_s of L, then $w_2(M, L)$ changes in general. The difference is described by $\delta[L_s]_2^*$, where $[L_s]_2^*$ is the dual to the homology class $[L_s]_2 \in H_1(L)$ represented by L_s with respect to the basis of $H_1(L)$ consisting of the homology classes represented by the components of L. This follows from the observation described in [7, pp. 520–521].

(However, we need to be careful, since if we change the framing of L_s, then the resulting framed link may not be framed null-cobordant any more.)

Remark 3.3. Let L be an oriented link in a closed oriented 3–manifold M. Then, we can easily show that it bounds a compact oriented surface in M if and only if L represents zero in $H_1(M;\mathbf{Z})$.

In order to apply Theorem 2.3 in practical situations, we have the following proposition which helps to identify the obstruction $w_2(M,L)$.

Proposition 3.4. *Let L be an oriented framed link which bounds a compact oriented surface V consistent with the framing. Let $\alpha \in H^1(L)$ be an element such that $\delta(\alpha) = w_2(M,L)$. Then, we have*

$$\begin{aligned} \langle w_2(M,L), [V,\partial V]_2\rangle &= \langle \delta(\alpha), [V,\partial V]_2\rangle \\ &= \langle \alpha, [L]_2\rangle \\ &\equiv \chi(V) \pmod 2 \\ &\equiv \sharp L \pmod 2, \end{aligned}$$

where $\langle\cdot\,,\cdot\rangle$ is the Kronecker product, $[V,\partial V]_2 \in H_2(M,L)$ is the fundamental class of V in $\mathbf{Z}_2$–coefficients, $\chi(V)$ denotes the Euler characteristic of V, and $\sharp L$ denotes the number of components of L.

The above proposition is similar to the Poincaré–Hopf theorem for vector fields. It can be proved by decomposing V into simplices, and by computing the contribution of each simplex. We omit the details.

The above proposition can also be proved as follows. First, we construct an excellent map $f : M \to \mathbf{R}^2$ such that for a regular value y, $f^{-1}(y)$ coincides with L as a framed link and that for a half line ℓ emanating from y in $\mathbf{R}^2$ transverse to f, we have $f^{-1}(\ell) = V$. Such an excellent map is constructed in [13]. Then, the map $f|_V : V \to \ell$ is a Morse function and its number of critical points coincides with the number of intersection points of V and $S(f)$. As $[S(f)]_2$ is Poincaré dual to $w_2(M,L)$, we see that this number modulo 2 coincides with $\langle w_2(M,L), [V,\partial V]_2\rangle$. Since the number of critical points of the Morse function is congruent modulo 2 to $\chi(V)$, we get the result. The congruence $\chi(V) \equiv \sharp L \pmod 2$ is obvious, since V is a compact orientable surface and $\partial V = L$.

Remark 3.5. The above proposition shows the following. If $f : M \to \mathbf{R}^2$ is an excellent map and $y \in \mathbf{R}^2$ is a regular value such that $L = f^{-1}(y)$ has an odd number of components, then every compact oriented surface V in M bounded by L compatible with the framing of L intersects with $S(f)$. If $H_1(M) = 0$, then this implies that the $\mathbf{Z}_2$ linking number of L and $S(f)$ in M does not vanish. Thus, in this case, the regular fiber L necessarily links with $S(f)$ (see Fig. 8). In particular, if a regular fiber is connected, then it is necessarily linked with at least one component of $S(f)$.

Let us now consider the case of a local knot component. Suppose that the oriented framed link L contains a component K that lies in the interior of a closed 3–disk D embedded in M. Set $U = \operatorname{Int} D$, which is an open set of M diffeomorphic to $\mathbf{R}^3$. In the following, let us identify U with $\mathbf{R}^3$. In this case, up to homotopy, we may assume that the framing τ for M over U is given by the standard framing of $\mathbf{R}^3$.

Let $\pi : \mathbf{R}^3 \to H$ be the orthogonal projection onto a generic hyperplane $H \cong \mathbf{R}^2$ in the sense that $\pi|_K$ is an immersion with normal crossings. On the other hand, we may assume that the first vector field defining the framing τ over K is tangent to K consistent with the orientation.

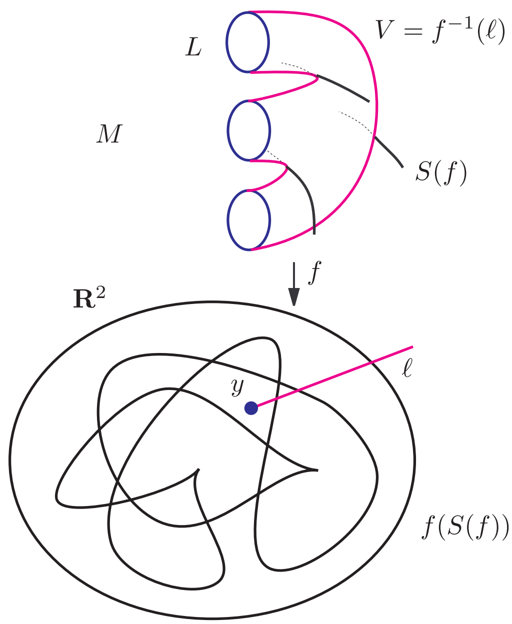

FIGURE 8. Regular fiber with an odd number of components links with the singular point set.

Since $\pi|_K$ is an immersion, we may assume that at each point x of K the remaining two vector fields give a basis for a 2–plane $N_x \subset T_x\mathbf{R}^3$ transverse to T_xK containing the direction $H^\perp$ perpendicular to H. Then, we count the number of times modulo 2 the 2–framing rotates in N_x with respect to a fixed positive direction of $H^\perp$ while $x \in K$ goes around K once. This number is denoted by $t_v(K)$, which is an element in $\mathbf{Z}_2$. Then, we have the following.

LEMMA 3.6. *Let $\alpha \in H^1(L)$ be an arbitrary element such that $\delta(\alpha) = w_2(M, L)$. Then, we have*

$$\langle \alpha, [K]_2 \rangle \equiv t_v(K) + c(K) + 1 \pmod 2,$$

where $c(K)$ denotes the number of crossings of the immersion $\pi|_K : K \to H$ with normal crossings.

Proof. Since the framing τ is standard on $U = \mathbf{R}^3$, in order for the obstruction to vanish on K, we need to have that the winding number of $\pi(K)$ on H is even as long as $t_v(K) = 0$. On the other hand, by [17], we have that the winding number has the same parity as $c(K)+1$. Thus, by the observation in [7, pp. 520–521], we have the conclusion. □

4. EXAMPLES

In this section, we give some explicit examples which imply that the answer to the problem posed in §1 for closed oriented 3–manifolds is negative in general.

EXAMPLE 4.1. Let L be a 2–component framed link $h^{-1}(\{y_1, y_2\})$ in S^3 that consists of two framed fibers of the positive Hopf fibration $h : S^3 \to S^2$, for $y_1 \neq y_2$ in S^2, where we reverse the orientation of one of the components and the framings are induced by h. By taking the inverse image $h^{-1}(a)$ of an embedded arc a in S^2 connecting y_1 and y_2, we see that L is framed null-cobordant (see Fig. 9). By Lemma 3.6, we have that $w_2(S^3, L)$ vanishes. This can also be proved as follows. Let us take two distinct points $p_1, p_2 \in S^2 \setminus \{y_1, y_2\}$. Since $S^2 \setminus \{p_1, p_2\}$ is

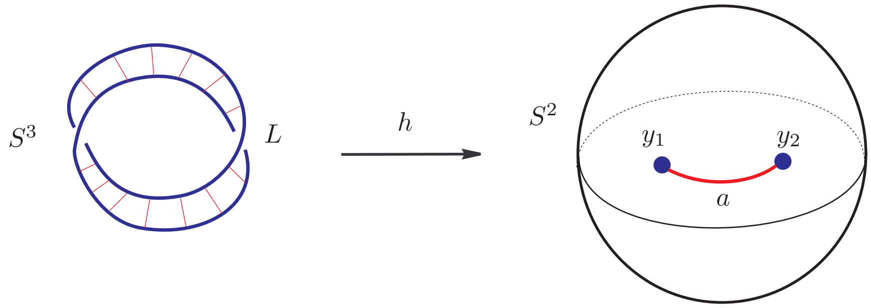

FIGURE 9. Framed Hopf link which is null-cobordant

diffeomorphic to an open annulus $S^1 \times (-1,1)$, it has a 2–framing. By pulling back this 2–framing by the Hopf fibration h, we see that the framing of $TS^3|_L$ naturally extends to $S^3 \smallsetminus h^{-1}(\{p_1, p_2\})$. This means that $w_2(S^3, L)$ is Poincaré dual to $h^{-1}(\{p_1, p_2\})$. Since

$$[h^{-1}(p_1)]_2 = [h^{-1}(p_2)]_2 \in H_1(S^3 \smallsetminus L),$$

we see that $w_2(S^3, L)$ vanishes.

Therefore, by Theorem 2.3, an arbitrary link J split from L can be realized as the singular point set of an excellent map $S^3 \to \mathbf{R}^2$ with L a framed regular fiber, since $[J]_2 = 0$ is Poincaré dual to $w_2(S^3, L) = 0$. In this example, the components of the regular fiber L do not link with the singular point set!

Note that L has an even number of components. This is consistent with the observation given in Remark 3.5.

Let M be an arbitrary closed oriented 3–manifold. By considering the above 2–component link L as embedded in $\mathbf{R}^3 \subset S^3$ and by embedding it to M, we get the same result for M as well. This gives counter examples to the question presented in §1 for closed oriented 3–manifolds.

We will give two explicit examples of excellent maps on S^3 which give counter examples.

EXAMPLE 4.2. Let $h : S^3 \to S^2$ be the (positive) Hopf fibration. Let $p_N = (0,0,1)$ and $p_S = (0,0,-1)$ be the north and the south poles of S^2, respectively, where we identify S^2 with the unit sphere in $\mathbf{R}^3$. We decompose S^2 as $S^2 = D_N \cup D_S \cup A$, where D_N (or D_S) is a small 2–disk neighborhood of p_N (resp. p_S) in S^2 with $D_N \cap D_S = \emptyset$, and A is the annulus obtained as the closure of $S^2 \smallsetminus (D_N \cup D_S)$.

Note that the fibration h is trivial on each of D_N, D_S and A. Let us fix a trivialization

$$h^{-1}(A) = S^1 \times A = S^1 \times ([-1,1] \times S^1) = (S^1 \times [-1,1]) \times S^1, \tag{4.1}$$

where we identify A with $[-1,1] \times S^1$ so that $\{1\} \times S^1$ (or $\{-1\} \times S^1$) coincides with ∂D_N (resp. ∂D_S). We take the trivialization of $h^{-1}(A)$ in such a way that it extends to a trivialization of h over $D_N \cup A$. Note that in (4.1), the first S^1–factor corresponds to the fibers of h and the last S^1–factor corresponds to the equatorial direction of S^2 in the target.

Let $k : S^1 \times [-1,1] \to [1, \infty)$ be a Morse function such that

(1) $k^{-1}(1) = S^1 \times \{-1, 1\}$,
(2) k has no critical point in a small neighborhood of $S^1 \times \{-1, 1\}$,
(3) k has exactly two critical points in such a way that one of them has index 1 and the other has index 2.

Using the above ingredients, let us now construct an excellent map $f : S^3 \to \mathbf{R}^2$ as follows. On $h^{-1}(D_N)$ (or on $h^{-1}(D_S)$), we define $f = i_N \circ h$ (resp. $f = i_S \circ h$), where $i_N : D_N \to \mathbf{R}^2$

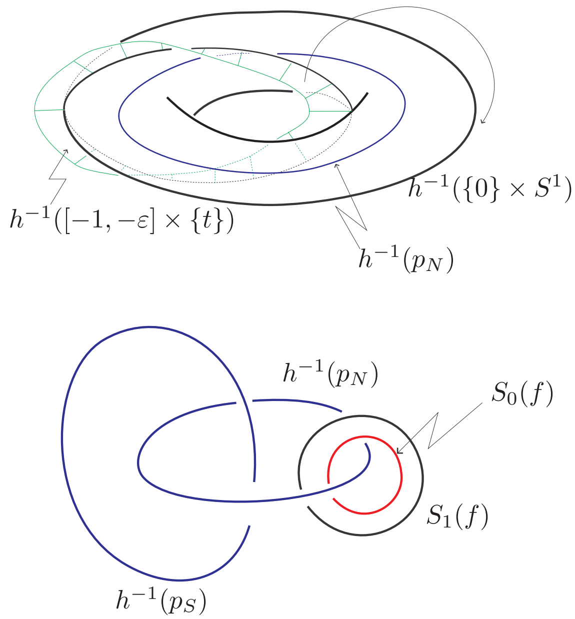

FIGURE 10. Framed regular fiber and the singular point set of the excellent map $f : S^3 \to \mathbf{R}^2$ in Example 4.2

(resp. $i_S : D_S \to \mathbf{R}^2$) is an orientation preserving (resp. reversing) embedding onto the unit disk in $\mathbf{R}^2$ such that $i_N(p_N) = i_S(p_S)$ coincides with the origin $\mathbf{0}$. Furthermore, we choose i_N and i_S such that for each $t \in S^1$, $i_N(1, t) = i_S(-1, t)$ holds for $(1, t)$ and $(-1, t) \in [-1, 1] \times S^1 = A$. On $h^{-1}(A) = (S^1 \times [-1, 1]) \times S^1$, we define f by $f(x, t) = \eta(k(x), t)$ for $x \in S^1 \times [-1, 1]$ and $t \in S^1$, where $\eta : [1, \infty) \times S^1 \to \mathbf{R}^2$ is an embedding such that its image is the complement of the open unit disk in $\mathbf{R}^2$ and that $\eta(\{1\} \times S^1)$ coincides with the unit circle in $\mathbf{R}^2$. We choose η consistently with i_N and i_S, i.e. we require the condition that $\eta(1, t) = i_N(1, t) = i_S(-1, t)$ for every $t \in S^1$. Then, the map $f : S^3 \to \mathbf{R}^2$ thus constructed is well-defined.

By modifying f near the attached tori $h^{-1}(\partial D_N \cup \partial D_S)$ appropriately, we may assume that f is a smooth excellent map. Furthermore, the origin $\mathbf{0}$ of $\mathbf{R}^2$ is a regular value and $f^{-1}(\mathbf{0})$ is a framed regular fiber as in Example 4.1. Note that $S(f)$ has two components: one consists of definite fold singularities and the other of indefinite fold singularities.

The situation is as depicted in Fig. 10. The torus in the top figure represents $h^{-1}(\{0\} \times S^1)$ for $\{0\} \times S^1 \subset [-1, 1] \times S^1 = A$, and it separates the regular fiber components $h^{-1}(p_N)$ and $h^{-1}(p_S)$ of f. The annulus depicts $h^{-1}([-1, -\varepsilon] \times \{t\})$ for some small $\varepsilon > 0$ and for some $t \in S^1$. We may assume that the critical points of k on $h^{-1}([-1, 1] \times \{t\})$ are contained in $h^{-1}([-1, -\varepsilon] \times \{t\})$. As t varies in S^1 in the positive direction, the annulus rotates as depicted in that figure. Therefore, the critical points of k on the annulus sweep out a 2–component link $S(f) = S_0(f) \cup S_1(f)$ as depicted in the bottom figure, where $S_0(f)$ (or $S_1(f)$) is the set of definite (resp. indefinite) fold singularities of f.

In this example, the regular fiber component $h^{-1}(p_S)$ of f does not link with $S(f)$.

EXAMPLE 4.3. We have yet another example $g : S^3 \to \mathbf{R}^2$ constructed as follows. In the following, we use the same notations as in Example 4.2. We define g on $h^{-1}(D_N \cup D_S)$ in exactly the same way as f. On the other hand, we replace f on $h^{-1}(A)$ with the map F defined

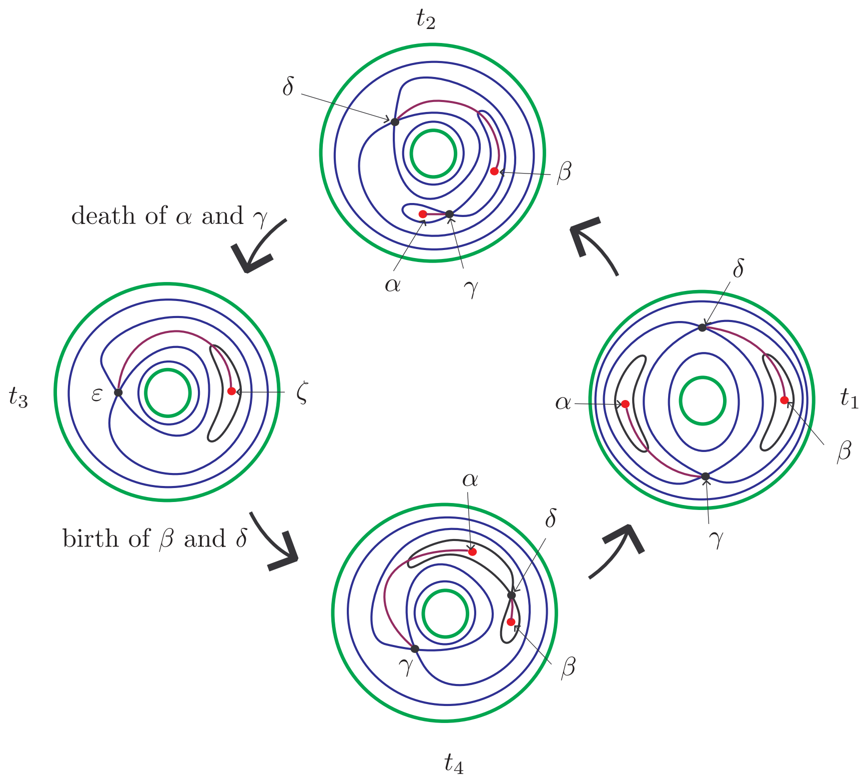

FIGURE 11. Level sets of $k_t : S^1 \times [-1, 1] \to [1, \infty)$ for $t = t_1, t_2, t_3$ and $t_4 \in S^1$, which correspond to those in Fig. 12.

by $F(x, t) = \eta(k_t(x), t)$ for $x \in S^1 \times [-1, 1]$ and $t \in S^1$, where $\eta : [1, \infty) \times S^1 \to \mathbf{R}^2$ is the embedding as in the above example, and $k_t : S^1 \times [-1, 1] \to [1, \infty)$, $t \in S^1$, is a generic 1–parameter family of functions on the annulus whose level sets are as depicted in Fig. 11, where the green circles depict the boundary components of the annulus and correspond to the level set $k_t^{-1}(1)$. Note that for $t \in S^1$, k_t is a Morse function, except for two values where a birth or a death of a pair of critical points occurs. In the figure, the red points depict critical points of index 2 and the black ones of index 1. The singular value set of F is as depicted in Fig. 12, and the critical points in Fig. 11 correspond to the curves $\alpha, \beta, \gamma, \delta, \varepsilon$ and ζ in Fig. 12.

In this way, we get an excellent map $g : S^3 \to \mathbf{R}^2$ with exactly two cusp singularities such that $S(g)$ consists of a circle. Furthermore, we see that $S(g)$ bounds a 2–disk in S^3 disjoint from the regular fiber $g^{-1}(\mathbf{0})$. Such a disk can be found by tracing the brown curves in Fig. 11. Therefore, $S(g)$ is an unknotted circle in S^3 and is split from the regular fiber over the origin $\mathbf{0}$. This again gives a desired counter example.

REMARK 4.4. The above examples show that the answer to the following question (see §1) is, in general, negative for excellent maps of S^3 into $\mathbf{R}^2$: *must every component of a regular fiber be linked by at least one component of the singular point set ?*

REMARK 4.5. Let $f : M \to \mathbf{R}^2$ be an excellent map of a closed oriented 3–manifold M. We assume that f is C^∞ stable, i.e. $f|_{S(f)}$ satisfies certain transversality conditions (for details, see [5, 10]). Such a C^∞ stable map f is *simple* if it has no cusp singularities and for every

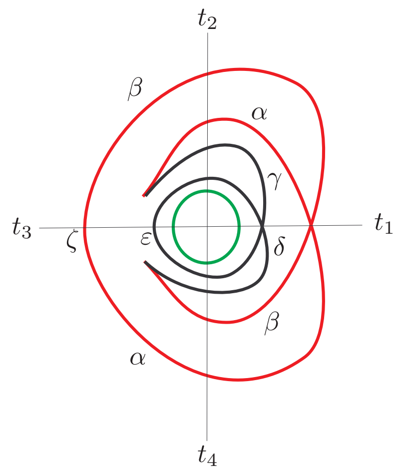

FIGURE 12. Singular value set of F, where the green circle in the center corresponds to the image of $\eta(\{1\} \times S^1)$, the red curve corresponds to the image of the definite fold singularities, and the black one to the image of the indefinite fold singularities. The values t_1, t_2, t_3 and $t_4 \in S^1$ correspond to those in Fig. 11.

$y \in f(S(f))$, each component of $f^{-1}(y)$ contains at most one singular point. In this case, by [15], regular fibers, the singular point set, or their unions are all graph links: i.e. their exteriors are unions of circle bundles over surfaces attached along their torus boundaries. The realization problem of graph links as regular fibers or the singular point set has been addressed in [15]. See also [12].

5. MAPS OF $\mathbf{R}^3$ INTO $\mathbf{R}^2$

Let us consider the following problem (see §1 and Remark 4.4).

PROBLEM 5.1. For a generic map $f : \mathbf{R}^3 \to \mathbf{R}^2$, must every component of a regular fiber be linked by at least one component of the singular point set $S(f)$?

In order to answer negatively to the above problem, we use the following theorem which is due to Hector and Peralta-Salas [6].

THEOREM 5.2 (Hector and Peralta-Salas, 2012). *Let $L = L_1 \cup L_2 \cup \cdots \cup L_\mu \subset \mathbf{R}^3$ be an oriented link. Then, there exist a submersion $f : \mathbf{R}^3 \to \mathbf{R}^2$ and a regular value $y \in \mathbf{R}^2$ such that $f^{-1}(y) = L$ if and only if for all i with $1 \leq i \leq \mu$, we have*

$$\sum_{j \neq i} \operatorname{lk}(L_i, L_j) \equiv 1 \pmod 2,$$

where lk *denotes the linking number.*

Now, let L be a link that satisfies the condition as described in Theorem 5.2 (for example, a Hopf link). Then, there exist a submersion $f : \mathbf{R}^3 \to \mathbf{R}^2$ and a regular value $y \in \mathbf{R}^2$ with $L = f^{-1}(y)$.

Take a point $p \in \mathbf{R}^3 \smallsetminus L$ and its small 3–disk neighborhood $N(p) \subset \mathbf{R}^3 \smallsetminus L$. Then, we can deform f in $N(p)$ so that the resulting map $g : \mathbf{R}^3 \to \mathbf{R}^2$ is excellent and $S(g)$ is an unknotted circle in $N(p)$ (use the move called "lip" or "birth". See [14, Lemma 3.1]). Then, no component of $L = g^{-1}(y)$ links with $S(g)$.

This gives a negative answer to Problem 5.1.

We finish this paper by posing some open problems.

PROBLEM 5.3. Can we generalize Theorem 2.3 for generic maps $f : M \to \mathbf{R}^2$ for closed non-orientable 3–manifolds? How about generic maps of general closed n–dimensional manifolds into $\mathbf{R}^p$ with $n > p > 1$?

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References

[1] M.V. Berry and M.R. Dennis, *Knotted and linked phase singularities in monochromatic waves*, Proc. R. Soc. Lond. A **457** (2001), 2251–2263. DOI: 10.1098/rspa.2001.0826

[2] M.V Berry and M.R. Dennis, *Topological events on wave dislocation lines: birth and death of loops, and reconnection*, J. Physics A: Mathematical and Theoretical **40** (2007), 65–74. DOI: 10.1088/1751-8113/40/1/004

[3] M. Cencelj, D. Repovš and M.B. Skopenkov, *Classification of framed links in* 3*–manifolds*, Proc. Indian Acad. Sci. (Math. Sci.) **117** (2007), 301–306. DOI: 10.1007/s12044-007-0025-x

[4] M.R. Dennis, *Local phase structure of wave dislocation lines: twist and twirl*, J. Opt. A: Pure Appl. Opt. **6** (2004), S202–S208. DOI: 10.1088/1464-4258/6/5/011

[5] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Grad. Texts in Math., Vol. 14, Springer–Verlag, New York, Heidelberg, Berlin, 1973.

[6] G. Hector and D. Peralta-Salas, *Integrable embeddings and foliations*, Amer. J. Math. **134** (2012), 773–825. DOI: 10.1353/ajm.2012.0018

[7] M.A. Kervaire, *Relative characteristic classes*, Amer. J. Math. **79** (1957), 517–558.

[8] H. Larocque, D. Sugic, D. Mortimer, A.J. Taylor, R. Fickler, R.W. Boyd, M.R. Dennis, and E. Karimi , *Reconstructing the topology of optical polarization knots*, Nature Physics (2018), https://doi.org/10.1038/s41567-018-0229-2.

[9] H.I. Levine, *Elimination of cusps*, Topology **3**, Suppl. 2 (1965), 263–296. DOI: 10.1007/bfb0075066

[10] H. Levine, *Classifying immersions into* $\mathbf{R}^4$ *over stable maps of* 3*-manifolds into* $\mathbf{R}^2$, Lecture Notes in Math., Vol. 1157, Springer–Verlag, Berlin, 1985.

[11] J. Milnor, *Spin structures on manifolds*, Enseignement Math. (2) **9** (1963), 198–203.

[12] O. Saeki, *Simple stable maps of* 3*–manifolds into surfaces II*, J. Fac. Sci. Univ. Tokyo **40** (1993), 73–124.

[13] O. Saeki, *Stable maps and links in* 3*–manifolds*, Kodai Math. J. **17** (1994), 518–529. DOI: 10.2996/kmj/1138040047

[14] O. Saeki, *Constructing generic smooth maps of a manifold into a surface with prescribed singular loci*, Ann. Inst. Fourier (Grenoble) **45** (1995), 1135–1162. DOI: 10.5802/aif.1489

[15] O. Saeki, *Simple stable maps of* 3*-manifolds into surfaces*, Topology **35** (1996), 671–698. DOI: 10.1016/0040-9383(95)00034-8

[16] R. Thom, *Les singularités des applications différentiables*, Ann. Inst. Fourier, Grenoble **6** (1955–1956), 43–87.

[17] H. Whitney, *On regular closed curves in the plane*, Compo. Math. **4** (1937), 276–284.

[18] H. Whitney, *On singularities of mappings of Euclidean spaces. I. Mappings of the plane into the plane*, Ann. of Math. (2) **62** (1955), 374–410. DOI: 10.1007/978-1-4612-2972-8_27

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