Journal *of* Singularities
Volume 17 (2018), 403-412

received: 20 March 2018
in revised form: 13 November 2018
DOI: 10.5427/jsing.2018.17r

RADIAL INDEX RELATED TO AN INTERSECTION INDEX

JULIE LAPEBIE

Abstract. On a closed semialgebraic set X with an isolated singularity at 0, we consider a vector field V with an isolated singularity at 0 and study its radial index at the singularity. We first prove a relation between this index and the radial index at 0 of the vector field $-V$ and give some corollaries of this formula. Then we express the radial index of V at 0 in terms of the intersection index of two submanifolds of a certain bundle defined over the link of the singularity.

1. Introduction

To work on manifolds with singularities, we cannot work with the Poincaré-Hopf index, which is only defined on smooth manifolds, we need to work with another one, the radial index (or Schwartz index depending on the authors). This index was introduced by Schwartz ([10], [11]) but only concerned radial vector fields. Later, a generalization to other vector fields was made by King and Trotman in [9] (see also [1], [5] and [12]). We will use the definition of Ebeling and Gusein-Zade given in [6].

The aim of this paper is to study some properties of the radial index of a vector field at an isolated singularity of a closed semialgebraic set. On a closed semialgebraic set X of dimension k in $\mathbb{R}^N$ with an isolated singularity at 0, we consider a vector field V. We first express the radial index of V at 0 in terms of the zeros of $V_{|lk(X)}$, the restriction of V to the link of the singularity (see Proposition 2.5). Then in Theorem 2.8, we relate the radial index of V at 0 to the one of $-V$ at 0. Namely we prove:

- if k is even, then
$$ind_{rad}(-V,0) = ind_{rad}(V,0),$$
- if k is odd, then
$$ind_{rad}(V,0) + ind_{rad}(-V,0) = 2 - \chi(lk(X)).$$

As a corollary (see Corollary 2.9), we obtain that:
$$ind_{rad}(V,0) + ind_{rad}(-V,0) \equiv \chi(lk(X)) \mod 2.$$

When $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ is an analytic function defined on a neighborhood of 0 such that the fiber $f^{-1}(0)$ admits an isolated singularity, Khimshiashvili (1977) proved the following formula:
$$\chi(f^{-1}(\delta) \cap B^n_\epsilon) = 1 - \mathrm{sign}(-\delta)^n \deg_0 \nabla f,$$
where δ is a regular value of f such that $0 < |\delta| \ll \epsilon \ll 1$ and $\deg_0 \nabla f$ is the degree of $\frac{\nabla f}{\|\nabla f\|} : S^{n-1}_\epsilon \to S^{n-1}$. As a corollary of Theorem 2.8, we obtain a singular version of the Khimshiashvili formula. Let $g : X \to \mathbb{R}$ be the restriction to X of a $\mathcal{C}^2$-semialgebraic function without any critical points on a neighborhood of 0 and such that $g(0) = 0$. For δ a regular value of g with $0 < \delta \ll \epsilon \ll 1$, we have (see Corollary 2.10):

2010 *Mathematics Subject Classification.* 14P10, 14P25, 58K05.
Key words and phrases. Radial index, semialgebraic set, Euler characteristic.

- if k is even,
$$\chi(g^{-1}(\delta) \cap B_\epsilon^N) = 1 - ind_{rad}(\nabla g, 0),$$
- if k is odd,
$$\chi(g^{-1}(\delta) \cap B_\epsilon^N) = \chi(lk(X)) - 1 + ind_{rad}(\nabla g, 0).$$

This gives us a direct connection between the Euler characteristic of the Milnor fiber of g and the radial index of ∇g at 0.

Finally, in the last section, we give a new characterization of the radial index of V at 0. Remarking that a coincidence point $x \in lk(X)$ of V and $-\nabla\rho$, where ρ is the restriction to X of the distance function to the origin and $\nabla\rho$ is its gradient vector field, is an inward zero (see Definition 2.4), we relate the radial index of V at 0 to an intersection index defined in terms of V and $\nabla\rho$. More precisely, we consider the normalized vector fields $\widehat{V}$ and $-\nabla\widehat{\rho}$ on $lk(X)$ and their respective graphs $G_{\widehat{V}}$ and $G_{-\nabla\widehat{\rho}}$. In Proposition 3.3, we prove that:
$$I(G_{-\nabla\hat{\rho}}, G_{\hat{V}}) = 1 - ind_{rad}(V, 0),$$
where $I(G_{-\nabla\hat{\rho}}, G_{\hat{V}})$ is the intersection index of the graphs.

The paper is organized as follows: in Section 2, we write the radial index of V at 0 as a sum involving only the zeros of $V_{|lk(X)}$. This enables us to prove Theorem 2.8 and the two corollaries presented above. In Section 3 we establish the formula relating the radial index of V at 0 to an intersection number.

2. First results on the radial index

Let us consider a k-dimensional closed semialgebraic set $X \subset \mathbb{R}^N$ with an isolated singularity at 0. We denote by B_ϵ^N the closed ball of dimension N, with radius ϵ and centered at 0, by $S_\epsilon^{N-1} = \partial B_\epsilon^N$ the sphere of dimension $N-1$ and by $\mathring{B}_\epsilon^N = B_\epsilon^N \setminus S_\epsilon^{N-1}$ the open ball. We denote by $lk(X)$ the link of X at 0, i.e., the intersection of X with S_ϵ^{N-1} where $\epsilon > 0$ is such that 0 is the only singularity of X in $X \cap B_\epsilon^N$ and $0 < \epsilon \ll 1$.

Before defining the radial index of a vector field at a singularity, we need to define what a vector field at a singularity is and what a radial vector field is.

Definition 2.1 ([3] p.32)**.** Let V be a vector field on $X \subset \mathbb{R}^N$. We say that V admits an isolated singularity at $0 \in X$ if V is a continuous section of $T\mathbb{R}^N_{|X}$ tangent to $X \setminus \{0\}$ and does not have any zero on a small neighborhood of 0.

Definition 2.2. A continuous vector field V_{rad} tangent to X is called **radial** (at 0) if it is transverse to the link of X. Moreover, we make the convention that it points outward, i.e., it does not point toward 0.

Definition 2.3 ([7] Definition 2.1, [5], [6] Section 1 and [3] Chapter 2)**.** Let us consider a continuous vector field V on X with an isolated singularity at 0. Let $\epsilon > 0$ be such that V does not have any zeros on $X \setminus \{0\} \cap B_\epsilon^N$ and let $0 < \epsilon' < \epsilon$. Let us set $X_{\epsilon\epsilon'} = (X \cap B_\epsilon^N) \setminus \mathring{B}_{\epsilon'}^N$ and $\partial X_{\epsilon\epsilon'} = X_\epsilon \cup X_{\epsilon'}$ where $X_\epsilon = X \cap S_\epsilon^{N-1}$ and $X_{\epsilon'} = X \cap S_{\epsilon'}^{N-1}$.
Let $\widetilde{V}$ be a vector field on $X_{\epsilon\epsilon'}$ such that:

- $\widetilde{V}_{|X_\epsilon} = V$,
- $\widetilde{V}_{|X_{\epsilon'}} = V_{rad}$ is a radial vector field,
- $\widetilde{V}_{|\mathring{X}_{\epsilon\epsilon'}}$ has a finite number of zeros $(p_i)_{1 \le i \le s}$ which are all non-degenerate.

Then we define the **radial index** of V at 0, denoted by $ind_{rad}(V, 0)$, by:
$$ind_{rad}(V, 0) := 1 + \sum_{i=1}^{s} ind_{PH}(\widetilde{V}, p_i).$$

For the next proposition, we need to define what a correct zero of a vector field is.

Definition 2.4. We say that a zero $p \in lk(X)$ of $V_{|lk(X)}$ is **correct** if it is a zero of $V_{|lk(X)}$ and not of V on X. Moreover we say that a correct zero $p \in lk(X)$ of $V_{|lk(X)}$ is an **inward** (resp. **outward**) zero if $V(p)$ points (resp. does not point) toward 0.

Proposition 2.5. *Let X be a closed semialgebraic set with an isolated singularity at 0 and let V be a continuous vector field with an isolated singularity at 0. We assume that V is defined on a neighborhood $\mathcal{U}_0$ of 0 in X. Let $\mathcal{U}_1 \subsetneq \mathcal{U}_0$ be a neighborhood of $lk(X)$. Let $\widetilde{V}$ be a $\mathcal{C}^1$-perturbation of V on $\mathcal{U}_0$ such that $\widetilde{V}_{|lk(X)}$ admits a finite number of isolated non-degenerate zeros $r_1, ..., r_t$ which are all correct, and such that $\widetilde{V}$ is equal to V on $\mathcal{U}_0 \setminus \mathcal{U}_1$. Then we have the following equality:*

$$ind_{rad}(V,0) = 1 - \sum_{\substack{i=1 \\ r_i \text{ inward zero}}}^{t} ind_{PH}(\widetilde{V}_{|lk(X)}, r_i).$$

Proof. Let $0 < \epsilon' < \epsilon$. We keep the notations of Definition 2.3. Let $\widetilde{\widetilde{V}}$ a perturbation of $\widetilde{V}$ on $\mathcal{U}_0$ such that:

- $\widetilde{\widetilde{V}}_{|\mathring{X}_{\epsilon\epsilon'}}$ admits a finite number of non-degenerate zeros $p_1, \ldots, p_s$,
- $\widetilde{\widetilde{V}}_{|X_\epsilon} = \widetilde{V}_{|X_\epsilon} = \widetilde{V}_{|lk(X)}$,
- $\widetilde{\widetilde{V}}_{|X_{\epsilon'}}$ is a $\mathcal{C}^1$-perturbation of V_{rad} on $X_{\epsilon'}$ having only nondegenerate zeros $q_1, ..., q_u$ which are all inward in $X_{\epsilon\epsilon'}$. This is possible for V_{rad} is radial so points inward $X_{\epsilon\epsilon'}$ at every point of $X_{\epsilon'}$.

Applying the Poincaré-Hopf theorem for manifolds with boundary (see for example [2] or [4]) to the vector field $\widetilde{\widetilde{V}}$ on the set $X_{\epsilon\epsilon'}$, we get the equality:

$$\chi(X_{\epsilon\epsilon'}) = \sum_{i=1}^{s} ind_{PH}(\widetilde{\widetilde{V}}, p_i) + \sum_{\substack{i=1 \\ q_i \text{ inward in } X_{\epsilon\epsilon'}}}^{u} ind_{PH}(\widetilde{\widetilde{V}}_{|X_{\epsilon'}}, q_i)$$

$$+ \sum_{\substack{i=1 \\ r_i \text{ inward in } X_{\epsilon\epsilon'}}}^{t} ind_{PH}(\widetilde{V}_{|X_\epsilon}, r_i).$$

As $\widetilde{\widetilde{V}}_{|X_{\epsilon'}}$ only admits inward zeros on $X_{\epsilon'}$, we have

$$\chi(X_{\epsilon'}) = \sum_{i=1}^{u} ind_{PH}(\widetilde{\widetilde{V}}_{|X_{\epsilon'}}, q_i),$$

and so

$$\chi(X_{\epsilon\epsilon'}) - \chi(X_{\epsilon'}) = \sum_{i=1}^{s} ind_{PH}(\widetilde{\widetilde{V}}, p_i) + \sum_{\substack{i=1 \\ r_i \text{ inward in} X_{\epsilon\epsilon'}}}^{t} ind_{PH}(\widetilde{V}_{|X_\epsilon}, r_i).$$

But $X_{\epsilon'}$ is a retract by deformation of $X_{\epsilon\epsilon'}$ so $\chi(X_{\epsilon\epsilon'}) = \chi(X_{\epsilon'})$. Then we find

$$\sum_{i=1}^{s} ind_{PH}(\widetilde{\widetilde{V}}, p_i) = - \sum_{\substack{i=1 \\ r_i \text{ inward in } X_{\epsilon\epsilon'}}}^{t} ind_{PH}(\widetilde{V}_{|X_\epsilon}, r_i),$$

and therefore,

$$\underbrace{1+\sum_{i=1}^{s} ind_{PH}(\widetilde{\widetilde{V}},p_i)}_{=\,ind_{rad}(V,0)} = 1 - \sum_{\substack{i=1 \\ r_i \text{ inward in } X_{\epsilon\epsilon'}}}^{t} ind_{PH}(\widetilde{V}|_{X_\epsilon},r_i),$$

which gives us the required result. ☐

Remark 2.6. This proposition is a particular case of Definition 5.5 in [9].

Remark 2.7. The proof of the next theorem is similar to the one of the Poincaré-Hopf theorem for manifolds with boundary.

Theorem 2.8. *Let X be a closed semialgebraic set of dimension k with an isolated singularity at 0 and let V be a vector field on X with an isolated singularity at 0. Then we have the following equalities:*

- *if k is even, then*
$$ind_{rad}(-V,0) = ind_{rad}(V,0),$$
- *if k is odd, then*
$$ind_{rad}(V,0) + ind_{rad}(-V,0) = 2 - \chi(lk(X)).$$

Proof. Let $\widetilde{V}$ be a $\mathcal{C}^1$-perturbation of V satisfying the conditions of Proposition 2.5. We denote by $(q_i)_{1\leq i\leq t}$ the inward zeros of $\widetilde{V}_{|lk(X)}$ and by $(p_i)_{1\leq i\leq s}$ its outward zeros. By Proposition 2.5, we can write

$$ind_{rad}(-V,0) = 1 - \sum_{r \text{ inward zero of } -\widetilde{V}_{|lk(X)}} ind_{PH}(-\widetilde{V}_{|lk(X)},r).$$

But we know that

$$\sum_{r \text{ inward zero of } -\widetilde{V}_{|lk(X)}} ind_{PH}(-\widetilde{V}_{|lk(X)},r) = \sum_{i=1}^{s} ind_{PH}(-\widetilde{V}_{|lk(X)},p_i).$$

As X has dimension k, the link $lk(X)$ has dimension $k-1$ and so

$$ind_{PH}(-\widetilde{V}_{|lk(X)},p_i) = (-1)^{k-1} ind_{PH}(\widetilde{V}_{|lk(X)},p_i).$$

Moreover,

$$\begin{array}{rcl} ind_{rad}(V,0) & = & 1 - \sum_{r \text{ inward zero of } \widetilde{V}_{|lk(X)}} ind_{PH}(\widetilde{V}_{|lk(X)},r) \\ & = & 1 - \sum_{i=1}^{t} ind_{PH}(\widetilde{V}_{|lk(X)},q_i). \end{array}$$

Thus we find that

$$\begin{array}{rcl} S & := & ind_{rad}(V,0) + ind_{rad}(-V,0) \\ & = & 2 - \sum_{i=1}^{t} ind_{PH}(\widetilde{V}_{|lk(X)},q_i) - (-1)^{k-1}\sum_{i=1}^{s} ind_{PH}(\widetilde{V}_{|lk(X)},p_i). \end{array}$$

We also have the equality

$$\chi(lk(X)) = \sum_{i=1}^{t} ind_{PH}(\widetilde{V}_{|lk(X)},q_i) + \sum_{i=1}^{s} ind_{PH}(\widetilde{V}_{|lk(X)},p_i).$$

Let us compare these two equalities according to the parity of k. If k is even, then we have

$$\begin{array}{rcl} S & = & 2 - \sum_{i=1}^{t} ind_{PH}(\widetilde{V}_{|lk(X)},q_i) + \sum_{i=1}^{s} ind_{PH}(\widetilde{V}_{|lk(X)},p_i) \\ & = & 2 - \sum_{i=1}^{t} ind_{PH}(\widetilde{V}_{|lk(X)},q_i) + \chi(lk(X)) - \sum_{i=1}^{t} ind_{PH}(\widetilde{V}_{|lk(X)},q_i) \\ & = & 2 - 2\sum_{i=1}^{t} ind_{PH}(\widetilde{V}_{|lk(X)},q_i) + \chi(lk(X)) \\ & = & 2\ ind_{rad}(V,0) + \chi(lk(X)). \end{array}$$

As the dimension of $lk(X)$ is odd, its Euler characteristic vanishes. Therefore

$$ind_{rad}(-V,0) = ind_{rad}(V,0).$$

If k is odd, then

$$\begin{array}{rcl} S & = & 2 - \sum_{i=1}^{t} ind_{PH}(\widetilde{V}_{|lk(X)}, q_i) - \sum_{i=1}^{s} ind_{PH}(\widetilde{V}_{|lk(X)}, p_i) \\ & = & 2 - \chi(lk(X)). \end{array}$$

□

The following result is straightforward consequence of Theorem 2.8.

Corollary 2.9. *Let X be a closed semialgebraic set with an isolated singularity at 0 and let V be a vector field on X with an isolated singularity at 0. Then*

$$ind_{rad}(V,0) + ind_{rad}(-V,0) \equiv \chi(lk(X)) \mod 2.$$

Proof. In the previous theorem, we have seen that if k is even, then

$$S = ind_{rad}(V,0) + ind_{rad}(-V,0) = 2\ ind_{rad}(V,0) \equiv 0 \mod 2.$$

Since in this case $\chi(lk(X)) = 0$, we get that

$$S \equiv \chi(lk(X)) \mod 2.$$

If k is odd, then

$$S = 2 - \chi(lk(X)),$$

and hence

$$S \equiv \chi(lk(X)) \mod 2.$$

□

Finally, applying these results to the case of a gradient vector field, we obtain a singular version of the Khimshiashvili formula.

Corollary 2.10. *Let $X \subset \mathbb{R}^N$ be a closed semialgebraic set of dimension k with an isolated singularity at 0 and let us consider a function $g : X \to \mathbb{R}$ restriction to X of a $\mathcal{C}^2$-semialgebraic function. Let us suppose that $g(0) = 0$ and that g does not have any critical point on a neighborhood of 0. Then for $0 < \delta \ll \epsilon \ll 1$, we have the following equalities:*

- *if k is even,*
$$\chi(g^{-1}(\delta) \cap B_\epsilon^N) = 1 - ind_{rad}(\nabla g, 0),$$
- *if k is odd,*
$$\chi(g^{-1}(\delta) \cap B_\epsilon^N) = \chi(lk(X)) - 1 + ind_{rad}(\nabla g, 0).$$

Proof. From [7] Example 2.6, we know that for $0 < \delta \ll \epsilon \ll 1$, we have

$$\chi(g^{-1}(-\delta) \cap B_\epsilon^N) = 1 - ind_{rad}(\nabla g, 0).$$

So

$$\chi(g^{-1}(\delta) \cap B_\epsilon^N) = 1 - ind_{rad}(-\nabla g, 0).$$

Thus by Theorem 2.8, we obtain that:

- if k is even, then
$$ind_{rad}(-\nabla g, 0) = ind_{rad}(\nabla g, 0),$$
and so
$$\chi(g^{-1}(\delta) \cap B_\epsilon^N) = 1 - ind_{rad}(\nabla g, 0),$$

- if k is odd, then

$$ind_{rad}(-\nabla g, 0) = 2 - \chi(lk(X)) - ind_{rad}(\nabla g, 0),$$

and therefore

$$\begin{array}{rcl}\chi(g^{-1}(\delta) \cap B_\epsilon^N) & = & 1 - ind_{rad}(-\nabla g, 0) \\ & = & \chi(lk(X)) - 1 + ind_{rad}(\nabla g, 0).\end{array}$$

□

3. A link with an intersection index

First, we recall the definition of the intersection index of two manifolds [8].

Definition 3.1. Let X, Y and Z be three oriented manifolds without boundary such that $X, Z \subset Y$. We suppose that X is compact, Z is closed and $\dim X + \dim Z = \dim Y$. If X and Z are transverse, then we define the **intersection index** $I(X, Z)$ of X and Z counting the points x of the intersection $X \cap Z$ and associating to each a plus sign $+$ or a minus sign $-$. More precisely, if for $x \in X \cap Z$ the given orientation of $T_xX \times T_xZ$ (where we denoted by T_xX the tangent plane of X at x) is the same as the one given by the tangent plane T_xY, then we associate a plus sign $+$ to x. We set $or(x) = 1$. Otherwise, we associate a minus sign $-$ and we set $or(x) = -1$. Let $C(X, Z)$ be the set of intersection points of X and Z. Then we define $I(X, Z)$ by:

$$I(X, Z) := \sum_{x \in C(X,Z)} (-1)^{or(x)}.$$

Remark 3.2. When X and Z are not transverse, we perturb X and Z in $\tilde{X}$ and $\tilde{Z}$ respectively so that $\tilde{X}$ and $\tilde{Z}$ are transverse. Then the intersection index $I(\tilde{X}, \tilde{Z})$ exists according to the previous definition. Moreover, as this index is invariant under perturbations, we define $I(X, Z)$ as $I(X, Z) = I(\tilde{X}, \tilde{Z})$.

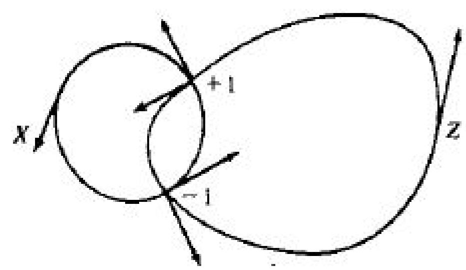

FIGURE 1. Example in dimension 2.

Let $X \subset \mathbb{R}^N$ be a closed semialgebraic set of dimension $k + 1$ with an isolated singularity at 0. We suppose that $X \setminus \{0\}$ is oriented. For example, we can consider a set of the form $X = \{f_1 = ... = f_l = 0\}$ such that $\operatorname{rank}(\nabla f_1(x), ..., \nabla f_l(x)) = l$ for all $x \in X \setminus \{0\}$ and where $l = N - k - 1$.

We consider a $\mathcal{C}^2$-tangent vector field V on $X \setminus \{0\}$ such that V has an isolated at 0. Let ρ be the restriction to X of the distance function to the origin and let $\nabla\rho$ be its gradient vector field. Then the vector field $\nabla\rho$ is a $\mathcal{C}^2$-radial vector field on X. We still denote by $lk(X)$ the link of X at 0.

The vector field V does not admit any zero on $lk(X)$. For simplicity, we set $Y := lk(X)$. We can remark that Y is an oriented submanifold of X of dimension k. Indeed, we know that there exists a $(k+1)$-dimension manifold with boundary Z such that the boundary ∂Z of Z is equal to Y. Here $Z = X \cap B_\epsilon^N \setminus \{0\}$. We recall that the orientation of Y is induced by the one of Z. Let $(v_1, ..., v_k)$ be a basis of $Y = \partial Z$ and let $x \in Y$. The space T_xY has codimension 1 in T_xZ so there exist exactly two unit vectors $\vec{v}$ and $-\vec{v}$ in T_xZ which are orthogonal to T_xY, $\vec{v}$ pointing outward and $-\vec{v}$ inward. We suppose that $(v_1, ..., v_k, \vec{v})$ is a positively oriented basis of T_xZ. Then the basis $(v_1, ..., v_k)$ is positively oriented for T_xY and so is for Y.

For all $x \in Y$, we introduce these two vector fields:

$$\widehat{V}(x) = \frac{V(x)}{\|V(x)\|} \quad \text{and} \quad -\nabla\widehat{\rho}(x) = -\frac{\nabla\rho(x)}{\|\nabla\rho(x)\|}.$$

We say that x is a **coincidence point** of $\widehat{V}$ and $-\nabla\widehat{\rho}$ if $\widehat{V}(x) = -\nabla\widehat{\rho}(x)$. We see that these coincidence points are related to the inward zeros of the vector field $V_{|Y}$. Indeed, we have

$$\begin{aligned} \widehat{V}(x) = -\nabla\widehat{\rho}(x) \quad &\Longleftrightarrow \quad V(x) = -\tfrac{\|V(x)\|}{\|\nabla\rho(x)\|}\nabla\rho(x) \\ &\Longleftrightarrow \quad x \text{ is an inward zero of } V_{|Y}. \end{aligned}$$

Let us introduce the following set:

$$TUY = \Big\{(x, \vec{v}) \mid x \in Y,\ \vec{v} \in T_xX \text{ and } \|\vec{v}\| = 1\Big\},$$

which is included in the unit tangent bundle of X, that is

$$TUX = \Big\{(x, \vec{v}) \mid x \in X,\ \vec{v} \in T_xX \text{ and } \|\vec{v}\| = 1\Big\}.$$

The bundle TUX is locally the product of an open set in $\mathbb{R}^k$ with the k-sphere S^k so has dimension $\dim X + \dim S^k = k + 1 + k = 2k + 1$.

The set TUY is an oriented manifold of dimension $2k$. Indeed,

$$TUY = TUX \cap \big(\{\rho(x) = \epsilon\} \times \mathbb{R}^{k+1}\big).$$

The tangent bundle TX of X is oriented so is TUX. Actually, TUX is the transverse intersection of TX with $\mathbb{R}^n \times \{\vec{v} \mid \|\vec{v}\| = 1\}$, which is also an oriented manifold. As the two manifolds TUX and $\{\rho(x) = \epsilon\} \times \mathbb{R}^{k+1}$ are oriented and their intersection is transverse, TUY is also oriented. Moreover, its dimension is equal to $2k$ for it is equal to $\dim TUX - 1$.

Let $G_{\widehat{V}}$ and $G_{-\nabla\widehat{\rho}}$ be the respective graphs of $\widehat{V}$ and $-\nabla\widehat{\rho}$:

$$G_{\widehat{V}} = \Big\{(x, \widehat{V}(x)) \mid x \in Y\Big\} \text{ and } G_{-\nabla\widehat{\rho}} = \Big\{(x, -\nabla\widehat{\rho}(x)) \mid x \in Y\Big\}.$$

These graphs are oriented submanifolds of TUY. A positive basis of $T_{(x,\widehat{V}(x))}G_{\widehat{V}}$ is of the form

$$\Big((\epsilon_1, D\widehat{V}(x)(\epsilon_1)), ..., (\epsilon_k, D\widehat{V}(x)(\epsilon_k))\Big)$$

for $(\epsilon_1, ..., \epsilon_k)$ a positive basis of T_xY.

The following result relates $ind_{rad}(V, 0)$ to $I(G_{-\nabla\widehat{\rho}}, G_{\widehat{V}})$.

Proposition 3.3. *Let $X \subset \mathbb{R}^N$ be a closed semialgebraic set of dimension $k+1$ with an isolated singularity at 0 such that $X \setminus \{0\}$ is oriented. Let V be a $\mathcal{C}^2$-vector field on X with an isolated singularity at the origin. Then we have the equality:*

$$I(G_{-\nabla\widehat{\rho}}, G_{\widehat{V}}) = 1 - ind_{rad}(V, 0).$$

Proof. The graphs $G_{\widehat{V}}$ and $G_{-\nabla\widehat{\rho}}$ of $\widehat{V}$ and $-\nabla\widehat{\rho}$ respectively are oriented compact submanifolds of TUY, each of dimension k. Then their intersection index $I(G_{-\nabla\widehat{\rho}}, G_{\widehat{V}})$ is well defined.

Let us consider a coincidence point $x \in Y$, where $Y = lk(X)$, of $\widehat{V}$ and $-\nabla\widehat{\rho}$. The point $(x, \widehat{V}(x))$ is in this case an intersection point of the two graphs $G_{\widehat{V}}$ and $G_{-\nabla\widehat{\rho}}$. We are going to compute the value of $or((x, \widehat{V}(x)))$, which we denote for simplicity by $or(x)$, in order to relate it with $ind_{PH}(V_{|Y}, x)$, the Poincaré-Hopf index of $V_{|Y}$ at x.

To compute $or(x)$, we must find a positive basis of the tangent plane to TUY at $(x, \widehat{V}(x))$ and a positive basis for the tangent planes $T_{(x,\widehat{V}(x))}G_{\widehat{V}}$ and $T_{(x,\widehat{V}(x))}G_{-\nabla\widehat{\rho}}$.

Let us begin with $T_{(x,\widehat{V}(x))}TUY$. As $X \setminus \{0\}$ is a manifold of dimension $k+1$, we know that there exists a neighborhood $\mathcal{U}_1$ of $x \in Y$ in X, and a diffeomorphism $\phi_1 : \mathcal{U}_1 \to \mathbb{R}^{k+1}$ such that $\phi_1(\mathcal{U}_1)$ is an open set of $\mathbb{R}^{k+1}$. Moreover, Y is a submanifold of X of dimension k so $\mathcal{U}_1 \cap Y$ is a neighborhood of x in Y and $\phi_1(\mathcal{U}_1 \cap Y) = \phi_1(\mathcal{U}_1) \cap \mathbb{R}^k$ is an open set of $\mathbb{R}^k$, where we identify $\mathbb{R}^k$ with $\mathbb{R}^k \times \{0\} \subset \mathbb{R}^{k+1}$. We will write $\mathbb{R}^k \times \{0\} = \{x_{k+1} = 0\}$. So

$$\phi_1(\mathcal{U}_1 \cap Y) = \phi_1(\mathcal{U}_1) \cap \{x_{k+1} = 0\}.$$

Moreover, as Z is a manifold with boundary included in X of dimension $k+1$ and such that $\partial Z = Y$, the open set $\mathcal{U}_1 \cap Z$ is a neighborhood of x in Z and

$$\phi_1(\mathcal{U}_1 \cap Z) = \phi_1(\mathcal{U}_1) \cap \{x_{k+1} \geq 0\},$$

which is an open set of $\mathbb{R}^k \times \mathbb{R}_+$, where $\{x_{k+1} \geq 0\}$ denotes $\mathbb{R}^k \times \mathbb{R}_+$.

Locally, if $x \in \mathcal{U}_1$, the bundle TX is diffeomorphic to $\phi_1(\mathcal{U}_1) \times \mathbb{R}^{k+1}$ and TUX to $\phi_1(\mathcal{U}_1) \times S^k$. Thus, TUY is diffeomorphic to $\phi_1(\mathcal{U}_1 \cap Y) \times S^k$. So, working with these local coordinates, we can suppose that $TX = \phi_1(\mathcal{U}_1) \times \mathbb{R}^{k+1}$ and $TY = \phi_1(\mathcal{U}_1 \cap Y) \times \mathbb{R}^{k+1}$.

We also know that

$$T_{(x,\widehat{V}(x))}G_{-\nabla\widehat{\rho}} = G_{-D\nabla\widehat{\rho}(x)} = \Big\{(u, -D\nabla\widehat{\rho}(x)(u)) \mid u \in T_xY\Big\},$$

and

$$T_{(x,\widehat{V}(x))}G_{\widehat{V}} = G_{D\widehat{V}(x)} = \Big\{(u, D\widehat{V}(x)(u)) \mid u \in T_xY\Big\}.$$

These sets are both submanifolds of dimension k.

The sphere S^k is locally diffeomorphic at $(0, \ldots, 0, 1)$ to an open set of $\mathbb{R}^k$ through the projection $(x_1, \ldots, x_{k+1}) \mapsto (x_1, \ldots, x_k)$. Then $T_{(x,\widehat{V}(x))}TUY$ is diffeomorphic to $\mathbb{R}^k \times \mathbb{R}^k$. Moreover we can choose the following basis for $T_{(x,\widehat{V}(x))}TUY$:

$$\mathcal{B} = \Big((e_1, 0), \ldots, (e_k, 0), (0, e_1), \ldots, (0, e_k)\Big),$$

where $(e_1, \ldots, e_k)$ is the canonical basis positively oriented of $\mathbb{R}^k$ and for all $i \in \{1, \ldots, k\}$, $(e_i, 0) \in \mathbb{R}^k \times \{0\}^k$.

The vector field $\nabla\widehat{\rho}$ is unit and radial on Y and so in local coordinates the vector field $-\nabla\widehat{\rho}$ is constant equal to $(0, \ldots, 0, 1) \in \mathbb{R}^{k+1}$. A basis of $T_{(x,\widehat{V}(x))}G_{-\nabla\widehat{\rho}}$ is given by

$$\Big((e_1, -D\nabla\widehat{\rho}(x)(e_1)), \ldots, (e_k, -D\nabla\widehat{\rho}(x)(e_k))\Big).$$

But $D\nabla\widehat{\rho}(x)(e_i) = 0$ for all $i \in \{1, \ldots, k\}$ so this basis is finally

$$\Big((e_1, 0), \ldots, (e_k, 0)\Big).$$

Similarly a basis of $T_{(x,\widehat{V}(x))}G_{\widehat{V}}$ is

$$\Big((e_1, D\widehat{V}(x)(e_1)), \ldots, (e_k, D\widehat{V}(x)(e_k))\Big).$$

Thus we obtain a basis of $T_{(x,\widehat{V}(x))}G_{-\nabla\widehat{\rho}} \times T_{(x,\widehat{V}(x))}G_{\widehat{V}}$:

$$\mathcal{B}' = \Big((e_1,0),\dots,(e_k,0),(e_1,D\widehat{V}(x)(e_1)),\dots,(e_k,D\widehat{V}(x)(e_k))\Big).$$

Let us compute $D\widehat{V}(x)(e_i)$ for all $i \in \{1,\dots,k\}$. We know that $\widehat{V}(x) = (0,\dots,0,1) \in S^k$ so in local coordinates, we have

$$\widehat{V}(x) = \Big(\widehat{V}_1(x),\dots,\widehat{V}_k(x)\Big) = \left(\frac{V_1(x)}{\|V(x)\|},\dots,\frac{V_k(x)}{\|V(x)\|}\right).$$

Then for all $i,j \in \{1,\dots,k\}$, we have

$$\frac{\partial \widehat{V}_i}{\partial x_j}(x) = \frac{\frac{\partial V_i}{\partial x_j}(x)\|V(x)\| - \frac{\partial \|V\|}{\partial x_j}(x)V_i(x)}{\|V(x)\|^2}.$$

But $V_i(x) = 0$ for all $i \in \{1,...,k\}$, thus

$$\frac{\partial \widehat{V}_i}{\partial x_j}(x) = \frac{1}{\|V(x)\|}\frac{\partial V_i}{\partial x_j}(x).$$

We finally have

$$D\widehat{V}(x)(e_i) = \frac{1}{\|V(x)\|}\left(\sum_{j=1}^{k}\frac{\partial V_1}{\partial x_j}(x)e_i^j,...,\sum_{j=1}^{k}\frac{\partial V_k}{\partial x_j}(x)e_i^j\right),$$

where we write $e_i = (e_i^1,\dots,e_i^k)$. Let A be the matrix $\Big[D\widehat{V}(x)(e_1) \quad \cdots \quad D\widehat{V}(x)(e_k)\Big]$. We have

$$A = \tfrac{1}{\|V(x)\|}\begin{bmatrix} \frac{\partial V_1}{\partial x_1}(x) & \cdots & \frac{\partial V_1}{\partial x_k}(x) \\ \vdots & & \vdots \\ \frac{\partial V_k}{\partial x_1}(x) & \cdots & \frac{\partial V_k}{\partial x_k}(x)\end{bmatrix} \times \underbrace{\begin{bmatrix} e_1^1 & \cdots & e_k^1 \\ \vdots & & \vdots \\ e_1^k & \cdots & e_k^k\end{bmatrix}}_{:=Id_{\mathbb{R}^k}} = \tfrac{1}{\|V(x)\|}DV_{|Y}(x).$$

Now the transition matrix P from $\mathcal{B}'$ to $\mathcal{B}$ is

$$P = \left[\begin{array}{c|c} Id_{\mathbb{R}^k} & Id_{\mathbb{R}^k} \\ \hline (0) & A \end{array}\right],$$

so

$$\det P = \det A = \frac{1}{\|V(x)\|}\det DV_{|Y}(x).$$

Perturbing V a little if necessary, we can assume that $V_{|Y}$ admits only non-degenerate zeros on Y. Therefore x is a non-degenerate zero of $V_{|Y}$, $\det DV_{|Y}(x) \neq 0$ and

$$\operatorname{sign}(\det P) = \operatorname{sign}(\det DV_{|Y}(x)).$$

Thus we have proved that for all coincidence point x of $-\nabla\widehat{\rho}$ and $\widehat{V}$, which gives us an intersection point $(x,\widehat{V}(x)) \in TUY$ of $G_{-\nabla\widehat{\rho}}$ and $G_{\widehat{V}}$, we have

$$\operatorname{sign}(\det DV_{|Y}(x)) = \operatorname{sign}(\det P),$$

where $\operatorname{sign}(\det P)$ is $or(x)$. Thus,

$$\sum_{x \in C(-\nabla\widehat{\rho},\widehat{V})}(-1)^{or(x)} = \sum_{x \text{ inward zero of } V_{|Y}}\operatorname{sign}(\det DV_{|Y}(x)).$$

But

$$\sum_{x \text{ inward zero of } V_{|Y}} \operatorname{sign}(\det DV_{|Y}(x)) = \sum_{x \text{ inward zero of } V_{|Y}} ind_{PH}(V_{|Y}, x),$$

and we have proved that

$$\sum_{x \text{ inward zero of } V_{|Y}} ind_{PH}(V_{|Y}, x) = 1 - ind_{rad}(V, 0).$$

By definition, we have

$$I(G_{-\nabla\widehat{\rho}}, G_{\widehat{V}}) = \sum_{x \in C(-\nabla\widehat{\rho}, \widehat{V})} (-1)^{or(x)}.$$

This finally gives that

$$I(G_{-\nabla\widehat{\rho}}, G_{\widehat{V}}) = 1 - ind_{rad}(V, 0).$$

□

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Julie Lapebie, Aix-Marseille Université, CNRS, Centre Gilles-Gaton Granger, Aix-en-Provence, France

Email address: julie.lapebie@univ-amu.fr