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DEFORMING MONOMIAL SPACE CURVES INTO SET-THEORETIC COMPLETE INTERSECTION SINGULARITIES

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Abstract. We deform monomial space curves in order to construct examples of set-theoretical complete intersection space curve singularities. As a by-product we describe an inverse to Herzog's construction of minimal generators of non-complete intersection numerical semigroups with three generators.

1. Introduction

It is a classical problem in algebraic geometry to determine the minimal number of equations that define a variety. The codimension is a lower bound for this number which is reached in case of set-theoretic complete intersections. Let I be an ideal in a polynomial ring or a regular analytic algebra over a field $\mathbb{K}$. Then I is called a set-theoretic complete intersection if $\sqrt{I} = \sqrt{I'}$ for some ideal I' generated by height I many elements. The subscheme or analytic subgerm X defined by I is also called a set-theoretic complete intersection in this case. It is hard to determine whether a given X is a set-theoretic complete intersection. We address this problem in the case $I \in \operatorname{Spec} \mathbb{K}\{x, y, z\}$ of irreducible analytic space curve singularities X over an algebraically closed (complete non-discretely valued) field $\mathbb{K}$.

Cowsik and Nori (see [CN78]) showed that over a perfect field $\mathbb{K}$ of positive characteristic any algebroid curve and, if $\mathbb{K}$ is infinite, any affine curve is a set-theoretic complete intersection. To our knowledge there is no example of an algebroid curve that is not a set-theoretic complete intersection. Over an algebraically closed field $\mathbb{K}$ of characteristic zero, Moh (see [Moh82]) showed that an irreducible algebroid curve $\mathbb{K}[\![\xi, \eta, \zeta]\!] \subset \mathbb{K}[\![t]\!]$ is a set-theoretic complete intersection if the valuations $\ell, m, n = \upsilon(\xi), \upsilon(\eta), \upsilon(\zeta)$ satisfy

$$\gcd(\ell, m) = 1, \quad \ell < m, \quad (\ell - 2)m < n. \tag{1.1}$$

We deform monomial space curves in order to find new examples of set-theoretic complete intersection space curve singularities. Our main result in Proposition 4.2 gives sufficient numerical conditions for the deformation to preserve both the value semigroup and the set-theoretic complete intersection property. As a consequence we obtain

Corollary 1.1. *Let C be the irreducible curve germ defined by*

$$\mathcal{O}_C = \mathbb{K}\{t^\ell, t^m + t^p, t^n + t^q\} \subset \mathbb{K}\{t\}$$

where $\gcd(\ell, m) = 1$, $p > m$, $q > n$ *and there are* $a, b \geq 2$ *such that*

$$\ell = b + 2, \quad m = 2a + 1, \quad n = ab + b + 1.$$

Let γ be the conductor of the semigroup $\Gamma = \langle \ell, m, n\rangle$ and set

$$d_1 = (a + 1)(b + 2), \quad \delta = \min\{p - m, q - n\}.$$

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(a) If $d_1 + \delta \geq \gamma$, then Γ is the value semigroup of C.
(b) If $d_1 + \delta \geq \gamma + \ell$, then C is a set-theoretic complete intersection.
(c) If $a, b \geq 3$ and $d_1 + q - n \geq \gamma + \ell$, then C defined by

$$p := \gamma - 1 - \ell > m$$

is a non-monomial set-theoretic complete intersection.

In the setup of Corollary 1.1 Moh's third condition in (1.1) becomes $ab < 1$ and is trivially false. Corollary 1.1 thus yields an infinite list of new examples of non-monomial set-theoretic complete intersection curve germs.

Let us explain our approach and its context in more detail. Let Γ be a numerical semigroup. Delorme (see [Del76]) characterized the complete intersection property of Γ by a recursive condition. The complete intersection property holds equivalently for Γ and its associated monomial curve $\mathrm{Spec}(\mathbb{K}[\Gamma])$ (see [Her70, Cor. 1.13]) and is preserved under flat deformations. For this reason we deform only non-complete intersection Γ. A curve singularity inherits the complete intersection property from its value semigroup since it is a flat deformation of the corresponding monomial curve (see Proposition 3.3). The converse fails as shown by a counter-example of Herzog and Kunz (see [HK71, p. 40-41]).

In case $\Gamma = \langle \ell, m, n \rangle$, Herzog (see [Her70]) described minimal relations of the generators ℓ, m, n. There are two cases (H1) and (H2) (see §2) with 3 and 2 minimal relations respectively. In the non-complete intersection case (H1) we describe an inverse to Herzog's construction (see Proposition 2.4). Bresinsky (see [Bre79b]) showed (for arbitrary $\mathbb{K}$) by an explicit calculation based on Herzog's case (H1) that any monomial space curve is a complete intersection. Our results are obtained by lifting his equations to a (flat) deformation with constant value semigroup. In section §3 we construct such deformations (see Proposition 3.3) following an approach using Rees algebras described by Teissier (see [Zar06, Appendix, Ch. I, §1]). In §4 we prove Proposition 4.2 by lifting Bresinsky's equations under the given numerical conditions. In §5 we derive Corollary 1.1 and give some explicit examples (see Example 5.2).

It is worth mentioning that Bresinsky (see [Bre79b]) showed (for arbitrary $\mathbb{K}$) that all monomial Gorenstein curves in 4-space are set-theoretic complete intersections.

2. Ideals of monomial space curves

Let $\ell, m, n \in \mathbb{N}$ generate a semigroup $\Gamma = \langle \ell, m, n \rangle \subset \mathbb{N}$.

$$d = \gcd(\ell, m).$$

We assume that Γ is numerical, that is, $\gcd(\ell, m, n) = 1$.

Let $\mathbb{K}$ be a field and consider the map

$$\varphi \colon \mathbb{K}[x, y, z] \to \mathbb{K}[t], \quad (x, y, z) \mapsto (t^\ell, t^m, t^n),$$

whose image $\mathbb{K}[\Gamma] = \mathbb{K}[t^\ell, t^m, t^n]$ is the semigroup ring of Γ.

Pick $a, b, c \in \mathbb{N}$ minimal such that

$$a\ell = b_1 m + c_2 n, \quad bm = a_2 \ell + c_1 n, \quad cn = a_1 \ell + b_2 m$$

for some $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{N}$. Herzog distinguished two cases and proved the following statements (see [Her70, Props. 3.3, 3.4, 3.5, Thm. 3.8]).

(H1) $0 \notin \{a_1, a_2, b_1, b_2, c_1, c_2\}$. Then

$$a = a_1 + a_2, \quad b = b_1 + b_2, \quad c = c_1 + c_2 \tag{2.1}$$

and the unique minimal relations of ℓ, m, n read

$$a\ell - b_1 m - c_2 n = 0, \tag{2.2}$$

$$-a_2\ell + bm - c_1 n = 0, \tag{2.3}$$

$$-a_1\ell - b_2 m + cn = 0. \tag{2.4}$$

Their coefficients form the matrix

$$\begin{pmatrix} a & -b_1 & -c_2 \\ -a_2 & b & -c_1 \\ -a_1 & -b_2 & c \end{pmatrix}. \tag{2.5}$$

Accordingly the ideal $I = \langle f_1, f_2, f_3 \rangle$ of maximal minors

$$f_1 = x^a - y^{b_1} z^{c_2}, \quad f_2 = y^b - x^{a_2} z^{c_1}, \quad f_3 = x^{a_1} y^{b_2} - z^c \tag{2.6}$$

of the matrix

$$M_0 = \begin{pmatrix} z^{c_1} & x^{a_1} & y^{b_1} \\ y^{b_2} & z^{c_2} & x^{a_2} \end{pmatrix}. \tag{2.7}$$

equals $\ker \varphi$, and the rows of this matrix generate the module of relations between f_1, f_2, f_3. Here $\mathbb{K}[\Gamma]$ is not a complete intersection.

(H2) $0 \in \{a_1, a_2, b_1, b_2, c_1, c_2\}$. One of the relations $(a, -b, 0)$, $(a, 0, -c)$, or $(0, b, -c)$ is a minimal relation of ℓ, m, n and, up to a permutation of the variables, the minimal relations are

$$a\ell = bm, \tag{2.8}$$

$$a_1\ell + b_2 m = cn. \tag{2.9}$$

Their coefficients form the matrix

$$\begin{pmatrix} a & -b & 0 \\ -a_1 & -b_2 & c \end{pmatrix}. \tag{2.10}$$

It is unique up to adding multiples of the first row to the second. Overall there are 3 cases and an overlap case described equivalently by 3 matrices

$$\begin{pmatrix} a & -b & 0 \\ a & 0 & c \end{pmatrix}, \quad \begin{pmatrix} a & -b & 0 \\ 0 & -b & c \end{pmatrix}, \quad \begin{pmatrix} a & 0 & -c \\ 0 & b & -c \end{pmatrix}. \tag{2.11}$$

Here $\mathbb{K}[\Gamma]$ is a complete intersection.

In the following we describe the image of Herzog's construction and give a left inverse:

(H1') Given $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{N} \setminus \{0\}$, define a, b, c by (2.1) and set

$$\ell' = b_1c_1 + b_1c_2 + b_2c_2 = b_1c + b_2c_2 = b_1c_1 + bc_2, \tag{2.12}$$

$$m' = a_1c_1 + a_2c_1 + a_2c_2 = ac_1 + a_2c_2 = a_1c_1 + a_2c, \tag{2.13}$$

$$n' = a_1b_1 + a_1b_2 + a_2b_2 = a_1b + a_2b_2 = a_1b_1 + ab_2, \tag{2.14}$$

and $e' = \gcd(\ell', m', n')$. Note that ℓ', m', n' are the submaximal minors of the matrix in (2.5).

(H2') Given $a, b, c \in \mathbb{N} \setminus \{0\}$ and $a_1, b_2 \in \mathbb{N}$, define ℓ', m', n', d' by

$$\ell' = bd', \tag{2.15}$$

$$m' = ad', \tag{2.16}$$

$$\frac{n'}{d'} = \frac{a_1b + ab_2}{c}, \quad \gcd(n', d') = 1. \tag{2.17}$$

Remark 2.1. In the overlap case (2.11) the formulas (2.15)-(2.16) yield

$$(\ell', m', n') = (bc, ac, ab).$$

Lemma 2.2. *In case* (H1)*, let* $\tilde{n} \in \mathbb{N}$ *be minimal with* $x^{\tilde{n}} - z^{\tilde{\ell}} \in I$ *for some* $\tilde{\ell} \in \mathbb{N}$*. Then* $\gcd(\tilde{\ell}, \tilde{n}) = 1$ *and* $(\tilde{n}, \tilde{\ell}) \cdot \gcd(b_1, b_2) = (n', \ell')$*.*

Proof. The first statement holds due to minimality. By Buchberger's criterion, the generators 2.6 form a Gröbner basis with respect to the reverse lexicographical ordering on x, y, z. Let g' denote a normal form of $g = x^{\tilde{n}} - z^{\tilde{\ell}}$ with respect to 2.6. Then $g \in I$ if and only if $g' = 0$. By (2.1), reductions by f_2 can be avoided in the calculation of g. If r_2 and r_1 many reductions by f_1 and f_3 respectively are applied, then

$$g' = x^{\tilde{n} - a_1 r_1 - a r_2} y^{b_1 r_2 - r_1 b_2} z^{r_1 c + r_2 c_2} - z^{\tilde{\ell}}$$

and $g' = 0$ is equivalent to

$$\tilde{\ell} = r_1 c + r_2 c_2, \quad b_1 r_2 = r_1 b_2, \quad \tilde{n} = a_1 r_1 + a r_2.$$

Then $r_i = \frac{b_i}{\gcd(b_1, b_2)}$ for $i = 1, 2$ and the claim follows. □

Lemma 2.3.

(a) In case (H1)*, equations* (2.12)-(2.14) *recover* ℓ, m, n*.*

(b) In case (H2)*, equations* (2.15)-(2.17) *recover* ℓ, m, n, d*.*

Proof.

(a) Consider $\tilde{n}, \tilde{\ell} \in \mathbb{N}$ as in Lemma 2.2. Then $x^{\tilde{n}} - z^{\tilde{\ell}} \in I = \ker \varphi$ means that $(t^\ell)^{\tilde{n}} = (t^n)^{\tilde{\ell}}$ and hence $\ell \tilde{n} = \tilde{\ell} n$. So the pair (ℓ, n) is proportional to $(\tilde{\ell}, \tilde{n})$ which in turn is proportional to (ℓ', n') by Lemma 2.2. Then the two triples (ℓ, m, n) and (ℓ', m', n') are proportional by symmetry. Since $\gcd(\ell, m, n) = 1$ by hypothesis $(\ell', m', n') = q \cdot (\ell, m, n)$ for some $q \in \mathbb{N}$. By Lemma 2.2, q divides $\gcd(b_1, b_2)$ and by symmetry also $\gcd(a_1, a_2)$ and $\gcd(c_1, c_2)$. By minimality of the relations (2.2)-(2.4), $\gcd(a_1, a_2, b_1, b_2, c_1, c_2) = 1$ and hence $q = 1$. The claim follows.

(b) By the minimal relation (2.8), $\gcd(a, b) = 1$ and hence $(\ell, m) = d \cdot (b, a)$. Substitution into equation (2.9) and comparison with (2.17) gives

$$\frac{n}{d} = \frac{a_1 b + a b_2}{c} = \frac{n'}{d'}$$

with $\gcd(n, d) = \gcd(\ell, m, n) = 1$ by hypothesis. We deduce that $(n, d) = (n', d')$ and then $(\ell, m) = (\ell', m')$. □

Proposition 2.4.

(a) In case (H1')*,* $a_1, a_2, b_1, b_2, c_1, c_2$ *arise through* (H1) *from some numerical semigroup* $\Gamma = \langle \ell, m, n \rangle$ *if and only if* $e' = 1$*. In this case,* $(\ell, m, n) = (\ell', m', n')$*.*

(b) In case (H2')*,* a, b, c, a_1, b_2 *arise through* (H2) *from some numerical semigroup* $\Gamma = \langle \ell, m, n \rangle$ *if and only if* (ℓ', m', n') *is in the corresponding subcase of* (H2)*,*

$$\gcd(a, b) = 1, \tag{2.18}$$

$$\forall q \in [-b_2/b, a_1/a] \cap \mathbb{N}\colon\ \gcd(-a_1 + qa, -b_2 - qb, c) = 1. \tag{2.19}$$

In this case, $(\ell, m, n) = (\ell', m', n')$*.*

Proof.

(a) By Lemma 2.3.(a), $e' = 1$ is a necessary condition. Conversely let $e' = 1$. By definition, (2.5) is a matrix of relations of (ℓ', m', n'). Assume that (ℓ', m', n') is in case (H2). By symmetry, we may assume that (ℓ', m', n') admits a matrix of minimal relations

$$\begin{pmatrix} a' & -b' & 0 \\ -a'_1 & -b'_2 & c' \end{pmatrix} \tag{2.20}$$

of type (2.10). By the choice of a', b', c', it follows that

$$a > a', \quad b > b', \quad c \geq c'.$$

By Lemma 2.3.(b), d' is the denominator of $\frac{a'_1 b' + a' b'_2}{c'}$ and

$$\ell' = b'd'.$$

In particular $c' \geq d'$. Then $b_1 \geq b'$ contradicts (2.12) since

$$\ell' = b_1 c + b_2 c_2 \geq b'c' + b_2 c_2 > b'c' \geq b'd' = \ell'.$$

We may thus assume that $b_1 < b'$. The difference of first rows of (2.20) and (2.5) is then a relation

$$\begin{pmatrix} a' - a & b_1 - b' & c_2 \end{pmatrix}$$

of (ℓ', m', n') with $a' - a < 0$, $b_1 - b' < 0$ and $c_2 > 0$. Then $c_2 \geq c' \geq d'$ by choice of c'. This contradicts (2.12) since

$$\ell' = b_1 c_1 + b c_2 \geq b_1 c_1 + b'd' > b'd' = \ell'.$$

We may thus assume that (ℓ', m', n') is in case (H1) with a matrix of unique minimal relations

$$\begin{pmatrix} a' & -b'_1 & -c'_2 \\ -a'_2 & b' & -c'_1 \\ -a'_1 & -b'_2 & c' \end{pmatrix} \tag{2.21}$$

of type (2.5) where

$$a' = a'_1 + a'_2, \quad b' = b'_1 + b'_2, \quad c' = c'_1 + c'_2.$$

as in (2.1). Then $(a, b, c) \geq (a', b', c')$ by choice of the latter and

$$\ell' = b'_1 c' + b'_2 c'_2 = b'_1 c'_1 + b' c'_2$$

by Lemma 2.3.(a). If $(a_i, b_i, c_i) \geq (a'_i, b'_i, c'_i)$ for $i = 1, 2$, then

$$\ell' = b_1 c + b_2 c_2 \geq b'_1 c' + b'_2 c'_2 = \ell'$$

implies $c = c'$ and hence $(a, b, c) = (a', b', c')$ by symmetry. By uniqueness of (2.21) then, $(a_1, a_2, b_1, b_2, c_1, c_2) = (a'_1, a'_2, b'_1, b'_2, c'_1, c'_2)$ and hence the claim. By symmetry, it remains to exclude the case $c'_2 > c_2$. The difference of first rows of (2.21) and (2.5) is then a relation

$$\begin{pmatrix} a' - a & b_1 - b'_1 & c_2 - c'_2 \end{pmatrix}$$

of (ℓ', m', n') with $a' - a \leq 0$, $c_2 - c'_2 < 0$ and hence $b_1 - b'_1 \geq b'$ by choice of the latter. This leads to the contradiction

$$\ell' = b_2 c_2 + b_1 c > b_1 c \geq b'c' + b'_1 c' > b'_2 c'_2 + b'_1 c' = \ell'.$$

(b) By Lemma 2.3.(b), the conditions are necessary. Conversely assume that the conditions hold true. By definition, (2.10) is a matrix of relations of (ℓ', m', n'). By hypothesis, (2.20) is a matrix of minimal relations of (ℓ', m', n'). By (2.18), $\gcd(\ell', m') = d'$ and hence by Lemma 2.3.(b)

$$b = \frac{\ell'}{d'} = b', \quad a = \frac{m'}{d'} = a'.$$

Writing the second row of (2.10) as a linear combination of (2.20) yields

$$\begin{pmatrix}-a_1+qa & -b_2-qb & c\end{pmatrix} = p\begin{pmatrix}-a_1' & -b_2' & c'\end{pmatrix}$$

with $p \in \mathbb{N}$ and $q \cap [-b_2/b, a_1/a] \cap \mathbb{N}$ and hence $p = 1$ by (2.19). The claim follows. $\square$

The following examples show some issues that prevent us from formulating stronger statement in Proposition 2.4.(b).

Example 2.5.

(a) Take $(a, -b, 0) = (3, -2, 0)$ and $(-a_1, -b_2, c) = (-1, -4, 4)$. Then $(\ell', m', n') = (4, 6, 7)$ which is in case (H2). The second minimal relation is $(-2, -1, 2) = \frac{1}{2}((-a_1, -b_2, c) - (a, -b, 0))$. The same (ℓ', m', n') is obtained from $(a, 0, -c) = (7, 0, -4)$ and $(-a_2, b, -c_1) = (-1, 3, -2)$. This latter satisfies (2.18) and (2.19), but $(a, 0, -c)$ is not minimal.

(b) Take $(a, -b, 0) = (4, -3, 0)$ and $(-a_1, -b_2, c) = (-2, -1, 2)$. Then $(\ell', m', n') = (3, 4, 5)$, but $(a, -b, 0)$ is not a minimal relation. In fact the corresponding complete intersection $\mathbb{K}[\Gamma]$ defined by the ideal $\langle x^3 - y^4, z^2 - x^2y\rangle$ is the union of two branches $x = t^3, y = t^4, z = \pm t^5$.

3. Deformation with constant semigroup

Let $\mathcal{O} = (\mathcal{O}, \mathfrak{m})$ be a local $\mathbb{K}$-algebra with $\mathcal{O}/\mathfrak{m} \cong \mathbb{K}$. Let $F_\bullet = \{F_i \mid i \in \mathbb{Z}\}$ be a decreasing filtration by ideals such that $F_i = \mathcal{O}$ for all $i \leq 0$ and $F_1 \subset \mathfrak{m}$. Consider the Rees ring

$$A = \bigoplus_{i\in\mathbb{Z}} F_i s^{-i} \subset \mathcal{O}[s^{\pm 1}].$$

It is a finite type graded $\mathcal{O}[s]$-algebra and flat (torsion free) $\mathbb{K}[s]$-algebra with retraction

$$A \twoheadrightarrow A/A \cap \mathfrak{m}[s^{\pm 1}] \cong \mathbb{K}[s].$$

For $u \in \mathcal{O}^*$ there are isomorphisms

$$A/(s-u)A \cong \mathcal{O}, \quad A/sA \cong \operatorname{gr}^F \mathcal{O}. \tag{3.1}$$

Geometrically A defines a flat morphism with section

$$\operatorname{Spec}(A) \underset{\iota}{\overset{\pi}{\rightleftarrows}} \mathbb{A}^1_{\mathbb{K}}$$

with fibers over $\mathbb{K}$-valued points

$$\begin{aligned}\pi^{-1}(x) &\cong \operatorname{Spec}(\mathcal{O}), \quad \iota(x) = \mathfrak{m}, \quad 0 \neq x \in \mathbb{A}^1_{\mathbb{K}},\\ \pi^{-1}(0) &\cong \operatorname{Spec}(\operatorname{gr}^F \mathcal{O}), \quad \iota(0) = \operatorname{gr}^F \mathfrak{m}.\end{aligned}$$

Let $\mathbb{K}$ be an algebraically closed complete non-discretely valued field. Let C be an irreducible $\mathbb{K}$-analytic curve germ. Its ring $\mathcal{O} = \mathcal{O}_C$ is a one-dimensional $\mathbb{K}$-analytic domain. Denote by Γ' its value semigroup. Pick a representative W such that $C = (W, w)$. We allow to shrink W suitably without explicit mention. Let $\overline{\mathcal{O}}_W$ be the normalization of $\mathcal{O}_W$. Then

$$\overline{\mathcal{O}}_{W,w} = (\overline{\mathcal{O}}, \overline{\mathfrak{m}}) \cong (\mathbb{K}\{t'\}, \langle t'\rangle) \xrightarrow{\upsilon} \mathbb{N} \cup \{\infty\}$$

is a discrete valuation ring. Denote by $\mathfrak{m}_W$ and $\overline{\mathfrak{m}}_W$ the ideal sheaves corresponding to $\mathfrak{m}$ and $\overline{\mathfrak{m}}$. There are decreasing filtrations by ideal (sheaves)

$$\mathcal{F}_\bullet = \overline{\mathfrak{m}}_W^\bullet \lhd \overline{\mathcal{O}}_W, \quad F_\bullet = \mathcal{F}_{\bullet,w} = \overline{\mathfrak{m}}^\bullet = \upsilon^{-1}[\bullet, \infty] \lhd \overline{\mathcal{O}}.$$

Setting $t = t'/s$ and identifying $\mathbb{K} \cong \overline{\mathcal{O}}_W/\overline{\mathfrak{m}}_W$ this yields a finite extension of finite type graded $\mathcal{O}_W$- and flat (torsion free) $\mathbb{K}[s]$-algebras

$$\mathcal{A} = \bigoplus_{i\in\mathbb{Z}}(\mathcal{F}_i \cap \mathcal{O}_W)s^{-i} \subset \bigoplus_{i\in\mathbb{Z}}\mathcal{F}_i s^{-i} = \overline{\mathcal{O}}_W[s,t] = \mathcal{B} \subset \overline{\mathcal{O}}_W[s^{\pm 1}] \tag{3.2}$$

with retraction defined by $\mathbb{K}[s] \cong \mathcal{B}/(\mathcal{B}_{<0} + \mathcal{B}\overline{\mathfrak{m}}_W)$. The stalk at w is

$$A = \mathcal{A}_w = \bigoplus_{i\in\mathbb{Z}}(F_i \cap \mathcal{O})s^{-i} \subset \bigoplus_{i\in\mathbb{Z}} F_i s^{-i} = \overline{\mathcal{O}}[s,t] = B \subset \overline{\mathcal{O}}[s^{\pm 1}].$$

At $w \neq w' \in W$ the filtration $\mathcal{F}_{w'}$ is trivial and the stalk becomes $\mathcal{A}_{w'} = \mathcal{O}_{W,w'}[s^{\pm}1]$. The graded sheaves $\operatorname{gr}^{\mathcal{F}} \mathcal{O}_W \subset \operatorname{gr}^{\mathcal{F}} \overline{\mathcal{O}}_W$ are thus supported at w and the isomorphism

$$\operatorname{gr}^{\mathcal{F}}(\overline{\mathcal{O}}_W)_w = \operatorname{gr}^F \overline{\mathcal{O}} \cong \mathbb{K}[t'] \cong \mathbb{K}[\mathbb{N}]$$

identifies

$$(\operatorname{gr}^{\mathcal{F}} \mathcal{O}_W)_w = \operatorname{gr}^F \mathcal{O} \cong \mathbb{K}[\Gamma'], \quad \Gamma' = \upsilon(\mathcal{O} \setminus \{0\}) \tag{3.3}$$

with the semigroup ring $\mathbb{K}[\Gamma']$ of $\mathcal{O}$.

The analytic spectrum $\operatorname{Spec}^{\mathrm{an}}_W(-) \to W$ applied to finite type $\mathcal{O}_W$-algebras represents the functor $T \mapsto \operatorname{Hom}_{\mathcal{O}_T}(-_T, \mathcal{O}_T)$ from $\mathbb{K}$-analytic spaces over W to sets (see [Car62, Exp. 19]). Note that

$$\operatorname{Spec}^{\mathrm{an}}_W(\mathbb{K}[s]) = \operatorname{Spec}^{\mathrm{an}}_{\{w\}}(\mathbb{K}[s]) = L$$

is the $\mathbb{K}$-analytic line. The normalization of W is

$$\nu\colon \overline{W} = \operatorname{Spec}^{\mathrm{an}}_W(\overline{\mathcal{O}}_W) \to W$$

and $\mathcal{B} = \nu_*\overline{\mathcal{B}}$ where $\overline{\mathcal{B}} = \mathcal{O}_{\overline{W}}[s,t]$. Applying $\operatorname{Spec}^{\mathrm{an}}_W$ to (3.2) yields a diagram of $\mathbb{K}$-analytic spaces (see [Zar06, Appendix])

$$\begin{array}{ccccc} X = \operatorname{Spec}^{\mathrm{an}}_W(\mathcal{A}) & & \xleftarrow{\ \rho\ } & & \operatorname{Spec}^{\mathrm{an}}_W(\mathcal{B}) = Y \\ & \searrow^{\pi} & & \nearrow^{\iota} & \\ & & L & & \end{array} \tag{3.4}$$

where π is flat with $\pi \circ \rho \circ \iota = \mathrm{id}$ and

$$\begin{aligned} \pi^{-1}(x) &\cong \operatorname{Spec}^{\mathrm{an}}_W(\mathcal{O}_W) = W, \quad \iota(x) = w, \quad 0 \neq x \in L, \\ \pi^{-1}(0) &\cong \operatorname{Spec}^{\mathrm{an}}_W(\operatorname{gr}^{\mathcal{F}} \mathcal{O}_W), \quad \iota(0) \leftrightarrow \operatorname{gr}^{\mathcal{F}} \mathfrak{m}_W. \end{aligned}$$

Remark 3.1. Teissier defines X as the analytic spectrum of $\mathcal{A}$ over $W \times L$ (see [Zar06, Appendix, Ch. I, §1]). This requires to interpret the $\mathcal{O}_W$-algebra $\mathcal{A}$ as an $\mathcal{O}_{W\times L}$-algebra.

Remark 3.2. In order to describe (3.4) in explicit terms, embed

$$L \supset \overline{W} \xrightarrow{\ \nu\ } W \subset L^n$$

with coordinates t' and $\underline{x} = x_1, \ldots, x_n$ and

$$\begin{aligned} X &= \overline{\{(\underline{x}, s) \mid (s^{\ell_1}x_1, \ldots, s^{\ell_n}x_n) \in W, s \neq 0\}} \subset L^n \times L, \\ Y &= \left\{(t,s) \,\middle|\, t' = st \in \overline{W}\right\} \cup L \times \{0\} \subset L \times L. \end{aligned}$$

This yields the maps $X \to W \leftarrow Y$. The map ρ in (3.4) becomes

$$\rho(t,s) = (x_1(t')/s^{\ell_1}, \ldots, x_n(t')/s^{\ell_n})$$

for $s \neq 0$ and the fiber $\pi^{-1}(0)$ is the image of the map

$$\rho(t,0) = ((\xi_1(t),\dots,\xi_n(t)),0),\quad \xi_k(t) = \lim_{s\to 0} x_k(st)/s^{\ell_k} = \sigma(x_k)(t).$$

Taking germs in (3.4) this yields the following.

Proposition 3.3. *There is a flat morphism with section*

$$S = (X,\iota(0)) \xrightarrow{\pi} (L,0), \qquad (L,0) \xrightarrow{\iota} S$$

with fibers

$$\begin{aligned}\pi^{-1}(x) &\cong (W,w) = C, \quad \iota(x) = w, \quad 0 \neq x \in L,\\ \pi^{-1}(0) &\cong \mathrm{Spec}^{\mathrm{an}}(\mathbb{K}[\Gamma']) = C_0, \quad \iota(0) \leftrightarrow \mathbb{K}[\Gamma'_+].\end{aligned}$$ □

The structure morphism factorizes through a flat morphism

$$X = \mathrm{Spec}^{\mathrm{an}}_W(\mathcal{A}) \xrightarrow{\hat f} (|W|,\mathcal{A}) \longrightarrow W, \qquad f\colon X \to W$$

and $\hat f^{\#}_{\iota(0)}\colon A \to \mathcal{O}_{X,\iota(0)}$ induces an isomorphism of completions (see [Car62, Exp. 19, §2, Prop. 4])

$$\widehat{A_{\iota(0)}} \cong \widehat{\mathcal{O}_{X,\iota(0)}}.$$

This yields the finite extension of $\mathbb{K}$-analytic domains

$$\mathcal{O}_S = \mathcal{O}_{X,\iota(0)} \subset \mathcal{O}_{Y,\iota(0)}.$$

We aim to describe $\mathcal{O}_{Y,\iota(0)}$ and $\mathbb{K}$-analytic algebra generators of $\mathcal{O}_S$. In explicit terms $\mathcal{O}_S$ is obtained from a presentation

$$I \to \mathcal{O}[\underline{x}] \to A \to 0$$

mapping $\underline{x} = x_1,\dots,x_n$ to $\iota(0) = A \cap \mathfrak{m}[s^{\pm 1}] + As$ as

$$\mathcal{O}_S = \mathcal{O}\{\underline{x}\}/\mathcal{O}\{\underline{x}\}I = \mathcal{O}\{\underline{x}\} \otimes_{\mathcal{O}[\underline{x}]} A,\quad \mathcal{O}\{\underline{x}\} = \mathcal{O}\widehat{\otimes}\mathbb{K}\{\underline{x}\}. \tag{3.5}$$

Any $\mathcal{O}_W$-module $\mathcal{M}$ gives rise to an $\mathcal{O}_X$-module

$$\widetilde{\mathcal{M}} = \mathcal{O}_X \otimes_{f^*\mathcal{A}} f^*\mathcal{M} = \hat f^*\mathcal{M}.$$

With $M = \mathcal{M}_w$, its stalk at $\iota(0)$ becomes

$$\widetilde{M} = \mathcal{O}_S \otimes_A M.$$

Lemma 3.4. $\mathrm{Spec}^{\mathrm{an}}_W(\mathcal{B}) = \mathrm{Spec}^{\mathrm{an}}_W(\overline{\mathcal{B}})$ *and hence* $\mathcal{O}_{Y,\iota(0)} = \mathbb{K}\{s,t\}$.

Proof. By finiteness of ν (see [Car62, Exp. 19, §3, Prop. 9]),

$$\overline{\mathcal{B}} = \widetilde{\nu_*\overline{\mathcal{B}}} = \widetilde{\mathcal{B}} = \mathcal{O}_{\overline{W}} \otimes_{\nu^*\mathcal{O}_W} \nu^*\mathcal{B}.$$

By the universal property of $\mathrm{Spec}^{\mathrm{an}}$, it follows that (see [Con06, Thm. 2.2.5.(2)])

$$\begin{aligned}\mathrm{Spec}^{\mathrm{an}}_W(\overline{\mathcal{B}}) &= \mathrm{Spec}^{\mathrm{an}}_{\overline{W}}(\mathcal{O}_{\overline{W}} \otimes_{\nu^*\overline{\mathcal{O}}_W} \nu^*\mathcal{B})\\ &= \mathrm{Spec}^{\mathrm{an}}_{\overline{W}}(\mathcal{O}_{\overline{W}}) \times_{\mathrm{Spec}^{\mathrm{an}}_{\overline{W}}(\nu^*\overline{\mathcal{O}}_W)} \mathrm{Spec}^{\mathrm{an}}_{\overline{W}}(\nu^*\mathcal{B})\\ &= \overline{W} \times_{\overline{W}\times_W \overline{W}} (\mathrm{Spec}^{\mathrm{an}}_W(\mathcal{B}) \times_W \overline{W})\\ &= \overline{W} \times_{\overline{W}} \mathrm{Spec}^{\mathrm{an}}_W(\mathcal{B})\\ &= \mathrm{Spec}^{\mathrm{an}}_W(\mathcal{B}).\end{aligned}$$ □

For $\xi' = \sum_{i\in\mathbb{N}} \xi_i {t'}^i \in \mathbb{K}[t']$ with $\ell = \upsilon(\xi')$ denote

$$\xi = \xi'/s^\ell = \sum_{i\geq\ell} \xi_i t^i s^{i-\ell} \in F_\ell s^{-\ell} = B_\ell. \tag{3.6}$$

Lemma 3.5. *Consider $\underline{\xi}' = \xi'_1,\dots,\xi'_n \in \mathfrak{m}\cap\mathbb{K}[t']$, define $\underline{\xi}$ by (3.6) and $\underline{\ell}$ by $\ell_i = \upsilon(\xi'_i)$ for $i = 1,\dots,n$. If $\Gamma' = \langle\underline{\ell}\rangle$, then $\mathcal{O} = \mathbb{K}\{\underline{\xi}'\}$ and $\mathcal{O}_S = \mathbb{K}\{\underline{\xi}, s\}$.*

Proof. By choice of $F_\bullet$, there is a cartesian square

$$\begin{array}{ccc} B = \overline{\mathcal{O}}[t,s] & \hookrightarrow & \overline{\mathcal{O}}[s^{\pm1}] \\ \uparrow & & \uparrow \\ A = \bigoplus_{i\in\mathbb{Z}}(F_i\cap\mathcal{O})s^{-i} & \hookrightarrow & \mathcal{O}[s^{\pm1}] \end{array}$$

of finite type graded $\mathcal{O}$-algebras. Thus $\xi \in A\cap\mathfrak{m}[s^{\pm1}]$ if $\xi' \in \mathfrak{m}\cap k[t']$.

By hypothesis and (3.3), the symbols $\sigma(\underline{\xi}')$ generate the graded $\mathbb{K}$-algebra $\mathrm{gr}^F\mathcal{O}$. Then $\overline{\sigma(\underline{\xi}')} = \sigma(\overline{\underline{\xi}}')$ generate

$$\mathrm{gr}^F\mathfrak{m}/\mathrm{gr}^F\mathfrak{m}^2 = \mathrm{gr}^F(\mathfrak{m}/\mathfrak{m}^2)$$

and hence $\overline{\underline{\xi}}'$ generate $\mathfrak{m}/\mathfrak{m}^2$ over $\mathbb{K}$. Then $\mathfrak{m} = \langle\underline{\xi}'\rangle_{\mathcal{O}}$ by Nakayama's lemma and hence $\mathcal{O} = \mathbb{K}\{\underline{\xi}'\}$ by the analytic inverse function theorem.

Under the graded isomorphism (3.1) with ξ as in (3.6)

$$\begin{array}{rcl} (A/As)_\ell & \xrightarrow{\cdot s^\ell} & \mathrm{gr}^F_\ell\mathcal{O}, \\ \overline{\xi} & \longmapsto & \sigma(\xi'). \end{array}$$

The graded $\mathbb{K}$-algebra A/sA is thus generated by $\overline{\underline{\xi}}$. Extend $F_\bullet$ to the graded filtration $F_\bullet[s^{\pm1}]$ on $\overline{\mathcal{O}}[s^{\pm1}]$. For $i\geq j$,

$$(A/As)_i = \mathrm{gr}^F_i A_i \xrightarrow[\cong]{\cdot s^{i-j}} \mathrm{gr}^F_i A_j.$$

Thus finitely many monomials in $\underline{\xi}, s$ generate any $A_j/F_iA_j \cong F_j/F_i$ over $\mathbb{K}$. With γ' the conductor of Γ' and $i = \gamma' + j$, $F_{\gamma'} \subset \overline{\mathfrak{m}}\cap\mathcal{O} = \mathfrak{m}$ and hence $F_i = F_{\gamma'}F_j \subset \mathfrak{m}F_j$. Therefore these monomials generate A_j as $\mathcal{O}$-module by Nakayama's lemma. It follows $A = \mathcal{O}[\underline{\xi}, s]$ as graded $\mathbb{K}$-algebra. Using $\mathcal{O} = \mathbb{K}\{\underline{\xi}'\}$ and $\underline{\xi}' = \underline{\xi}s^{\underline{\ell}}$ then $\mathcal{O}_S = \mathbb{K}\{\underline{\xi}', \underline{\xi}, s\} = \mathbb{K}\{\underline{\xi}, s\}$ (see (3.5)). □

We now reverse the above construction to deform generators of a semigroup ring. Let Γ be a numerical semigroup with conductor γ generated by $\underline{\ell} = \ell_1,\dots,\ell_n$. Pick corresponding indeterminates $\underline{x} = x_1,\dots,x_n$. The weighted degree $\deg(-)$ defined by $\deg(\underline{x}) = \underline{\ell}$ makes $\mathbb{K}[\underline{x}]$ a graded $\mathbb{K}$-algebra and induces on $\mathbb{K}\{\underline{x}\}$ a weighted order $\mathrm{ord}(-)$ and initial part $\mathrm{inp}(-)$. The assignment $x_i \mapsto \ell_i$ defines a presentation of the semigroup ring of Γ (see (3.3))

$$\mathbb{K}[\underline{x}]/I \cong \mathbb{K}[\Gamma] \subset \mathbb{K}[t'] \subset \mathbb{K}\{t'\} = \overline{\mathcal{O}}.$$

The defining ideal I is generated by homogeneous binomials $\underline{f} = f_1,\dots,f_m$ of weighted degrees $\deg(\underline{f}) = \underline{d}$. Consider elements $\underline{\xi} = \xi_1,\dots,\xi_n$ defined by

$$\xi_j = t^{\ell_j} + \sum_{i\geq\ell_j+\Delta\ell_j} \xi_{j,i}t^i s^{i-\ell_j} \in \mathbb{K}[t,s] \subset \overline{\mathcal{O}}[t,s] = B \tag{3.7}$$

with $\Delta\ell_j \in \mathbb{N}\setminus\{0\}\cup\{\infty\}$. Set

$$\delta = \min\{\Delta\underline{\ell}\}, \quad \Delta\underline{\ell} = \Delta\ell_1,\dots,\Delta\ell_n.$$

With $\deg(t) = 1 = -\deg(s)$ $\underline{\xi}$ defines a map of graded $\mathbb{K}$-algebras $\mathbb{K}[\underline{x}, s] \to \mathbb{K}[t, s]$ and a map of analytically graded $\mathbb{K}$-analytic domains $\mathbb{K}\{\underline{x}, s\} \to \mathbb{K}\{t, s\}$ (see [SW73] for analytic gradings).

Remark 3.6. Converse to (3.6), any homogeneous $\xi \in \mathbb{K}\{t, s\}$ of weighted degree ℓ can be written as $\xi = \xi'/s^\ell$ for some $\xi' \in \mathbb{K}\{t'\}$. It follows that $\xi(t, 1) = \xi'(t) \in \mathbb{K}\{t\}$.

Consider the curve germ C with $\mathbb{K}$-analytic ring

$$\mathscr{O} = \mathscr{O}_C = \mathbb{K}\{\underline{\xi}'\}, \quad \underline{\xi}' = \underline{\xi}(t, 1), \tag{3.8}$$

and value semigroup $\Gamma' \supset \Gamma$.

We now describe when (3.7) generate the flat deformation in Proposition 3.3.

Proposition 3.7. *The deformation* (3.7) *satisfies* $\Gamma' = \Gamma$ *if and only if there is a* $\underline{f}' \in \mathbb{K}\{\underline{x}, s\}^m$ *with homogeneous components such that*

$$\underline{f}(\underline{\xi}) = \underline{f}'(\underline{\xi}, s)s \tag{3.9}$$

and $\operatorname{ord}(f_i'(\underline{x}, 1)) \geq d_i + \min\{\Delta\underline{\ell}\}$. *The flat deformation in Proposition 3.3 is then defined by*

$$\mathscr{O}_S = \mathbb{K}\{\underline{\xi}, s\} = \mathbb{K}\{\underline{x}, s\}/\langle \underline{F}\rangle, \quad \underline{F} = \underline{f} - \underline{f}'s. \tag{3.10}$$

Proof. First let $\Gamma' = \Gamma$. Then Lemma 3.5 yields the first equality in (3.10). By flatness of π in Proposition 3.3, the relations $\underline{f}$ of $\underline{\xi}(t, 0) = t^{\underline{\ell}}$ lift to relations $\underline{F} \in \mathbb{K}\{\underline{x}, s\}^m$ of $\underline{\xi}$. That is, $\underline{F}(\underline{x}, 0) = \underline{f}$ and $\underline{F}(\underline{\xi}, s) = 0$. Since $\underline{f}$ and $\underline{\xi}$ have homogeneous components of weighted degrees $\underline{d}$ and $\underline{\ell}$, $\underline{F}$ can be written as $\underline{F} = \underline{f} - \underline{f}'s$ where $\underline{f}' \in \mathbb{K}\{\underline{x}, s\}^m$ has homogeneous components of weighted degrees $\underline{d} + \underline{1}$. This proves in particular the last claim. Since $f_i(t^{\underline{\ell}}) = 0$, any term in $f_i'(\underline{\xi}, s)s = f_i(\underline{\xi})$ involves a term of the tail of ξ_j for some j. Such a term is divisible by $t^{d_i + \Delta\ell_j}$ which yields the bound for $\operatorname{ord}(f_i'(\underline{x}, 1))$.

Conversely let $\underline{f}'$ with homogeneous components satisfy (3.9). Suppose that there is a $k' \in \Gamma' \setminus \Gamma$. Take $h \in \mathbb{K}\{\underline{x}\}$ of maximal weighted order k such that $\upsilon(h(\underline{\xi}')) = k'$. In particular, $k < k'$ and $\operatorname{inp} h(t^{\underline{\ell}}) = 0$. Then $\operatorname{inp} h \in I = \langle \underline{f}\rangle$ and $\operatorname{inp} h = \sum_{i=1}^m q_i f_i$ for some $\underline{q} \in \mathbb{K}[\underline{x}]^m$. Set

$$h' = h - \sum_{i=1}^m q_i F_i(\underline{x}, 1) = h - \operatorname{inp} h + \sum_{i=1}^m q_i f_i'(\underline{x}, 1).$$

Then $h'(\underline{\xi}') = h(\underline{\xi}')$ by (3.9) and hence $\upsilon(h'(\underline{\xi}')) = k'$. With (3.9) and homogeneity of $\underline{f}'$ it follows that $\operatorname{ord}(h') > k$ contradicting the maximality of k. □

Remark 3.8. The proof of Proposition 3.7 shows in fact that the condition $\Gamma' = \Gamma$ is equivalent to the flatness of a homogeneous deformation of the parametrization as in (3.7). These Γ-constant deformations are a particular case of δ-constant deformations of germs of complex analytic curves (see [Tei77, §3, Cor. 1]).

The following numerical condition yields the hypothesis of Proposition 3.7.

Lemma 3.9. *If* $\min\{\underline{d}\} + \delta \geq \gamma$ *then* $\Gamma' = \Gamma$.

Proof. Any $k \in \Gamma'$ is of the form $k = \upsilon(p(\underline{\xi}'))$ for some $p \in \mathbb{K}\{\underline{x}\}$ with $p_0 = \operatorname{inp}(p) \in \mathbb{K}[\underline{x}]$. If $p_0(t^{\underline{\ell}}) \neq 0$, then $k \in \Gamma$. Otherwise, $p_0 \in \langle \underline{f}\rangle$ and hence $k \geq \min\{\underline{d}\} + \min\{\underline{\ell}'\}$. The second claim follows. □

4. Set-theoretic complete intersections

We return to the special case $\Gamma = \langle \ell, m, n\rangle$ of §2. Recall Bresinsky's method to show that $\operatorname{Spec}(\mathbb{K}[\Gamma])$ is a set-theoretic complete intersection (see [Bre79a]). Starting from the defining equations (2.6) in case (H1) he computes

$$\begin{aligned} f_1^c = (x^a - y^{b_1}z^{c_2})^c &= x^a g_1 \pm y^{b_1 c} z^{c_2 c} \\ &= x^a g_1 \pm y^{b_1 c} z^{(c_2-1)c}(x^{a_1}y^{b_2} - f_3) \\ &= x^{a_1} g_2 \mp y^{b_1 c} z^{(c_2-1)c} f_3 \\ &\equiv x^{a_1} g_2 \mod \langle f_3\rangle \end{aligned}$$

where $g_1 \in \langle x, z\rangle$ and

$$g_2 = x^{a-a_1} g_1 \pm y^{b_1 c + b_2} z^{(c_2-1)c}.$$

He shows that if $c_2 \geq 2$, then further reducing g_2 by f_3 yields

$$\begin{aligned} g_2 &= x^{a-a_1} g_1 \pm y^{b_1 c + b_2} z^{(c_2-2)c}(x^{a_1}y^{b_2} - f_3) \\ &\equiv x^{a-a_1} g_1 \pm x^{a_1} y^{b_1 c + 2b_2} z^{(c_2-2)c} \mod \langle f_3\rangle \\ &\equiv x^{a_1}\left(\tilde{g}_1 + y^{b_1 c + 2b_2} z^{(c_2-2)c}\right) \mod \langle f_3\rangle \\ &\equiv x^{a_1} g_3 \mod \langle f_3\rangle \end{aligned}$$

for some $\tilde{g}_1 \in \mathbb{K}[x, y, z]$. Iterating c_2 many times yields a relation

$$f_1^c = q f_3 + x^k g, \quad k = a_1 c_2, \tag{4.1}$$

where $g \equiv y^{\ell'} \mod \langle x, z\rangle$ with ℓ' from (2.12). One computes that

$$x^{a_1} f_2 = y^{b_1} f_3 - z^{c_1} f_1, \quad z^{c_2} f_2 = x^{a_2} f_3 - y^{b_2} f_1.$$

Bresinsky concludes that

$$Z(x, z) \not\subset Z(g, f_3) \subset Z(f_1, f_3) = Z(f_1, f_2, f_3) \cup Z(x, z) \tag{4.2}$$

making $\operatorname{Spec}(\mathbb{K}[\Gamma]) = Z(g, f_3)$ a set-theoretic complete intersection.

As a particular case of (3.7) consider three elements

$$\begin{aligned} \xi &= t^\ell + \sum_{i \geq \ell + \Delta\ell} \xi_i s^{i-\ell} t^i, \\ \eta &= t^m + \sum_{i \geq m + \Delta m} \eta_i s^{i-m} t^i, \\ \zeta &= t^n + \sum_{i \geq n + \Delta n} \zeta_i s^{i-n} t^i \in \mathbb{K}[t, s]. \end{aligned} \tag{4.3}$$

Consider the curve germ C in (3.8) with $\mathbb{K}$-analytic ring

$$\mathcal{O} = \mathcal{O}_C = \mathbb{K}\{\xi', \eta', \zeta'\}, \quad (\xi', \eta', \zeta') = (\xi, \eta, \zeta)(t, 1), \tag{4.4}$$

and value semigroup $\Gamma' \supset \Gamma$. We aim to describe situations where C is a set-theoretic complete intersection under the hypothesis that $\Gamma' = \Gamma$. By Proposition 3.7, (ξ, η, ζ) then generate the flat deformation of $C_0 = \operatorname{Spec}^{\mathrm{an}}(\mathbb{K}[\Gamma])$ in Proposition 3.3. Let F_1, F_2, F_3 be the defining equations from Proposition 3.7.

Lemma 4.1. *If g in (4.1) deforms to $G \in \mathbb{K}\{x, y, z, s\}$ such that*

$$F_1^c = q F_3 + x^k G, \quad G(x, y, z, 0) = g, \tag{4.5}$$

then

$$C = S \cap Z(s-1) = Z(G, F_3, s-1)$$

is a set-theoretic complete intersection.

Proof. Consider a matrix of indeterminates

$$M = \begin{pmatrix} Z_1 & X_1 & Y_1 \\ Y_2 & Z_2 & X_2 \end{pmatrix}$$

and the system of equations defined by its maximal minors

$$F_1 = X_1X_2 - Y_1Z_2,$$
$$F_2 = Y_1Y_2 - X_2Z_1,$$
$$F_3 = X_1Y_2 - Z_1Z_2.$$

By Schaps' theorem (see [Sch77]), there is a solution with coefficients in $\mathbb{K}\{x, y, z\}[\![s]\!]$ that satisfies $M(x, y, z, 0) = M_0$. By Grauert's approximation theorem (see [Gra72]), the coefficients can be taken in $\mathbb{K}\{x, y, z, s\}$. Using the fact that M is a matrix of relations, we imitate in Bresinsky's argument in (4.2),

$$Z(G, F_3) \subset Z(F_1, F_3) = Z(F_1, F_2, F_3) \cup Z(X_1, Z_2).$$

The $\mathbb{K}$-analytic germs $Z(G, F_3)$ and $Z(G, X_1, Z_2)$ are deformations of the complete intersections $Z(g, f_3)$ and $Z(g, x^{a_1}, z^{c_2})$, and are thus of pure dimensions 2 and 1 respectively. It follows that $Z(G, F_3)$ does not contain any component of $Z(X_1, Z_2)$ and must hence equal $Z(F_1, F_2, F_3) = S$. The claim follows. ☐

Proposition 4.2. *Set $\delta = \min(\Delta\ell, \Delta m, \Delta n)$ and $k = a_1c_2$. Then the curve germ C defined by (4.3) is a set-theoretic complete intersection if*

$$\min(d_1, d_2, d_3) + \delta \geq \gamma,$$
$$\min(d_1, d_3) + \delta \geq \gamma + k\ell,$$

or, equivalently,

$$\min(d_1, d_2 + k\ell, d_3) + \delta \geq \gamma + k\ell.$$

Proof. By Lemma 3.9, the first inequality yields the assumption $\Gamma' = \Gamma$ on (4.3). The conductor of $\xi^k\mathcal{O}$ equals $\gamma + k\ell$ and contains $(F_i - f_i)(\xi', \eta', \zeta')$, $i = 1, 3$, by the second inequality. This makes $F_i - f_i$, $i = 1, 3$, divisible by x^k. Substituting into (4.1) yields (4.5) and by Lemma 4.1 the claim. ☐

Remark 4.3. We can permute the roles of the f_i in Bresinsky's method. If the role of (f_1, f_3) is played by (f_1, f_2), we obtain a formula similar to (4.1), $f_1^b = qf_2 + x^kg$ with $k = a_2b_1$. Instead of x^k, there is a power of y if we use instead (f_2, f_1) or (f_2, f_3) and a power of z if we use (f_3, f_1) or (f_3, f_1). The calculations are the same. In the examples we favor powers of x in order to minimize the conductor $\gamma + k\ell$.

5. Series of examples

Redefining a, b suitably, we specialize to the case where the matrix in (2.7) is of the form

$$M_0 = \begin{pmatrix} z & x & y \\ y^b & z & x^a \end{pmatrix}.$$

By Proposition 2.4.(a), these define $\operatorname{Spec}(\mathbb{K}[\langle \ell, m, n\rangle])$ if and only if

$$\ell = b + 2, \quad m = 2a + 1, \quad n = ab + b + 1 (= (a + 1)\ell - m), \quad \gcd(\ell, m) = 1.$$

We assume that $a, b \geq 2$ and $b+2 < 2a+1$ so that $\ell < m < n$. The maximal minors (2.6) of M_0 are then

$$f_1 = x^{a+1} - yz, \quad f_2 = y^{b+1} - x^a z, \quad f_3 = z^2 - xy^b$$

with respective weighted degrees

$$d_1 = (a+1)(b+2), \quad d_2 = (2a+1)(b+1), \quad d_3 = 2ab + 2b + 2$$

where $d_1 < d_3 < d_2$. In Bresinsky's method (4.1) with $k = 1$ reads

$$f_1^2 - y^2 f_3 = xg, \quad g = x^{2a+1} - 2x^a yz + y^{b+2}.$$

We reduce the inequality in Proposition 4.2 to a condition on d_1.

Lemma 5.1. *The conductor of $\xi\mathcal{O}$ is bounded by*

$$\gamma + \ell \leq d_2 - \left\lfloor \frac{m}{\ell} \right\rfloor \ell < d_3.$$

In particular, $d_2 \geq \gamma + 2\ell$ and $d_3 > \gamma + \ell$.

Proof. The subsemigroup $\Gamma_1 = \langle \ell, m \rangle \subset \Gamma$ has conductor

$$\gamma_1 = (\ell - 1)(m - 1) = 2a(b+1) = n + (a-1)\ell + 1 \geq \gamma.$$

To obtain a sharper upper bound for γ we think of Γ as obtained from Γ_1 by filling gaps of Γ_1. Since $2n \geq \gamma_1$,

$$\Gamma \setminus \Gamma_1 = (n + \Gamma_1) \setminus \Gamma_1.$$

The smallest elements of Γ_1 are $i\ell$ where $i = 0, \ldots, \lfloor \frac{m}{\ell} \rfloor$. By symmetry of Γ_1 (see [Kun70]), the largest elements of $\mathbb{N} \setminus \Gamma_1$ are

$$\gamma_1 - 1 - i\ell = n + (a - 1 - i)\ell, \quad i = 0, \ldots, \left\lfloor \frac{m}{\ell} \right\rfloor,$$

and contained in $n + \Gamma_1$ since the minimal coefficient $a - 1 - i$ is non-negative by

$$a - 1 - \left\lfloor \frac{m}{\ell} \right\rfloor \geq a - 1 - \frac{m}{\ell} = \frac{(a-1)b - 3}{b+2} > -1.$$

They are thus the largest elements of $\Gamma \setminus \Gamma_1$. Their minimum attained at $i = \lfloor \frac{m}{\ell} \rfloor$ then bounds

$$\gamma \leq \gamma_1 - 1 - \left\lfloor \frac{m}{\ell} \right\rfloor \ell.$$

Substituting $\gamma_1 + \ell - 1 = d_2$ yields the first particular inequality. The second one follows from

$$d_2 - d_3 = 2a - b - 1 = m - \ell < \left\lfloor \frac{m}{\ell} \right\rfloor \ell. \qquad \square$$

Proof of Corollary 1.1.

(a) This follows from Lemma 3.9.

(b) By Lemma 5.1, the inequality in Proposition 4.2 simplifies to $d_1 + \delta \geq \gamma + \ell$. The claim follows.

(c) Suppose that

$$d_1 + q - n \geq \gamma + \ell$$

for some $q > n$ and $a, b \geq 3$. Set $p = \gamma - 1 - \ell$. Then $n > m + \ell$ and $\Gamma \cap (m + \ell, m + 2\ell)$ can include at most n and some multiple of ℓ. Since $\ell \geq 4$ it follows that $(m + \ell, m + 2\ell)$ contains a gap of Γ and hence $\gamma - 1 > \ell + m$ and $p > m$. Moreover $(a-1)b \geq 4$ is equivalent to

$$d_1 + p - m \geq \gamma + \ell.$$

By (b), C is a set-theoretic complete intersection.

It remains to show that $C \ncong C_0$. This follows from the fact that $\Omega^1_{C_0} \to \mathbb{K}\{t\}dt$ has valuations $\Gamma \setminus \{0\}$ whereas the 1-form

$$\omega = mydx - \ell xdy = \ell(m-p)t^{p+\ell-1}dt \in \Omega^1_C \to \mathbb{K}\{t\}dt$$

has valuation $p + \ell = \gamma - 1 \notin \Gamma$. □

Example 5.2. We discuss a list of special cases of Corollary 1.1.

(a) $a = b = 2$. The monomial curve C_0 defined by $(x, y, z) = (t^4, t^5, t^7)$ has conductor $\gamma = 7$. Its only admissible deformation is

$$(x, y, z) = (t^4, t^5 + st^6, t^7).$$

However, this deformation is trivial and our method does not yield a new example. To see this, we adapt a method of Zariski (see [Zar06, Ch. III, (2.5), (2.6)]). Consider the change of coordinates

$$\tilde{x} = x + \frac{4s}{5}y = t^4 + \frac{4s}{5}t^5 + \frac{4s^2}{5}t^6$$

and the change of parameters of the form $\tau = t+O(t^2)$ such that $\tilde{x} = \tau^4$. Then $\tau = t+\frac{s}{5}t^2+O(t^3)$ and hence $y = \tau^5 + O(t^7)$ and $z = \tau^7 + O(t^8)$. Since $O(t^7)$ lies in the conductor, it follows that $C \cong C_0$.

In all other cases, Corollary 1.1 yields an infinite list of new examples.

(b) $a = 3$, $b = 2$. Consider the monomial curve C_0 defined by $(x, y, z) = (t^4, t^7, t^9)$. By Zariski's method from (a), we reduce to considering the deformation

$$(x, y, z) = (t^4, t^7, t^9 + st^{10}).$$

While part (c) of Corollary 1.1 does not apply, $C \ncong C_0$ remains valid. To see assume that $C_0 \cong C$ induced by an automorphism φ of $\mathbb{C}\{t\}$. Then $\varphi(x) \in \mathcal{O}_C$ shows that φ has no quadratic term. This, however, contradicts $\varphi(z) \in \mathcal{O}_C$.

(c) $a = b = 3$. The monomial curve C_0 defined by $(x, y, z) = (t^5, t^7, t^{13})$ has conductor $\gamma = 17$. We want to satisfy $p \geq \gamma + \ell - d_1 + m = 9$. The most general deformation of y thus reads

$$y = t^7 + s_1 t^9 + s_2 t^{11} + s_3 t^{16}.$$

The parameter s_1 can be again eliminated by Zariski's method as in (a). This leaves us with the deformation

$$(x, y, z) = (t^5, t^7 + s_2 t^{11} + s_3 t^{16}, t^{13} + s_4 t^{16})$$

which is non-trivial due to part (c) of Corollary 1.1 with $p = 11$.

(d) $a = 8$, $b = 3$. The monomial curve C_0 defined by $(x, y, z) = (t^5, t^{17}, t^{28})$ has conductor $\gamma = 47$. The condition in part (b) of Corollary 1.1 requires $p \geq \gamma - d_1 + m = 19$. In fact, the deformation

$$(x, y, z) = (t^5, t^{17} + st^{18}, t^{28})$$

is not flat since C has value semigroup $\Gamma' = \Gamma \cup \{46\}$. However, C is isomorphic to the general fiber of the flat deformation in 4-space

$$(x, y, z, w) = (t^5, t^{17} + st^{18}, t^{28}, t^{46}).$$

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