Journal *of* Singularities
Volume 17 (2018), 70-80

received: 9 November 2017
in revised form: 19 February 2018
DOI: 10.5427/jsing.2018.17d

ON VERTICES AND INFLECTIONS OF PLANE CURVES

FABIO SCALCO DIAS AND FARID TARI

Abstract. We count the number of inflections (I) and vertices (V) concentrated at the singularity of a germ of a parametrised plane curve and show that $V = I + \mu - 2$, where μ is the Milnor number of a certain singularity.

1. Introduction

There exist various counting formulae relating special points on a smooth (C^∞) closed plane curve. There are also lower bounds giving the minimum number of such special points on the curves. For instance, the 4-vertex theorem states that any simple closed curve in the Euclidean plane has at least 4 vertices (see [4] for a survey article). Formulae relating the number of bitangencies, the number of inflections and the number of cusps on a closed plane curve are obtained in [10, 11]. Such points are in fact projective invariant and a similar formula is obtained in [15] for curves in the projective plane. In [8] the authors considered curves defined on an interval and proved that, under some conditions, the formula in [11] still holds. Further work in [7] includes cusp points. Other local and global formulae for curves parametrised by polynomials are given in [7]. There is also work on inflections and vertices of sections of surfaces, flattenings and Darboux vertices of space curves in, for example, [1, 9, 12, 17, 18].

Here we are concerned with vertices and inflections of germs of smooth, but not necessarily regular, curves in the Euclidean plane. For a regular plane curve γ, a vertex is a point where the derivative of its curvature function vanishes. Vertices can be captured by the evolute of γ which is the locus of centres of its osculating circles (i.e., the centres of circles that have at least 3-point contact with γ). Vertices are points where the osculating circles have at least 4-point contact with γ and the centres of such circles are where the evolute is singular. Another way of characterising vertices is via the singularities of the distance squared functions on γ. A point p on γ is a vertex if and only if the distance squared function from the centre of curvature associated to p has an A_k-singularity with $k \geq 3$ (i.e., it can be reduced by a change of variable to $\pm t^{k+1}$, $k \geq 3$). Varying the centres of the circles in the plane gives the family of distance squared functions. The evolute is the bifurcation set of this family (see for example [3] and §2).

Our study is local in nature, so we consider germs of plane curves at a given point which we take to be the origin. When the germ of the curve γ is singular, we call the closure of the centres of the osculating circles of γ with its singular point removed the *proper evolute* of γ. The proper evolute intersects the limiting normal line at the singular point of γ at a point c_0 which could be at infinity. In fact, any point on the limiting normal line is a centre of an osculating circle at the singular point of γ, and the bifurcation set of the family of distance squared functions on γ (the *full evolute*) consists of the proper evolute together with the limiting normal line counted with multiplicity l. It turns out that when c_0 is not at infinity, the number l coincides with the

2010 *Mathematics Subject Classification.* 53A04, 58K05, 58K60.

Key words and phrases. bifurcations, evolutes, plane curves, inflections, vertices, singularities.

The first author is partially supported by the grant FAPEMIG APQ-01086-15. The second author is partially supported by the grants FAPESP 2014/00304-2, CNPq 302956/2015-8.

number I of inflections concentrated at the singular point (Theorem 3.5). The point c_0 is also where the singularity at the origin of the distance squared function from c_0 is more degenerate than that from any other point on the limiting normal line (Proposition 3.3). Let μ be the Milnor number of this singularity. We prove in Theorem 3.4 that the number V of vertices of γ concentrated at the singular point satisfies

$$V = I + \mu - 2.$$

We give in §4 examples of deformations of singularities where I simple inflections and V simple vertices appear on the deformed curve.

2. Preliminaries

Let $\gamma : I \to \mathbb{R}^2$ be a smooth curve in the Euclidean plane where I is an open interval of $\mathbb{R}$. Suppose that γ is a regular curve parametrised by arc length s and write $\mathbf{t}(s) = \gamma'(s)$ for its unit tangent vector. We denote by $\mathbf{n}(s)$ the unit normal vector to γ obtained by rotating $\mathbf{t}(s)$ anti-clockwise by an angle of $\pi/2$. Then $\mathbf{t}'(s) = \kappa(s)\mathbf{n}(s)$, where $\kappa(s)$ is the curvature of γ at s. The *evolute* of γ is the curve parametrised by

$$e(s) = \gamma(s) + \frac{1}{\kappa(t)}\mathbf{n}(s).$$

When the parameter t of $\gamma(t) = (x(t), y(t))$ is not necessarily the arc length parameter, the curvature is given at any t by

$$\kappa = \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{\frac{3}{2}}}.$$

We observe that if t_0 is a singular point of γ, then $(x'y'' - x''y')(t_0) = 0$. Away from the singular points of γ the denominator of κ is strictly positive, so if we denote the numerator of κ by $\iota = x'y'' - x''y'$, a point t_0 is an *inflection* if and only of

$$\iota(t_0) = (x'y'' - x''y')(t_0) = 0. \tag{1}$$

Similarly, by differentiating the curvature function, a point t_0 is a vertex if and only if the numerator $\upsilon(t)$ of $\kappa'(t)$ vanishes at t_0, that is

$$\upsilon(t_0) = \left((x'^2 + y'^2)(x'y''' - x'''y') + 3(x'x'' + y'y'')(x''y' - x'y'')\right)(t_0) = 0. \tag{2}$$

Remark 2.1. When the curve is parametrised in Monge-form $\gamma(t) = (t, f(t))$, a point t_0 is an inflection if and only if $f''(t_0) = 0$ and a vertex if and only if

$$\left((1 + f'^2)f''' - 3f'(f'')^2\right)(t_0) = 0.$$

The vertices of γ and its evolute can be picked up by the family of distance squared functions $D : I \times \mathbb{R}^2 \to \mathbb{R}$ on γ given by

$$D(t, c) = \langle \gamma(t) - c, \gamma(t) - c \rangle,$$

where $\langle , \rangle$ is the scalar product in $\mathbb{R}^2$. For c fixed, we write $D_c(t) = D(t, c)$. The function D_c measures the contact of the curve γ with circles centred at c (see for example [3]). When γ is a regular curve, $D_c'(t_0) = D_c''(t_0) = 0$ if and only if c is the point $e(t_0)$ on the evolute of γ. Thus, the

$$\text{Evolute} = \left\{c \in \mathbb{R}^2 \,:\, D_c \text{ has an } A_{\geq 2} - \text{singularity at some } t \in I\right\},$$

which is precisely the (local) bifurcation set of the family D. When γ is singular, we call the bifurcation set of the family D the *full evolute* of γ and define the *proper evolute* of γ as the closure of the centres of the osculating circles of γ with its singular point removed.

Inflections can be picked up by the family of height functions $H : I \times S^1 \to \mathbb{R}$ on γ given by

$$H(t,v) = \langle \gamma(t), v \rangle,$$

where S^1 denotes the unit circle in $\mathbb{R}^2$. The height function $H_v(t) = H(t,v)$ is singular at t_0 if and only if v is parallel to the normal direction of γ at t_0. When γ is regular, the singularity is of type $A_{\geq 2}$ if furthermore the point $\gamma(t_0)$ is an inflection point. (The discriminant of the family H gives the dual of the curve γ.)

We denote by $\mathrm{ord}(f)$ the order of a smooth function f at $t = 0$, i.e., the first integer k such that the kth order derivative $f^{(k)}(0) \neq 0$. Thus, f has an A_k-singularity at $t = 0$ if and only if $\mathrm{ord}(f) = k + 1$. The Milnor number of f at $t = 0$ is $\mu = \dim_{\mathbb{R}} \mathcal{E}_1/\langle f' \rangle = \mathrm{ord}(f) - 1$, where $\mathcal{E}_1$ the ring of germs of smooth functions $(\mathbb{R}, 0) \to \mathbb{R}$.

3. Counting vertices and inflections

Let γ be a germ of a possibly singular smooth (C^∞) plane curve. We choose a system of coordinates in $\mathbb{R}^2$, assume that the point of interest is the origin and write $\gamma(t) = (x(t), y(t))$. We make the following assumptions in this paper:

Assumption 1: $x(t)$ and $y(t)$ are not germs of flat-functions,
i.e. $\mathrm{ord}(x)(0) < \infty$ and $\mathrm{ord}(y)(0) < \infty$.
Assumption 2: $y(t)$ is not a multiple of $x(t)$ by a non-zero constant.

The set of germs of smooth plane curves can be identified with $\mathcal{E}_1^2 = \mathcal{E}_1 \times \mathcal{E}_1$ and given the Whitney topology. The subset of germs of plane curves satisfying Assumptions 1 and 2 above form an open and dense subset of $\mathcal{E}_1^2$.

With the above assumptions, we can reparametrise the curve, make isometric changes of coordinates in the plane and write

$$\gamma(t) = (t^m, \sum_{i=m+1}^{k} a_i t^i + O(t^{k+1})). \tag{3}$$

Again, with the above assumptions, the curve γ has a limiting tangent and normal directions at $t = 0$. We can define the multiplicity of γ as the order of the height function along the limiting normal direction. The integer m in (3) is precisely the multiplicity of the curve at its singular point.

We define, for reasons that will be apparent later,

$$n_1 := \min\{i : a_i \neq 0, i \neq 2m\}.$$

If γ is analytic and $n_1 < 2m$, then n_1 is the first Puiseux exponent β_1.

We define, in the usual way, the number of inflections I and the number of vertices V concentrated at the singular point of γ by

$$I = \dim_{\mathbb{R}} \frac{\mathcal{E}_1}{\langle x'y'' - x''y' \rangle},$$
$$V = \dim_{\mathbb{R}} \frac{\mathcal{E}_1}{\langle (x'^2 + y'^2)(x'y''' - x'''y') + 3(x'x'' + y'y'')(x''y' - x'y'') \rangle}.$$

Clearly, $I = \mathrm{ord}(\iota)(0)$ and $V = \mathrm{ord}(\upsilon)(0)$.

Theorem 3.1. *For γ as in (3),*

$$I = \begin{cases} 3m-3 & \text{if } n_1 > 2m \text{ and } a_{2m} \neq 0 \\ m+n_1-3 & \text{otherwise} \end{cases}$$

$$V = \begin{cases} 3m+n_1-6 & \text{if } n_1 < 4m, \text{ or } \ n_1 > 4m \text{ and } a_{2m} = 0 \\ 7m-6 & \text{if } n_1 = 4m \text{ and } a_{4m} - a_{2m}^3 \neq 0, \text{ or } \ n_1 > 4m \text{ and } a_{2m} \neq 0 \end{cases}$$

Proof. The proof follows by calculating the order of the functions ι and υ at the origin. For γ as in (3), we have

$$\iota(t) = \begin{cases} 2a_{2m}m^3t^{3m-3} + O(t^{3m-2}) & \text{if } a_{2m} \neq 0 \text{ and } n_1 > 2m \\ a_{n_1}n_1m(n_1-m)t^{m+n_1-3} + O(t^{m+n_1-2}) & \text{otherwise} \end{cases}$$

and the value of I follows. Similarly, for calculating the vertices V we have

$$\upsilon(t) = \begin{cases} a_{n_1}m^3n_1(2m-n_1)(m-n_1)t^{3m+n_1-6} + O(t^{3m+n_1-5}) & \text{if } n_1 < 4m, \text{ or } \ n_1 > 4m \text{ and } a_{2m} = 0 \\ 24m^6(a_{4m} - a_{2m}^3)t^{7m-6} + O(t^{7m-5}) & \text{if } n_1 = 4m \text{ and } a_{4m} - a_{2m}^3 \neq 0 \\ -24a_{2m}^3m^6t^{7m-6} + O(t^{7m-5}) & \text{if } n_1 > 4m \text{ and } a_{2m} \neq 0 \end{cases}$$

□

Remarks 3.2. 1. One can also calculate the number of vertices when $a_{4m} - a_{2m}^3 = 0$. A condition depending on the non-vanishing of a polynomial in the coefficients a_i, $i \geq 2m$, is then required.
2. Theorem 3.1 is also valid for regular curves. In this case $m = 1$ and

$$I = \begin{cases} 0 & \text{if } n_1 > 2 \text{ and } a_2 \neq 0 \\ n_1 - 2 & \text{otherwise} \end{cases} \qquad V = \begin{cases} 0 & \text{if } n_1 = 3 \\ 1 & \text{if } n_1 \geq 4 \text{ and } a_2^3 - a_4 \neq 0 \end{cases}$$

Guided by the example of the cusp (see Figure 1), we shall seek an interpretation of the formula of V in Theorem 3.1 using the bifurcation set of the family of distance squared function D on γ, given by

$$D(t,c) = (t^m - a)^2 + \left(\sum_{i=m+1}^{k} a_it^i + O(t^{k+1}) - b \right)^2,$$

with $c = (a,b)$.

Proposition 3.3. *Let γ be a germ of a singular curve parametrised as in (3). Then the distance squared function D_c on γ has a degenerate singularity at $t=0$ for any point c on the limiting normal line $a = 0$ of γ at $t = 0$. There is a unique point c_0 on this line where the singularity of D_{c_0} at $t = 0$ is more degenerate than any other singularity of D_c at $t = 0$, with $c \neq c_0$. The Milnor number μ of D_{c_0} is given by*

$$\mu = \begin{cases} 2m-1 & \text{if } n_1 < 2m \\ n_1 - 1 & \text{if } 2m < n_1 < 4m \\ 4m-1 & \text{if } n_1 = 4m \text{ and } a_{4m} - a_{2m}^3 \neq 0, \text{or } n_1 > 4m \text{ and } a_{2m} \neq 0 \end{cases}$$

The point c_0 is at infinity when $n_1 > 2m$ and $a_{2m} = 0$, and the associated circle becomes a line. The Milnor number of the height function on γ along a normal direction to this line is $n_1 - 1$.

When c_0 is not at infinity, it is precisely the point of intersection of the proper evolute with the limiting normal line to γ at its singular point.

Proof. We divide the proof in two cases: $n_1 < 2m$ and $n_1 > 2m$.

For the case $n_1 < 2m$ we have $\gamma(t) = (t^m, a_{n_1}t^{n_1} + O(t^{n_1+1}))$, so

$$\begin{array}{rcl} D(t,c) & = & (t^m - a)^2 + (a_{n_1}t^{n_1} + O(t^{n_1+1}) - b)^2 \\ & = & a^2 + b^2 - 2at^m - 2ba_{n_1}t^{n_1} + t^{2m} - 2bO(t^{n_1+1}) + O(t^{2n_1}). \end{array}$$

On the limiting normal line $a = 0$, D_c has an A_{n_1-1}-singularity except when $b = 0$, that is when $c_0 = (0,0)$. The function D_{c_0} has an A_{2m-1}-singularity, so its Milnor number is $\mu = 2m-1$.

For the second case $n_1 > 2m$, we write $\gamma(t) = (t^m, a_{2m}t^{2m} + a_{n_1}t^{n_1} + O(t^{n_1+1}))$. Thus,

$$\begin{array}{rcl} D(t,c) & = & (t^m - a)^2 + (a_{2m}t^{2m} + a_{n_1}t^{n_1} + O(t^{n_1+1}) - b)^2 \\ & = & a^2 + b^2 - 2at^m + (1 - 2ba_{2m})t^{2m} - 2ba_{n_1}t^{n_1} + a_{2m}^2t^{4m} \\ & & -2bO(t^{n_1+1}) + O(t^{2m+n_1}). \end{array}$$

Suppose that $a_{2m} \neq 0$. Then, on the limiting normal line $a = 0$, the function D_c has an A_{2m-1}-singularity for all b except when $b = b_0 = 1/(2a_{2m})$. If we write $c_0 = (0, b_0)$, then D_{c_0} has A_{n_1-1}-singularity (so $\mu = n_1 - 1$) if $n_1 < 4m$. When $n_1 = 4m$, we have

$$D_{c_0}(t) = b_0^2 - \frac{1}{a_{2m}}(a_{4m} - a_{2n}^3)t^{4m} + O(t^{4m+1})$$

and this is an A_{4m-1}-singularity (so $\mu = 4m - 1$) provided $a_{4m} - a_{2n}^3 \neq 0$. If $n_1 > 4m$, the singularity of D_{c_0} is of type A_{4m-1}.

Suppose now that $a_{2m} = 0$. This means that c_0 above is at infinity, so the circle of centre c_0 becomes the horizontal line and the distance squared function becomes the height function along the direction $(0,1)$. This height function is given by $a_{n_1}t^{n_1} + O(t^{n_1+1})$ and has an A_{n_1-1}-singularity.

The evolute is parametrised by $e(t) = \gamma(t) + (1/\kappa(t))\mathbf{n}(t)$, $t \neq 0$. If we write $\gamma = (x,y)$, then $\mathbf{n} = (x'^2 + y'^2)^{-\frac{1}{2}}(-y', x')$ and $\kappa = (x'^2 + y'^2)^{-\frac{3}{2}}\iota$. It follows that

$$e = (x - \frac{x'^2 + y'^2}{\iota}y', y + \frac{x'^2 + y'^2}{\iota}x').$$

Using the expression for $\iota(t)$ in the proof of Theorem 3.1, we conclude that

$$c_0 = \left\{ \begin{array}{ll} (0,0) & \text{if } n_1 < 2m \\ (0, \frac{1}{2a_{2m}}) & \text{if } n_1 > 2m \text{ and } a_{2m} \neq 0 \end{array} \right.$$

above is precisely the limit of $e(t)$ as $t \to 0$. (When $n_1 > 2m$ and $a_{2m} = 0$, $e(t)$ goes to infinity on the line $a = 0$ as $t \to 0$.) $\square$

We can now relate the number of vertices V, the number of inflections I and the Milnor number μ of D_{c_0} (Proposition 3.3) as follows.

Theorem 3.4. *Let γ be a germ of a smooth curve satisfying Assumptions* 1 *and* 2*. Suppose further that the point c_0 where the proper evolute intersects the limiting normal line of γ at its singular point is not at infinity. Then*

$$V = I + \mu - 2.$$

Proof. We take γ as in (3) and compute the values of V, μ and I using Theorem 3.1 and Propositions 3.3 and these are as in Table 1. The proof becomes a matter of checking the formula for these values. Observe that when $n_1 = 4m$, we need the condition $a_{4m} - a_{2n}^3 \neq 0$.

When c_0 is at infinity ($a_{2m} = 0$ and $n_1 > 2m$), we have $I = m + n_1 - 3$ and $\mu = n_1 - 1$, so $I + \mu - 2 = m + 2n_1 - 6 \neq V$. (Here $V = 3m + n_1 - 6$ if $n_1 \neq 4m$ or to $7m - 6$ if $n_1 = 4m$ and $a_{4m} \neq 0$; see Theorem 3.1.) $\square$

TABLE 1. Values of V, μ and I.

Conditions	I	μ	V
$n_1 < 2m$	$m + n_1 - 3$	$2m - 1$	$3m + n_1 - 6$
$2m < n_1 < 4m$	$3m - 3$	$n_1 - 1$	$3m + n_1 - 6$
$4m < n_1$	$3m - 3$	$4m - 1$	$7m - 6$
$n_1 = 4m,\ a_{4m} - a_{2m}^3 \neq 0$	$3m - 3$	$4m - 1$	$7m - 6$

We turn now to a geometric interpretation of the number I. The idea is suggested by Figure 1. When the cusp of γ_0 is deformed in a family γ_u, we get in Figure 1 (right) a birth of two inflections. We know that the evolute goes to infinity at an inflection asymptotically along the normal line to the curve at the inflection (see for example [14]). As u gets closer to 0, the branches of the evolute that are close to the two normal lines at the inflections are dragged to the point c_0 (which coincides in this case with the singular point of γ) to form the limiting normal line to γ_0 at the cusp counted with multiplicity $l = 2$. Thus, $l = I = 2$. We have the following result when c_0 is not at infinity.

Theorem 3.5. *Let γ be a germ of a smooth curve parametrised as in (3). Then the limiting normal line $a = 0$ is a part of the bifurcation set of the family D of distance squared functions on γ and has multiplicity l, with*

$$l = \begin{cases} n_1 + m - 3 & \text{if} \quad n_1 < 2m \\ 3m - 3 & \text{if} \quad n_1 > 2m \end{cases}$$

Suppose that the proper evolute intersects the limiting normal line $a = 0$ at a point c_0 which is not at infinity (i.e., $a_{2m} \neq 0$ or $n_1 < 2m$). Then $l = I$.

Proof. The bifurcation set of the family D is given by

$$\mathcal{B}_1 = \{c = (a, b) \in \mathbb{R}^2 \mid \exists\, t \in I \text{ such that } D'_c(t) = D''_c(t) = 0\}.$$

Following the proof of Proposition 3.3, when $n_1 < 2m$ we have

$$\begin{aligned} D'(t, c) &= 2t^{m-1}\big(- am - ba_{n_1} n_1 t^{n_1 - m} + mt^m + O(t^{n_1 - m})\big), \\ D''(t, c) &= 2t^{m-2}\big(- am(m-1) - ba_{n_1} n_1 (n_1 - 1) t^{n_1 - m} + m(2m-1)t^m + O(t^{n_1 - m})\big). \end{aligned}$$

Denote by $m(f, g)$ the multiplicity of the germ $(f, g) : (\mathbb{R}^2, 0) \to \mathbb{R}^2$ of a smooth map in (a, b), so $m(f, g) = \dim_{\mathbb{R}} \mathcal{E}_2 / \langle f, g\rangle$ and gives the maximum number of (complex) solutions that appear in a deformation of the system of equations $f(a, b) = 0$, $g(a, b) = 0$. Given $f, g, h : (\mathbb{R}^2, 0) \to \mathbb{R}$, we have $m(f, hg) = m(f, g) + m(f, h)$.

If we write $D'(t, c) = t^{m-1}\phi_1$ and $D''(t, c) = t^{m-2}\phi_2$, then the multiplicity l of the limiting normal line $a = 0$ as a solution of $D'(t, c) = D''(t, c) = 0$ (considered as equations in (t, a)) is

$$\begin{aligned} l &= m(t^{m-1}, \phi_2) + m(t^{m-2}, \phi_1) + m(\phi_1, \phi_2) \\ &= (m-1) + (m-2) + (n_1 - m) \\ &= m + n_1 - 3. \end{aligned}$$

For computing $m(\phi_1, \phi_2)$, we need to assume that the point on the limiting normal line is a generic one, that is $b \neq 0$, equivalently we are away from the point c_0 where the proper evolute intersects the limiting normal line.

When $n_1 > 2m$ (and by hypothesis $a_{2m} \neq 0$), we have

$$\begin{aligned} D'(t, c) &= 2t^{m-1}\big(- am - m(2ba_{2m} - 1)t^m + O(t^{m+1})\big), \\ D''(t, c) &= 2t^{m-2}\big(- am(m-1) - m(2m-1)(2ba_{2m} - 1)t^m + O(t^{m+1})\big). \end{aligned}$$

Following the same arguments as above and taking points away from $c_0 = (0, \frac{1}{2a_{2m}})$, we get

$$\begin{aligned} l &= (m-1)+(m-2)+m \\ &= 3m-3. \end{aligned}$$

□

Remark 3.6. When $n_1 > 2m$ and $a_{2m} = 0$ we have $I = m + n_1 - 3$ and $l = 3m - 3$ so $l < I$. In this case the proper evolute goes to infinity, that is c_0 goes to infinity. The geometric explanation for why $l = I$ given before Theorem 3.5 when c_0 is not at infinity does not work in this case as there does not exist a point on the limiting normal line to drag to the branches of the evolute near the inflections. Thus, these branches may not collapse to the limiting normal line and contribute to the multiplicity of this line as part of the bifurcation set of the family of distance squared functions on γ. The fact that $l < I$ indicates that indeed some of these branches do not collapse to the limiting normal line.

4. Examples of realisations

In the previous section, we counted the number of vertices V and inflections I concentrated at the singular point of a germ of a plane curve. The numbers V and I are equal to 1 when the curve is regular and the vertex and inflection are ordinary ones. Therefore, V and I represent the maximum number of vertices and inflections that can appear in a deformation of the curve. A natural question follows: is there always a (real) deformation of the curve where V vertices and I inflections appear on the deformed curve? We consider this question in some examples.

A_2-singularity (ordinary cusp).
The curve can be parametrised by $\gamma(t) = (t^2, a_3t^3 + O(t^4))$, with $a_3 \neq 0$, so its defining equation has an A_2-singularity (see [2] for a relation between the singularity of a parametrisation and that of a defining equation). Here $l = I = 2$, $\mu = 3$ and $V = 3$ ([14]), and any $\mathcal{A}_e$-versal deformation of γ_u, with $\gamma_0 = \gamma$ exhibits V vertices and I inflections for $u > 0$ or $u < 0$ (as in the model $\gamma_u(t) = (t^2, ut + t^3)$, with $u > 0$), see [14] and Figure 1.

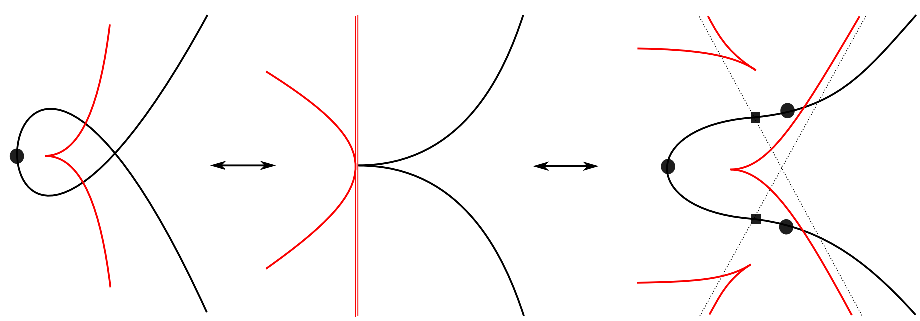


Figure 1. A generic deformation of the cusp and of its evolute ([14]). The full squares represent the inflections and the full discs the vertices.

A_4-singularity (ramphoid cusp).
The curve can be parametrised by $\gamma(t) = (t^2, a_4t^4 + a_5t^5 + O(t^6))$, with $a_5 \neq 0$. In [6, 13, 16], ways to obtain models of the singularities of a plane curve which take into consideration its contact with lines at the singular point are proposed. The equivalence relation in [6] is denoted by $\mathcal{A}_h$ and it is shown there that there are three $\mathcal{A}_h$-models of ramphoid cusps represented by

(t^2, t^4+t^5), (t^2, t^5+t^6), (t^2, t^5) ([6]). The proposed methods in [6, 13, 16] can handle well a single curve. However, finding a theory that explains the deformations of the singularities of a curve as well as the changes in its geometry (appearance and configuration of its inflections and vertices) is still an open problem. An approach to deal with this problem is proposed in [14].

Consider the case $\gamma(t) = (t^2, a_4t^4 + a_5t^5 + a_6t^6 + O(t^7))$, with $a_4 \neq 0$, $a_5 \neq 0$ and $a_6 \neq 0$. (In particular $\gamma \sim_{\mathcal{A}_h} (t^2, t^4+t^5)$, which is the least degenerate $\mathcal{A}_h$-singularity within the $\mathcal{A}$-class A_4; see [6, 13, 16]. The geometric meaning of $a_6 \neq 0$ can be found in [14].) Here $V = 5$, $I = 3$ and $\mu = 4$. It is shown in [14] that any $\mathcal{A}_e$-versal deformation $\gamma_{\mathbf{u}}$, with $\gamma_0 = \gamma$, exhibits V vertices and I inflections. Such deformations can be written in the form

$$\gamma_{\mathbf{u}=(u,v)}(t) = (t^2, vt + \bar{a}_2(\mathbf{u})t^2 + ut^3 + \bar{a}_4(\mathbf{u})t^4 + \bar{a}_5(\mathbf{u})t^5 + O_{\mathbf{u}}(t^6)),$$

with $\bar{a}_2(0) = 0$, $\bar{a}_4(0) = a_4 \neq 0$, $\bar{a}_5(0) = a_5 \neq 0, \bar{a}_6(0) = a_6 \neq 0$. We have a stratification of the parameter space $\mathbf{u} = (u, v)$ into codimension 1 phenomena given by curves (Figure 2, central figure) and stable phenomena in open sets delimited by these curves. In the region delimited by the curve $C : v = 0$ (where $\gamma_{\mathbf{u}}$ is singular) and the curve $SI : v = \dfrac{1}{16b_4^2}u^3 + O(u^4)$ (where $\gamma_{\mathbf{u}}$ has a second order inflection) with $v > 0$ there appear $V = 5$ simple vertices and $I = 3$ simple inflections with relative position as in Figure 2③ ([14]).

E_6-singularity.
The curve can be parametrised by $\gamma(t) = (t^3, a_4t^4 + O(t^5))$, with $a_4 \neq 0$. (Existence of a Morsification of this singularity and of others are studied in [5].) In this case $V = 7$ and $I = 4$. This case shows how hard the situation gets as the singularity becomes more degenerate. We shall not attempt here to obtain a model deformation of γ which takes into consideration singularities, vertices and inflections, but only exhibit a 1-parameter family of curves γ_α, α small enough with $\gamma_0 = \gamma$, where $V = 7$ vertices and $I = 4$ inflections appear on γ_α for $\alpha \neq 0$. For simplicity, we take $\gamma(t) = (t^3, t^4)$ and $\gamma_{\mathbf{u}}(t) = (t^3 + ut, t^4 + vt^2 + wt)$, which is an $\mathcal{A}_e$-versal deformation of γ, where $\mathbf{u} = (u, v, w)$. Changes of the number of inflections or vertices occur when $\gamma_{\mathbf{u}}$ is singular, has a second or higher order inflection or a second or higher order vertex.

The stratum in the $\mathbf{u}$-space where $\gamma_{\mathbf{u}}$ is singular is a (germ at the origin of a) surface with a cross-cap singularity and is given by

$$C : 27w^2 + 4u(2u - 3v)^2 = 0.$$

We have

$$\iota_{\mathbf{u}}(t) = 12t^4 + 6(2u - v)t^2 - 6wt + 2uv,$$

and calculating its resultant with $\iota'_{\mathbf{u}}(t)$ with respect to t gives the second order inflections stratum, which is the union of the surface C together with another cross-cap surface given by

$$SI : 27w^2 - 2v(6u - v)^2 = 0.$$

We have

$$\begin{aligned}\upsilon_{\mathbf{u}}(t) = \quad & -960t^9 - 24(9 + 56u - 12v)t^7 + 720wt^6 - 24(15u - 9v + 44uv - 18v^2)t^5 \\ & +30w(9 - 8u + 24v)t^4 - 24(u^2 + 3uv - 9w^2 + 10uv^2 - v^3)t^3 \\ & +12w(6u - 4uv + 3v^2)t^2 + 24u(u^2 - 2vu + w^2 - v^3)t - 6w(u^2 + w^2 + 2uv^2)\end{aligned}$$

with $\mathrm{ord}(\upsilon_0)(0) = 7$. The resultant of $\upsilon_{\mathbf{u}}(t)$ and $\upsilon'_{\mathbf{u}}(t)$ with respect to t, which gives the second order vertices stratum SV, is too lengthy to reproduce here (one of its factors is the left hand side of the equation of C, with multiplicity 3).

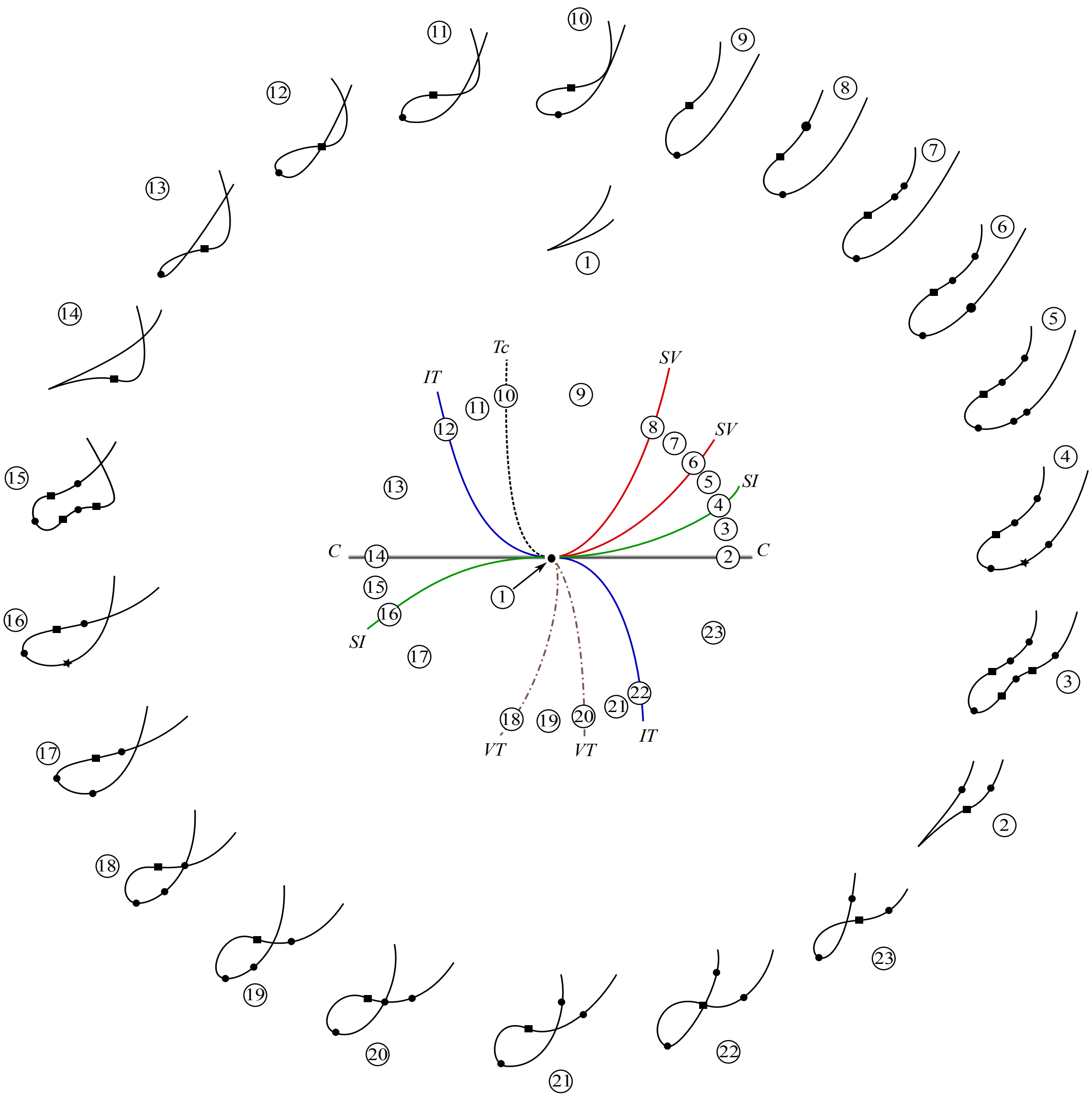
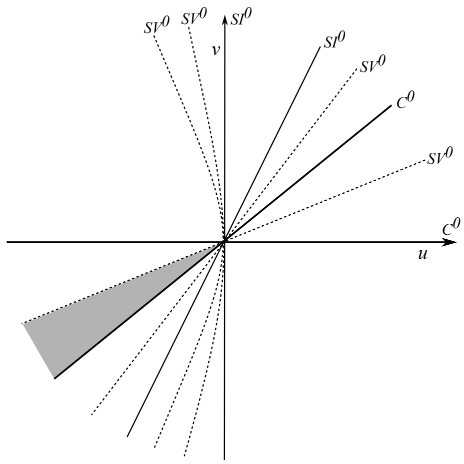


FIGURE 2. From [14]: deformation of a ramphoid cusp of $\mathcal{A}_h$-type $(t^2, t^4 + t^5)$. In stratum ③ (middle figure), we have a realisation of $V = 5$ vertices (full squares) and $I = 3$ inflections (full discs) as in figure ③.

We restrict to a plane through the origin in the $\mathbf{u}$-space. It turns out that a good choice is the plane $w = 0$. Observe that on such a plane $\iota_{\mathbf{u}}$ becomes an even polynomial and $\upsilon_{\mathbf{u}}$ an odd polynomial in t. In particular, their roots come in pairs $\pm t_i$, together with $t = 0$ for $\upsilon_{\mathbf{u}}$.

The trace of C on the plane $w = 0$ is

$$C^0 : 4u(2u - 3v)^2 = 0$$



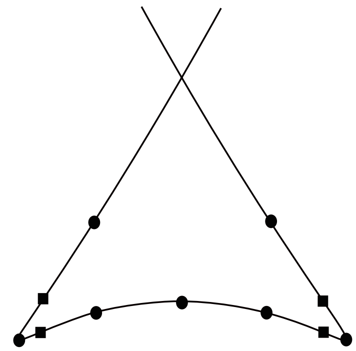


FIGURE 3. Left: traces of the strata C, SI, SV on the plane $w = 0$. The shaded region is where $V = 7$ vertices and $I = 4$ inflections appear on the deformed curve. Right: the curve $\gamma_1 = (t^3 - 0.018t, t^4 - 0.01t^2)$ with $V = 7$ vertices (full discs) and $I = 4$ inflections (full squares).

and that of SI is

$$SI^0 : 2v(6u - v)^2 = 0$$

(together with C^0).

The trace SV^0 of SV on $w = 0$ consists of C^0 (with multiplicity 3) and three other curves given by $27u - 108u^2 + 108uv + 9v^2 + 32v^3 = 0$ (with multiplicity 2), $u^2 - 2uv - v^3 = 0$ (with multiplicity 3) and $243u(14u - 3v) + P_3^5 = 0$ (with multiplicity 2), where P_3^5 is a polynomial of degree 5 and order 3 in (u, v). The curves C^0, SI^0, SV^0 are as in Figure 3 (left).

The number of inflections and vertices of $\gamma_{\mathbf{u}}$ are constant in the open region in Figure 3 (left), delimited by the curves C^0, SI^0, SV^0. It turns out that in the shaded region in Figure 3 (left), $\gamma_{\mathbf{u}}$ has $V = 7$ vertices and $I = 4$ inflections. In particular, if we take $u = -0.018\alpha$ and $v = -0.01\alpha$, the curve $\gamma_\alpha = (t^3 - 0.018\alpha t, t^4 - 0.01\alpha t^2)$ for α near zero, has $V = 7$ vertices and $I = 4$ inflections (Figure 3 (right)). Indeed, $\iota_\alpha(t) = 12t^4 - 0.156\alpha t^2 + 0.00036\alpha^2$ has four real roots when $\alpha > 0$ which are given by $\pm 0.1\sqrt{\alpha}, \pm 0.05477225575\sqrt{\alpha}$. We solve $\upsilon_\alpha(t) = 0$ for $\alpha = 1$ (this is not a restriction as the number of vertices is constant in the shaded region in Figure 3 (left)) and get the following roots

$$0, \pm 0.03007832731, \pm 0.07738799248, \pm 0.1142378731.$$

Acknowledgment: We thank the referee for valuable suggestions and for a thorough reading of the paper.

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Fabio Scalco Dias, Instituto de Matemática e Computação, Universidade Federal de Itajubá, Avenida BPS 1303, Pinheirinho, CEP 37.500–903, Itajubá, MG, Brazil

E-mail address: `scalco@unifei.edu.br`

Farid Tari, Instituto de Ciências Matemáticas e de Computação - USP, Avenida Trabalhador são-carlense, 400 - Centro, CEP: 13566-590 - São Carlos - SP, Brazil

E-mail address: `faridtari@icmc.usp.br`