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ABOUT THE ALGEBRAIC CLOSURE OF THE FIELD OF POWER SERIES IN SEVERAL VARIABLES IN CHARACTERISTIC ZERO

GUILLAUME ROND

Abstract. We begin this paper by constructing different algebraically closed fields containing an algebraic closure of the field of power series in several variables over a characteristic zero field. Each of these fields depends on the choice of an Abhyankar valuation and is constructed via a generalization of the Newton-Puiseux method for this valuation.

Then we study the Galois group of a polynomial with power series coefficients. In particular by examining more carefully the case of monomial valuations we are able to give several results concerning the Galois group of a polynomial whose discriminant is a weighted homogeneous polynomial times a unit. One of our main results is a generalization of Abhyankar-Jung Theorem for such polynomials, classical Abhyankar-Jung Theorem being devoted to polynomials whose discriminant is a monomial times a unit.

Contents

1. Introduction

When $\mathbb{k}$ is an algebraically closed field of characteristic zero, we can always express the roots of a polynomial with coefficients in the field of power series over $\mathbb{k}$, denoted by $\mathbb{k}(\!(t)\!)$, as formal Laurent series in $t^{\frac{1}{k}}$ for some positive integer k. This result was known by Newton (at least formally see [BK] p. 372) and had been rediscovered by Puiseux in the complex analytic case [Pu1], [Pu2] (see [BK] or [Cu] for a presentation of this result). A modern way to reformulate

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this fact is to say that an algebraic closure of $\Bbbk((t))$ is the field of Puiseux power series $\mathbb{P}$ defined in the following way:

$$\mathbb{P} := \bigcup_{k\in\mathbb{N}} \Bbbk\left(\left(t^{\frac{1}{k}}\right)\right).$$

The proof of this result, called the Newton-Puiseux method, consists essentially in constructing the roots of a polynomial $P(Z) \in \Bbbk[\![t]\!][Z]$ by successive approximations in a similar way to Newton method in numerical analysis. These approximations converge since $\Bbbk\left(\left(t^{\frac{1}{k}}\right)\right)$ is a complete field with respect to the Krull topology.

This result, applied to a polynomial with coefficients in $\Bbbk[\![t]\!]$ defining a germ of algebroid plane curve $(X, 0)$, provides an uniformization of this germ, i.e., a parametrization of this germ.

On the other hand this description of the algebraic closure of $\Bbbk((t))$ describes very easily the Galois group of $\Bbbk((t)) \longrightarrow \mathbb{P}$, since this one is generated by the multiplication of the k-th roots of unity by $t^{\frac{1}{k}}$ for any positive integer k. In particular if an irreducible monic polynomial $P(Z) \in \mathbb{C}[\![t]\!][Z]$ has a root which is a convergent power series in $t^{\frac{1}{k}}$, i.e., an element of $\mathbb{C}\{t^{\frac{1}{k}}\}$, then its other roots are also in $\mathbb{C}\{t^{\frac{1}{k}}\}$ and the coefficients of $P(Z)$ are convergent power series.

When $\Bbbk$ is a characteristic zero field (but not necessarily algebraically closed), we can prove in the same way that an algebraic closure of $\Bbbk((t))$ is

$$\mathbb{P} := \bigcup_{\Bbbk'} \bigcup_{k\in\mathbb{N}} \Bbbk'\left(\left(t^{\frac{1}{k}}\right)\right). \tag{1}$$

where the first union runs over all finite field extensions $\Bbbk \longrightarrow \Bbbk'$.

The aim of this work is double: the first one consists in finding representations of the roots of a polynomial whose coefficients are power series in several variables over a characteristic zero field. Our main results regarding these representations are Theorem 4.2 for Abhyankar valuations and its stronger version for monomial valuations (see Theorem 5.12). The second goal is to describe the Galois group of such polynomials. In particular we concentrate our study to irreducible polynomials that remain irreducible as polynomials with coefficients in the completion of the valuation ring associated to a monomial valuation. Our main result regarding this problem is a generalization of Abhyankar-Jung Theorem to polynomials whose discriminant is weighted homogeneous (see Theorems 7.5 and 7.7).

But let us present in more details the situation, the problems and the results given in this paper. It is tempting to find such a similar expression to (1) for the algebraic closure of the field of power series in n variables, $\Bbbk((x_1, \ldots, x_n))$, for $n \geq 2$. But it appears easily that the algebraic closure of this field admits a really more complicated description and considering only power series depending on $x_1^{\frac{1}{k}}, \ldots, x_n^{\frac{1}{k}}$ is not sufficient. For instance it is easy to see that a square root of $x_1 + x_2$ can not be expressed as such a power series.

Nevertheless there exist positive results in some specific cases, the most famous one being the Abhyankar-Jung theorem:

Theorem (Abhyankar-Jung Theorem)**.** *If $\Bbbk$ is an algebraically closed field of characteristic zero, then any polynomial with coefficients in $\Bbbk[\![x_1, \ldots, x_n]\!]$, whose discriminant has the form $ux_1^{\alpha_1} \cdots x_n^{\alpha_n}$ where $u \in \Bbbk[\![x_1, \ldots, x_n]\!]$ is a unit and $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}_{\geq 0}$, has its roots in $\Bbbk[\![x_1^{\frac{1}{k}}, \ldots, x_n^{\frac{1}{k}}]\!]$ for some positive integer k.*

Such a polynomial is called a quasi-ordinary polynomial and this theorem asserts that the roots of quasi-ordinary polynomials are Puiseux power series in several variables. It provides not only a description of the roots of a quasi-ordinary polynomial but also a description of its Galois group. This result has first being proven by Jung in the complex analytic case, then by Abhyankar in the general case ([Ju], [Ab]).

In the general case, a naive approach involves the use of Newton-Puiseux theorem n times (i.e., the formula (1) for the algebraic closure of $\Bbbk(\!(t)\!)$). For example in the case where $n = 2$ and $\Bbbk$ is an algebraically closed field of characteristic zero, this means that the algebraic closure of $\Bbbk(\!(x_1, x_2)\!)$ is included in

$$\mathbb{L} := \bigcup_{k_2\in\mathbb{N}} \bigcup_{k_1\in\mathbb{N}} \Bbbk\left(\left(x_1^{\frac{1}{k_1}}\right)\right)\left(\left(x_2^{\frac{1}{k_2}}\right)\right).$$

But this field, which is algebraically closed, is very much larger than the algebraic closure of $\Bbbk(\!(x_1, x_2)\!)$ (see [Sa] for some thoughts about this). Moreover the action of the k_1-th and k_2-th roots of unity are not sufficient to generate the Galois group of the algebraic closure since there exist elements of $\Bbbk(\!(x_1)\!)(\!(x_2)\!)$ which are algebraic over $\Bbbk(\!(x_1, x_2)\!)$ but are not in $\Bbbk(\!(x_1, x_2)\!)$. For instance consider

$$x_1\sqrt{1+\frac{x_2}{x_1}} = \sum_{i\in\mathbb{Z}_{\geq 0}} a_i \frac{1}{x_1^{i-1}} x_2^i \in \mathbb{Q}(\!(x_1)\!)(\!(x_2)\!)\backslash\mathbb{Q}(\!(x_1, x_2)\!)$$

for some well chosen rational numbers $a_i \in \mathbb{Q}$, $i \in \mathbb{Z}_{\geq 0}$.

Nevertheless a deeper analysis of the Newton-Puiseux method leads to the fact that it is enough to consider the field of fractions of the ring of elements

$$f = \sum_{(l_1,l_2)\in\mathbb{Z}^2} a_{l_1,l_2} x_1^{\frac{l_1}{k_1}} x_2^{\frac{l_2}{k_2}} \in \mathbb{L}$$

for some k_1, $k_2 \in \mathbb{N}$ whose support is included in a rational strongly convex cone of $\mathbb{R}^2$. Here the support of f is the set

$$\text{Supp}(f) := \{(l_1, l_2) \in \mathbb{Z}^2 \ / \ a_{l_1,l_2} \neq 0\}.$$

This result has been proven by MacDonald [McD] (see also [Go], [Aro], [AI], [SV]). But once more, for any rational strongly convex cone of $\mathbb{R}^2$, denoted by σ, $\mathbb{R}^2_{\geq 0} \subsetneq \sigma$, there exist elements whose support is in σ but that are not algebraic over $\Bbbk(\!(x_1, x_2)\!)$.

One of the main difficulties comes from the fact that $\Bbbk(\!(x_1, \ldots, x_n)\!)$ is not a complete field with respect to the topology induced by the maximal ideal of $\Bbbk[\![x_1, \ldots, x_n]\!]$ (called the Krull topology; it is induced by the following norm $\left|\frac{f}{g}\right| := e^{\text{ord}(g)-\text{ord}(f)}$ for any f, $g \in \Bbbk[\![x_1, \ldots, x_n]\!]$, $g \neq 0$, where $\text{ord}(f)$ is the order of the series f in the usual sense). Indeed, in order to apply the Newton-Puiseux method we have to work with a complete field since the roots are constructed by successive approximations. A very natural idea is to replace $\Bbbk(\!(x_1, \ldots, x_n)\!)$ by its completion. But the completion of $\Bbbk(\!(x_1, \ldots, x_n)\!)$ is not algebraic over $\Bbbk(\!(x_1, \ldots, x_n)\!)$, thus the fields we construct in this way are bigger than the algebraic closure of $\Bbbk(\!(x_1, \ldots, x_n)\!)$. In fact we need to replace the completion of $\Bbbk(\!(x_1, \ldots, x_n)\!)$ by its henselization in the completion. The problem is that there is no general criterion to distinguish elements of the henselization from other elements of the completion. In some sense this problem is analogous to the fact that there is no general criterion to determine if a real number is algebraic or not over the rationals. One more issue is that choosing the Krull topology is arbitrary and we may replace this one by any topology induced by an other norm (or valuation) on this field.

In this paper, we first investigate the use of the Newton-Puiseux method with respect to "tame" valuations (i.e., replace $\Bbbk(\!(x_1, \ldots, x_n)\!)$ by its completion for this valuation). By a "tame" valuation we mean a rank one (or real valued) valuation that satisfies the equality in the Abhyankar inequality (see Definition 2.1). These valuations are called Abhyankar valuations (cf. [ELS]) or quasi-monomial valuations (cf. [FJ]) and, essentially, these are monomial valuations after some sequence of blowing-ups. This is the first part of this work.

If ν is such a valuation, we denote by $\widehat{\mathbb{K}}_\nu$ the completion of $\Bbbk(\!(x_1,\ldots,x_n)\!)$ for the topology induced by this valuation. This field will play the role of $\Bbbk(\!(t)\!)$ in the classical Newton-Puiseux method. Then we have to define the elements that will play the role of $t^{\frac{1}{k}}$. This is where the first difficulty appears, since instead of working over $\widehat{\mathbb{K}}_\nu$, we need to work over the graded ring associated to ν. Both are isomorphic but there is no canonical isomorphism between them. In the case of $\Bbbk(\!(t)\!)$ where t is a single variable, such an isomorphism is defined by identifying the $\Bbbk$-vector space of homogeneous elements of degree i of the graded ring with the $\Bbbk$-vector space of homogeneous polynomials of degree i, i.e., $\Bbbk.t^i$. But this identification depends on the choice of an uniformizer of $\Bbbk[\![t]\!]$. In the case of $\Bbbk(\!(x_1,\ldots,x_n)\!)$ an isomorphism will be determined by the choice of "coordinates" such that the valuation ν is monomial in these coordinates since Abhyankar valuations are monomial valuations after a sequence of blow-ups (cf. Remark 3.6). This is the reason why we restrict our study to these valuations.

Nevertheless when such an isomorphism is chosen, we are able to define the elements that will play the role of $t^{\frac{1}{k}}$, this the aim of Section 3. These elements are called *homogeneous elements with respect to* ν (cf. Definitions 3.15 and 3.17). These are defined as being the roots of weighted homogeneous polynomials with coefficients in the graded ring of $\Bbbk[\![x_1,\ldots,x_n]\!]$ for the valuation ν. If $\Bbbk$ is the field of complex numbers and the weights of the monomial valuation are positive integers, we can think about these homogeneous elements as weighted homogeneous algebraic (multivalued) functions. In fact we can replace $\widehat{\mathbb{K}}_\nu$ by a smaller field, the subfield of $\widehat{\mathbb{K}}_\nu$ whose elements have support included in a finitely generated sub-semigroup of $\mathbb{R}_{\geq 0}$. Let us remark that this field is similar to the field of generalized power series $\cup_\Gamma \mathbb{C}(\!(t^\Gamma)\!)$ where the sum runs over all finitely generated semigroups Γ of $\mathbb{R}_{\geq 0}$ (see [Ri] for instance). Our first result is that the inductive limit of the extensions of $\widehat{\mathbb{K}}_\nu$ by homogeneous elements with respect to ν is algebraically closed (see Theorem 4.2). This field is $\varinjlim_{\gamma_1,\ldots,\gamma_s} \widehat{\mathbb{K}}_\nu[\gamma_1,\ldots,\gamma_s]$ where the limit runs over all subsets $\{\gamma_1,\ldots,\gamma_s\}$ of homogeneous elements with respect to ν and is denoted by $\overline{\mathbb{K}}_\nu$. The field extension $\Bbbk(\!(x_1,\ldots,x_n)\!) \longrightarrow \overline{\mathbb{K}}_\nu$ factors through the field extension $\Bbbk(\!(x_1,\ldots,x_n)\!) \longrightarrow \widehat{\mathbb{K}}_\nu$. While the Galois group of the field extension $\widehat{\mathbb{K}}_\nu \longrightarrow \overline{\mathbb{K}}_\nu$ is easily described by the Galois group of weighted homogeneous polynomials, the Galois group of the algebraic closure of $\Bbbk(\!(x_1,\ldots,x_n)\!)$ in $\widehat{\mathbb{K}}_\nu$ is more complicated. So it is very natural to study irreducible polynomials over $\Bbbk(\!(x_1,\ldots,x_n)\!)$ which remain irreducible over $\widehat{\mathbb{K}}_\nu$, since their Galois groups are described by the Galois groups of weighted homogeneous polynomials. Proposition 4.14 shows that this property is an open property with respect to the topology induced by the chosen valuation. Let us mention that these polynomials are called ν-analytically irreducible polynomials in [Te] and their study is motivated by the construction of key polynomials for Abhyankar valuations (not necessarily of rank 1) in order to prove local uniformization.

Then we investigate more deeply the particular case of monomial valuations. In Section 5, using an idea of Tougeron [To] based on a work of Gabrielov [Ga], for any monomial valuation ν we construct a field, smaller than the ones constructed previously using the Newton-Puiseux method, and containing an algebraic closure of $\Bbbk(\!(x_1,\ldots,x_n)\!)$. The main result (see Theorem 5.12) is a non-archimedean version of Eisenstein Theorem (classical Eisenstein Theorem concerns algebraic power series over $\mathbb{Q}$). The tool we use here is an effective version of the Implicit Function Theorem (see Proposition 5.10). The elements we need to consider are of the form

$$\sum_{i\in\Lambda} \frac{a_i}{\delta^{m(i)}} \tag{2}$$

where the a_i and δ are weighted homogeneous polynomials for the weights corresponding to the given monomial valuation, Λ is a finitely sub-semigroup of $\mathbb{R}_{\geq 0}$, $\nu\left(\frac{a_i}{\delta^{m(i)}}\right) = i$ for all $i \in \Lambda$ and

$i \longmapsto m(i)$ is bounded by a an affine function. In the particular case where the weights are $\mathbb{Q}$-linearly independent this corresponds to the result of MacDonald (see Theorem 6.9).

In Section 7, we use this description of the roots of polynomials with coefficients in $\mathbb{C}\{x_1, \ldots, x_n\}$ to make a topological and complex analytical study of such polynomials whose discriminant is a weighted homogeneous polynomial multiplied by a unit. This study has been inspired by the work of Tougeron in [To] and more particularly by Remarque 2.7 of [To] where it is noticed that the elements of the form (2) define analytic functions on an open domain of $\mathbb{C}^n$ which is the complement of some hornshaped neighborhood of $\{\delta = 0\}$ (see Definition 7.1). This study is possible in the case of monomial valuations whose weights are positive integers. To obtain the same results in the case of general monomial valuations we need to approximate general monomial valuations by divisorial monomial valuations, i.e., monomial valuations whose weights are positive integers. This is the subject of Section 6.

One of the main results we obtain in Section 7 is the following theorem which gives a criterion for an irreducible polynomial over $\Bbbk(\!(x_1, \ldots, x_n)\!)$ to remain irreducible over $\widehat{\mathbb{K}}_\nu$:

Theorem. 7.5 *Let $\Bbbk$ be a field of characteristic zero and $\alpha \in \mathbb{R}^n_{>0}$. Let $\mathbf{x}$ denotes the set of variables $(x_1, \ldots, x_n)$ and let ν_α be the monomial valuation given by the weights α_i. Let $P(Z) \in \Bbbk[\![\mathbf{x}]\!][Z]$ be a monic polynomial whose discriminant is equal to δu where $\delta \in \Bbbk[\mathbf{x}]$ is a weighted homogeneous polynomial for the weights $\alpha_1, \ldots, \alpha_n$ and $u \in \Bbbk[\![\mathbf{x}]\!]$ is a unit. If $P(Z)$ factors as $P(Z) = P_1(Z) \ldots P_s(Z)$ where $P_i(Z)$ is an irreducible monic polynomial of $\Bbbk[\![\mathbf{x}]\!][Z]$, then $P_i(Z)$ is irreducible in $\widehat{V}_\alpha[Z]$ where $\widehat{V}_\alpha$ denotes the completion of the valuation ring of ν_α.*

Then we show that Abhyankar-Jung Theorem is in fact a generalization of this result when the α_i are $\mathbb{Q}$-linearly independent (see Corollary 7.9) and we give the following generalization of Abhyankar-Jung Theorem for polynomials whose discriminant is weighted homogeneous with respect to weights $\alpha_1, \ldots, \alpha_n \in \mathbb{R}_{>0}$:

Theorem. 7.7 *We assume that the hypothesis of Theorem 7.5 are satisfied. Let us set*

$$N := \dim_{\mathbb{Q}}(\mathbb{Q}\alpha_1 + \cdots + \mathbb{Q}\alpha_n).$$

Then there exist $\gamma_1, \ldots, \gamma_N$ integral homogeneous elements with respect to ν_α and a weighted homogeneous polynomial for the weights $\alpha_1, \ldots, \alpha_n$ denoted by $c(\mathbf{x}) \in \Bbbk[\mathbf{x}]$ such that the roots of $P(Z)$ are in $\frac{1}{c(\mathbf{x})}\Bbbk'[\![\mathbf{x}]\!][\gamma_1, \ldots, \gamma_N]$ where $\Bbbk \longrightarrow \Bbbk'$ is a finite field extension.

Indeed in the case $N = n$, i.e., $\alpha_1, \ldots, \alpha_n$ are $\mathbb{Q}$-linearly independent, the only weighted homogeneous polynomials are the monomials and the integral homogeneous elements with respect to ν_α are of the form $\mathbf{x}^\beta$ where $\beta \in \mathbb{Q}^n_{\geq 0}$ (see Remark 3.18). Abhyankar-Jung Theorem simply asserts that we may choose $c(\mathbf{x}) = 1$, a fact that we are able to prove in this case (see Corollary 7.9).

We remark that this result (along with Theorem 7.5) shows that the Galois group of an irreducible monic polynomial with coefficients in $\Bbbk[\![x_1, \ldots, x_n]\!]$ whose discriminant is weighted homogeneous is generated by the Galois group of one weighted homogeneous polynomial (see Remark 7.8).

Finally in Section 8 we give a result of Diophantine approximation (it is just an direct generalization of [Ro1] and [II]) that gives a necessary condition for an element of $\widehat{\mathbb{K}}_\nu$ to be algebraic over $\Bbbk(\!(x_1, \ldots, x_n)\!)$.

At the end we give a list of notations for the convenience of the reader.

Let us mention that this work has been motivated by the understanding of the paper [To] of Tougeron where the study we make for monomial valuations is made in the case of the $(x_1, \ldots, x_n)$-adic valuation of $\Bbbk(\!(x_1, \ldots, x_n)\!)$.

I would like to thank Guy Casale and Adam Parusiński for their answers to my questions regarding the proofs of Lemma 7.4 and Lemma 7.2 respectively. I also thank H. Mourtada for the valuable discussions we had on these problems and his comments that helped to improve the presentation fo this paper. I also thank the referees for their valuable suggestions.

2. Notations and Abhyankar valuations

Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{Z}_{\geq 0}$ the set of non-negative integers. Let $\mathbf{x}$ denote the multi-variable $(x_1, \ldots, x_n)$ where $n \geq 2$. Let $\mathbb{k}$ denote a characteristic zero field. Then $\mathbb{k}[\![\mathbf{x}]\!] = \mathbb{k}[\![x_1, \ldots, x_n]\!]$ denotes the ring of formal power series in n variables over $\mathbb{k}$ and we denote by $\mathbb{K}_n$ its fraction field and by $\mathfrak{m}$ its maximal ideal.

When $(A, \mathfrak{m})$ is a local domain, a *valuation* on A is a function $\nu : A \backslash \{0\} \longrightarrow \Gamma^+$, where Γ is an ordered subgroup of $\mathbb{R}$ and $\Gamma^+ := \Gamma \cap \mathbb{R}_{\geq 0}$, such that

$$\nu(fg) = \nu(f) + \nu(g) \text{ and } \nu(f+g) \geq \min\{\nu(f), \nu(g)\} \quad \forall f, g \in A.$$

We will also impose that $\nu(f) > 0$ if and only if $f \in \mathfrak{m}$. We set $\nu(0) = \infty$ where $\infty > i$ for any $i \in \Gamma$.

Such valuation ν extends to $\mathbb{K}_A$, the fraction field of A, by

$$\nu\left(\frac{f}{g}\right) := \nu(f) - \nu(g)$$

for any f, $g \in A$, $g \neq 0$. We will always assume that $\nu : \mathbb{K}_A \longrightarrow \Gamma$ is surjective. In this case Γ is called the *value group* of ν. The image of $A \backslash \{0\}$ by ν is called the *semigroup of* ν and we denote it by Σ. Then Γ is the group generated by Σ. Let us denote by V_ν the valuation ring of ν:

$$V_\nu := \left\{ \frac{f}{g} \;/\; f, g \in A,\ \nu(f) \geq \nu(g) \right\}.$$

This is a local ring whose maximal ideal, denoted by $\mathfrak{m}_V$, is the set of elements f/g such that $\nu(f/g) > 0$. Its residue field $\frac{V_\nu}{\mathfrak{m}_V}$ is denoted by $\mathbb{k}_\nu$.

Let us denote by $\widehat{V}_\nu$ the completion of V_ν which is defined as follows: For any $\lambda \in \Gamma$ let us set $I_\lambda := \{v \in V_\nu \;/\; \nu(v) \geq \lambda\}$. The family of ideals $\{I_\lambda\}_{\lambda \in \Gamma}$ as a system of neighbourhoods of 0 makes V_ν into a topological ring. Then $\widehat{V}_\nu$ is the completion of V_ν for this topology. We can also remark that the family $\{V_\nu / I_\lambda\}_\lambda$ is an inverse system and its inverse limit is exactly $\widehat{V}_\nu$.

Then $\widehat{V}_\nu$ is an equicharacteristic complete valuation ring and its residue field is isomorphic to $\mathbb{k}_\nu$.

In this paper we will only consider a particular case of valuations, called Abhyankar valuations:

Definition 2.1. A valuation ν is called an *Abhyankar valuation* if the following equality holds:

$$\operatorname{tr.deg}_{\mathbb{k}} \mathbb{k}_\nu + \dim_{\mathbb{Q}} \Gamma \otimes_{\mathbb{Z}} \mathbb{Q} = n.$$

This equality is called the *Abhyankar's Equality.*

Remark 2.2. If $\dim_{\mathbb{Q}} \Gamma \otimes \mathbb{Q} = 1$, then $\Gamma \simeq \mathbb{Z}$. Otherwise Γ is a dense subgroup of $\mathbb{R}$.

Example 2.3. The first example is the $\mathfrak{m}$-adic valuation denoted by ord on the ring $A = \mathbb{k}[\![x]\!]$, and defined by

$$\operatorname{ord}(f) := \max\{n \in \mathbb{N} \;/\; f \in \mathfrak{m}^n\} \qquad \forall f \in \mathbb{k}[\![\mathbf{x}]\!] \backslash \{0\}.$$

In this case its value group Γ is equal to $\mathbb{Z}$ and its semigroup Σ is equal to $\mathbb{Z}_{\geq 0}$.

Example 2.4. Let $\alpha := (\alpha_1, \ldots, \alpha_n) \in (\mathbb{R}_{>0})^n$. Let us denote by ν_α the monomial valuation on $A = \mathbb{k}[\![x]\!]$ defined by $\nu_\alpha(x_i) := \alpha_i$ for $1 \leq i \leq n$. For instance $\nu_{(1,\ldots,1)} = \operatorname{ord}$.

Here we have $\Gamma = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n$ and $\Sigma = \mathbb{Z}_{\geq 0}\alpha_1 \oplus \cdots \oplus \mathbb{Z}_{\geq 0}\alpha_n$.

Example 2.5. If Γ is isomorphic to $\mathbb{Z}$ and ν is an Abhyankar valuation, then ν is a *divisorial valuation*. For such valuation there exists a proper birational dominant map $\pi : X \longrightarrow \operatorname{Spec}(\Bbbk[\![\mathbf{x}]\!])$ and E an irreducible component of the exceptional locus of π such that ν is the composition of π^* with the $\mathfrak{m}_E$-adic valuation of the ring $\mathcal{O}_{X,E}$.

Remark 2.6. Geometrically, an Abhyankar valuation is a monomial valuation at a point lying on the exceptional divisor E of some proper birational map $(Y, E) \longrightarrow (\Bbbk^n, 0)$. More precisely we have the following:

The restriction of ν to $\Bbbk[\mathbf{x}]$ is an Abhyankar valuation with the same value group as ν. We denote it by $\widetilde{\nu}$. By Proposition 2.8 [ELS] there exists a regular local domain $(A, \mathfrak{m}_A)$, an injective morphism

$$\pi : \Bbbk[\mathbf{x}] \longrightarrow A$$

inducing an isomorphism between the fields of fractions and a regular system of parameter z_1, ..., z_r of A such that $\widetilde{\nu}(z_1)$, ..., $\widetilde{\nu}(z_r)$ freely generate the value group of $\widetilde{\nu}$ (or the value group of ν since both are equal). Let us denote by μ the restriction of $\widetilde{\nu}$ to A. Then π induces an isomorphism between V_ν and V_μ. Thus it induces an isomorphism between $\widehat{V}_{\widetilde{\nu}}$ and $\widehat{V}_\mu$. Moreover the completion of A is isomorphic to $\mathbb{L}[\![z_1, \ldots, z_r]\!]$ where $\Bbbk \longrightarrow \mathbb{L}$ is a field extension of transcendence degree $n-r$ (here $\mathbb{L} = \frac{A}{\mathfrak{m}_A}$) and μ extends to a valuation on $\widehat{A}$ which is exactly the monomial valuation that sends z_i onto $\nu(z_i)$ for all i.

Remark 2.7. If $n = 2$, in fact any discrete valuation (i.e., $\Gamma = \mathbb{Z}$) is an Abhyankar valuation [HOV].

Definition 2.8. Let $\alpha \in \mathbb{R}^n_{>0}$. A polynomial $f \in \Bbbk[\![\mathbf{x}]\!]$ is called (α)-homogeneous of degree i is every nonzero monomial $c\mathbf{x}^\beta$ of f satisfies

$$\sum_{k=0}^{n} \alpha_k \beta_k = i$$

or equivalently $\nu_\alpha(c\mathbf{x}^\beta) = i$. This means that f is weighted homogeneous of degree i where x_j has weight α_j for every j.

Example 2.9. Let ν_α be a monomial valuation as before. Any power series $g \in \Bbbk[\![\mathbf{x}]\!]$ can be written $g = \sum_{i\in\Sigma} g_i$ where g_i is a (α)-homogeneous polynomial of degree $i \in \Sigma$. Let us denote by i_0 the least $i \in \Sigma$ such that $g_i \neq 0$. Then we can write formally

$$g = g_{i_0}\left(1 + \sum_{i>i_0} \frac{g_i}{g_{i_0}}\right)$$

and this equality is satisfied in $\widehat{V}_{\nu_\alpha}$. Now if $f \in \Bbbk[\![\mathbf{x}]\!]$, $g \neq 0$ and $\nu(f) \geq \nu(g)$ we can write

$$\frac{f}{g} = \left(\sum_i \frac{f_i}{g_{i_0}}\right)\left(1 + \sum_{i>i_0} \frac{g_i}{g_{i_0}}\right)^{-1}$$

where $f = \sum_i f_i$ where f_i is (α)-homogeneous of degree $i \in \Sigma$.

Thus any element of V_{ν_α} is of the form $\displaystyle\sum_{i\geq 0, i+i_0\in\Sigma} \frac{a_i(\mathbf{x})}{b_i(\mathbf{x})}$ for some $i_0 \in \Sigma$, where $a_i(\mathbf{x})$ and $b_i(\mathbf{x})$ are (α)-homogeneous and $\nu_\alpha\left(\frac{a_i(\mathbf{x})}{b_i(\mathbf{x})}\right) = i$ for any $i \in \mathbb{R}$.

On the other hand $\widehat{V}_{\nu_\alpha}$ is the set of elements of the form $\displaystyle\sum_{i\in\Lambda}\frac{a_i(\mathbf{x})}{b_i(\mathbf{x})}$ where Λ is a finite or countable subset of Γ^+ with no accumulation point, where $a_i(\mathbf{x})$ and $b_i(\mathbf{x})$ are (α)-homogeneous and $\nu_\alpha\left(\frac{a_i(\mathbf{x})}{b_i(\mathbf{x})}\right)=i$ for any $i\in\mathbb{R}$.

Let us denote by $\widehat{\mathbb{K}}_\nu$ the fraction field of $\widehat{V}_\nu$. The valuation ν defines an ultrametric norm on $\widehat{\mathbb{K}}_\nu$, denoted by $|\ |_\nu$, defined by

$$\left|\frac{f}{g}\right|_\nu = e^{\nu(g)-\nu(f)} \qquad \forall f\in\mathbb{k}[\![\mathbf{x}]\!], g\in\mathbb{k}[\![\mathbf{x}]\!]\backslash\{0\}.$$

Then $\widehat{\mathbb{K}}_\nu$ is the completion of $\mathbb{K}_n$ for the topology induced by this norm and this norm (thus the valuation ν) extends canonically on $\widehat{\mathbb{K}}_\nu$. We shall also denote by ν the extension of ν to $\mathbb{K}_\nu$.

Let us denote by $\mathbb{K}_\nu^{\text{alg}}$ the algebraic closure of $\mathbb{K}_n$ in $\widehat{\mathbb{K}}_\nu$. We also denote by V_ν^{alg} the ring of elements of $\widehat{V}_\nu$ which are algebraic over $\mathbb{K}_n$: $V_\nu^{\text{alg}}:=\mathbb{K}_\nu^{\text{alg}}\cap\widehat{V}_\nu$. We have the following lemma:

Lemma 2.10. *The ring V_ν^{alg} is a valuation ring (associated to the valuation ν) and $\mathbb{K}_\nu^{\text{alg}}$ is its fraction field. Moreover $V_\nu\longrightarrow V_\nu^{\text{alg}}$ is the henselization of V_ν in $\widehat{V}_\nu$.*

Proof. If f, $g\in V_\nu^{\text{alg}}$ and $\nu(f)\geq\nu(g)$, then $\frac{f}{g}\in\mathbb{K}_\nu^{\text{alg}}\cap\widehat{V}_\nu=V_\nu^{\text{alg}}$ so V_ν^{alg} is a valuation ring. For $f\in\mathbb{K}_\nu^{\text{alg}}$ there exists $N\in\mathbb{N}$ such that $x_1^Nf\in\mathbb{K}_\nu^{\text{alg}}\cap\widehat{V}_\nu=V_\nu^{\text{alg}}$ since $\nu(x_1^N)>0$. Thus $\mathbb{K}_\nu^{\text{alg}}$ is the fraction field of V_ν^{alg}.

By construction the elements of the henselization of V_ν are algebraic over V_ν. On the other hand every element of $\widehat{V}_\nu$ which is algebraic over V_ν is in the Henselization of V_ν (see Corollary 1.2.1 [M-B]). $\square$

Thus we can summarize the situation with the following commutative diagram, where the bottom part corresponds to the quotient fields of the rings of the upper part:

$$\begin{tikzcd}
\mathbb{k}[\![\mathbf{x}]\!] \arrow[r] & V_\nu \arrow[rr] \arrow[dr] \arrow[dd] & & \widehat{V}_\nu \arrow[dd] \\
& & V_\nu^{\text{alg}} \arrow[ur] \arrow[dd] & \\
& \mathbb{K}_n \arrow[rr] \arrow[dr] & & \widehat{\mathbb{K}}_\nu \\
& & \mathbb{K}_\nu^{\text{alg}} \arrow[ur] &
\end{tikzcd}$$

3. Homogeneous elements with respect to an Abhyankar valuation

3.1. **Graded ring of an Abhyankar valuation and support.** Let A be an integral domain and let $\nu: A\longrightarrow\Gamma^+$ be a valuation where Γ is a subgroup of $\mathbb{R}$. We define $\text{Gr}_\nu A=\bigoplus_{i\in\Gamma^+}\frac{\mathfrak{p}_{\nu,i}}{\mathfrak{p}_{\nu,i}^+}$ where $\mathfrak{p}_{\nu,i}:=\{f\in A/\ \nu(f)\geq i\}$ and $\mathfrak{p}_{\nu,i}^+:=\{f\in A\ /\ \nu(f)>i\}$.

Definition 3.1. Let Γ^+ be a sub-semigroup of $\mathbb{R}_{\geq 0}$. A Γ^+-*graded ring* is a ring A that has a direct sum of abelian groups, $A=\bigoplus_{i\in\Gamma^+}A_i$, such that $A_iA_j\subset A_{i+j}$ for any i, $j\in\Gamma^+$.

For any $j \in \Gamma^+$, $\bigoplus_{i\in\Gamma^+, i\geq j} A_i$ is an ideal of A. This family of ideals as a system of neighborhoods of 0 makes A into a topological whose completion is denoted by $\widehat{A}$ or $\widehat{\bigoplus}_{i\in\Gamma^+} A_i$. The completion of A is the set of elements that are written as a series $\sum_{i\in\Lambda} a_i$ where $\Lambda \subset \Gamma^+$ is either a finite set, either a countable subset of $\mathbb{R}_{>0}$ with no accumulation point, and $a_i \in A_i$ for any $i \in \Lambda$.

A *complete* $(\Gamma^+$-)*graded ring* is the completion of a $(\Gamma^+$-)graded ring.

Remark 3.2. Let A be a complete graded ring. If A_0 is a field then A is a local ring and its maximal ideal is $\mathfrak{m} := \widehat{\bigoplus}_{i>0} A_i$.

For any $a \in A$ we can write $a = \sum_{i\in\Lambda} a_i$ where $a_i \in A_i$ for any i. If $a \neq 0$ let us set $\nu(a) := \min\{i \in \Gamma^+ \ / \ a_i \neq 0\}$. Set $\nu(0) = \infty$. Then ν is an order function, i.e., $\nu(ab) \geq \nu(a)+\nu(b)$ and $\nu(a+b) \geq \min\{\nu(a), \nu(b)\}$. Moreover ν is a valuation if and only if A is an integral domain. The order function ν is called the *order function of* A.

Example 3.3. For a given Abhyankar valuation ν on $\Bbbk[[\mathbf{x}]]$ the rings $\mathrm{Gr}_\nu \Bbbk[[\mathbf{x}]]$ and $\mathrm{Gr}_\nu V_\nu$ are Γ^+-graded rings and $\widehat{\mathrm{Gr}_\nu \Bbbk[[\mathbf{x}]]}$ and $\widehat{\mathrm{Gr}_\nu V_\nu}$ are complete Γ^+-graded rings.

Remark 3.4. The ring $\widehat{\mathrm{Gr}_\nu V_\nu}$ is isomorphic to the ring of generalized power series $\Bbbk_\nu[[t^{\Gamma^+}]]$ where t is a single variable.

Remark 3.5. The elements of $\widehat{\mathrm{Gr}_\nu V_\nu}$ are the elements of the form $\sum_{i\in\Lambda} a_i$ where $a_i \in \frac{\mathfrak{p}_{\nu,i}}{\mathfrak{p}^+_{\nu,i}}$ for all $i \in \Lambda$ where Λ is either a finite set, either a countable subset of $\mathbb{R}_{\geq 0}$ with no accumulation point.

Remark 3.6. Let us consider a monomial valuation ν on $\Bbbk[[\mathbf{x}]]$, let us say $\nu := \nu_\alpha$ where $\alpha \in \mathbb{R}^n_{>0}$.

In this case $\frac{\mathfrak{p}_{\nu,i}}{\mathfrak{p}^+_{\nu,i}}$ is isomorphic to the $\Bbbk$-vector space of rational fractions $\frac{a(\mathbf{x})}{b(\mathbf{x})}$ where $a(\mathbf{x})$ and $b(\mathbf{x})$ are (α)-homogeneous polynomials and $\nu_\alpha\left(\frac{a(\mathbf{x})}{b(\mathbf{x})}\right) = i$. Thus, by Example 2.9 $\widehat{\mathrm{Gr}_\nu V_\nu}$ and $\widehat{V}_\nu$ are $\Bbbk$-isomorphic.

Let us now consider a general Abhyankar valuation ν on $\Bbbk[[\mathbf{x}]]$. By Remark 2.6 there exist a regular local domain $(A, \mathfrak{m}_A)$, an injective morphism

$$\pi : \Bbbk[\mathbf{x}] \longrightarrow A$$

inducing an isomorphism between the fields of fractions and such that, if we denote by μ the restriction of ν to A, the following properties hold:

The extension of μ to $\widehat{A}$ is a monomial valuation (denoted by $\widehat{\mu}$) and π induces isomorphisms $V_\nu \simeq V_\mu$ and $\widehat{V}_\nu \simeq \widehat{V}_{\widehat{\mu}}$.

We have $\widehat{V}_\mu = \widehat{V}_{\widehat{\mu}}$ and $\mathrm{Gr}_\nu V_\nu \simeq \mathrm{Gr}_\mu V_\mu = \mathrm{Gr}_\mu \widehat{V}_\mu$. Thus $\widehat{\mathrm{Gr}_\nu V_\nu}$ and $\widehat{V}_\nu$ are $\Bbbk$-isomorphic by the monomial case.

We can summarize this in the following proposition:

Proposition 3.7. *The choice of a proper birational map π and parameters $z_1, \ldots, z_r$ as in Remark 2.6 yields an isomorphism*

$$\widehat{Gr_\nu V_\nu} \simeq \widehat{V}_\nu.$$

Remark 3.8. A different choice of π and $z_1, \ldots, z_r$ would give an other isomorphism between these two rings.

Definition 3.9. Let $A = \widehat{\bigoplus}_{i\in\Gamma^+} A_i$ be a complete Γ^+-graded ring. Let $a \in A$, $a = \sum_{i\in\Gamma^+} a_i$, $a_i \in A_i$ for any i. The support of a is the subset I of Γ^+ defined by $i \in I$ if and only if $a_i \neq 0$. We denote this set I by $\mathrm{Supp}(a)$.

Definition 3.10. Let ν be an Abhyankar valuation defined on $\Bbbk[\![\mathbf{x}]\!]$. Let us fix a $\Bbbk$-isomorphism φ between $\widehat{\mathrm{Gr}_\nu V_\nu}$ and $\widehat{V}_\nu$ as in Proposition 3.7. Let $a \in \widehat{V}_\nu$ and let us write $\varphi(a) = \sum_{i\in\Gamma^+} a_i$ with $a_i \in \frac{\mathfrak{p}_{\nu,i}}{\mathfrak{p}^+_{\nu,i}}$. The ν-support with respect to φ of a is the subset of Γ^+ defined as

$$\mathrm{Supp}_{\nu,\varphi}(a) := \{i \in \Gamma^+ \ / \ a_i \neq 0\}.$$

When the isomorphism is clear from the context we will skip the mention of φ and denote the ν-support of a by $\mathrm{Supp}_\nu(a)$.

Proposition 3.11. *Let ν be an Abhyankar valuation on $\Bbbk[\![\mathbf{x}]\!]$ and let φ be a $\Bbbk$-isomorphism between $\widehat{Gr_\nu V_\nu}$ and $\widehat{V}_\nu$ as in Proposition 3.7. Then there exists a finitely generated sub-semigroup of $\mathbb{R}_{\geq 0}$, denoted by Λ, such that the ν-support of any element of $\Bbbk[\![\mathbf{x}]\!]$ with respect to φ is included in Λ.*

Proof. By Remark 2.6, we may assume that ν is a monomial valuation. Thus the proposition comes from the following lemma applied to $\Sigma = \mathbb{Z}^n_{\geq 0}$: □

Lemma 3.12. *Let Σ be a strongly convex rational cone of $\mathbb{R}^n$. Let $\alpha \in \mathbb{R}^n_{>0}$ such that $\langle \alpha, \beta\rangle > 0$ for any $\beta \in \Sigma$, $\beta \neq 0$. Then there exists a finitely generated subgroup of $\mathbb{R}_{\geq 0}$, denoted by Λ, such that $\mathrm{Supp}_{\nu_\alpha}(f) \subset \Lambda$ for any $f \in \Bbbk[\![x^\beta, \beta \in \Sigma \cap \mathbb{Z}^n]\!]$ where $\Bbbk[\![x^\beta, \beta \in \Sigma \cap \mathbb{Z}^n]\!]$ denotes the ring of formal Laurent series whose support is included in $\Sigma \cap \mathbb{Z}^n$.*

Proof. By Gordan Lemma, $\Sigma\cap\mathbb{Z}^n$ is a finitely generated semigroup, let us say $\Sigma\cap\mathbb{Z}^n$ is generated by $u_1, \ldots, u_k$. Let us set $r_i := \langle \alpha, u_i\rangle$, $1 \leq i \leq k$. Since any element of $\Sigma \cap \mathbb{Z}^n$ is a $\mathbb{Z}_{\geq 0}$-linear combination of $u_1, \ldots, u_k$, then $\langle \alpha, \beta\rangle$ is a $\mathbb{Z}_{\geq 0}$-linear combination of $r_1, \ldots, r_k$ for any $\beta \in \Sigma \cap \mathbb{Z}^n$. Let us denote by Λ the semigroup of $\mathbb{R}_{\geq 0}$ generated by $r_1, \ldots, r_k$. Then $\mathrm{Supp}_{\nu_\alpha}(f) \subset \Lambda$. □

Remark 3.13. Proposition 3.11 does not imply that the semigroup Σ of ν is finitely generated, which is not true in general for Abhyankar valuations which are not monomial valuations.

3.2. **Homogeneous elements.** From now on we fix an Abhyankar valuation ν on $\Bbbk[\![\mathbf{x}]\!]$ and a $\Bbbk$-isomorphism φ between $\widehat{\mathrm{Gr}_\nu V_\nu}$ and $\widehat{V}_\nu$ induced by an injective birational morphism π as in Remark 3.6 and we will skip to mention it in the following. There are several reasons for that. The first one is that we are interested in effective results on the algebraic elements over $\Bbbk[\![\mathbf{x}]\!]$, thus we are interested by valuations which are given effectively and this will be the case essentially through a map π as in Remark 2.6. In particular we will investigate more deeply the case of monomial valuations and, in this case, the set of variables $x_1, \ldots, x_n$ will be fixed from the beginning, thus φ is quite natural in this case. The last reason is that we will give properties on the ν-support of algebraic elements, and Proposition 3.11 will allow us to consider only elements whose ν-support is included in a finitely generated sub-semigroup of $\mathbb{R}_{>0}$, and this fact does not depend on φ.

Definition 3.14. Let ν be an Abhyankar valuation defined on $\Bbbk[\![\mathbf{x}]\!]$. We will denote by V_ν^{fg} the subset of $\widehat{V}_\nu$ of elements whose ν-support is included in a finitely generated sub-semigroup of $\mathbb{R}_{\geq 0}$ (when we identify $\widehat{V}_\nu$ and $\widehat{\mathrm{Gr}_\nu V_\nu}$ via φ). It is straightforward to check that V_ν^{fg} is a valuation ring. We denote by $\mathbb{K}_\nu^{\mathrm{fg}}$ its fraction field.

Definition 3.15. Let A be a complete Γ^+-graded domain and let ν be its order function (which is a valuation since A is a domain). A *homogeneous element with respect to* ν is an element γ of a finite extension of A such that its minimal polynomial $Q(Z)$ is irreducible in $A[Z]$ and has the following form:

$$Z^q + g_1 Z^{q-1} + \cdots + g_q$$

where $g_k \in A_{i(k)}$ with $i(k) \in \Gamma$ for $1 \leq k \leq q$ such that $k.i(l) = l.i(k)$ for all k and l. In this case $d := \frac{i(k)}{k} \in \frac{1}{q!}\Gamma$ is called the *order* of γ.

Example 3.16. Let $\alpha \in \mathbb{R}^n_{>0}$ such that $\dim_{\mathbb{Q}}(\mathbb{Q}\alpha_1 + \cdots + \mathbb{Q}\alpha_n) = n$ i.e., the α_i are $\mathbb{Q}$-linearly independent. Then the value group of ν_α is the following group:

$$\Gamma = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n$$

and for any $i \in \Gamma$ there exists a unique $(\beta_{i,1}, \ldots, \beta_{i,n}) \in \mathbb{Z}^n$ such that

$$i = \beta_{i,1}\alpha_1 + \cdots + \beta_{i,n}\alpha_n.$$

Thus if $i \in \Gamma^+$ this means that $\frac{\mathfrak{p}_{\nu_\alpha,i}}{\mathfrak{p}^+_{\nu_\alpha,i}}$ is isomorphic to the one dimensional $\Bbbk_{\nu_\alpha}$-vector space generated by $x_1^{\beta_{i,1}} \cdots x_n^{\beta_{i,n}}$. Let us remark here that $\Bbbk_{\nu_\alpha}$ is equal to $\Bbbk$ since the α_i are $\mathbb{Q}$-linearly independent. Thus if $g_k \in \frac{\mathfrak{p}_{\nu_\alpha,dk}}{\mathfrak{p}^+_{\nu_\alpha,dk}}$ for $1 \leq k \leq q$ we have that

$$Z^q + g_1 Z^{q-1} + \cdots + g_q = x_1^{\beta_{qd,1}} \cdots x_n^{\beta_{qd,n}} \left(T^q + g'_1 T^{q-1} + \cdots + g'_q\right)$$

where $Z = x_1^{\beta_{d,1}} \cdots x_n^{\beta_{d,n}} T$ and $g'_1, \ldots, g'_q \in \Bbbk$. If $g_q \neq 0$ then $\beta_{qd,j} \in \mathbb{Z}$ for any j but $\beta_{d,j} = \frac{\beta_{qd,j}}{d}$ may not be an integer. Then the roots of $T^q + g'_1 T^{q-1} + \cdots + g'_q$ are algebraic over $\Bbbk$. Thus homogeneous elements with respect to ν_α are of the form $c\mathbf{x}^\beta$ where c is algebraic over $\Bbbk$ and $\beta \in \mathbb{Q}^n$ with $\langle \alpha, \beta \rangle := \alpha_1\beta_1 + \cdots + \alpha_n\beta_n \geq 0$.

Definition 3.17. Let ν be an Abhyankar valuation on $\Bbbk[\![\mathbf{x}]\!]$. Let $A = \widehat{\mathrm{Gr}_\nu V_\nu}$ and γ be a homogeneous element with respect to ν. Let $Q(Z)$ be its minimal polynomial:

$$Q(Z) = Z^q + g_1 Z^{q-1} + \cdots + g_q$$

with $g_k \in \frac{\mathfrak{p}_{\nu,dk}}{\mathfrak{p}^+_{\nu,dk}}$ for $1 \leq k \leq q$. We say that γ is an *integral homogeneous element with respect to* ν if g_k is the image of an element of $\Bbbk[\![\mathbf{x}]\!] \cap \mathfrak{p}_{\nu,dk}$ for all k.

Example 3.18. Let $\alpha \in \mathbb{R}^n_{>0}$ such that $\dim_{\mathbb{Q}}(\mathbb{Q}\alpha_1 + \cdots + \mathbb{Q}\alpha_n) = n$ and let us keep the notations of Example 3.16. Then γ is an integral homogeneous element with respect to ν_α if $g_k \in \frac{\Bbbk[\![\mathbf{x}]\!] \cap \mathfrak{p}_{\nu_\alpha,dk}}{\Bbbk[\![\mathbf{x}]\!] \cap \mathfrak{p}^+_{\nu_\alpha,dk}}$ for $1 \leq k \leq q$. Since $g_q \neq 0$ this means that $\beta_{qd,j} \in \mathbb{Z}_{\geq 0}$ for all j. Thus integral homogeneous elements with respect to ν_α are of the form $c\mathbf{x}^\beta$ where c is algebraic over $\Bbbk$ and $\beta \in \mathbb{Q}^n_{\geq 0}$.

Example 3.19. Let ν be an Abhyankar valuation on $\Bbbk[\![\mathbf{x}]\!]$ and let us assume that $\Bbbk$ is not algebraically closed. Let c be in the algebraic closure of $\Bbbk$, $c \notin \Bbbk$. Then c is a root of a polynomial equation with coefficients in $\Bbbk$ and since $\Bbbk$ is a subfield of $\Bbbk_\nu$, this shows that c is an integral homogeneous element of order 0 with respect to ν.

Remark 3.20. Let ν be an Abhyankar valuation on $\Bbbk[\![\mathbf{x}]\!]$ and let γ be a homogeneous element of order d with respect to ν. Let us denote by $Q(Z)$ its minimal polynomial, say

$$Q(Z) = Z^q + g_1 Z^{q-1} + \cdots + g_q$$

where $g_k \in \frac{\mathfrak{p}_{\nu_\alpha,dk}}{\mathfrak{p}^+_{\nu_\alpha,dk}}$ for $1 \leq k \leq q$. Each g_k is the image in $\widehat{\mathrm{Gr}_\nu V_\nu}$ of some fraction $\frac{f_k}{h_k}$ where f_k, $h_k \in \Bbbk[\![\mathbf{x}]\!]$. Set $h := h_1 \ldots h_k$, let h_0 be the image of h in $\widehat{\mathrm{Gr}_\nu V_\nu}$ and set $\gamma' := h_0\gamma$. Then γ' is a homogeneous element annihilating $Z^q + g'_1 Z^{q-1} + \cdots + g'_q$ where g'_k is the image of $\frac{f_k}{h_k} h^{k-1} \in \Bbbk[\![\mathbf{x}]\!]$ in $\widehat{\mathrm{Gr}_\nu V_\nu}$, thus it is an integral homogeneous element with respect to ν. Moreover we have

$$\mathrm{Frac}(\widehat{\mathrm{Gr}_\nu V_\nu})[\gamma] = \mathrm{Frac}(\widehat{\mathrm{Gr}_\nu V_\nu})[\gamma'].$$

Definition 3.21. Let ν be an Abhyankar valuation on $\mathbb{k}[\![\mathbf{x}]\!]$,

$$P(Z_1,\ldots,Z_m)\in\widehat{V}_\nu[Z_1,\ldots,Z_m]$$

and $\mathbf{d}:=(d_1,\ldots,d_m)\in\mathbb{R}^m_{>0}$. One says that $P(Z_1,\ldots,Z_m)$ is $(\nu,\mathbf{d})$-homogeneous of degree $d\in\mathbb{R}$ if for every nonzero monomial $gZ_1^{\beta_1}\ldots Z_m^{\beta_m}$ of $P(Z)$ one has $g\in\frac{\mathfrak{p}_{\nu,k}}{\mathfrak{p}^+_{\nu,k}}$ with

$$k+\beta_1d_1+\cdots+\beta_md_m=d.$$

Remark 3.22. Let ν be an Abhyankar valuation on $\mathbb{k}[\![\mathbf{x}]\!]$. Let γ be a homogeneous element of order d with respect to ν. Let us denote by $P(Z)$ its minimal monic polynomial. Then $P(Z)$ is (ν,d)-homogeneous.

Conversely if $P(Z)\in\widehat{V}_\nu[Z]$ satisfies $P(\gamma)=0$ for some element γ algebraic over $\widehat{V}_\nu$, and if $P(Z)$ is a nonzero (ν,d)-homogeneous, then the divisors of P in $\widehat{V}_\nu[Z]$ are also (ν,d)-homogeneous, thus the minimal polynomial of γ is (ν,d)-homogeneous. Hence γ is a homogeneous element of order d with respect to ν.

Lemma 3.23. *Let γ_1 and γ_2 be two homogeneous elements of order d_1 and d_2 respectively with respect to the valuation ν and let $k\in\mathbb{Z}$. Then*

i) *γ_1^k is homogeneous of order kd_1,*
ii) *if $e_1d_1=e_2d_2$ with $e_1,e_2\in\mathbb{N}$, then $\gamma_1^{e_1}+\gamma_2^{e_2}$ is homogeneous of order d_1e_1,*
iii) *$\gamma_1\gamma_2$ is homogeneous of order d_1+d_2.*

Proof. If γ is homogeneous of order $d\in\mathbb{Q}$, then γ^k, $k\in\mathbb{N}$, is homogeneous of order kd. Indeed a polynomial having γ^k as a root is $Q(Z):=\mathrm{Res}_X(P(X),Z-X^k)$ where P is the minimal monic polynomial of γ over $\mathbb{k}(\mathbf{x})$. But $P(X)$ is (ν,d)-homogeneous and $Z-X^k$ is (ν,d,kd)-homogeneous. Thus $Q(Z)$ is (ν,d,kd)-homogeneous, hence (ν,kd)-homogeneous since it does not depend on X. This proves that γ^k is homogeneous of order kd.

In order to show ii) we may assume, by i), that γ_1 and γ_2 are homogeneous of same order $d=e_1d_1=e_2d_2$. Let us denote by $P_1(Z)$ and $P_2(Z)$ the minimal monic polynomials of γ_1 and γ_2 respectively. Then $Q(Z):=\mathrm{Res}_X(P_1(Z-X),P_2(X))$ is (ν,d,d)-homogeneous, thus (ν,d)-homogeneous since it does not depend on X. Since $Q(\gamma_1+\gamma_2)=0$, $\gamma_1+\gamma_2$ is homogeneous of order d.

In order to show iii) let us denote by $P_1(X)$ the minimal monic polynomial of γ_1 (this is a (ν,d_1)-homogeneous polynomial) and $P_2(Z)$ the minimal monic polynomial of γ_2 ((ν,d_2)-homogeneous). Let us denote by k the degree in Z of $P_1(Z)$ and set $R(X,Y):=X^kP_1(Y/X)$. Then $\gamma_1\gamma_2$ is a root of $Q(Z):=\mathrm{Res}_X(R(X,Z),P_2(X))$. Moreover $R(X,Z)$ is (ν,d_2,d_1+d_2)-homogeneous. Thus $Q(Z)$ is (ν,d_1+d_2)-homogeneous, which proves that $\gamma_1\gamma_2$ is homogeneous of order d_1+d_2. □

Lemma 3.24. *Let $P(T,Z)$ be a nonzero (ν,d_1,d_2)-homogeneous polynomial of $\widehat{V}_\nu[T,Z]$ and let γ_1 be a homogeneous element of order d_1 with respect to ν. If an element γ_2 belonging to a finite extension of $\mathbb{k}(\mathbf{x})$ satisfies $P(\gamma_1,\gamma_2)=0$, then γ_2 is a homogeneous element of order d_2 with respect to ν.*

Proof. Let $Q(T)\in\widehat{V}_\nu[T]$ be a nonzero (ν,d_1)-homogeneous polynomial such that $Q(\gamma_1)=0$. Let us denote $R(Z)=\mathrm{Res}_T(P(T,Z),Q(T))$. Then $R(Z)$ is a (ν,d_2)-homogeneous polynomial such that $R(\gamma_2)=0$. This proves the result. □

Remark 3.25. Let A be a complete Γ^+-graded integral domain, let say A is the completion of $A':=\bigoplus_{i\in\Gamma^+}A_i$, and let ν be its order valuation. Let $Q(Z)$ be an irreducible polynomial of $A[Z]$ having the following form:

$$Z^q+g_1Z^{q-1}+\cdots+g_q$$

where $g_k \in A_{dk}$ for $1 \leq k \leq q$ and $d \in \frac{1}{q!}\Gamma^+$. The ring $B := \frac{A[Z]}{(Q(Z))}$ is an integral domain and ν extends to a valuation of this ring by defining $\nu(Z) := d$ and

$$\nu\left(\sum_{i=0}^{q-1} a_i Z^i\right) := \inf_i \{\nu(a_i) + di\}.$$

Let us set $B' := A'[Z]/(Q(Z))$. Then B is a complete $\frac{1}{q!}\Gamma$-graded domain since B is the completion of

$$B' = \bigoplus_{i\in\Gamma^+} \bigoplus_{\substack{0\leq j\leq \min\{\lfloor \frac{i}{d}\rfloor, q\} \\ j\in\frac{1}{d!}\Gamma^+}} A_{i-dj}Z^j.$$

Definition 3.26. Let γ be an algebraic element over A whose minimal polynomial is the polynomial $Q(Z)$ as in the previous remark. Then the integral domain B constructed in the previous remark is denoted by $A[\gamma]$.

By induction, we can define $A[\gamma_1, \ldots, \gamma_s]$, where γ_{i+1} is a homogeneous element over $A[\gamma_1, \ldots, \gamma_i]$ for $1 \leq i < s$. When ν is an Abhyankar valuation on $\Bbbk[\![\mathbf{x}]\!]$ and $A = \widehat{V}_\nu$, V_ν^{alg} or V_ν^{fg}, the valuation ν extends to $A[\gamma_1, \ldots, \gamma_i]$ as in Remark 3.25. Then we denote by $A[\langle\gamma_1, \ldots, \gamma_s\rangle]$ the valuation ring associated to the order valuation of $A[\gamma_1, \ldots, \gamma_s]$. In this case the elements of $A[\langle\gamma_1, \ldots, \gamma_s\rangle]$ are the elements which are finite sums of terms of the form $b\gamma_1^{j_1}...\gamma_s^{j_s}$ where $b \in \mathrm{Frac}(A)$ and $\nu(b) \geq -(j_1\nu(\gamma_1) + \cdots + j_s\nu(\gamma_s))$.

Definition 3.27. If ν is an Abhyankar valuation we denote by

$$\overline{V}_\nu := \varinjlim_{\gamma_1,\ldots,\gamma_s} \widehat{V}_\nu[\langle\gamma_1, \ldots, \gamma_s\rangle]$$

the direct limit over all subsets $\{\gamma_1, \ldots, \gamma_s\}$ of homogeneous elements with respect to ν and by $\overline{\mathbb{K}}_\nu$ its fraction field. By Remark 3.20 we may restrict the limit over the subsets of integral homogeneous elements.

In the same way we define

$$\overline{V}_\nu^{\mathrm{fg}} := \varinjlim_{\gamma_1,\ldots,\gamma_s} V_\nu^{\mathrm{fg}}[\langle\gamma_1, \ldots, \gamma_s\rangle],$$

$$\overline{V}_\nu^{\mathrm{alg}} := \varinjlim_{\gamma_1,\ldots,\gamma_s} V_\nu^{\mathrm{alg}}[\langle\gamma_1, \ldots, \gamma_s\rangle],$$

the limits being taken over all subsets $\{\gamma_1, ..., \gamma_s\}$ of (integral) homogeneous elements with respect to ν, and we denote by $\overline{\mathbb{K}}_\nu^{\mathrm{fg}}$ and $\overline{\mathbb{K}}_\nu^{\mathrm{alg}}$ their respective fraction fields.

The following result provides an upper bound on the number of homogeneous elements we need to consider:

Proposition 3.28. *Let ν be an Abhyankar valuation on $\Bbbk[\![\mathbf{x}]\!]$ and let Γ denote its value group. Set $N := \dim_{\mathbb{Q}} \Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ and let $\gamma_1, \ldots, \gamma_s$ be homogeneous elements with respect to ν. Then there exist integral homogeneous elements $\gamma_1', \ldots, \gamma_N'$ with respect to ν such that*

$$\widehat{V}_\nu[\langle\gamma_1, \ldots, \gamma_s\rangle] = \widehat{V}_\nu[\langle\gamma_1', \ldots, \gamma_N'\rangle].$$

This equality remains true if we replace $\widehat{V}_\nu$ by V_ν^{alg} or V_ν^{fg}.

Proof. We will prove this proposition by induction on s. Let $\gamma_1, \ldots, \gamma_{N+1}$ be nonzero homogeneous elements with respect to ν. Let d_i be the order of γ_i, for $1 \leq i \leq N+1$. By assumption on

N the d_i are $\mathbb{Q}$-linearly dependent. Thus, after a permutation of the g_i , there exists an integer $1\leq l\leq N$ and integers $p_i\in\mathbb{Z}_{\geq 0}$, $q_i\in\mathbb{N}$ for all $1\leq i\leq N+1$, such that

$$\frac{p_1}{q_1}d_1+\cdots+\frac{p_l}{q_l}d_l=\frac{p_{l+1}}{q_{l+1}}d_{l+1}+\cdots+\frac{p_{N+1}}{q_{N+1}}d_{N+1}. \tag{3}$$

Set $r_i:=\frac{p_1\cdots p_{N+1}}{p_i}$ for $1\leq i\leq N+1$. Let us denote $\gamma_i':=\gamma_i^{\frac{1}{q_ir_i}}$. Then we have

$$\widehat{V}_\nu[\langle\gamma_1,\ldots,\gamma_{N+1}\rangle]\subset\widehat{V}_\nu[\langle\gamma_1',\ldots,\gamma_{N+1}'\rangle].$$

By (3) and Lemma 3.23, $\gamma_1'\cdots\gamma_l'$ and $\gamma_{l+1}'\cdots\gamma_{N+1}'$ are homogeneous elements of same order. By the Primitive Element Theorem there exists $c\in\Bbbk$ such that

$$\Bbbk(\mathbf{x})[\gamma_1'\cdots\gamma_l',\gamma_{l+1}'\cdots\gamma_{N+1}']=\Bbbk(\mathbf{x})[\gamma_1'\cdots\gamma_l'+c\gamma_{l+1}'\cdots\gamma_{N+1}'].$$

Moreover $\gamma:=\gamma_1'\cdots\gamma_l'+c\gamma_{l+1}'\cdots\gamma_{N+1}'$ is a homogeneous element with respect to ν of same order as $\gamma_1'\cdots\gamma_l'$ and $\gamma_{l+1}'\cdots\gamma_{N+1}'$ by Lemma 3.23. Since

$$\Bbbk(\mathbf{x})[\gamma_1',\ldots,\gamma_l']=\Bbbk(\mathbf{x})[\gamma_1',\ldots,\gamma_{l-1}',\gamma_1'\cdots\gamma_l']$$

and

$$\Bbbk(\mathbf{x})[\gamma_{l+1}',\ldots,\gamma_{N+1}']=\Bbbk(\mathbf{x})[\gamma_{l+1}',\ldots,\gamma_N',\gamma_{l+1}'\cdots\gamma_{N+1}'],$$

we have

$$\Bbbk(\mathbf{x})[\gamma_1',\ldots,\gamma_{N+1}']=\Bbbk(\mathbf{x})[\gamma_1',\ldots,\gamma_{l-1}',\gamma_{l+1}',\ldots,\gamma_N',\gamma].$$

Thus γ_l' is a finite sum of products of elements $a_i(\mathbf{x})\in\Bbbk(\mathbf{x})$ and powers of γ_1', $\ldots$, γ_{l-1}', γ_{l+1}', $\ldots$, γ_N', γ and by homogeneity we may assume that $a_i(\mathbf{x})$ are (ν)-homogeneous. Thus

$$\widehat{V}_\nu[\langle\gamma_1',\ldots,\gamma_{N+1}'\rangle]=\widehat{V}_\nu[\langle\gamma_1',\ldots,\gamma_{l-1}',\gamma_{l+1}',\ldots,\gamma_N',\gamma\rangle].$$

By Remark 3.20 we may assume that the γ_i' are integral homogeneous elements.

The proof is the same if we replace $\widehat{V}_\nu$ by V_ν^{alg} or V_ν^{fg}.

□

4. Newton method and algebraic closure of $\Bbbk[[\mathbf{x}]]$ with respect to an Abhyankar valuation

4.1. Newton method.

Lemma 4.1. *Let $(A,\mathfrak{m})$ be a complete graded local ring. Let B be the set of the elements of A whose support is included in a finitely generated sub-semigroup of $\mathbb{R}_{\geq 0}$. Then B is a Henselian local domain.*

Proof. Let us prove that B is a ring: let b_1 and b_2 be two elements of B whose supports are included in Λ_1 and Λ_2 respectively. Thus we can write $b_i=\sum_{j\in\Lambda_i}b_{i,j}$ where $b_{i,j}$ is a homogeneous element of degree j for any $i=1,2$ and $j\in\Lambda_1$ or Λ_2. Let Λ be the finitely generated sub-semigroup of $\mathbb{R}_{\geq 0}$ generated by Λ_1 and Λ_2. Then $\operatorname{Supp}(b_1+b_2)$ and $\operatorname{Supp}(b_1b_2)$ are included in Λ. This proves that B is a ring. Since $B\subset A$, B is a domain.

It is clear that $\mathfrak{m}\cap B$ is an ideal of B. If $b\in B\backslash(\mathfrak{m}\cap B)$, then there exists $a\in A$ such that $ab=1$. Let us write $b=\sum_{i\in\Lambda}b_i$ where b_i is homogeneous of degree i and Λ is a finitely generated sub-semigroup of $\mathbb{R}_{\geq 0}$. Since $b\notin\mathfrak{m}$, then $b_0\neq 0$. In this case we have

$$a=b^{-1}=\frac{1}{b_0}\left(1+\sum_{i\in\Lambda\backslash\{0\}}\frac{b_i}{b_0}\right)^{-1}=\frac{1}{b_0}\sum_{k=1}^{\infty}(-1)^k\left(\sum_{i\in\Lambda\backslash\{0\}}\frac{b_i}{b_0}\right)^k.$$

Thus $\operatorname{Supp}(a)\subset\Lambda$. This proves that B is a local ring with maximal ideal $\mathfrak{m}\cap B$.

Now let $P(Z) \in B[Z]$, such that $P(0) \in \mathfrak{m} \cap B$ and $P'(0) \notin \mathfrak{m}$. We denote by ν the order function of A, i.e., if $a \in A$, $a \neq 0$, $a = \sum_i a_i$ where a_i is homogeneous of degree i, $\nu(a) := \inf\{i \ / \ a_i \neq 0\}$ and the initial term of a is $\mathrm{in}(a) := a_{\nu(a)}$. Since A is a complete local ring it is a Henselian local ring and there exists $a \in \mathfrak{m}$ such that $P(a) = 0$. We can construct a by using the fact that

$$P(Z) = P(0) + P'(0)Z + Q(Z)Z^2 \tag{4}$$

where $Q(Z) \in B[Z]$. Indeed, let Λ denote a finitely generated sub-semigroup of $\mathbb{R}_{\geq 0}$ containing the supports of all the coefficients of $P(Z)$. In this case $a_1 := \mathrm{in}(a) = -\frac{\mathrm{in}(P(0))}{\mathrm{in}(P'(0))}$ is a homogeneous element of degree $d_1 \in \Lambda$, $d_1 > 0$. If we set $P_1(Z) := P(Z + a_1)$, we see that

$$\nu(P_1(0)) = \nu(P(a_1)) > d_1,$$

$P_1'(0) = P'(0) = 0$ and $a - a_1$ is the solution of $P_1(Z) = 0$ given by the Hensel Lemma. Then we replace P by P_1 in Equation (4) and repeat the same argument, using the fact that the coefficients of $P_1(Z)$ have support included in Λ. Thus we see that $\mathrm{in}(a - a_1) = -\frac{\mathrm{in}(P_1(0))}{\mathrm{in}(P'(0))}$ is a homogeneous element of degree $d_2 \in \Lambda$, $d_2 > d_1$. We repeat this operation a countable number of times (since Λ is countable) in order to construct a and we see that $\mathrm{Supp}(a) \subset \Lambda$.

□

Now we can prove the following theorem:

Theorem 4.2. *Let $\Bbbk$ be a field of characteristic zero and ν be an Abhyankar valuation of $\Bbbk[\![\mathbf{x}]\!]$. Let $N := \dim_{\mathbb{Q}} \Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$. Let*

$$P(Z) \in V_\nu^{\mathrm{fg}}[\langle \gamma_1, \ldots, \gamma_N \rangle][Z]$$

(resp. $\widehat{V}_\nu[\langle \gamma_1, \ldots, \gamma_N \rangle][Z]$) be a monic polynomial of degree d where γ_i is a homogeneous element with respect to ν for $1 \leq i \leq N$. Then there exist integral homogeneous elements γ_1', ..., γ_N' such that the roots of $P(Z)$ are in $V_\nu^{\mathrm{fg}}[\langle \gamma_1', \ldots, \gamma_N' \rangle]$ (resp. $\widehat{V}_\nu[\langle \gamma_1', \ldots, \gamma_N' \rangle]$).

Proof. Let us prove the case $P(Z) \in V_\nu^{\mathrm{fg}}[\langle \gamma_1, \ldots, \gamma_N \rangle][Z]$. We write

$$P(Z) = Z^d + a_1 Z^{d-1} + \cdots + a_d.$$

By replacing Z by $Z - \frac{1}{d}a_1$ we can assume that $a_1 = 0$. Let i_0 be an integer such that

$$\frac{\nu(a_{i_0})}{i_0} \leq \frac{\nu(a_i)}{i}, \quad \text{for every } 2 \leq i \leq d.$$

Let γ be a i_0th root of $\mathrm{in}_\nu(a_{i_0})$, i.e., γ is a homogeneous element such that $\gamma^{i_0} = \mathrm{in}_\nu(a_{i_0})$. By the definition of i_0, for every $2 \leq i \leq d$ we can write

$$a_i = \gamma^i a_i'$$

with $a_i' \in V_\nu^{\mathrm{fg}}[\langle \gamma_1, \ldots, \gamma_N, \gamma \rangle]$. Then we have

$$P(\gamma Z) = \gamma^d Z^d + \gamma^{d-2} a_2 Z^{d-2} + \cdots + a_d = \gamma^d \left(Z^d + a_2' Z^{d-2} + \cdots + a_d'\right)$$

Let $S(Z) := Z^d + a_2' Z^{d-2} + \cdots + a_d'$ and let $\overline{S}(Z)$ be the image of $S(Z)$ in the residue field

$$\mathbb{L} = V_\nu^{\mathrm{fg}}[\langle \gamma_1, \ldots, \gamma_s, \gamma \rangle]/\mathfrak{m}$$

where $\Bbbk_\nu \longrightarrow \mathbb{L}$ is finite and $\mathfrak{m}$ is the maximal ideal of $V_\nu^{\mathrm{fg}}[\langle \gamma_1, \ldots, \gamma_s, \gamma \rangle]$. If $\overline{S}(Z) = (Z + \overline{a})^d$ where $\overline{a} \in \mathbb{L}$, since $a_1 = 0$ and $\mathrm{char}(\mathbb{L}) = 0$, this would imply $\overline{a} = 0$. But $\overline{S}(Z) \neq Z^d$ since its coefficient of Z^{d-i_0} is nonzero . Thus we can factor $\overline{S}(Z) = \overline{S}_1(Z)\overline{S}_2(Z)$ such that $\overline{S}_1(Z)$ and $\overline{S}_2(Z)$ are coprime monic polynomials in $\mathbb{L}[\gamma'][Z]$ where γ' is algebraic over $\mathbb{L}$, i.e., γ' is a homogeneous element of degree 0 with respect to ν. Since $V_\nu^{\mathrm{fg}}[\langle \gamma_1, \ldots, \gamma_N, \gamma, \gamma' \rangle]$ is a Henselian local ring by Lemma 4.1, by Hensel Lemma the polynomial $S(Z)$ factors as $S(Z) = S_1(Z)S_2(Z)$

where the images of $S_1(Z)$ and $S_2(Z)$ in $V_\nu^{\mathrm{fg}}[\langle\gamma_1,\ldots,\gamma_N,\gamma,\gamma'\rangle]$ are $\overline{S}_1(Z)$ and $\overline{S}_2(Z)$ and the ν-support of the coefficients of $S_1(Z)$ and $S_2(Z)$ are contained in a finitely generated sub-semigroup of $\mathbb{R}_{\geq 0}$.

Since $\deg_Z(S_1(Z))$, $\deg_Z(S_2(Z)) < d = \deg_Z(P(Z))$, the theorem is proven by induction on d by using Proposition 3.28 and Remark 3.20.

The case $P(Z) \in \widehat{V}_\nu[\langle\gamma_1,\ldots,\gamma_N\rangle][Z]$ is proven in a similar way by using the fact that $\widehat{V}_\nu[\langle\gamma_1,\ldots,\gamma_N,\gamma,\gamma'\rangle]$ is a complete local ring, thus a Henselian local ring. □

Remark 4.3. The proof of this theorem is what we call the *Newton-Puiseux method*. Usually the term of Newton-Puiseux method is used when one compute the roots of a monic polynomial with coefficients in the ring of power series in one variable: one root is constructed by computing step by step its coefficients. The fact that the ring of formal power series is a complete local ring allows to conclude that this process converges. But when we want to find roots of a polynomial in a local ring that is not complete but only Henselian, it is more convenient to use the Hensel Lemma as we have done here. The proof we used here appeared for the first time in [BM] (to the knowledge of the author). Of course if ν is a divisorial valuation $\widehat{V}_\nu$ is isomorphic to the ring of formal power series in one variable over the residue field $\Bbbk_\nu$ and the previous theorem may be proven by using the classical Newton-Puiseux method.

Corollary 4.4. *The field $\overline{\mathbb{K}}_\nu^{\mathrm{fg}}$ (resp. $\overline{\mathbb{K}}_\nu$) is algebraically closed and it is the algebraic closure of $\mathbb{K}_\nu^{\mathrm{fg}}$ (resp. $\widehat{\mathbb{K}}_\nu$).*

Proof. Let $P(Z) \in \overline{\mathbb{K}}_\nu^{\mathrm{fg}}[Z]$ be an irreducible polynomial. By multiplying $P(Z)$ by an element of V_ν^{fg}, we may assume that

$$P(Z) \in V_\nu^{\mathrm{fg}}[\langle\gamma_1,\ldots,\gamma_N\rangle][Z]$$

for some homogeneous elements $\gamma_1, \ldots, \gamma_N$ with respect to ν. We write $P(Z) = a_dZ^d + \cdots + a_0$, $a_i \in V_\nu^{\mathrm{fg}}[\langle\gamma_1,\ldots,\gamma_N\rangle]$, $0 \leq i \leq d$. We set $Q(Z) := a_d^{d-1}P(Z/a_d)$. Then $Q(Z)$ is a monic polynomial of $V_\nu^{\mathrm{fg}}[\langle\gamma_1,\ldots,\gamma_N\rangle][Z]$ and if z is a root of $Q(Z)$, then $\frac{z}{a_d}$ is a root of $P(Z)$. Hence the result comes from Theorem 4.2.

The assertion concerning $\overline{\mathbb{K}}_\nu$ is proven similarly. □

We have the similar result for $\overline{\mathbb{K}}^{\mathrm{alg}}$:

Lemma 4.5. *The algebraic closure of $\mathbb{K}_n$ in $\overline{\mathbb{K}}_\nu$ is equal to $\overline{\mathbb{K}}_\nu^{\mathrm{alg}}$. In particular $\overline{\mathbb{K}}_\nu^{\mathrm{alg}}$ is algebraically closed.*

Proof. Let $\gamma_1, \ldots, \gamma_s$ be homogeneous elements with respect to ν. Let us denote by q_{i+1} the degree of the minimal polynomial of γ_{i+1} over $\mathbb{K}_n[\gamma_1,\ldots,\gamma_i]$ for $0 \leq i \leq s-1$. Thus any element z of $\widehat{\mathbb{K}}_\nu[\gamma_1,\ldots,\gamma_s]$ can be uniquely written as $z = \sum_{i\in I} A_{i_1,\ldots,i_s}\gamma_1^{i_1}\cdots\gamma_s^{i_s}$ where $A_{i_1,\ldots,i_s} \in \widehat{\mathbb{K}}_\nu$ for all $i \in I$ and $I = \{0,\ldots,q_1-1\}\times\cdots\times\{0,\ldots,q_s-1\}$.

In order to prove the lemma we need to show that $A_{i_1,\ldots,i_s} \in \mathbb{K}_\nu^{\mathrm{alg}}$ for any $i_1, \ldots, i_s$ when z is algebraic over $\Bbbk[\![\mathbf{x}]\!]$. In this case let $\mathbb{L} := \widehat{\mathbb{K}}_\nu[\gamma_1,\ldots,\gamma_{s-1}]$ and let us write $z := \sum_{i=0}^{q_s-1} B_i\gamma_s^i$ where $B_i \in \mathbb{L}$ for all i. Let us set $\zeta_1 := \gamma_s$ and let $\zeta_2, \ldots, \zeta_{q_s}$ be the conjugates of ζ_1 over

$\overline{\mathbb{K}}_\nu[\gamma_1, \dots, \gamma_{s-1}]$. Let us define $z_j = \sum_{i=0}^{q_s-1} B_i \zeta_j^i$ for $1 \leq j \leq q_s$. Then we have

$$\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{q_s} \end{pmatrix} = \begin{pmatrix} 1 & \zeta_1 & \cdots & \zeta_1^{q_s-1} \\ 1 & \zeta_2 & \cdots & \zeta_2^{q_s-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \zeta_{q_s} & \cdots & \zeta_{q_s}^{q_s-1} \end{pmatrix} \begin{pmatrix} B_0 \\ B_1 \\ \vdots \\ B_{q_s-1} \end{pmatrix}.$$

The matrix $\begin{pmatrix} 1 & \zeta_1 & \cdots & \zeta_1^{q_s-1} \\ 1 & \zeta_2 & \cdots & \zeta_2^{q_s-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \zeta_{q_s} & \cdots & \zeta_{q_s}^{q_s-1} \end{pmatrix}$ is invertible and its entries are algebraic over $\Bbbk(\mathbf{x})$, z_j is algebraic over $\Bbbk[\![\mathbf{x}]\!]$ for all j, hence B_j is algebraic over $\Bbbk[\![\mathbf{x}]\!]$ for all j. By induction on s we see that $A_{i_1,\dots,i_s} \in \mathbb{K}_\nu^{\text{alg}}$ for any $i_1, \dots, i_s$. $\square$

We can summarize the situation with the following commutative diagram where the bottom part corresponds to the quotient fields of the rings of the upper part and all the morphisms are injective:

$$\begin{tikzcd}
\Bbbk[\![\mathbf{x}]\!] \subset V_\nu \arrow[r] \arrow[dd] & V_\nu^{\text{alg}} \arrow[rr] \arrow[dr] \arrow[dd] & & V_\nu^{\text{fg}} \arrow[rr] \arrow[dr] \arrow[dd] & & \widehat{V}_\nu \arrow[dr] \arrow[dd] & \\
& & \overline{V}_\nu^{\text{alg}} \arrow[rr] \arrow[dd] & & \overline{V}_\nu^{\text{fg}} \arrow[rr] \arrow[dd] & & \overline{V}_\nu \arrow[dd] \\
\mathbb{K}_n \arrow[r] & \mathbb{K}_\nu^{\text{alg}} \arrow[rr] \arrow[dr] & & \mathbb{K}_\nu^{\text{fg}} \arrow[rr] \arrow[dr] & & \widehat{\mathbb{K}}_\nu \arrow[dr] & \\
& & \overline{\mathbb{K}}_\nu^{\text{alg}} \arrow[rr] & & \overline{\mathbb{K}}_\nu^{\text{fg}} \arrow[rr] & & \overline{\mathbb{K}}_\nu
\end{tikzcd}$$

Example 4.6. Let $g(T) = \sum_{i=1}^\infty c_i T^i \in \mathbb{Q}[\![T]\!]$ be a formal power series which is not algebraic over $\mathbb{Q}[T]$. Let $\alpha := (\alpha_1, \alpha_2) \in \mathbb{N}^n$. Let us set

$$f := g\left(\frac{x_2^{\alpha_1}}{x_1^{\alpha_2}}\right) = \sum_{i=1}^\infty c_i \frac{x_2^{\alpha_1 i}}{x_1^{\alpha_2 i}} \in \Bbbk(\!(x_1)\!)(\!(x_2)\!).$$

But $f \notin \overline{\mathbb{K}}_{\nu_\alpha}$: let $P(Z) = a_0(\mathbf{x})Z^d + \cdots + a_d(\mathbf{x}) \in \widehat{V}_{\nu_\alpha}[Z]$ be a polynomial such that $P(f) = 0$. Let us write $a_i(\mathbf{x}) = \sum_{k=0}^\infty a_{i,k}(\mathbf{x})$ where $a_{i,k}(\mathbf{x})$ is a (α_1, α_2)-homogeneous rational fraction of degree k. By homogeneity we have

$$a_{0,k} f^d + a_{1,k} f^{d-1} + \cdots + a_{d,k} = 0 \quad \forall k \in \mathbb{N}.$$

This implies that

$$a_{0,k}(1,T) g(T^{\alpha_1})^d + a_{1,k}(1,T) g(T^{\alpha_1})^{d-1} + \cdots + a_{d,k}(1,T) = 0 \quad \forall k \in \mathbb{N}.$$

Thus $a_{i,k}(\mathbf{x}) = 0$ for all $0 \leq i \leq d$ and $0 \leq k$. Hence $P(Z) = 0$ and $f \notin \overline{\mathbb{K}}_{\nu_\alpha}$.

On the other hand, $h := g\left(\frac{x_1^{2\alpha_2}}{x_2^{\alpha_1}}\right) = \sum_{i=1}^\infty c_i \frac{x_1^{2\alpha_2 i}}{x_2^{\alpha_1 i}} \in \widehat{\mathbb{K}}_{\nu_\alpha}$ but h is not algebraic over $\Bbbk(\!(x_1)\!)(\!(x_2)\!)$.

4.2. **Analytically irreducible polynomials.**

Proposition 4.7. *Let $P(Z)\in V_\nu^{\mathrm{fg}}[Z]$ (resp. $V_\nu^{\mathrm{alg}}[Z]$) be an irreducible monic polynomial. Then $P(Z)$ is irreducible in $\widehat{V}_\nu[Z]$.*

Proof. By Corollary 4.4, $P(Z)$ splits in $V_\nu^{\mathrm{fg}}[\langle\gamma_1,\ldots,\gamma_s\rangle]$ for some homogeneous elements $\gamma_1,\ldots,\gamma_s$ with respect to ν. Since

$$V_\nu^{\mathrm{fg}}[\langle\gamma_1,\ldots,\gamma_s\rangle]\cap\widehat{V}_\nu=V_\nu^{\mathrm{fg}}$$

the result follows.

The proof is the same for V_ν^{alg}. □

Lemma 4.8. *Let σ be a $\widehat{\mathbb{K}}_\nu$-automorphism of $\overline{\mathbb{K}}_\nu$. For any $z\in\overline{\mathbb{K}}_\nu$ we have $\nu(\sigma(z))=\nu(z)$.*

Proof. Let $z\in\widehat{\mathbb{K}}_\nu[\gamma_1,\ldots,\gamma_s]$ where $\gamma_1,\ldots,\gamma_s$ are homogeneous elements with respect to ν. Let us write $z:=\sum_{i\in\Lambda}z_i$ where z_i is homogeneous of degree i for every i and Λ is a countable subset of $\mathbb{R}$ with no accumulation point (see Remark 3.5). If $i_0=\nu(z)$, then $z_{i_0}\neq 0$ and $\nu(z_i)=0$ for all $i<i_0$. Since σ acts only on the homogeneous elements $\gamma_1,\ldots,\gamma_s$, we have $\sigma(z)=\sum_i\sigma(z_i)$. For all i, $\sigma(z_i)$ is homogeneous of degree i and $\sigma(z_i)=0$ if and only if $z_i=0$. This proves that $i_0=\nu(\sigma(z))$. □

Definition 4.9. Let $P(Z)\in A[Z]$ where A is an integral domain. We write

$$P(Z)=a_0Z^d+a_1Z^{d-1}+\cdots+a_d.$$

Let $\nu:A\longrightarrow\mathbb{R}_{\geq 0}$ be a valuation. The *Newton polygon* of P is the convex hull of the set

$$\left\{(\nu(a_i),d-i)\in\mathbb{R}^2_{\geq 0}\ /\ i=0,\ldots,d\right\}+\mathbb{R}^2_{\geq 0}.$$

Corollary 4.10. *Let $P(Z)\in\widehat{V}_\nu[Z]$ be an irreducible monic polynomial. Then the Newton polygon of $P(Z)$ has only one edge. The result remains valid if we replace $\widehat{V}_\nu$ by V_ν^{alg} or V_ν^{fg}.*

Proof. Let z be a root of $P(Z)$ in $\overline{V}_\nu$. Let σ be a $\widehat{\mathbb{K}}_\nu$-automorphism of $\overline{\mathbb{K}}_\nu$. Then $\nu(\sigma(z))=\nu(z)$ by Lemma 4.8. The finite product of the distinct linear forms $Z-\sigma(z)$ obtained in this way is a monic polynomial with coefficients in $\widehat{\mathbb{K}}_\nu$ and divides $P(Z)$. Since $P(Z)$ is irreducible, both polynomials are equal. This proves that all the roots of $P(Z)$ have same valuation, hence the Newton polygon of $P(Z)$ has only one edge.

The cases V_ν^{alg} and V_ν^{fg} are deduced from Lemma 4.7. □

Example 4.11. Let $P(Z):=Z^3+3x_1x_2Z-2x_1^4\in\Bbbk[\![x_1,x_2]\!][Z]$. We see that $P(Z)$ has one root of order 2 and two roots of order 1 in $\overline{V}_{\mathrm{ord}}^{\mathrm{fg}}$. By Corollary 4.10, $P(Z)$ has at least one root in $V_{\mathrm{ord}}^{\mathrm{alg}}$ of order 2.

Let $\sqrt{1+U}:=1+\sum_{i\geq 1}a_iU^i$, $a_i\in\mathbb{Q}$ for all i, the formal powers series whose square is equal to $1+U$, and let $\sqrt[3]{1+U}:=1+\sum_{i\geq 1}b_iU^i$, $b_i\in\mathbb{Q}$ for all i, the formal power series whose cube is equal to $1+U$. Then the roots of $P(Z)$ are

$$a\sqrt[3]{q+\sqrt{q^2+p^3}}+b\sqrt[3]{q-\sqrt{q^2+p^3}}$$

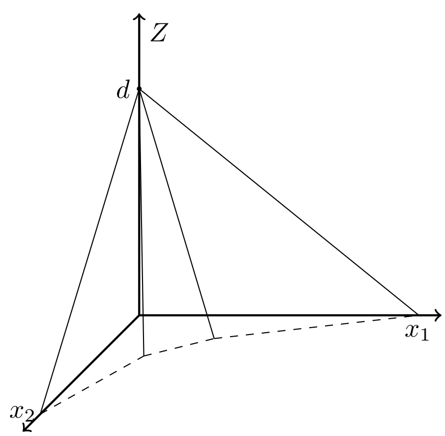
with $(a,b)=(1,1)$, (j,j^2) or (j^2,j) and $p=x_1x_2$ and $q=x_1^4$. But

$$\sqrt[3]{q+\varepsilon\sqrt{q^2+p^3}}=\sqrt[3]{x_1^4+\varepsilon\sqrt{x_1^3x_2^3+x_1^8}}=\sqrt[3]{\varepsilon}\sqrt{x_1x_2}+\eta$$

where $\varepsilon=1$ or -1 and $\mathrm{ord}(\eta)>1$. Both order 1 roots of $P(Z)$ have initial term of the form $\alpha\sqrt{x_1x_2}$ where $\alpha\in\mathbb{C}^*$. Thus $P(Z)$ has only one root in $V_{\mathrm{ord}}^{\mathrm{alg}}$ and even in $\mathbb{K}_{\mathrm{ord}}^{\mathrm{fg}}$.

Let z be the only root of $P(Z)$ in $V_{\text{ord}}^{\text{alg}}$. If $z \in \mathbb{K}_n$, since $P(Z)$ is monic and $\Bbbk[\![\mathbf{x}]\!]$ is an integral domain, then $z \in \Bbbk[\![\mathbf{x}]\!]$. But $\text{in}(z) = \frac{2}{3}\frac{x_1^3}{x_2} \notin \Bbbk[\![\mathbf{x}]\!]$. Thus $z \notin \mathbb{K}_n$, hence $P(Z)$ is irreducible in $\mathbb{K}_n[Z]$. This shows that $\mathbb{K}_n \longrightarrow \mathbb{K}_{\text{ord}}^{\text{alg}}$ is not a normal extension in general.

Corollary 4.12. *Let $P(Z) := Z^d + a_1(\mathbf{x})Z^{d-1} + \cdots + a_d(\mathbf{x}) \in \Bbbk[\![\mathbf{x}]\!][Z]$ be an irreducible polynomial having its roots in $\Bbbk[\![x_1^{\frac{1}{e}}, \ldots, x_n^{\frac{1}{e}}]\!]$ for some positive integer e. Then the Newton polyhedron of $P(Z)$ is the convex hull of the cone of $\mathbb{N}^{n+1}$ centered in $(0, \ldots, 0, d)$ and generated by the convex hull of the Newton polyhedra of $a_d(\mathbf{x})$ in $\mathbb{N}^n$.*



Proof. Let $\alpha \in \mathbb{N}^n$. Let

$$z_1, \ldots, z_d \in \Bbbk[\![x_1^{\frac{1}{e}}, \ldots, x_n^{\frac{1}{e}}]\!]$$

be the roots of $P(Z)$. Then $z_i \in \widehat{V}_{\nu_\alpha}[x_1^{\frac{1}{e}}, \ldots, x_n^{\frac{1}{e}}]$ for any i, the $x_i^{\frac{1}{e}}$ being homogeneous elements with respect to ν_α. Let $G \simeq (\mathbb{Z}/e\mathbb{Z})^n$ be the Galois group of the extension

$$\widehat{V}_{\nu_\alpha} \longrightarrow \widehat{V}_{\nu_\alpha}[x_1^{\frac{1}{e}}, \ldots, x_n^{\frac{1}{e}}].$$

The z_i are conjugated under the action of G, thus $P(Z) := \prod_{i=1}^d (Z - z_i)$ is irreducible in $\widehat{V}_{\nu_\alpha}[Z]$. This being true for any $\alpha \in \mathbb{N}^n$, the result follows from Corollary 4.10.

□

We finish this section by giving two results relating the roots of a polynomial $P(Z)$ to the roots of polynomials approximating $P(Z)$. First of all we give the following definition:

Definition 4.13. Let $P(Z) \in A[Z]$ where A is an integral domain and let ν be a valuation on A. We define

$$\nu(P(Z)) := \min_a \nu(a)$$

where a runs over all the coefficients of $P(Z)$.

The following proposition is the analogue of Proposition 2.6 of [To]:

Proposition 4.14. *Let $P(Z) \in V_\nu^{\text{fg}}[Z]$ be a monic polynomial with no multiple factor. Let us write $P(Z) = P_1(Z)...P_r(Z)$ where $P_i(Z) \in V_\nu^{\text{fg}}[Z]$, $1 \leq i \leq r$, are irreducible monic polynomials. Let $Q(Z) \in V_\nu^{\text{fg}}[Z]$ be a monic polynomial and let $z_1, \ldots, z_d$ be the roots of $P(Z)$. If*

$$\deg(Q(Z)) = \deg(P(Z))$$

and

$$\nu(Q(Z)-P(Z))>d\max_{i\neq j}\{\nu(z_i-z_j)\}$$

then we may factor $Q(Z)=Q_1(Z)...Q_r(Z)$ *such that* $Q_i(Z)\in V_\nu^{\mathrm{fg}}[Z]$ *is an irreducible monic polynomial,* $1\leq i\leq r$*, and*

$$\nu(Q_i(Z)-P_i(Z))\geq\frac{\nu(Q(Z)-P(Z))}{d}.$$

The result is still valid if we replace V_ν^{fg} *by* V_ν^{alg} *or* $\widehat{V}_\nu$*.*

Proof. Since $P(Z)$ has no multiple factor and since $\mathrm{char}(\Bbbk)=0$, we have $z_i\neq z_j$ for all $i\neq j$. Let us set $r:=\max_{i\neq j}\{\nu(z_i-z_j)\}$. Let z'_i, $1\leq i\leq d$, be the roots of $Q(Z)$. Let z be a root of $P(Z)$ in $V_\nu^{\mathrm{fg}}[\langle\gamma_1,\dots,\gamma_N\rangle]$. Let us write $P(Z)=Z^d+a_1Z^{d-1}+\cdots+a_d$ and $Q(Z)=Z^d+b_1Z^{d-1}+\cdots+b_d$. Then

$$\prod_{1\leq i\leq d}(z-z'_i)=Q(z)=Q(z)-P(z)=\sum_{i=1}^d(b_i-a_i)z^{d-i}.$$

Thus there exists at least one i such that

$$\nu(z'_i-z)\geq\frac{\min_{1\leq i\leq d}\{\nu(a_i-b_i)\}}{d}=\frac{\nu(Q(Z)-P(Z))}{d}>r.$$

Let t be another root of $P(Z)$. Then

$$\nu(z'_i-t)=\nu(z'_i-z+z-t)=\nu(z-t)\leq r$$

since $\nu(z'_i-z)\geq\frac{\min_{1\leq i\leq d}\{\nu(a_i-b_i)\}}{d}>r\geq\nu(z-t)$. Thus for any root of $P(Z)$ denoted by z, there is only one i such that

$$\nu(z-z'_i)\geq\frac{\min_{1\leq i\leq d}\{\nu(a_i-b_i)\}}{d}.$$

Let $\sigma_1(z)$, ..., $\sigma_e(z)$ be the conjugates of z over $\mathbb{K}_\nu^{\mathrm{fg}}$. Set

$$R(Z):=(Z-z)\prod_{j=1}^e(Z-\sigma_j(z))\in V_\nu^{\mathrm{fg}}[Z].$$

Then $R(Z)$ is an irreducible factor of $P(Z)$. Moreover $\sigma_1(z'_i)$, ..., $\sigma_e(z'_i)$ are conjugates of z'_i over $\mathbb{K}_\nu^{\mathrm{fg}}$. Let σ be a $\mathbb{K}_\nu^{\mathrm{fg}}$-automorphism of $\overline{\mathbb{K}}_\nu^{\mathrm{fg}}$. Then $\sigma(z)$ is a conjugate of z thus there exists j such that $\sigma(z)=\sigma_j(z)$. Moreover $\sigma(z)$ is a root of $P(Z)$ and $\nu(\sigma(z'_i)-\sigma(z))\geq\frac{\min_{1\leq i\leq d}\{\nu(a_i-b_i)\}}{d}$ by Lemma 4.8. Thus we have

$$\nu(\sigma(z'_i)-\sigma_j(z))=\nu(\sigma(z'_i)-\sigma(z))=\nu(z'_i-z)=$$
$$\nu(\sigma_j(z'_i)-\sigma_j(z))\geq\frac{\min_{1\leq i\leq d}\{\nu(a_i-b_i)\}}{d}$$

and since there is only one root of $Q(Z)$ whose difference with $\sigma_j(z)$ has valuation greater than $\frac{\min_{1\leq i\leq d}\{\nu(a_i-b_i)\}}{d}$, we necessarily have $\sigma(z'_i)=\sigma_j(z'_i)$. Thus $\sigma_1(z'_i)$, ..., $\sigma_e(z'_i)$ are the conjugates of z'_i over $\mathbb{K}_\nu^{\mathrm{fg}}$. Thus the polynomial

$$S(Z):=(Z-z'_i)\prod_{j=1}^e(Z-\sigma_j(z'_i))$$

is irreducible in $V_\nu^{\mathrm{fg}}[Z]$ and

$$\nu(S(Z)-R(Z))\geq\frac{\min_{1\leq i\leq d}\{\nu(a_i-b_i)\}}{d}.$$

The proof for $\widehat{V}_\nu$ is the same and the case V_ν^{alg} is proven with the help of Lemma 4.7. □

Remark 4.15. Let us remark that $\nu(Q(Z)-P(Z)) > \frac{d}{2}\nu(\Delta_P)$, where Δ_P is the discriminant of $P(Z)$, implies that
$$\nu(Q(Z)-P(Z)) > d\max_{i\neq j}\{\nu(z_i-z_j)\}.$$

Remark 4.16. This result is not true if $P(Z)$ has multiple factors. For example, let ν be a divisorial valuation and let us consider $P(Z)=Z^2$ and let $Q(Z)=X^2+a$ where $\nu(a)=2k+1$ and $k\in\mathbb{N}$. Since $\nu(a)$ is odd and since the value group of ν is $\mathbb{Z}$, then it is not a square in $\widehat{V}_\nu$ and $Q(Z)$ is irreducible but $P(Z)$ is not irreducible.

Proposition 4.17. *Let ν be an Abhyankar valuation and let*
$$N := \dim_{\mathbb{Q}}\Gamma\otimes_{\mathbb{Z}}\mathbb{Q}.$$
Let $P(Z)\in\widehat{V}_\nu[\langle\gamma_1,\ldots,\gamma_N\rangle][Z]$ be a monic polynomial where $\gamma_1,\ \ldots,\ \gamma_N$ are homogeneous elements with respect to ν. Then there exist integral homogeneous elements with respect to ν, denoted by $\gamma_1',\ \ldots,\ \gamma_N'$, and $c\in\mathbb{R}_{>0}$ such that the roots of $P(Z)$ are in $\widehat{V}_\nu[\langle\gamma_1',\ldots,\gamma_N'\rangle]$ and for any monic polynomial $Q(Z)\in\widehat{V}_\nu[\langle\gamma_1,\ldots,\gamma_N\rangle][Z]$ such that $\deg(Q(Z))=\deg(P(Z))$ and $\nu(P(Z)-Q(Z))\geq c$, the roots of $Q(Z)$ are in $\widehat{V}_\nu[\langle\gamma_1',\ldots,\gamma_N'\rangle]$.

Proof. The proof of this proposition is based on the proof of Theorem 4.2. So let us use the notations of that proof. Let us write $Q(Z)=Z^d+b_1Z^{d-1}+\cdots+b_d$ and let us define
$$R(Z):=Z^d+b_1'Z^{d-1}+\cdots+b_d'$$
where $b_i':=\frac{b_i}{\gamma^i}$ for $1\leq i\leq d$. We have $Q(\gamma Z)=\gamma^dR(Z)$. Let us assume that $\nu(b_i'-a_i')>0$ for all $1\leq i\leq d$ (i.e., if $\nu(b_i-a_i)>\nu(\gamma^i)$ for all i, thus we assume here that $c>d\nu(\gamma)$).

Then $\overline{R}(Z)=\overline{S}(Z)$ ($\overline{R}(Z)$ denotes the image of $R(Z)$ in $\mathbb{L}[Z]$) and the factorization $\overline{R}(Z)=\overline{S}_1(Z)\overline{S}_2(Z)$ lifts to a factorization $R(Z)=R_1(Z)R_2(Z)$ of $R(Z)$ as the product of two monic polynomials as in the proof of Theorem 4.2.

Lemma 4.18. *In the previous situation there exist two constants $a>0$, $b\geq 0$ depending only on $S_1(Z)$ and $S_2(Z)$ such that for any $c>\max\{b,\nu(\gamma^d)\}$, we have $\nu(R_i(Z)-S_i(Z))>\frac{c-b}{a}$ for $i=1,2$.*

Proof of Lemma 4.18. Let us denote by $r_{i,k}$ the coefficient of Z^k of the polynomial $R_i(Z)$, for $i=1,2$ and $0\leq k\leq\deg_Z(R_i(Z))$, and let us denote by r the vector whose coordinates are the $r_{i,k}$. The coefficient of Z^k of $R_1(Z)R_2(Z)-S_1(Z)S_2(Z)$, for $0\leq k\leq d$, is a polynomial $f_k(r)$ whose coefficients are in $\widehat{V}_\nu[\langle\gamma_1,\ldots,\gamma_N,\gamma,\gamma'\rangle]$ and depend themselves on the coefficients of $S(Z)$. By Theorem 1.2 [M-B], there exist $a>0$, $b\geq 0$ such that
$$\forall c>b,\ \forall r\in\widehat{V}_\nu[\langle\gamma_1,\ldots,\gamma_N,\gamma,\gamma'\rangle]^{d+2}\ \text{ such that }\ \nu(f_k(r))\geq c\ \ \forall k$$
$$\exists r'\in\widehat{V}_\nu[\langle\gamma_1,\ldots,\gamma_N,\gamma,\gamma'\rangle]^{d+2}\ \text{ such that }\ f_k(r')=0\ \ \forall k$$
$$\text{and } \nu(r_{i,j}'-r_{i,j})\geq\frac{c-b}{a}\ \ \forall i,j.$$
Let us denote by $R_i'(Z)$ the polynomial whose coefficients are the $r_{i,j}'$ where $0\leq j\leq\deg(R_i)$. Then $R_1'(Z)R_2'(Z)=S_1(Z)S_2(Z)$. Moreover $\overline{R}_i'(Z)=\overline{R}_i(Z)=\overline{S}_i(Z)$ if $\frac{c-b}{a}>0$. Since the roots of $\overline{S}_1(Z)$ and $\overline{S}_2(Z)$ are different, and since $\widehat{V}_\nu[\langle\gamma_1,\ldots,\gamma_N,\gamma,\gamma'\rangle][Z]$ is a GCD domain, then $R_i'(Z)=S_i(Z)$ for $i=1,2$. This proves the lemma. □

Here we remark that, the constants a, b, $\nu(\gamma)$ depend only on $P(Z)$. Thus the result is proven by induction on the degree of $P(Z)$ (since $\deg(S_i(Z)) < \deg(P(Z))$ for $i = 1, 2$) and using Proposition 3.28 and Remark 3.20.

□

5. Monomial valuation case: Eisenstein Theorem

We will first construct a subring of V_ν^{fg} containing V_ν^{alg} when ν is a monomial valuation.

Definition 5.1. Let $\alpha \in \mathbb{R}^n_{>0}$ and let δ be a (α)-homogeneous polynomial of degree d. We define

$$\begin{aligned}\mathcal{V}_{\alpha,\delta} := \Big\{ A \in \widehat{V}_{\nu_\alpha} \;/\; \exists \Lambda \text{ a finitely generated sub-semigroup of } \mathbb{R}_{\geq 0},\\ \forall i \in \Lambda\ \exists a_i \in \Bbbk[\mathbf{x}]\ (\alpha)\text{-homogeneous},\\ \exists a \geq 0, b \in \mathbb{R}\ \forall i \in \Lambda\ \exists m(i) \in \mathbb{N} \text{ s.t. } m(i) \leq ai + b,\\ \nu_\alpha\left(\frac{a_i}{\delta^{m(i)}}\right) = i \text{ and } A = \sum_{i\in\Lambda} \frac{a_i}{\delta^{m(i)}} \Big\}.\end{aligned}$$

With this notation we say that $i \longmapsto ai + b$ is a *bounding function* for $\sum_{i\in\Lambda} \frac{a_i}{\delta^{m(i)}}$.

By Lemma 3.12 we have $\Bbbk[\![\mathbf{x}]\!] \subset \mathcal{V}_{\alpha,\delta} \subset V_{\nu_\alpha}^{\text{fg}}$, by identifying a formal power series $\sum_{\beta\in\mathbb{Z}^n_{\geq 0}} c_\beta x^\beta$ to $\sum_{i\in\Lambda} \frac{a_i(x)}{\delta(x)^{m(i)}}$ with $a_i(x) := \sum_{\alpha_1\beta_1+\cdots+\alpha_n\beta_n = i} c_\beta x^\beta$ et $m(i) = 0$ for all $i \in \Lambda$. We extend in an obvious way the addition and multiplication of $\Bbbk[\![x]\!]$ to $\mathcal{V}_{\alpha,\delta}$: this defines a $\Bbbk$-algebra structure over $\mathcal{V}_{\alpha,\delta}$. We have easily the following lemma:

Lemma 5.2. *If $i \longmapsto ai + b$ is a bounding function of A and $B \in \mathcal{V}_{\alpha,\delta}$ then it is also a bounding function of $A + B$ and the function $i \longmapsto ai + 2b$ is a bounding function of AB.*

Proof. Let us write

$$A = \sum_{i\in\Lambda} \frac{a_i}{\delta^{ai+b}}, \quad B = \sum_{i\in\Lambda} \frac{b_i}{\delta^{ai+b}}$$

where Λ is a semigroup and the a_i and b_i are (α)-homogeneous polynomials and

$$\nu_\alpha\left(\frac{a_i}{\delta^{ai+b}}\right) = \nu_\alpha\left(\frac{b_i}{\delta^{ai+b}}\right) = i \quad \forall i \in \Lambda.$$

Then we have

$$A + B = \sum_{i\in\Lambda} \frac{a_i + b_i}{\delta^{ai+b}}$$

$$\text{and } AB = \sum_{i\in\Lambda} \sum_{j\in\Lambda, j\leq i} \frac{a_j b_{i-j}}{\delta^{aj+b}\delta^{a(i-j)+b}} = \sum_{i\in\Lambda} \sum_{j\in\Lambda, j\leq i} \frac{a_j b_{i-j}}{\delta^{ai+2b}}.$$

This proves the lemma. □

Remark 5.3. If $A \in \mathcal{V}_{\alpha,\delta}$ satisfies $\nu_\alpha(A) > 0$ then A admits a bounding function which is linear. Indeed let $i \longmapsto ai + b$ be a bounding function of A and let $i_0 := \nu_\alpha(A)$. Then $i \longmapsto \left(a + \frac{b}{i_0}\right) i$ is a bounding function of A.

Definition 5.4. Let $A := \sum_{i\in\Lambda} \frac{a_i}{\delta^{m(i)}} \in \mathcal{V}_{\alpha,\delta}$, $A \neq 0$. Let i_0 be the least element of Λ such that $a_{i_0} \neq 0$. We say that $\frac{a_{i_0}}{\delta^{m(i_0)}}$ is the *initial term* of A with respect to ν_α or its (α)-*initial term*. We denote it by $\mathrm{in}_\alpha(A)$.

Lemma 5.5. *Let δ and δ' be two (α)-homogeneous polynomials. We have the following properties:*

i) *The ((α)-homogeneous) irreducible divisors of δ divide δ' if and only if $\mathcal{V}_{\alpha,\delta'} \subset \mathcal{V}_{\alpha,\delta}$. We denote by $\mathcal{V}_\alpha$ the inductive limit of the $\mathcal{V}_{\alpha,\delta}$.*

ii) *The valuation ν_α is well defined on $\mathcal{V}_{\alpha,\delta}$ and extends to $\mathcal{V}_\alpha$. Its valuation ring is exactly $\mathcal{V}_\alpha$.*

Proof. It is clear that if the irreducible divisors of δ divide δ' then $\mathcal{V}_{\alpha,\delta} \subset \mathcal{V}_{\alpha,\delta'}$. On the other hand if $\mathcal{V}_{\alpha,\delta} \subset \mathcal{V}_{\alpha,\delta'}$, then $\frac{1}{\delta} \in \mathcal{V}_{\alpha,\delta'}$, thus there exist a (α)-homogeneous polynomial $a \in \Bbbk[x]$ and an integer $m \in \mathbb{N}$ such that $\frac{1}{\delta} = \frac{a}{\delta'^m}$, hence $a\delta = \delta'^m$. This proves i).

If $A \in \mathcal{V}_{\alpha,\delta}$ and $B \in \mathcal{V}_{\alpha,\delta'}$ satisfy $\nu_\alpha(B) \geq \nu_\alpha(A)$, let $\frac{a_k(x)}{\delta(x)^{m(k)}}$ denote the first nonzero term in the expansion of A. Then we can check easily that $\frac{B}{A} \in \mathcal{V}_{a,\delta\delta' a_k}$. This proves ii). $\square$

Definition 5.6. For any $\alpha \in \mathbb{R}^n_{>0}$ we denote by $\mathcal{K}_\alpha$ the fraction field of $\mathcal{V}_\alpha$ and

$$\overline{\mathcal{K}}_\alpha := \varinjlim_{\gamma_1,\dots,\gamma_s} \mathcal{K}_\alpha[\gamma_1,\dots,\gamma_s]$$

the limit being taken over all subsets $\{\gamma_1, ..., \gamma_s\}$ of (integral) homogeneous elements with respect to ν.

If $\gamma_1, \dots, \gamma_s$ are homogeneous elements with respect to ν_α we denote by $\mathcal{V}_{\alpha,\delta}[\langle\gamma_1,\dots,\gamma_s\rangle]$ the ring of elements $\sum_{\underline{k}} A_{\underline{k}}\gamma^{\underline{k}}$ where the sum is finite, $\underline{k} := (k_1,\dots,k_s)$, $A_{\underline{k}} = \sum_{i\in\Lambda} \frac{a_i}{\delta^{m(i)}}$ where $a_i \in \Bbbk[x]$ is (α)-homogeneous, there exist two constants $a \geq 0$, $b \in \mathbb{R}$ such that $m(i) \leq ai + b$ for all i and there exists $i_0 \in \Lambda$ such that $\nu_\alpha\left(\frac{a_i}{\delta^{m(i)}}\right) = i - i_0$ and $\nu(\gamma^{\underline{k}}) \geq i_0$.

This means that $\mathcal{V}_{\alpha,\delta}[\langle\gamma_1,\dots,\gamma_s\rangle]$ is the subring of $\mathcal{K}_\alpha[\gamma_1,\dots,\gamma_s]$ whose elements have non negative valuation ν_α. In the same way we denote by $\mathcal{V}_\alpha[\langle\gamma_1,\dots,\gamma_s\rangle]$ the ring of elements of $\mathcal{K}_\alpha[\gamma_1,\dots,\gamma_s]$ having a non negative valuation ν_α. The field of fractions of $\mathcal{V}_\alpha[\langle\gamma_1,\dots,\gamma_s\rangle]$ is exactly $\mathcal{K}_\alpha[\gamma_1,\dots,\gamma_s]$.

Remark 5.7. We will see later (see Remark 6.10) that these fields $\overline{\mathcal{K}}_\alpha$ coincide with those introduced in [AI] when $\dim_{\mathbb{Q}}(\alpha_1\mathbb{Q}+\dots+\alpha_n\mathbb{Q}) = n$ where it is proven that they are algebraically closed.

Remark 5.8. For any $\alpha \in \mathbb{R}^n_{>0}$ it is clear that $\mathcal{V}_\alpha \subset V^{\mathrm{fg}}_{\nu_\alpha}$ but both rings are never equal if $\dim_{\mathbb{Q}}(\alpha_1\mathbb{Q}_1 + \dots + \alpha_n\mathbb{Q}) < n$. For instance, let $n = 2$ and $\alpha = (1,1)$ and set

$$z = \sum_{i\in\mathbb{N}} \frac{x_1^{(i+1)^2}}{x_2^{i^2}} \text{ or } \sum_{i\in\mathbb{N}} \frac{x_1^i}{x_1 + ix_2}.$$

Then obviously $z \in V^{\mathrm{fg}}_{\nu_\alpha}$ but $z \notin \mathcal{V}_\alpha$.

Proposition 5.9. *If $\alpha_1, \dots, \alpha_n$ are linearly independent over $\mathbb{Q}$ then $\mathcal{V}_\alpha = V^{\mathrm{fg}}_{\nu_\alpha}$.*

Proof. Let us denote by $\alpha^* : \mathbb{Q}^n \longrightarrow \mathbb{R}$ the $\mathbb{Q}$-linear map defined by $\alpha^*(u) = \langle\alpha, u\rangle$ for any $u \in \mathbb{Q}^n$. Since the α_i are $\mathbb{Q}$-linearly independent then α^* is injective.

If Λ is a finitely generated sub-semigroup of $\mathbb{Z}\alpha_1 + \dots + \mathbb{Z}\alpha_n$ let $\beta_1, \dots, \beta_s$ be generators of Λ. Then $\alpha^{*-1}(\Lambda)$ is a finitely generated semigroup whose generators are

$$b_1 = \alpha^{*-1}(\beta_1), \dots, b_s = \alpha^{*-1}(\beta_s) \in \mathbb{Z}^n.$$

If the support of $z\in V_{\nu_\alpha}^{\text{fg}}$ is in Λ, since α^* is injective z can be written as

$$z=\sum_{\underline{k}\in\mathbb{Z}_{\geq 0}^s} a_{\underline{k}}\mathbf{x}^{k_1b_1+\cdots+k_sb_s}$$

where $a_{\underline{k}}\in\Bbbk$ for all $\underline{k}=(k_1,\cdots,k_s)$. Let us remark that the monomial $a_{\underline{k}}\mathbf{x}^{k_1b_1+\cdots+k_sb_s}$ is (α)-homogeneous of degree $k_1\beta_1+\cdots+k_s\beta_s$.

Let us write $b_i=b_{1,i}-b_{2,i}$ where $b_{1,i},\ b_{2,i}\in\mathbb{Z}^n_{\geq 0}$. Then we have

$$\begin{aligned}\mathbf{x}^{k_1b_1+\cdots+k_sb_s}&=\frac{\mathbf{x}^{k_1b_{1,1}+\cdots+k_sb_{1,s}}}{\mathbf{x}^{k_1b_{2,1}+\cdots+k_sb_{2,s}}}\\&=\frac{\mathbf{x}^{k_1b_{1,1}+\cdots+k_sb_{1,s}+(\max_i\{k_i\}-k_1)b_{2,1}+\cdots+(\max_i\{k_i\}-k_s)b_{2,s}}}{\mathbf{x}^{(b_{2,1}+\cdots+b_{2,s})\max_i\{k_i\}}}.\end{aligned}$$

Moreover

$$\max_i\{k_i\}\leq\max_j\left\{\frac{1}{\beta_j}\right\}(k_1\beta_1+\cdots+k_s\beta_s).$$

This shows that $z\in\mathcal{V}_{\alpha,\mathbf{x}^{b_{2,1}+\cdots+b_{2,s}}}$ and

$$i\longmapsto\max_j\left\{\frac{1}{\beta_j}\right\}i$$

is a bounding function of z. $\square$

Then we give the following version of the Implicit Function Theorem inspired by Lemma 1.2 [Ga] (see also Lemma 2.2. [To]):

Proposition 5.10. *Let $\alpha\in\mathbb{R}^n_{>0}$ and let $P(Z)\in\mathcal{V}_{\alpha,\delta}[\langle\gamma_1,\ldots,\gamma_s\rangle][Z]$, $P(Z)=\sum_{k=0}^d a_kZ^k$, where γ_i is homogeneous for all i with respect to ν_α and $d\geq 2$.*

Let $u\in\mathcal{V}_{\alpha,\delta}[\langle\gamma_1,\ldots,\gamma_s\rangle]$ such that $\nu_\alpha(P(u))>2\nu_\alpha(P'(u))$. Let $\frac{\widetilde{\delta}}{\delta^m}$ denote the initial term of $P'(u)$ with respect to ν_α.

Then there exists a unique solution $\overline{u}$ in $\mathcal{V}_{\alpha,\delta\widetilde{\delta}}[\langle\gamma_1,\ldots,\gamma_s\rangle]$ of $P(Z)=0$ such that

$$\nu_\alpha(\overline{u}-u)\geq\nu_\alpha(P(u))-\nu_\alpha(P'(u)).$$

Proof. • By replacing $P(Z)$ by $P(u+Z)$ we can assume that $u=0$. In this case we have that $P(u)=P(0)=a_0$ and $P'(u)=P'(0)=a_1$.

The valuation ν_α is defined on the ring $\mathcal{V}_{\alpha,\delta}[\langle\gamma_1,\ldots,\gamma_s\rangle]$ and we denote by V its valuation ring. We denote by $\widehat{V}$ the completion of V. Let V^{fg} be the subring of $\widehat{V}$ of all elements of $\widehat{V}$ whose ν_α-support is included in a finitely generated semigroup. Then V^{fg} is a Henselian local ring by Lemma 4.1. We set $Z=\frac{\widetilde{\delta}}{\delta^m}Y$. Thus we are looking for solving the following equation:

$$\widetilde{P}(Y):=\frac{\delta^{2m}}{\widetilde{\delta}^2}P\left(\frac{\widetilde{\delta}}{\delta^m}Y\right)=a_0\frac{\delta^{2m}}{\widetilde{\delta}^2}+a_1\frac{\delta^m}{\widetilde{\delta}}Y+a_2Y^2+\cdots+a_d\frac{\widetilde{\delta}^{d-2}}{\delta^{(d-2)m}}Y^d=0.$$

From now on we denote by $\widetilde{a}_k$ the coefficients of $\widetilde{P}(Y)$:

$$\widetilde{a}_k:=a_k\frac{\widetilde{\delta}^{k-2}}{\delta^{(k-2)m}}\quad k=0,\ldots,d.$$

Since

$$\nu_\alpha(a_0)=\nu_\alpha(P(0))>2\nu_\alpha(P'(0))=2\nu_\alpha(a_1)=\nu_\alpha\left(\frac{\widetilde{\delta}^2}{\delta^{2m}}\right)$$

we have that $\widetilde{a}_0 \in V^{\text{fg}}$. By assumption we have that $\nu_\alpha(\widetilde{a}_1) = 0$ thus $\widetilde{a}_1 \in V^{\text{fg}}$. Since $\nu_\alpha\left(\frac{\widetilde{\delta}}{\delta^m}\right) \geq 0$ we have that $\widetilde{a}_k \in V^{\text{fg}}$ for all $k \geq 2$. Moreover we have

$$\nu_\alpha(\widetilde{P}(0)) > 0 \text{ and } \nu_\alpha(\widetilde{P}'(0)) = 0.$$

Thus by Hensel Lemma this equation has a unique solution $y \in V^{\text{fg}}$ such that

$$\nu_\alpha(y) = \nu_\alpha\left(a_0 \frac{\delta^{2m}}{\widetilde{\delta}^2}\right) = \nu_\alpha(P(0)) - 2\nu_\alpha(P'(0)) > 0.$$

Hence there exists a unique solution $z := \frac{\widetilde{\delta}}{\delta^m} y \in V^{\text{fg}}$ of the equation $P(Z) = 0$ such that

$$\nu_\alpha(z) \geq \nu_\alpha(P(u)) - \nu_\alpha(P'(u)).$$

Now we have to show that z or $y \in \mathcal{V}_{\alpha,\delta\widetilde{\delta}}[\langle\gamma_1, \ldots, \gamma_s\rangle]$.

- We can write $y = \widetilde{a}_0 \widetilde{y}$ where $\widetilde{y} \in V^{\text{fg}}$ and $\nu_\alpha(\widetilde{y}) = 0$. Then $\widetilde{y}$ is a root of the polynomial

$$\begin{aligned}\widetilde{P}(\widetilde{a}_0 Y) &= \widetilde{a}_0 + \widetilde{a}_1 \widetilde{a}_0 Y + \widetilde{a}_2 \widetilde{a}_0^2 Y^2 + \cdots + \widetilde{a}_d \widetilde{a}_0^d Y^d \\ &= \widetilde{a}_0 \left(1 + \widetilde{a}_1 Y + \widetilde{a}_2 \widetilde{a}_0 Y^2 + \cdots + \widetilde{a}_d \widetilde{a}_0^{d-1} Y^d\right)\end{aligned}$$

and $y \in \mathcal{V}_{\alpha,\delta\widetilde{\delta}}[\langle\gamma_1, \ldots, \gamma_s\rangle]$ if and only if $\widetilde{y} \in \mathcal{V}_{\alpha,\delta\widetilde{\delta}}[\langle\gamma_1, \ldots, \gamma_s\rangle]$.

Since $\nu_\alpha(\widetilde{a}_0) > 0$, by replacing $\widetilde{P}$ (resp. y) by $1 + \widetilde{a}_1 Y + \widetilde{a}_2 \widetilde{a}_0 Y^2 + \cdots + \widetilde{a}_d \widetilde{a}_0^{d-1} Y^d$ (resp. $\widetilde{y}$), we may assume that

$$\nu_\alpha(\widetilde{a}_i) > 0 \text{ for } i \geq 2.$$

In this case we have $\widetilde{a}_0 = 1$, $\text{in}_\alpha(\widetilde{a}_1) = 1$ and $\text{in}_\alpha(y) = -1$.

Let Λ be a finitely generated sub-semigroup of $\mathbb{R}_{\geq 0}$ containing the ν_α-supports of y and the $\widetilde{a}_k$. We denote by λ_l, $l \in \mathbb{Z}_{\geq 0}$, its elements ordered as follows:

$$\lambda_0 := 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_l < \lambda_{l+1} < \cdots.$$

Let us expand the coefficients of $\widetilde{P}(Y)$ as

$$\widetilde{a}_k = \sum_{l \in \mathbb{Z}_{\geq 0}} \widetilde{a}_{k,\lambda_l}$$

where $\widetilde{a}_{k,\lambda_l}$ is homogeneous of degree λ_l with respect to ν_α. For every $l \in \mathbb{N}$ let Y_{λ_l} be a new variable and set $Y^* := \sum_{l \in \mathbb{N}} Y_{\lambda_l}$. We extend the valuation ν_α to $V^{\text{fg}}[Y_{\lambda_1}, \ldots, Y_{\lambda_l}, ...]$ by setting $\nu_\alpha(Y_{\lambda_l}) := \lambda_l$ for any $l \in \mathbb{N}$. We may write formally $\widetilde{P}(Y^*) = \sum_l \widetilde{P}_{\lambda_l}(Y^*)$ where $\widetilde{P}_{\lambda_l}(Y^*) \in \mathbb{Z}[\widetilde{a}_{k,\lambda_i}][Y_{\lambda_j}]$ is the homogeneous term of degree λ_l with respect to ν_α.

Since $\text{in}_\alpha(\widetilde{a}_1) = 1$ the equation

$$\widetilde{P}(Y) = \widetilde{a}_0 + \widetilde{a}_1 Y + \widetilde{a}_2 Y^2 + \cdots + \widetilde{a}_d Y^d = 0, \tag{5}$$

where Y is replaced by Y^*, yields the following equation, for every $l \in \mathbb{Z}_{\geq 0}$:

$$\widetilde{P}_\lambda(Y^*) = Y_{\lambda_l} + Q_{\lambda_l}(Y^*) = 0. \tag{6}$$

where $Q_{\lambda_l}(Y^*) \in \mathbb{Z}[\widetilde{a}_{k,\lambda_i}][Y_{\lambda_j}]$ is a polynomial depending only on the variables $\widetilde{a}_{k,\lambda_i}$ $(\lambda_i \leq \lambda_l)$ and Y_{λ_j} $(j < l)$. Since y is a solution of Equation (5), by replacing Y^* by y we have $\widetilde{P}_{\lambda_l}(y) = 0$, hence

$$y_{\lambda_l} = -Q_{\lambda_l}(y_{\lambda_j}, j < l) \quad \forall l \in \mathbb{N}.$$

So by induction on l we see that we may write

$$y_{\lambda_l} = \frac{c_l}{(\delta\widetilde{\delta})^{m(\lambda_l)}}$$

for some $c_l \in \Bbbk[x][\gamma_1,\ldots,\gamma_s]$ and $m(\lambda_l)\in\mathbb{N}$ for all l.

Let $i \longmapsto ai+b$ be a common bounding functions of the coefficients of $\widetilde{a}_0$, $\widetilde{a}_1$, $\widetilde{a}_2$, …, $\widetilde{a}_d$ seen as elements of $\mathcal{V}_{\alpha,\delta\widetilde{\delta}}[\langle\gamma_1,\ldots,\gamma_s\rangle]$. By Remark 5.3 we may assume that $b=0$ since $\nu_\alpha(\widetilde{a}_k)>0$ for $k\geq 2$ and $\mathrm{in}_\alpha(\widetilde{a}_0)=\mathrm{in}_\alpha(\widetilde{a}_1)=1$.

Thus we have

$$(\delta\widetilde{\delta})^{a\lambda_i}\widetilde{a}_{k,\lambda_i}\in\Bbbk[x][\gamma_1,\ldots,\gamma_s]\quad\forall i.$$

Let $m(\lambda_l)$ be the least integer such that $(\delta\widetilde{\delta})^{m(\lambda_l)}y_{\lambda_l}\in\Bbbk[x][\gamma_1,\ldots,\gamma_s]$. We will show by induction on l that

$$m(\lambda_l)\leq a\lambda_l. \tag{7}$$

This inequality is satisfied for $l=0$ since $\mathrm{in}_\alpha(y)=-1$ implies that $m(\lambda_0)=0$.

We fix an integer $l>0$ and we assume that (7) is satisfied for any integer less than l.

Let Q be a monomial of $Q_i(Y^*)$. We may write

$$Q=\widetilde{a}_{k,\lambda_i}y_{\lambda_{j_1}}\cdots y_{\lambda_{j_k}}$$

where $k\leq d$, $j_1\leq\cdots\leq j_k<l$ and $\lambda_i+\lambda_{j_1}+\cdots+\lambda_{j_k}=\lambda_l$.

Then

$$(\delta\widetilde{\delta})^{a\lambda_i+a(\lambda_{j_1}+\cdots+\lambda_{j_k})}Q=(\delta\widetilde{\delta})^{a\lambda_l}Q\in\Bbbk[\![x]\!][\gamma_1,\cdots,\gamma_s].$$

This proves (7). So $y\in\mathcal{V}_{\alpha,\delta\widetilde{\delta}}[\langle\gamma_1,\ldots,\gamma_s\rangle]$. □

We deduce from this proposition the main result of this part (Theorem 5.12) which is a general version of Eisenstein Theorem for algebraic power series over $\mathbb{Q}$. First we recall the classical Eisenstein Theorem:

Theorem 5.11. [Ei] *Let* $\sum_{k\in\mathbb{Z}_{\geq 0}} a_kT^k\in\mathbb{Q}[\![T]\!]$ *be a power series algebraic over* $\mathbb{Q}[T]$. *Then there exists an integers* $a\in\mathbb{N}$ *such that*

$$a^{k+1}a_k\in\mathbb{Z}$$

for every integer k.

Theorem 5.12 (Eisenstein Theorem)**.** *Let* $\Bbbk$ *be a field of characteristic zero. Let* $\alpha\in\mathbb{R}^n_{>0}$ *and let us set* $N=\dim_{\mathbb{Q}}(\mathbb{Q}\alpha_1+\cdots+\mathbb{Q}\alpha_n)$. *Let*

$$P(Z)\in\mathcal{V}_\alpha[\langle\gamma_1,\ldots,\gamma_s\rangle][Z]$$

be a monic polynomial where γ_1, …, γ_s *are homogeneous elements with respect to* ν_α. *Then there exist integral homogeneous elements with respect to* ν_α, *denoted by* γ'_1,… γ'_N, *such that* $P(Z)$ *has all its roots in* $\mathcal{V}_\alpha[\langle\gamma'_1,\ldots,\gamma'_N\rangle]$.

Proof. By replacing $P(Z)$ by one of its irreducible factors we may assume that $P(Z)$ is irreducible. Let

$$z\in V^{\mathrm{fg}}_{\nu_\alpha}[\langle\gamma'_1,\ldots,\gamma'_N\rangle]$$

be a root of $P(Z)$ where γ'_i is an integral homogeneous with respect to ν_α (by Theorem 4.2 such a z exists). Since $P(Z)$ is irreducible, then $P'(z)\neq 0$. Let us set $i_0:=\max\{\nu_\alpha(z-z')\}$ where the maximum is taken over all the roots z' of P different from z. Let us take $\widetilde{z}\in\mathcal{V}_\alpha[\langle\gamma'_1,\ldots,\gamma'_N\rangle]$ such that

$$\nu_\alpha(\widetilde{z}-z)>\max\{2\nu_\alpha(P'(z)),i_0+\nu_\alpha(P'(z))\}. \tag{8}$$

For instance if we expand $z = \sum_{i\in\Lambda} z_i$ where Λ is a finitely generated sub-semigroup of $\mathbb{R}_{\geq 0}$ and z_i is a homogeneous element of degree i with respect to ν_α we can choose

$$\widetilde{z} := \sum_{i\leq \max\{2\nu_\alpha(P'(z)), i_0+\nu_\alpha(P'(z))\}} z_i.$$

By replacing $P(Z)$ by $P(Z+\widetilde{z})$ we may assume that $\widetilde{z}=0$. In this case $P'(\widetilde{z}) = P'(0) = a_{d-1}$ if we write

$$P(Z) = Z^d + a_1Z^{d-1} + \cdots + a_{d-1}Z + a_d.$$

Now $\text{in}_{\nu_\alpha}(a_{d-1}) \in \mathbb{k}(\mathbf{x})[\gamma'_1,\ldots,\gamma'_N]$ so if we denote by a the product of the conjugates of $\text{in}_{\nu_\alpha}(a_{d-1})$ over $\mathbb{k}(\mathbf{x})$ different from $\text{in}_{\nu_\alpha}(a_{d-1})$ we have $\text{in}_{\nu_\alpha}(aa_{d-1}) \in \mathbb{k}(\mathbf{x})$ and a is a homogeneous element with respect to ν_α by Lemma 4.8. Let b be a homogeneous element such that $b^{d-1}=a$. By Proposition 3.28 we may assume that $b\in V^{\text{fg}}_{\nu_\alpha}[\langle\gamma'_1,\ldots,\gamma'_N\rangle]$. We have that

$$\begin{aligned} b^dP\left(\frac{Z}{b}\right) &= b^d\left(\frac{1}{b^d}Z^d + \frac{a_1}{b^{d-1}}Z^{d-1} + \cdots + \frac{a_{d-1}}{b}Z + a_d\right) \\ &= Z^d + a_1bZ^{d-1} + \cdots + b^{d-1}a_{d-1}Z + b^da_d. \end{aligned}$$

By replacing $P(Z)$ by $b^dP\left(\frac{Z}{b}\right)$ we may assume that $\text{in}_{\nu_\alpha}(P'(\widetilde{z})) = \text{in}_{\nu_\alpha}(a_{d-1}) \in \mathbb{k}(\mathbf{x})$.

Since $P(\widetilde{z}) - P(z) \in (\widetilde{z}-z)$ then by Inequality (8)

$$\nu_\alpha(P(\widetilde{z})) > 2\nu_\alpha(P'(\widetilde{z}))$$

and

$$\nu_\alpha(P(\widetilde{z})) > i_0 + \nu_\alpha(P'(z)).$$

In the same way, since $P'(\widetilde{z}) - P'(z) \in (\widetilde{z}-z)$, Inequality (8) yields

$$\nu_\alpha(P'(\widetilde{z})) = \nu_\alpha(P'(z)).$$

Then we apply Proposition 5.10 (with $u := \widetilde{z} = 0$), and we get a root $\overline{z} \in \mathcal{V}_\alpha[\langle\gamma'_1,\ldots,\gamma'_N\rangle]$ of $P(Z)$ such that

$$\nu_a(\widetilde{z}-\overline{z}) \geq \nu_\alpha(P(\widetilde{z})) - \nu_\alpha(P'(\widetilde{z})) > i_0.$$

Thus

$$\nu_\alpha(z-\overline{z}) = \nu_\alpha(z-\widetilde{z}+\widetilde{z}-\overline{z}) > i_0 = \max_{\substack{z'\neq z\\ P(z')=0}}\{\nu_\alpha(z-z')\}.$$

Hence $z = \overline{z} \in \mathcal{V}_\alpha[\langle\gamma'_1,\ldots,\gamma'_N\rangle]$. □

Corollary 5.13. *The field $\mathbb{K}^{\text{alg}}_{\nu_\alpha}$ is a subfield of $\mathcal{K}_\alpha$.*

Proof. Let $z\in\mathbb{K}^{\text{alg}}_{\nu_\alpha}$ and let $P(Z) = a_0Z^d+\cdots+a_d \in \mathbb{k}[\![\mathbf{x}]\!][Z]$ be a polynomial such that $P(z)=0$. Then $a_0z\in\mathbb{K}^{\text{alg}}_{\nu_\alpha}$ is a root of the polynomial $a_0^{d-1}P(Z/a_0) = Z^d+a_1Z^{d-1}+a_2a_0Z^{d-2}+\cdots+a_da_0^{d-1}$ which is a monic polynomial. Hence $a_0z\in\mathcal{V}_\alpha$ by Theorem 5.12 and $z\in\mathcal{K}_\alpha$. □

Example 5.14. Let us assume that $\text{Disc}_Z(P(Z))$ is normal crossing after a formal change of coordinates and let us assume that $\mathbb{k}$ is algebraically closed. This means that there exist power series $x_i(\mathbf{y}) \in (\mathbf{y})\mathbb{k}[\![\mathbf{y}]\!]$ ($\mathbf{y} = (y_1,\ldots,y_n)$), for $1\leq i\leq n$, such that the morphism of $\mathbb{k}$-algebras $\varphi : \mathbb{k}[\![\mathbf{x}]\!] \longrightarrow \mathbb{k}[\![\mathbf{y}]\!]$ defined by $\varphi(f(\mathbf{x})) = f(x_1(\mathbf{y}),\ldots,x_n(\mathbf{y}))$ is an isomorphism, and such that

$$\varphi(\text{Disc}_Z(P(Z)))\mathbb{k}[\![\mathbf{y}]\!] = y_1^{e_1}\cdots y_m^{e_m}\mathbb{k}[\![\mathbf{y}]\!],\ m\leq n.$$

By Abhyankar-Jung Theorem [Ab] (or [KV], [PR], [MS]), the roots of $P(Z)$ can be written as

$$t_k = \sum_{l=0}^{d} t_{k,l}(\mathbf{y})\mathbf{w}^l$$

where $\mathbf{w} = \mathbf{y}^\beta$ for some $\beta \in \mathbb{Q}_{\geq 0}^m \times \{0\}^{n-m}$, $d \in \mathbb{Z}_{\geq 0}$ and the $t_{k,l}(\mathbf{y})$ are power series with coefficients in $\mathbb{k}$. Let us write:

$$\beta = \left(\frac{b_1}{e}, \ldots, \frac{b_m}{e}, 0, \ldots, 0\right)$$

for some non negative integers $b_1, \ldots, b_m$ and $e \in \mathbb{N}$. Let us denote by $f_i(\mathbf{x})$, $1 \leq i \leq n$, the power series satisfying $\varphi(f_i(\mathbf{x})) = y_i$.

Let $\alpha \in \mathbb{N}^n$ and write $f_i(\mathbf{x}) = l_{i,\alpha}(\mathbf{x}) + \varepsilon_{i,\alpha}(\mathbf{x})$ where $l_{i,\alpha}(\mathbf{x})$ is (α)-homogeneous and $\nu_\alpha(\varepsilon_i(\mathbf{x})) > \nu_\alpha(l_{i,\alpha}(\mathbf{x}))$ for any i. Thus we have for $1 \leq i \leq m$:

$$y_i^{\frac{1}{e}} = l_{i,\alpha}(\mathbf{x})^{\frac{1}{e}} \left(1 + \frac{\varepsilon_{i,\alpha}(\mathbf{x})}{l_{i,\alpha}(\mathbf{x})}\right)^{\frac{1}{e}} = l_{i,\alpha}(\mathbf{x})^{\frac{1}{e}} \left(1 + \sum_{k\geq 1} c_k \frac{\varepsilon_{i,\alpha}(\mathbf{x})^k}{l_{i,\alpha}(\mathbf{x})^k}\right)$$

where $c_k \in \mathbb{Q}$ for all k - here

$$c_k = \frac{\frac{1}{e}\left(\frac{1}{e}-1\right)\cdots\left(\frac{1}{e}-k+1\right)}{k!}.$$

Hence

$$\begin{aligned}\mathbf{w} =& y_1^{\frac{b_1}{e}} \cdots y_m^{\frac{b_m}{e}} = \\ & l_{1,\alpha}(\mathbf{x})^{\frac{b_1}{e}} \cdots l_{m,\alpha}(\mathbf{x})^{\frac{b_m}{e}} \prod_{j=1}^m \left(1 + \sum_{k\geq 1} c_k \frac{\varepsilon_{j,\alpha}(\mathbf{x})^k \prod_{p\neq j} l_{p,\alpha}(\mathbf{x})^k}{(\prod_{p=1}^m l_p(\mathbf{x}))^k}\right)^{b_j}.\end{aligned}$$

We remark that $\mathrm{Disc}_Z(P(Z)) = \prod_{p=1}^m l_{p,\alpha}(\mathbf{x})^{e_p} + \varepsilon(\mathbf{x})$ with

$$\nu_\alpha(\varepsilon(\mathbf{x})) > \nu_\alpha(\prod_{p=1}^m l_{p,\alpha}(\mathbf{x})^{e_p}).$$

Let $\gamma := \prod_{j=1}^m l_{j,\alpha}(\mathbf{x})^{\frac{b_j}{e}}$ be a root of the polynomial

$$Z^e - \prod_{j=1}^m l_{j,\alpha}(\mathbf{x})^{b_j}$$

(in particular it is an integral homogeneous element with respect to ν_α), and set

$$\delta := \prod_{j=1}^m l_{j,\alpha}(\mathbf{x})^{e_p}.$$

Here δ is the (α)-initial term of the discriminant of $P(Z)$. Hence we obtain the following three cases:

i) If φ is a linear change of coordinates (i.e., $\alpha = (1, \ldots, 1)$ and $\varepsilon_{i,\alpha} = 0\ \forall i$), then the roots of $P(Z)$ are in $\mathbb{k}[\![\mathbf{x}]\!][\gamma]$ (since in this case $\mathbf{w} = \gamma$).

ii) If φ is a quasi-linear change of variables (i.e., $\alpha \in \mathbb{N}^n$ and $\varepsilon_{i,\alpha} = 0\ \forall i$), then the roots of $P(Z)$ are still in $\mathbb{k}[\![\mathbf{x}]\!][\gamma]$ (since in this case we also have $\mathbf{w} = \gamma$).

iii) If (at least) one of the $\varepsilon_{i,\alpha}$ is not zero, then the roots of $P(Z)$ are in $\mathcal{V}_{\alpha,\delta}[\langle\gamma\rangle]$.

This example will be generalized later (see Theorem 7.7).

Example 5.15. Let $P(Z) = Z^2 + 2aZ + b$ where a and b are power series over $\mathbb{k}$ and let $\alpha \in \mathbb{Q}_{>0}^n$. Let δ denote the (α)-initial term of the discriminant of $P(Z)$, i.e., the (α)-initial term of $a^2 - b$. Then the roots of $P(Z)$ are of the form $-a + \sqrt{a^2 - b} \in \mathcal{V}_{\alpha,\delta}[\langle\gamma\rangle]$ where γ is a root square of δ.

Example 5.16. Let $P(Z)=Z^3+3x_2^2Z-2(x_1^3+\varepsilon)$ where ε is a homogeneous polynomial of degree greater or equal to 4. Its discriminant is $D:=x_1^6+x_2^6+2x_1^3\varepsilon+\varepsilon^2$ whose initial term is $x_1^6+x_2^6$. The roots of P are

$$a\sqrt[3]{x_1^3+\varepsilon+\sqrt{D}}+b\sqrt[3]{x_1^3+\varepsilon-\sqrt{D}}$$

with $(a,b)=(1,1)$, (j,j^2) or (j^2,j). But we have

$$\sqrt[3]{x_1^3+\varepsilon+\sqrt{D}}=\gamma_1\sqrt[3]{1+\varepsilon+\frac{\gamma_2}{x_1^3+\gamma_2}\sqrt{1+\frac{2x_1^3\varepsilon+\varepsilon^2}{\delta}}-\frac{\gamma_2}{x_1^3+\gamma_2}}$$

with $\gamma_2^2=x_1^6+x_2^6$, $\gamma_1^3=x_1^3+\gamma_2$ and $\delta=x_1^6+x_2^6$ is the initial term of D. Thus

$$\sqrt[3]{x_1^3+\varepsilon+\sqrt{D}}\in\mathcal{V}_{(1,1),\delta}\left[\left\langle\gamma_1,\gamma_2,\frac{\gamma_2\varepsilon}{x_1^3+\gamma_2}\right\rangle\right].$$

By doing the same remark for $\sqrt[3]{x_1^3+\varepsilon-\sqrt{D}}$, we see that there exist $\gamma_1,\ldots,\gamma_5$ homogeneous elements with respect to ord such that the roots of $P(Z)$ are in $\mathcal{V}_{(1,1),\delta}[\langle\gamma_1,\ldots,\gamma_5\rangle]$. But there is no reason that the roots of $P(Z)$ are in $\mathcal{V}_{\alpha,\delta}[\langle\gamma\rangle]$ where γ is one (integral) homogeneous element with respect to ν_α.

6. Approximation of monomial valuations by divisorial monomial valuations

In several cases, it will be easier to work with a monomial valuation ν_α which is divisorial, i.e., such that $\dim_{\mathbb{Q}}(\mathbb{Q}\alpha_1+\cdots+\mathbb{Q}\alpha_n)=1$. In order to extend some results which are proven for divisorial monomial valuations to general monomial valuations, we will approximate monomial valuations by divisorial monomial valuations. The aim of this section is to explain how this can be done.

Definition 6.1. Let $\alpha\in\mathbb{R}_{>0}^n$. Let $\alpha^*:\mathbb{Q}^n\longrightarrow\mathbb{R}$ be the $\mathbb{Q}$-linear morphism defined by $\alpha^*(q_1,\ldots,q_n):=\sum_i\alpha_iq_i$. We denote by Rel_α the kernel of this morphism.

For any $\varepsilon>0$ and $q\in\mathbb{N}$, we define the following set:

$$\mathrm{Rel}(\alpha,q,\varepsilon):=\left\{\alpha'\in\mathbb{N}^n\ /\ \mathrm{Rel}_\alpha\subset\mathrm{Rel}_{\alpha'}\ \text{ and }\ \max_i\left|q-\frac{\alpha_i'}{\alpha_i}\right|<q\varepsilon\right\}.$$

Example 6.2. If $n=4$, and $\alpha_1=\sqrt{2}$, $\alpha_2=\sqrt{3}$, $\alpha_3=13\sqrt{2}+\sqrt{3}$, $\alpha_4=\sqrt{2}+757\sqrt{3}$, then any α' of the form $(n_1,n_2,13n_1+n_2,n_1+757n_2)$, where n_1, $n_2\in\mathbb{N}_{>0}$, will satisfy $\mathrm{Rel}_\alpha\subset\mathrm{Rel}_{\alpha'}$.

Remark 6.3. For α and $\beta\in\mathbb{R}_{>0}^n$ we have

$$\mathrm{Rel}_\alpha\subset\mathrm{Rel}_\beta\Longleftrightarrow\beta\in V\otimes_{\mathbb{Q}}\mathbb{R}$$

where $V:=(\operatorname{Ker}\alpha^*)^\perp\subset\mathbb{Q}^n$. By definition we have that $\alpha\in V\otimes_{\mathbb{Q}}\mathbb{R}$. Since V is dense in $V\otimes_{\mathbb{Q}}\mathbb{R}$ there exists $\beta\in V$ such that

$$\max_{1\leq i\leq n}\left|1-\frac{\beta_i}{\alpha_i}\right|<\varepsilon.$$

Let us write $\beta_i=\frac{\alpha_i'}{q}$ where the α_i' and q are positive integers. This implies that

$$\max_{1\leq i\leq n}\left|q-\frac{\alpha_i'}{\alpha_i}\right|<q\varepsilon.$$

Since $\beta\in V$ we have that $\alpha'\in V$ thus $\mathrm{Rel}_\alpha\subset\mathrm{Rel}_{\alpha'}$. This shows that for any given $\alpha\in\mathbb{R}_{>0}^n$ and $\varepsilon>0$ there always exists $q\in\mathbb{N}$ such that $\mathrm{Rel}(\alpha,q,\varepsilon)\neq\emptyset$.

Moreover if $\alpha\in\mathbb{N}^n$ then $\mathrm{Rel}(\alpha,q,\varepsilon)=\{q\alpha\}$ if $0<\varepsilon<\frac{1}{q\max\{\alpha_i\}}$. Indeed in this case the only $\alpha'\in\mathbb{N}^n$ satisfying $\max\limits_i|q\alpha_i-\alpha'_i|<q\alpha_i\varepsilon$ is $\alpha'=q\alpha$.

Lemma 6.4. *Let α, $\alpha'\in\mathbb{R}^n_{>0}$. Then $\mathrm{Rel}_\alpha\subset\mathrm{Rel}_{\alpha'}$ if and only if every (α)-homogeneous polynomial is a (α')-homogeneous polynomial.*

Moreover if $\alpha'\in\mathrm{Rel}(\alpha,q,\varepsilon)$ and if $a(\mathbf{x})$ is a (α)-homogeneous polynomial then

$$q(1-\varepsilon)\nu_\alpha(a(\mathbf{x}))\leq\nu_{\alpha'}(a(\mathbf{x}))\leq q(1+\varepsilon)\nu_\alpha(a(\mathbf{x})).$$

Proof. First let us assume that $\mathrm{Rel}_\alpha\subset\mathrm{Rel}_{\alpha'}$ and let $a(\mathbf{x})$ be a (α)-homogeneous polynomial. This means that for any p, $q\in\mathbb{N}^n$, if $\mathbf{x}^p$ and $\mathbf{x}^q$ are two nonzero monomials of $a(\mathbf{x})$, then $\sum_i\alpha_ip_i=\sum_i\alpha_iq_i$. In particular $p-q\in\mathrm{Ker}(\alpha^*)$, thus $\sum_i\alpha'_ip_i=\sum_i\alpha'_iq_i$. Thus $a(\mathbf{x})$ is a (α')-homogeneous.

On the other hand let us assume that every (α)-homogeneous polynomial is a (α')-homogeneous polynomial. Let $r\in\mathrm{Rel}_\alpha$. We can write $r=p-q$ where p, $q\in\mathbb{Q}^n_{>0}$. By multiplying r by a positive integer m, we may assume that mp, $mq\in\mathbb{N}^n$. By assumption on r, the polynomial $\mathbf{x}^{mp}+\mathbf{x}^{mq}$ is (α)-homogeneous. Thus it is (α')-homogeneous. This means that $\sum_i\alpha'_imp_i=\sum_i\alpha'_imq_i$. Hence $\sum_i\alpha'_i(p_i-q_i)=0$ and $r=p-q\in\mathrm{Rel}_{\alpha'}$.

Now let $\mathbf{x}^p$ be a monomial. Then

$$\nu_{\alpha'}(\mathbf{x}^p)=\sum_i\alpha'_ip_i.$$

But $q(1-\varepsilon)\alpha_i\leq\alpha'_i\leq q(1+\varepsilon)\alpha_i$ for any $1\leq i\leq n$. This proves both inequalities. □

Example 6.5. Let $\alpha\in\mathbb{N}^n$ and $\alpha'\in\mathbb{R}^n_{>0}$. Then $\mathrm{Rel}_\alpha\subset\mathrm{Rel}_{\alpha'}$ if and only if there exists $\lambda\in\mathbb{R}$ such that $\alpha'=\lambda\alpha$. Indeed we have $\dim_{\mathbb{Q}}(\mathrm{Rel}_\alpha)=n-1$ hence either $\dim_{\mathbb{Q}}(\mathrm{Rel}_{\alpha'})=n$ and $\alpha'=0$, either $\dim_{\mathbb{Q}}(\mathrm{Rel}_{\alpha'})=n-1$ and there exists $\lambda\in\mathbb{R}^*$ such that $\alpha'=\lambda\alpha$.

Lemma 6.6. *Let $\alpha\in\mathbb{R}^n_{>0}$ and let $A\in\mathcal{V}_\alpha$. Let us write*

$$A=\sum_{i\in\Lambda}\frac{a_i(\mathbf{x})}{\delta(\mathbf{x})^{m(i)}}$$

where Λ is a finitely generated sub-semigroup of $\mathbb{R}_{\geq0}$ and $i\longmapsto m(i)$ is bounded by an affine function. Then there exists $\varepsilon_A>0$ such that for all $0<\varepsilon\leq\varepsilon_A$, for all $q\in\mathbb{N}$, for all $\alpha'\in\mathrm{Rel}(\alpha,q,\varepsilon)$, the element $\displaystyle\sum_{i\in\Lambda}\frac{a_i(\mathbf{x})}{\delta(\mathbf{x})^{m(i)}}$ is in the fraction field of $\mathcal{V}_{\alpha'}$.

Moreover if $A\in\mathcal{V}_\alpha$ is not invertible, i.e., $\nu_\alpha(A)>0$, then we may even choose $\varepsilon_A>0$ such that for all $0\leq\varepsilon\leq\varepsilon_A$, for all $q\in\mathbb{N}$, for all $\alpha'\in\mathrm{Rel}(\alpha,q,\varepsilon)$, $\displaystyle\sum_{i\in\Lambda}\frac{a_i(\mathbf{x})}{\delta(\mathbf{x})^{m(i)}}\in\mathcal{V}_{\alpha'}$ and this element is not invertible in $\mathcal{V}_{\alpha'}$.

Proof. Let a, $b\geq0$ such that $m(i)\leq ai+b$ for any $i\in\Lambda$. By Lemma 6.4 we have

$$\nu_{\alpha'}\left(\frac{a_i(\mathbf{x})}{\delta(\mathbf{x})^{m(i)}}\right)=\nu_{\alpha'}(a_i(\mathbf{x}))-m(i)\nu_{\alpha'}(\delta(\mathbf{x}))\geq q(1-\varepsilon)\nu_\alpha(a_i(\mathbf{x}))-q(1+\varepsilon)m(i)\nu_\alpha(\delta(\mathbf{x}))$$
$$=q(1-\varepsilon)i-2q\varepsilon m(i)\nu_\alpha(\delta(\mathbf{x})).$$

Let ε_A be a positive real number such that $\varepsilon_A<\frac{1}{1+2a\nu_\alpha(\delta(\mathbf{x}))}$ and set

$$\eta:=1-\varepsilon_A(1+2a\nu_\alpha(\delta(\mathbf{x})))>0.$$

Then for any $0 \leq \varepsilon \leq \varepsilon_A$, any $q \in \mathbb{N}$ and any $\alpha' \in \mathrm{Rel}(\alpha, q, \varepsilon)$ we have

$$\nu_{\alpha'}\left(\frac{a_i(\mathbf{x})}{\delta(\mathbf{x})^{m(i)}}\right) \geq \eta q i - 2qb\varepsilon\nu_\alpha(\delta(\mathbf{x})) \ \ \forall i \in \Lambda.$$

This proves that $\displaystyle\sum_{i\in\Lambda} \frac{a_i(\mathbf{x})}{\delta(\mathbf{x})^{m(i)}}$ is in the fraction field of $\mathcal{V}_{\alpha'}$.

If $\nu_\alpha(A) > 0$, then $a_0(\mathbf{x}) = 0$. Let $i_0 := \nu_\alpha(A)$. Let $\varepsilon \geq 0$ be such that $\varepsilon \leq \varepsilon_A$ and

$$i_0 > \varepsilon\left((1 + 2a\nu_\alpha(\delta(\mathbf{x})))i_0 + 2b\nu_\alpha(\delta(\mathbf{x}))\right).$$

In this case $\nu_{\alpha'}\left(\frac{a_i(\mathbf{x})}{\delta(\mathbf{x})^{m(i)}}\right) > 0$ for any $i \in \Lambda$, $i \geq i_0$. This proves the second assertion. □

Definition 6.7. Let $\alpha \in \mathbb{R}^n_{>0}$ and $\alpha' \in \mathrm{Rel}_\alpha \cap \mathbb{N}^n$. Then every (α)-homogeneous polynomial $p(\mathbf{x})$ is (α')-homogenous by Lemma 6.4. In particular if $\delta(\mathbf{x})$ is an other (α)-homogeneous polynomial and $s \in \mathbb{N}$ then

$$p(x_1\delta(\mathbf{x})^{\alpha'_1 s}, \ldots, x_n\delta(\mathbf{x})^{\alpha'_n s}) = p(\mathbf{x})\delta(\mathbf{x})^{s\nu_{\alpha'}(p(\mathbf{x}))}$$

is also a (α)-homogeneous polynomial.

If $A = \sum_{i\in\Lambda} \frac{a_i(\mathbf{x})}{\delta(\mathbf{x})^{m(i)}} \in \mathcal{V}_{\alpha,\delta}$ and $\alpha' \in \mathrm{Rel}_\alpha \cap \mathbb{N}^n$, we will set

$$\varphi_{\alpha',s}(A) := \sum_{i\in\Lambda} \frac{a_i}{\delta_i^{m(i)}}(x_1\delta(\mathbf{x})^{\alpha'_1 s}, \ldots, x_n\delta(\mathbf{x})^{\alpha'_n s}).$$

Then $\varphi_{\alpha,s} : \mathcal{V}_{\alpha,\delta} \longrightarrow \mathcal{V}_{\alpha,\delta}$ is a ring morphism. We also define

$$\psi_{\alpha',s}(A) := \delta^s\varphi_{\alpha',s}(A) \ \ \forall A \in \mathcal{V}_{\alpha,\delta}.$$

Lemma 6.8. *Let $\alpha \in \mathbb{R}^n_{>0}$ and $A \in \mathcal{V}_{\alpha,\delta}$. For any $\varepsilon > 0$ small enough there exists $s(\varepsilon) \in \mathbb{N}$ such that for every $q \in \mathbb{N}$, $\alpha' \in \mathrm{Rel}(\alpha, q, \varepsilon)$ and $s \geq s(\varepsilon)$:*

$$\psi_{\alpha',s}(A) \in \Bbbk[\![\mathbf{x}]\!].$$

If $\nu_\alpha(A) > 0$ we may even assume that $\varphi_{\alpha',s}(A) \in \Bbbk[\![\mathbf{x}]\!]$ for every $q \in \mathbb{N}$, $\alpha' \in \mathrm{Rel}(\alpha, q, \varepsilon)$ and $s \geq s(\varepsilon)$.

Proof. Let $a(\mathbf{x})$, $\delta(\mathbf{x}) \in \Bbbk[\mathbf{x}]$ be (α)-homogeneous polynomials and let $m \in \mathbb{N}$ be such that $\nu_\alpha\left(\frac{a(\mathbf{x})}{\delta(\mathbf{x})^m}\right) = i$. Let $s \in \mathbb{N}$ and $a' \in \mathbb{N}^n$ such that $\mathrm{Rel}_\alpha \subset \mathrm{Rel}_{\alpha'}$. By Lemma 6.4 we have

$$\frac{a(x_1\delta(\mathbf{x})^{\alpha'_1 s}, \ldots, x_n\delta(\mathbf{x})^{\alpha'_n s})}{\delta(x_1\delta(\mathbf{x})^{\alpha'_1 s}, \ldots, x_n\delta(\mathbf{x})^{\alpha'_n s})^m} = a(\mathbf{x})\delta(\mathbf{x})^{s[\nu_{\alpha'}(a(\mathbf{x}))-\nu_{\alpha'}(\delta(\mathbf{x}))m]-m}. \tag{9}$$

Now let $A = \displaystyle\sum_{i\in\Lambda} \frac{a_i}{\delta^{m(i)}} \in \mathcal{V}_{\alpha,\delta}$ with $m(i) \leq ai + b$ for any $i \in \Lambda$, Λ being a finitely generated sub-semigroup of $\mathbb{R}_{\geq 0}$. Set $d_\alpha := \nu_\alpha(\delta)$. Thus $\nu_\alpha(a_i) = d_\alpha m(i) + i$ for any $i \in \Lambda$. Hence by Lemma 6.4 we have that

$$\nu_{\alpha'}(a_i) - m(i)\nu_{\alpha'}(\delta) \geq q(1-\varepsilon)\left[d_\alpha m(i) + i\right] - q(1+\varepsilon)m(i)d_\alpha$$

$$\nu_{\alpha'}(a_i) - m(i)\nu_{\alpha'}(\delta) \geq q(1-\varepsilon)i - 2q\varepsilon d_\alpha m(i). \tag{10}$$

Since $(1-\varepsilon)i - 2\varepsilon d_\alpha m(i) \geq (1-\varepsilon)i - 2\varepsilon d_\alpha(ai+b)$, for every ε small enough there exists $a_\varepsilon > 0$ such that

$$\nu_{\alpha'}(a_i) - m(i)\nu_{\alpha'}(\delta) \geq qa_\varepsilon i$$

for all $q \in \mathbb{N}$, all $\alpha' \in \mathrm{Rel}(\alpha, q, \varepsilon)$ and all $i \in \Lambda$, $i > 0$. Thus for $s \in \mathbb{N}$ and $i \in \Lambda \backslash \{0\}$ we have that

$$s(\nu_{\alpha'}(a_i) - m(i)\nu_{\alpha'}(\delta)) - m(i) \geq sqa_\varepsilon i - m(i) \geq (sqa_\varepsilon - a)i - b \geq \left(sqa_\varepsilon - a - \frac{b}{\min \Lambda\backslash\{0\}}\right) i.$$

In particular if $s \geq \left(a + \frac{b}{\min \Lambda\backslash\{0\}}\right)/a_\varepsilon$ then

$$s(\nu_{\alpha'}(a_i) - m(i)\nu_{\alpha'}(\delta)) - m(i) \geq 0$$

and $\frac{a_i}{\delta_i^{m(i)}}(x_1\delta(\mathbf{x})^{\alpha'_1 s}, \ldots, x_n\delta(\mathbf{x})^{\alpha'_n s}) \in \Bbbk[\![\mathbf{x}]\!]$ for all $i > 0$. Thus if $\nu_\alpha(A) > 0$, $a_0 = 0$ and $\varphi_{\alpha',s}(A) \in \Bbbk[\![\mathbf{x}]\!]$ for $s \geq \left(a + \frac{b}{\min \Lambda\backslash\{0\}}\right)/a_\varepsilon$.

In the general case where $a_0 \neq 0$, if we assume moreover that $s \geq b$, we have that

$$\delta(x)^s \frac{a_0}{\delta_0^{m(0)}}(x_1\delta(\mathbf{x})^{\alpha'_1 s}, \ldots, x_n\delta(\mathbf{x})^{\alpha'_n s}) \in \Bbbk[\![\mathbf{x}]\!].$$

This proves the lemma. □

When the components of α are $\mathbb{Q}$-linearly independent, by using Lemma 6.8, Theorem 5.12 gives the following generalization of the main result of [McD]:

Theorem 6.9. [McD] *Let $\Bbbk$ be a field of characteristic zero and $\alpha \in \mathbb{R}^n_{>0}$ such that*

$$\dim_{\mathbb{Q}}(\mathbb{Q}\alpha_1 + \cdots + \mathbb{Q}\alpha_n) = n.$$

Then

$$\mathbb{K}^{\mathrm{alg}}_{\nu_\alpha} \subset \bigcup_\sigma \Bbbk\left(\!\left(x^\beta, \beta \in \sigma \cap \mathbb{Z}^n\right)\!\right)$$

where the first union runs over all rational strongly convex cones σ such that $\langle \alpha, \tau \rangle > 0$ for any $\tau \in \sigma$, $\tau \neq 0$. Moreover we have:

$$\overline{\mathbb{K}}^{\mathrm{alg}}_{\nu_\alpha} \subset \bigcup_\sigma \bigcup_{\Bbbk'} \bigcup_{q \in \mathbb{N}} \Bbbk'\left(\!\left(x^\beta, \beta \in \sigma \cap \frac{1}{q}\mathbb{Z}^n\right)\!\right)$$

where the first union runs over all rational strongly convex cones σ such that $\langle \alpha, \tau \rangle > 0$ for any $\tau \in \sigma$, $\tau \neq 0$, and the second union runs over all the fields $\Bbbk'$ finite over $\Bbbk$.

Proof. In order to prove the first inclusion, by Corollary 5.13 it is enough to prove that

$$\mathcal{K}_\alpha \subset \bigcup_\sigma \Bbbk(\!(x^\beta, \beta \in \sigma \cap \mathbb{Z}^n)\!)$$

or $\mathcal{V}_\alpha \subset \bigcup_\sigma \Bbbk\left[\!\left[x^\beta, \beta \in \sigma \cap \mathbb{Z}^n\right]\!\right]$.

Since the α_i are $\mathbb{Q}$-linearly independent the only (α)-homogeneous polynomials are the monomials. Let $\omega \in \mathbb{N}^n$ and A be an element of $\mathcal{V}_{\alpha,\mathbf{x}^\omega}$: $A = \sum_{i\in\Lambda} \frac{\mathbf{x}^{p(i)}}{\mathbf{x}^{m(i)\omega}}$ where Λ is a finitely generated sub-semigroup of $\mathbb{R}_{\geq 0}$. We have to prove that $A \in \bigcup_\sigma \Bbbk\left[\!\left[x^\beta, \beta \in \sigma \cap \mathbb{Z}^n\right]\!\right]$. Since $x_1 A \in \bigcup_\sigma \Bbbk\left[\!\left[x^\beta, \beta \in \sigma \cap \mathbb{Z}^n\right]\!\right]$ implies that $A \in \bigcup_\sigma \Bbbk\left[\!\left[x^\beta, \beta \in \sigma \cap \mathbb{Z}^n\right]\!\right]$ we may assume that $\nu_\alpha(A) > 0$.

By Lemma 6.8, we see that the monomial map $\varphi_{\alpha',s}$ defined by $x_j \longmapsto x_j\mathbf{x}^{s\alpha'_j\omega}$ maps A onto an element of $\Bbbk[\![\mathbf{x}]\!]$ for $\alpha' \in \mathrm{Rel}(\alpha, q, \varepsilon)$, $\varepsilon > 0$ small enough and s large enough. Such a monomial map is induced by a linear map on the set of monomials and its matrix is

$$M_1 := \begin{pmatrix} 1+s\omega_1\alpha'_1 & s\omega_1\alpha'_2 & s\omega_1\alpha'_3 & \cdots & s\omega_1\alpha'_n \\ s\omega_2\alpha'_1 & 1+s\omega_2\alpha'_2 & s\omega_2\alpha'_3 & \cdots & s\omega_2\alpha'_n \\ s\omega_3\alpha'_1 & s\omega_3\alpha'_2 & 1+s\omega_3\alpha'_3 & \cdots & s\omega_3\alpha'_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s\omega_n\alpha'_1 & s\omega_n\alpha'_2 & s\omega_n\alpha'_3 & \cdots & 1+s\omega_n\alpha'_n \end{pmatrix}$$

Set

$$M_2 := \begin{pmatrix} -s\omega_1\alpha'_1 & -s\omega_1\alpha'_2 & -s\omega_1\alpha'_3 & \cdots & -s\omega_1\alpha'_n \\ -s\omega_2\alpha'_1 & -s\omega_2\alpha'_2 & -s\omega_2\alpha'_3 & \cdots & -s\omega_2\alpha'_n \\ -s\omega_3\alpha'_1 & -s\omega_3\alpha'_2 & -s\omega_3\alpha'_3 & \cdots & -s\omega_3\alpha'_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -s\omega_n\alpha'_1 & -s\omega_n\alpha'_2 & -s\omega_n\alpha'_3 & \cdots & -s\omega_n\alpha'_n \end{pmatrix}$$

and let $\chi(t)$ be the characteristic polynomial of M_2. Then $\chi(1) = \det(M_1)$. If $\chi(1) = 0$, then the vector $\omega := (\omega_1, \ldots, \omega_n)$ is an eigenvector of M_2 with eigenvalue 1 since the image of M_2 is generated by ω. Thus $-s(\omega_1\alpha'_1 + \cdots + \omega_n\alpha'_n) = 1$ which is not possible since $\omega_i \geq 0$ and $\alpha'_i > 0$ for any i. Thus $\det(M_1) \neq 0$ and M_1 is invertible. In particular $\sigma := M_1^{-1}(\mathbb{R}^n_{\geq 0})$ is a rational strongly convex cone. Moreover, since $A \in \mathcal{V}_{\alpha,\delta}$, we have $\langle \alpha, \tau \rangle > 0$ for any $\tau \in \sigma$, $\tau \neq 0$. Hence $A \in \Bbbk\left[\!\left[x^\beta, \beta \in \sigma \cap \mathbb{Z}^n\right]\!\right]$.

By Example 3.18 integral homogeneous elements with respect to ν_α are either finite over $\Bbbk$, either of the form $cx_1^{\frac{n_1}{q}} \cdots x_n^{\frac{n_n}{q}}$ for some integers $n_1, \ldots, n_n \in \mathbb{Z}_{\geq 0}$, $q \in \mathbb{N}$ such that $\sum_{j=1}^n \alpha_j n_j > 0$. Using Theorem 5.12 and since $\overline{\mathbb{K}}_{\nu_\alpha} = \varinjlim_{\gamma_1,\ldots,\gamma_s} \mathbb{K}^{\text{alg}}_{\nu_\alpha}[\gamma_1, \ldots, \gamma_s]$ where the γ_i are homogeneous with respect to ν_α, we have the second inclusion by replacing σ by the rational strongly convex cone generated by σ and the n-uples $(n_1, \ldots, n_n)$ corresponding to the homogeneous elements $\gamma_1, \ldots, \gamma_s$.

□

Remark 6.10. In fact the proof shows that the field $\overline{\mathcal{K}}_\alpha$, as soon as

$$\dim_{\mathbb{Q}}(\mathbb{Q}\alpha_1 + \cdots + \mathbb{Q}\alpha_n) = n,$$

is the field of Puiseux power series with support in rational strongly convex cones σ such that $\langle \alpha, \gamma \rangle \geq 0$ for all $\gamma \in \sigma$. Thus $\overline{\mathcal{K}}_\alpha$ is the field of *α-positive Puiseux series* according to [AI].

Lemma 6.11. *Let $\alpha \in \mathbb{R}^n_{>0}$ and $\alpha' \in \mathrm{Rel}_\alpha \cap \mathbb{N}^n$. Then*

$$\psi_{\alpha',t} \circ \psi_{\alpha',s} = \psi_{\alpha',\nu_{\alpha'}(\delta)st+s+t} \quad \forall s,t \in \mathbb{Z}_{\geq 0}.$$

Proof. Let $A = \sum_{i\in\Lambda} \frac{a_i}{\delta^{m(i)}} \in \mathcal{V}_{\alpha,\delta}$. Then we have (see Equation (9) in the proof of Lemma 6.8):

$$\delta^s \varphi_{\alpha',s}(A) = \sum_{i\in\Lambda} a_i(\mathbf{x})\delta(\mathbf{x})^{s(1+\nu_{\alpha'}(a_i(\mathbf{x}))-\nu_{\alpha'}(\delta(\mathbf{x}))m(i))-m(i)}.$$

If $t \in \mathbb{Z}_{\geq 0}$ and $l \in \mathbb{Z}_{\geq 0}$, and $\alpha(\mathbf{x})$ and $\delta(x)$ are (α')-homogeneous, we have that

$$\varphi_{\alpha',t}(a(\mathbf{x})\delta(\mathbf{x})^l) = a(\mathbf{x})\delta(\mathbf{x})^{t\nu_{\alpha'}(a)+l(t\nu_{\alpha'}(\delta)+1)}.$$

Thus by denoting

$$p_{\alpha'}(i) := \nu_{\alpha'}(a_i(\mathbf{x})) - \nu_{\alpha'}(\delta(\mathbf{x}))m(i) \text{ and } d_{\alpha'} := \nu_{\alpha'}(\delta(\mathbf{x}))$$

we obtain

$\delta^t\varphi_{\alpha',t}(\delta^s\varphi_{\alpha',s}(A))$

$= \delta^t \sum_{i\in\Lambda} a_i(x_1\delta^{\alpha'_1 t},\ldots,x_n\delta^{\alpha'_n t}) \times \delta(x_1\delta^{\alpha'_1 t},\ldots,x_n\delta^{\alpha'_n t})^{s(1+\nu_{\alpha'}(a_i(\mathbf{x}))-\nu_{\alpha'}(\delta)m(i))-m(i)}$

$= \sum_{i\in\Lambda} a_i\delta^{t+t\nu_{\alpha'}(a_i)+(s+sp_{\alpha'}(i)-m(i))(td_{\alpha'}+1)}$

$= \sum_{i\in\Lambda} a_i\delta^{d_{\alpha'}tsp_{\alpha'}(i)+tp_{\alpha'}(i)+sp_{\alpha'}(i)-m(i)+d_{\alpha'}ts+t+s}.$

In particular we have

$$\delta^t\varphi_{\alpha',t}(\delta^s\varphi_{\alpha',s}(A)) = \delta^{d_{\alpha'}st+s+t}\varphi_{\alpha',d_{\alpha'}st+s+t}(A) \quad \forall t\in\mathbb{Z}_{\geq 0}. \tag{11}$$

□

Lemma 6.12. *Let $\alpha\in\mathbb{R}^n_{>0}$ and $\alpha'\in \operatorname{Rel}_\alpha\cap\mathbb{N}^n$. For all s_1, $s_2\in\mathbb{N}$ there exist t_1, $t_2\in\mathbb{N}$ such that*

$$\psi_{\alpha',t_1}\circ\psi_{\alpha',s_1} = \psi_{\alpha',t_2}\circ\psi_{\alpha',s_2}.$$

Proof. Let d denote $\nu_{\alpha'}(\delta)$. Let p be a prime number and $k\in\mathbb{N}$ such that p^k divides ds_1+1 and ds_2+1. Then $\gcd(p,d)=1$ and p^k divides ds_1-ds_2. Thus p^k divides s_1-s_2. This proves that $\gcd(ds_1+1,ds_2+1)$ divides s_1-s_2. Thus there exist $t_1\in\mathbb{Z}$ and $t_2\in\mathbb{Z}$ such that $(ds_1+1)t_1-(ds_2+1)t_2=s_2-s_1$. If $t_1t_2<0$, let say $t_1>0$ and $t_2<0$, then

$$(ds_1+1)t_1-(ds_2+1)t_2 > s_1+s_2 > |s_1-s_2|$$

which is not possible. Thus we have that $t_1t_2\geq 0$. If $t_1\leq 0$ and $t_2\leq 0$, we can replace t_1 (resp. t_2) by $t_1+k(ds_2+1)$ (resp. by $t_2+k(ds_1+1)$) for some positive integer k large enough. This will allows to assume that t_1 and t_2 are positive integers. Hence

$$\exists t_1,t_2\in\mathbb{N},\ ds_1t_1+s_1+t_1 = ds_2t_2+s_2+t_2.$$

This proves the lemma by Lemma 6.11.

□

Definition 6.13. Now we consider a subring R of $\Bbbk[\![\mathbf{x}]\!]$ that is an excellent Henselian local ring with maximal ideal $\mathfrak{m}_R$ and satisfying the following properties:

(A) $\Bbbk[x_1,\ldots,x_n]_{(\mathbf{x})}\subset R$,
(B) $\mathfrak{m}_R=(\mathbf{x})R$ and $\widehat{R}=\Bbbk[\![\mathbf{x}]\!]$,
(C) if $p(\mathbf{x})\in\Bbbk[\mathbf{x}]$ is (α)-homogeneous for some $\alpha\in\mathbb{R}^n_{>0}$ then

$$f(\mathbf{x})\in R \implies f(p(\mathbf{x})x_1,\ldots,p(\mathbf{x})x_n)\in R.$$

Remark 6.14. If $\Bbbk$ is a field, the ring of algebraic power series $\Bbbk\langle\mathbf{x}\rangle$ is an excellent Henselian local ring satisfying Properties (A), (B) and (C). If $\Bbbk$ is a valued field, then the field of convergent power series $\Bbbk\{\mathbf{x}\}$ does also.

For a field $\Bbbk$, the ring $\Bbbk[\![x_1,\ldots,x_r]\!]\langle x_{r+1},\ldots,x_n\rangle$ for formal power series algebraic over $\Bbbk[\![x_1,\ldots,x_r]\!][x_{r+1},\ldots,x_n]$ is also an excellent Henselian local ring satisfying Properties (A), (B) and (C).

Definition 6.15. Let $\alpha\in\mathbb{R}^n_{>0}$ and let δ be a (α)-homogeneous polynomial. Let R be a ring satisfying Definition 6.13. We set

$$\begin{aligned}\mathcal{V}^R_{\alpha,\delta}:=\Big\{&A\in\widehat{V}_{\nu_\alpha}\ /\ \exists\Lambda\ \text{ a finitely generated sub-semigroup of }\mathbb{R}_{\geq 0},\\ &\forall i\in\Lambda\ \exists a_i\in\Bbbk[\mathbf{x}]\ (\alpha)\text{-homogeneous},\exists a,b\geq 0\ \forall i\in\Lambda\ \exists m(i)\in\mathbb{N}\text{ s.t.}\\ &\qquad m(i)\leq ai+b,\ \nu_\alpha\left(\frac{a_i}{\delta^{m(i)}}\right)=i,\ A=\sum_{i\in\Lambda}\frac{a_i}{\delta^{m(i)}}\\ &\text{and }\ \exists\varepsilon>0\ \forall q\in\mathbb{N}\ \forall\alpha'\in\operatorname{Rel}(\alpha,q,\varepsilon)\ \ \exists s\in\mathbb{N}\ \text{ such that }\psi_{\alpha',s}(A)\in R\Big\}.\end{aligned}$$

Then $\mathcal{V}^R_\alpha$ is the union of the sets $\mathcal{V}^R_{\alpha,\delta}$ when δ runs over all the (α)-homogeneous polynomials.

Lemma 6.16. *The sets $\mathcal{V}^R_{\alpha,\delta}$ and $\mathcal{V}^R_\alpha$ are subrings of $\mathcal{V}_{\alpha,\delta}$ and $\mathcal{V}_\alpha$.*

Proof. Let $A=\sum_{i\in\Lambda}\frac{a_i}{\delta^{m(i)}}$ and $B=\sum_{i\in\Lambda}\frac{b_i}{\delta^{n(i)}}\in\mathcal{V}^R_{\alpha,\delta}$. Then there exists $\varepsilon>0$ such that $\forall q\in\mathbb{N}$, $\forall\alpha'\in\operatorname{Rel}(\alpha,q,\varepsilon)$, there exist $s_1,s_2\in\mathbb{N}$ such that

$$\psi_{\alpha',s_1}(A),\psi_{\alpha',s_2}(B)\in R.$$

Then by Lemma 6.11, Lemma 6.12 and condition (C) of Definition 6.13 there exists $s\in\mathbb{N}$ such that

$$\psi_{\alpha',s}(A),\psi_{\alpha',s}(B)\in R.$$

This shows that $\psi_{\alpha',s}(A+B)=\psi_{\alpha',s}(A)+\psi_{\alpha',s}(B)\in R$ and $A+B\in\mathcal{V}^R_{\alpha,\delta}$.

Now by Lemma 6.8 we can assume that there exists $s(\varepsilon)\in\mathbb{N}$ such that $\psi_{\alpha',s}(AB)\in\Bbbk[\![\mathbf{x}]\!]$ for all $s>s(\varepsilon)$, for all $q\in\mathbb{N}$ and all $\alpha'\in\operatorname{Rel}(\alpha,q,\varepsilon)$. On the other hand since $\psi_{\alpha',s}(A),\psi_{\alpha',s}(B)\in R$ then $\psi_{\alpha',\nu_{\alpha'}(\delta)st+s+t}(A)$, $\psi_{\alpha',\nu_{\alpha'}(\delta)st+s+t}(B)\in R$ for all $t\in\mathbb{N}$ by Lemma 6.11 and Condition (C) of Definition 6.13. Thus there exists $s\in\mathbb{N}$ such that

$$\psi_{\alpha',s}(A),\psi_{\alpha',s}(B)\in R\text{ and }\psi_{\alpha',s}(AB)\in\Bbbk[\![\mathbf{x}]\!].$$

But we have that

$$\psi_{\alpha',s}(A)\psi_{\alpha',s}(B)=\delta^s\psi_{\alpha',s}(AB)\in R.$$

Hence by Artin Approximation Theorem (cf. [Po], [Sp2]) $\psi_{\alpha',s}(AB)\in R$.

Thus $AB\in\mathcal{V}^R_{\alpha,\delta}$. This proves that $\mathcal{V}^R_{\alpha,\delta}$ is a ring.

Since $\mathcal{V}^R_\alpha$ is the direct limit of the $\mathcal{V}^R_{\alpha,\delta}$ it is also a ring. □

Example 6.17. If $\alpha\in\mathbb{N}^n$ and $R=\mathbb{C}\{\mathbf{x}\}$ is the ring of convergent power series over $\mathbb{C}$, we claim that

$$\begin{aligned}\mathcal{V}^{\mathbb{C}\{\mathbf{x}\}}_{\alpha,\delta}=\Bigg\{&\sum_{i\in\mathbb{Z}_{\geq 0}}\frac{a_i}{\delta^{a(i+1)}}\ /\ \forall i\ \ a_i\in\mathbb{C}[\mathbf{x}]\text{ is }(\alpha)\text{-homogeneous},\\ &\qquad\nu_\alpha\left(\frac{a_i}{\delta^{a(i+1)}}\right)=i,\ a\in\mathbb{Z}_{\geq 0}\\ &\text{and }\ \exists C,r>0\text{ such that }\ |a_i(z)|\leq Cr^i\|z\|_\alpha^{\nu_\alpha(a_i)}\ \ \forall z\in\mathbb{C}^n\Big\}\end{aligned}$$

where $\|z\|_\alpha:=\max_{j=1,\dots,n}\left|z_j^{\frac{1}{\alpha_j}}\right|$ for any $z\in\mathbb{C}^n$.

First of all every element A of $\mathcal{V}_{\alpha,\delta}$ is of the form

$$A=\sum_{i\in\mathbb{Z}_{\geq 0}}\frac{a_i}{\delta^{m(i)}}$$

where $\nu_\alpha\left(\frac{a_i}{\delta^{m(i)}}\right)=i$ and $m(i)\leq ai+b$ for some $a,\,b\in\mathbb{Z}_{\geq 0}$. By multiplying the numerator and the denominator of $\frac{a_i}{\delta^{m(i)}}$ by $\delta^{ai+b-m(i)}$ and replacing a_i by $a_i\delta^{ai+b-m(i)}$, we may assume that $m(i)=ai+b$. If $a>b$, we may replace $\frac{a_i}{\delta^{ai+b}}$ by $\frac{a_i\delta^{a-b}}{\delta^{ai+a}}$, if $a<b$ we may replace $\frac{a_i}{\delta^{ai+b}}$ by $\frac{a_i\delta^{(b-a)i}}{\delta^{bi+b}}$. Thus any element of $\mathcal{V}_{\alpha,\delta}$ is of the form $\displaystyle\sum_{i\in\mathbb{Z}_{\geq 0}}\frac{a_i}{\delta^{a(i+1)}}$ where $\nu_\alpha\left(\frac{a_i}{\delta^{a(i+1)}}\right)=i$ for all $i\in\mathbb{Z}_{\geq 0}$. In this case $\nu_\alpha(a_i)=(a\nu_\alpha(\delta)+1)i+a\nu_\alpha(\delta)$ for any $i\in\mathbb{N}$.

By Remark 6.3 $\mathrm{Rel}(\alpha,q,\varepsilon)=\{q\alpha\}$ for $\varepsilon>0$ small enough since $\alpha\in\mathbb{N}^n$. Then we have (with $s=a$ in Lemma 6.8):

$$f(\mathbf{x}):=\psi_{\alpha,a}(A)=\delta(\mathbf{x})^a\sum_{i\in\mathbb{Z}_{\geq 0}}\frac{a_i}{\delta^{a(i+1)}}(x_1\delta(\mathbf{x})^{\alpha_1 a},\ldots,x_n\delta(\mathbf{x})^{\alpha_n a})=\sum_{i\in\mathbb{Z}_{\geq 0}}a_i(\mathbf{x})$$

and $f(\mathbf{x})\in\mathbb{C}[\![\mathbf{x}]\!]$. Moreover we have for every $q\in\mathbb{N}$

$$f_q(\mathbf{x}):=\psi_{q\alpha,a}(A)=\delta(\mathbf{x})^a\sum_{i\in\mathbb{Z}_{\geq 0}}\frac{a_i}{\delta^{a(i+1)}}(x_1\delta(\mathbf{x})^{\alpha_1 qa},\ldots,x_n\delta(\mathbf{x})^{\alpha_n qa})=\sum_{i\in\mathbb{Z}_{\geq 0}}a_i(\mathbf{x})\delta(\mathbf{x})^{a(q-1)i}$$

Thus $f\in\mathbb{C}\{\mathbf{x}\}$ if and only if this power series is convergent on a neighborhood of the origin. This neighborhood may be chosen of the form:

$$B_\alpha(0,r):=\{z\in\mathbb{C}^n\ /\ |z_j|\leq r^{\alpha_j},\ j=1,\ldots,n\}.$$

For any $z\in B_\alpha(0,r)$ set $t_j^{\alpha_j}=z_j$ for $j=1,\ldots,n$ and $b_i(t)=a_i(z)$ for any $i\in\mathbb{N}$. Then f is convergent on $B_\alpha(0,r)$ if and only $\displaystyle\sum_{i\in\mathbb{Z}_{\geq 0}}b_i(t)$ is convergent on

$$B(0,r):=\{t\in\mathbb{C}^n\ /\ |t_j|\leq r,\ j=1,\ldots,n\}.$$

But this series is convergent if and only if there exist $c\geq 0$ and $\rho<1$ such that $|b_i(t)|\leq c\rho^i$ for all $i\in\mathbb{Z}_{\geq 0}$ and all $t\in B(0,r)$. Since $b_i(t)$ is a homogeneous polynomial of degree

$$\nu_\alpha(a_i)=(ad+1)i+ad$$

where $d:=\nu_\alpha(\delta)$, we have

$$\sup_{|t_j|\leq r,j=1,\ldots,n}|b_i(t)|=r^{(ad+1)i+ad}\sup_{|t_j|\leq 1,j=1,\ldots,n}|b_i(t)|.$$

We see that f is convergent if and only if there exist $C\geq 0$ and $R>0$ such that

$$\sup_{|z_j|\leq 1,j=1,\ldots,n}|a_i(z)|=\sup_{|t_j|\leq 1,j=1,\ldots,n}|b_i(t)|\leq CR^i.$$

This is equivalent to the following inequality for any $z\in\mathbb{C}^n$:

$$|a_i(z)|=|b_i(t)|\leq\max_{j=1,\ldots,n}|t_j|\sup_{|t_j|\leq 1,j=1,\ldots,n}|b_i(t)|\leq CR^i\|z\|_\alpha^{\nu_\alpha(a_i)}. \tag{12}$$

On the other hand if $f\in\mathbb{C}\{\mathbf{x}\}$ we have seen that there exist $C\geq 0$ and $R>0$ such that

$$\sup_{|z_j|\leq 1,j=1,\ldots,n}|a_i(z)|\leq CR^i.$$

Thus

$$\sup_{|z_j|\leq 1,j=1,\ldots,n}|a_i(z)\delta(z)^{a(q-1)i}|\leq C(RS)^i$$

where $S:=\max_{|z_j|\leq 1,j=1,\ldots,n}|\delta(z)|^{a(q-1)}$. Hence $f_q\in\mathbb{C}\{\mathbf{x}\}$ for every $q\in\mathbb{N}$. This proves the claim.

We have the following analogue of Theorem 5.12 in the Henselian case:

Theorem 6.18. *Let $\Bbbk$ be a field of characteristic zero and let R be a subring of $\Bbbk[\![\mathbf{x}]\!]$ satisfying Definition 6.13. Let $\alpha \in \mathbb{R}^n_{>0}$ and let us set $N = \dim_{\mathbb{Q}}(\mathbb{Q}\alpha_1 + \cdots + \mathbb{Q}\alpha_n)$.*

Let $P(Z) \in \mathcal{V}^R_\alpha[\langle\gamma_1, \ldots, \gamma_s\rangle][Z]$ be a distinguished polynomial of degree d where the γ_i are homogeneous elements with respect to ν_α. Then the roots of $P(Z)$ are in $\mathcal{V}^R_\alpha[\langle\gamma'_1, \ldots, \gamma'_N\rangle]$ for some integral homogeneous elements $\gamma'_1, \ldots, \gamma'_N$ with respect to ν_α.

Proof. Let $P(Z) = Z^d + a_1 Z^{d-1} + \cdots + a_d$ with $a_j \in \mathcal{V}^R_\alpha[\langle\gamma_1, \ldots, \gamma_s\rangle]$ for $1 \leq j \leq d$. By Theorem 5.12 we may assume that $P(Z)$ has a root $z \in \mathcal{V}_{\alpha,\delta}[\langle\gamma_1, \ldots, \gamma_N\rangle]$. We denote

$$a_i = \sum_{i_1,\cdots,i_N} A_{i,i_1,\ldots,i_N}\gamma_1^{i_1}\cdots\gamma_N^{i_N} \text{ with } A_{i,i_1,\ldots,i_N} \in \mathcal{V}_{\alpha,\delta},$$

$$z = \sum_{i_1,\ldots,i_N} z_{i_1,\ldots,i_N}\gamma_1 t^{i_1}\cdots\gamma_N^{i_N} \text{ with } z_{i_1,\ldots,i_N} \in \mathcal{V}_{\alpha,\delta}.$$

Let us fix $\varepsilon > 0$, $q \in \mathbb{N}$, $\alpha' \in \mathrm{Rel}(\alpha, q, \varepsilon)$ and s satisfying Lemma 6.8 for the $A_{i,i_1,\ldots,i_N}$ and for the $z_{i_1,\ldots,i_N}$. For convenience we denote by φ the morphism $\varphi_{\alpha',s}$ defined in Definition 6.7. Then if A denotes one of the $A_{i,i_1,\ldots,i_N}$ or the $Z_{i_1,\ldots,i_N}$ we have $\varphi(A) \in \mathcal{V}_{\alpha',\delta}$ by Lemma 6.8. We set

$$R := \mathcal{V}_{\alpha,\delta} \cap \varphi^{-1}(\mathcal{V}_{\alpha',\delta})$$

and R' denotes the subring of $\mathcal{V}_{\alpha,\gamma}[\langle\gamma_1, \ldots, \gamma_s\rangle]$ of elements $\sum_{i_1,\ldots,i_N} A_{i_1,\ldots,i_N}\gamma_1^{i_1}\cdots\gamma_N^{i_N}$ whose coefficients $A_{i_1,\ldots,i_N}$ are in R.

Of course φ induces a morphism $R \longrightarrow \mathcal{V}_{\alpha',\delta}$ but we have the following lemma:

Lemma 6.19. *Let γ_i be homogeneous elements with respect to ν_α for $1 \leq i \leq N$. Then there exist homogeneous element γ'_i with respect to $\nu_{\alpha'}$, $1 \leq i \leq N$, such that, for any finite number of elements $A_{i_1,\ldots,i_N} \in \mathcal{V}_{\alpha,\delta}$,*

$$\varphi\left(\sum_{i_1,\ldots,i_N} A_{i_1,\ldots,i_N}{\gamma_1}^{i_1}\cdots{\gamma_N}^{i_N}\right) := \sum_{i_1,\ldots,i_N}\varphi(A_{i_1,\ldots,i_N}){\gamma'_1}^{i_1}\cdots{\gamma'_N}^{i_N}$$

defines an extension of φ from R' to $\mathcal{V}_{\alpha',\delta}[\langle\gamma'_1, \ldots, \gamma'_N\rangle]$.

Proof of Lemma 6.19. Let us assume that γ_i is a homogeneous element of degree e_i with respect to ν_α. Let

$$Q_i(Z) := g_{i,0}(\mathbf{x})Z^{q_i} + g_{i,1}(\mathbf{x})Z^{q_i-1} + \cdots + g_{i,q_i}(\mathbf{x})$$

be a polynomial such that $Q_i(\gamma'_i) = 0$ and such that $g_{i,j}(\mathbf{x})$ is a (α)-homogeneous polynomial of degree $d_i + je_i$ for some d_i.

Then $g_{i,j}(\mathbf{x})$ is a (α')-homogeneous polynomial of degree $d'_i + je'_i$ for some constants d'_i and e'_i. Indeed, if a, b and c are (α)-homogeneous polynomials and $\nu_\alpha(a) - \nu_\alpha(b) = \nu_\alpha(b) - \nu_\alpha(c)$, then ac and b^2 are two (α)-homogeneous polynomials of same degree, i.e., $ac - b^2$ is (α)-homogeneous. Then, by Lemma 6.4, $ac - b^2$ is (α')-homogeneous, thus $\nu_{\alpha'}(a) - \nu_{\alpha'}(b) = \nu_{\alpha'}(b) - \nu_{\alpha'}(c)$.

Set $\overline{Q}_i(Z) = \delta^{se'_iq_i}Q_i\left(\frac{Z}{\delta^{se'_i}}\right)$. We have

$$\overline{Q}_i(Z) := g_{i,0}(\mathbf{x})Z^{q_i} + g_{i,1}(\mathbf{x})\delta(\mathbf{x})^{se'_i}Z^{q_i-1} + \cdots + g_{q_i}(\mathbf{x})\delta(\mathbf{x})^{se'_iq_i}.$$

For any i let γ'_i denote a root of $\overline{Q}_i(Z)$. So γ'_i is a homogeneous element of degree $e'_i(1+\nu_{\alpha'}(\delta(\mathbf{x}))s)$ with respect to $\nu_{\alpha'}$. Then it is straightforward to check that

$$\varphi\left(\sum A_{i_1,\ldots,i_N}{\gamma_1}^{i_1}\cdots{\gamma_N}^{i_N}\right) = \sum\varphi(A_{i_1,\ldots,i_N}){\gamma'_1}^{i_1}\cdots{\gamma'_N}^{i_N}$$

defines an extension of φ from R' to $\mathcal{V}_{\alpha',\delta}[\langle\gamma'_1, \ldots, \gamma'_N\rangle]$. $\square$

By Lemmas 6.19, 6.11 and 6.12, and Property (C) we can assume that s is large enough for having that

$$\delta^{js}\varphi(A_j) \in R[\gamma'_1, \dots, \gamma'_N]$$

for $1 \leq j \leq d$. Again by applying Lemmas 6.19, 6.11 and 6.12 we may even assume that

$$\delta^{s}\varphi(z) \in \Bbbk[\![\mathbf{x}]\!][\gamma'_1, \dots, \gamma'_N]$$

by taking s large enough. Thus $z' := \delta^s\varphi(z) \in \Bbbk[\![\mathbf{x}]\!][\gamma'_1, \dots, \gamma'_N]$ is a root of the polynomial

$$\overline{P}(Z) := Z^d + \delta^s\varphi(A_1)Z^{d-1} + \cdots + \delta^{ds}\varphi(A_d) \in R[Z].$$

Let us write

$$z' := \sum_{i_1,\dots,i_N} z'_{i_1,\dots,i_N} {\gamma'_1}^{i_1} \cdots {\gamma'_N}^{i_r}$$

with $z'_{i_1,\dots,i_N} \in \Bbbk[\![x]\!]$ for any $i_1, \dots, i_N$. Let us set

$$Z := \sum_{i_1,\dots,i_N} Z_{i_1,\dots,i_N} {\gamma'_1}^{i_1} \cdots {\gamma'_N}^{i_N}$$

where $Z_{i_1,\dots,i_N}$ are new variables. Solving $P(Z) = 0$ is equivalent to solve a finite system $(\mathcal{S})$ of polynomial equations in the variables $Z_{i_1,\dots,i_N}$ with coefficients in R, just by replacing Z by $\sum_{i_1,\dots,i_N} Z_{i_1,\dots,i_N}{\gamma_1}^{i_1}\cdots{\gamma_N}^{i_N}$ and replacing the high powers of the γ_i by smaller ones using the division by the $Q_i(Z_i)$. By Artin Approximation Theorem (cf. [Po], [Sp2]), the set of solutions of $(\mathcal{S})$ in R is dense in the set of solutions in $\Bbbk[\![\mathbf{x}]\!]$, but since $P(Z) = 0$ has a finite number of solutions, then $(\mathcal{S})$ has a finite number of solutions and they are in R. Thus $z'_{i_1,\dots,i_N} \in R$ for all $i_1, \dots, i_N$, hence $z' \in R[\gamma'_1, \dots, \gamma'_N]$. This proves that $z \in \mathcal{V}^R_{\alpha,\delta}[\langle\gamma_1, \dots, \gamma_N\rangle]$. □

7. A generalization of Abhyankar-Jung Theorem

Definition 7.1. Let $\alpha \in \mathbb{N}^n$ and let $\theta \in \mathbb{C}[\mathbf{x}]$ be a (α)-homogeneous polynomial. Let $a > 0$, $C > 0$ and $\eta > 0$. Set :

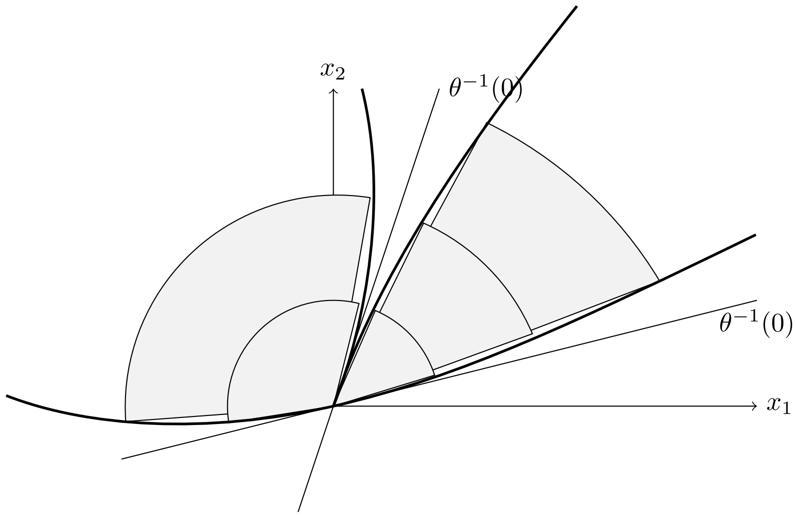
$$\mathcal{D}_{\theta,C,a,\eta} := \left(\bigcup_{\substack{K>0,\varepsilon>0\\ \varepsilon<K^aC}} C_{K,\varepsilon} \right) \bigcap B(0,\eta)$$

where $B(0,\eta)$ is the open ball centered in 0 and of radius η and

$$C_{K,\varepsilon} := \left\{ x \in \mathbb{C}^n \ / \ d_\alpha(x, \theta^{-1}(0)) > K\|x\|_\alpha \text{ and } \|x\|_\alpha < \varepsilon \right\}$$

where $\|.\|_\alpha$ is defined in Example 6.17 and d_α is defined as follows: for any x, $y \in \mathbb{C}^n$ let us denote by $x_i^{\frac{1}{\alpha_i}}$ (resp. $y_i^{\frac{1}{\alpha_i}}$) a complex α_i-th root of x_i (resp. y_i) and let $\mathbb{U}_i$ be the set of α_i-roots of unity. Then we define $d_\alpha(x,y) := \max\limits_i \inf\limits_{\xi\in\mathbb{U}_i} \left| x_i^{\frac{1}{\alpha_i}} - \xi y_i^{\frac{1}{\alpha_i}} \right|$ and $d_\alpha(x, \theta^{-1}(0)) := \inf\limits_{x'\in\theta^{-1}(0)} d_\alpha(x, x')$.

Then $\mathcal{D}_{\theta,C,a,\eta}$ is the complement of a hornshaped neighborhood of $\{\theta = 0\}$ as we can see on the following picture (here $n = 2$ and $\alpha = (1, 1)$):



Lemma 7.2. *Let $a \in \mathbb{N}^n$ and $A \in \mathcal{V}_{\alpha,\theta}^{\mathbb{C}\{\mathbf{x}\}}$. Then there exist constants $a > 0$ and $C > 0$ such that A is analytic on $\mathcal{D}_{\theta,C,a,\eta}$ for every $\eta > 0$.*

Proof. We write $A = \sum_i \frac{a_i}{\theta^{m(i)}}$ where a_i is (α)-homogeneous for every $i \in \mathbb{N}$. By multiplying a_i by a convenient power of θ we may even assume that there exist positive constants a and b such that $m(i) = ai + b$ for every i.

If $\nu_\alpha(a_i) = d_i$ there exist $C > 0$ and $r > 0$ such that

$$|a_i(x)| \leq Cr^i \|x\|_\alpha^{d_i} \qquad \forall x \in \mathbb{C}^n \tag{13}$$

by Theorem 6.18, Example 6.17 and Inequality (12) of Example 6.17. On the other hand we claim that there exists a constant $C' > 0$ such that

$$|\theta(x)| \geq C' d_\alpha(x, \theta^{-1}(0))^{\nu_\alpha(\theta)} \qquad \forall x \in \mathbb{C}^n. \tag{14}$$

Indeed if we embed $\mathbb{C}\{\mathbf{x}\}$ in $\mathbb{C}\{\mathbf{y}\}$ by sending x_i onto $y_i^{\alpha_i}$, we have

$$\theta(\mathbf{x}) = \theta(y_1^{\alpha_1}, \ldots, y_n^{\alpha_n}) = \tau(y_1, \ldots, y_n)$$

and τ is a homogeneous polynomial of degree $\nu_\alpha(\theta)$. After a linear change of coordinates, we may assume that τ is a monic polynomial in y_n of degree $\nu_\alpha(\theta)$ multiplied by a constant. Then, for all $y_1, \ldots, y_n \in \mathbb{C}^n$, we have

$$|\tau(y_1, \ldots, y_n)| = C' \left| \prod_{i=1}^{\nu_\alpha(\theta)} (y_n - \varphi_i(y_1, \ldots, y_{n-1})) \right|$$

where φ_i is a homogeneous function which is locally analytic outside the discriminant locus of τ, for some constant $C' > 0$. Thus

$$|\tau(y_1, \ldots, y_n)| \geq C' \min_i |y_n - \varphi_i(y_1, \ldots, y_{n-1})|^{\nu_\alpha(\theta)}$$

$$\geq C' \inf_{y' \in \tau^{-1}(0)} \max_k |y_k - y'_k|^{\nu_\alpha(\theta)} = C' d(y, \tau^{-1}(0))^{\nu_\alpha(\theta)}$$

since $(y_1, \ldots, y_{n-1}, \varphi_i(y_1, \ldots, y_{n-1})) \in \tau^{-1}(0)$ for any i. This proves (14).

Hence we have (for positive constants ε, K and $x \in C_{K,\varepsilon}$):

$$\left|\frac{a_i(x)}{\theta^{m(i)}(x)}\right| \leq \frac{C}{C'^{m(i)}} \frac{r^i \|x\|_\alpha^{d_i}}{d_\alpha(x,\theta^{-1}(0))^{\nu_\alpha(\theta)m(i)}} = \frac{C}{C'^{m(i)}} \frac{r^i \|x\|_\alpha^{i+\nu_\alpha(\theta)m(i)}}{d_\alpha(x,\theta^{-1}(0))^{\nu_\alpha(\theta)m(i)}}$$

$$\leq \frac{Cr^i\|x\|_\alpha^i}{C'^{m(i)}K^{\nu_\alpha(\theta)m(i)}} \leq \frac{C(r\varepsilon)^i}{C'^{m(i)}K^{\nu_\alpha(\theta)m(i)}} = \frac{C}{C'^b K^{\nu_\alpha(\theta)b}}\left(\frac{r\varepsilon}{C'^a K^{\nu_\alpha(\theta)a}}\right)^i.$$

Then if $\varepsilon < K^{a\nu_\alpha(\theta)}\left(\frac{C'^a}{r}\right)$, A defines an analytic function on the domain $C_{K,\varepsilon}$. Thus A defines an analytic function on the domain $\mathcal{D}_{\theta,C'^a/r,a\nu_\alpha(\theta),\eta}$ for every $\eta > 0$. □

This following proposition has been proven by Tougeron in the case $\alpha = (1,\ldots,1)$ (see Proposition 2.8 [To]):

Proposition 7.3. *Let $\alpha \in \mathbb{N}^n$ and let $P(Z) \in \mathbb{C}\{\mathbf{x}\}[Z]$ be a monic polynomial whose discriminant is equal to δu where $\delta \in \mathbb{C}[\mathbf{x}]$ is (α)-homogeneous and $u \in \mathbb{C}\{\mathbf{x}\}$ is invertible. If $P(Z)$ factors as $P(Z) = P_1(Z)\cdots P_r(Z)$ where $P_i(Z) \in \mathbb{C}\{\mathbf{x}\}[Z]$ is an irreducible monic polynomial of $\mathbb{C}\{\mathbf{x}\}[Z]$ for all i, then $P_i(Z)$ is irreducible in $\mathcal{V}_\alpha^{\mathbb{C}\{\mathbf{x}\}}[Z]$.*

Proof. Let $Q(Z)$ be an irreducible monic factor of $P(Z)$ in $\mathcal{V}_\alpha[Z]$. By Theorem 5.12 there exists a (α)-homogeneous polynomial $\theta \in \mathbb{C}[x]$ such that the coefficients of $Q(Z)$ are in $\mathcal{V}_{\alpha,\theta}$. Let us denote by A one of these coefficients.

Since $\mathcal{V}_{\alpha,\theta} \subset \mathcal{V}_{\alpha,\theta\delta}$ we may assume that δ divides θ, thus

$$\delta^{-1}(0) \cap B(0,\varepsilon) \subset \theta^{-1}(0) \cap B(0,\varepsilon)$$

for every $\varepsilon > 0$.

Let $\eta > 0$ small enough such that the roots of $P(Z)$ are locally analytic on the domain

$$D_{\theta,\eta} := B(0,\eta)\backslash\theta^{-1}(0) \subset B(0,\eta)\backslash\delta^{-1}(0).$$

Since A is a polynomial depending on the roots of $P(Z)$ it is locally analytic on $D_{\theta,\eta}$.

On the other hand by Lemma 7.2 A defines an analytic function on a domain $\mathcal{D}_{\theta,C,a,\eta}$.

Thus by Lemma 7.4 given below A is global analytic on $D_{\theta,\eta}$. Since the roots of $P(Z)$ are bounded near the origin, A is bounded near the origin, thus A extends to an analytic function near the origin. This proves that A is analytic on a neighborhood of the origin and $Q(Z) \in \mathbb{C}\{\mathbf{x}\}[Z]$. □

Lemma 7.4. *Set $C > 0$, $a > 0$ and $\eta > 0$ and let $\theta \in \mathbb{C}[\mathbf{x}]$ be a (α)-homogeneous polynomial. Let $A : D_{\theta,\eta} \longrightarrow \mathbb{C}$ be a multivalued function. Let us assume that A is analytic on $\mathcal{D}_{\theta,C,a,\eta}$ and locally analytic on $D_{\theta,\eta}$. Then A is analytic on $D_{\theta,\eta}$.*

Proof. Since A is locally analytic on $D_{\theta,\eta}$, then A extends to an analytic function on a small neighborhood of every path in $D_{\theta,\eta}$. If A is not analytic on $D_{\theta,\eta}$, then there exists a loop based at a point p of $D_{\theta,\eta}$, denoted by $\varphi : [0,1] \longrightarrow D_{\theta,\eta}$ with $\varphi(0) = \varphi(1) = p$, such that A extends to an analytic function on a neighborhood of φ but $A\circ\varphi(0) \neq A\circ\varphi(1)$. Let us write $\varphi(t) = (\varphi_1(t),\ldots,\varphi_n(t))$ and let us define $\Phi : [0,1]\times S \longrightarrow \mathbb{C}^n$ by

$$\Phi(t,s) := (s^{\alpha_1}\varphi_1(t),\ldots,s^{\alpha_n}\varphi_n(t))$$

where $S := \{z \in \mathbb{C} \;/\; |z| \leq 1, \Re(z) > 0\}$.

Then we have that

$$\delta(\Phi(t,s)) = s^{\nu_\alpha(\delta)}\delta(\varphi(t)) \neq 0$$

for any $(t,s) \in [0,1]\times S$ since $\mathrm{Im}(\varphi) \subset D_{\theta,\eta}$ and $s \neq 0$. Thus the image of Φ is included in $D_{\theta,\eta}$. Moreover, for any $t \in [0,1]$, let $\Phi_t : S \longrightarrow D_{\theta,\eta}$ be the function defined by $\Phi_t(s) := \Phi(t,s)$. Its image is simply connected since S is simply connected and Φ_t is analytic. Thus $A \circ \Phi_t$, which is locally analytic, extends to an analytic function on S by the Monodromy Theorem.

Let us denote by h the holomorphic function on S defined by

$$h(s) := A \circ \Phi(0, s) - A \circ \Phi(1, s)$$

for any $s \in S$.

For any $s \in S$ and any $t \in [0, 1]$ we have

$$\|\Phi(t, s)\|_\alpha = |s| \|\varphi(t)\|_\alpha$$

and

$$d_\alpha \left(\Phi(t, s), \theta^{-1}(0)\right) = |s| d_\alpha \left(\varphi(t), \theta^{-1}(0)\right).$$

Let us set

$$K := \frac{1}{2} \min_{t \in [0,1]} \frac{d_\alpha \left(\Phi(t, s), \theta^{-1}(0)\right)}{\|\Phi(t, s)\|_\alpha} = \frac{1}{2} \min_{t \in [0,1]} \frac{d_\alpha \left(\varphi(t), \theta^{-1}(0)\right)}{\|\varphi(t)\|_\alpha} > 0.$$

Thus for any s belonging to the domain $S \cap \{|s| < K^a C\}$, we have $\Phi(t, s) \in \mathcal{D}_{\theta,C,a,\eta}$. Since $\Phi(t, s) \in \mathcal{D}_{\theta,C,a,\eta}$ and A is analytic on $\mathcal{D}_{\theta,C,a,\eta}$, then $A \circ \Phi(0, s) = A \circ \Phi(1, s)$, thus $h(s) = 0$ on $S \cap \{s < K^a C\}$. Since h is holomorphic on the connected domain S, then $h \equiv 0$ on S. This contradicts the assumption. Hence A is analytic on $D_{\theta,\eta}$. □

Then we can extend Proposition 7.3 to the formal setting over any field of characteristic zero:

Theorem 7.5. *Let $\Bbbk$ be a field of characteristic zero and $\alpha \in \mathbb{R}^n_{>0}$. Let $P(Z) \in \Bbbk[\![\mathbf{x}]\!][Z]$ be a monic polynomial whose discriminant is equal to δu where $\delta \in \Bbbk[\mathbf{x}]$ is (α)-homogeneous and $u \in \Bbbk[\![\mathbf{x}]\!]$ is a unit. If $P(Z)$ factors as $P(Z) = P_1(Z) \cdots P_s(Z)$ where the $P_i(Z)$ are irreducible monic polynomials of $\Bbbk[\![\mathbf{x}]\!][Z]$, then the $P_i(Z)$ remain irreducible in $\mathcal{V}_\alpha[Z]$.*

Proof. Let us prove this theorem when $P(Z) \in \mathbb{C}\{\mathbf{x}\}[Z]$. If $\alpha \in \mathbb{N}^n$, this is exactly Proposition 7.3. If $\alpha \notin \mathbb{N}^n$, then by Lemma 6.6, any decomposition $P(Z) = Q_1(Z) \cdots Q_r(Z)$ in $\mathcal{V}_\alpha[Z]$ is also a decomposition in $\mathcal{V}_{\alpha'}[Z]$ for $\alpha' \in \mathrm{Rel}(\alpha, q, \varepsilon)$ where ε is small enough. Then every irreducible monic factor of $Q_i(Z)$ in $\mathcal{V}_{\alpha'}[Z]$ is in $\mathbb{C}\{\mathbf{x}\}[Z]$ by Proposition 7.3, thus $Q_i(Z) \in \mathbb{C}\{\mathbf{x}\}[Z]$ for every i. In particular since the $Q_i(Z)$ are irreducible in $\mathcal{V}_\alpha[Z]$ then they are irreducible polynomials of $\mathbb{C}\{\mathbf{x}\}[Z]$.

Now let us consider the general case. Let

$$P(Z) = Z^d + a_{d-1}(\mathbf{x}) Z^{d-1} + \cdots + a_0(\mathbf{x})$$

be a polynomial satisfying the hypothesis of the theorem with $a_k(\mathbf{x}) \in \Bbbk[\![\mathbf{x}]\!]$ for $0 \leq k \leq d-1$. Since $P(Z)$ is defined over a field extension of $\mathbb{Q}$ generated by countably many elements and since such a field extension embeds in $\mathbb{C}$, we may assume that $\mathbb{C}$ is a field extension of $\Bbbk$ and $P(Z) \in \mathbb{C}[\![\mathbf{x}]\!]$.

The discriminant of $P(Z)$ is a polynomial depending on the coefficients $a_0(\mathbf{x}), \ldots, a_{d-1}(\mathbf{x})$ that we denote by $D(a_0(\mathbf{x}), \ldots, a_{d-1}(\mathbf{x}))$. Let

$$R(A_0, \ldots, A_{d-1}, U) := D(A_0, \ldots, A_{d-1}) - \delta(\mathbf{x}) U \in \mathbb{C}[\mathbf{x}][A_0, \ldots, A_{d-1}, U].$$

Then $R(a_0(\mathbf{x}), \ldots, a_{d-1}(\mathbf{x}), u(\mathbf{x})) = 0$.

On the other hand, saying that $P(Z)$ factors as $P = P_1 \cdots P_s$ is equivalent to

$$\exists b_1(\mathbf{x}), \ldots, b_r(\mathbf{x}) \text{ such that } a_i(\mathbf{x}) = R_i(b_1(\mathbf{x}), \ldots, b_r(\mathbf{x})) \quad \forall i$$

for some polynomials $R_i(B_1, \ldots, B_r) \in \mathbb{Q}[B_1, \ldots, B_r]$, $0 \leq i \leq d-1$ (these R_i are the coefficients of Z^i in the product $P_1(Z) \cdots P_s(Z)$ and the b_j are the coefficients of the $P_k(Z)$).

By Artin Approximation Theorem [Art], for any integer $c>0$ there exist convergent power series

$$\overline{a}_{0,c}(\mathbf{x}),\dots,\overline{a}_{d-1,c}(\mathbf{x}),\overline{u}_c(\mathbf{x}),\overline{b}_{1,c}(\mathbf{x}),\dots,\overline{b}_{r,c}(\mathbf{x})\in\mathbb{C}\{\mathbf{x}\}$$

such that

$$R(\overline{a}_{0,c}(\mathbf{x}),\dots,\overline{a}_{d-1,c}(\mathbf{x}),\overline{u}_c(\mathbf{x}))=0, \tag{15}$$

$$\overline{a}_{i,c}(\mathbf{x})-R_i(\overline{b}_{1,c}(\mathbf{x}),\dots,\overline{b}_{r,c}(\mathbf{x}))=0 \text{ for } 0\leq i\leq d-1 \tag{16}$$

and

$$\overline{a}_{k,c}(\mathbf{x})-a_k(\mathbf{x}),\ \overline{u}_c(\mathbf{x})-u(\mathbf{x}),\ \overline{b}_{l,c}(\mathbf{x})-b_l(\mathbf{x})\in(\mathbf{x})^c$$

for $0\leq k\leq d, 1\leq l\leq r$. Set

$$P_{(c)}(Z):=Z^d+\overline{a}_{d-1,c}(\mathbf{x})Z^{d-1}+\cdots+\overline{a}_{0,c}(\mathbf{x}).$$

Then $P_{(c)}(Z)$ factors as

$$P_{(c)}(Z)=P_{1,(c)}(Z)\cdots P_{s,(c)}(Z)$$

in $\mathbb{C}\{\mathbf{x}\}[Z]$ because of Equation (16) (the coefficients of the $P_{i,c}(Z)$ are the $b_{k,c}$) , and

$$P_{i,(c)}(Z)-P_i(Z)\in(\mathbf{x})^c\Bbbk[\![\mathbf{x}]\!][Z]$$

for $1\leq i\leq s$. Moreover the discriminant of $P_{(c)}(Z)$ is of the form $\delta(\mathbf{x})u_{(c)}$ where $u_{(c)}$ is a unit in $\mathbb{C}\{\mathbf{x}\}$ if $c\geq 1$ by Equation (15). Since $P_i(Z)$ is irreducible in $\Bbbk[\![\mathbf{x}]\!][Z]$, then $P_{i,(c)}(Z)$ is irreducible in $\Bbbk[\![\mathbf{x}]\!][Z]$ for all i for c large enough (let us say for $c\geq c_0$). Moreover we can remark that $\nu_\alpha(a)\geq\min_i\{\alpha_i\}\operatorname{ord}(a)$ for any $a\in\Bbbk[\![\mathbf{x}]\!]$, thus $\nu_\alpha(\overline{b}_{k,c}(\mathbf{x})-b_k(\mathbf{x}))\geq\min_i\{\alpha_i\}c$.

Let $c\geq c_0$ and let us assume that $P_{i,(c)}(Z)$ is not irreducible in $\mathcal{V}_\alpha[Z]$. Thus it is the product of two monic polynomials: let us say

$$P_{i,(c)}(Z)=P_{i,(c),1}(Z)P_{i,(c),2}(Z)$$

with $P_{i,(c),1}(Z)$, $P_{i,(c),2}(Z)\in\mathcal{V}_\alpha[Z]$ and $\deg_Z(P_{i,(c),k}(Z))>0$ for $k=1,2$. In fact by Theorem 6.18 we may assume that $P_{i,(c),1}(Z)$, $P_{i,(c),2}(Z)\in\mathcal{V}_\alpha^{\mathbb{C}\{\mathbf{x}\}}[Z]$. By Proposition 7.3 we see that $P_{i,(c),1}(Z)$, $P_{i,(c),2}(Z)\in\mathbb{C}\{x\}[Z]$, and by Proposition 7.6 $P_{i,(c)1}(Z)$, $P_{i,(c),2}(Z)\in\mathbb{L}\{x\}[Z]$ where $\mathbb{L}$ is a subfield of $\mathbb{C}$ which is finite over $\Bbbk$. Thus $\mathbb{L}=\Bbbk[\gamma]$ by the Primitive Element Theorem where γ is a homogeneous element of degree 0 with respect to ν_α by Example 3.19. But we have $\mathcal{V}_\alpha\bigcap\Bbbk[\gamma]=\Bbbk$. Thus $P_{i,(c),1}(Z)$, $P_{i,(c),2}(Z)\in\Bbbk\{\mathbf{x}\}[Z]\subset\Bbbk[\![\mathbf{x}]\!][Z]$ which contradicts the assumption that $P_{i,(c)}$ is irreducible in $\Bbbk[\![\mathbf{x}]\!][Z]$. Thus $P_{i,(c)}(Z)$ is irreducible in $\mathcal{V}_\alpha[Z]$. Hence, by Corollary 4.14, $P_i(Z)$ is irreducible in $\mathcal{V}_\alpha[Z]$ since $\nu_\alpha(\overline{b}_{k,c}(\mathbf{x})-b_k(\mathbf{x}))$ increases at least linearly with c.

□

The next proposition is a generalization of a result of S. Cutkosky and O. Kashcheyeva [CK] (see also Proposition 1 [AM]) and we will use it to prove Theorem 7.7. It is again an application of Theorem 5.12.

Proposition 7.6. *Let $\Bbbk\longrightarrow\Bbbk'$ be a characteristic zero field extension. Let $f\in\Bbbk'[\![\mathbf{x}]\!]$ be algebraic over $\Bbbk[\![\mathbf{x}]\!]$ and let $\mathbb{L}$ be the field extension of $\Bbbk$ generated by all the coefficients of f. Then $\Bbbk\longrightarrow\mathbb{L}$ is a finite field extension.*

Proof. Let $\alpha\in\mathbb{R}^n_{>0}$ such that $\dim_{\mathbb{Q}}(\mathbb{Q}\alpha_1+\cdots+\mathbb{Q}\alpha_n)=n$. By Theorem 5.12 the roots of the minimal polynomial of f are in $\mathcal{V}_\alpha[\langle\gamma_1,\dots,\gamma_n\rangle]$ for some homogeneous elements $\gamma_1,\dots,\gamma_n$ with respect to ν_α. Let us denote by $\mathcal{V}'_\alpha$ the ring defined in Definition 5.1 and Lemma 5.5 where $\Bbbk$ is replaced by $\Bbbk'$. Then $\Bbbk'[\![\mathbf{x}]\!]$ and $\mathcal{V}_\alpha[\langle\gamma_1,\dots,\gamma_n\rangle]$ are subrings of $\mathcal{V}'_\alpha[\langle\gamma_1,\dots,\gamma_n\rangle]$. Thus by unicity of the roots of the minimal polynomial of f we have that $f \in \mathcal{V}_\alpha[\langle\gamma_1, \ldots, \gamma_n\rangle]$. By Example 3.16 the homogeneous elements γ_i may be written as $\gamma_i = c_i \mathbf{x}^{\beta_i}$ where c_i is algebraic over $\Bbbk'$ (and so over $\Bbbk$) and $\beta_i \in \mathbb{Q}^n$ for $1 \leq i \leq n$.

By expanding f either as a formal power series of $\Bbbk'[\![\mathbf{x}]\!]$, $f = \sum_i b_i(\mathbf{x})$ where $b_i(\mathbf{x}) \in \Bbbk'[\mathbf{x}]$ is a (α)-homogeneous polynomial for any i, either as an element of $\mathcal{V}_\alpha[\langle\gamma_1, \ldots, \gamma_n\rangle]$,

$$f = \sum_i \frac{a_i(\mathbf{x})}{\delta(\mathbf{x})^{m(i)}} \gamma_1^{k_1(i)} \ldots \gamma_n^{k_n(i)},$$

and by identifying the homogeneous terms of same valuation (which are monomials by Example 3.16), we obtain a countable number of relations of the following form:

$$b(\mathbf{x})\delta^m(\mathbf{x}) = \sum a_{n_1,\ldots,n_s}(\mathbf{x})\gamma_1^{n_1} \ldots \gamma_n^{n_n} \tag{17}$$

where $b(\mathbf{x})$ (corresponding to the $b_i(\mathbf{x})$), $a_{n_1,\ldots,n_s}(\mathbf{x})$ (corresponding to the $a_i(\mathbf{x})$) and δ are monomials, $b(\mathbf{x}) \in \Bbbk'[\mathbf{x}]$, $a_{n_1,\ldots,n_s}(\mathbf{x}) \in \Bbbk[\mathbf{x}]$, $m \in \mathbb{N}$, and the sum is finite. By dividing Equality (17) by $\mathbf{x}^\beta$ for β well chosen, we see that the coefficient of $b(\mathbf{x})$ is in $\Bbbk[c_1, \ldots, c_n]$ and $\mathbb{L}$ is a subfield of $\Bbbk[c_1, \ldots, c_n]$. $\square$

We can strengthen Theorem 7.5 as follows:

Theorem 7.7. *Let $\alpha \in \mathbb{R}^n_{>0}$ and let $P(Z) \in \Bbbk[\![\mathbf{x}]\!][Z]$ be a monic polynomial such that its discriminant $\Delta = \delta u$ where $\delta \in \Bbbk[\mathbf{x}]$ is (α)-homogeneous and $u \in \Bbbk[\![\mathbf{x}]\!]$ is a unit. Let us set $N := \dim_{\mathbb{Q}}(\mathbb{Q}\alpha_1 + \cdots + \mathbb{Q}\alpha_n)$. Then there exist $\gamma_1, \ldots, \gamma_N$ integral homogeneous elements with respect to ν_α and a (α)-homogeneous polynomial $c(\mathbf{x}) \in \Bbbk[\mathbf{x}]$ such that the roots of $P(Z)$ are in $\frac{1}{c(\mathbf{x})}\Bbbk'[\![\mathbf{x}]\!][\gamma_1, \ldots, \gamma_N]$ where $\Bbbk \longrightarrow \Bbbk'$ is finite.*

Remark 7.8. This result shows that for a given root z of the polynomial $P(Z)$ the other roots of $P(Z)$ are obtained from z by the action of the elements of the Galois groups of the elements $\gamma_1, \ldots, \gamma_N$ on z. For instance if $\alpha \in \mathbb{N}^n$ (so $N = 1$ – we can always assume this by Lemma 6.4), then the Galois group of $P(Z)$ is a quotient of the Galois group of the minimal polynomial of γ_1, i.e., the Galois group of one weighted homogeneous polynomial.

Proof of Theorem 7.7. If $Q(Z)$ is a monic polynomial dividing $P(Z)$ in $\Bbbk[\![\mathbf{x}]\!][Z]$, then the discriminant of $Q(Z)$ divides the discriminant of $P(Z)$. Thus we may assume that $P(Z)$ is irreducible.

We will consider three cases: first the case where the coefficients of $P(Z)$ are complex analytic with $\alpha \in \mathbb{N}^n$, then with $\alpha \in \mathbb{R}^n_{>0}$, and finally the general case.

• Let us assume that $\alpha \in \mathbb{N}^n$ and that $P(Z) \in \mathbb{C}\{\mathbf{x}\}[Z]$. By Theorem 5.12 the roots of $P(Z)$ are of the form

$$\sum_{i_1,\ldots,i_s} A_{i_1,\ldots,i_s} \gamma_1^{i_1} \cdots \gamma_s^{i_s}$$

where $\gamma_1, \ldots, \gamma_s$ are integral homogeneous elements with respect to ν_α and $A_{i_1,\ldots,i_s} \in \mathcal{K}_\alpha^{\mathbb{C}\{\mathbf{x}\}}$ for any $i_1, \ldots, i_s$. We may even choose $s = 1$ by Proposition 3.28, but we treat here the general case $s \geq 1$ that will be used in the sequel.

We replace $\gamma_1, \ldots, \gamma_s$ by other integral homogeneous elements with respect to ν_α as follows: let us denote by $\gamma_{1,1} := \gamma_1, \ldots, \gamma_{1,q_1}$ the conjugates of γ_1 over $\mathbb{K}_n$. If $\gamma_2 \notin \mathbb{K}_n[\gamma_{1,1}, \ldots, \gamma_{1,q_1}]$ we denote by $\gamma_{2,1} := \gamma_2, \ldots, \gamma_{2,q_2}$ its conjugates over $\mathbb{K}_n[\gamma_{1,1}, \ldots, \gamma_{1,q_1}]$ and so on. So for $1 \leq l \leq s$, q_l denotes the degree of the minimal polynomial of γ_l over $\mathbb{K}_n[\gamma_{i,j}]_{1\leq i<l, 1\leq j\leq q_i}$, and for $1 \leq l \leq s$, $\gamma_{l,1}, \ldots, \gamma_{l,q_l}$ denote the conjugates of $\gamma_l = \gamma_{l,1}$ over $\mathbb{K}_n[\gamma_{i,j}]_{1\leq i<l,1\leq j\leq q_i}$. Then we may assume

that the roots of $P(Z)$ are of the form

$$\sum_{\substack{0\leq i_1<q_1\\ \cdots\\ 0\leq i_s<q_s}} A_{i_1,\ldots,i_s}\gamma_{1,j_1}^{i_1}\cdots\gamma_{s,j_s}^{i_s}$$

where $A_{i_1,\ldots,i_s}\in\mathcal{K}_\alpha^{\mathbb{C}\{\mathbf{x}\}}$, $\nu_\alpha\left(A_{i_1,\ldots,i_s}\gamma_{1,j_1}^{i_1}\cdots\gamma_{s,j_s}^{i_s}\right)\geq 0$ for any i_1, ..., i_s, and $1\leq j_i\leq q_i$ for any i.

Let us assume that $P(Z)$ factors into a product of monic irreducible polynomials as

$$P(Z)=P_1(Z)\cdots P_r(Z)$$

in $\mathcal{K}_\alpha^{\mathbb{C}\{\mathbf{x}\}}[\gamma_{i,j}]_{1\leq i<s,1\leq j\leq q_i}[Z]$. We write the roots of $P_1(Z)$ as $z_j=\displaystyle\sum_{i=0}^{q_s-1}B_i\gamma_{s,j}^i$ where

$$B_i\in\mathcal{K}_\alpha^{\mathbb{C}\{\mathbf{x}\}}[\gamma_{i,j}]_{1\leq i<s,1\leq j\leq q_i}$$

for all i. Then the roots of the other $P_l(Z)$ are $\displaystyle\sum_{i=0}^{q_s-1}B'_i\gamma_{s,j}^i$ where $(B'_0,\ldots,B'_{q_s-1})$ is the image $(B_0,\ldots,B_{q_s-1})$ by a $\mathcal{K}_\alpha^{\mathbb{C}\{x\}}$-automorphism of $\mathcal{K}_\alpha^{\mathbb{C}\{\mathbf{x}\}}[\gamma_{i,j}]_{1\leq i<s,0\leq j<q_i}$. If the roots of $P_1(Z)$ satisfy the theorem, then we see that the roots of the other $P_l(Z)$ will also satisfy the theorem since they are conjugates of the roots of $P_1(Z)$ by $\mathcal{K}_\alpha^{\mathbb{C}\{x\}}$-automorphisms of $\mathcal{K}_\alpha^{\mathbb{C}\{\mathbf{x}\}}[\gamma_{i,j}]_{1\leq i<s,0\leq j<q_i}$. Thus it is enough to prove the result for the roots of $P_1(Z)$. We have

$$\begin{pmatrix} z_1\\ z_2\\ \vdots\\ z_{q_s}\end{pmatrix}=\begin{pmatrix}1 & \gamma_{s,1} & \cdots & \gamma_{s,1}^{q_s-1}\\ 1 & \gamma_{s,2} & \cdots & \gamma_{s,2}^{q_s-1}\\ \vdots & \vdots & \vdots & \vdots\\ 1 & \gamma_{s,q_s} & \cdots & \gamma_{s,q_s}^{q_s-1}\end{pmatrix}\begin{pmatrix} B_0\\ B_1\\ \vdots\\ B_{q_s-1}\end{pmatrix}.$$

Let us set $M:=\begin{pmatrix}1 & \gamma_{s,1} & \cdots & \gamma_{s,1}^{q_s-1}\\ 1 & \gamma_{s,2} & \cdots & \gamma_{s,2}^{q_s-1}\\ \vdots & \vdots & \vdots & \vdots\\ 1 & \gamma_{s,q_s} & \cdots & \gamma_{s,q_s}^{q_s-1}\end{pmatrix}$. The determinant of M is a homogeneous element c with respect to ν_α where $\nu_\alpha(c)=\frac{1}{2}q_s(q_s-1)\nu_\alpha(\gamma_s)$. Thus we have

$$B_i=\frac{1}{c}\left(R_{i,1}(\gamma_{s,1},\ldots,\gamma_{s,q_s})z_1+\cdots+R_{i,s}(\gamma_{s,1},\ldots,\gamma_{s,q_s})z_{q_s}\right)$$

where the $R_{i,j}$ are polynomials with coefficients in $\mathbb{Q}$ and the element $R_{i,j}(\gamma_{s,1},\ldots,\gamma_{s,q_s})$ is homogeneous with respect to ν_α. By multiplying c and

$$R_{i,1}(\gamma_{s,1},\ldots,\gamma_{s,q_s})z_1+\cdots+R_{i,s}(\gamma_{s,1},\ldots,\gamma_{s,q_s})z_{q_s}$$

by the conjugates of c over $\Bbbk[\mathbf{x}]$ we may assume that $c=c(\mathbf{x})\in\Bbbk[\mathbf{x}]$ is a (α)-homogeneous polynomial. The z_i and the $\gamma_{s,j}$ are locally analytic on $D_{\theta,\eta}:=B(0,\eta)\backslash\theta^{-1}(0)$ and bounded near the origin, where $\{\theta=0\}$ contains the discriminant locus of $P(Z)$ and of the minimal polynomials of the γ_i and η is small enough. Thus $c(\mathbf{x})B_i$ is locally analytic on $D_{\theta,\eta}$ for $1\leq i\leq q_s$ and is bounded near the origin. Moreover $c(\mathbf{x})B_i$ is algebraic over $\Bbbk[\![\mathbf{x}]\!]$ since the $g_{s,j}$ and the z_k are algebraic over $\Bbbk[\![\mathbf{x}]\!]$. By induction on s (we replace z_1, ..., z_{q_s} by $c(\mathbf{x})B_0$, ..., $c(\mathbf{x})B_{q_s-1}$ - here we just used the fact that the roots of $P(Z)$ are algebraic over $\Bbbk[\![\mathbf{x}]\!]$ and locally analytic over a domain of the form $D_{\theta,\eta}$) we see that there exists a (α)-homogeneous polynomial $c(\mathbf{x})$ such that $c(\mathbf{x})A_{\underline{i}}$ is locally analytic on $D_{\theta,\eta}$ and bounded near the origin for any $\underline{i}:=(i_1,\ldots,i_s)$.

Since $c(\mathbf{x})A_{\underline{i}} \in \mathcal{K}_\alpha^{\mathbb{C}\{\mathbf{x}\}}$ and it is bounded near the origin, we see that $c(\mathbf{x})A_{\underline{i}} \in \mathcal{V}_\alpha^{\mathbb{C}\{\mathbf{x}\}}$. Thus it is analytic on $\mathcal{D}_{\theta,C,a,\eta}$ for C, a and η well chosen (see Lemma 7.2). Hence by Lemma 7.4 it is analytic on $D_{\theta,\eta}$ and since it is bounded near the origin, $c(\mathbf{x})A_{\underline{i}} \in \mathbb{C}\{\mathbf{x}\}$ for any $\underline{i}$.

• Now let us consider any $\alpha \in \mathbb{R}^n_{>0}$ and $P(Z) \in \mathbb{C}\{\mathbf{x}\}[Z]$. Then the roots of $P(Z)$ are in $\mathcal{V}_\alpha[\langle\gamma_1, \ldots, \gamma_s\rangle]$ for some integral homogeneous elements with respect to ν_α, which we denote by $\gamma_1, \ldots, \gamma_s$. Let us denote these roots by $z_1, \ldots, z_d$. For any $\alpha' \in \mathbb{N}^n$ such that $\mathrm{Rel}_\alpha \subset \mathrm{Rel}_{\alpha'}$, $\gamma_1, \ldots, \gamma_s$ are integral homogeneous elements with respect to $\nu_{\alpha'}$. Thus, for any $\varepsilon > 0$ small enough (say $\varepsilon < \varepsilon_0$), for any $q \in \mathbb{N}$ and any $\alpha' \in \mathrm{Rel}(\alpha, q, \varepsilon)$, $z_1, \ldots, z_d \in \mathcal{V}_{\alpha'}[\langle\gamma_1, \ldots, \gamma_s\rangle]$ by Proposition 6.6. Moreover, by the previous case, we see that

$$\forall \varepsilon \in]0, \varepsilon_0[, \ \forall q \in \mathbb{N}, \ \forall \alpha' \in \mathrm{Rel}(\alpha, q, \varepsilon),$$
$$\exists c_{\alpha'}(\mathbf{x}) \text{ an } (\alpha')\text{-homogeneous polynomial such that}$$
$$c_{\alpha'}(\mathbf{x})z_1, \ldots, \ c_{\alpha'}(\mathbf{x})z_d \in \mathbb{C}\{\mathbf{x}\}[\gamma_1, \ldots, \gamma_s].$$

Moreover we see that that $c_\alpha(\mathbf{x})$ may be chosen as being the product of the determinants of Vandermonde matrices as M depending only on $\gamma_1, \ldots, \gamma_s$, thus $c_{\alpha'}(\mathbf{x})$ does not depend on α'. Let us denote $c(\mathbf{x}) := c_{\alpha'}(\mathbf{x})$. Since $c(\mathbf{x})$ is a (α')-homogeneous polynomial for all $\alpha' \in \mathrm{Rel}(\alpha, q, \varepsilon)$ then $c(\mathbf{x})$ is a (α)-homogeneous polynomial. This proves the result.

• Now let us consider the general case, $\alpha \in \mathbb{R}^n_{>0}$ and $P(Z) \in \Bbbk[\![\mathbf{x}]\!][Z]$ where $\Bbbk$ is a field of characteristic zero.

Let us write $P(Z) = Z^d + a_{d-1}(\mathbf{x})Z^{d-1} + \cdots + a_0(\mathbf{x})$. Exactly as in the proof of Theorem 7.5 we may assume that $\mathbb{C}$ is a field extension of $\Bbbk$. Let us use the notations of the proof of Theorem 7.5. Let

$$R(A_0, \ldots, A_{d-1}, U) := D(A_0, \ldots, A_{d-1}) - \delta(\mathbf{x})U \in \mathbb{C}[\mathbf{x}][A_0, \ldots, A_{d-1}, U]$$

where D is the universal discriminant of a monic polynomial of degree d. Then

$$R(a_0(\mathbf{x}), \ldots, a_{d-1}(\mathbf{x}), u(\mathbf{x})) = 0.$$

By Artin Approximation Theorem [Art], for any integer $c > 0$, there exist convergent power series $\overline{a}_{0,c}(\mathbf{x}), \ldots, \overline{a}_{d-1,c}(\mathbf{x}), \overline{u}_c(\mathbf{x}) \in \mathbb{C}\{\mathbf{x}\}$ such that

$$R(\overline{a}_{0,c}(\mathbf{x}), \ldots, \overline{a}_{d-1,c}(\mathbf{x}), \overline{u}_c(\mathbf{x})) = 0, \tag{18}$$

and

$$\overline{a}_{k,c}(\mathbf{x}) - a_k(\mathbf{x}), \ \overline{u}_c(\mathbf{x}) - u(\mathbf{x}) \in (\mathbf{x})^c \quad \text{for } 0 \leq k \leq d.$$

Let $P_{(c)}(Z) := Z^d + \overline{a}_{d-1,c}(\mathbf{x})Z^{d-1} + \cdots + \overline{a}_{0,c}(\mathbf{x})$. Then $P_{(c)}(Z)$ is irreducible for c large enough (say $c \geq c_0$). Moreover the discriminant of $P_{(c)}(Z)$ is of the form $\delta(\mathbf{x})u_{(c)}$ where $u_{(c)}$ is a unit in $\mathbb{C}\{\mathbf{x}\}$ if $c \geq 1$ by Equation (18). By the previous case, the roots of $P_{(c)}(Z)$ are in $\frac{1}{c_c(\mathbf{x})}\mathbb{C}\{\mathbf{x}\}[\gamma_{1,c}, \ldots, \gamma_{N,c}]$ where $\gamma_{1,c}, \ldots, \gamma_{N,c}$ are integral homogeneous elements with respect to ν_α and $c_c(\mathbf{x})$ is a (α)-homogeneous polynomial. By Proposition 4.17 and the previous cases, we may assume that the $\gamma_{i,c}$ does not depend on c, thus let us denote $\gamma_{i,c}$ by γ_i. Moreover $c_c(\mathbf{x})$ may be chosen as being the product of the determinants of Vandermonde matrices as M depending only on $\gamma_1, \ldots, \gamma_s$, thus $c_c(\mathbf{x})$ does not depend on c. Let us denote by $c(\mathbf{x})$ this common (α)-homogeneous polynomial.

Thus, when c goes to infinity, we see that the roots of $P(Z)$ are in $\frac{1}{c(\mathbf{x})}\mathbb{C}[\![\mathbf{x}]\!][\gamma_1, \ldots, \gamma_N]$. Such a root has the form $\sum_{i_1,\ldots,i_N} A_{i_1,\ldots,i_N}\gamma_1^{i_1}\cdots\gamma_N^{i_N}$ where i_k runs from 0 to $q_k - 1$. In this case $c(\mathbf{x})A_{i_1,\ldots,i_N} \in \mathbb{C}[\![\mathbf{x}]\!]$ is algebraic over $\Bbbk[\![\mathbf{x}]\!]$, thus $c(\mathbf{x})A_{i_1,\ldots,i_N} \in \Bbbk'[\![\mathbf{x}]\!]$ where $\Bbbk \longrightarrow \Bbbk'$ is finite by Proposition 7.6. Thus the roots of $P(Z)$ are in $\frac{1}{c(\mathbf{x})}\Bbbk'[\![\mathbf{x}]\!][\gamma_1, \ldots, \gamma_N]$.

□

In the case where the α_i are linearly independent over $\mathbb{Q}$, we can choose $c(\mathbf{x}) = 1$. This is exactly the Abhyankar-Jung Theorem:

Corollary 7.9 (Abhyankar-Jung Theorem)**.** *Let $P(Z) \in \Bbbk[\![\mathbf{x}]\!][Z]$ be a monic polynomial whose discriminant has the form $\mathbf{x}^\beta u(\mathbf{x})$ where $\beta \in \mathbb{N}^n$ and $u(0) \neq 0$. Then there exist an integer $q \in \mathbb{N}$ and a finite field extension $\Bbbk \longrightarrow \Bbbk'$ such that the roots of $P(Z)$ are in $\Bbbk'[\![x_1^{\frac{1}{q}}, \ldots, x_n^{\frac{1}{q}}]\!]$.*

Proof. By the previous theorem applied to any $\alpha \in \mathbb{R}^n_{>0}$ satisfying $\dim_{\mathbb{Q}}(\mathbb{Q}\alpha_1 + \cdots + \mathbb{Q}\alpha_n) = n$, the roots of $P(Z)$ are in $\frac{1}{\mathbf{x}^\gamma}\Bbbk'[\![x_1^{\frac{1}{q}}, \ldots, x_n^{\frac{1}{q}}]\!]$ for some $\beta \in \mathbb{N}^n$, $q \in \mathbb{N}$ and $\Bbbk \longrightarrow \Bbbk'$ a finite field extension. Since the discriminant of any monic factor of $P(Z)$ in $\Bbbk'[\![x_1, \ldots, x_n]\!][Z]$ divides the discriminant of $P(Z)$, we may assume that $P(Z)$ is irreducible in $\Bbbk'[\![x_1, \ldots, x_n]\!][Z]$, thus we assume that $\Bbbk' = \Bbbk$.

Let z be a root of $P(Z)$ and let us denote by $\mathrm{NP}(z)$ its Newton polyhedron. Then

$$\mathrm{NP}(z) \subset -\gamma + \mathbb{R}^n_{\geq 0}.$$

Let us assume that $\mathrm{NP}(z) \not\subset \mathbb{R}^n_{\geq 0}$. This means that there exists $\gamma' \in \mathrm{NP}(z)$ such that one its coordinates, let us say γ'_n, is negative. But since z is a root of $P(Z)$ that is a monic polynomial with coefficients in $\Bbbk[\![\mathbf{x}]\!]$ then $\nu_\alpha(z) \geq 0$ for any $\alpha \in \mathbb{R}^n_{>0}$. But in this case there exists $\alpha \in \mathbb{R}^n_{>0}$ such that $\langle \alpha, \gamma' \rangle < 0$ which is a contradiction. Thus $\mathrm{NP}(z) \subset \mathbb{R}^n_{\geq 0}$ which proves the corollary. $\square$

Let us finish this part by giving a few results which are analogous to the fact that if $z \in \mathbb{C}\{t^{\frac{1}{k}}\}$ for some $k \in \mathbb{N}$, t being a single variable, then its minimal polynomial over $\mathbb{C}[\![t]\!]$ is a polynomial with convergent power series. The next result can also be seen as the converse of Theorem 6.18:

Corollary 7.10. *Let $P(Z) \in \Bbbk[\![\mathbf{x}]\!][Z]$ be an irreducible monic polynomial whose discriminant has the form $\delta(\mathbf{x})u(\mathbf{x})$, where $\delta(\mathbf{x})$ is a (α)-homogeneous polynomial, $\alpha \in \mathbb{R}^n_{>0}$, and $u(\mathbf{x}) \in \Bbbk[\![\mathbf{x}]\!]$ is invertible. Let us assume that $P(Z)$ has a root in $\mathcal{V}^R_\alpha[\langle \gamma_1, \ldots, \gamma_s \rangle]$ where R is an excellent Henselian local ring satisfying Properties (A), (B) and (C) and γ_1, $\ldots$, γ_s are homogeneous elements with respect to ν_α. Then the coefficients of $P(Z)$ are in R.*

Proof. By Theorem 7.5, $P(Z)$ is irreducible in $\mathcal{V}_\alpha[Z]$. Let

$$z \in \mathcal{V}^R_\alpha[\langle \gamma_1, \ldots, \gamma_s \rangle]$$

be a root of $P(Z)$ as given in the statement. We can write $z = \sum A_{i_1,\ldots,i_s} \gamma_1^{i_1} \cdots \gamma_s^{i_s}$ where the sum is finite and $A_{i_1,\ldots,i_s} \in \mathcal{V}^R_\alpha$. Then the others roots of $P(Z)$ are of the form

$$\sum A_{i_1,\ldots,i_s} \sigma(\gamma_1)^{i_1} \cdots \sigma(\gamma_s)^{i_s}$$

where σ is a $\mathbb{K}^{\mathrm{alg}}_{\nu_\alpha}$-automorphism of $\overline{\mathbb{K}}^{\mathrm{alg}}_\nu$. Thus all the roots of $P(Z)$ are in $\overline{\mathcal{V}}^R_\alpha$. Hence the coefficients of $P(Z)$ are in $\overline{\mathcal{V}}^R_\alpha \cap \Bbbk[\![\mathbf{x}]\!] = R$. $\square$

Definition 7.11. Let $\Bbbk$ be a valued field and let σ be a strongly convex rational cone of $\mathbb{R}^n$ containing $\mathbb{R}^n_{\geq 0}$. There exists an invertible $n \times n$ matrix $M = (m_{i,j})_{1 \leq i,j \leq n}$ such that $M\gamma \in \mathbb{R}^n_{\geq 0}$ for any $\gamma \in \sigma$. We denote by $\Bbbk\{\mathbf{x}^\beta, \beta \in \sigma \cap \mathbb{Z}^n\}$ the subring of $\Bbbk[\![\mathbf{x}^\beta, \beta \in \sigma \cap \mathbb{Z}^n]\!]$ of power series $f(\mathbf{x})$ such that $f(\tau(\mathbf{x})) \in \Bbbk\{\mathbf{x}\}$ where τ is the map defined by

$$(\tau(x_1), \ldots, \tau(x_n)) = (x_1^{m_{1,1}} \cdots x_n^{m_{1,n}}, \ldots, x_1^{m_{n,1}} \cdots x_n^{m_{n,n}}).$$

By Example 6.17 $\Bbbk\{\mathbf{x}^\beta, \beta \in \sigma \cap \mathbb{Z}^n\}$ is a subring of $\mathcal{V}^{\Bbbk\{x\}}_{\alpha,\delta}$ for any α such that $\langle \alpha, \gamma \rangle > 0$ for all $\gamma \in \sigma \backslash \{0\}$.

Let us mention the following theorem proven by A. Gabrielov and J.-Cl. Tougeron by using transcendental methods (they use in a crucial way the maximum principle for analytic functions):

Theorem 7.12. [Ga][To] *Let $P(Z) \in \mathbb{C}[\![\mathbf{x}]\!][Z]$ be an irreducible monic polynomial. If one root of $P(Z)$ is in $\mathbb{C}\{\mathbf{x}^\beta, \beta \in \sigma \cap \frac{1}{q}\mathbb{Z}^n\}$ where σ is a strongly convex rational cone and $q \in \mathbb{N}$, then $P(Z) \in \mathbb{C}\{\mathbf{x}\}[Z]$.*

Using what we have done we extend this theorem to any algebraically closed valued field of characteristic zero under the assumption that the discriminant of $P(Z)$ is close to be weighted homogeneous. First we need the following lemma:

Lemma 7.13. *Let $P(Z) \in \Bbbk[\![\mathbf{x}]\!][Z]$ be an irreducible monic polynomial where $\Bbbk$ is a characteristic zero algebraically closed valued field. Let $\alpha \in \mathbb{R}^n_{>0}$ such that $\dim_{\mathbb{Q}}(\mathbb{Q}\alpha_1 + \cdots + \mathbb{Q}\alpha_n) = n$ and $P(Z)$ is irreducible in $\mathcal{V}_\alpha[Z]$. By Theorem 6.9, the roots of $P(Z)$ are in $\Bbbk[\![\mathbf{x}^\beta, \beta \in \sigma \cap \frac{1}{q}\mathbb{Z}^n]\!]$ where σ is a strongly convex rational cone such that $\langle \alpha, \gamma\rangle > 0$ for any $\gamma \in \sigma$, $\gamma \neq 0$, and $q \in \mathbb{N}$. If one root of $P(Z)$ is in $\Bbbk\{\mathbf{x}^\beta, \beta \in \sigma \cap \frac{1}{q}\mathbb{Z}^n\}$, then the others roots of $P(Z)$ are in $\Bbbk\{\mathbf{x}^\beta, \beta \in \sigma \cap \frac{1}{q}\mathbb{Z}^n\}$ and $P(Z) \in \Bbbk\{\mathbf{x}\}[Z]$.*

Proof. Let $z \in \Bbbk\{\mathbf{x}^\beta, \beta \in \sigma \cap \frac{1}{q}\mathbb{Z}^n\}$ be a root of $P(Z)$. For any $\xi = (\xi_1, \ldots, \xi_n)$ vector of q-th roots of unity let us denote by z_ξ the element of $\Bbbk\{\mathbf{x}^\beta, \beta \in \sigma \cap \frac{1}{q}\mathbb{Z}^n\}$ obtain from z by replacing $(x_1^{\frac{1}{q}}, \ldots, x_n^{\frac{1}{q}})$ by $(\xi_1 x_1^{\frac{1}{q}}, \ldots, \xi_n x_n^{\frac{1}{q}})$. In particular $z_\xi \in \Bbbk\{\mathbf{x}^\beta, \beta \in \sigma \cap \frac{1}{q}\mathbb{Z}^n\}$. Then for any ξ, z_ξ is a root of $P(Z)$. Let I be a subset of $\mathbb{U}_q^n$, where $\mathbb{U}_q$ is the group of q-th root of unity, such that

$$z_\xi \neq z_{\xi'} \text{ for any } \xi,\, \xi' \in I,\, \xi \neq \xi',$$
$$\text{and } \forall \xi \in \mathbb{U}_q^n,\, \exists \xi' \in I,\; z_{\xi'} = z_\xi.$$

Let us set $Q(Z) = \prod_{\xi \in I}(Z - z_\xi)$. Then $Q(Z)$ is a monic polynomial of $\mathcal{V}_\alpha[Z]$ whose roots are roots of $P(Z)$. Thus it divides $P(Z)$ in $\mathcal{V}_\alpha[Z]$ hence, since $P(Z)$ is irreducible, $Q(Z) = P(Z)$. Thus the other roots of $P(Z)$ are in $\Bbbk\{\mathbf{x}^\beta, \beta \in \sigma \cap \frac{1}{q}\mathbb{Z}^n\}$ and $P(Z) \in \Bbbk\{\mathbf{x}\}[Z]$. □

Corollary 7.14. *Let $P(Z) \in \Bbbk[\![\mathbf{x}]\!][Z]$ be an irreducible monic polynomial of degree d where $\Bbbk$ is a characteristic zero algebraically closed valued field. Let $\alpha \in \mathbb{R}^n_{>0}$ such that*

$$\dim_{\mathbb{Q}}(\mathbb{Q}\alpha_1 + \cdots + \mathbb{Q}\alpha_n) = n.$$

Let us assume that there exists an irreducible monic polynomial $Q(Z) \in \Bbbk[\![\mathbf{x}]\!][Z]$ of degree d whose discriminant Δ_Q is a monomial times a unit and such that

$$\nu_\alpha(P(Z) - Q(Z)) \geq \frac{d}{2}\nu_\alpha(\Delta_Q).$$

Let us assume moreover that one of the roots of $P(Z)$ is in $\Bbbk\{\mathbf{x}^\beta, \beta \in \sigma \cap \frac{1}{q}\mathbb{Z}^n\}$ for some strongly convex rational cone σ, where $\langle \alpha, \gamma\rangle > 0$ for any $\gamma \in \sigma \backslash \{0\}$, and $q \in \mathbb{N}$. Then the coefficients of $P(Z)$ are in $\Bbbk\{\mathbf{x}\}$.

Proof. By Remark 4.15 and Proposition 4.14, the polynomial $P(Z)$ is irreducible in $\mathcal{V}_\alpha[Z]$. Thus we can apply the previous Lemma. □

8. Diophantine Approximation

Here we give a necessary condition for an element of $\widehat{\mathbb{K}}_\nu$ to be algebraic over $\mathbb{K}_n$:

Theorem 8.1. [Ro1][II] *Let ν be an Abhyankar valuation and let $z \in \mathbb{K}_\nu^{\text{alg}}$. Then there exist two constants $C > 0$ and $a \geq 1$ such that*

$$\left| z - \frac{f}{g} \right|_\nu \geq C|g|_\nu^a \quad \forall f, g \in \Bbbk[\![\mathbf{x}]\!].$$

Proof. Let $P(Z) := a_0Z^d + a_1Z^{d-1} + \cdots + a_d \in \mathbb{K}_n[Z]$ be an irreducible polynomial such that $P(z) = 0$. Let $h \in \Bbbk[\![\mathbf{x}]\!]$ and set

$$P_h(Z) := h^d a_0^{d-1} P\left(\frac{Z}{ha_0}\right).$$

Then $P_h(Z) = Z^d + a_1hZ^{d-1} + a_2a_0h^2Z^{d-2} + \cdots + a_da_0^{d-1}h^d$ and zha_d is a root of $P_h(Z)$. It is straightforward to check that z satisfies the theorem if and only if zha_d does. Thus we may assume that $P(Z)$ is a monic polynomial and $\nu(z) > 0$ by choosing h such that $\nu(h)$ is large enough. Let us set $Q(Z_1, Z_2) := Z_1^d P(Z_2/Z_1)$. By Theorem 3.1 [Ro1] there exist two constants $a \geq d$ and $b \geq 0$ such that

$$\mathrm{ord}(Q(f, g)) \leq a \min\{\mathrm{ord}(f), \mathrm{ord}(g)\} + b \quad \forall f, g \in \Bbbk[\![\mathbf{x}]\!].$$

Moreover, by Izumi's Theorem ([Iz], [Re], [ELS]), there exists a constant $c \geq 1$ such that for all $f \in \Bbbk[\![\mathbf{x}]\!]$, $\mathrm{ord}(f) \leq \nu(f) \leq c\,\mathrm{ord}(f)$. Thus

$$\nu(Q(f, g)) \leq ac \min\{\nu(f), \nu(g)\} + bc \quad \forall f, g \in \Bbbk[\![\mathbf{x}]\!].$$

Since $P(Z)$ is irreducible in $\mathbb{K}_n[Z]$ and $\mathbb{K}_n$ is a characteristic zero field, $P(Z)$ has no multiple roots in $\widehat{V}_\nu$ and we may write

$$P(Z) = R(Z)(Z - z)$$

where $R(Z) \in \widehat{V}_\nu[Z]$ and $R(z) \neq 0$. Set $r := \nu(z)$. Let $f, g \in \Bbbk[\![\mathbf{x}]\!]$ with $g \neq 0$. Two cases may occur: either

$$\left| z - \frac{f}{g} \right|_\nu \geq e^{-r} \tag{19}$$

either $\nu\left(z - \frac{f}{g}\right) > r$. In the last case we have $\nu\left(\frac{f}{g}\right) = \nu(z) > 0$. In particular $\nu\left(R\left(\frac{f}{g}\right)\right) \geq 0$ and $\nu(f) > \nu(g)$. Thus

$$(ac - d)\nu(g) + bc \geq \nu\left(P\left(\frac{f}{g}\right)\right) \geq \nu\left(\frac{f}{g} - z\right).$$

Thus we have

$$A\nu(g) + B \geq \nu\left(\frac{f}{g} - z\right) \quad \text{or} \quad \left| z - \frac{f}{g} \right|_\nu \geq e^{-B}|g|_\nu \tag{20}$$

with $A = ac - d$ and $B = bc$. Then (19) and (20) prove the theorem.

□

Example 8.2. Let $\sigma := (-1, 1)\mathbb{R}_{\geq 0} + (1, 0)\mathbb{R}_{\geq 0} \subset \mathbb{R}^2$. This is a rational strongly convex cone of $\mathbb{R}^2$. Let $f(x_1, x_2)$ be a power series, $f(x_1, x_2) \in \Bbbk[\![x_1, x_2]\!]$. Let us set

$$g(x_1, x_2) := \sum_{i=0}^{\infty} \left(\frac{x_2}{x_1}\right)^{i!} + f(x_1, x_2) \in \Bbbk[\![x^\beta, \beta \in \sigma \cap \mathbb{Z}]\!].$$

Then $g \in \mathcal{V}_\alpha$ for any $\alpha \in \mathbb{R}^2_{>0}$ such that $\alpha_2 > \alpha_1$. Moreover

$$\nu_\alpha\left(g - f - \sum_{i=0}^{n} \left(\frac{x_2}{x_1}\right)^{i!}\right) = (n + 1)!(\alpha_2 - \alpha_1) = \frac{\alpha_2 - \alpha_1}{\alpha_1}(n + 1)\nu_\alpha(x_1^{n!}).$$

Thus there do not exist constants A and B such that

$$A\nu_\alpha(x_1^{n!}) + B \geq \nu_\alpha\left(g - f - \sum_{i=0}^{n} \left(\frac{x_2}{x_1}\right)^{i!}\right) \quad \forall n \in \mathbb{N}.$$

Hence $g(x_1,x_2)$ is not algebraic over $\mathcal{F}_2$ by Theorem 8.1.

Notations

- ν_α is the monomial valuation defined by $\nu_\alpha(x_i):=\alpha_i$ for any i (cf. Example 2.4).
- V_ν is the valuation ring associated to ν.
- $\widehat{V}_\nu$ is the completion of V_ν.
- $\mathbb{K}_n$ is the fraction field of $\Bbbk[\![\mathbf{x}]\!]$ and V_ν.
- $\widehat{\mathbb{K}}_\nu$ is the fraction field of $\widehat{V}_\nu$.
- $\mathrm{Gr}_\nu V_\nu$ is the graded ring associated to V_ν (cf. Part 3).
- V_ν^{alg} is the algebraic closure (or the Henselization) of V_ν in $\widehat{V}_\nu$ (see Lemma 2.10).
- $\mathbb{K}_\nu^{\mathrm{alg}}$ is the fraction field of V_ν^{alg}.
- V_ν^{fg} is the subring of $\widehat{V}_\nu$ whose elements have ν-support included in a finitely generated sub-semigroup of $\mathbb{R}_{\geq 0}$ (cf. Definition 3.14).
- $\mathbb{K}_\nu^{\mathrm{fg}}$ is the fraction field of V_ν^{fg}.
- For any $\alpha\in\mathbb{R}^n_{>0}$, a (α)-homogeneous polynomial is a weighted homogeneous polynomial for the weights $\alpha_1,\ldots,\alpha_n$ (see Definition 2.8).
- $A[\langle\gamma_1,\ldots,\gamma_s\rangle]$ is the valuation ring associated to $A[\gamma_1,\ldots,\gamma_s]$ when $A=\widehat{V}_\nu$, V_ν^{fg} or V_ν^{alg} (cf. Definition 3.26).
- $\overline{V}_\nu$ is the direct limit of the rings $\widehat{V}_\nu[\langle\gamma_1,\ldots,\gamma_s\rangle]$ where the γ_i are homogeneous elements with respect to ν (cf. Definition 3.27).
- $\overline{\mathbb{K}}_\nu$ is the fraction field of $\overline{V}_\nu$.
- $\overline{V}_\nu^{\mathrm{alg}}$ is the direct limit of the rings $V_\nu^{\mathrm{alg}}[\langle\gamma_1,\ldots,\gamma_s\rangle]$ where the γ_i are homogeneous elements with respect to ν.
- $\overline{\mathbb{K}}_\nu^{\mathrm{alg}}$ is the fraction field of $\overline{V}_\nu^{\mathrm{alg}}$.
- $\overline{V}_\nu^{\mathrm{fg}}$ is the direct limit of the rings $V_\nu^{\mathrm{fg}}[\langle\gamma_1,\ldots,\gamma_s\rangle]$ where the γ_i are homogeneous elements with respect to ν.
- $\overline{\mathbb{K}}_\nu^{\mathrm{fg}}$ is the fraction field of $\overline{V}_\nu^{\mathrm{fg}}$.
- $\mathcal{V}_{\alpha,\delta}$ is the subring of $V_{\nu_\alpha}^{\mathrm{fg}}$ of elements of the form $\displaystyle\sum_{i\in\Lambda}\frac{a_i}{\delta^{m(i)}}$ where $\Lambda\subset\mathbb{R}$ is a finitely generated semigroup, $\nu_\alpha\left(\frac{a_i}{\delta^{m(i)}}\right)=i$ and $i\longmapsto m(i)$ is bounded by an affine function (see Definition 5.1).
- $\mathcal{V}_\alpha$ is the direct limit of the $\mathcal{V}_{\alpha,\delta}$ over all the (α)-homogeneous polynomials δ. It is a valuation ring (cf. Proposition 5.5).
- $\mathcal{K}_\alpha$ is the fraction field of $\mathcal{V}_\alpha$ (cf. Definition 5.6).
- $\overline{\mathcal{K}}_\alpha$ is the direct limit of the fields $\mathcal{K}[\langle\gamma_1,\ldots,\gamma_s\rangle]$ where the γ_i are homogeneous elements with respect to ν (cf. Definition 5.6).
- $\mathcal{V}_{\alpha,\delta}^R$ is the subring $\mathcal{V}_{\alpha,\delta}$ whose elements are in the Henselian ring R after a suitable transform (cf. Definition 6.15).
- $\mathcal{V}_\alpha^R$ is the direct limit of the $\mathcal{V}_{\alpha,\delta}^R$ over all the (α)-homogeneous polynomials δ.

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E-mail address: guillaume.rond@univ-amu.fr

Aix-Marseille Université, CNRS, Centrale Marseille, I2M, Marseille, France

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THÉORÈME DE COMPARAISON POUR LES CYCLES PROCHES PAR UN MORPHISME SANS PENTE

MATTHIEU KOCHERSPERGER

Résumé. Le but de cet article est de démontrer le théorème de comparaison entre les cycles proches algébriques et topologiques associés à un morphisme sans pente. Nous obtenons en particulier que dans le cas d'une famille de fonctions holomorphes sans pente, l'itération des isomorphismes de comparaison des cycles proches associés à chacune de ces fonctions ne dépend pas de l'ordre d'itération.

Abstract. The goal of this article is to prove the comparison theorem between algebraic and topological nearby cycles of a morphism whitout slopes. We prove in particular that for a family of holomorphic functions whitout slopes, if we iterate comparison isomorphisms for nearby cycles of each function the result is independent of the order of iteration.

1. Introduction

1.1. **Théorème de comparaison pour une fonction.** Soit X une variété analytique complexe et $f : X \to \mathbb{C}$ une fonction holomorphe. Soit $(\mathcal{F}, \mathcal{M})$ la donnée d'un faisceau pervers sur X et d'un $\mathcal{D}_X$-module holonome régulier associés par la correspondance de Riemann-Hilbert, c'est-à-dire $\mathcal{F} = \mathbf{DR}_X(\mathcal{M})$. Le foncteur *cycles proches topologiques* Ψ_f de P. Deligne associe à $\mathcal{F}$ un faisceau pervers à support $f^{-1}(0)$ muni d'un automorphisme de monodromie. Prolongeant une construction de B. Malgrange [Mal83], M. Kashiwara introduit dans [Kas83] le foncteur *cycles proches algébriques* Ψ_f^{alg} (voir aussi [MM04]) qui associe à $\mathcal{M}$ un $\mathcal{D}_X$-module holonome régulier à support $f^{-1}(0)$ muni d'un automorphisme de monodromie. Ces deux foncteurs sont reliés par un isomorphisme de comparaison qui commute à la monodromie :

$$\Psi_f(\mathcal{F}) \simeq \mathbf{DR}_X \Psi_f^{\mathrm{alg}}(\mathcal{M}). \tag{1}$$

1.2. **Théorème de comparaison pour plusieurs fonctions.** Soit maintenant $p \geqslant 2$ et $f_1, ..., f_p$ des fonctions holomorphes sur X. Notons $\boldsymbol{f} = (f_1, ..., f_p) : X \to \mathbb{C}^p$ le morphisme associé. En général, les foncteurs Ψ_{f_i} $(i = 1, ..., p)$ ne commutent pas entre eux, de même que les foncteurs $\Psi_{f_i}^{\mathrm{alg}}$.

Dans [Mai13] Ph. Maisonobe montre que sous la condition *sans pente* pour le couple $(\boldsymbol{f}, \mathrm{car}(\mathcal{F}))$ on peut définir les foncteurs cycles proches topologiques et algébriques associés à $\boldsymbol{f}$. Il montre alors l'existence d'isomorphismes

$$\Psi_{\boldsymbol{f}}\mathcal{F} \simeq \Psi_{f_{\sigma(1)}} ... \Psi_{f_{\sigma(p)}}\mathcal{F}$$

et

$$\Psi_{\boldsymbol{f}}^{\mathrm{alg}}\mathcal{M} \simeq \Psi_{f_{\sigma(1)}}^{\mathrm{alg}} ... \Psi_{f_{\sigma(p)}}^{\mathrm{alg}}\mathcal{M}$$

pour toute permutation σ de $\{1, ..., p\}$. Ceci assure la commutativité des foncteurs cycles proches associés aux fonctions f_i pour $1 \leqslant i \leqslant p$. Dans l'introduction Ph. Maisonobe mentionne que, par

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itération de l'isomorphisme (1), ses résultats permettent d'obtenir pour tout σ des isomorphismes de comparaison

$$\Psi_{\boldsymbol{f}}\mathcal{F} \simeq \Psi_{f_{\sigma(1)}}...\Psi_{f_{\sigma(p)}}(\mathcal{F}) \simeq \mathbf{DR}_X\Psi^{\mathrm{alg}}_{f_{\sigma(1)}}...\Psi^{\mathrm{alg}}_{f_{\sigma(p)}}(\mathcal{M}) \simeq \mathbf{DR}_X\Psi^{\mathrm{alg}}_{\boldsymbol{f}}\mathcal{M}. \tag{2}$$

Dans cet article, nous montrerons (corollaire 3.7) que cet isomorphisme ne dépend pas de la permutation σ. Pour ce faire, nous exhibons un morphisme de comparaison entre $\Psi_{\boldsymbol{f}}\mathcal{F}$ et $\mathbf{DR}_X\Psi^{\mathrm{alg}}_{\boldsymbol{f}}\mathcal{M}$ et nous montrons qu'il coïncide avec les isomorphismes de comparaison itérés (2) pour toute permutation σ.

1.3. **Un exemples de morphisme sans pente.** On appelle singularité quasi-ordinaire un germe d'espace analytique réduit admettant une projection finie sur $\mathbb{C}^p$ dont le lieu de ramification est contenu dans un diviseur à croisements normaux. Si S est une hypersurface de $\mathbb{C}^n$ à singularité quasi-ordinaire définie par une fonction holomorphe f, il existe une projection $\pi : \mathbb{C}^n \to \mathbb{C}^{n-1}$ quasi-ordinaire pour S. Le faisceau $\Psi_f\underline{\mathbb{C}}_{\mathbb{C}^n}$ est pervers et dans cette situation le couple $(\pi, \mathrm{car}(\Psi_f\underline{\mathbb{C}}_{\mathbb{C}^n}))$ est sans pente.

Les singularités quasi-ordinaires apparaissent en particulier dans la méthode de Jung de résolution des surfaces singulières (voir [Lip75]).

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2. V-multifiltration canonique et foncteurs cycles proches

Dans cette section on définit les cycles proches algébriques à l'aide de la V-multifiltration canonique d'un $\mathcal{D}_X$-module sans pente. On démontre des propriétés de cette multifiltration ainsi que de ses gradués. On définit ensuite les cycles proches topologiques associés à plusieurs fonctions. Enfin on introduit les fonctions de classe de Nilsson à plusieurs variables et on en montre des propriétés utilisées dans la section suivante pour établir un lien entre cycles proches algébriques et cycles proches topologiques.

2.1. **V-multifiltration canonique d'un $\mathcal{D}_X$-module sans pente.** On notera dans la suite

- $d_x := \dim_{\mathbb{C}} X$
- $\partial_i := \partial_{t_i}$
- $E_i := t_i\partial_i$
- $\boldsymbol{x} := (x_1, ..., x_{d_X-p})$
- $\mathbf{1}_i := (0, ..., 0, 1, 0, ..., 0)$ où le 1 est en position i.
- $\boldsymbol{\alpha} := (\alpha_1, ..., \alpha_p)$
- $\boldsymbol{\alpha}_I := (\alpha_i)_{i\in I}$ pour $I \subset \{1, ..., p\}$
- $\boldsymbol{t} := t_1...t_p$
- $\boldsymbol{t}^{\boldsymbol{s}} := t_1^{s_1}...t_p^{s_p}$
- $\mathcal{D}_X[\boldsymbol{s}] := \mathcal{D}_X[s_1, ..., s_p]$
- $\boldsymbol{H} = \{H_1, ..., H_p\}$ où les H_i sont des hypersurfaces lisses dont la réunion définit un diviseur à croisements normaux. On se place ici dans le cas où il existe localement des coordonnées $(\boldsymbol{x}, t_1, ..., t_p)$ telles que

$$\begin{array}{cccc} \boldsymbol{f} : & X & \to & \mathbb{C}^p \\ & (\boldsymbol{x}, t_1, ..., t_p) & \mapsto & (t_1, ..., t_p) \end{array}$$

et $H_i = f_i^{-1}(0)$.

Définition 2.1. Notons, pour tout $1 \leqslant i \leqslant p$, $\mathcal{I}_i$ l'idéal de l'hypersurface H_i et $\mathcal{I}^{\boldsymbol{k}} := \prod_{i=1}^{p} \mathcal{I}_i^{k_i}$. Pour tout $\boldsymbol{k} \in \mathbb{Z}^p$ et pour tout $x \in X$ on définit :

$$(V_{\boldsymbol{k}}\mathcal{D}_X)_x := \{P \in \mathcal{D}_{X,x} \mid \forall \boldsymbol{m} \in \mathbb{Z}^p, P(\mathcal{I}_x^{\boldsymbol{k}+\boldsymbol{m}}) \subset \mathcal{I}_x^{\boldsymbol{k}+\boldsymbol{m}}\},$$

ceci permet de définir une filtration croissante de $\mathcal{D}_X$ indexée par $\mathbb{Z}^p$.

Soit $\mathcal{M}$ un $\mathcal{D}_X$-module cohérent. Une *V-multifiltration* $U_\bullet\mathcal{M}$ de $\mathcal{M}$ est une filtration croissante indexée par $\mathbb{Z}^p$ satisfaisant à $V_{\boldsymbol{k}}\mathcal{D}_X \cdot U_{\boldsymbol{k}'}\mathcal{M} \subset U_{\boldsymbol{k}+\boldsymbol{k}'}\mathcal{M}$ pour tout $\boldsymbol{k}$ et $\boldsymbol{k}'$ dans $\mathbb{Z}^p$. Une telle V-multifiltration est *bonne* si elle est engendrée localement par un nombre fini de sections $(m_j)_{j\in J}$, c'est-à-dire que pour tout $j \in J$ il existe $\boldsymbol{k}_j \in \mathbb{Z}^p$ tel que pour tout $\boldsymbol{k} \in \mathbb{Z}^p$

$$U_{\boldsymbol{k}}\mathcal{M} = \sum_{j\in J} V_{\boldsymbol{k}+\boldsymbol{k}_j}\mathcal{D}_X \cdot m_j.$$

Lorsque des inégalités entre nombres complexes apparaîtront, l'ordre considéré sera toujours l'ordre lexicographique sur $\mathbb{C}$, c'est-à-dire

$$x + iy \leqslant a + ib \iff x < a \text{ ou } (x = a \text{ et } y \leqslant b).$$

En suivant [Mai13] on commence par donner les conditions pour qu'un couple $(\boldsymbol{H}, \mathcal{M})$ soit sans pente puis on définit la V-multifiltration de Malgrange-Kashiwara.

Définition 2.2. Soit $\mathcal{M}$ un $\mathcal{D}_X$-module cohérent.

(1) On dit que le couple $(\boldsymbol{H}, \mathcal{M})$ est *multispécialisable sans pente* si au voisinage de tout point de X, il existe une bonne V-multifiltration $U_\bullet(\mathcal{M})$ de $\mathcal{M}$ et des polynômes $b_i(s) \in \mathbb{C}[s]$ pour tout $1 \leqslant i \leqslant p$ tels que pour tout $\boldsymbol{k} \in \mathbb{Z}^p$, $b_i(E_i + k_i)U_{\boldsymbol{k}}\mathcal{M} \subset U_{\boldsymbol{k}-\boldsymbol{1}_i}\mathcal{M}$.

(2) On dit que le couple $(\boldsymbol{H}, \mathcal{M})$ est *multispécialisable sans pente par section* si, pour toute section locale m de $\mathcal{M}$, il existe des polynômes $b_i(s) \in \mathbb{C}[s]$ pour tout $1 \leqslant i \leqslant p$ tels que $b_i(E_i)m \in V_{-\boldsymbol{1}_i}\mathcal{D}_X \cdot m$.

Rappelons la proposition 1 de [Mai13] :

Proposition 2.3. *Les deux définitions précédentes sont équivalentes et si la première est satisfaite pour une bonne V-multifiltration de $\mathcal{M}$, elle l'est pour toute. On dit alors que le couple $(\boldsymbol{H}, \mathcal{M})$ est* sans pente.

On fixe $\mathcal{M}$ un $\mathcal{D}_X$-module cohérent tel que le couple $(\boldsymbol{H}, \mathcal{M})$ soit sans pente.

Définition 2.4. Le polynôme unitaire de plus bas degré vérifiant la propriété 1. de la définition pour l'indice i est appelé *polynôme de Bernstein-Sato d'indice i de la V-multifiltration $U_\bullet(\mathcal{M})$*, on le note $b_{i,U_\bullet(\mathcal{M})}$.

Le polynôme unitaire de plus bas degré vérifiant la propriété 2. de la définition pour l'indice i est appelé *polynôme de Bernstein-Sato d'indice i de la section m*, on le note $b_{i,m}$.

Proposition 2.5. *Soient, pour $1 \leqslant i \leqslant p$, des sections $\sigma_i : \mathbb{C}/\mathbb{Z} \to \mathbb{C}$ de la projection naturelle $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$. Il existe une unique bonne V-multifiltration $V_\bullet^\sigma(\mathcal{M})$ de $\mathcal{M}$ telle que pour tout i les racines de $b_{i,V_\bullet^\sigma(\mathcal{M})}$ soient dans l'image de σ_i.*

La démonstration de cette proposition et de la proposition 2.7 est identique à celle du théorème 1. de [Mai13].

Définition 2.6. On définit la multifiltration $V_\bullet(\mathcal{M})$ indexée par $\mathbb{C}^p$ et vérifiant :

$$\forall x \in X,\ V_{\boldsymbol{\alpha}}(\mathcal{M})_x := \{m \in \mathcal{M}_x;\ s_i \geqslant -\alpha_i - 1,\ \forall s_i \in b_{i,m}^{-1}(0) \text{ et } 1 \leqslant i \leqslant p\}.$$

Cette V-multifiltration est appelée *V-multifiltration canonique* ou *V-multifiltration de Malgrange-Kashiwara.*

Si on considère l'ordre partiel standard sur $\mathbb{C}^p$

$$\boldsymbol{\alpha} \leqslant \boldsymbol{\beta} \iff \alpha_i \leqslant \beta_i \text{ pour tout } 1 \leqslant i \leqslant p$$

on peut définir

$$V_{<\boldsymbol{\alpha}}(\mathcal{M}) := \sum_{\boldsymbol{\beta}<\boldsymbol{\alpha}} V_{\boldsymbol{\beta}}(\mathcal{M})$$

et

$$\mathrm{gr}_{\boldsymbol{\alpha}}(\mathcal{M}) := V_{\boldsymbol{\alpha}}(\mathcal{M})/V_{<\boldsymbol{\alpha}}(\mathcal{M}).$$

Soit $I \subset \{1,...,p\}$ et I^c son complémentaire, on définit

$$V_{<\boldsymbol{\alpha}_I,\boldsymbol{\alpha}_{I^c}}(\mathcal{M}) := \sum_{\boldsymbol{\beta}_I<\boldsymbol{\alpha}_I} V_{\boldsymbol{\beta}_I,\boldsymbol{\alpha}_{I^c}}(\mathcal{M}).$$

Proposition 2.7. *On a l'égalité des V-multifiltrations $V_{(<\boldsymbol{\alpha}_I,\boldsymbol{\alpha}_{I^c})+\boldsymbol{k}}(\mathcal{M}) = V_{\boldsymbol{k}}^{\sigma_{<\boldsymbol{\alpha}_I,\boldsymbol{\alpha}_{I^c}}}(\mathcal{M})$ où $\sigma_{<\boldsymbol{\alpha}_I,\boldsymbol{\alpha}_{I^c}}$ est la section dont l'image est l'ensemble*

$$\left\{\begin{array}{lll} \boldsymbol{a} \in \mathbb{C}^p & \textit{tel que} & -\alpha_i - 1 \leqslant a_i < -\alpha_i \ \forall\ i \in I^c \\ & \textit{et} & -\alpha_i - 1 < a_i \leqslant -\alpha_i \ \forall\ i \in I \end{array}\right\}.$$

Il existe un ensemble fini $A \subset [-1,0[^p$ tel que la V-multifiltration canonique soit indexée par $A+\mathbb{Z}^p$. Ainsi la V-multifiltration canonique est cohérente.

Soit $I \subset \{1,...,p\}$ et $J \subset I^c$. Comme pour les $\mathcal{D}_X$-modules cohérents, on a une notion de $V_{\mathbf{0}_I}^{\mathbf{H}_I}\mathcal{D}_X$-module multispécialisable sans pente le long des hypersurfaces $\mathbf{H}_J := (H_i)_{i\in J}$.

Définition 2.8. Soit $\mathcal{M}$ un $V_{\mathbf{0}_I}^{\mathbf{H}_I}\mathcal{D}_X$-module cohérent et $J \subset I^c$, on note $q := \#J$.

(1) On dit que le couple $(\boldsymbol{H}_J, \mathcal{M})$ est *multispécialisable sans pente* (ou *spécialisable* si $q = 1$) si au voisinage de tout point de X, il existe une bonne V-multifiltration $U_\bullet(\mathcal{M})$ de $\mathcal{M}$ et des polynômes $b_i(s) \in \mathbb{C}[s]$ pour tout $i \in J$ tels que pour tout $\boldsymbol{k} \in \mathbb{Z}^q$,

$$b_i(E_i + k_i)U_{\boldsymbol{k}}\mathcal{M} \subset U_{\boldsymbol{k}-\mathbf{1}_i}\mathcal{M}.$$

(2) On dit que le couple $(\boldsymbol{H}_J, \mathcal{M})$ est *multispécialisable sans pente par section* (ou *spécialisable par section* si $q = 1$) si, pour toute section locale m de $\mathcal{M}$, il existe des polynômes $b_i(s) \in \mathbb{C}[s]$ pour tout $i \in J$ tels que $b_i(E_i)m \in V_{-\mathbf{1}_i}^{\mathbf{H}_J}(V_{\mathbf{0}_I}^{\mathbf{H}_I}\mathcal{D}_X)\cdot m = V_{-\mathbf{1}_i}\mathcal{D}_X \cdot m$.

Remarque 2.9. Comme pour les $\mathcal{D}_X$-modules (proposition 2.3) les deux définitions sont équivalentes et si elle sont satisfaites on dira que le couple $(\boldsymbol{H}_J, \mathcal{M})$ est sans pente (ou spécialisable si $q = 1$). Les analogues des propositions 2.5 et 2.7 sont vraies pour les $V_{\mathbf{0}_I}^{\mathbf{H}_I}\mathcal{D}_X$-modules sans pente.

Proposition 2.10. *Soit $I \subset \{1,...,p\}$ et $\mathcal{M}$ un $\mathcal{D}_X$-module cohérent tel que le couple $(\boldsymbol{H}, \mathcal{M})$ soit sans pente. Alors le couple $(\boldsymbol{H}_I, \mathcal{M})$ est sans pente et pour tout $\boldsymbol{\alpha}_I$ le couple $(\boldsymbol{H}_{I^c}, V_{\boldsymbol{\alpha}_I}^{\mathbf{H}_I}\mathcal{M})$ est sans pente. De plus, pour $I, J \subset \{1,...,p\}$ disjoints, les V-multifiltrations de Malgrange-Kashiwara satisfont à :*

$$V_{\boldsymbol{\alpha}_I,\boldsymbol{\alpha}_J}^{\boldsymbol{H}_I\cup\boldsymbol{H}_J}(\mathcal{M}) = V_{\boldsymbol{\alpha}_I}^{\boldsymbol{H}_I}(\mathcal{M}) \cap V_{\boldsymbol{\alpha}_J}^{\boldsymbol{H}_J}(\mathcal{M}) = V_{\boldsymbol{\alpha}_I}^{\boldsymbol{H}_I}\left(V_{\boldsymbol{\alpha}_J}^{\boldsymbol{H}_J}(\mathcal{M})\right). \tag{3}$$

On a également l'analogue de [MM04, corollaire 4.2-7]

Proposition 2.11. *Pour tout $\alpha \in \mathbb{C}$ et tout $j \in I^c$, l'application $\mathcal{M} \mapsto V_\alpha^{H_j}(\mathcal{M})$ définit un foncteur exact de la catégorie des $V_{\mathbf{0}_I}^{\mathbf{H}_I}\mathcal{D}_X$-modules spécialisables le long de H_j vers la catégorie des $V_0^{H_j}(V_{\mathbf{0}_I}^{\mathbf{H}_I})\mathcal{D}_X$-modules.*

Sachant que la V-multifiltration canonique est indexée par $A + \mathbb{Z}^p$ avec $A \subset [-1,0[^p$ fini, quitte à renuméroter ces indices on peut la supposer indexée par $\mathbb{Z}^p$ et appliquer la définition B.3 de l'appendice B aux V-filtrations canoniques de $\mathcal{M}$.

La condition *sans pente* s'interprète de manière naturelle comme une condition de compatibilité des V-filtrations relatives aux différentes hypersurfaces considérées.

Proposition 2.12. *Si le couple* $(\mathbf{H}, \mathcal{M})$ *est sans pente alors les filtrations* $V_\bullet^{H_1}(\mathcal{M}), ..., V_\bullet^{H_p}(\mathcal{M})$ *de* $\mathcal{M}$ *sont* compatibles *au sens de la définition B.3.*

Démonstration. Soit $\boldsymbol{\alpha} < \boldsymbol{\beta} \in \mathbb{C}^p$ et notons $I_q := \{1, ..., q\}$. On va construire par récurrence sur l'entier p le p-hypercomplexe X_p correspondant à la compatibilité des sous-objets

$$V_{\alpha_1}^{H_1}(V_{\boldsymbol{\beta}_{I_p}}^{\boldsymbol{H}_{I_p}}\mathcal{M}), ..., V_{\alpha_p}^{H_p}(V_{\boldsymbol{\beta}_{I_p}}^{\boldsymbol{H}_{I_p}}\mathcal{M}) \subseteq V_{\boldsymbol{\beta}_{I_p}}^{\boldsymbol{H}_{I_p}}\mathcal{M}.$$

D'après la remarque B.2, deux filtrations sont toujours compatibles. Supposons construit le q-hypercomplexe X_q. D'après la proposition 2.10 la propriété sans pente assure que les objets qui apparaissent dans X_q sont des $V_{\mathbf{0}_{I_q}}^{\boldsymbol{H}_{I_q}}\mathcal{D}_X$-modules cohérents spécialisables le long de H_{q+1}. On déduit alors de la proposition 2.11 que l'application de $V_{\alpha_{q+1}}^{H_{q+1}}(.)$ et $V_{\beta_{q+1}}^{H_{q+1}}(.)$ à de tels objets sont deux foncteurs exacts munis d'un monomorphisme de foncteurs donné par l'inclusion naturelle déduite de l'inégalité $\alpha_{q+1} \leqslant \beta_{q+1}$. On applique alors ces deux foncteurs à X_q, la fonctorialité fournit un $(q+1)$-hypercomplexe

$$0 \longrightarrow V_{\alpha_{q+1}}^{H_{q+1}}(X^q) \overset{i}{\hookrightarrow} V_{\beta_{q+1}}^{H_{q+1}}(X^q) \longrightarrow \mathrm{Coker}(i) \longrightarrow 0.$$

C'est le $(q+1)$-hypercomplexe X_{q+1} voulu. L'exactitude des différentes suites courtes provient de l'exactitude des suite courtes de X^q, de l'exactitude des foncteurs $V^{H_{q+1}}$-filtration ainsi que de l'exactitude du foncteur Coker(.) appliqué à des inclusions (lemme du serpent). On utilise également ici les identifications (3). Ceci nous donne par récurrence le p-hypercomplexe X_p. En prenant alors la limite inductive des p-hypercomplexes X_p sur $\boldsymbol{\beta} \in \mathbb{C}^p$ on obtient le p-hypercomplexe correspondant à la compatibilité des sous-objets

$$V_{\alpha_1}^{H_1}(\mathcal{M}), ..., V_{\alpha_p}^{H_p}(\mathcal{M}) \subseteq \mathcal{M}.$$

Ceci étant vérifié pour tout $\boldsymbol{\alpha} \in \mathbb{C}^p$ la proposition est démontrée. □

La proposition B.5 fournit le corollaire suivant

Corollaire 2.13. *Si le couple* $(\mathbf{H}, \mathcal{M})$ *est sans pente alors l'objet obtenu en appliquant successivement les gradués* $\mathrm{gr}_{\alpha_i}^{H_i}$ *par rapport aux V-filtrations canoniques* $V_\bullet^{H_i}$ *ne dépend pas de l'ordre dans lequel on applique ces foncteurs et est égal à* $\mathrm{gr}_{\boldsymbol{\alpha}}(\mathcal{M})$.

Proposition 2.14. *Soit $\mathcal{M}$ un $\mathcal{D}_X$-module cohérent tel que $(\boldsymbol{H}, \mathcal{M})$ soit sans pente et soit $1 \leqslant i \leqslant p$. Alors le $\mathcal{D}_X$-module $\mathcal{M}(*H_i)$ est cohérent et le couple $(\boldsymbol{H}, \mathcal{M}(*H_i))$ est sans pente. De plus, pour tout $\boldsymbol{\alpha}$ vérifiant $\alpha_i < 0$, le morphisme naturel de $V_{\mathbf{0}}\mathcal{D}_X$-modules :*

$$V_{\boldsymbol{\alpha}}(\mathcal{M}) \to V_{\boldsymbol{\alpha}}(\mathcal{M}(*H_i))$$

est un isomorphisme.

Démonstration. Comme $(\boldsymbol{H}, \mathcal{M})$ est sans pente, $\mathcal{M}$ est spécialisable le long de H_i et on peut appliquer [MM04, proposition 4.4-3] qui assure que $\mathcal{M}(*H_i)$ est cohérent, spécialisable le long de H_i et que pour $\alpha_i < 0$,

$$V_{\alpha_i}^{H_i}(\mathcal{M}) \to V_{\alpha_i}^{H_i}(\mathcal{M}(*H_i))$$

est un isomorphisme.

Montrons que le couple $(\boldsymbol{H}, \mathcal{M}(*H_i))$ est sans pente. C'est un problème local, on peut supposer que $H_i = \{t_i = 0\}$. Soit m' une section de $\mathcal{M}(*H_i)$, on a $m' = m/{t_i}^k$ où m est dans l'image de $\mathcal{M} \to \mathcal{M}[1/t_i]$ et $k \in \mathbb{N}$. Le couple $(\boldsymbol{H}, \mathcal{M})$ étant sans pente, pour tout $1 \leqslant j \leqslant p$ il existe un polynôme non nul $b_j(s_j)$ satisfaisant à

$$b_j(E_j)m \in V_{-\mathbf{1}_j}(\mathcal{D}_X)m.$$

On a alors

$$\begin{aligned} b_j(E_j)t_i^k m' &\in V_{-\mathbf{1}_j}(\mathcal{D}_X)t_i^k m' \\ t_i^k b_j(E_j + \delta_{ij}k)m' &\in t_i^k V_{-\mathbf{1}_j}(\mathcal{D}_X)m'. \end{aligned}$$

En divisant par t_i^k on obtient, $b_j(E_j + \delta_{ij}k_i)m' \in V_{-\mathbf{1}_j}(\mathcal{D}_X)m'$, ce qui permet de conclure que $(\boldsymbol{H}, \mathcal{M}(*H_i))$ est sans pente.

D'après la proposition 2.10 $V^{H_i}_{\alpha_i}(\mathcal{M})$ et $V^{H_i}_{\alpha_i}(\mathcal{M}(*H_i))$ sont des $V^{H_i}_0\mathcal{D}_X$-modules sans pente le long de $\boldsymbol{H}_{\{i\}^c}$ donc, si $\boldsymbol{\alpha}$ satisfait à $\alpha_i < 0$, on a un isomorphisme

$$V_{\boldsymbol{\alpha}}(\mathcal{M}) \simeq V^{\boldsymbol{H}_{\{i\}^c}}_{\boldsymbol{\alpha}_{\{i\}^c}}\left(V^{H_i}_{\alpha_i}(\mathcal{M})\right) \xrightarrow{\sim} V^{\boldsymbol{H}_{\{i\}^c}}_{\boldsymbol{\alpha}_{\{i\}^c}}\left(V^{H_i}_{\alpha_i}(\mathcal{M}(*H_i))\right) \simeq V_{\boldsymbol{\alpha}}(\mathcal{M}(*H_i))$$

ce qui conclut la démonstration de la proposition.

□

Corollaire 2.15. *Soit $\mathcal{M}$ un $\mathcal{D}_X$-module cohérent tel que $(\boldsymbol{H}, \mathcal{M})$ soit sans pente. Alors le $\mathcal{D}_X$-module $\mathcal{M}(*(H_1 \cup \ldots \cup H_p))$ est cohérent et le couple $(\boldsymbol{H}, \mathcal{M}(*(H_1 \cup \ldots \cup H_p)))$ est sans pente. De plus pour tout $\boldsymbol{\alpha}$ vérifiant $\alpha_i < 0$ pour tout $1 \leqslant i \leqslant p$, le morphisme naturel de $V_{\mathbf{0}}\mathcal{D}_X$-modules :*

$$V_{\boldsymbol{\alpha}}(\mathcal{M}) \to V_{\boldsymbol{\alpha}}(\mathcal{M}(*(H_1 \cup \ldots \cup H_p))$$

est un isomorphisme.

Démonstration. On effectue une récurrence sur le nombre d'hypersurfaces par rapport auxquelles on localise $\mathcal{M}$ et le corollaire est une conséquence immédiate de la proposition précédente.

□

2.2. **Gradués d'un $\mathcal{D}_X$-module sans pente et cycles proches algébriques.** Ici on démontre des propriétés des gradués de la V-multifiltration de Malgrange-Kashiwara et on définit les cycles proches algébriques.

Proposition 2.16. *Soit $\mathcal{M}$ un $\mathcal{D}_X$-module tel que $(\boldsymbol{H}, \mathcal{M})$ soit sans pente. Pour tout $\beta \in \mathbb{C}$ et tout $1 \leqslant i \leqslant p$, l'endomorphisme $(E_i + \beta + 1)$ de*

$$V_{\beta, \boldsymbol{\alpha}_{\{i\}^c}}(\mathcal{M})/V_{<\beta, \boldsymbol{\alpha}_{\{i\}^c}}(\mathcal{M})$$

est nilpotent.

Démonstration. Notons $\sigma := \sigma_{\beta, \boldsymbol{\alpha}_{\{i\}^c}}$ et $b_i(s)$ le polynôme de Bernstein-Sato d'indice i de la multifiltration correspondant à la section σ. Les racines de b_i sont donc dans l'intervalle $[-\beta - 1, -\beta[$. Soit ℓ la multiplicité de la racine $-\beta - 1$ de b_i. On pose $b_i(s) = b'_i(s)(s + \beta + 1)^\ell$. On considère comme dans la preuve de [Kas83, Théorème 1] la V-multifiltration de $\mathcal{M}$ suivante :

$$U_{\boldsymbol{k}}(\mathcal{M}) := V^\sigma_{\boldsymbol{k}-\mathbf{1}_i}(\mathcal{M}) + (E_i + k_i + \beta + 1)^\ell V^\sigma_{\boldsymbol{k}}(\mathcal{M}).$$

On peut montrer que c'est une bonne V-multifiltration, que ses polynômes de Bernstein-Sato d'indice $j \neq i$ divisent ceux de $V^\sigma_\bullet$ et que son polynôme de Bernstein-Sato d'indice i divise $b'(s)(s + \beta)^\ell$. Les racines de $b'(s)(s + \beta)^\ell$ sont dans $] - \beta - 1, -\beta]$, par unicité la multifiltration

$U_\bullet(\mathcal{M})$ est égale à la multifiltration $V_\bullet^{\widetilde{\sigma}}(\mathcal{M})$ où $\widetilde{\sigma} = \sigma_{<\beta,\boldsymbol{\alpha}_{\{i\}^c}}$. On a donc $U_{\mathbf{0}}(\mathcal{M}) = V_{<\beta,\boldsymbol{\alpha}_{\{i\}^c}}(\mathcal{M})$ et on en déduit que $(E_i + \beta + 1)^\ell$ annule

$$V_{\beta,\boldsymbol{\alpha}_{\{i\}^c}}(\mathcal{M})/V_{<\beta,\boldsymbol{\alpha}_{\{i\}^c}}(\mathcal{M}).$$

□

Étant donnée la définition de $\mathrm{gr}_{\boldsymbol{\alpha}}(\mathcal{M})$ on déduit immédiatement de cette proposition le corollaire suivant :

Corollaire 2.17. *Soit $\mathcal{M}$ un $\mathcal{D}_X$-module tel que $(\boldsymbol{H}, \mathcal{M})$ soit sans pente. Pour tout $\boldsymbol{\alpha} \in \mathbb{C}^p$ et tout $1 \leqslant i \leqslant p$, l'endomorphisme $(E_i + \alpha_i + 1)$ de $\mathrm{gr}_{\boldsymbol{\alpha}}(\mathcal{M})$ est nilpotent.*

Définition 2.18. Étant donné un couple $(\boldsymbol{H}, \mathcal{M})$ sans pente, on définit les *cycles proches algébriques* de $\mathcal{M}$ relatifs à la famille d'hypersurfaces $\boldsymbol{H}$ de la manière suivante

$$\Psi_{\boldsymbol{H}}\mathcal{M} := \bigoplus_{\boldsymbol{\alpha}\in[-1,0[^p} \mathrm{gr}_{\boldsymbol{\alpha}}(\mathcal{M}).$$

C'est un $\mathrm{gr}_{\mathbf{0}}^V\mathcal{D}_X$-modules cohérent. Or, si l'on note $X_0 := \bigcap_{1\leqslant i\leqslant p} H_i$, on a

$$\mathrm{gr}_{\mathbf{0}}^V\mathcal{D}_X \simeq \mathcal{D}_{X_0}[E_1, ..., E_p].$$

Le corollaire 2.17 implique ainsi que $\Psi_{\boldsymbol{H}}\mathcal{M}$ est un $\mathcal{D}_{X_0}$-module cohérent. Les cycles proches algébriques sont munis d'endomorphismes de monodromie pour $1 \leqslant i \leqslant p$

$$T_i := \exp(-2i\pi E_i).$$

La proposition suivante est une conséquence du corollaire 2.13

Proposition 2.19. *Soit $I \subset \{1, ..., p\}$, on a alors un morphisme naturel, fonctoriel en $\mathcal{M}$, de $\mathrm{gr}_{\mathbf{0}}^V\mathcal{D}_X$-modules*

$$\Psi_{\mathbf{H}}\mathcal{M} \to \Psi_{\mathbf{H}_I}(\Psi_{\mathbf{H}_{I^c}}\mathcal{M})$$

qui est un isomorphisme si le couple $(\boldsymbol{H}, \mathcal{M})$ est sans pente.

Dans le cas général $\boldsymbol{f} : X \to \mathbb{C}^p$, l'inclusion du graphe de $\boldsymbol{f}$ permet de définir les cycles proches algébriques.

Définition 2.20. Considérons le diagramme

$$\begin{tikzcd} X \arrow[r, "i_{\boldsymbol{f}}"] \arrow[dr, "\boldsymbol{f}"'] & X \times \mathbb{C}^p \arrow[d, "\boldsymbol{\pi}=(\pi_1,\dots,\pi_p)"] \\ & \mathbb{C}^p. \end{tikzcd}$$

où $i_{\boldsymbol{f}}$ est le graphe de $\boldsymbol{f}$. Soit $H_i := \pi_i^{-1}(0)$. D'après ce qui précède, si le couple $(\boldsymbol{H}, i_{\boldsymbol{f}+}\mathcal{M})$ est sans pente, alors $\Psi_{\boldsymbol{H}} i_{\boldsymbol{f}+}\mathcal{M}$ est un $\mathcal{D}_{X\times 0}$-module cohérent à support $\{(\boldsymbol{x}, 0) | \boldsymbol{f}(\boldsymbol{x}) = 0\}$. On peut le voir comme un $\mathcal{D}_X$-module cohérent à support $\boldsymbol{f}^{-1}(0)$, on le note alors $\Psi_{\boldsymbol{f}}^{\mathrm{alg}}\mathcal{M}$.

On déduit de la proposition 2.19 l'isomorphisme

$$\Psi_{\boldsymbol{f}}^{\mathrm{alg}}\mathcal{M} \to \Psi_{\boldsymbol{f}_I}^{\mathrm{alg}}(\Psi_{\boldsymbol{f}_{I^c}}^{\mathrm{alg}}\mathcal{M}).$$

2.3. **Cycles proches topologiques.** Ici on définit le foncteur cycles proches topologiques associé à une fonction $f : X \to \mathbb{C}^p$ et appliqué à la catégorie des complexes de faisceaux à cohomologie $\mathbb{C}$-constructible.

Définition 2.21. Considérons le diagramme suivant où les carrés sont cartésiens :

$$\begin{tikzcd} f^{-1}(0) \arrow[r, "i"] \arrow[d] & X \arrow[d, "f"] & X^* \arrow[l, "j"'] \arrow[d, "f_{|X^*}"] & \widetilde{X} \arrow[l, "p"'] \arrow[d, "\widetilde{f}"] \\ \{0\} \arrow[r, "i"] & \mathbb{C}^p & (\mathbb{C}^*)^p \arrow[l, "j"'] & \widetilde{(\mathbb{C}^*)^p}. \arrow[l, "p"'] \end{tikzcd}$$

Ici $X^* = X - F^{-1}(0)$ avec $F = f_1 ... f_p$ et $\widetilde{(\mathbb{C}^*)^p}$ est le revêtement universel de $(\mathbb{C}^*)^p$.

Si $\mathcal{F}$ est un complexe de faisceaux à cohomologie $\mathbb{C}$-constructible, on définit :

$$\Psi_f \mathcal{F} := i^{-1} \boldsymbol{R} j_* p_* p^{-1} j^{-1} \mathcal{F}$$

c'est le *foncteur cycles proches*. On peut identifier le morphisme $\widetilde{(\mathbb{C}^*)^p} \to (\mathbb{C}^*)^p$ à

$$\begin{array}{rccc} \exp : & \mathbb{C}^p & \to & (\mathbb{C}^*)^p \\ & (z_1, ..., z_p) & \mapsto & (e^{2i\pi z_1}, ..., e^{2i\pi z_p}). \end{array}$$

Pour $1 \leqslant i \leqslant p$ les translations

$$\begin{array}{rccc} \tau_i : & \widetilde{(\mathbb{C}^*)^p} & \to & \widetilde{(\mathbb{C}^*)^p} \\ & (z_1, ..., z_i, ..., z_p) & \mapsto & (z_1, ..., z_i + 1, ..., z_p). \end{array}$$

permettent d'induire des endomorphismes de monodromie $T_i : \Psi_f \mathcal{F} \to \Psi_f \mathcal{F}$.

Supposons que les f_i définissent un diviseur à croisements normaux $\mathbf{H}$ où $H_i = \{f_i = 0\}$ et que $\mathcal{F} = \mathbf{DR}(\mathcal{M})$. Dans [Mai13] Ph. Maisonobe démontre la proposition suivante

Proposition 2.22. *Soit $I \subset \{1, ..., p\}$, il existe un morphisme naturel*

$$\Psi_f \mathcal{F} \to \Psi_{f_I}(\Psi_{f_{I^c}} \mathcal{F}). \tag{4}$$

De plus si le couple $(\mathbf{H}, \mathcal{M})$ est sans pente alors ce morphisme est un isomorphisme.

2.4. **Fonctions de classe de Nilsson.** On se place ici dans le cas d'une famille d'hypersurfaces qui forment un diviseur à croisements normaux, quitte à diminuer X, on suppose qu'il existe un système de coordonnées $(\boldsymbol{x}, t_1, ..., t_p)$ tel que pour tout $1 \leqslant i \leqslant p$, l'hypersurface H_i ait pour équation $t_i = 0$. On note

$$\begin{array}{rccc} \pi : & X & \to & \mathbb{C}^p \\ & (\boldsymbol{x}, t_1, ..., t_p) & \mapsto & (t_1, ..., t_p). \end{array}$$

Définition 2.23. Soit $\boldsymbol{\alpha} \in [-1, 0[^p$ et $\boldsymbol{k} \in \mathbb{N}^p$. On note $\mathcal{N}_{\boldsymbol{\alpha},\boldsymbol{k}}$ la connexion méromorphe sur $\mathbb{C}^p$:

$$\mathcal{N}_{\boldsymbol{\alpha},\boldsymbol{k}} = \bigoplus_{0 \leqslant \boldsymbol{\ell} \leqslant \boldsymbol{k}} \mathcal{O}_{\mathbb{C}^p}[\frac{1}{z_1 ... z_p}] e_{\boldsymbol{\alpha},\boldsymbol{\ell}}$$

avec la structure de $\mathcal{D}$-module donnée par la formule

$$z_i \partial_{z_i} e_{\boldsymbol{\alpha},\boldsymbol{\ell}} = (\alpha_i + 1) e_{\boldsymbol{\alpha},\boldsymbol{\ell}} + e_{\boldsymbol{\alpha},\boldsymbol{\ell} - \mathbf{1}_i}.$$

On définit T_i le morphisme de monodromie d'indice i par la formule

$$T_i e_{\boldsymbol{\alpha},\boldsymbol{\ell}} = \exp(2i\pi(\alpha_i + 1)) \sum_{0 \leqslant m \leqslant \ell_i} \frac{(2i\pi)^m}{m!} e_{\boldsymbol{\alpha},\boldsymbol{\ell} - m.\mathbf{1}_i}.$$

Remarque 2.24. Pour se souvenir de la structure de $\mathcal{D}$-module et de la monodromie il faut remarquer que la section $e_{\boldsymbol{\alpha},\boldsymbol{\ell}}$ se comporte comme la fonction multiforme $\boldsymbol{z}^{\boldsymbol{\alpha}+\mathbf{1}}\frac{\log^{\ell_1} z_1}{\ell_1!}\ldots\frac{\log^{\ell_p} z_p}{\ell_p!}$.

Définition 2.25. Soit $\mathcal{M}$ un $\mathcal{D}_X$-module tel que le couple $(\boldsymbol{H},\mathcal{M})$ soit sans pente. On définit :

$$\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}} = \mathcal{M}\otimes_{\pi^{-1}\mathcal{O}_{\mathbb{C}^p}} \pi^{-1}\left(\mathcal{N}_{\boldsymbol{\alpha},\boldsymbol{k}}\right) = \mathcal{M}[\frac{1}{t_1\ldots t_p}]\otimes_{\pi^{-1}\mathcal{O}_{\mathbb{C}^p}} \pi^{-1}\left(\mathcal{N}_{\boldsymbol{\alpha},\boldsymbol{k}}\right).$$

D'autre part on a

$$\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}} = \mathcal{M}\otimes_{\mathcal{O}_X} \pi^{+}\left(\mathcal{N}_{\boldsymbol{\alpha},\boldsymbol{k}}\right)$$

où π^+ est l'image inverse dans la catégorie des $\mathcal{D}$-modules. Ceci permet de munir $\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}}$ d'une structure naturelle de $\mathcal{D}_X$-module. Notons $Y := \bigcap_{1\leqslant i\leqslant p} H_i$. La restriction de $\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}}$ à Y est munie d'endomorphismes T_i induits par les morphismes de monodromie de $\mathcal{N}_{\boldsymbol{\alpha},\boldsymbol{k}}$ et définis par :

$$T_i(m\otimes e_{\boldsymbol{\alpha},\boldsymbol{\ell}}) = m\otimes T_i e_{\boldsymbol{\alpha},\boldsymbol{\ell}}.$$

Proposition 2.26. *Soit $\boldsymbol{\alpha}\in[-1,0[^p$ et $\boldsymbol{k}\in\mathbb{N}^p$ et $\mathcal{M}$ un $\mathcal{D}_X$-module tel que le couple $(\boldsymbol{H},\mathcal{M})$ soit sans pente. Alors le couple $(\boldsymbol{H},\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}})$ est sans pente. De plus, pour tout $\boldsymbol{\beta}\in\mathbb{C}^p$, on a :*

$$V_{\boldsymbol{\beta}}(\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}}) = \bigoplus_{\mathbf{0}\leqslant\boldsymbol{\ell}\leqslant\boldsymbol{k}} V_{\boldsymbol{\alpha}+\boldsymbol{\beta}+\mathbf{1}}\left(\mathcal{M}[\frac{1}{t_1\ldots t_p}]\right)e_{\boldsymbol{\alpha},\boldsymbol{\ell}}.$$

On commence par un lemme qui sera utile dans la démonstration de cette proposition.

Définition 2.27. Soit $(\boldsymbol{x},t_1,\ldots,t_p)$ un système de coordonnées locales où $t_i=0$ est une équation de H_i. Soit $\mathcal{M}[1/\boldsymbol{t},\boldsymbol{s}]\boldsymbol{t}^{\boldsymbol{s}}$ le $\mathcal{O}_X[\boldsymbol{s}]$-module isomorphe à $\mathcal{M}[1/\boldsymbol{t},\boldsymbol{s}]$ par l'application $m\mapsto m\boldsymbol{t}^{\boldsymbol{s}}$. Il est muni d'une structure naturelle de $\mathcal{D}_X[\boldsymbol{s}]$-module par la formule :

$$\partial_i(m\boldsymbol{t}^{\boldsymbol{s}}) := (\partial_i m)\boldsymbol{t}^{\boldsymbol{s}} + (\frac{s_i m}{t_i})\boldsymbol{t}^{\boldsymbol{s}}$$

Lemme 2.28. *Soit m une section locale de $\mathcal{M}[1/\boldsymbol{t}]$ et $b(s)\in\mathbb{C}[s]$. Les conditions suivantes sont équivalentes :*

(1) $b(E_i)m\in V_{-\mathbf{1}_i}(\mathcal{D}_X)m$
(2) $b(-s_i-1)m\boldsymbol{t}^{\boldsymbol{s}}\in\mathcal{D}_X[\boldsymbol{s}]t_i m\boldsymbol{t}^{\boldsymbol{s}}$

Démonstration. Montrons que *1* implique *2*. Dans $\mathcal{M}[1/\boldsymbol{t},\boldsymbol{s}]\boldsymbol{t}^{\boldsymbol{s}}$ on a l'égalité

$$(t_i\partial_i m)\boldsymbol{t}^{\boldsymbol{s}} = (-s_i-1)m\boldsymbol{t}^{\boldsymbol{s}} + \partial_i(t_i m\boldsymbol{t}^{\boldsymbol{s}}).$$

On montre alors par récurrence que pour tout k

$$((t_i\partial_i)^k m)\boldsymbol{t}^{\boldsymbol{s}} - (-s_i-1)^k m\boldsymbol{t}^{\boldsymbol{s}}\in\mathcal{D}_X[\boldsymbol{s}]t_i m\boldsymbol{t}^{\boldsymbol{s}}.$$

On a donc pour tout polynôme $b(s)\in\mathbb{C}[s]$

$$(b(E_i)m)\boldsymbol{t}^{\boldsymbol{s}} - b(-s_i-1)m\boldsymbol{t}^{\boldsymbol{s}}\in\mathcal{D}_X[\boldsymbol{s}]t_i m\boldsymbol{t}^{\boldsymbol{s}}.$$

D'autre part, si $b(E_i)m\in V_{-\mathbf{1}_i}(\mathcal{D}_X)m$ une récurrence permet de montrer que $(b(E_i)m)\boldsymbol{t}^{\boldsymbol{s}}\in\mathcal{D}_X[\boldsymbol{s}]t_i m\boldsymbol{t}^{\boldsymbol{s}}$ et on en déduit *2*.

Montrons que *2* implique *1*. D'une part, on peut montrer par récurrence que pour tout $k\in\mathbb{N}$ et tout $1\leqslant\ell\leqslant k$, il existe $m_{k,\ell}\in\mathcal{M}[1/\boldsymbol{t}]$ satisfaisant à :

$$s_i^k m\boldsymbol{t}^{\boldsymbol{s}} = ((-\partial_i t_i)^k m)\boldsymbol{t}^{\boldsymbol{s}} + \sum_{\ell=1}^{k}\partial_i^{\ell}(m_{k,\ell}\boldsymbol{t}^{\boldsymbol{s}}). \tag{5}$$

D'autre part, en faisant opérer les $\partial^{\boldsymbol{\alpha}} = \partial_1^{\alpha_1}...\partial_p^{\alpha_p}$ et en annulant les coefficients du polynôme en les s_i que l'on obtient, on peut montrer le résultat suivant :

$$\left[\sum_{\boldsymbol{\alpha}} \partial^{\boldsymbol{\alpha}}(m_{\boldsymbol{\alpha}}\boldsymbol{t}^{\boldsymbol{s}}) = 0\right] \Rightarrow [m_{\boldsymbol{\alpha}} = 0 \quad \forall \boldsymbol{\alpha}] \tag{6}$$

pour une somme finie sur les $\boldsymbol{\alpha}$. Enfin, si l'on regarde plus précisément la récurrence faite dans la première partie de la démonstration on obtient

$$(b(E_i)m)\boldsymbol{t}^{\boldsymbol{s}} - b(-s_i-1)m\boldsymbol{t}^{\boldsymbol{s}} \in \partial_i\mathcal{D}_X[\boldsymbol{s}]t_im\boldsymbol{t}^{\boldsymbol{s}}.$$

L'hypothèse *2* implique

$$b(-s_i-1)m\boldsymbol{t}^{\boldsymbol{s}} = \sum_{\boldsymbol{\alpha},\boldsymbol{k}} \partial^{\boldsymbol{\alpha}}\boldsymbol{s}^{\boldsymbol{k}}A_{\boldsymbol{\alpha},\boldsymbol{k}}t_im\boldsymbol{t}^{\boldsymbol{s}}$$

où $A_{\boldsymbol{\alpha},\boldsymbol{k}}$ est un opérateur différentiel indépendant des ∂_i pour tout $1 \leqslant i \leqslant p$. En utilisant l'égalité (5) on peut substituer les s_j et on obtient

$$(b(E_i)m)\boldsymbol{t}^{\boldsymbol{s}} - \sum_{\boldsymbol{k}}\left[(-t_1\partial_1-1)^{k_1}...(-t_p\partial_p-1)^{k_p}A_{\boldsymbol{0},\boldsymbol{k}}t_im\right]\boldsymbol{t}^{\boldsymbol{s}} = \sum_{\boldsymbol{\alpha}>\boldsymbol{0}} \partial^{\boldsymbol{\alpha}}(m_{\boldsymbol{\alpha}}\boldsymbol{t}^{\boldsymbol{s}})$$

avec $m_{\boldsymbol{\alpha}} \in \mathcal{M}[1/\boldsymbol{t}]$. En utilisant (6) et le fait que $(-t_1\partial_1-1)^{k_1}...(-t_p\partial_p-1)^{k_p}A_{\boldsymbol{0},\boldsymbol{k}}t_i \in V_{-\boldsymbol{1}_i}(\mathcal{D}_X)$ on conclut que $b(E_i)m \in V_{-\boldsymbol{1}_i}(\mathcal{D}_X)m$.

□

Démonstration de la proposition 2.26. On commence par montrer que le couple $(\boldsymbol{H}, \mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}})$ est sans pente. Quelque soit $1 \leqslant i \leqslant p$, le $\mathcal{D}_{\mathbb{C}^p}$-module $\mathcal{N}_{\boldsymbol{\alpha},\boldsymbol{k}}/\mathcal{N}_{\boldsymbol{\alpha},\boldsymbol{k}-\boldsymbol{1}_i}$ s'identifie à $\mathcal{N}_{\boldsymbol{\alpha},\boldsymbol{k}-k_i.\boldsymbol{1}_i}$ On a donc la suite exacte :

$$0 \to \mathcal{N}_{\boldsymbol{\alpha},\boldsymbol{k}-\boldsymbol{1}_i} \to \mathcal{N}_{\boldsymbol{\alpha},\boldsymbol{k}} \to \mathcal{N}_{\boldsymbol{\alpha},\boldsymbol{k}-k_i.\boldsymbol{1}_i} \to 0.$$

Pour tout $\boldsymbol{k} \in \mathbb{N}^p$ le $\pi^{-1}\mathcal{O}_{\mathbb{C}^p}$-module $\pi^{-1}\mathcal{N}_{\boldsymbol{\alpha},\boldsymbol{k}}$ est à fibres plates car libres, il est donc acyclique pour le foncteur de produit tensoriel par $\mathcal{M}[\frac{1}{t_1...t_p}]$ et on a la suite exacte :

$$0 \to \mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}-\boldsymbol{1}_i} \to \mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}} \to \mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}-k_i.\boldsymbol{1}_i} \to 0. \tag{7}$$

Le module central est sans pente si et seulement si les deux autres modules le sont. En effet, comme dans le cas des bonnes V-filtration pour $p = 1$ (cf [MM04]), une bonne V-multifiltration du terme central induit des bonnes V-multifiltration des termes extrêmes. On considère alors la suite exacte

$$0 \to U_{\boldsymbol{\ell}}\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}-\boldsymbol{1}_i} \to U_{\boldsymbol{\ell}}\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}} \to U_{\boldsymbol{\ell}}\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}-k_i.\boldsymbol{1}_i} \to 0$$

et on observe que la condition multispécialisable sans pente de la définition 2.2 est satisfaite pour le module central si et seulement si elle l'est pour les deux autres modules. Par récurrence on est alors ramené à montrer que $(\boldsymbol{H}, \mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{0}})$ est sans pente. Soit m une section locale de $\mathcal{M}[\frac{1}{t_1...t_p}]$. D'après la proposition 2.15 le couple $(\boldsymbol{H}, \mathcal{M}[\frac{1}{t_1...t_p}])$ est sans pente et par conséquent le lemme 2.28 fournit localement, pour $1 \leqslant i \leqslant p$, des polynômes b_i non nuls vérifiant :

$$b_i(s_i)m\boldsymbol{t}^{\boldsymbol{s}} \in \mathcal{D}_X[\boldsymbol{s}]t_im\boldsymbol{t}^{\boldsymbol{s}}.$$

Par définition du $\mathcal{D}_X[s]$-module $\mathcal{M}[1/\boldsymbol{t}, \boldsymbol{s}]\boldsymbol{t}^{\boldsymbol{s}}$, on obtient les équations :

$$b_i(s_i+\alpha_i+1)(m\otimes e_{\boldsymbol{\alpha},\boldsymbol{0}})\boldsymbol{t}^{\boldsymbol{s}} \in \mathcal{D}_X[\boldsymbol{s}]t_i(m\otimes e_{\boldsymbol{\alpha},\boldsymbol{0}})\boldsymbol{t}^{\boldsymbol{s}}. \tag{8}$$

Soit $\boldsymbol{k}_0 \in \mathbb{N}^p$ tel que pour tout $k_i \in \mathbb{N}$ vérifiant $k_i \geqslant k_{0,i}+1$, l'entier $-k_i$ n'est pas racine de $b_i(s_i+\alpha_i+1) \in \mathbb{C}[s_i]$. En remplaçant les s_i par les entiers k_i dans la relation (8) et en multipliant éventuellement par des t_i on obtient que pour tout $\boldsymbol{k} \in \mathbb{Z}^p$

$$(m\otimes e_{\boldsymbol{\alpha},\boldsymbol{0}})\boldsymbol{t}^{\boldsymbol{k}} \in \mathcal{D}_X((m\otimes e_{\boldsymbol{\alpha},\boldsymbol{0}})\boldsymbol{t}^{-\boldsymbol{k}_0}).$$

De plus pour tout $1 \leqslant i \leqslant p$, l'égalité $(\partial_i(m \otimes e_{\boldsymbol{\alpha},\mathbf{0}}))\boldsymbol{t}^{\boldsymbol{k}} = \partial_i((m \otimes e_{\boldsymbol{\alpha},\mathbf{0}})\boldsymbol{t}^{\boldsymbol{k}}) + k_i(m \otimes e_{\boldsymbol{\alpha},\mathbf{0}})\boldsymbol{t}^{\boldsymbol{k}-\mathbf{1}_i}$ montre que $(\partial_i(m \otimes e_{\boldsymbol{\alpha},\mathbf{0}}))\boldsymbol{t}^{\boldsymbol{k}} \in \mathcal{D}_X((m \otimes e_{\boldsymbol{\alpha},\mathbf{0}})\boldsymbol{t}^{-\boldsymbol{k}_0})$ pour tout $\boldsymbol{k} \in \mathbb{Z}^p$. Comme $\mathcal{M}$ est engendré par un nombre fini de sections, en utilisant des extensions successives on peut supposer que m engendre $\mathcal{M}$. On a donc $\mathcal{M}_{\boldsymbol{\alpha},\mathbf{0}} = \mathcal{D}_X((m \otimes e_{\boldsymbol{\alpha},\mathbf{0}})\boldsymbol{t}^{-\boldsymbol{k}_0})$. La filtration $\mathcal{D}_X(l)((m \otimes e_{\boldsymbol{\alpha},\mathbf{0}})\boldsymbol{t}^{-\boldsymbol{k}_0}))$ étant une bonne filtration du $\mathcal{D}_X$-module $\mathcal{M}_{\boldsymbol{\alpha},\mathbf{0}}$, celui-ci est cohérent. Les équations (8) ainsi que le lemme 2.28 permettent alors de conclure que $(\boldsymbol{H}, \mathcal{M}_{\boldsymbol{\alpha},\mathbf{0}})$ est sans pente et donc par ce qui précède que $(\boldsymbol{H}, \mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}})$ l'est.

Pour démontrer la deuxième partie de la proposition on commence par noter

$$U_{\boldsymbol{\beta}}(\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}}) := \bigoplus_{\mathbf{0}\leqslant\boldsymbol{\ell}\leqslant\boldsymbol{k}} V_{\boldsymbol{\alpha}+\boldsymbol{\beta}+\mathbf{1}}\left(\mathcal{M}[\frac{1}{t_1\ldots t_p}]\right) e_{\boldsymbol{\alpha},\boldsymbol{\ell}}$$

et on va montrer que c'est une bonne V-multifiltration qui satisfait à toutes les propriétés caractéristiques de la multifiltration de Malgrange-Kashiwara. Soit $m \in \mathcal{M}$, $\boldsymbol{\ell} \in \mathbb{N}^p$, $\beta \in \mathbb{C}$ et $1 \leqslant i \leqslant p$. On a localement

$$(t_i\partial_i + \beta)(m \otimes e_{\boldsymbol{\alpha},\boldsymbol{\ell}}) = ((t_i\partial_i + \beta + \alpha_i + 1)m) \otimes e_{\boldsymbol{\alpha},\boldsymbol{\ell}} + m \otimes e_{\boldsymbol{\alpha},\boldsymbol{\ell}-\mathbf{1}_i} \tag{9}$$

et pour tout $\boldsymbol{n} \in \mathbb{Z}^p$

$$\boldsymbol{t}^{\boldsymbol{n}}(m \otimes e_{\boldsymbol{\alpha},\boldsymbol{\ell}}) = (\boldsymbol{t}^{\boldsymbol{n}}m) \otimes e_{\boldsymbol{\alpha},\boldsymbol{\ell}}.$$

Ceci permet de montrer que $U_\bullet(\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}})$ est une V-multifiltration de $\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}}$ (c'est-à-dire que cette multifiltration vérifie $V_{\boldsymbol{\ell}}\mathcal{D}_X.U_{\boldsymbol{\beta}}(\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}}) \subset U_{\boldsymbol{\beta}+\boldsymbol{\ell}}(\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}})$ pour tout $\boldsymbol{\beta} \in \mathbb{C}^p$ et pour tout $\boldsymbol{\ell} \in \mathbb{Z}^p$).

Pour montrer que c'est une bonne V-multifiltration on fixe $\boldsymbol{\beta} \in \mathbb{C}^p$ et on montre que la V-multifiltration indexée par $\mathbb{Z}^p$, $U_{\boldsymbol{\beta}+\bullet}(\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}})$, est une bonne V-multifiltration de $\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}}$. Comme la V-multifiltration indexée par $\mathbb{Z}^p$, $V_{\boldsymbol{\alpha}+\boldsymbol{\beta}+\bullet+\mathbf{1}}\left(\mathcal{M}[\frac{1}{t_1\ldots t_p}]\right)$, est une bonne V-multifiltration elle est engendrée localement par un nombre fini de sections $\{m_j\}_{j\in J}$. Si $\boldsymbol{k} = \mathbf{0}$ l'égalité (9) permet de montrer que les sections $\{m_j \otimes e_{\boldsymbol{\alpha},\mathbf{0}}\}_{j\in J}$ engendrent la V-multifiltration $U_{\boldsymbol{\beta}+\bullet}(\mathcal{M}_{\boldsymbol{\alpha},\mathbf{0}})$. On peut alors montrer par récurrence, en considérant la suite exacte (7) et l'égalité (9), que pour tout $\boldsymbol{k} \in \mathbb{N}^p$ les sections $m_j \otimes e_{\boldsymbol{\alpha},\boldsymbol{\ell}}$, pour $j \in J$ et $0 \leqslant \boldsymbol{\ell} \leqslant \boldsymbol{k}$, engendrent la V-multifiltration $U_{\boldsymbol{\beta}+\bullet}(\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}})$. C'est donc une bonne V-multifiltration de $\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}}$.

On fixe maintenant $\boldsymbol{\beta} \in \mathbb{C}^p$ et on va construire, pour tout $1 \leqslant i \leqslant p$, un polynôme $b_i(s)$ qui satisfait à

$$b_i(t_i\partial_i)U_{\boldsymbol{\beta}}(\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}}) \subset U_{\boldsymbol{\beta}-\mathbf{1}_i}(\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}}).$$

Par définition de la multifiltration de Malgrange-Kashiwara on peut choisir, pour tout $1 \leqslant i \leqslant p$, un polynôme $c_i(s)$ vérifiant

$$c_i(t_i\partial_i + \alpha_i + \beta_i + 1)V_{\boldsymbol{\alpha}+\boldsymbol{\beta}+\mathbf{1}}\left(\mathcal{M}[\frac{1}{t_1\ldots t_p}]\right) \subset V_{\boldsymbol{\alpha}+\boldsymbol{\beta}+\mathbf{1}-\mathbf{1}_i}\left(\mathcal{M}[\frac{1}{t_1\ldots t_p}]\right)$$

et ayant ses racines dans l'intervalle $[-1, 0[$. Soit $m \in V_{\boldsymbol{\alpha}+\boldsymbol{\beta}+\mathbf{1}}\left(\mathcal{M}[\frac{1}{t_1\ldots t_p}]\right)$, l'égalité (9) permet de montrer que

$$c_i(t_i\partial_i + \beta_i)(m \otimes e_{\boldsymbol{\alpha},\boldsymbol{\ell}}) = (c_i(t_i\partial_i + \beta_i + \alpha_i + 1)m) \otimes e_{\boldsymbol{\alpha},\boldsymbol{\ell}} + \widetilde{m}$$

où $\widetilde{m} \in U_{\boldsymbol{\beta}}(\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}-\mathbf{1}_i})$ si on pose $U_{\boldsymbol{\beta}}(\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{\ell}}) = 0$ pour $l_i < 0$. On peut donc construire par récurrence un polynôme $b_{i,m}(s)$ ayant ses racines dans l'intervalle $[-1, 0[$ et vérifiant

$$b_{i,m}(t_i\partial_i + \beta_i)(m \otimes e_{\boldsymbol{\alpha},\boldsymbol{\ell}}) \in U_{\boldsymbol{\beta}-\mathbf{1}_i}(\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}}).$$

Comme $U_{\boldsymbol{\beta}}(\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}})$ est localement engendré par un nombre fini de sections de la forme $m \otimes e_{\boldsymbol{\alpha},\boldsymbol{\ell}}$ pour $0 \leqslant \boldsymbol{\ell} \leqslant \boldsymbol{k}$ on peut construire $b_i(s)$ ayant ses racines dans $[-1, 0[$ tel que

$$b_i(t_i\partial_i + \beta_i)U_{\boldsymbol{\beta}}(\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}}) \subset U_{\boldsymbol{\beta}-\mathbf{1}_i}(\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}}).$$

Les racines du polynôme de Bernstein-Sato de la V-multifiltration $U_\bullet(\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}-\mathbf{1}_i})$ sont donc dans l'intervalle $[-1,0[$, ce qui permet de conclure que c'est bien la V-multifiltration de Malgrange-Kashiwara :

$$V_{\boldsymbol{\beta}}(\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}}) = \bigoplus_{\mathbf{0}\leqslant\boldsymbol{\ell}\leqslant\boldsymbol{k}} V_{\boldsymbol{\alpha}+\boldsymbol{\beta}+\mathbf{1}}\left(\mathcal{M}[\frac{1}{t_1...t_p}]\right) e_{\boldsymbol{\alpha},\boldsymbol{\ell}}.$$

□

3. Morphisme de comparaison

On va construire un morphisme de comparaison entre les cycles proches algébriques de $\mathcal{M}$ et les cycles proches topologiques de $\mathbf{DR}(\mathcal{M})$ relativement à l'application

$$\begin{array}{rccc} \pi : & X & \to & \mathbb{C}^p \\ & (\boldsymbol{x}, t_1, ..., t_p) & \mapsto & (t_1, ..., t_p). \end{array}$$

On établira le lien avec la composition du morphisme de comparaison relatif aux r premières coordonnées t_i et de celui relatif aux $p-r$ coordonnées t_i suivantes pour $1 < r < p$.

3.1. Comparaison avec les gradués.

Commençons par donner deux définitions.

Définition 3.1. Soit $\mathcal{M}$ un $\mathcal{D}_X$-module tel que le couple $(\boldsymbol{H},\mathcal{M})$ soit sans pente. On considère la famille $\{\mathrm{gr}_{\boldsymbol{k}}(\mathcal{M}),\partial_i\}_{\boldsymbol{k}\in\{0,1\}^p,1\leqslant i\leqslant p}$ composée des objets $\mathrm{gr}_{\boldsymbol{k}}(\mathcal{M})$ pour $\boldsymbol{k}\in\{0,1\}^p$ et des morphismes $\partial_i : \mathrm{gr}_{\boldsymbol{k}}(\mathcal{M}) \to \mathrm{gr}_{\boldsymbol{k}+\mathbf{1}_i}(\mathcal{M})$. On définit

$$i^\dagger\mathcal{M} := s(\mathrm{Cube}(\mathrm{gr}_\bullet(\mathcal{M})))|_{X_0}$$

où $s(.)$ et $\mathrm{Cube}(.)$ sont les foncteurs définis dans l'appendice A.2 et A.5 et $X_0 = \pi^{-1}(0)$.

Par exemple pour $p = 2$ on a

$$\begin{array}{rccccccc} i^\dagger\mathcal{M} = & 0 \to & \mathrm{gr}_{-1,-1}(\mathcal{M})|_{X_0} & \to & \mathrm{gr}_{0,-1}(\mathcal{M})|_{X_0} \bigoplus \mathrm{gr}_{-1,0}(\mathcal{M})|_{X_0} & \to & \mathrm{gr}_{0,0}(\mathcal{M})|_{X_0} & \to 0 \\ & & m & \mapsto & (\partial_1 m, -\partial_2 m) & & & \\ & & & & (m_1,m_2) & \mapsto & \partial_2 m_1 + \partial_1 m_2 & . \end{array}$$

Définition 3.2. De la même manière que pour la définition précédente on considère la famille $\{V_{\boldsymbol{k}}(\mathcal{M}),\partial_i\}_{\boldsymbol{k}\in\{0,1\}^p,1\leqslant i\leqslant p}$ composée des objets $V_{\boldsymbol{k}}(\mathcal{M})$ pour $\boldsymbol{k}\in\{0,1\}^p$ et des morphismes

$$\partial_i : V_{\boldsymbol{k}}(\mathcal{M}) \to V_{\boldsymbol{k}+\mathbf{1}_i}(\mathcal{M}).$$

On définit

$$i^\#\mathcal{M} := s(\mathrm{Cube}(V_\bullet(\mathcal{M})))|_{X_0}$$

où $X_0 = \pi^{-1}(0)$.

Remarque 3.3.

1. Notons que si on considère la famille $\mathscr{M} := \{\mathcal{M},\partial_i\}_{\boldsymbol{k}\in\{0,1\}^p,1\leqslant i\leqslant p}$ on a
$$s(\mathrm{Cube}(\mathscr{M}))|_{X_0} \simeq \mathbf{DR}_{X/X_0}(\mathcal{M})|_{X_0}$$
où l'on considère la projection
$$\begin{array}{rccc} \tau : & X & \to & X_0 \\ & (\boldsymbol{x}, t_1, ..., t_p) & \mapsto & (\boldsymbol{x}, 0, ..., 0). \end{array}$$
2. On étend ces définitions aux complexes en commençant par appliquer $\mathrm{Cube}(.)$ en chaque degré puis en prenant le complexe simple associé à l'hypercomplexe obtenu. On note encore $i^\#$ et $i^\dagger$ ces foncteurs appliqués aux complexes.

D'après la remarque précédente les morphismes naturels pour tout $\boldsymbol{k} \in \{0,1\}^p$

$$\mathrm{gr}_{\boldsymbol{k}}(\mathcal{M}) \leftarrow V_{\boldsymbol{k}}(\mathcal{M}) \rightarrow \mathcal{M}$$

induisent les morphismes de complexes

$$\boxed{i^{\dagger}\mathcal{M} \leftarrow i^{\#}\mathcal{M} \rightarrow \mathbf{DR}_{X/X_0}(\mathcal{M})} \tag{10}$$

où l'on omet de noter la restriction de $\mathbf{DR}_{X/X_0}(\mathcal{M})$ à X_0. Soit $I = \{1, ..., r\} \subset \{1, ..., p\}$ les r premiers entiers pour $r < p$, on note

$$\begin{array}{cccc} \pi_I : & X & \rightarrow & \mathbb{C}^r \\ & (\boldsymbol{x}, t_1, ..., t_p) & \mapsto & (t_1, ..., t_r) \end{array}$$

et $X_0^I := \pi_I^{-1}(0)$. On note $V_{\bullet}^I$ la V-multifiltration par rapport aux fonctions $t_1, ..., t_r$. La V-multifiltration de Malgrange-Kashiwara de $\mathcal{M}$ induit une V^{I^c}-multifiltration du $\mathcal{D}_{X_0^I}$-module $\mathrm{gr}^I_{\boldsymbol{\alpha}_I}(\mathcal{M})$ pour tout $\boldsymbol{\alpha}_I \in \mathbb{C}^r$. Pour tout $\boldsymbol{\alpha} \in \mathbb{C}^p$ on a le diagramme commutatif suivant

$$\begin{tikzcd} \mathrm{gr}_{\boldsymbol{\alpha}}\mathcal{M} \arrow[d] & & V_{\boldsymbol{\alpha}}\mathcal{M} \arrow[ll] \arrow[d, equal] \\ \mathrm{gr}^{I^c}_{\boldsymbol{\alpha}_{I^c}}\left(\mathrm{gr}^I_{\boldsymbol{\alpha}_I}\mathcal{M}\right) & V^{I^c}_{\boldsymbol{\alpha}_{I^c}}\left(\mathrm{gr}^I_{\boldsymbol{\alpha}_I}\mathcal{M}\right) \arrow[l] & V^{I^c}_{\boldsymbol{\alpha}_{I^c}}\left(V^I_{\boldsymbol{\alpha}_I}\mathcal{M}\right). \arrow[l] \end{tikzcd} \tag{11}$$

On définit les foncteurs $i_I^{\dagger}$ et $i_I^{\#}$ en considérant respectivement les familles

$$\{\mathrm{gr}_{\boldsymbol{k}_I}(\mathcal{M}'), \partial_i\}_{\boldsymbol{k}_I \in \{0,1\}^r, 1 \leqslant i \leqslant r}$$

et $\{V_{\boldsymbol{k}_I}(\mathcal{M}'), \partial_i\}_{\boldsymbol{k}_I \in \{0,1\}^r, 1 \leqslant i \leqslant r}$. On définit de manière analogue les foncteurs $i_{I^c}^{\dagger}$ et $i_{I^c}^{\#}$ appliqués à la catégorie des $\mathcal{D}_{X_0^I}$-modules en considérant la projection

$$\begin{array}{cccc} \pi_{I^c|X_0^I} : & X_0^I & \rightarrow & \mathbb{C}^{p-r} \\ & (\boldsymbol{x}, t_{p-r}, ..., t_p) & \mapsto & (t_{p-r}, ..., t_p). \end{array}$$

Les propriétés des hypercomplexes, du foncteur $s(.)$ et le diagramme commutatif (11) pour $\boldsymbol{\alpha} \in \{0,1\}^p$ fournissent le diagramme commutatif suivant

$$\begin{tikzcd} i^{\dagger}\mathcal{M} \arrow[d] & & i^{\#}\mathcal{M} \arrow[ll] \arrow[rr] \arrow[d, equal] & & \mathbf{DR}_{X/X_0}\mathcal{M} \arrow[d, equal] \\ i^{\dagger}_{I_c}(i^{\dagger}_I\mathcal{M}) & i^{\#}_{I_c}(i^{\dagger}_I\mathcal{M}) \arrow[l] & i^{\#}_{I_c}(i^{\#}_I\mathcal{M}) \arrow[l] \arrow[r] & i^{\#}_{I^c}(\mathbf{DR}_{X/X_0^I}\mathcal{M}) \arrow[r] & \mathbf{DR}_{X_0^I/X_0}(\mathbf{DR}_{X/X_0^I}\mathcal{M}). \end{tikzcd} \tag{12}$$

3.2. **Le morphisme «Nils».** D'après la proposition 2.26 on a

$$\mathrm{gr}_{-\mathbf{1}}(\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}}) = \bigoplus_{\mathbf{0} \leqslant \boldsymbol{\ell} \leqslant \boldsymbol{k}} \mathrm{gr}_{\boldsymbol{\alpha}}\left(\mathcal{M}[\frac{1}{t_1 ... t_p}]\right) e_{\boldsymbol{\alpha},\boldsymbol{\ell}}.$$

La proposition 2.15 assure que pour $\boldsymbol{\alpha} \in [-1, 0[^p$ on a l'isomorphisme

$$\mathrm{gr}_{\boldsymbol{\alpha}}(\mathcal{M}) \simeq \mathrm{gr}_{\boldsymbol{\alpha}}\left(\mathcal{M}[\frac{1}{t_1 ... t_p}]\right).$$

On définit alors le morphisme suivant

$$\begin{array}{cccc} \Phi : & \mathrm{gr}_{\boldsymbol{\alpha}}(\mathcal{M}) & \longrightarrow & \mathrm{gr}_{-\mathbf{1}}(\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}}) \\ & m & \longmapsto & \displaystyle\sum_{0\leqslant \boldsymbol{\ell}\leqslant \boldsymbol{k}} \left[(-1)^{\ell_1+\dots+\ell_p}(t_1\partial_1+\alpha_1+1)^{\ell_1}\dots(t_p\partial_p+\alpha_p+1)^{\ell_p} m\right]\otimes e_{\boldsymbol{\alpha},\boldsymbol{\ell}} \end{array}$$

qui induit un morphisme de complexes

$$\boxed{\mathbf{Nils} : \mathrm{gr}_{\boldsymbol{\alpha}}(\mathcal{M}) \to i^{\dagger}\mathcal{M}_{\boldsymbol{\alpha}}}$$

où l'on identifie $\mathrm{gr}_{\boldsymbol{\alpha}}(\mathcal{M})$ avec un complexe concentré en degré zéro et où $\mathcal{M}_{\boldsymbol{\alpha}}$ est la limite inductive des $\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}}$ prise sur $\boldsymbol{k}\in\mathbb{N}^p$.

Remarque 3.4. Remarquons ici que $\mathcal{M}_{\boldsymbol{\alpha}}$ n'est pas un $\mathcal{D}_X$-module de type fini. Mais le fait qu'il soit limite des $\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}}$ et que les couples $(\boldsymbol{H},\mathcal{M}_{\boldsymbol{\alpha},\boldsymbol{k}})$ soient sans pente suffit pour le reste de la construction et pour le théorème de comparaison.

En utilisant la définition 2.23 on obtient

$$\mathcal{O}_X\otimes_{\pi^{-1}\mathcal{O}_{\mathbb{C}^p}}\pi^{-1}(\mathcal{N}_{\boldsymbol{\alpha},\boldsymbol{k}}) \simeq \Big(\mathcal{O}_X\otimes_{\pi_I^{-1}\mathcal{O}_{\mathbb{C}^r}}\pi_I^{-1}(\mathcal{N}_{\boldsymbol{\alpha}_I,\boldsymbol{k}_I})\Big)\otimes_{\mathcal{O}_X}\Big(\mathcal{O}_X\otimes_{\pi_{I^c}^{-1}\mathcal{O}_{\mathbb{C}^{p-r}}}\pi_{I^c}^{-1}(\mathcal{N}_{\boldsymbol{\alpha}_{I^c},\boldsymbol{k}_{I^c}})\Big).$$

On déduit de cet isomorphisme et de la définition du morphisme Φ le diagramme commutatif suivant

$$\begin{tikzcd} \mathrm{gr}_{\boldsymbol{\alpha}}\mathcal{M} \arrow[rr, "\mathbf{Nils}"] \arrow[dd] & & i^{\dagger}\mathcal{M}_{\boldsymbol{\alpha}} \arrow[d] \\ & & i^{\dagger}_{I_c}(i^{\dagger}_I\mathcal{M}_{\boldsymbol{\alpha}}) \arrow[d, equal] \\ \mathrm{gr}^{I^c}_{\boldsymbol{\alpha}_{I^c}}(\mathrm{gr}^I_{\boldsymbol{\alpha}_I}\mathcal{M}) \arrow[r] & \mathrm{gr}^{I^c}_{\boldsymbol{\alpha}_{I^c}}\left(i^{\dagger}_I\mathcal{M}_{\boldsymbol{\alpha}_I}\right) \arrow[r] & i^{\dagger}_{I^c}\left[(i^{\dagger}_I\mathcal{M}_{\boldsymbol{\alpha}_I})_{\boldsymbol{\alpha}_{I^c}}\right]. \end{tikzcd} \tag{13}$$

3.3. **Le morphisme «Topo».** Rappelons le diagramme commutatif utilisé pour définir les cycles proches topologiques :

$$\begin{tikzcd} \pi^{-1}(0) \arrow[r, "i"] \arrow[d] & X \arrow[d, "\pi"] & X^* \arrow[l, "j"'] \arrow[d, "\pi_{\mid X^*}"] & \widetilde{X} \arrow[l, "p"'] \arrow[d, "\widetilde{\pi}"] \\ \{0\} \arrow[r, "i"] & \mathbb{C}^p & (\mathbb{C}^*)^p \arrow[l, "j"'] & \widetilde{(\mathbb{C}^*)^p}. \arrow[l, "p"'] \end{tikzcd}$$

Lemme 3.5. *Soit $\boldsymbol{\alpha}\in\mathbb{C}^p$, il existe un morphisme naturel*

$$\boxed{\mathbf{Topo} : \mathbf{DR}_X(\mathcal{M}_{\boldsymbol{\alpha}}) \to \Psi_\pi\mathbf{DR}_X(\mathcal{M}).}$$

Démonstration. Par définition, $\mathcal{M}_{\boldsymbol{\alpha}} = \mathcal{M}\otimes_{\pi^{-1}\mathcal{O}_{\mathbb{C}^p}}\pi^{-1}\mathcal{N}_{\boldsymbol{\alpha}}$, or on a une inclusion $\mathcal{N}_{\boldsymbol{\alpha}}\subset j_*p_*p^{-1}\mathcal{O}_{(\mathbb{C}^*)^p}$ dans le faisceau des fonctions holomorphes multiformes. Par fonctorialité on a donc le morphisme :

$$\mathbf{DR}_X(\mathcal{M}_{\boldsymbol{\alpha}}) \to \mathbf{DR}_X(\mathcal{M}\otimes\pi^{-1}j_*p_*p^{-1}\mathcal{O}_{(\mathbb{C}^*)^p}).$$

L'adjonction des foncteurs image inverse et image directe fournit un morphisme de foncteurs $\pi^{-1}(j\circ p)_* \to (j\circ p)_*\widetilde{\pi}^{-1}$. Ceci donne le morphisme :

$$\begin{array}{rcl}\mathbf{DR}_X(\mathcal{M}\otimes\pi^{-1}j_*p_*p^{-1}\mathcal{O}) & \to & \mathbf{DR}_X(\mathcal{M}\otimes j_*p_*\widetilde{\pi}^{-1}p^{-1}\mathcal{O}) \\ & & =\mathbf{DR}_X(\mathcal{M}\otimes j_*p_*p^{-1}\pi_{|X^*}{}^{-1}\mathcal{O}).\end{array}$$

Par adjonction on a le morphisme :

$$\begin{array}{rcl}\mathbf{DR}_X(\mathcal{M}\otimes j_*p_*p^{-1}\pi_{|X^*}{}^{-1}\mathcal{O}) & \to & \boldsymbol{R}j_*j^{-1}\mathbf{DR}_X(\mathcal{M}\otimes j_*p_*p^{-1}\pi_{|X^*}{}^{-1}\mathcal{O}) \\ & = & \boldsymbol{R}j_*\mathbf{DR}_X(j^{-1}\mathcal{M}\otimes j^{-1}j_*p_*p^{-1}\pi^{-1}\mathcal{O}) \\ & = & \boldsymbol{R}j_*\mathbf{DR}_X(j^{-1}\mathcal{M}\otimes p_*p^{-1}\pi^{-1}\mathcal{O}).\end{array}$$

On applique ensuite le morphisme (2.3.21) de [KS94] (formule de projection) à la fonction p, en considérant le fait que p_* est un foncteur exact car p est à fibres discrètes. Par fonctorialité on a alors le morphisme suivant :

$$\begin{array}{rcl}\boldsymbol{R}j_*\mathbf{DR}_X(j^{-1}\mathcal{M}\otimes p_*p^{-1}\pi^{-1}\mathcal{O}) & \to & \boldsymbol{R}j_*\mathbf{DR}_X(p_*p^{-1}(j^{-1}\mathcal{M}\otimes\pi^{-1}\mathcal{O})) \\ & = & \boldsymbol{R}j_*\mathbf{DR}_X(p_*p^{-1}j^{-1}\mathcal{M}).\end{array}$$

Sachant que $\mathbf{DR}_X\mathcal{M}=\Omega^n\overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X}\mathcal{M}$, on peut appliquer le morphisme (2.6.21) de [KS94] à p (formule de projection) et on obtient le morphisme :

$$\begin{array}{rcl}\boldsymbol{R}j_*\mathbf{DR}_X(p_*p^{-1}j^{-1}\mathcal{M}) & \to & \boldsymbol{R}j_*p_*\mathbf{DR}_X(p^{-1}j^{-1}\mathcal{M}) \\ & = & \boldsymbol{R}j_*p_*p^{-1}j^{-1}\mathbf{DR}_X(\mathcal{M}).\end{array}$$

Si l'on compose tous les morphismes naturels que l'on vient de construire on obtient bien le morphisme naturel attendu :

$$\mathbf{DR}_X(\mathcal{M}_{\boldsymbol{\alpha}})\to\Psi_\pi\mathbf{DR}_X(\mathcal{M}).$$

□

La naturalité de ce morphisme ainsi que la définition du morphisme (4)

$$\Psi_\pi\mathbf{DR}_X(\mathcal{M})\to\Psi_{\pi_{I^c}}(\Psi_{\pi_I}\mathbf{DR}_X(\mathcal{M}))$$

permettent de montrer que le diagramme suivant est commutatif

$$\begin{tikzcd}[column sep=small] & \mathbf{DR}_X\mathcal{M}_{\boldsymbol{\alpha}} \arrow[rr, "\mathbf{Topo}"] \arrow[d, equal] & & \Psi_\pi\mathbf{DR}_X\mathcal{M} \arrow[d] \\ (14) & \mathbf{DR}_X\left[(\mathcal{M}_{\boldsymbol{\alpha}_I})_{\boldsymbol{\alpha}_{I^c}}\right] \arrow[r] & \Psi_{\pi_{I^c}}(\mathbf{DR}_X\mathcal{M}_{\boldsymbol{\alpha}_I}) \arrow[r] & \Psi_{\pi_{I^c}}(\Psi_{\pi_I}\mathbf{DR}_X\mathcal{M}).\end{tikzcd}$$

3.4. **Le morphisme de comparaison.** En combinant les morphismes (10), **Nils** et **Topo** on obtient la suite de morphismes suivante

$$\text{(15)}\quad \mathbf{DR}_{X_0}\Psi_{\mathbf{H}}(\mathcal{M})\xrightarrow{\mathbf{Nils}}\bigoplus_{\boldsymbol{\alpha}\in[-1,0[^p}\mathbf{DR}_{X_0}i^\dagger\mathcal{M}_{\boldsymbol{\alpha}}\leftarrow\bigoplus_{\boldsymbol{\alpha}\in[-1,0[^p}\mathbf{DR}_{X_0}i^\#\mathcal{M}_{\boldsymbol{\alpha}}\to$$
$$\bigoplus_{\boldsymbol{\alpha}\in[-1,0[^p}\mathbf{DR}_X(\mathcal{M}_{\boldsymbol{\alpha}})\xrightarrow{\mathbf{Topo}}\Psi_\pi\mathbf{DR}_X(\mathcal{M}).$$

On a appliqué les morphisme (10) à $\mathcal{M}_{\boldsymbol{\alpha}}$, on a ensuite appliqué le foncteur $\mathbf{DR}_{X_0}$ et on a pris la somme sur $\boldsymbol{\alpha}\in]-1,0]^p$ en utilisant la définition

$$\Psi_{\mathbf{H}}(\mathcal{M}):=\bigoplus_{\boldsymbol{\alpha}\in[-1,0[^p}\mathrm{gr}_{\boldsymbol{\alpha}}(\mathcal{M}).$$

Théorème 3.6. *Si le couple $(\mathbf{H}, \mathcal{M})$ est sans pente alors les morphismes (15) sont des isomorphismes qui commutent aux endomorphismes de monodromie T_i, on obtient l'isomorphisme de comparaison*

$$\mathbf{DR}_{X_0}\Psi_{\mathbf{H}}(\mathcal{M}) \simeq \Psi_\pi \mathbf{DR}_X(\mathcal{M}).$$

De plus si $I = \{1, ..., r\} \subset \{1, ..., p\}$ et si l'on applique successivement cet isomorphisme de comparaison par rapport aux familles d'hypersurfaces $\mathbf{H}_I$ et $\mathbf{H}_{I^c}$ le résultat ne dépend pas de l'ordre dans lequel on applique l'isomorphisme. Autrement dit le diagramme suivant est commutatif

$$\begin{tikzcd}
\mathbf{DR}_{X_0}\Psi_{\mathbf{H}_{I^c}}(\Psi_{\mathbf{H}_I}\mathcal{M}) \arrow[d, "\simeq"] & \mathbf{DR}_{X_0}\Psi_{\mathbf{H}}(\mathcal{M}) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\simeq"] & \mathbf{DR}_{X_0}\Psi_{\mathbf{H}_I}(\Psi_{\mathbf{H}_{I^c}}\mathcal{M}) \arrow[d, "\simeq"] \\
\Psi_{\pi_{I^c}}(\Psi_{\pi_I}\mathbf{DR}_X(\mathcal{M})) & \Psi_\pi \mathbf{DR}_X(\mathcal{M}) \arrow[l, "\sim"'] \arrow[r, "\sim"] & \Psi_{\pi_I}(\Psi_{\pi_{I^c}}\mathbf{DR}_X(\mathcal{M})).
\end{tikzcd}$$

Démonstration. On raisonne par récurrence sur le nombre p d'hypersurfaces dans $\mathbf{H}$, le cas $p = 1$ est traité par Ph. Maisonobe et Z. Mebkhout dans [MM04, théorème 5.3-2] ou par Morihiko Saito dans [Sai88, lemmes 3.4.4 et 3.4.5].

Pour $p > 1$, soit $I = \{1, ..., r\} \subset \{1, ..., p\}$ avec $1 < r < p$, on va considérer les diagrammes commutatifs (12), (13) et (14). L'hypothèse sans pente permet d'appliquer la proposition 2.19 (resp. 2.22) qui assure que les flèches verticales des diagrammes (12) et (13) (resp. (14)) sont des isomorphismes. La commutativité de ces diagrammes permet de se ramener aux cas de r et $p - r$ hypersurfaces en appliquant successivement les deux isomorphismes de comparaison obtenus par récurrence. La commutativité donne alors également directement la deuxième partie du théorème.

□

Pour un morphisme $\boldsymbol{f} : X \to \mathbb{C}^p$, l'inclusion du graphe de $\boldsymbol{f}$ permet de donner une version générale de ce théorème :

Corollaire 3.7. *Soit $\boldsymbol{f} : X \to \mathbb{C}^p$ un morphisme d'espaces analytiques complexes réduits et $\mathcal{M}$ un $\mathcal{D}_X$-module holonome régulier tel que le couple $(\boldsymbol{H}, i_{\boldsymbol{f}+}\mathcal{M})$ soit sans pente. On a un isomorphisme de comparaison*

$$\mathbf{DR}_X\Psi^{\mathrm{alg}}_{\boldsymbol{f}}(\mathcal{M}) \simeq \Psi_{\boldsymbol{f}}\mathbf{DR}_X(\mathcal{M}).$$

De plus si $I = \{1, ..., r\} \subset \{1, ..., p\}$ et si l'on applique successivement cet isomorphisme de comparaison par rapport aux fonctions $\boldsymbol{f}_I$ et $\boldsymbol{f}_{I^c}$ le résultat ne dépend pas de l'ordre dans lequel on applique l'isomorphisme. Autrement dit le diagramme suivant est commutatif

$$\begin{tikzcd}
\mathbf{DR}_X\Psi^{\mathrm{alg}}_{\boldsymbol{f}_{I^c}}(\Psi^{\mathrm{alg}}_{\boldsymbol{f}_I}\mathcal{M}) \arrow[d, "\simeq"] & \mathbf{DR}_X\Psi^{\mathrm{alg}}_{\boldsymbol{f}}(\mathcal{M}) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\simeq"] & \mathbf{DR}_X\Psi^{\mathrm{alg}}_{\boldsymbol{f}_I}(\Psi^{\mathrm{alg}}_{\boldsymbol{f}_{I^c}}\mathcal{M}) \arrow[d, "\simeq"] \\
\Psi_{\boldsymbol{f}_{I^c}}(\Psi_{\boldsymbol{f}_I}\mathbf{DR}_X(\mathcal{M})) & \Psi_{\boldsymbol{f}}\mathbf{DR}_X(\mathcal{M}) \arrow[l, "\sim"'] \arrow[r, "\sim"] & \Psi_{\boldsymbol{f}_I}(\Psi_{\boldsymbol{f}_{I^c}}\mathbf{DR}_X(\mathcal{M})).
\end{tikzcd}$$

Démonstration. On applique le théorème 3.6 à $i_{\boldsymbol{f}+}\mathcal{M}$, on obtient l'isomorphisme

$$\mathbf{DR}_{X_0}\Psi_{\mathbf{H}}\left(i_{\boldsymbol{f}+}\mathcal{M}\right) \simeq \Psi_\pi \mathbf{DR}_{X\times\mathbb{C}^p}(i_{\boldsymbol{f}+}\mathcal{M}).$$

où $\pi : X \times \mathbb{C}^p \to \mathbb{C}^p$ est la projection. On applique le foncteur $i_{\boldsymbol{f}}{}^{-1}$ à cette isomorphisme. On observe qu'un théorème de changement de base propre donne l'isomorphisme de foncteur $\Psi_{\boldsymbol{f}} i_{\boldsymbol{f}}{}^{-1} \simeq i_{\boldsymbol{f}}{}^{-1}\Psi_\pi$. On en déduit l'isomorphisme

$$i_{\boldsymbol{f}}{}^{-1}\mathbf{DR}_{X_0}\Psi_{\mathbf{H}}\left(i_{\boldsymbol{f}+}\mathcal{M}\right) \simeq \Psi_{\boldsymbol{f}} i_{\boldsymbol{f}}{}^{-1}\mathbf{DR}_{X\times\mathbb{C}^p}(i_{\boldsymbol{f}+}\mathcal{M}).$$

On déduit enfin de l'équivalence de Kashiwara appliquée à l'injection du graphe de $\boldsymbol{f}$ dans $X \times \mathbb{C}^p$ l'isomorphisme attendu

$$\mathbf{DR}_X \Psi^{\mathrm{alg}}_{\boldsymbol{f}}(\mathcal{M}) \simeq \Psi_{\boldsymbol{f}} \mathbf{DR}_X(\mathcal{M}).$$

La suite du corollaire se démontre de la même manière.

□

On déduit en particulier de ce corollaire que, dans le cas sans pente, si l'on applique l'isomorphisme de comparaison par rapport aux fonctions $f_1, ..., f_p$ l'une après l'autre l'isomorphisme

$$\mathbf{DR}_X \left(\Psi^{\mathrm{alg}}_{f_{\sigma(p)}} \left(... \Psi^{\mathrm{alg}}_{f_{\sigma(2)}} \left(\Psi^{\mathrm{alg}}_{f_{\sigma(1)}} \mathcal{M} \right) \right) \right) \simeq \Psi_{f_{\sigma(p)}} \left(... \Psi_{f_{\sigma(2)}} \left(\Psi_{f_{\sigma(1)}} \mathbf{DR}_X(\mathcal{M}) \right) \right)$$

ne dépend pas de la permutation σ de $\{1, ..., p\}$.

Annexe A. Hypercomplexes

On définit ici les n-hypercomplexes qui correspondent aux complexes n^{uple} naïfs introduits par P. Deligne au paragraphe 0.4 de [Del73].

Définition A.1. Soit $\mathcal{C}$ une catégorie abélienne, on définit par induction la catégorie abélienne des *n-hypercomplexes* de la façon suivante :

- Les 1-hypercomplexes sont les complexes d'objets de $\mathcal{C}$.
- Les n-hypercomplexes sont les complexes de (n-1)-hypercomplexes.

On notera $\boldsymbol{C}^n(\mathcal{C})$ la catégorie abélienne des n-hypercomplexes d'objets de $\mathcal{C}$. Par exemple les 2-hypercomplexes sont les complexes doubles. Un n-hypercomplexe est donc la donnée pour tout $\boldsymbol{k} \in \mathbb{Z}^n$ d'un objet $X^{\boldsymbol{k}}$ de $\mathcal{C}$ et, pour tout $1 \leqslant i \leqslant n$ de morphismes $d^{(i)\boldsymbol{k}} : X^{\boldsymbol{k}} \to X^{\boldsymbol{k}+\boldsymbol{1}_i}$ vérifiant les propriétés suivantes :

$$\begin{array}{lll} d^{(i)} \circ d^{(i)} = 0 & \text{pour tout} & i \\ d^{(i)} \circ d^{(j)} = d^{(j)} \circ d^{(i)} & \text{pour tout} & (i,j) \end{array}$$

pour les exposants $\boldsymbol{k}$ convenables.

Soit X un n-hypercomplexe, pour tout $1 \leqslant i \leqslant n$ et tout $m \in \mathbb{Z}$ on note X_i^m le $(n-1)$-hypercomplexe composé des $X^{\boldsymbol{k}}$ avec $k_i = m$ et des différentielles correspondantes. Les différentielles $d^{(i)\boldsymbol{k}}$ avec $k_i = m$ définissent un morphisme :

$$d_i^m : X_i^m \to X_i^{m+1}$$

qui vérifie $d_i^{m+1} \circ d_i^m = 0$ par définition d'un n-hypercomplexe. On a donc pour tout $1 \leqslant i \leqslant n$ un foncteur :

$$\begin{array}{cccc} F_i : & \boldsymbol{C}^n(\mathcal{C}) & \to & \boldsymbol{C}(\boldsymbol{C}^{(n-1)}(\mathcal{C})) \\ & X & \mapsto & \{X_i^m, d_i^m\}_{m \in \mathbb{Z}} \end{array}$$

de la catégorie des n-hypercomplexes dans la catégorie des complexes de $(n-1)$-hypercomplexes. On introduit alors le $(n-1)$-hypercomplexe :

$$H_i^p(X) := H^p(F_i(X)),$$

et le n-hypercomplexe :

$$H_i(X) := ... \to H_i^p(X) \xrightarrow{0} H_i^{p+1}(X) \to ...$$

où toutes les flèches horizontales sont nulles.

Définition A.2. Si un n-hypercomplexe X vérifie la propriété de finitude suivante :

(16) pour tout $m \in \mathbb{Z}$ l'ensemble $\{(k_1, ..., k_n) \in \mathbb{Z}^n \mid k_1 + ... + k_n = m, X^{\boldsymbol{k}} \neq 0\}$ est fini,

alors on peut associer à X un complexe simple $s(X)$. On pose

$$s(X)^m := \bigoplus_{k_1+...+k_n=m} X^{\boldsymbol{k}}.$$

Soit $\boldsymbol{k} \in \mathbb{Z}^n$ tel que $k_1 + ... + k_n = m$. On note $i_{\boldsymbol{k}} : X^{\boldsymbol{k}} \to s(X)^m$ et $p_{\boldsymbol{k}} : s(X)^m \to X^{\boldsymbol{k}}$ les morphismes naturels. On peut alors définir la différentielle $d^m_{s(X)} : s(X)^m \to s(X)^{m+1}$ du complexe $s(X)$ par :

$$p_{\boldsymbol{l}} \circ d^m_{s(X)} \circ i_{\boldsymbol{k}} = \begin{cases} (-1)^{k_1+...+k_{j-1}} d^{(j)\boldsymbol{k}} & \text{si} \quad \#\{i \mid k_i \neq l_i\} = 1 \quad \text{où } j \text{ vérifie } k_j \neq l_j \\ 0 & \text{sinon} \end{cases}$$

pour tout $\boldsymbol{k}$ et $\boldsymbol{l}$ vérifiant $k_1 + ... + k_n = m$ et $l_1 + ... + l_n = m + 1$. On peut alors vérifier que $d^{m+1}_{s(X)} \circ d^m_{s(X)} = 0$ et $(s(X), d_{s(X)})$ est donc bien un complexe. On a défini un foncteur

$$\begin{array}{rccc} s: & \boldsymbol{C}^n_f(\mathcal{C}) & \to & \boldsymbol{C}(\mathcal{C}) \\ & X & \mapsto & (s(X), d_{s(X)}) \end{array}$$

où $\boldsymbol{C}^n_f(\mathcal{C})$ est la catégorie des n-hypercomplexes vérifiant la propriété (16). De plus on observe facilement que $s(.)$ est un foncteur exact.

Théorème A.3. *Soit $f : X \to Y$ un morphisme de n-hypercomplexes où X et Y vérifient la propriété* (16) *et supposons que f induise un isomorphisme :*

$$f : H_1(H_2(...H_n(X)...)) \simeq H_1(H_2(...H_n(Y)...)).$$

Alors $s(f) : s(X) \to s(Y)$ est un quasi-isomorphisme.

Démonstration. On raisonne par récurrence sur l'entier n. Pour $n = 1$ c'est la définition d'un quasi-isomorphisme, pour $n = 2$ c'est le théorème 1.9.3 de [KS94]. On suppose que $n \geqslant 3$. Pour tout $p \in \mathbb{Z}$, on a deux $(n-1)$-hypercomplexes, $H^p_n(X)$ et $H^p_n(Y)$, qui vérifient les hypothèses du théorème et donc par hypothèse de récurrence f induit un quasi-isomorphisme entre $s\,(H^p_n(X))$ et $s\,(H^p_n(Y))$. Or $H^p_n(X) = H^p(F_n(X))$ et $H^p(.)$ est un foncteur additif, il commute donc avec le foncteur $s(.)$ et f induit un quasi-isomorphisme entre

$$H^p(\{s(X^m_n), s(d^m_n)\}_{m\in\mathbb{Z}}) \qquad \text{et} \qquad H^p(\{s(Y^m_n), s(d^m_n)\}_{m\in\mathbb{Z}})$$

pour tout $p \in \mathbb{Z}$. Mais ce quasi-isomorphisme correspond aux conditions du théorème pour les complexes doubles $\{s(X^m_i), s(d^m_i)\}_{m\in\mathbb{Z}}$ et $\{s(Y^m_i), s(d^m_i)\}_{m\in\mathbb{Z}}$, les complexes simples associés à ces deux complexes doubles sont donc quasi-isomorphes par hypothèse de récurrence pour $n = 2$. En appliquant la définition du foncteur s on montre alors que ces deux derniers complexes simples sont en fait les complexes simples associés à X et à Y ce qui conclut la démonstration du théorème.

□

Corollaire A.4. *Soit X un n-hypercomplexe tel qu'il existe un indice i pour lequel le complexe $F_i(X)$ soit exact, alors $s(X)$ est quasi-isomorphe au complexe nul.*

Démonstration. Le théorème précédent est évidemment vérifié si l'on permute les indices des H_i. Si le complexe $F_i(X)$ est exact alors $H_i(X) \simeq H_i(0_n)$ où 0_n est le n-hypercomplexe nul. On a donc

$$H_1(...H_{i-1}(H_{i+1}(...H_n(H_i(X))...) \simeq H_1(...H_{i-1}(H_{i+1}(...H_n(H_i(0_n))...)$$

et on peut appliquer le théorème précédent, $s(X) \simeq s(0_n)$, $s(X)$ est quasi-isomorphe au complexe nul.

$\square$

Définition A.5. Soit $\{X^{\boldsymbol{k}}, f^{(i)\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^n, 1\leqslant i\leqslant n}$ une famille d'objets de $\mathcal{C}$ et de morphismes $f^{(i)\boldsymbol{k}} : X^{\boldsymbol{k}} \to X^{\boldsymbol{k}+\boldsymbol{1}_i}$, on appelle *hypercube associé à* X le n-hypercomplexe noté $\mathrm{Cube}(X)^\bullet$ vérifiant

$$\mathrm{Cube}(X)^{k_1,\dots,k_n} = \begin{cases} X^{k_1-1,\dots,k_n-1} & \text{si} \quad \boldsymbol{k} \in \{0,1\}^n \\ 0 & \text{sinon} \end{cases}$$

les morphismes étant ceux donnés par les $f^{(i)\boldsymbol{k}}$. On vérifie facilement que $\mathrm{Cube}(.)$ définit un foncteur exact.

Par exemple, pour $n = 3$ on a

$$\mathrm{Cube}(X) = \begin{tikzcd} & X^{-1,0,0} \arrow[rr] & & X^{0,0,0} \\ X^{-1,-1,0} \arrow[ur] \arrow[rr] & & X^{0,-1,0} \arrow[ur] & \\ & X^{-1,0,-1} \arrow[uu] \arrow[rr] & & X^{0,0,-1} \arrow[uu] \\ X^{-1,-1,-1} \arrow[uu] \arrow[ur] \arrow[rr] & & X^{0,-1,-1} \arrow[uu] \arrow[ur] & \end{tikzcd}$$

où le reste de l'hypercomplexe est nul et $X^{-1,-1,-1}$ est en degré $(0,0,0)$.

Annexe B. Filtrations compatibles

Les définitions qui suivent ont été introduites par Morihiko Saito dans [Sai88]

Définition B.1. Soit A un objet de la catégorie abélienne $\mathcal{C}$ et $A_1, \dots, A_n \subseteq A$ des sous-objets de A. On dit que $A_1, \dots, A_n$ sont des *sous-objets compatibles* de A si il existe un n-hypercomplexe X satisfaisant à :

(1) $X^{\boldsymbol{k}} = 0$ si $\boldsymbol{k} \notin \{-1,0,1\}^n$.
(2) $X^{\boldsymbol{0}} = A$.
(3) $X^{\boldsymbol{0}-\boldsymbol{1}_i} = A_i$ pour $1 \leqslant i \leqslant n$.
(4) Pour tout $1 \leqslant i \leqslant n$ et tout $\boldsymbol{k} \in \{-1,0,1\}^n$ tel que $k_i = 0$, la suite

$$0 \to X^{\boldsymbol{k}-\boldsymbol{1}_i} \to X^{\boldsymbol{k}} \to X^{\boldsymbol{k}+\boldsymbol{1}_i} \to 0$$

est une suite exacte courte.

Remarque B.2.
- En utilisant les propriétés universelles fournies par les suites exactes courtes on observe que si les sous-objets $A_1, \dots, A_n$ sont compatibles, alors le n-hypercomplexe X est déterminé de manière unique. Par exemple si $\boldsymbol{k} \in \{-1,0\}^n$ et si

$$I = \{i; k_i = -1\} \subset \{1, \dots, n\}$$

alors

$$X^{\boldsymbol{k}} = \bigcap_{i\in I} A_i.$$

- Si $n = 1$, le complexe X est la suite exacte courte
$$0 \to A_1 \to A \to A/A_1 \to 0.$$
- Si $n = 2$ deux sous-objets A_1 et A_2 sont toujours compatibles et X est le complexe double suivant
$$\begin{tikzcd} A_1/(A_1 \cap A_2) \arrow[r] & A/A_2 \arrow[r] & A/(A_1 + A_2) \\ A_1 \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & A/A_1 \arrow[u] \\ A_1 \cap A_2 \arrow[r] \arrow[u] & A_2 \arrow[r] \arrow[u] & A_2/(A_1 \cap A_2). \arrow[u] \end{tikzcd}$$
- Si $n \geqslant 3$ des sous-objets $A_1, ..., A_n$ ne sont pas compatibles en général.
- Par définition si $A_1, ..., A_n \subseteq A$ sont compatibles alors pour tout $I \subset \{1, ..., n\}$ les sous-objets $(A_i)_{i \in I} \subseteq A$ sont compatibles et l'hypercomplexe correspondant est le $\#I$-hypercomplexe X_I dont les objets sont les $X^{\boldsymbol{k}}$ tels que $k_i = 0$ pour tout $i \in I^c$.

Définition B.3. Soient $F^1_\bullet, ..., F^n_\bullet$ des filtrations croissantes indexées par $\mathbb{Z}$ d'un objet A, on dit que ces filtrations sont *compatibles* si pour tout $\boldsymbol{\ell} \in \mathbb{Z}^n$ les sous-objets $F^1_{\ell_1}, ..., F^n_{\ell_n}$ de A sont compatibles.

Remarque B.4.
- D'après la remarque précédente toute sous famille d'une famille de filtrations compatibles est compatible.
- On peut montrer que si $F^1_\bullet, ..., F^n_\bullet$ sont compatibles alors pour tout $\ell \in \mathbb{Z}$ les filtrations induites par $F^1_\bullet, ..., F^{n-1}_\bullet$ sur $\mathrm{gr}^{F_n}_\ell$ sont compatibles.
- Si $F^1_\bullet, ..., F^n_\bullet$ sont compatibles alors les filtrations induites sur $F^1_{\ell_1} \cap ... \cap F^n_{\ell_n}$ sont compatibles.

La proposition suivante correspond à [Sai88, corollaire 1.2.13]

Proposition B.5. *Soit $F^1_\bullet, ..., F^n_\bullet$ des filtrations compatibles d'un objet A. L'objet obtenu en appliquant successivement les gradués $\mathrm{gr}^{F_{\sigma(j)}}_{\ell_{\sigma(j)}}$ par rapport aux filtrations $F_{\sigma(j)}$ induites sur $\mathrm{gr}^{F_{\sigma(j-1)}}_{\ell_{\sigma(j-1)}} ... \mathrm{gr}^{F_{\sigma(1)}}_{\ell_{\sigma(1)}} A$ pour $1 \leqslant j \leqslant n$ ne dépend pas de la permutation σ de $\{1, ..., n\}$ et est égal à*
$$\frac{F^1_{\ell_1} A \cap ... \cap F^n_{\ell_n} A}{\sum_j F^1_{\ell_1} A \cap ... \cap F^1_{\ell_j - 1} A \cap ... \cap F^n_{\ell_n} A}.$$

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UMR 7640 du CNRS, Centre de Mathématiques Laurent Schwartz, École polytechnique, F–91128 Palaiseau cedex, France

E-mail address: `matthieu.kochersperger@polytechnique.edu`

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FLAT SURFACES ALONG CUSPIDAL EDGES

SHYUICHI IZUMIYA, KENTARO SAJI, AND NOBUKO TAKEUCHI

ABSTRACT. We consider developable surfaces along the singular set of a cuspidal edge surface which are regarded as flat approximations of the cuspidal edge surface. For the study of singularities of such developable surfaces, we introduce the notion of Darboux frames along cuspidal edges, and introduce invariants. As a by-product, we introduce the notion of higher-order helices which are generalizations of previous notions of generalized helices (i.e., slant helices and clad helices). We use this notion to characterize special cuspidal edges.

1. Introduction

In recent decades, there have appeared several articles concerning the differential geometry of singular surfaces in Euclidean 3-space [5, 6, 19, 20, 21, 25, 27, 28, 32]. Wave fronts are particularly interesting singular surfaces which always have normal directions, even along singularities. A cuspidal edge surface is one of the generic wave fronts in Euclidean 3-space. In this paper, we consider developable surfaces along the singular curve of a cuspidal edge surface in Euclidean 3-space. Such a developable surface is called a *developable surface along* the cuspidal edge. Actually there are infinitely many developable surfaces along a cuspidal edge. Since a cuspidal edge surface has the normal direction at any point (even at a singular point), we focus on two typical developable surfaces along the cuspidal edge. One of them is a developable surface which is tangent to the cuspidal edge surface and the other is normal to the cuspidal edge surface. These two developable surfaces are considered to be flat approximations of the cuspidal edge surface along the cuspidal edge. We investigate the singularities of these developable surfaces along the cuspidal edge and introduce new invariants for the cuspidal edge.

For this purpose, we introduce the notion of Darboux frames along cuspidal edges, which is analogous to the notion of Darboux frames along curves on regular surfaces (cf. [7, 8, 14]). Since the Darboux frame along a cuspidal edge is an orthonormal frame along the cuspidal edge, we can obtain structure equations and invariants (cf. Proposition 3.1). We show that these invariants are equal to the invariants which are known as basic invariants of a cuspidal edge in [20, 21, 27], in which the normal form of the cuspidal edge was used for the study of geometric properties. The normal form of the cuspidal edge is a very strong tool from a singularity theory viewpoint. However, it is rather difficult to understand the geometric meanings intuitively. Here, we emphasize that we use the Darboux frame instead of the normal form of the cuspidal edge. By using the Darboux frame, we can directly and intuitively understand geometric properties of the cuspidal edge.

The precise definition of the cuspidal edge (surface) is given as follows: The unit cotangent bundle $T_1^*\mathbb{R}^3$ of $\mathbb{R}^3$ has a canonical contact structure and can be identified with the unit tangent bundle $T_1\mathbb{R}^3$. Let α denote that canonical contact form. Let M be a 2-dimensional manifold. A map $i : M \to T_1\mathbb{R}^3$ is said to be *isotropic* if the pull-back $i^*\alpha$ vanishes identically. We call

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the image of $\pi \circ i$ the *wave front set* of i, where $\pi : T_1\mathbb{R}^3 \to \mathbb{R}^3$ is the canonical projection and we denote it by $W(i)$. Moreover, i is called the *Legendrian lift* of $W(i)$. With this framework, we define the notion of fronts as follows: A map-germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is called a *frontal* if there exists a unit vector field ν (called a *unit normal of* f) of $\mathbb{R}^3$ along f such that

$$L = (f, \nu) : (\mathbb{R}^2, 0) \to (T_1\mathbb{R}^3, 0)$$

is an isotropic map by an identification $T_1\mathbb{R}^3 = \mathbb{R}^3 \times S^2$, where S^2 is the unit sphere in $\mathbb{R}^3$ (cf. [1], see also [18]). A frontal f is a *front* if the above L can be taken as an immersion. A point $q \in (\mathbb{R}^2, 0)$ is a singular point if f is not an immersion at q. A map $f : M \to N$ between M and a 3-dimensional manifold N is called a frontal (respectively, a front) if for every $p \in M$, the map-germ f at p is a frontal (respectively, a front). A singular point p of a map f is called a *cuspidal edge* if the map-germ f at p is $\mathcal{A}$-equivalent to $(u, v) \mapsto (u, v^2, v^3)$ at 0. (Two map-germs $f_1, f_2 : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$ are $\mathcal{A}$-*equivalent* if there exist diffeomorphisms $S : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $T : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ such that $f_2 \circ S = T \circ f_1$.) Therefore if the singular point p of f lies on a cuspidal edge, then f is a front at p, and furthermore, they are one of two possible types of generic singularities of fronts (the other one is a *swallowtail* which is a singular point p of f satisfying that f at p is $\mathcal{A}$-equivalent to $(u, v) \mapsto (u, u^2v + 3u^4, 2uv + 4u^3)$ at 0).

On the other hand, a developable surface is known to be a frontal, so that the normal direction is well-defined at any point. We say that a developable surface is an *osculating developable surface* along the cuspidal edge if it contains the singular set of the cuspidal edge such that the normal direction of the developable surface coincides with the normal direction of the cuspidal edge at any point of the singular set. We also say that a developable surface is a *normal developable surface* along the cuspidal edge if it contains the singular set of the cuspidal edge such that the normal direction of the developable surface belongs to the tangent plane of the cuspidal edge at any point of the singular set. In this paper, we study the geometric properties of cuspidal edges using these two developable surfaces along cuspidal edges. In particular, we show that the singular values of those developable surfaces characterize some cuspidal edges with special geometric properties. As a by-product, we introduce the notion of higher order helices which is a generalization of previous notions of generalized helices (i.e., slant helices and clad helices) in [13, 30, 31].

This paper is organized as follows: We describe basic properties of cuspidal edges in §2. The Darboux frame along a cuspidal edge is introduced in §3. Associated to the Darboux frame, we introduce three basic invariants, which are the same as those of cuspidal edges, as in [20, 21, 27]. We also introduce two vector fields along a cuspidal edge which will play critical roles in this paper. In §4, definitions and basic properties of (general) developable surfaces are described. Moreover, the notion of higher order helices is introduced and characterizations of those generalized helices by the curvature and the torsion are given (cf. Proposition 4.4, the Lancret type theorem). We also consider a tangent developable surface of a curve such that the curve is a kth-order helix. We give a characterization of such tangent developable surfaces as a corollary of Proposition 4.4 (cf. Theorem 4.6). Returning to the study of cuspidal edges, we introduce two developable surfaces along a cuspidal edge in §5. In order to classify the singularities of those two developable surfaces, we introduce four new invariants represented by the three basic invariants of a cuspidal edge. The classifications are give by those four invariants (cf. Theorems 5.1 and 5.3). Moreover, if one of the three basic invariants is identically equal to zero, we have special developable surfaces alone the cuspidal edge, whose singularities are classified in Corollaries 5.2 and 5.4. If two of these three basic invariants are identically equal to zero, the cuspidal edge is a subset of a plane (cf. §5.3). If the all three basic invariants are identically equal to zero, the cuspidal edge is a line. In §6 we investigate cuspidal edges with special properties. We compare the properties of cuspidal edges with those of curves on regular surfaces in §7. In particular, we give a geometric interpretation of the cuspidal torsion. Finally we briefly describe definitions and properties of support functions of a cuspidal edge in the appendix. By using support functions, we give geometric interpretations of singularities from the contact viewpoint.

2. Cuspidal edges

Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a frontal with a unit normal vector field ν. For a coordinate system (u, v) on $(\mathbb{R}^2, 0)$, we define a function λ by $\lambda = \det(f_u, f_v, \nu)$ and call it the *signed area density of* f. We say that a singular point $0 \in (\mathbb{R}^2, 0)$ is a *non-degenerate singular point* if $d\lambda(0) \neq 0$. Let 0 be a non-degenerate singular point. Then there exists a vector field germ η on $(\mathbb{R}^2, 0)$ such that $\langle \eta(p) \rangle_{\mathbb{R}} = \ker df_p$ for any $p \in S(f)$, where $S(f)$ is the set germ of the singular points of f. We call η a *null vector field*. We say that $0 \in (\mathbb{R}^2, 0)$ is a *singular point of the first kind* if it is non-degenerate and $\eta(0)$ is transversal to $S(f)$ at 0. The following lemma is well-known.

Lemma 2.1. ([28, Corollary 2.5, p.735], see also [18]) *Let* 0 *be a singular point of a front* $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$. *Then* 0 *is a cuspidal edge* (*respectively, swallowtail*) *if and only if* $\eta\lambda \neq 0$ (*respectively,* $\eta\lambda = 0$ $\eta\eta\lambda \neq 0$ *and* $d\lambda \neq 0$) *at* 0, *where* $\eta\lambda$ *stands for the directional derivative of* λ *by* η.

By this lemma, if f is a front, then the singular point of the first kind is a cuspidal edge. The cuspidal cross cap ($(u, v) \mapsto (u, v^2, uv^3)$) is a singular point of the first kind, which is not a front. For details see [27].

On the other hand, it is known [20, 21, 27] that there exist several geometric invariants for cuspidal edges in $\mathbb{R}^3$. In [21], these invariants are defined and studied for cuspidal edges in any Riemannian 3-manifold. See [21] for details.

Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a frontal and ν the unit normal vector field. Suppose that 0 is a singular point of the first kind. Then one can easily see that there exists a coordinate system (u, v) of $(\mathbb{R}^2, 0)$ with the following properties:

(1) $S(f) = \{v = 0\}$,
(2) u is an arc-length parameter of the curve given by $f(u, 0)$,
(3) $\ker df_{(u,0)}$ is generated by $\partial/\partial v$,
(4) (u, v) is compatible with the orientation of $\mathbb{R}^2$.

We call a coordinate system satisfying these properties an *adapted coordinate system centered at* $(u, v) = (0, 0)$. On an adapted coordinate system, since $\partial/\partial u$ is tangent to $S(f)$, it holds that $\lambda_u = 0$. Thus $d\lambda(0) \neq 0$ implies $\lambda_v \neq 0$. Since $f_v(0) = 0$, we see that

$$\det(f_u, f_{vv}, \nu)(0) = \lambda_v(0) \neq 0.$$

Hence one can choose the direction of ν such that $\det(f_u, f_{vv}, \nu)(0) > 0$. We always choose the unit normal vector ν of f on an adapted coordinate system centered at a singular point of the first kind so that it satisfies $\det(f_u, f_{vv}, \nu)(0) > 0$.

We define three invariants for f as follows on an adapted coordinate system (u, v):

$$\kappa_s(u) = \det\left(\gamma'(u), \gamma''(u), \nu(u, 0)\right), \quad \kappa_\nu(u) = \langle \gamma''(u), \nu(u, 0) \rangle,$$

$$\kappa_t(u) = \left[\frac{\det\left(\gamma', f_{vv}, f_{uvv}\right)}{|\gamma' \times f_{vv}|^2} - \frac{\det\left(\gamma', f_{vv}, f_{uu}\right) \langle \gamma', f_{vv} \rangle}{|\gamma'|^2 |\gamma' \times f_{vv}|^2} \right]_{v=0},$$

where $\gamma(u) = f(u, 0)$ and $\langle \ , \ \rangle$ is the canonical inner product of $\mathbb{R}^3$. We call $\kappa_s(u)$ the *singular curvature*, $\kappa_\nu(u)$ the *normal curvature* and $\kappa_t(u)$ the *cuspidal torsion* of f at $(u, 0)$, respectively. The singular curvature measures convexity or concavity of a cuspidal edge and the cuspidal torsion measures the rate of revolution of the direction of incidence of a cusp along a cuspidal edge. See [20, 27] for details. See [9, 24, 21] for other studies of geometric invariants of cuspidal edges.

3. Darboux frames along cuspidal edges

Let $f : I \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ be a frontal with a unit normal vector ν, where I is an open interval or a circle, and $\varepsilon > 0$. Assume that $I \times \{0\}$ consists of singular points of the first kind, and we take a coordinate system (u, v) of $I \times (-\varepsilon, \varepsilon)$ satisfying that

(1) u is an arc-length parameter of the curve given by $f(u, 0)$,
(2) $\ker df_{(u,0)}$ is generated by $\partial/\partial v$,
(3) (u, v) is compatible with the orientation of $\mathbb{R}^2$.

We also call this coordinate system *adapted*. In this paper we always choose the unit normal vector ν of f on an adapted coordinate system so that it satisfies $\det(f_u, f_{vv}, \nu)(u, 0) > 0$.

We now set $\boldsymbol{\gamma}(u) = f(u, 0)$ and consider unit vector fields $\boldsymbol{e}(u) = f_u(u, 0) = \boldsymbol{\gamma}'(u)$, $\boldsymbol{\nu}(u) = \nu(u, 0)$ and $\boldsymbol{b}(u) = -\boldsymbol{e}(u) \times \boldsymbol{\nu}(u)$ along $\boldsymbol{\gamma}$. Here, $\boldsymbol{a}_1 \times \boldsymbol{a}_2$ is the exterior product of $\boldsymbol{a}_1, \boldsymbol{a}_2$ in $\mathbb{R}^3$. Then $\{\boldsymbol{e}, \boldsymbol{b}, \boldsymbol{\nu}\}$ is a orthonormal frame along $\boldsymbol{\gamma}$. We call $\{\boldsymbol{e}, \boldsymbol{b}, \boldsymbol{\nu}\}$ the *Darboux frame* along the cuspidal edge $\boldsymbol{\gamma}$. As the structure equations for the Darboux frame along the cuspidal edge, we have the following proposition.

Proposition 3.1 (Frenet-Serret type formulae)**.**

$$
\left\{
\begin{aligned}
\boldsymbol{e}'(u) &= \kappa_s(u)\boldsymbol{b}(u) + \kappa_\nu(u)\boldsymbol{\nu}(u),\\
\boldsymbol{b}'(u) &= -\kappa_s(u)\boldsymbol{e}(u) + \kappa_t(u)\boldsymbol{\nu}(u),\\
\boldsymbol{\nu}'(u) &= -\kappa_\nu(u)\boldsymbol{e}(u) - \kappa_t(u)\boldsymbol{b}(u).
\end{aligned}
\right. \tag{3.1}
$$

By using the matrix representation, we have

$$
\begin{pmatrix} \boldsymbol{e}' \\ \boldsymbol{b}' \\ \boldsymbol{\nu}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_s & \kappa_\nu \\ -\kappa_s & 0 & \kappa_t \\ -\kappa_\nu & -\kappa_t & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{b} \\ \boldsymbol{\nu} \end{pmatrix}.
$$

Proof. Since $\{\boldsymbol{e}, \boldsymbol{b}, \boldsymbol{\nu}\}$ is an orthonormal frame along $\boldsymbol{\gamma}$, we have

$$
\begin{pmatrix} \boldsymbol{e}' \\ \boldsymbol{b}' \\ \boldsymbol{\nu}' \end{pmatrix} = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \delta \\ -\beta & -\delta & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{b} \\ \boldsymbol{\nu} \end{pmatrix},
$$

where $\alpha = \langle \boldsymbol{e}', \boldsymbol{b} \rangle$, $\beta = \langle \boldsymbol{e}', \boldsymbol{\nu} \rangle$ and $\delta = -\langle \boldsymbol{\nu}', \boldsymbol{b} \rangle$. By a straightforward calculation, we have

$$
\alpha = \langle \boldsymbol{e}', \boldsymbol{b} \rangle = -\langle \boldsymbol{e}', \boldsymbol{e} \times \boldsymbol{\nu} \rangle = \det(\boldsymbol{e}, \boldsymbol{e}', \boldsymbol{\nu}) = \det(\boldsymbol{\gamma}', \boldsymbol{\gamma}'', \boldsymbol{\nu}).
$$

Since $\det(f_u, f_{vv}, \nu) > 0$, we have $\alpha = \kappa_s$. It follows from $\beta = \langle \boldsymbol{e}', \boldsymbol{\nu} \rangle$ that $\beta = \kappa_\nu$. Since f has a singular point of the first kind at $0 \in (\mathbb{R}^2, 0)$, f_{vv}, f_u are linearly independent. We set

$$
(\tilde{u}, \tilde{v}) = \phi(u, v) = (u + a(u)v^2, v), \quad a(u) = \langle f_u(u, 0), f_{vv}(u, 0) \rangle /2.
$$

Then we see that

$$
\begin{pmatrix} u_{\tilde{u}} & u_{\tilde{v}} \\ v_{\tilde{u}} & v_{\tilde{v}} \end{pmatrix} = \frac{1}{1 + a'(u)v^2} \begin{pmatrix} 1 & -2a(u)v \\ 0 & 1 + a'(u)v^2 \end{pmatrix} \circ \phi^{-1}(\tilde{u}, \tilde{v})
$$

$f_{\tilde{u}} = f_u$. Moreover, since

$$
f_{\tilde{v}} = f_u u_{\tilde{u}} + f_v = f_u \frac{-2a(u)v}{1 + a'(u)v^2} + f_v,
$$

it holds that

$$\begin{aligned} f_{\tilde v\tilde v}(\tilde u,0) &= f_{\tilde v v}(u,0)=\Big(f_u v\frac{-2a(u)v}{1+a'(u)v^2}+f_u\frac{-2a(u)}{1+a'(u)v^2}+f_u\frac{-4a(u)a'(u)v^2}{(1+a'(u)v^2)^2}+f_{vv}\Big)(u,0)\\ &= -2a(u)f_u(u,0)+f_{vv}(u,0). \end{aligned}$$

By the definition of $a(u)$, it holds that $\langle f_{\tilde u}, f_{\tilde v\tilde v}\rangle(\tilde u,0)=0$. Therefore we can choose an adapted coordinate system (u,v) such that f_u, f_{vv} are orthogonal, namely $\boldsymbol{\nu}=f_u\times f_{vv}/|f_u\times f_{vv}|$ on the u-axis. Moreover, we have $-\boldsymbol{b}=\boldsymbol{e}\times\boldsymbol{\nu}=f_u\times(f_u\times f_{vv})/|f_u\times f_{vv}|=-f_{vv}/|f_u\times f_{vv}|$, so that

$$-\delta=\langle\boldsymbol{\nu}',\boldsymbol{b}\rangle=\frac{\langle f_u\times f_{uvv}, f_{vv}\rangle}{|f_u\times f_{vv}|^2}=\frac{\det(f_u,f_{uvv},f_{vv})}{|f_u\times f_{vv}|^2}=-\frac{\det(f_u,f_{vv},f_{uvv})}{|f_u\times f_{vv}|^2}=-\kappa_t,$$

on the u-axis. □

We define a vector field $D_o(u)$ along $\boldsymbol{\gamma}$ by

$$D_o(s)=\kappa_t(u)\boldsymbol{e}(u)-\kappa_\nu(u)\boldsymbol{b}(u),$$

which is called an *osculating Darboux vector field* along $\boldsymbol{\gamma}$. If $\kappa_\nu^2+\kappa_t^2\neq 0$, we can define the *unit osculating Darboux vector field* by

$$\overline{D_o}(u)=\frac{\kappa_t(u)\boldsymbol{e}(u)-\kappa_\nu(u)\boldsymbol{b}(u)}{\sqrt{\kappa_\nu(u)^2+\kappa_t(u)^2}}. \tag{3.2}$$

We also define a vector field $D_r(u)$ along $\boldsymbol{\gamma}$ by

$$D_r(s)=\kappa_t(u)\boldsymbol{e}(u)+\kappa_s(u)\boldsymbol{\nu}(u),$$

which is called a *normal Darboux vector field* along $\boldsymbol{\gamma}$. If $\kappa_t^2+\kappa_s^2\neq 0$, we can also define the *unit normal Darboux vector field* by

$$\overline{D}_r(u)=\frac{\kappa_t(u)\boldsymbol{e}(u)+\kappa_s(u)\boldsymbol{\nu}(u)}{\sqrt{\kappa_t(u)^2+\kappa_s(u)^2}}. \tag{3.3}$$

We now define the notion of contour edges of cuspidal edges. For a unit vector $\boldsymbol{k}\in S^2$, we say that the cuspidal edge $S(f)$ is the *tangential contour edge of the orthogonal projection* with direction $\boldsymbol{k}$ if

$$S(f)=\{(u,0)\in(\mathbb{R}^2,0)\mid\langle\boldsymbol{\nu}(u),\boldsymbol{k}\rangle=0\}.$$

We also say that the cuspidal edge $S(f)$ is the *normal contour edge of the orthogonal projection* with direction $\boldsymbol{k}$ if

$$S(f)=\{(u,0)\in(\mathbb{R}^2,0)\mid\langle\boldsymbol{b}(u),\boldsymbol{k}\rangle=0\}.$$

Moreover, for a point $\boldsymbol{c}\in\mathbb{R}^3$, say that the cuspidal edge $S(f)$ is the *tangential contour edge of the central projection* (respectively, *normal contour edge of the central projection*) with center $\boldsymbol{c}$ if

$$\begin{aligned} S(f) &= \{(u,0)\in(\mathbb{R}^2,0)\mid\langle f(u,0)-\boldsymbol{c},\boldsymbol{\nu}(u)\rangle=0\ \}.\\ \Big(\text{respectively, } S(f) &= \{(u,0)\in(\mathbb{R}^2,0)\mid\langle f(u,0)-\boldsymbol{c},\boldsymbol{b}(u)\rangle=0\ \}.\Big) \end{aligned}$$

For a regular surface, the notion of contour edges corresponds to the notion of contour generators [3].

On the other hand, there is a notion of isophotic curves on a regular surfaces. An isophotic curve of a surface is a curve consisting of points which have the same light intensity from a given light source. If the light source is infinitely far from the surface, the light rays might be considered as parallel lines. In this case, an *isophotic curve* is a curve on a regular surface such that the normal of the surface along the curve makes a constant angle with a fixed direction. Therefore, we can define the notion of isophotic curves on the cuspidal edge exactly the same way as the definition for curves on a regular surface. In particular, the cuspidal edge $S(f)$ is said to be a *normally isophotic edge* if there exists a unit vector $\boldsymbol{d}$ such that $\langle \boldsymbol{d}, \boldsymbol{\nu}(u)\rangle$ is constant. We also say that $S(f)$ is a *tangential isophotic edge* if there exists a unit vector $\boldsymbol{d}$ such that $\langle \boldsymbol{d}, \boldsymbol{b}(u)\rangle$ is constant.

We emphasize that notions of contour generators and isophotic curves on regular surfaces play important roles in the vision theory and visual psychophysics (cf. [3, 15, 16, 17]).

4. Developable surfaces and generalizations of helices

We briefly review the notions and basic properties of ruled surfaces and developable surfaces. Let $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^3$ and $\boldsymbol{\xi} : I \longrightarrow \mathbb{R}^3 \setminus \{\boldsymbol{0}\}$ be C^∞-maps, where I is an open interval or a circle. Then we define a map $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})} : I \times \mathbb{R} \longrightarrow \mathbb{R}^3$ by

$$F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}(u,t) = \boldsymbol{\gamma}(u) + t\boldsymbol{\xi}(u).$$

We call the image of $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ a *ruled surface*, the map $\boldsymbol{\gamma}$ a *base curve* and the map $\boldsymbol{\xi}$ a *director curve*. The line defined by $\boldsymbol{\gamma}(u) + t\boldsymbol{\xi}(u)$ for a fixed $u \in I$ is called a *ruling*. If the direction of the director curve $\boldsymbol{\xi}$ is constant, we call $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ a (*generalized*) *cylinder*. Using the notation $\overline{\boldsymbol{\xi}}(u) = \boldsymbol{\xi}(u)/\|\boldsymbol{\xi}(u)\|$, we have $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}(I\times\mathbb{R}) = F_{(\boldsymbol{\gamma},\overline{\boldsymbol{\xi}})}(I\times\mathbb{R})$. In this case $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is a cylinder if and only if $\dot{\overline{\boldsymbol{\xi}}}(u) \equiv 0$, where $\equiv$ means that equality holds identically. We say that $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is *non-cylindrical* if $\dot{\overline{\boldsymbol{\xi}}}(u) \neq 0$ for any $u \in I$. Suppose that $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is non-cylindrical. Then a *striction curve* is defined to be

$$\boldsymbol{s}(u) = \boldsymbol{\gamma}(u) - \frac{\langle \dot{\boldsymbol{\gamma}}(u), \dot{\overline{\boldsymbol{\xi}}}(u)\rangle}{\langle \dot{\overline{\boldsymbol{\xi}}}(u), \dot{\overline{\boldsymbol{\xi}}}(u)\rangle}\overline{\boldsymbol{\xi}}(u). \tag{4.1}$$

It is known that a singular point of the non-cylindrical ruled surface is located on the striction curve. We call the ruled surface with vanishing Gaussian curvature on the regular part a *developable surface*. It is known that a ruled surface $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is a developable surface if and only if

$$\det\left(\dot{\boldsymbol{\gamma}}(u), \boldsymbol{\xi}(u), \dot{\boldsymbol{\xi}}(u)\right) = 0, \tag{4.2}$$

where $\dot{\boldsymbol{\gamma}}(u) = (d\boldsymbol{\gamma}/du)(u)$(cf., [12]). The set of singular points of a non-cylindrical developable surface coincides with the striction curve[11]. A non-cylindrical ruled surface $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is a *cone* if the striction curve $\boldsymbol{s}$ is constant. It is known (cf., [12]) that a non-cylindrical developable surface $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is a wave front if and only if

$$\psi(u) = \det\left(\boldsymbol{\xi}(u), \dot{\boldsymbol{\xi}}(u), \ddot{\boldsymbol{\xi}}(u)\right) \neq 0. \tag{4.3}$$

In this case we call $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ a (*non-cylindrical*) *developable front*. Let $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}(u,t)$ be a non-cylindrical developable surface. Then by (4.2), there exist $\alpha(u)$ and $\beta(u)$ such that $\dot{\gamma}(u) = \alpha(u)\xi(u) + \beta(u)\dot{\xi}(u)$. The striction curve of $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is written as $\boldsymbol{s}(u) = \gamma(u) - \beta(u)\boldsymbol{\xi}(u)$, and we see that the signed area density of $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is proportional to $\lambda = t + \beta(u)$. Thus a singular point of $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is always non-degenerate. By Lemma 2.1, we have the following:

Proposition 4.1. *With the above notations, a singular point $(u, -\beta(u))$ of $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is a cuspidal edge* (*respectively, a swallowtail*) *if and only if $\psi(u) \neq 0$ and $\beta'(u) - \alpha(u) \neq 0$* (*respectively, $\psi(u) \neq 0$, $\beta'(u) - \alpha(u) = 0$ and $\beta''(u) - \alpha'(u) \neq 0$*).

On the other hand, by [4, Corollary 1.5], we have the following:

Proposition 4.2. *With the same notations as in Proposition 4.1, a singular point $(u, -\beta(u))$ of $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is a cuspidal cross cap if and only if $\beta'(u) - \alpha(u) \neq 0$, $\psi(u) = 0$ and $\psi'(u) \neq 0$.*

See [23] for other investigations of developable surfaces with singularities.

Remarkable generalizations of helices in $\mathbb{R}^3$ were introduced and investigated in [13, 30, 31]. Let $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ be a space curve with an arc-length parameter u. We call $\boldsymbol{\gamma}$ a *Frenet curve* if $\kappa(u) = \|\boldsymbol{\gamma}''(u)\| \neq 0$. For a Frenet curve $\boldsymbol{\gamma}$, let $\{\boldsymbol{t}, \boldsymbol{n}_{\boldsymbol{\gamma}}, \boldsymbol{b}_{\boldsymbol{\gamma}}\}$ be the Frenet frame along $\boldsymbol{\gamma}$, and κ, τ the curvature and torsion, respectively. Then $\boldsymbol{\gamma}$ is said to be a *cylindrical helix* (or, a *generalized helix*) if there exists a constant vector $\boldsymbol{v}$ such that $\boldsymbol{t}(u)$ makes a constant angle with $\boldsymbol{v}$. By the Frenet-Serret formulae, this condition is equivalent to the condition that $\boldsymbol{n}_{\boldsymbol{\gamma}}(u)$ is orthogonal to $\boldsymbol{v}$. Moreover, $\boldsymbol{\gamma}$ is called a *slant helix* if there exists a constant vector $\boldsymbol{v}$ such that $\boldsymbol{n}_{\boldsymbol{\gamma}}(u)$ makes a constant angle with $\boldsymbol{v}$ [13]. By definition, $\boldsymbol{\gamma}$ is a slant helix if and only if $\boldsymbol{n}_{\boldsymbol{\gamma}}(u)$ is a circle in the unit sphere. Recently, the notion of clad helices have been introduced in [30, 31]. We say that $\boldsymbol{\gamma}$ is a *clad helix* if $\boldsymbol{n}_{\boldsymbol{\gamma}}(u)$ is a cylindrical helix. Since $\boldsymbol{n}_{\boldsymbol{\gamma}}(u)$ is a curve in the unit sphere, it is a spherical cylindrical helix. It is classically known that $\boldsymbol{\gamma}$ is cylindrical helix if and only if τ/κ is constant (i.e., the Lancret theorem). If both of τ and κ are constant, $\boldsymbol{\gamma}$ is a circular helix (i.e., an ordinary helix). Therefore, a cylindrical helix is a generalization of circular helix. A curve $\boldsymbol{\gamma}$ is a slant helix if and only if

$$\theta(u) = \frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)'(u)$$

is constant [13]. Moreover, $\boldsymbol{\gamma}$ is a clad helix if and only if

$$\eta(u) = \frac{\theta'}{(\kappa^2+\tau^2)^{1/2}(1+\theta^2)^{3/2}}(u)$$

is constant [30, 31]. See [13, 30, 31] for details. Motivated by the results in [13, 30, 31], we consider generalizations of these notions of helices. For a Frenet curve $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^3$, we say that $\boldsymbol{\gamma}$ is a *0th-order helix* if it is a cylindrical helix, $\boldsymbol{\gamma}$ is a *1st-order helix* if it is a slant helix and $\boldsymbol{\gamma}$ is a *2nd-order helix* if it is a clad helix, respectively. For $k \geq 1$, we inductively define the notion of kth-order helices. We say that $\boldsymbol{\gamma}$ is a *kth-order helix* if $\boldsymbol{t}$ is a $(k-1)$th-order helix.

Proposition 4.3. *A Frenet curve $\boldsymbol{\gamma}$ is a kth-order helix if and only if $\boldsymbol{n}_{\boldsymbol{\gamma}}$ is a $(k-2)$th-order helix.*

Proof. For $k = 2$, $\boldsymbol{\gamma}$ is a 2nd-order helix if and only if $\boldsymbol{\gamma}$ is a clad helix. Therefore, $\boldsymbol{n}_{\boldsymbol{\gamma}}$ is a cylindrical helix. By definition, it means that $\boldsymbol{n}_{\boldsymbol{\gamma}}$ is a 0th-order helix. The assertion holds for $k = 2$. For $k > 2$, $\boldsymbol{\gamma}$ is a kth-order helix if and only if $\boldsymbol{t}$ is a $(k-1)$th-order helix. This means that $\boldsymbol{n}_{\boldsymbol{\gamma}} = \boldsymbol{t}'/\|\boldsymbol{t}'\|$ is a $(k-2)$th-order helix. This completes the proof. $\square$

We remark that a cylindrical helix is also called a *constant slope curve* because its tangent vector has a constant angle with a constant direction. We can interpret a constant slope as a 0th-order slope. In this sense, we also call a kth-order helix a *kth-order slope curve*.

On the other hand, we now give a characterization of kth-order helices by the curvature and the torsion (i.e., the Lancret-type theorem). We define $\mathscr{H}[\boldsymbol{\gamma}]_0(u) = \tau(u)/\kappa(u)$, which is called a *0th-order helical curvature* of $\boldsymbol{\gamma}$. We have

$$\theta(u) = \frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)'(u) = \frac{1}{\kappa}\frac{\left(\frac{\tau}{\kappa}\right)'}{\left(1+\left(\frac{\tau}{\kappa}\right)^2\right)^{3/2}}(u) = \frac{1}{\kappa}\frac{\mathscr{H}[\boldsymbol{\gamma}]_0'}{\left(1+\left(\mathscr{H}[\boldsymbol{\gamma}]_0\right)^2\right)^{3/2}}(u).$$

We set $\mathscr{H}[\boldsymbol{\gamma}]_1(u) = \theta(u)$, which is called a *1st-order helical curvature.* Moreover, the *2nd-order helical curvature* of $\boldsymbol{\gamma}$ is defined to be

$$\mathscr{H}[\boldsymbol{\gamma}]_2(u) = \eta(u) = \frac{1}{\kappa(1+(\mathscr{H}[\boldsymbol{\gamma}])_0^2)^{1/2}}\frac{\mathscr{H}[\boldsymbol{\gamma}]_1'}{\left(1+\left(\mathscr{H}[\boldsymbol{\gamma}]_1\right)^2\right)^{3/2}}(u).$$

For $r \geq 2$, we inductively define

$$\mathscr{H}[\boldsymbol{\gamma}]_{2r-1}(u) = \frac{1}{(1+(\mathscr{H}[\boldsymbol{\gamma}]_{2r-3})^2)^{1/2}} \frac{\mathscr{H}[\boldsymbol{\gamma}]'_{2r-2}}{\left(1+\left(\mathscr{H}[\boldsymbol{\gamma}]_{2r-2}\right)^2\right)^{3/2}}(u),$$

which is called a $(2r-1)$*st-order helical curvature*, and

$$\mathscr{H}[\boldsymbol{\gamma}]_{2r}(u) = \frac{1}{(1+(\mathscr{H}[\boldsymbol{\gamma}]_{2r-3})^2)^{1/2}(1+(\mathscr{H}[\boldsymbol{\gamma}]_{2r-2})^2)^{1/2}} \frac{\mathscr{H}[\boldsymbol{\gamma}]'_{2r-1}}{\left(1+\left(\mathscr{H}[\boldsymbol{\gamma}]_{2r-1}\right)^2\right)^{3/2}}(u),$$

which is called a $2r$*th-order helical curvature*. On the other hand, let $\kappa_n(u)$ and $\tau_n(u)$ be the curvature and the torsion of the principal normal $\boldsymbol{n}(u)$, respectively. Then we can calculate that

$$\kappa_n(u) = \sqrt{1+(\mathscr{H}[\boldsymbol{\gamma}]_1)^2}(u),\ \tau_n(u) = \left(\frac{\mathscr{H}[\boldsymbol{\gamma}]'_1}{(1+(\mathscr{H}[\boldsymbol{\gamma}]_1)^2)(\kappa^2+\tau^2)^{1/2}}\right)(u).$$

By using these formulae, we can show that the above inductive definitions are well-defined. Then we have the following characterization of higher-order helices.

Proposition 4.4. *Let $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^3$ be a Frenet curve. Then the following conditions are equivalent*:

(1) *$\boldsymbol{\gamma}$ is a kth-order helix,*
(2) *$\mathscr{H}[\boldsymbol{\gamma}]_k(u)$ is constant,*
(3) *$\mathscr{H}[\boldsymbol{\gamma}]_{k+1}(u)$ is identically equal to zero.*

Proof. By definition (2) and (3) are equivalent. It follows from [12, 30, 31] that conditions (1) and (2) are equivalent for $k \leq 2$. Let us write $\mathscr{H}[\boldsymbol{n}]_k(u)$ as the kth-order helical curvature of the principal normal curve $\boldsymbol{n}(u)$ of $\boldsymbol{\gamma}(u)$. By Proposition 4.3, $\boldsymbol{\gamma}(u)$ is a 3rd-order helix if and only if $\boldsymbol{n}(u)$ is a 1st-order helix. By the result in [12], this is equivalent to

$$\mathscr{H}[\boldsymbol{n}]_1(u) = \frac{1}{\kappa_n} \frac{(\mathscr{H}[\boldsymbol{n}]_0)'}{\left(1+(\mathscr{H}[\boldsymbol{n}]_0)^2\right)^{3/2}}(u)$$

being constant. If we substitute $\kappa_n(u) = \sqrt{1+(\mathscr{H}[\boldsymbol{\gamma}]_1)^2}(u)$ and $\mathscr{H}[\boldsymbol{n}]_0 = \tau_n/\kappa_n = \mathscr{H}[\boldsymbol{\gamma}]_2$, we have $\mathscr{H}[\boldsymbol{\gamma}]_3(u) = \mathscr{H}[\boldsymbol{n}]_1(u)$, so that conditions (1) and (2) are equivalent for $k=3$. By Proposition 4.3, $\boldsymbol{\gamma}(u)$ is a 4th-order helix if and only if $\boldsymbol{n}(u)$ is a 2nd-order helix. This condition is equivalent to the condition that

$$\mathscr{H}[\boldsymbol{n}]_2(u) = \frac{1}{\kappa_n(1+(\mathscr{H}[\boldsymbol{n}]_0)^2)^{1/2}} \frac{(\mathscr{H}[\boldsymbol{n}]_1)'}{\left(1+(\mathscr{H}[\boldsymbol{n}]_1)^2\right)^{3/2}}(u)$$

is constant. If we substitute $\kappa_n(u) = \sqrt{1+(\mathscr{H}[\boldsymbol{\gamma}]_1)^2}(u)$, $\mathscr{H}[\boldsymbol{n}]_0 = \mathscr{H}[\boldsymbol{\gamma}]_2$ and $\mathscr{H}[\boldsymbol{n}]_1 = \mathscr{H}[\boldsymbol{\gamma}]_3$ into the above formulae, then the above condition is equivalent to the condition that

$$\mathscr{H}[\boldsymbol{\gamma}]_4(u) = \frac{1}{(1+(\mathscr{H}[\boldsymbol{\gamma}]_1)^2)(1+(\mathscr{H}[\boldsymbol{\gamma}]_2)^2)^{1/2}} \frac{\mathscr{H}[\boldsymbol{\gamma}]'_3}{(1+(\mathscr{H}[\boldsymbol{\gamma}]_3)^2)^{3/2}}(u)$$

is constant. Therefore, conditions (1) and (2) are equivalent for $k=4$. We can show that condition (1) and (2) are equivalent by inductive arguments similar to the above cases. $\square$

We now consider the tangent surface $F_{(\boldsymbol{\gamma},\boldsymbol{t})}(u,t) = \boldsymbol{\gamma}(u) + t\boldsymbol{t}(u)$ for a Frenet curve $\boldsymbol{\gamma}(u)$. We remark that a tangent surface is a developable surface. Here, we consider tangent surfaces of special curves in $\mathbb{R}^3$. We also remark that $F_{(\boldsymbol{\gamma},\boldsymbol{t})}$ is non-cylindrical if and only if $\boldsymbol{\gamma}$ is a Frenet curve. We assume that $\boldsymbol{\gamma}$ is a Frenet curve and $F_{(\boldsymbol{\gamma},\boldsymbol{t})}$ is said to be a *developable surface with kth-order slope* if $\boldsymbol{\gamma}$ is a kth-order helix. In particular, a developable surface with 0th-order slope is called a *constant angle surface* [22] (or, a *developable surface of constant slope* [26, 6.3]). By Proposition 4.3, $F_{(\boldsymbol{\gamma},\boldsymbol{t})}$ is a developable surface with kth-order slope if and only if $\boldsymbol{n}_{\boldsymbol{\gamma}}(u)$ is a $(k-2)$th-order helix. By the Frenet-Serret formula $\boldsymbol{b}'_{\boldsymbol{\gamma}} = -\tau \boldsymbol{n}_{\boldsymbol{\gamma}}$, this implies that $\boldsymbol{b}_{\boldsymbol{\gamma}}$ is a $(k-1)$th-order helix. If $\tau \neq 0$, then the converse holds. Let $\boldsymbol{v} : I \longrightarrow S^2 \subset \mathbb{R}^3$ be a smooth unit vector field. For a unit constant vector $\boldsymbol{c}$, we say that $\boldsymbol{v}(u)$ has a 1st-order angle with $\boldsymbol{c}$ if $\langle \boldsymbol{v}(u), \boldsymbol{c}\rangle$ is constant. For $k \geq 2$, we say that $\boldsymbol{v}(u)$ has a kth-order angle with $\boldsymbol{c}$ if $\boldsymbol{v}'(u)/\|\boldsymbol{v}'(u)\|$ has a $(k-1)$th-order angle with $\boldsymbol{c}$. We have the following lemma.

Lemma 4.5. *Let $\boldsymbol{v} : I \longrightarrow S^2 \subset \mathbb{R}^3$ be a smooth unit vector field. For $k \geq 2$, there exists a unit constant vector $\boldsymbol{c}$ such that $\boldsymbol{v}(u)$ has a kth-order angle with $\boldsymbol{c}$ if and only if $\boldsymbol{v}(u)$ is a $(k-2)$th-order helix.*

Proof. We prove this by induction. Since a 0th-order helix is a cylindrical helix, which is equivalent to the condition that $\langle \boldsymbol{v}'(u)/\|\boldsymbol{v}'(u)\|, \boldsymbol{c}\rangle$ is constant for a unit vector $\boldsymbol{c}$. This means that $\boldsymbol{v}(u)$ has a 1st-order angle with $\boldsymbol{c}$. This completes the proof for $k = 2$. Suppose that the assertion holds for $k-1$. If $\boldsymbol{v}(u)$ has a kth-order angle with $\boldsymbol{c}$ for a unit vector $\boldsymbol{c}$. By definition, $\boldsymbol{v}'(u)/\|\boldsymbol{v}'(u)\|$ has a $(k-1)$th-order angle with $\boldsymbol{c}$ for a unit vector $\boldsymbol{c}$, by the inductive assumption, $\boldsymbol{v}'(u)/\|\boldsymbol{v}'(u)\|$ is a $(k-3)$th-order helix. By definition, $\boldsymbol{v}$ is a $(k-2)$th-order helix. The converse also holds. □

We have the following theorem.

Theorem 4.6. *Let $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^3$ be a Frenet curve. Then the following conditions are equivalent*:
(1) *$F_{(\boldsymbol{\gamma},\boldsymbol{t})}$ is a developable surface with kth-order slope,*
(2) *$\mathscr{H}[\boldsymbol{\gamma}]_k(u)$ is constant,*
(3) *$\mathscr{H}[\boldsymbol{\gamma}]_{k+1}(u) \equiv 0$,*
(4) *$\boldsymbol{t}$ is a $(k-1)$th-order helix,*
(5) *$\boldsymbol{n}_{\boldsymbol{\gamma}}$ is a $(k-2)$th-order helix.*

If $\tau(u) \neq 0$, then the following condition is equivalent to the above:
(6) *The restriction of the unit normal vector field of $F_{(\boldsymbol{\gamma},\boldsymbol{t})}$ on the striction curve $\boldsymbol{\gamma}$ has a $(k-1)$th-order angle with a constant unit vector.*

Proof. By Propositions 4.3 and 4.4, conditions (1), (2), (3), (4) and (5) are equivalent. Suppose $\tau(u) \neq 0$. By a straightforward calculation, the restriction of the unit normal vector field of $F_{(\boldsymbol{\gamma},\boldsymbol{t})}$ on the striction curve $\boldsymbol{\gamma}(u)$ is the binormal vector field $\boldsymbol{b}_{\boldsymbol{\gamma}}(u)$ of $\boldsymbol{\gamma}(u)$. Suppose that $k = 2$. Since $\mathscr{H}[\boldsymbol{\gamma}]_2(u)$ is constant, $\boldsymbol{\gamma}(u)$ is a clad helix (i.e., 2nd-order helix), which is equivalent to the condition that $\boldsymbol{n}_{\boldsymbol{\gamma}}(u)$ is a cylindrical helix. Since $\boldsymbol{b}'_{\boldsymbol{\gamma}} = -\tau \boldsymbol{n}_{\boldsymbol{\gamma}}$, this condition is equivalent to the condition that $\boldsymbol{b}'_{\boldsymbol{\gamma}}(u)/\|\boldsymbol{b}'_{\boldsymbol{\gamma}}(u)\|$ is a cylindrical helix. By definition, $\boldsymbol{b}_{\boldsymbol{\gamma}}(u)$ has a 1st-order angle with a unit vector $\boldsymbol{c}$. For $k > 2$, by Lemma 4.5, condition (5) is equivalent to the condition that $\boldsymbol{n}_{\boldsymbol{\gamma}}(u)$ has a kth-order angle with a unit vector $\boldsymbol{c}$. By the relation $\boldsymbol{b}'_{\boldsymbol{\gamma}} = -\tau \boldsymbol{n}_{\boldsymbol{\gamma}}$ and definition, $\boldsymbol{b}_{\boldsymbol{\gamma}}(u)$ has a $(k-1)$th-order angle with $\boldsymbol{c}$. □

In the above theorem, we do not consider condition (4) for $k = 0$ and condition (5) for $k = 0, 1$ respectively.

5. Developable surfaces along cuspidal edges

In this section we introduce two kinds of flat surfaces along a cuspidal edge. Let $f : I \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ be a frontal with a unit normal vector ν, where I is an open interval or a circle, and $\varepsilon > 0$. Assume that $I \times \{0\}$ consists of singular points of the first kind, and we take an adapted coordinate system (u, v) on $I \times (-\varepsilon, \varepsilon)$.

5.1. **Osculating developable surfaces along cuspidal edges.** If $(\kappa_\nu(u), \kappa_t(u)) \neq (0,0)$ on $u \in I$, we define a map $OD_f : I \times \mathbb{R} \longrightarrow \mathbb{R}^3$ by

$$OD_f(u,t) = f(u,0) + t\overline{D_o}(u) = f(u,0) + t\frac{\kappa_t(u)\boldsymbol{e}(u) - \kappa_\nu(u)\boldsymbol{b}(u)}{\sqrt{\kappa_t(u)^2 + \kappa_\nu(u)^2}}.$$

This is a ruled surface. Setting

$$\delta_o = \kappa_s(\kappa_\nu^2 + \kappa_t^2) - \kappa_t\kappa_\nu' + \kappa_\nu\kappa_t', \tag{5.1}$$

where $' = d/du$, by (3.1), we have

$$\overline{D_o}' = \frac{\delta_o}{(\kappa_t^2 + \kappa_\nu^2)^{3/2}}(\kappa_\nu\boldsymbol{e} + \kappa_t\boldsymbol{b}). \tag{5.2}$$

Here and in what follows, we omit "(u)" if it does not create misunderstandings. By (5.2), we have $\det\left(\boldsymbol{\gamma}', \overline{D_o}, \overline{D_o}'\right) = 0$. This means that $OD_f(I \times \mathbb{R})$ is a developable surface. We call OD_f an *osculating developable surface* of f along $S(f)$. By (5.2), OD_f is non-cylindrical if and only if $\delta_o \neq 0$. The osculating developable surface of f approximates f along $S(f)$ as a developable surface, and it has common tangent planes with f along $S(f)$ (see Figure 1). Let $\boldsymbol{s}_{OD}$ be the

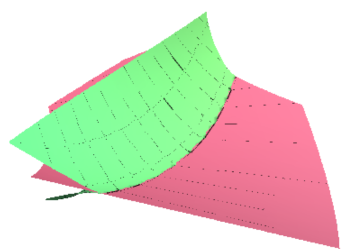

FIGURE 1. A cuspidal edge (green) with its osculating developable surface (purple)

striction curve of OD_f, which is defined by $\boldsymbol{s}_{OD}(u) = OD_f\left(u, -\sqrt{\kappa_\nu(u)^2 + \kappa_t(u)^2}\kappa_\nu(u)/\delta_o(u)\right)$. By a straightforward calculation, we see that

$$\boldsymbol{s}_{OD}' = \frac{\sigma_o}{\delta_o^2}(\kappa_t\boldsymbol{e} - \kappa_\nu\boldsymbol{b}), \tag{5.3}$$

where we set

$$\begin{aligned}\sigma_o &= \kappa_\nu\delta_o' + (\kappa_s\kappa_t - 2\kappa_\nu')\delta_o \\ &= \kappa_t(\kappa_\nu^2 + \kappa_t^2)\kappa_s^2 + 3\kappa_t(-\kappa_t\kappa_\nu' + \kappa_\nu\kappa_t')\kappa_s \\ &\qquad + \kappa_s'\kappa_\nu^3 + \kappa_t''\kappa_\nu^2 + (\kappa_t^2\kappa_s' - 2\kappa_\nu'\kappa_t' - \kappa_t\kappa_\nu'')\kappa_\nu + 2\kappa_t(\kappa_\nu')^2.\end{aligned}$$

By Propositions 4.1 and 4.2, we have the following theorem:

Theorem 5.1. *Suppose that OD_f is non-cylindrical. Then a singular point $(u, -\kappa_\nu(u)/\delta_o(u))$ of OD_f is*

(1) *a cuspidal edge if and only if $\delta_o(u) \neq 0$ and $\sigma_o(u) \neq 0$,*
(2) *a swallowtail if and only if $\delta_o(u) \neq 0$, $\sigma_o(u) = 0$ and $\sigma_o'(u) \neq 0$.*

Moreover, cuspidal cross caps never appear.

Proof. Since $D_o' = (\kappa_\nu\kappa_s + \kappa_t')\boldsymbol{e} + (\kappa_s\kappa_t - \kappa_\nu')\boldsymbol{b}$, and $D_o'' = *\boldsymbol{e} + *\boldsymbol{b} + \delta_0\boldsymbol{\nu}$, we see that $\psi = \delta_o^2$, where $*$ stands for some function. On the other hand, since

$$\boldsymbol{e} = \frac{1}{\delta}\Big((\kappa_s\kappa_t - \kappa_\nu')D_o + \kappa_\nu D_o'\Big),$$

α, β in Proposition 4.1 can be taken as $(\alpha, \beta) = (\kappa_s\kappa_t - \kappa_\nu', \kappa_\nu)/\delta_o$. Thus we see that $\beta' - \alpha = \sigma/\delta_o^2$. By Proposition 4.1, we see assertions (1) and (2). Since $\psi = \delta_o^2$, if $\psi(u) = 0$ then $\psi'(u) = 0$ for $u \in I$. This proves the last assertion. $\square$

Since OD_f is a developable surface, the striction curve $\boldsymbol{s}_{OD}$ coincides with $OD_f|_{S(OD_f)}$, and is a curve in $\mathbb{R}^3$. By (5.3), $\boldsymbol{s}_{OD}$ is regular if $\sigma_o \neq 0$. We denote by κ_{OD} (respectively, τ_{OD}) the curvature (respectively, the torsion) of $\boldsymbol{s}_{OD}$ the torsions of $OD_f|_{S(OD_f)}$ and $ND_f|_{S(OD_f)}$, respectively. By (5.3) and

$$\boldsymbol{s}_{OD}'' = \frac{1}{\delta_o^3}\bigg[\Big(\delta_o(\sigma_o'\kappa_t + \sigma_o\kappa_t' + \sigma_o\kappa_s\kappa_\nu) - 2\kappa_t\sigma_o\delta_o'\Big)\boldsymbol{e} + \Big(\delta_o(-\sigma_o'\kappa_\nu - \sigma_o\kappa_\nu' + \sigma_o\kappa_s\kappa_t) + 2\kappa_\nu\sigma_o\delta_o'\Big)\boldsymbol{b}\bigg],$$

$\boldsymbol{s}_{OD}''' = *\boldsymbol{e} + *\boldsymbol{b} + \dfrac{\sigma_o}{\delta_o^2}\Big(\kappa_s(\kappa_\nu^2 + \kappa_t^2) - \kappa_\nu'\kappa_t + \kappa_\nu\kappa_t'\Big)\boldsymbol{\nu}$, if $\sigma_o \neq 0$, then it holds that

$$\kappa_{OD} = \frac{|\delta_o|^3}{(\kappa_\nu^2 + \kappa_t^2)^{3/2}|\sigma_o|}, \quad \tau_{OD} = \frac{\delta_o^2}{\sigma_o}. \tag{5.4}$$

Therefore, $\boldsymbol{s}_{OD}$ is a Frenet curve if $\sigma_o \neq 0$ and $\delta_o \neq 0$. If $\kappa_\nu \equiv 0$, then $\boldsymbol{s}_{OD}$ is equal to $f(S(f))$. Moreover, if the cuspidal edge f is a tangent developable surface $F_{(\boldsymbol{\gamma},\boldsymbol{t})}$, then $\boldsymbol{e} = \boldsymbol{t}$, $\boldsymbol{b} = \boldsymbol{n}_{\boldsymbol{\gamma}}$ and $\boldsymbol{\nu} = \boldsymbol{b}_{\boldsymbol{\gamma}}$. By the Frenet-Serret formulae, we have $\kappa_\nu \equiv 0$, $\kappa_s = \kappa$ and $\kappa_t = \tau$. Then $\overline{D}_o(u) = \pm\boldsymbol{e}(u)$ and the image of $\boldsymbol{s}_{OD}$ coincides with $f(S(f))$. If $\kappa_t \equiv 0$ and $\kappa_\nu \neq 0$, then $\overline{D}_o(u) = \mp\boldsymbol{b}(u)$. We have the following corollary of Theorem 5.1.

Corollary 5.2. *Let f be a cuspidal edge. Then we have the following*:
(A) *Suppose that $\kappa_\nu \equiv 0$ and $\kappa_t \neq 0$. Then $\boldsymbol{s}_{OD}(I) = f(S(f))$ (i.e., OD_f is the tangent developable of $S(f)$) and a singular point $(u, 0) \in S(f)$ of OD_f is a cuspidal edge if and only if $\kappa_s(u) \neq 0$. Moreover, swallowtails never appear.*
(B) *Suppose that $\kappa_t \equiv 0$ and $\kappa_\nu \neq 0$. Then $OD_f(u,t) = f(u,0) + t\boldsymbol{b}(u)$. If $\kappa_s(u_0) = 0$, then OD_f is cylindrical at u_0. If OD_f is non-cylindrical (i.e., $\kappa_s \neq 0$), then*

$$\boldsymbol{s}_{OD}(u) = OD_f(u, -|\kappa_\nu(u)|/\kappa_\nu(u)\kappa_s(u))$$

and a singular point $(u, -|\kappa_\nu(u)|/\kappa_\nu(u)\kappa_s(u))$ of OD_f is
(1) *a cuspidal edge if and only if $\kappa_s'(u) \neq 0$,*
(2) *a swallowtail if and only if $\kappa_s'(u) = 0$ and $\kappa_s''(u) \neq 0$.*

Proof. (A) Since $\kappa_\nu \equiv 0$, $\delta_o = \kappa_s\kappa_t^2$ and $\sigma_o = \kappa_t^3\kappa_s^2$, and then the results follow from Theorem 5.1.

(B) Since $\kappa_t \equiv 0$ and $\kappa_\nu \neq 0$, $\delta_o = \kappa_s\kappa_\nu^2$ and $\sigma_o = \kappa_\nu^3\kappa_s'$, so that $\sigma_o' = 3\kappa_\nu^2\kappa_\nu'\kappa_s' + \kappa_\nu^3\kappa_s''$, and then the results follow from Theorem 5.1. $\square$

Let f be a cuspidal edge with $\kappa_\nu \equiv 0$. Then by Corollary 5.2, $S(f) = S(OD_f)$. If $\kappa_s > 0$ (respectively, $\kappa_s < 0$), then $S(OD_f)$ locates the opposite side across the $f(S(f))$ (respectively, the same side with f with respect to $f(S(f))$). See Figure 2. For a cuspidal edge f with $\kappa_\nu \neq 0$, this is investigated in [24], and a cuspidal edge $\hat{f}$ which is isometric to f and satisfies $f(S(f)) = \hat{f}(S(\hat{f}))$. See [24] for detail.

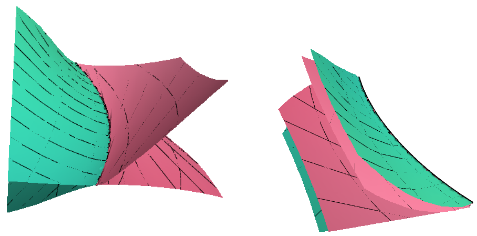

FIGURE 2. Left(respectively, right): Cuspidal edge f with $\kappa_\nu \equiv 0$ and $\kappa_s > 0$ (respectively, $\kappa_s < 0$) (green), and OD_f (purple).

5.2. **Normal developable surfaces along cuspidal edges.** If $(\kappa_t(u), \kappa_s(u)) \neq (0,0)$, we define a map $ND_f : I \times \mathbb{R} \longrightarrow \mathbb{R}^3$ by

$$ND_f(u,t) = f(u,0) + t\overline{D}_r(u) = f(u,0) + t\frac{\kappa_t(u)\boldsymbol{e}(u) + \kappa_s(u)\boldsymbol{\nu}(u)}{\sqrt{\kappa_t(u)^2 + \kappa_s(u)^2}}.$$

Since

$$\overline{D}_r' = \frac{\delta_n}{(\kappa_t^2 + \kappa_s^2)^{3/2}}(-\kappa_s \boldsymbol{e} + \kappa_t \boldsymbol{\nu}), \tag{5.5}$$

where

$$\delta_n = \kappa_\nu(\kappa_s^2 + \kappa_t^2) - \kappa_s\kappa_t' + \kappa_t\kappa_s', \tag{5.6}$$

we can also show that $ND_f(I \times \mathbb{R})$ is a developable surface (See Figure 3). By (5.5), ND_f is

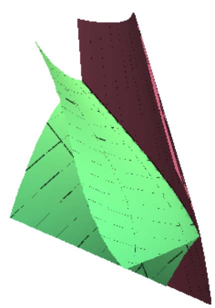

FIGURE 3. A cuspidal edge (green) with its normal developable surface (purple)

non-cylindrical if and only if $\delta_n \neq 0$. Let $\boldsymbol{s}_{ND}$ be the striction curve of ND_f, which is defined by $\boldsymbol{s}_{ND}(u) = ND_f\big(u, -\sqrt{\kappa_s(u)^2 + \kappa_t(u)^2}\kappa_s(u)/\delta_n(u)\big)$. Again by a straightforward calculation, we have

$$\boldsymbol{s}_{ND}' = \frac{\sigma_n}{\delta_n^2}(\kappa_t \boldsymbol{e} + \kappa_s \boldsymbol{\nu}), \tag{5.7}$$

where we set

$$\begin{aligned}
\sigma_n &= -\kappa_s\delta_n' + (\kappa_\nu\kappa_t + 2\kappa_s')\delta_n \\
&= \kappa_t(\kappa_s^2 + \kappa_t^2)\kappa_\nu^2 + 3\kappa_t(\kappa_t\kappa_s' - \kappa_s\kappa_t')\kappa_\nu \\
&\qquad -\kappa_s\kappa_\nu'\kappa_t^2 + (2\kappa_s'^2 - \kappa_s\kappa_s'')\kappa_t + \kappa_s(-\kappa_s^2\kappa_\nu' - 2\kappa_s'\kappa_t' + \kappa_s\kappa_t'').
\end{aligned}$$

Similar to Section 5.1, by Propositions 4.1 and 4.2, we have the following theorem:

Theorem 5.3. *Suppose that ND_f is non-cylindrical. Then a singular point $(u, -\kappa_s(u)/\delta_n(u))$ of ND_f is*

(1) *a cuspidal edge if and only if $\delta_n(u) \neq 0$ and $\sigma_n(u) \neq 0$,*
(2) *a swallowtail if and only if $\delta_n(u) \neq 0$, $\sigma_n(u) = 0$ and $\sigma_n'(u) \neq 0$.*

Moreover, cuspidal cross caps never appear.

If $\kappa_s \equiv 0$, then $\overline{D}_n(u) = \pm \boldsymbol{e}(u)$ and the image of $\boldsymbol{s}_{ND}$ coincides with $f(S(f))$. If $\kappa_t \equiv 0$ and $\kappa_s \neq 0$, then $\overline{D}_n(u) = \pm \boldsymbol{\nu}(u)$.

Therefore we have the following corollary of Theorem 5.3.

Corollary 5.4. *Let f be a cuspidal edge. Then we have the following*:
(A) *Suppose that $\kappa_s \equiv 0$ and $\kappa_t \neq 0$. Then $\boldsymbol{s}_{ND}(I) = f(S(f))$ (i.e., ND_f is the tangent developable of $S(f)$) and a singular point $(u, 0) \in S(f)$ of ND_f is a cuspidal edge if and only if $\kappa_\nu(u) \neq 0$. Moreover, swallowtails never appear.*
(B) *Suppose that $\kappa_t \equiv 0$ and $\kappa_s \neq 0$. Then $ND_f(u,t) = f(u,0) + t\boldsymbol{\nu}(u)$. If $\kappa_\nu(u_0) = 0$, then ND_f is cylindrical at u_0. If ND_f is non-cylindrical (i.e., $\kappa_\nu \neq 0$), then*

$$\boldsymbol{s}_{ND}(u) = ND_f(u, -|\kappa_\nu(u)|/\kappa_\nu(u)\kappa_s(u))$$

and a singular point $(u, -|\kappa_\nu(u)|/\kappa_\nu(u)\kappa_s(u))$ of ND_f is

(1) *a cuspidal edge if and only if $\kappa_\nu(u) \neq 0$,*
(2) *a swallowtail if and only if $\kappa_\nu \neq 0$, $\kappa_\nu' = 0$ and $\kappa_\nu''(u) \neq 0$.*

Proof. (A) Since $\kappa_s \equiv 0$, $\delta_n = \kappa_\nu \kappa_t^2$ and $\sigma_n = \kappa_t^3 \kappa_\nu^2$. Then the results follow from Theorem 5.1.
(B) If $\kappa_t \equiv 0$, then we have $\delta_n = \kappa_\nu \kappa_s^2$ and $\sigma_n = -\kappa_s^3 \kappa_\nu'$, so that $\sigma_n' = -3\kappa_s^2\kappa_s'\kappa_\nu' - \kappa_s^3\kappa_\nu''$. $\square$

On the other hand, also similar to Section 5.1, if $\sigma_n \neq 0$, then the curvature κ_{ND} and the torsion τ_{ND} of $\boldsymbol{s}_{ND}$ are given by

$$\kappa_{ND} = \frac{|\delta_n|^3}{(\kappa_s^2 + \kappa_t^2)^{3/2}|\sigma_n|}, \quad \tau_{ND} = \frac{\delta_n^2}{\sigma_n}. \tag{5.8}$$

We close this subsection giving examples of OD_f and ND_f having cuspidal edges and swallowtails.

Example 5.5. Let us consider a space curve

$$\boldsymbol{\gamma}(u) = \left(\cos \frac{u}{\sqrt{2}},\ \sin \frac{u}{\sqrt{2}},\ \frac{u}{\sqrt{2}} \right). \tag{5.9}$$

Let $\boldsymbol{e}_{\boldsymbol{\gamma}}, \boldsymbol{n}_{\boldsymbol{\gamma}}, \boldsymbol{b}_{\boldsymbol{\gamma}}$ be the Frenet frame of γ. We set

$$f(u,v) = \boldsymbol{\gamma} + v^2\Big(\cos\theta(u)\boldsymbol{n}_{\boldsymbol{\gamma}} - \sin\theta(u)\boldsymbol{b}_{\boldsymbol{\gamma}} \Big) + v^3\Big(\sin\theta(u)\boldsymbol{n}_{\boldsymbol{\gamma}} - \cos\theta(u)\boldsymbol{b}_{\boldsymbol{\gamma}} \Big), \tag{5.10}$$

for a function $\theta(u)$. Then we see that $S(f) = \{v = 0\}$ and it consists of cuspidal edges. If $\theta(u) = \pi/4$, then

$$\boldsymbol{s}_{OD}(0) = OD_f(0, -2\sqrt{2/3}), \quad \boldsymbol{s}_{ND}(0) = ND_f(0, 2\sqrt{2/3}), \text{ and } \ \sigma_o(0) = \sigma_n(0) = 3/128.$$

Thus singular points of OD_f near $(0, -2\sqrt{2/3})$ and ND_f near $(0, 2\sqrt{2/3})$ consist of a cuspidal edge. See Figures 4 and 5. In these pictures, f is colored in green, and OD_f and ND_f are colored in purple.

Example 5.6. Let us consider the case $\theta = \pi/4 + u/4$ in (5.10) of Example 5.5. We see that $\boldsymbol{s}_{OD}(0) = OD_f(0, -2\sqrt{3})$, $\boldsymbol{s}_{ND}(0) = ND_f(0, 2\sqrt{3})$, $\sigma_o(0) = \sigma_n(0) = 0$, and $\sigma_o'(0) = -1/256$, $\sigma_n'(0) = 1/256$. Thus each singular point of OD_f at $(0, -2\sqrt{3})$ and ND_f at $(0, 2\sqrt{3})$ is a

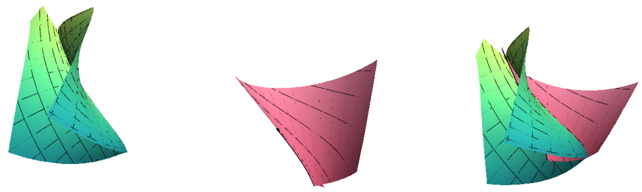

FIGURE 4. Left to right: Cuspidal edge f of $\theta = \pi/4$, OD_f and combined picture of f and OD_f

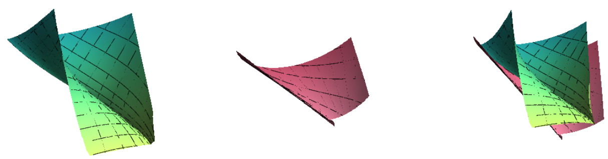

FIGURE 5. Left to right: Cuspidal edge f of $\theta = \pi/4$, ND_f and combined picture of f and ND_f

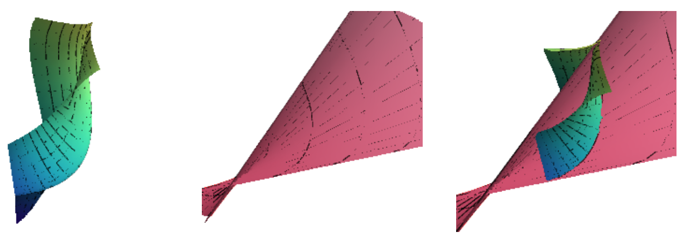

FIGURE 6. Left to right: Cuspidal edge f of $\theta = \pi/4 + u/4$, OD_f and combined picture of f and OD_f

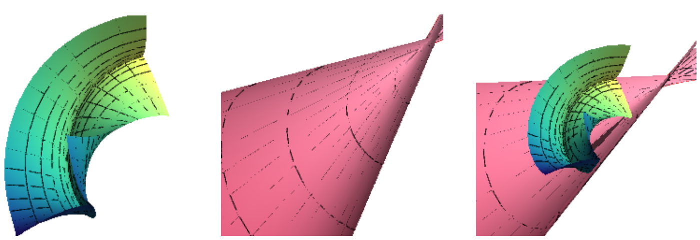

FIGURE 7. Left to right: Cuspidal edge f of $\theta = \pi/4 + u/4$, ND_f and combined picture of f and ND_f

swallowtail. See Figures 6 and 7. In these pictures, f is colored in green, and OD_f and ND_f are colored in purple.

5.3. **Planer cuspidal edges.** In the previous subsections we investigated the singularities of OD_f and ND_f with the condition $(\kappa_\nu(u), \kappa_t(u)) \neq (0,0)$ and $(\kappa_t(u), \kappa_s(u)) \neq (0,0)$ for any $u \in I$. Moreover, we also investigated the case when one of κ_s, κ_ν and κ_t is identically equal to zero as special cases (cf. Corollaries 5.2 and 5.4). Here, we study cuspidal edges with

$(\kappa_\nu(u),\kappa_t(u))=(0,0)$ and $(\kappa_t(u),\kappa_s(u))=(0,0)$ for any $u\in I$. With the same setting to the above subsections, let us assume $(\kappa_\nu(u),\kappa_t(u))=(0,0)$ and $\kappa_s\neq 0$ for any $u\in I$. Since the curvature κ and the torsion τ of the curve $f(u,0)$ as a curve in $\mathbb{R}^3$ satisfy

$$\kappa^2=\kappa_s^2+\kappa_\nu^2,\quad \tau=\frac{\kappa_s\kappa_\nu'-\kappa_\nu\kappa_s'}{\kappa_s^2+\kappa_\nu^2}+\kappa_t, \tag{5.11}$$

(see [20]) and $\boldsymbol{\nu}'(u)\equiv 0$, we see that $f(u,0)$ lies on a plane which is perpendicular to the constant vector $\boldsymbol{\nu}$. In this case, OD_f can be considered as a subset of this plane and

$$ND_f(u,t)=f(u,0)+t\boldsymbol{\nu}$$

is a cylinder. By the same argument as the above, we see that if $(\kappa_t(u),\kappa_s(u))\equiv(0,0)$ and $\kappa_\nu\neq 0$, then $f(u,0)$ lies on a plane which is perpendicular to the constant vector $\boldsymbol{b}$. In this case, ND_f can be considered as a subset of this plane and $ND_f(u,t)=f(u,0)+t\boldsymbol{b}$ is a cylinder. Moreover, if we assume $(\kappa_s(u),\kappa_\nu(u),\kappa_t(u))\equiv(0,0,0)$, then $f(u,0)$ is a straight line, and $\boldsymbol{\nu}'\equiv\boldsymbol{b}'\equiv 0$. In this case, OD_f should be defined as the plane perpendicular to $\boldsymbol{\nu}$ and ND_f as the plane perpendicular to $\boldsymbol{b}$. Since OD_f and ND_f intersect orthogonally, the cuspidal edge $S(f)$ is a line in this case.

5.4. **Normalized derivate director curves and derivate striction curves.** We set

$$\overline{\overline{D_o}'}=\frac{(\overline{D_o})'}{|(\overline{D_o})'|}=\frac{\kappa_\nu\boldsymbol{e}+\kappa_t\boldsymbol{b}}{\sqrt{\kappa_\nu^2+\kappa_t^2}},\quad \overline{\overline{D}_r'}=\frac{(\overline{D}_r)'}{|(\overline{D}_r)'|}=\frac{-\kappa_s\boldsymbol{e}+\kappa_t\boldsymbol{b}}{\sqrt{\kappa_s^2+\kappa_t^2}},$$

and call them the *normalized* $\overline{D_o}'$ and *normalized* $\overline{D}_r'$, respectively. They are curves in the unit sphere in $\mathbb{R}^3$. Here, we calculate their geodesic curvatures. Since

$$\begin{aligned}
\left(\overline{\overline{D_o}'}\right)' &= \frac{\delta_o}{\kappa_\nu^2+\kappa_t^2}\Big(-\kappa_t\boldsymbol{e}+\kappa_\nu\boldsymbol{b}\Big)+\frac{\boldsymbol{\nu}}{\sqrt{\kappa_\nu^2+\kappa_t^2}},\\
\left(\overline{\overline{D_o}'}\right)'' &= \frac{1}{\sqrt{\kappa_\nu^2+\kappa_t^2}^5}\Bigg\{-\Bigg[\Big(\kappa_\nu^3\kappa_s+\kappa_\nu^2\kappa_t'+\kappa_\nu\kappa_t(\kappa_s\kappa_t-3\kappa_\nu')-2\kappa_t^2\kappa_t'\Big)\delta_o\\
&\qquad +(\kappa_\nu^2+\kappa_t^2)(\kappa_\nu^5+2\kappa_\nu^3\kappa_t^2+\kappa_\nu\kappa_t^4+\kappa_t\delta_o')\Bigg]\boldsymbol{e}\\
&\qquad -\Bigg[\Big(\kappa_t^3\kappa_s-\kappa_t^2\kappa_\nu'+\kappa_t\kappa_\nu(\kappa_\nu\kappa_s+3\kappa_t')+2\kappa_\nu^2\kappa_\nu'\Big)\delta_o\\
&\qquad +(\kappa_\nu^2+\kappa_t^2)(\kappa_\nu^4\kappa_t+2\kappa_\nu^2\kappa_t^3+\kappa_t^5-\kappa_\nu\delta_o')\Bigg]\boldsymbol{b}\\
&\qquad +(\kappa_\nu^2+\kappa_t^2)^2(\kappa_\nu\kappa_\nu'+\kappa_t\kappa_t')\boldsymbol{\nu}\Bigg\},
\end{aligned}$$

we obtain the geodesic curvature of $\overline{\overline{D_o}'}$ as follows:

$$\left(\frac{\delta_o^2+1}{\kappa_\nu^2+\kappa_t^2}\right)^{3/2}\Big(-(\kappa_\nu^2+\kappa_t^2)\delta_o'+3(\kappa_\nu\kappa_\nu'+\kappa_t\kappa_t')\delta_o\Big),$$

and in a similar manner, we obtain the geodesic curvature $\overline{\overline{D}_r'}$ as follows:

$$\left(\frac{\delta_n^2+1}{\kappa_s^2+\kappa_t^2}\right)^{3/2}\Big(-(\kappa_s^2+\kappa_t^2)\delta_n'+3(\kappa_s\kappa_s'+\kappa_t\kappa_t')\delta_n\Big).$$

Next we consider normalized striction curves. By (3.2), (5.3), and (3.3), (5.7), we see that

$$\overline{\boldsymbol{s}'_{OD}} = \frac{\boldsymbol{s}'_{OD}}{|\boldsymbol{s}'_{OD}|} = \overline{D}_o, \quad \overline{\boldsymbol{s}'_{ND}} = \frac{\boldsymbol{s}'_{ND}}{|\boldsymbol{s}'_{ND}|} = \overline{D}_r.$$

Thus the normalized derivate striction curves coincide with the normalized director curves. Moreover, since $\overline{D_o}$ and $\boldsymbol{\nu}$ (respectively, $\overline{D}_r$ and $\boldsymbol{b}$) are dual to each other as curves in the unit sphere in $\mathbb{R}^3$, $\overline{\boldsymbol{s}'_{OD}}$ and $\boldsymbol{\nu}$ (respectively, $\overline{\boldsymbol{s}'_{ND}}$ and $\boldsymbol{b}$) are dual to each other.

6. Special cuspidal edges

In this section we consider the case when the singular values of OD_f and ND_f are special curves in $\mathbb{R}^3$. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a cuspidal edge and $\{\boldsymbol{e}, \boldsymbol{b}, \boldsymbol{\nu}\}$ Darboux frame along the cuspidal edge $\boldsymbol{\gamma}$, where $\boldsymbol{\gamma} = f|_{S(f)}$.

6.1. **Contour edges.** In this subsection we give characterizations of contour edges by using the invariants of cuspidal edges. We have the following theorem.

Theorem 6.1. *With the same notations as the previous sections, we have the following*:
(A) *Suppose that* $\kappa_\nu^2 + \kappa_t^2 \neq 0$. *Then the following properties are equivalent*:

(1) OD_f *is a cylinder,*
(2) $\delta_o \equiv 0$,
(3) $\boldsymbol{\nu}$ *is a part of a great circle in* S^2.
(4) $S(f)$ *is a tangential contour edge with respect to an orthogonal projection.*
(5) $\overline{D_o}$ *is a constant vector.*

(B) *Suppose that* $\kappa_s^2 + \kappa_t^2 \neq 0$. *Then the following properties are equivalent*:

(1) ND_f *is a cylinder,*
(2) $\delta_n(u) \equiv 0$,
(3) $\boldsymbol{b}$ *is a part of a great circle in* S^2,
(4) $S(f)$ *is a normal contour edge with respect to an orthogonal projection.*
(5) $\overline{D}_r$ *is a constant vector.*

Proof. We show the assertion (A). By (5.2), we see the equivalency of (1) and (2). The condition $\kappa_t^2 + \kappa_\nu^2 \neq 0$ means that $\boldsymbol{\nu}$ is a non-singular spherical curve. Moreover, since

$$\nu'' = (\kappa_s \kappa_t - \kappa'_\nu)\boldsymbol{e} + (-\kappa_\nu \kappa_s - \kappa'_t)\boldsymbol{b},$$

we see that $\det(\nu, \nu', \nu'') = \delta_o$. This implies that the geodesic curvature of $\boldsymbol{\nu}$ is $\delta_o(\kappa_t^2 + \kappa_\nu^2)^{-3/2}$, and it shows that the equivalency of (2) and (3). We assume (2). Then $\overline{D_o}(u)$ is a constant vector $\overline{D_o}$. Thus $\langle \boldsymbol{\nu}(u), \overline{D_o} \rangle = 0$ for any u. This implies that $S(f)$ is a tangential contour edge with respect to $\overline{D_o}$. This implies (4). Conversely, we assume (4). Then there exists a vector $\boldsymbol{k}$ such that $\langle \boldsymbol{\nu}(u), \boldsymbol{k} \rangle = 0$ holds for any u. This implies that $\boldsymbol{\nu}(u)$ belongs to the normal plane of $\boldsymbol{k}$ passing through the origin, and it implies (3). Since $\boldsymbol{\nu}$ and $\overline{D_o}$ are dual each other as spherical curves by (3.2) and (5.2), we see that the equivalency of (3) and (5). Thus the assertion (A) holds. One can show the assertion (B) by the same method as in the proof of (A), using (3.3) and (5.5) instead of (3.2) and (5.2). □

Theorem 6.2. *With the same notations as above, we have the following*:
(A) *Suppose that* $\kappa_t^2 + \kappa_\nu^2 \neq 0$ *and* $\delta_o \neq 0$ *for any* $u \in I$. *Then the following properties are equivalent*:

(1) OD_f *is a cone,*
(2) $\sigma_o \equiv 0$,

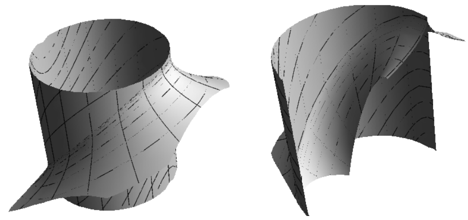

FIGURE 8. Cuspidal edge whose osculating developable surface is a cylinder

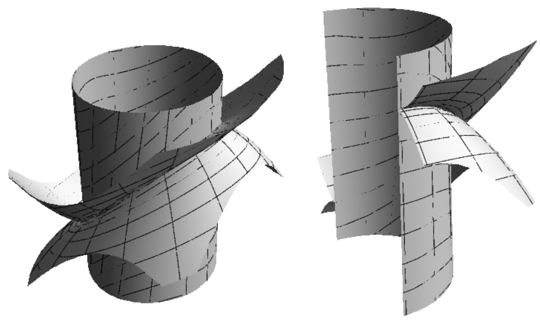

FIGURE 9. Cuspidal edge whose normal developable surface is a cylinder

(3) *$S(f)$ is a tangential contour edge with respect to a central projection.*
(4) *$\boldsymbol{s}_{OD}$ is a constant vector.*

(B) *Suppose that $\kappa_t^2 + \kappa_s^2 \neq 0$ and $\delta_n \neq 0$ for any $u \in I$. Then the following properties are equivalent*:

(1) *ND_f is a cone,*
(2) *$\sigma_n \equiv 0$,*
(3) *$S(f)$ is a normal contour edge with respect to a central projection.*
(4) *$\boldsymbol{s}_{ND}$ is a constant vector.*

Proof. By (5.3), we see that the equivalency of (1) and (2). We assume (2). Then $\boldsymbol{s}_{OD}(u)$ is a constant vector for any u. We set $\boldsymbol{c} = \boldsymbol{s}_{OD}(u)$. Then by (4.1), $f(u,0) - \boldsymbol{c}$ is parallel to $\overline{D_o}(u)$. Thus $\langle f(u,0) - \boldsymbol{c}, \boldsymbol{\nu}(u)\rangle = \langle \overline{D_o}(u), \boldsymbol{\nu}(u)\rangle = 0$ holds for any u. This implies (3). Conversely, we assume (3). Then there exists a vector $\boldsymbol{c}$ such that $\langle f(u,0) - \boldsymbol{c}, \boldsymbol{\nu}(u)\rangle \equiv 0$. By (4.1), $\boldsymbol{s}_{OD}(u) - f(u,0)$ is parallel to $\overline{D_o}(u)$, $\langle \boldsymbol{s}_{OD}(u) - \boldsymbol{c}, \boldsymbol{\nu}(u)\rangle \equiv 0$. Differentiating this equation by u, and noticing $\langle \boldsymbol{s}'_{OD}(u), \boldsymbol{\nu}(u)\rangle \equiv 0$ by (5.3), we have $\langle \boldsymbol{s}_{OD}(u), \boldsymbol{\nu}'(u)\rangle \equiv 0$. By (5.3) and (3.1), we see that $\langle \boldsymbol{s}'_{OD}(u), \boldsymbol{\nu}'(u)\rangle \equiv 0$. Thus, differentiating $\langle \boldsymbol{s}_{OD}(u), \boldsymbol{\nu}'(u)\rangle \equiv 0$ by u, we have $\langle \boldsymbol{s}_{OD}(u), \boldsymbol{\nu}''(u)\rangle \equiv 0$. On the other hand, by (3.1), the three vectors $\boldsymbol{\nu}(u), \boldsymbol{\nu}'(u), \boldsymbol{\nu}''(u)$ are linearly independent if and only if $\delta_o(u) \neq 0$. Hence

$$\langle \boldsymbol{s}_{OD}(u) - \boldsymbol{c}, \boldsymbol{\nu}(u)\rangle \equiv \langle \boldsymbol{s}_{OD}(u) - \boldsymbol{c}, \boldsymbol{\nu}'(u)\rangle \equiv \langle \boldsymbol{s}_{OD}(u) - \boldsymbol{c}, \boldsymbol{\nu}''(u)\rangle \equiv 0$$

implies $\boldsymbol{s}_{OD}(u) - \boldsymbol{c} \equiv 0$, and this implies (1). Thus the assertion (A) holds. One can show the assertion (B) by the same method as in the proof of (A) using (5.7) instead of (5.3). □

6.2. **Isophotic edges.** Recall that the curve γ is called the (normal) isophotic edge (respectively, the tangent isophotic edge) if there exists a constant vector $\boldsymbol{v}$ such that $\boldsymbol{\nu}$ (respectively, $\boldsymbol{b}$) makes a constant angle with $\boldsymbol{v}$.

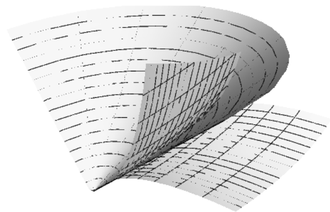

FIGURE 10. Cuspidal edge whose osculating developable surface is a cone

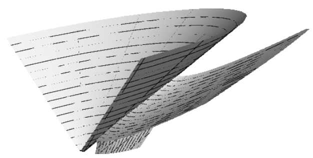

FIGURE 11. Cuspidal edge whose normal developable surface is a cone

Let us turn to our setting. With the same notations as those of Section 5, by a straightforward calculation, we have

$$\left(\frac{\tau_{OD}}{\kappa_{OD}}\right)^2 = \frac{(\kappa_\nu^2+\kappa_t^2)^3}{\delta_o^2} \quad \text{and} \quad \left(\frac{\tau_{ND}}{\kappa_{ND}}\right)^2 = \frac{(\kappa_s^2+\kappa_t^2)^3}{\delta_n^2}. \tag{6.1}$$

These are squares of the geodesic curvatures of $\boldsymbol{\nu}$ and $\boldsymbol{b}$, respectively. Thus we obtain:

Theorem 6.3. *With the same notations as those of Section 5, we have the following*:
(A) *Suppose that $\kappa_t^2+\kappa_\nu^2 \neq 0$, $\delta_o \neq 0$ and $\sigma_o \neq 0$ for any $u \in I$. Then the following properties are equivalent*:

(1) *OD_f is a constant angle surface,*
(2) *$\boldsymbol{\nu}$ is a part of a small circle,*
(3) *$S(f)$ is a normal isophotic edge,*
(4) *$\overline{D_o}$ is a part of a small circle,*
(5) *$\overline{\boldsymbol{s}'_{OD}}$ is a part of a small circle,*
(6) *$\delta_o/(\kappa_\nu^2+\kappa_t^2)^{3/2}$ is constant,*
(7) *$\boldsymbol{s}_{OD}$ is a cylindrical helix.*

(B) *Suppose that $\kappa_s^2+\kappa_\nu^2 \neq 0$, $\delta_n \neq 0$ and $\sigma_n \neq 0$ for any $u \in I$. Then the following properties are equivalent*:

(1) *ND_f is a constant angle surface,*
(2) *$\boldsymbol{b}$ is a part of a small circle,*
(3) *$\boldsymbol{\gamma}$ is a tangent isophotic edge,*
(4) *$\overline{D_r}$ is a part of a small circle,*
(5) *$\overline{\boldsymbol{s}'_{ND}}$ is a part of a small circle,*
(6) *$\delta_n/(\kappa_s^2+\kappa_t^2)^{3/2}$ is constant,*
(7) *$\boldsymbol{s}_{ND}$ is a cylindrical helix.*

Proof. By the definition and (6.1), the equivalency of (1) and (6) is obvious. By the proof of Theorem 5.3, $\delta_o/(\kappa_\nu^2+\kappa_t^2)^{3/2}$ is the geodesic curvature of $\boldsymbol{\nu}$, so that (2) and (6) are equivalent. Since $\boldsymbol{\nu}$ is a curve on the unit sphere, we see the equivalency of (2) and (3). By (5.2), $\boldsymbol{\nu}$ and $\overline{D_o}$ are spherical dual each other. Hence we see equivalency of (2) and (4). Equivalency of (2) and

(5) is obvious since $\overline{D_o}$ and $\overline{\boldsymbol{s}'_{OD}}$ are parallel. By definition, (5) and (7) are equivalent. Thus the assertion (A) holds.

One can show the assertion (B) by arguments similar to those for (A). □

6.3. **General order sloped edges.** In this subsection we consider cuspidal edges such that the osculating or the normal developables of cuspidal edges are general order sloped, where we say that $S(f)$ is a *k-th order sloped edge with respect to* $\overline{D_o}$ (respectively, $\overline{D_r}$) if $\overline{D_o}$ (respectively, $\overline{D_r}$) is a $(k-1)$th-order (spherical) helix. We denote the kth-order helical curvature of $\boldsymbol{s}_{OD}(u)$ (respectively, $\boldsymbol{s}_{ND}(u)$) by $\mathscr{H}[\boldsymbol{s}_{OD}]_k(u)$ (respectively, $\mathscr{H}[\boldsymbol{s}_{ND}]_k(u)$). By (6.1), we have

$$\mathscr{H}[\boldsymbol{s}_{OD}]_0(u) = \frac{(\kappa_\nu^2+\kappa_t^2)^{3/2}}{|\delta_o|},$$

$$\mathscr{H}[\boldsymbol{s}_{ND}]_0(u) = \frac{(\kappa_s^2+\kappa_t^2)^{3/2}}{|\delta_n|},$$

$$\mathscr{H}[\boldsymbol{s}_{OD}]_1(u) = \frac{\sqrt{\kappa_\nu^2+\kappa_t^2}}{\delta_o^2+(\kappa_\nu^2+\kappa_t^2)^3}\big(3\kappa_\nu\kappa_\nu' + 3\kappa_t\kappa_t' - (\kappa_\nu^2+\kappa_t^2)\delta_o'\big),$$

$$\mathscr{H}[\boldsymbol{s}_{ND}]_1(u) = \frac{\sqrt{\kappa_s^2+\kappa_t^2}}{\delta_n^2+(\kappa_s^2+\kappa_t^2)^3}\big(3\kappa_s\kappa_s' + 3\kappa_t\kappa_t' - (\kappa_s^2+\kappa_t^2)\delta_n'\big)$$

and

$$\mathscr{H}[\boldsymbol{s}_{OD}]_2(u) = \frac{\sigma_o(\kappa_\nu^2+\kappa_t^2)^{3/2}\theta_{OD}'}{\delta_o\sqrt{\delta_o^2+(\kappa_\nu^2+\kappa_t^2)^3}(1+\theta_{OD}^2)^{3/2}},$$

$$\mathscr{H}[\boldsymbol{s}_{ND}]_2(u) = \frac{\sigma_n(\kappa_s^2+\kappa_t^2)^{3/2}\theta_{ND}'}{\delta_n\sqrt{\delta_n^2+(\kappa_s^2+\kappa_t^2)^3}(1+\theta_{ND}^2)^{3/2}}.$$

Higher order helical curvatures of $\boldsymbol{s}_{OD}(u)$ and $\boldsymbol{s}_{ND}(u)$ are inductively defined. However, these are very complicated, so we omit explanations by using basic invariants for the cuspidal edge. Then we have the following theorem as a simple corollary of Theorem 4.6.

Theorem 6.4. *With the same notations as those of Sections 4 and 5, we have the following*:
(A) *Suppose that $\kappa_t^2+\kappa_\nu^2 \neq 0$, $\delta_o \neq 0$ and $\sigma_o \neq 0$ for any $u \in I$. Then the following properties are equivalent*:

(1) *OD_f is a developable surface with kth-order slope,*
(2) *$\boldsymbol{s}_{OD}$ is a kth-order helix,*
(3) *$\overline{\overline{D_o}'}$ is a $(k-2)$th-order (spherical) helix,*
(4) *$\overline{\boldsymbol{s}'_{OD}}$ is a $(k-1)$th-order (spherical) helix,*
(5) *$\mathscr{H}[\boldsymbol{s}_{OD}]_k$ is constant,*
(6) *$\mathscr{H}[\boldsymbol{s}_{OD}]_{k+1} \equiv 0$,*
(7) *$S(f)$ is a k-th order sloped edge with respect to $\overline{D_o}$.*

(B) *Suppose that $\kappa_t^2+\kappa_s^2 \neq 0$, $\delta_n \neq 0$ and $\sigma_n \neq 0$ for any $u \in I$. Then the following properties are equivalent*:

(1) *ND_f is a developable surface with kth-order slope,*
(2) *$\boldsymbol{s}_{ND}$ is a kth-order helix,*
(3) *$\overline{\overline{D_r}'}$ is a $(k-2)$th-order (spherical) helix,*
(4) *$\overline{D_r}$ is a $(k-1)$th-order (spherical) helix,*
(5) *$\overline{\boldsymbol{s}'_{ND}}$ is a $(k-1)$th-order (spherical) helix,*
(6) *$\mathscr{H}[\boldsymbol{s}_{ND}]_k$ is constant,*
(7) *$\mathscr{H}[\boldsymbol{s}_{ND}]_{k+1} \equiv 0$.*

(8) *$S(f)$ is a k-th order sloped edge with respect to $\overline{D_r}$.*

Proof. (A) With assumptions $\kappa_t^2 + \kappa_\nu^2 \neq 0$ and $\delta_o \neq 0$, $\boldsymbol{s}_{OD}$ is a Frenet curve. By definition, $\overline{D_o}'$ is the unit principal normal vector field of $\boldsymbol{s}_{OD}$. Since $\boldsymbol{s}_{OD}$ is the striation curve of OD_f, the director curve of OD_f is equal to $\boldsymbol{s}'_{OD}$, so that we can apply Theorem 4.6 to $\boldsymbol{s}_{OD}$. By definition, (4) and (8) are equivalent. For (B), we have arguments similar to the case (A) and apply Theorem 4.6 to $\boldsymbol{s}_{ND}$. □

If we consider the case when one of $\kappa_\nu,\ \kappa_t,\ \kappa_s$ is identically equal to zero, we have the following representations of helical curvatures of $\boldsymbol{s}_{OD}$ and $\boldsymbol{s}_{ND}$, respectively:
(1) Suppose that $\kappa_\nu \equiv 0$ and $\kappa_t \neq 0$. Then $\delta_o = \kappa_s\kappa_t^2$ and $\sigma_o = \kappa_t^3\kappa_s^2$. If $\kappa_s \neq 0$, then $\overline{D}_o(u) = \pm\boldsymbol{e}(u)$ and $\boldsymbol{s}_{OD}(I) = f(S(f))$. If we denote by κ and τ the curvature and the torsion of $S(f)$ respectively, then $\kappa(u) = |\kappa_s(u)|$ and $\tau(u) = \kappa_t(u)$. Therefore we have

$$\mathscr{H}[\boldsymbol{s}_{OD}]_0(u) = \mathscr{H}[S(f)]_0(u) = \kappa_t(u)/|\kappa_s(u)|.$$

Moreover, we have

$$\mathscr{H}[S(f)]_1(u) = \frac{1}{|\kappa_s(u)|}\frac{\mathscr{H}[S(f)]_0'(u)}{(1+(\mathscr{H}[S(f)]_0(u))^2)^{3/2}},$$

$$\mathscr{H}[S(f)]_2(u) = \frac{1}{|\kappa_s(u)|(1+(\mathscr{H}[S(f)]_0(u))^2)^{1/2}}\frac{\mathscr{H}[S(f)]_1'(u)}{(1+(\mathscr{H}[S(f)]_1(u))^2)^{3/2}}.$$

Higher order helical curvatures of $S(f)$ are inductively defined. Moreover, OD_f is the tangent developable of $f(S(f))$.
(2) Suppose that $\kappa_t \equiv 0$ and $\kappa_\nu \neq 0$. Then $\delta_o(u) = \kappa_s(u)\kappa_\nu(u)^2$ and $\sigma_o(u) = \kappa_\nu(u)^3\kappa_s'(u)$. If $\kappa_s \neq 0$ and $\kappa_s' \neq 0$, then $\overline{D}_o(u) = \pm\boldsymbol{b}(u)$ and $\boldsymbol{s}_{OD}(u) = OD_f(u, -|\kappa_\nu(u)|/\kappa_\nu(u)\kappa_s(u))$. Moreover, we have

$$\kappa_{OD}(u) = \frac{|\kappa_s(u)|^3|\kappa_\nu(u)|^3}{|\sigma_o(u)|} \text{ and } \tau_{OD}(u) = \frac{\kappa_s(u)^2\kappa_\nu(u)^4}{\sigma_o(u)},$$

so that $\mathscr{H}[\boldsymbol{s}_{OD}]_0(u) = \tau_{OD}(u)/\kappa_{OD}(u) = |\kappa_t(u)|/\kappa_s(u)$. We can define kth-order helical curvature $\mathscr{H}[\boldsymbol{s}_{OD}]_k(u)$ inductively. In this case $ND_f(u,t) = f(u,0) + t\boldsymbol{b}(u)$.
(3) Suppose that $\kappa_s \equiv 0$ and $\kappa_t \neq 0$. Then $\delta_n = \kappa_\nu\kappa_t^2$ and $\sigma_n = \kappa_t^3\kappa_\nu^2$. If $\kappa_s \neq 0$, then $\overline{D}_r(u) = \pm\boldsymbol{e}(u)$, $\boldsymbol{s}_{ND}(I) = f(S(f))$ and $\kappa(u) = |\kappa_\nu(u)|$ and $\tau(u) = \kappa_t(u)$. Therefore we have $\mathscr{H}[\boldsymbol{s}_{ND}]_0(u) = \mathscr{H}[S(f)]_0(u) = \kappa_t(u)/|\kappa_\nu(u)|$. We can define kth-order helical curvature $\mathscr{H}[S(f)]_k(u)$ inductively. In this case ND_f is the tangent developable of $f(S(f))$.
(4) Suppose that $\kappa_t \equiv 0$ and $\kappa_s \neq 0$. Then $\delta_n = \kappa_\nu\kappa_s^2$ and $\sigma_n = -\kappa_s^3\kappa_\nu'$. If $\kappa_\nu \neq 0$ and $\kappa_\nu' \neq 0$, then $\overline{D}_o(u) = \pm\boldsymbol{n}(u)$ and $\boldsymbol{s}_{ND}(u) = ND_f(u, -|\kappa_\nu(u)|/\kappa_\nu(u)\kappa_s(u))$. Moreover, we have

$$\kappa_{ND}(u) = \frac{|\kappa_\nu(u)|^3|\kappa_s(u)|^3}{|\sigma_n(u)|} \text{ and } \tau_{ND}(u) = \frac{\kappa_\nu(u)^2\kappa_t(u)^4}{\sigma_n(u)},$$

so that $\mathscr{H}[\boldsymbol{s}_{ND}]_0(u) = |\kappa_s(u)|/\kappa_\nu(u)$. We can define kth-order helical curvature $\mathscr{H}[\boldsymbol{s}_{ND}]_k(u)$ inductively. In this case $ND_f(u,t) = f(u,0) + t\boldsymbol{\nu}(u)$.

Corollary 6.5. *With the same notations as those in the above theorem, we have the following:*
(A) *Suppose that $\kappa_\nu \equiv 0$, $\kappa_t \neq 0$, and $\kappa_s \neq 0$. Then OD_f is the tangent developable of $S(f)$ and the following properties are equivalent:*

(1) *OD_f is a developable surface with kth-order slope,*
(2) *$S(f)$ is a kth-order helix,*
(3) *$\boldsymbol{b}$ is a $(k-2)$th-order (spherical) helix,*
(4) *$\boldsymbol{e}$ is a $(k-1)$th-order (spherical) helix,*
(5) *$\mathscr{H}[S(f)]_k$ is constant,*

(6) $\mathscr{H}[S(f)]_{k+1} \equiv 0$.

(B) *Suppose that* $\kappa_t \equiv 0$, $\kappa_\nu \neq 0$, $\kappa_s \neq 0$ *and* $\kappa_s' \neq 0$. *Then* OD_f *is the tangent developable of* $S(f)$ *and the following properties are equivalent*:

(1) OD_f *is a developable surface with* k*th-order slope,*
(2) $\boldsymbol{s}_{OD}$ *is a* k*th-order helix,*
(3) $\boldsymbol{e}$ *is a* $(k-2)$*th-order* (*spherical*) *helix,*
(4) $\boldsymbol{b}$ *is a* $(k-1)$*th-order* (*spherical*) *helix,*
(5) $\overline{\boldsymbol{s}_{OD}'}$ *is a* $(k-1)$*th-order* (*spherical*) *helix,*
(6) $\mathscr{H}[\boldsymbol{s}_{OD}]_k$ *is constant,*
(7) $\mathscr{H}[\boldsymbol{s}_{OD}]_{k+1} \equiv 0$.
(8) $S(f)$ *is a* k*-th order sloped edge with respect to* $\boldsymbol{b}$.

(C) *Suppose that* $\kappa_s \equiv 0$, $\kappa_t \neq 0$, *and* $\kappa_\nu \neq 0$. *Then* ND_f *is the tangent developable of* $S(f)$ *and the following properties are equivalent*:

(1) ND_f *is a developable surface with* k*th-order slope,*
(2) $S(f)$ *is a* k*th-order helix,*
(3) $\boldsymbol{\nu}$ *is a* $(k-2)$*th-order* (*spherical*) *helix,*
(4) $\boldsymbol{e}$ *is a* $(k-1)$*th-order* (*spherical*) *helix,*
(5) $\mathscr{H}[S(f)]_k$ *is constant,*
(6) $\mathscr{H}[S(f)]_{k+1} \equiv 0$.

(D) *Suppose that* $\kappa_t \equiv 0$, $\kappa_s \neq 0$, $\kappa_\nu \neq 0$ *and* $\kappa_\nu' \neq 0$. *Then* $ND_f(u,t) = f(u,0) + t\boldsymbol{\nu}(u)$ *and the following properties are equivalent*:

(1) ND_f *is a developable surface with* k*th-order slope,*
(2) $\boldsymbol{s}_{ND}$ *is a* k*th-order helix,*
(3) $\boldsymbol{e}$ *is a* $(k-2)$*th-order* (*spherical*) *helix,*
(4) $\boldsymbol{\nu}$ *is a* $(k-1)$*th-order* (*spherical*) *helix,*
(5) $\overline{\boldsymbol{s}_{ND}'}$ *is a* $(k-1)$*th-order* (*spherical*) *helix,*
(6) $\mathscr{H}[\boldsymbol{s}_{ND}]_k$ *is constant,*
(7) $\mathscr{H}[\boldsymbol{s}_{ND}]_{k+1} \equiv 0$.
(8) $S(f)$ *is a* k*-th order sloped edge with respect to* $\boldsymbol{\nu}$.

7. Curves on regular surfaces and relationships with cuspidal edges

In this section we consider curves on regular surfaces and investigate the relationship with the previous results on cuspidal edges. In [8, 14], developable surfaces along a curve on a regular surface are investigated. We consider a regular surface M parametrized by an embedding $\boldsymbol{X} : U \to \mathbb{R}^3$ with a unit normal vector field $\boldsymbol{n}$ (i.e., $M = \boldsymbol{X}(U)$). For a curve $c : I \to U$, we define $\boldsymbol{\gamma} = \boldsymbol{X} \circ c$ as a curve on M. We assume that $\boldsymbol{\gamma}$ is parametrized by the arc-length parameter s. The Darboux frame $\{\boldsymbol{t}, \boldsymbol{d}, \boldsymbol{n}\}$ along $\boldsymbol{\gamma}$ is defined to be the unit tangent vector $\boldsymbol{t}$ of $\boldsymbol{\gamma}$, $\boldsymbol{n} = \boldsymbol{n} \circ \boldsymbol{\gamma}$, and $\boldsymbol{d} = -\boldsymbol{t} \times \boldsymbol{n}$. Then we have

$$\left\{\begin{array}{lcl} \boldsymbol{t}' & = & \kappa_g \boldsymbol{d} + \kappa_n \boldsymbol{n} \\ \boldsymbol{d}' & = & -\kappa_g \boldsymbol{t} + \tau_g \boldsymbol{n} \\ \boldsymbol{n}' & = & -\kappa_n \boldsymbol{t} - \tau_g \boldsymbol{d}. \end{array}\right.$$

The invariants κ_g, κ_n and τ_g are called the *geodesic curvature*, the *normal curvature* and the *geodesic torsion* respectively. It is known that $\boldsymbol{\gamma}$ is a geodesic of M if and only if $\kappa_g \equiv 0$, $\boldsymbol{\gamma}$ is an asymptotic curve of M if and only if $\kappa_n \equiv 0$ and $\boldsymbol{\gamma}$ is a principal curve of M if and only if $\tau_g \equiv 0$. Here, $\boldsymbol{\gamma}$ is said to be a *geodesic* if the curvature vector $\boldsymbol{t}'(s)$ has only a normal component of the surface M, an *asymptotic curve* if $\boldsymbol{t}'(s)$ has only a tangential component of the surface M and a *line of curvature* if $\boldsymbol{\nu}'(s)$ is parallel to $\boldsymbol{t}(s)$, respectively.

In [14], an invariant $\tilde{\delta}_o = \kappa_g + (\kappa_n\tau_g' - \kappa_n'\tau_g)(\kappa_n^2 + \tau_g^2)^{-1}$ is introduced[1] and it is shown that $\tilde{\delta}_o \equiv 0$ if and only if $(\tau_g\boldsymbol{t} - \kappa_n\boldsymbol{d})(\kappa_n^2 + \tau_g^2)^{-1/2}$ is a constant vector. Moreover, it is shown that $\tilde{\delta}_o \equiv 0$ if and only if $\boldsymbol{\gamma}$ is a contour generator (i.e., singular set) with respect to an orthogonal projection such that its kernel is generated by $\tau_g\boldsymbol{t} - \kappa_n\boldsymbol{d}$. Furthermore, in [7], it is shown that $\tilde{\delta}_o(\kappa_n^2 + \tau_g^2)^{-1/2}$ is constant if and only if γ is an isophotic curve (i.e., $\boldsymbol{n} \circ \boldsymbol{\gamma}$ makes a constant angle with a constant vector $(\tau_g\boldsymbol{t} + \kappa_g\boldsymbol{n})(\kappa_g^2 + \tau_g^2)^{-1/2}$.).

On the other hand, in [7], an invariant $\tilde{\delta}_r = \kappa_n + (\kappa_g'\tau_g - \kappa_g\tau_g')(\kappa_g^2 + \tau_g^2)^{-1}$ is introduced[2] and it is shown that $\tilde{\delta}_r \equiv 0$ if and only if $(\tau_g\boldsymbol{t} + \kappa_g\boldsymbol{n})(\kappa_g^2 + \tau_g^2)^{-1/2}$ is a constant.

Actually, $(\tau_g\boldsymbol{t} - \kappa_n\boldsymbol{d})(\kappa_n^2 + \tau_g^2)^{-1/2}$ (respectively, $(\tau_g\boldsymbol{t} + \kappa_g\boldsymbol{n})(\kappa_g^2 + \tau_g^2)^{-1/2}$) is called a *normalized osculating Darboux vector* (respectively, a *normalized rectifying Darboux vector*) along $\boldsymbol{\gamma}$ in [7, 14]. Therefore, the osculating Darboux vector and the rectifying Darboux vector along a cuspidal edge are the notions analogous to those of the case for a regular curve on a regular surface. In this section we compare their properties along regular curves on regular surfaces with those along cuspidal edges.

On the other hand, with the same setting as in Section 5, $S(f)$ is not only a curve on f but also a curve on OD_f and ND_f. In particular, if $\kappa_\nu \neq 0$, then $S(f)$ is a regular curve on the regular part of OD_f. Moreover, $S(f)$ is always a regular curve on the regular part of ND_f. Therefore, we consider the invariants of $S(f)$ as a regular curve on OD_f and ND_f, respectively. Let $\tilde{\kappa}_g$, $\tilde{\kappa}_\nu$ and $\tilde{\tau}_g$ be the geodesic curvature, normal curvature and geodesic torsion of

$$S(f) = \{f(u,0) = OD_f(u,0) \; |u \in I\}$$

as a curve on OD_f, respectively. Also let $\overline{\kappa}_g$, $\overline{\kappa}_\nu$ and $\overline{\tau}_g$ denote the geodesic curvature, normal curvature and geodesic torsion of $S(f) = \{f(u,0) = ND_f(u,0)| \; u \in I\}$ as a curve on ND_f, respectively.

Since $\boldsymbol{\nu}$ is a unit normal vector of OD_f, we see that $\tilde{\kappa}_g = \kappa_s$, $\tilde{\kappa}_n = \kappa_\nu$ and $\tilde{\tau}_g = \kappa_t$. Also, since $\boldsymbol{b}$ is a unit normal vector of ND_f, we see that $\overline{\kappa}_g = -\kappa_\nu$, $\overline{\kappa}_n = \kappa_s$ and $\overline{\tau}_g = \kappa_t$. Hence we see that the invariants $\tilde{\delta}_o$ and $\tilde{\delta}_r$ of $f(u,0) = OD_f(u,0)$ as a curve on OD_f are

$$\tilde{\delta}_o = \frac{\delta_o}{\kappa_\nu^2 + \kappa_t^2}, \quad \tilde{\delta}_r = \frac{\delta_n}{\kappa_s^2 + \kappa_t^2},$$

respectively. On the other hand, the invariants $\tilde{\delta}_o$ and $\tilde{\delta}_r$ of $f(u,0) = ND_f(u,0)$ as a curve on ND_f are

$$\tilde{\delta}_o = -\frac{\delta_n}{\kappa_s^2 + \kappa_t^2}, \quad \tilde{\delta}_r = \frac{\delta_o}{\kappa_\nu^2 + \kappa_t^2}.$$

For the invariants $\kappa_g, \kappa_n, \tau_g$ of a curve $\boldsymbol{\gamma}$ on a regular surface, $\boldsymbol{\gamma}$ is an asymptotic curve of f if and only if $\kappa_n \equiv 0$, $\boldsymbol{\gamma}$ is a geodesic of f if and only if $\kappa_g \equiv 0$, and $\boldsymbol{\gamma}$ is a line of curvature of f if and only if $\tau_g \equiv 0$. It is natural to expect this type of explanation about invariants $\kappa_s, \kappa_\nu, \kappa_t$ of cuspidal edge. The singular curvature κ_s (respectively, the limiting normal curvature κ_ν) is defined as a limit of the geodesic curvatures with sign (respectively, the normal curvatures) of curves approaching the singular set of the cuspidal edge, and one can see the same explanation about κ_s and κ_ν [27, 20]. Here, we study κ_t from this point of view. For a regular curve $c : I \longrightarrow U$, it is classically known that $\boldsymbol{\gamma} = \boldsymbol{X} \circ c$ is a line of curvature if and only if the ruled surface with the normal director curve $\boldsymbol{\gamma}(s) + t\boldsymbol{n}(s)$ is a developable surface (i.e., Theorem of

[1] In [14], $\tilde{\delta}_o$ is denoted by δ.

[2] In [7], $\tilde{\delta}_r$ is denoted by δ_r.

Bonnet [29, Page 295]). On the other hand, let $f : I \times \mathbb{R} \longrightarrow \mathbb{R}^3$ be a frontal, and suppose $S(f) = I \times \{0\}$ consists of singular points of the first kind. Assume that $\kappa_t \equiv 0$ on I. Then $\overline{D}_r(u) = \pm\boldsymbol{\nu}(u)$, so that ND_f is a ruled surface with base curve $f|_{S(f)}$ and director curve $\boldsymbol{\nu}$, and it is, by definition, developable. Thus it is natural to expect that $S(f)$ of a frontal with vanishing κ_t can be considered as a line of curvature.

Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a map-germ and 0 a cuspidal edge. Suppose that (u, v) is an adapted coordinate system. Since $f_v(u, 0) = 0$, there exists a vector $h(u, v)$ such that $f_v(u, v) = vh(u, v)$. Set

$$\widetilde{E} = \langle f_u, f_u\rangle\,,\ \widetilde{F} = \langle f_u, h\rangle\,,\ \widetilde{G} = \langle h, h\rangle\,,\ \widetilde{L} = -\langle f_u, \nu_u\rangle\,,\ \widetilde{M} = -\langle h, \nu_u\rangle\,,\ \widetilde{N} = -\langle h, \nu_v\rangle\,.$$

Then

$$E = \widetilde{E},\quad F = v\widetilde{F},\quad G = v^2\widetilde{G},\quad L = \widetilde{L},\quad M = v\widetilde{M},\quad N = v\widetilde{N} \tag{7.1}$$

holds, where E, F, G (respectively, L, M, N) are the coefficients of the first fundamental form (respectively, the second fundamental form). Consider the equation

$$(EM - FL)\,du^2 + (EN - GL)\,dudv + (FN - GM)\,dv^2 = 0 \tag{7.2}$$

for a tangent vector $a(u, v)\partial_u + b(u, v)\partial_v \in T_{(u,v)}\mathbb{R}^2$. It is known that if $u'(t)\partial_u + v'(t)\partial_v$ satisfies (7.2), then the curve $(u(t), v(t))$ is a principal curve of f. Substituting (7.1) to (7.2) and factoring v out, we obtain the equation

$$(\widetilde{E}\widetilde{M} - \widetilde{F}\widetilde{L})\,du^2 + (\widetilde{E}\widetilde{N} - v\widetilde{G}\widetilde{L})\,dudv + (v\widetilde{F}\widetilde{N} - v^2\widetilde{G}\widetilde{M})\,dv^2 = 0.$$

Thus if $(\widetilde{E}\widetilde{M} - \widetilde{F}\widetilde{L})(u, 0) \equiv 0$, then we can regard the curve $(u, 0)$ as a line of curvature. By (5.1) of [20], $\kappa_t(u)$ is proportional to $(\widetilde{E}\widetilde{M} - \widetilde{F}\widetilde{L})(u, 0)$. Summarizing the above arguments, $S(f)$ can be regarded as a line of curvature if $\kappa_t \equiv 0$ holds.

Appendix A. Support functions

In this appendix we study invariants of a cuspidal edge using a family of functions on a curve. It is well-known that this method is useful for studying singular curves on singular surfaces. Although the results are the same as we have obtained above, we believe that it is worth mentioning that one can get the same result as Theorems 5.1 and 5.3 by this method.

For a unit speed curve $\boldsymbol{\gamma} : I \longrightarrow M \subset \mathbb{R}^3$ and a vector field $k : I \to TM$ along $\boldsymbol{\gamma}$, we define a function $G_{\boldsymbol{k}} : I \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ by $G_{\boldsymbol{k}}(u, \boldsymbol{x}) = \langle \boldsymbol{x} - \boldsymbol{\gamma}(u), \boldsymbol{k}(u)\rangle$. We call $G_{\boldsymbol{k}}$ a *support function on* $\boldsymbol{\gamma}$ *with respect to* $\boldsymbol{k}$. We denote that $g_{\boldsymbol{k},\boldsymbol{x}_0}(u) = G_{\boldsymbol{k}}(u, \boldsymbol{x}_0)$ for any $\boldsymbol{x}_0 \in \mathbb{R}^3$.

Let $f : I \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ be a frontal with a unit normal vector $\boldsymbol{\nu}$, where I is an open interval or a circle, and $\varepsilon > 0$. Assume that $I \times \{0\}$ consists of singular points of the first kind, and we take an adapted coordinate system (u, v) of $I \times (-\varepsilon, \varepsilon)$. Let $\boldsymbol{e}, \boldsymbol{b}, \boldsymbol{\nu}$ be the Darboux frame of $S(f)$. We consider

$$G_{\boldsymbol{\nu}}(u, \boldsymbol{x}),\quad g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u),\quad G_{\boldsymbol{b}}(u, \boldsymbol{x}),\quad g_{\boldsymbol{b},\boldsymbol{x}_0}(u)$$

We have the following propositions.

Proposition A.1. *Under the above setting, we have the following*:

(A) *Suppose that* $\kappa_\nu(u)^2 + \kappa_t(u)^2 \neq 0$. *Then*

- (A1) $g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u) = 0$ *if and only if there exist* $\alpha(u)$ *and* $\beta(u)$ *such that*
$$\boldsymbol{x}_0 - f(u, 0) = \alpha\boldsymbol{e}(u) + \beta\boldsymbol{b}(u).$$
- (A2) $g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u) = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = 0$ *if and only if there exists* $l(u)$ *such that*
$$\boldsymbol{x}_0 - f(u, 0) = -l(u)\overline{D_o}(u).$$
- (AI) *Suppose that* $\delta_o(u) \neq 0$. *Then*

(A3) $g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u) = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = 0$ *if and only if*

$$x_0 - f(u,0) = -\frac{\kappa_\nu}{\delta_o}\overline{D_o}(u). \tag{A.1}$$

(A4) $g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u) = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)'' = 0$ *if and only if* (A.1) *and* $\sigma_o = 0$.

(A5) $g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u) = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)''' = 0$ *if and only if* (A.1) *and* $\sigma_o = \sigma_o' = 0$.

(AII) *Suppose that* $\delta_o(u) = 0$. *Then*

(A3') $g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u) = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)'' = 0$ *if and only if* $\kappa_\nu = 0$. *We remark that under this condition,* $\delta_o = \kappa_s\kappa_t - \kappa_\nu'$.

(AII-1) *Set* $\delta_{\nu 1} = \kappa_t\kappa_s' + 2\kappa_s\kappa_t' - \kappa_\nu''$ *and suppose that* $\delta_o(u) = 0, \delta_{\nu 1}(u) \neq 0$. *Then*

(A4') $g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u) = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)''' = 0$ *if and only if* $\kappa_s = 0$ *and* $x_0 - f(u,0) = -\kappa_s\kappa_t\boldsymbol{e}(u)/\delta_{\nu 1}$.

(A5') $g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u) = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)''' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)'''' = 0$ *if and only if* $\kappa_s = 0$, $x_0 - f(u,0) = -\kappa_\nu\kappa_t\boldsymbol{e}(u)/\delta_{\nu 1}$ *and*

$$-2\kappa_s^4\kappa_t^2 - (2\kappa_t\kappa_s' - 3\kappa_\nu'')(\kappa_t\kappa_s' - \kappa_\nu'') - 3\kappa_s^2(2(\kappa_t')^2 + \kappa_t\kappa_t'') - \kappa_s(\kappa_t^2\kappa_s'' - 9\kappa_t'\kappa_\nu'' - \kappa_t(-10\kappa_s'\kappa_t' + \kappa_\nu''')) = 0.$$

(AII-2) *Suppose that* $\delta_o(u) = 0, \delta_{\nu 1}(u) = 0$. *Then*

(A4") $g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u) = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)''' = 0$ *if and only if* $\kappa_s = \kappa_\nu = 0$ *and there exists* $l(u)$ *such that* $x_0 - f(u,0) = l(u)\boldsymbol{e}(u)$. *We remark that under this condition,* $\delta_{\nu 1} = -\kappa_t\kappa_s' + \kappa_\nu''$.

(AII-2-1) *Set* $\delta_{\nu 2} = 3\kappa_s'\kappa_t' + \kappa_t\kappa_s'' - \kappa_\nu'''$, *and suppose that* $\delta_o(u) = 0, \delta_{\nu 1}(u) = 0, \delta_{\nu 2}(u) \neq 0$. *Then*

(A5") $g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u) = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)''' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)'''' = 0$ *if and only if* $\kappa_s = \kappa_\nu = 0$ *and* $x_0 - f(u,0) = -\kappa_t\kappa_s'\boldsymbol{e}(u)/\delta_{\nu 2}$.

(AII-2-2) *Suppose that* $\delta_o(u) = 0, \delta_{\nu 1}(u) = 0, \delta_{\nu 2}(u) = 0$. *Then*

(A5"') $g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u) = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)''' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)'''' = 0$ *if and only if* $\kappa_s = \kappa_\nu = \kappa_s' = 0$ *and there exists* $l(u)$ *such that* $x_0 - f(u,0) = l(u)\boldsymbol{e}(u)$. *We remark that under this condition,* $\delta_{\nu 2} = \kappa_t\kappa_s'' - \kappa_\nu'''$.

(B) *Suppose that* $\kappa_s(u)^2 + \kappa_t(u)^2 \neq 0$. *Then*

(B1) $g_{\boldsymbol{b},\boldsymbol{x}_0}(u) = 0$ *if and only if there exist* $\alpha(u)$ *and* $\beta(u)$ *such that*

$$\boldsymbol{x}_0 - f(u,0) = \alpha\boldsymbol{e}(u) + \beta\boldsymbol{\nu}(u).$$

(B2) $g_{\boldsymbol{b},\boldsymbol{x}_0}(u) = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)' = 0$ *if and only if there exists* $l(u)$ *such that*

$$\boldsymbol{x}_0 - f(u,0) = l(u)\overline{D}_r(u).$$

(BI) *Suppose that* $\delta_n(u) \neq 0$. *Then*

(B3) $g_{\boldsymbol{b},\boldsymbol{x}_0}(u) = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'' = 0$ *if and only if*

$$x_0 - f(u,0) = \frac{-\kappa_s}{\delta_n}D_r(u). \tag{A.2}$$

(B4) $g_{\boldsymbol{b},\boldsymbol{x}_0}(u) = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)''' = 0$ *if and only if* (A.2) *and* $\sigma_n(u) = 0$.

(B5) $g_{\boldsymbol{b},\boldsymbol{x}_0}(u) = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)''' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'''' = 0$ *if and only if* (A.2) *and* $\sigma_n(u) = \sigma_n'(u) = 0$.

(BII) *Suppose that* $\delta_n(u) = 0$. *Then*

(B3') $g_{\boldsymbol{b},\boldsymbol{x}_0}(u) = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'' = 0$ *if and only if* $\kappa_s = 0$. *We remark that under this condition,* $\delta_n = \kappa_\nu\kappa_t^2 + \kappa_t\kappa_s'$.

(BII-1) *Set* $\delta_{b1} = \kappa_t\kappa_\nu' + 2\kappa_\nu\kappa_t' + \kappa_s''$ *and suppose that* $\delta_n(u) = 0, \delta_{b1}(u) \neq 0$. *Then*

(B4') $g_{\boldsymbol{b},\boldsymbol{x}_0}(u) = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)''' = 0$ *if and only if* $\kappa_s = 0$ *and* $x_0 - f(u,0) = -\kappa_\nu\kappa_t\boldsymbol{e}/\delta_{b1}$.

(B5') $g_{\boldsymbol{b},\boldsymbol{x}_0}(u) = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)''' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'''' = 0$ *if and only if* $\kappa_s = 0$, $x_0 - f(u,0) = -\kappa_\nu \kappa_t \boldsymbol{e}/\delta_{b1}$ *and*

$$2\kappa_\nu^4\kappa_t^2 + (\kappa_t\kappa_\nu' + \kappa_s'')(2\kappa_t\kappa_\nu + 3\kappa_s'') + 3\kappa_\nu^2(2(\kappa_t')^2 + \kappa_t\kappa_t'') + \kappa_\nu\big(9\kappa_t'\kappa_s'' + \kappa_t^2\kappa_\nu'' - \kappa_t(-10\kappa_\nu'\kappa_t' - \kappa_s''')\big) = 0.$$

(BII-2) *Suppose that* $\delta_n(u) = 0, \delta_{b1}(u) = 0$. *Then*

(B4") $g_{\boldsymbol{b},\boldsymbol{x}_0}(u) = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)''' = 0$ *if and only if* $\kappa_s(u) = \kappa_\nu(u) = 0$. *We remark that under this condition,* $\delta_{b1} = \kappa_t\kappa_\nu' + \kappa_s''$.

(BII-2-1) *Set* $\delta_{b2} = 3\kappa_\nu'\kappa_t' + \kappa_t\kappa_\nu'' + \kappa_s'''$, *and suppose that* $\delta_n(u) = 0, \delta_{b1}(u) = 0$, $\delta_{b2}(u) \neq 0$. *Then*

(B5") $g_{\boldsymbol{b},\boldsymbol{x}_0}(u) = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)''' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'''' = 0$ *if and only if* $\kappa_s = \kappa_\nu = 0$ *and* $x_0 - f(u,0) = -\kappa_\nu'\kappa_t\boldsymbol{e}(u)/\delta_{b2}$.

(BII-2-2) *Suppose that* $\delta_n(u) = 0, \delta_{b1}(u) = 0, \delta_{b2}(u) = 0$. *Then*

(B5"') $g_{\boldsymbol{b},\boldsymbol{x}_0}(u) = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)''' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'''' = 0$ *if and only if* $\kappa_s = \kappa_\nu = \kappa_\nu' = 0$ *and, there exists* $l(u)$ *such that*

$$x_0 - f(u,0) = l(u)\boldsymbol{e}(u).$$

We remark that under this condition, $\delta_{b2} = \kappa_t\kappa_\nu'' + \kappa_s'''$.

If $g_{\boldsymbol{b},\boldsymbol{x}_0}(u) = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'' = 0$, $g_{\boldsymbol{b},\boldsymbol{x}_0}(u)''' \neq 0$ *or*

$$g_{\boldsymbol{b},\boldsymbol{x}_0}(u) = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)''' = 0,$$

$g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'''' = 0$ *hold, then* $G_{\boldsymbol{b}}$ *is a* $\mathcal{K}$*-versal unfolding of* $g_{\boldsymbol{b},\boldsymbol{x}_0}$ *at* $(u, \boldsymbol{x}_0)$.

If $g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u) = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)'' = 0$, $g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)''' \neq 0$ *or*

$$g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u) = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)''' = 0,$$

$g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)'''' \neq 0$ *hold, then* $G_{\boldsymbol{\nu}}$ *is a* $\mathcal{K}$*-versal unfoldings of* $g_{\boldsymbol{\nu},\boldsymbol{x}_0}$ *at* $(u, \boldsymbol{x}_0)$.

See [1] or [10, Appendix] for $\mathcal{K}$-versal unfolding (written as $\mathcal{K}$-versal deformations). Using Proposition A.1, and by some general results for the singularity theory for families of function germs, one can also show Theorems 5.1 and 5.3. Detailed descriptions on general results in the singularity theory are found in the book[2].

On the other hand, the calculations by using support functions are rather complicated comparing with the direct use of the criteria for frontals in the proof of Theorems 5.1 and 5.3. However, one of the advantages of the method using the support functions is that we can clarify the geometric meanings of the singularities from the contact viewpoint. Let $\boldsymbol{\Gamma} : I \longrightarrow \mathbb{R}^3 \times S^2$ be a regular curve and $F : \mathbb{R}^3 \times S^2 \longrightarrow \mathbb{R}$ a submersion. We say that $\boldsymbol{\Gamma}$ and $F^{-1}(0)$ have contact of *at least order* k for $t = t_0$ if the function $g(t) = F \circ \boldsymbol{\Gamma}(t)$ satisfies $g(t_0) = g'(t_0) = \cdots = g^{(k)}(t_0) = 0$. If $\boldsymbol{\gamma}$ and $F^{-1}(0)$ have contact of at least order k for $t = t_0$ and satisfy the condition that $g^{(k+1)}(t_0) \neq 0$, then we say that $\boldsymbol{\Gamma}$ and $F^{-1}(0)$ have *contact of order* k for $t = t_0$. For any $\boldsymbol{x} \in \mathbb{R}^3$, we define a function $\mathfrak{g}_{\boldsymbol{x}} : \mathbb{R}^3 \times S^2 \longrightarrow \mathbb{R}$ by $\mathfrak{g}_{\boldsymbol{x}}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{x} - \boldsymbol{u}, \boldsymbol{v} \rangle$. Then we have

$$\mathfrak{g}_{\boldsymbol{x}}^{-1}(0) = \{(\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{R}^3 \times S^2 \ |\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{x}, \boldsymbol{v} \rangle\}.$$

If we fix $\boldsymbol{v} \in S^2$, then $\mathfrak{g}_{\boldsymbol{x}}^{-1}(0)|\mathbb{R}^3 \times \{\boldsymbol{v}\}$ is an affine plane defined by $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = c$, where $c = \langle \boldsymbol{x}, \boldsymbol{v} \rangle$. Since this plane is orthogonal to $\boldsymbol{v}$, it is parallel to the tangent plane $T_{\boldsymbol{v}}S^2$ at $\boldsymbol{v}$. Here we have a representation of the tangent bundle of S^2 as follows:

$$TS^2 = \{(\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{R}^3 \times S^2 \ |\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 1\}.$$

We consider the canonical projection $\pi_2|\mathfrak{g}_{\boldsymbol{x}}^{-1}(0) : \mathfrak{g}_{\boldsymbol{x}}^{-1}(0) \longrightarrow S^2$, where $\pi_2 : \mathbb{R}^3 \times S^2 \longrightarrow S^2$. Then $\pi_2|\mathfrak{g}_{\boldsymbol{x}}^{-1}(0) : \mathfrak{g}_{\boldsymbol{x}}^{-1}(0) \longrightarrow S^2$ is a plane bundle over S^2. Moreover, we define a map

$$\Psi : \mathfrak{g}_{\boldsymbol{x}}^{-1}(0) \longrightarrow TS^2$$

by $\Phi(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{u}/\langle \boldsymbol{x},\boldsymbol{v}\rangle,\boldsymbol{v})$. Then Φ is a bundle isomorphism. Therefore, we write $TS^2(\boldsymbol{x}) = \mathfrak{g}_{\boldsymbol{x}}^{-1}(0)$ and call it an *affine tangent bundle over S^2 through $\boldsymbol{x}$*. With the same notations as above, we distinguish two cases.

(A) Suppose that $(\kappa_\nu(u),\kappa_t(u)) \neq (0,0)$ and $\delta_o(u) \neq 0$. We consider

$$\boldsymbol{s}_{OD}(u) = f(u,0) - \frac{\kappa_\nu(u)}{\delta_o(u)}\overline{D}_o(u).$$

By (5.3), we have

$$\boldsymbol{s}'_{OD}(u) = \frac{\sigma_o(u)}{\delta_o(u)}(\kappa_t(u)\boldsymbol{e}(u) - \kappa_\nu(u)\boldsymbol{b}(u)).$$

If we assume that $\sigma_o(u) \equiv 0$, then $\boldsymbol{s}_{OD}$ is a constant vector $\boldsymbol{x}_0$. Then

$$f(u,0) - \boldsymbol{x}_0 = \frac{\kappa_\nu(u)}{\delta_o(s)}\overline{D}_o(u)$$

Therefore

$$\mathfrak{g}_{\boldsymbol{x}_0}(f(u,0),\boldsymbol{\nu}(u)) = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(s) = \langle \boldsymbol{x}_0 - f(u,0),\boldsymbol{\nu}(u)\rangle = 0.$$

If there exists $\boldsymbol{x}_0 \in \mathbb{R}^3$ such that $\mathfrak{g}_{\boldsymbol{x}_0}(f(u,0),\boldsymbol{\nu}(u)) = 0$, then we have

$$f(u,0) - \boldsymbol{x}_0 = \frac{\kappa_\nu(u)}{\delta_o(s)}\overline{D}_o(u).$$

and $\sigma_o(u) \equiv 0$. We consider a regular curve $(f|_{S(f)},\boldsymbol{\nu}) : I \longrightarrow \mathbb{R}^3 \times S^2$.

(B) Suppose that $(\kappa_s(u),\kappa_t(u)) \neq (0,0)$ and $\delta_n(u) \neq 0$. Then we have similar results to case (A), so that we have the following proposition.

Proposition A.2. *With the same notations as above, we have the following*:
(A) *Suppose that $(\kappa_\nu(u),\kappa_t(u) \neq (0,0)$ and $\delta_o(u) \neq 0$. Then there exists $\boldsymbol{x}_0 \in \mathbb{R}^3$ such that $(f|_{S(f)},\boldsymbol{\nu})(I) \subset TS^2(\boldsymbol{x}_0)$ if and only if $\sigma_o \equiv 0$.*
(B) *Suppose that $(\kappa_s(u),\kappa_t(u) \neq (0,0)$ and $\delta_n(u) \neq 0$. Then there exists $\boldsymbol{x}_0 \in \mathbb{R}^3$ such that $(f|_{S(f)},\boldsymbol{b})(I) \subset TS^2(\boldsymbol{x}_0)$ if and only if $\sigma_n \equiv 0$.*

The results of Proposition A.1 can be interpreted from the contact viewpoint as follows.

Proposition A.3. *With the same notations as above, we have the following*:
(A) *Suppose that $(\kappa_\nu(u),\kappa_t(u) \neq (0,0)$ and $\delta_o(u) \neq 0$. For $\boldsymbol{x}_0 = OD_f(u_0,t_0)$, we have the following*:
(1) *The order of contact of $(f|_{S(f)},\boldsymbol{\nu})$ with $TS^2(\boldsymbol{x}_0)$ at $u = u_0$ is two if and only if*

$$t_0 = -\frac{\kappa_\nu(u_0)}{\delta_o(u_0)} \tag{A.3}$$

and $\sigma_o(u_0) \neq 0$. In this case OD_f is a cuspidal edge at (u_0,t_0).
(2) *The order of contact of $(f|_{S(f)},\boldsymbol{\nu})$ with $TS^2(\boldsymbol{x}_0)$ at $u = u_0$ is three if and only if (A.3) and $\sigma_o(u_0) = 0$ and $\sigma'_o(u_0) \neq 0$. In this case OD_f is a swallowtail at (u_0,t_0).*
(B) *Suppose that $(\kappa_s(u),\kappa_t(u) \neq (0,0)$ and $\delta_n(u) \neq 0$. For $\boldsymbol{x}_0 = ND_f(u_0,t_0)$, we have the following*:
(1) *The order of contact of $(f|_{S(f)},\boldsymbol{b})$ with $TS^2(\boldsymbol{x}_0)$ at $u = u_0$ is two if and only if*

$$t_0 = -\frac{\kappa_s(u_0)}{\delta_n(u_0)} \tag{A.4}$$

and $\sigma_n(u_0) \neq 0$. In this case ND_f is a cuspidal edge at (u_0,t_0).
(2) *The order of contact of $(f|_{S(f)},\boldsymbol{b})$ with $TS^2(\boldsymbol{x}_0)$ at $u = u_0$ is three if and only if (A.4) and $\sigma_n(u_0) = 0$ and $\sigma'_n(u_0) \neq 0$. In this case ND_f is a swallowtail at (u_0,t_0).*

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Shyuichi Izumiya, Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan
E-mail address: `izumiya@math.sci.hokudai.ac.jp`

Kentaro Saji, Department of Mathematics, Kobe University, Rokko 1-1, Nada, Kobe 657-8501, Japan
E-mail address: `saji@math.kobe-u.ac.jp`

Nobuko Takeuchi, Department of Mathematics, Tokyo Gakugei University, Koganei, Tokyo, 184-8501, Japan
E-mail address: `nobuko@u-gakugei.ac.jp`

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A REMARK ON THE IRREGULARITY COMPLEX

CLAUDE SABBAH

Abstract. We prove that, for a good meromorphic flat bundle with poles along a divisor with normal crossings, the restriction of the irregularity complex to each natural stratum of this divisor only depends on the formal flat bundle along this stratum. This answers a question raised by J.-B. Teyssier.

1. Statement of the results

Let X be a complex manifold of dimension n and let $D=\bigcup_{i\in J}D_i$ be a divisor with normal crossings. We assume that each irreducible component D_i of D is smooth. For any subset $I\subset J$, we set $D_I=\bigcap_{i\in I}D_i$ and $D_I^\circ=D_I\smallsetminus\bigcup_{j\notin I}D_j$. We denote the codimension of D_I° by ℓ, that we regard as a locally constant function on D_I° (which can have many connected components), and by $\iota_I:D_I^\circ\hookrightarrow D$ the inclusion. Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module such that

(1) $\mathscr{M}=\mathscr{M}(*D)$,
(2) $\mathscr{M}_{X\smallsetminus D}$ is locally $\mathscr{O}_X$-free of finite rank.

We then say that $\mathscr{M}$ is a meromorphic flat bundle with poles along D. *In this note, we assume that $\mathscr{M}$ has a good formal structure along D* (we simply say that $\mathscr{M}$ is a *good D-meromorphic flat bundle*, or a *good meromorphic flat bundle on* (X,D)). This notion, together with the Riemann-Hilbert correspondence, will be recalled in Section 2. Recall also that, given any meromorphic flat bundle on (X',D') (where D' is an arbitrary reduced divisor in X'), there exists, locally on X', a projective modification $X\to X'$ such that the pullback of D' by this modification is a divisor with simple normal crossings D and the pullback meromorphic flat bundle is a D-meromorphic flat bundle having a good formal structure along D (see [Ked10, Ked11], and [Moc09, Moc11a] in the algebraic case; see also [Sab00] for special cases when $\dim X=2$).

For every $I\subset J$, we consider the sheaf $\mathscr{O}_{\widehat{X|D_I^\circ}}$ on D_I°, also denoted by $\mathscr{O}_{\widehat{D_I^\circ}}$, defined as the formalization of $\mathscr{O}_X$ along D_I°. We also regard it as a sheaf on X by extending it by zero. We then set $\mathscr{D}_{\widehat{D_I^\circ}}=\mathscr{O}_{\widehat{D_I^\circ}}\otimes_{\mathscr{O}_X}\mathscr{D}_X$, and $\mathscr{M}_{\widehat{D_I^\circ}}:=\mathscr{D}_{\widehat{D_I^\circ}}\otimes_{\mathscr{D}_X}\mathscr{M}$.

For any holonomic $\mathscr{D}_X$-module $\mathscr{N}$, the irregularity complexes $\mathrm{Irr}_D\,\mathscr{N}$ and $\mathrm{Irr}^*_D\,\mathscr{N}$, as defined by Mebkhout [Meb90], are constructible complexes supported on D, and only depend on $\mathscr{N}(*D)$. For a *good* D-meromorphic flat bundle $\mathscr{M}$ as above, the cohomology of $\mathrm{Irr}_D\,\mathscr{M}$ and $\mathrm{Irr}^*_D\,\mathscr{M}$ is locally constant along each stratum D_I°: this follows from [Tey13, Th. 12.2.7] if $\#I=1$ and from Corollary 3.4 together with the case $\#I=1$ otherwise. On the other hand, Mebkhout has shown that the complexes $\mathrm{Irr}_D\,\mathscr{M}[\dim X]$, $\mathrm{Irr}^*_D\,\mathscr{M}[\dim X]$ are a perverse sheaves (see loc. cit.).

Our aim in this note is to compare the irregularity complexes of $\mathscr{M}$ restricted to D_I° and those of the formalized module $\mathscr{M}_{\widehat{D_I^\circ}}$. However, the irregularity complexes of $\mathscr{M}_{\widehat{D_I^\circ}}$ are not defined by

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the procedure of [Meb90]. To give a meaning to the question, we start by proving in Section 2.f the following proposition.

Proposition 1.1. *For every $I\subset J$, there exists a unique good D-meromorphic flat bundle $\mathscr{M}_I^\circ$ in the neighbourhood of D_I° which satisfies the following two properties.*

1. $\mathscr{D}_{\widehat{D_I^\circ}}\otimes_{\mathscr{D}_X}\mathscr{M}_I^\circ\simeq\mathscr{M}_{\widehat{D_I^\circ}}$.
2. *At each point of D_I°, the local formal decomposition of $\mathscr{M}_I^\circ$ (after a local ramification around D) into elementary formal D-meromorphic flat bundles already holds without taking formalization.*

The main result of this note can now be stated.

Theorem 1.2. *For every $I\subset J$, we have*

$$\iota_I^{-1}\operatorname{Irr}_D\mathscr{M}\simeq\iota_I^{-1}\operatorname{Irr}_D(\mathscr{M}_I^\circ),\quad\textit{and}\quad\iota_I^{-1}\operatorname{Irr}_D^*\mathscr{M}\simeq\iota_I^{-1}\operatorname{Irr}_D^*(\mathscr{M}_I^\circ).$$

In other words, the complexes $\iota_I^{-1}\operatorname{Irr}_D\mathscr{M},\iota_I^{-1}\operatorname{Irr}_D^*\mathscr{M}$ only depend (up to isomorphism) on the formalization $\mathscr{M}_{\widehat{D_I^\circ}}$ of $\mathscr{M}$ along D_I°.

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2. Good formal structure and the Riemann-Hilbert correspondence

2.a. **Notation.** We keep the notation of the introduction. If Z is any locally closed analytic subspace of the complex analytic manifold X, we denote by $\mathscr{O}_{\widehat{Z}}$, the formal completion of $\mathscr{O}_X$ with respect to the ideal sheaf $\mathscr{I}_Z$. We regard $\mathscr{O}_{\widehat{Z}}$ as a sheaf on Z.

Given $x_o\in D$, there exists a unique $I\subset J$ such that $x_o\in D_I^\circ$, and we will be mostly interested in the case where Z is the point $x_o\in D$ and the case where Z is equal to D_I°. We will denote by $\mathscr{O}_{\widehat{Z}}(*D)$ the sheaf $\mathscr{O}_{X|Z}(*D)\otimes_{\mathscr{O}_{X|Z}}\mathscr{O}_{\widehat{Z}}$, where as usual $\mathscr{O}_{X|Z}$ (resp. $\mathscr{O}_{X|Z}(*D)$) denotes the sheaf-theoretic restriction to Z of the sheaf $\mathscr{O}_X$ of holomorphic functions on X (resp. the sheaf $\mathscr{O}_X(*D)$ of meromorphic functions on X with poles at most on D).

If φ (resp. $\widehat{\varphi}$) is a section of $\mathscr{O}_X(*D)$ (resp. of $\mathscr{O}_{\widehat{Z}}(*D)$), we denote by $\mathscr{E}^\varphi$ (resp. $\mathscr{E}^{\widehat{\varphi}}$) the module with connection $(\mathscr{O}_X(*D),\mathrm{d}+\mathrm{d}\varphi)$ (resp. $(\mathscr{O}_{\widehat{Z}}(*D),\mathrm{d}+\widehat{\varphi})$). It only depends on the class, also denoted by φ (resp. $\widehat{\varphi}$), of φ (resp. $\widehat{\varphi}$) modulo $\mathscr{O}_X$ (resp. $\mathscr{O}_{\widehat{Z}}$).

2.b. **Good formal structure.** We say that the D-meromorphic flat bundle $\mathscr{M}$ has a *good formal structure* if, for any $x_o\in D$, there exists a local ramification $\rho_{\boldsymbol{d}_I}$ of multi-degree $\boldsymbol{d}_I$ around the branches $(D_i)_{i\in I}$ passing through x_o (hence inducing an isomorphism above D_I° in the neighbourhood of x_o) such that the pullback of the formal flat bundle $\mathscr{M}_{\widehat{x_o}}:=\mathscr{O}_{\widehat{x}_o}\otimes_{\mathscr{O}_{X,x_o}}\mathscr{M}_{x_o}$ by this ramification decomposes as the direct sum of formal elementary D-meromorphic connections $\mathscr{E}^{\widehat{\varphi}}\otimes\widehat{\mathscr{R}}_{\widehat{\varphi}}$, as defined below.

We denote by $\mathrm{nb}(x_o)$ a small open neighbourhood of x_o in X above which the ramification is defined, and we denote by x'_o the pre-image of x_o, so the ramification is a finite morphism $\rho_{\boldsymbol{d}_I}:\mathrm{nb}(x'_o)\to\mathrm{nb}(x_o)$. It induces a one-to-one map above $D_I^\circ\cap\mathrm{nb}(x_o)$. We also set

$$D'=\rho_{\boldsymbol{d}_I}^{-1}(D\cap\mathrm{nb}(x_o)),$$

so that D'_I maps isomorphically to $D_I\cap\mathrm{nb}(x_o)=D_I^\circ\cap\mathrm{nb}(x_o)$.

In the above decomposition, $\widehat{\varphi}$ varies in a *good* finite subset $\widehat{\Phi}_{x_o} \subset \mathscr{O}_{\widehat{x'_o}}(*D')/\mathscr{O}_{\widehat{x'_o}}$ and $\widehat{\mathscr{R}}_{\widehat{\varphi}}$ is a free $\mathscr{O}_{\widehat{D'_I}}(*D')$-module with an integrable connection having a regular singularity along D'. In other words, we do not distinguish between $\widehat{\varphi}$ and $\widehat{\psi}$ in $\mathscr{O}_{\widehat{x'_o}}(*D')$ if their difference has no poles along D'. Goodness means here that for any pair $\widehat{\varphi} \neq \widehat{\psi} \in \widehat{\Phi}_{x_o} \cup \{0\}$, the difference $\widehat{\varphi} - \widehat{\psi}$ can be written as $x^{-\boldsymbol{m}}\widehat{\eta}(x)$, with $\boldsymbol{m} \in \mathbb{N}^{\#I}$ and $\widehat{\eta} \in \mathscr{O}_{\widehat{x'_o}}$ satisfying $\widehat{\eta}(0) \neq 0$ (see [Sab00, §I.2.1].[1] By [Ked11, Prop. 4.4.1 & Def. 5.1.1] (see also [Sab00, §I.2.4] and [Moc11b, Prop. 2.19]), the $\widehat{\varphi}$'s are convergent, i.e., the set $\widehat{\Phi}_{x_o}$ is the formalization at x_o of a finite subset

$$\Phi_{x_o} \subset \Gamma(\mathrm{nb}(x'_o), \mathscr{O}_{\mathrm{nb}(x'_o)}(*D')/\mathscr{O}_{\mathrm{nb}(x'_o)}),$$

and the decomposition extends in a neighbourhood of x'_o, that is, it holds for the pullback by $\rho_{\boldsymbol{d}_I}$ of $\mathscr{M}_{\widehat{D^\circ_I},x_o}$ and induces the original one after taking formalization at x'_o.[2]

2.c. **Stratified $\mathcal{I}$-covering.** The set $\bigsqcup_{x_o \in D^\circ_I}(\Phi_{x_o} \cup \{0\})$ has a natural structure of a finite non-ramified covering of D°_I (in particular, it is a Hausdorff topological space), that we denote by $\Sigma^\circ_I \to D^\circ_I$. Locally, it is described as follows. Given a germ $\varphi_{x'_o} \in \Phi_{x_o} \cup \{0\}$, it extends locally as a section of $\mathscr{O}_{\mathrm{nb}(x'_o)}(*D')/\mathscr{O}_{\mathrm{nb}(x'_o)}$ and thus defines a germ in $\Phi_{y_o} \cup \{0\}$ for any $y_o \in D^\circ_I \cap \mathrm{nb}(x_o)$. This defines the local branch of Σ°_I passing through $\varphi_{x'_o}$. (This construction is nothing but that of the sheaf space, or étalé space, of a sheaf.)

By a similar procedure, the set $\Sigma(\mathscr{M}) := \bigsqcup_I \Sigma_I$ can be endowed with a natural topology as a sheaf space, but the topology can be non-Hausdorff: this occurs if some difference $\varphi_{x'_o} - \psi_{x'_o}$ does not have poles along *all* the components of D' passing through x'_o.

In order to state the Riemann-Hilbert correspondence, we will lift these objects to the *real oriented blowing-up* $\varpi : \widetilde{X} := \widetilde{X}(D_{i\in J}) \to X$ along the components D_i of D in X. We set $\partial\widetilde{X} := \varpi^{-1}(D)$ and $\partial\widetilde{X}^\circ_I := \varpi^{-1}(D^\circ_I)$. The fibre of ϖ over a point in D°_I is diffeomorphic to $(S^1)^\ell$, making $\partial\widetilde{X}^\circ_I$ a $(S^1)^\ell$-bundle on D°_I. We consider the sheaf $\mathcal{I}$ on $\partial\widetilde{X}$ as constructed in [Sab13, §9.3].

By considering the fiber product

$$\begin{array}{ccc} \widetilde{\Sigma}^\circ_I & \longrightarrow & \Sigma^\circ_I \\ \downarrow & & \downarrow \\ \partial\widetilde{X}^\circ_I & \xrightarrow{\varpi} & D^\circ_I \end{array}$$

we obtain a finite covering $\widetilde{\Sigma}^\circ_I$ of $\partial\widetilde{X}^\circ_I$ which is naturally contained in the étalé space $\mathcal{I}^{\text{ét}}$ of $\mathcal{I}$. By a similar procedure, we get a *good stratified $\mathcal{I}$-covering* $\bigsqcup_I \widetilde{\Sigma}^\circ_I =: \widetilde{\Sigma}(\mathscr{M}) \to \partial\widetilde{X}$ of $\partial\widetilde{X}$, in the sense of [Sab13, Rem. 11.12]. As before, $\widetilde{\Sigma}(\mathscr{M})$ can be non-Hausdorff.

2.d. **The Riemann-Hilbert correspondence (local theory).** Let us fix a *good stratified $\mathcal{I}$-covering* $\widetilde{\Sigma}$. Let $x_o \in D^\circ_I$. The local Riemann-Hilbert correspondence ([Moc11a, Moc11b], [Sab13]) is an equivalence between the category of germs at x_o of good D-meromorphic flat bundles $\mathscr{M}_{x_o}$ with stratified $\mathcal{I}$-covering $\widetilde{\Sigma}(\mathscr{M})$ contained in $\widetilde{\Sigma}$, and that of germs at $\varpi^{-1}(x_o)$ of good Stokes-filtered local systems $(\mathscr{L}^\circ_I, \mathscr{L}^\circ_{I,\bullet})$ on $\partial\widetilde{X}^\circ_I$ (see e.g. [Sab13, §9.5]) with $\mathcal{I}$-covering contained in $\widetilde{\Sigma}^\circ_I$ (see [Moc11b, Th. 4.11] and [Sab13, Th. 12.16]).

[1] Note that, here, the goodness condition is assumed for $\widehat{\Phi} \cup \{0\}$ and not only for $\widehat{\Phi}$, because of [Sab13, Cor. 12.7]. This is unfortunately not made precise in [Sab13, Th. 12.16] and should be corrected.

[2] I thank J.-B. Teyssier for pointing this out to me. In [Moc11a, Moc11b] (see also [Sab13, §11.3]), this is shown to hold only if one assumes the good formal structure at all points of $D^\circ_I \cap \mathrm{nb}(x_o)$.

More precisely, we have a commutative diagram of functors

$$
\begin{tikzcd}
\mathscr{M}_{x_o} \arrow[r, mapsto, "\sim"] \arrow[d, mapsto] & (\mathscr{L}_I^\circ, \mathscr{L}_{I,\bullet}^\circ)_{\varpi^{-1}(x_o)} \arrow[d, mapsto, "\mathrm{gr}"] \\
\mathscr{M}_{\widehat{D_I^\circ}, x_o} \arrow[r, mapsto, "\sim"] & (\mathrm{gr}\,\mathscr{L}_I^\circ, \mathrm{gr}\,\mathscr{L}_{I,\bullet}^\circ)_{\varpi^{-1}(x_o)}
\end{tikzcd}
\tag{2.1}
$$

similar to that of [Mal91, p. 58], where gr means grading with respect to the Stokes filtration and the horizontal functors are equivalences of categories. Recall that grading a Stokes-filtered local system is well-defined only when one restricts to $\widetilde{\Sigma}_I^\circ$, which is Hausdorff (see [Sab13, Chap. 1]). In order to give a meaning to grading in general, one needs to control the extension from D_I° to a small neighbourhood $\mathrm{nb}(D_I^\circ)$. Locally, this is provided by the following equivalence.

Proposition 2.2 (see [Moc11b, Lem. 3.17]). *The restriction functor to $\partial\widetilde{X}_I^\circ$ induces an equivalence between the category of germs at $\varpi^{-1}(x_o)$ of Stokes-filtered local systems $(\mathscr{L}, \mathscr{L}_\bullet)$ on $\partial\widetilde{X}$ with associated stratified $\mathfrak{I}$-covering contained in $\widetilde{\Sigma}$ and the category of germs at $\varpi^{-1}(x_o)$ of Stokes-filtered local systems $(\mathscr{L}_I^\circ, \mathscr{L}_{I,\bullet}^\circ)$ on $\partial\widetilde{X}_I^\circ$ with associated $\mathfrak{I}$-covering contained in $\widetilde{\Sigma}_I^\circ$.*

2.e. **The Riemann-Hilbert correspondence (global theory).** We now consider the previous correspondence all along D_I°. We consider a covering $\mathscr{U}$ of D_I° by open subsets U_α which are the intersection of D_I° with a local chart on X. Any germ $\mathscr{M}$ of D-meromorphic flat bundle along D_I° gives rise to gluing data $((\mathscr{M}_\alpha), (\sigma_{\alpha\beta}))$, where

$$\mathscr{M}_\alpha = \mathscr{M}_{|U_\alpha}, \quad \sigma_{\alpha\beta} : \mathscr{M}_{\alpha|U_\alpha\cap U_\beta} \longrightarrow \mathscr{M}_{\beta|U_\alpha\cap U_\beta}$$

is an isomorphism, and the family $(\sigma_{\alpha\beta})$ satisfies the cocycle property. Any germ $\mathscr{M}$ of good D-meromorphic flat bundle along D_I° admits a covering $\mathscr{U}$ such that one can apply the local Riemann-Hilbert correspondence of Section 2.d to its restriction $\mathscr{M}_\alpha$ to every U_α. Given such a covering $\mathscr{U}$, we can consider the category of such good gluing data $\big((\mathscr{M}_\alpha), (\sigma_{\alpha\beta})\big)$. The local Riemann-Hilbert correspondence gives rise to a commutative diagram of functors between gluing data

$$
\begin{tikzcd}
\big((\mathscr{M}_\alpha), (\sigma_{\alpha\beta})\big) \arrow[r, mapsto] \arrow[d, mapsto] & \big((\mathscr{L}_I^\circ, \mathscr{L}_{I,\bullet}^\circ)_\alpha, (\eta_{\alpha\beta})\big) \arrow[d, mapsto, "\mathrm{gr}"] \\
\big((\mathscr{M}_{\alpha, \widehat{D_I^\circ}}), (\widehat{\sigma}_{\alpha\beta})\big) \arrow[r, mapsto] & \big((\mathrm{gr}\,\mathscr{L}_I^\circ, \mathrm{gr}\,\mathscr{L}_{I,\bullet}^\circ)_\alpha, (\mathrm{gr}\,\eta_{\alpha\beta})\big)
\end{tikzcd}
\tag{2.3}
$$

and the horizontal functors remain equivalences, due to the full faithfulness of the horizontal functors in (2.1).

Arguing similarly with the equivalence of Proposition 2.2, we obtain the Riemann-Hilbert correspondence.

Theorem 2.4. *The category $\mathsf{Mod}_{\mathrm{hol}}\big((X, D_I^\circ), D, \widetilde{\Sigma}\big)$ of germs along D_I° of good D-meromorphic flat bundles with stratified $\mathfrak{I}$-covering contained in $\widetilde{\Sigma}$ is equivalent to that of germs along $\partial\widetilde{X}_I^\circ$ of Stokes-filtered local systems $(\mathscr{L}, \mathscr{L}_\bullet)$ on $\partial\widetilde{X}$ with associated stratified $\mathfrak{I}$-covering contained in $\widetilde{\Sigma}$ and, by restriction, to that of Stokes-filtered local systems $(\mathscr{L}_I^\circ, \mathscr{L}_{I,\bullet}^\circ)$ on $\partial\widetilde{X}_I^\circ$ with associated $\mathfrak{I}$-covering contained in $\widetilde{\Sigma}_I^\circ$.* $\square$

2.f. **Proof of Proposition 1.1.** By Theorem 2.4, there exists a germ $\mathscr{M}_I^\circ$ along D_I° of good D-meromorphic flat bundle whose associated Stokes-filtered local system is $(\operatorname{gr}\mathscr{L}_I^\circ, \operatorname{gr}\mathscr{L}_{I,\bullet}^\circ)$, and it is unique up to isomorphism with respect to this property. A covering $\mathscr{U}$ adapted to $\mathscr{M}$ is also adapted to $\mathscr{M}_I^\circ$, and the diagram (2.3) shows that the gluing data of $\mathscr{M}_{\widehat{D}_I^\circ}$ and of $\mathscr{M}^\circ_{I,\widehat{D}_I^\circ}$ are isomorphic, since they correspond to the same Stokes-filtered gluing data $\big((\operatorname{gr}\mathscr{L}_I^\circ, \operatorname{gr}\mathscr{L}_{I,\bullet}^\circ)_\alpha, (\operatorname{gr}\eta_{\alpha\beta})\big)$. The uniqueness of $\mathscr{M}_I^\circ$ is proved similarly. $\square$

Remark 2.5. The construction of $\mathscr{M}_I^\circ$ is functorial with respect to $\mathscr{M}_{|D_I^\circ}$.

2.g. **An equivalence of categories.** Let A be a category and let G be a group. The category G-A is the category whose objects are G-objects of A, that is, pairs (M,ρ) where M is an object of A and ρ is a morphism $G\to \operatorname{Aut}(M)$, and for which

$$\operatorname{Hom}_{G\text{-}\mathsf{A}}((M,\rho_M),(N,\rho_N))\subset \operatorname{Hom}_{\mathsf{A}}(M,N)$$

is the subset consisting of morphisms $\varphi: M\to N$ such that, for every $g\in G$, $\varphi\circ\rho_M(g)=\rho_N(g)$.

Let $\widetilde{\Sigma}\to\partial\widetilde{X}$ be a good stratified $\mathcal{I}$-covering and let $\mathsf{Mod}_{\mathrm{hol}}(X,D,\widetilde{\Sigma})$ denote the full subcategory of that of holonomic $\mathscr{D}_X$-modules whose objects consist of good meromorphic flat bundles on (X,D) with associated stratified $\mathcal{I}$-covering contained in $\widetilde{\Sigma}$.

Let us fix a nonempty subset $I\subset J$, let D_I° the corresponding stratum of D, let $x_o\in D_I^\circ$ and let $D_I^\circ(x_o)$ the connected component of D_I° containing x_o. Let us fix a local holomorphic decomposition

$$\big(\operatorname{nb}(x_o,X),\operatorname{nb}(x_o,D)\big)=(\Omega,D_\Omega)\times\operatorname{nb}(x_o,D_I^\circ),$$

where Ω is an open neighbourhood of 0 in $\mathbb{C}^\ell$ and D_Ω is the union of the coordinate hyperplanes in $\mathbb{C}^\ell$. The category $\mathsf{Mod}_{\mathrm{hol}}\big((X,D_I^\circ(x_o)),D,\widetilde{\Sigma}\big)$ has been defined in Section 2, and we have the similar category $\mathsf{Mod}_{\mathrm{hol}}((\Omega,0),D_\Omega,\widetilde{\Sigma}_{x_o})$, where $\widetilde{\Sigma}_{x_o}$ is the restriction of $\widetilde{\Sigma}$ above

$$\partial\widetilde{\Omega}:=\varpi^{-1}(D_\Omega).$$

Theorem 2.6. *Set $G=\pi_1\big(D_I^\circ(x_o),x_o\big)$. There is a natural equivalence of categories:*

$$\mathsf{Mod}_{\mathrm{hol}}\big((X,D_I^\circ(x_o)),D,\widetilde{\Sigma}\big)\simeq G\text{-}\mathsf{Mod}_{\mathrm{hol}}\big((\Omega,0),D_\Omega,\widetilde{\Sigma}_{x_o}\big).$$

Proof. We set $\partial\widetilde{X}_I^\circ(x_o):=\varpi^{-1}(D_I^\circ(x_o))$ and we denote similarly by $\widetilde{\Sigma}_I^\circ(x_o)$ the restriction of $\widetilde{\Sigma}$ above this set.

(1) By the Riemann-Hilbert correspondence (Theorem 2.4), we can replace the category on the left-hand side with that of Stokes-filtered local systems on $\partial\widetilde{X}_I^\circ(x_o)$ with associated $\mathcal{I}$-covering contained in $\widetilde{\Sigma}_I^\circ(x_o)$.

(2) Let $\pi:(E_I^\circ(x_o),y_o)\to(D_I^\circ(x_o),x_o)$ be a universal covering of $D_I^\circ(x_o)$ with base-point y_o above x_o and let $G=\operatorname{Gal}(\pi)$ be the corresponding Galois group. We consider the fibre-product diagram

$$\begin{tikzcd}
\partial\widetilde{Y}_I^\circ(x_o) \arrow[r] \arrow[d] & \partial\widetilde{X}_I^\circ(x_o) \arrow[d, "\varpi"] \\
(E_I^\circ(x_o),y_o) \arrow[r, "\pi"] & (D_I^\circ(x_o),x_o)
\end{tikzcd}$$

and we denote by $\pi^{-1}\widetilde{\Sigma}_I^\circ(x_o)$ the corresponding pullback $\pi^{-1}\mathcal{I}$-covering of $\partial\widetilde{Y}_I^\circ(x_o)$. Then the category considered in (1) is equivalent to the category of G-Stokes-filtered local systems $(\mathscr{L}_I^\circ,\mathscr{L}_{I,\bullet}^\circ)$ on $\partial\widetilde{Y}_I^\circ(x_o)$ with associated $\pi^{-1}\mathcal{I}$-covering contained in $\pi^{-1}\widetilde{\Sigma}_I^\circ(x_o)$. This is a standard argument.

(3) (See [Moc11b, Th. 4.13] and Remark A.11) The sheaf-theoretic restriction functor is an equivalence from the latter category to the category of G-Stokes-filtered local systems $(\mathscr{L}, \mathscr{L}_\bullet)$ on $(\partial\widetilde{\Omega})_0 \simeq (S^1)^\ell$ with associated $\mathcal{I}_{x_o}$-covering contained in $\widetilde{\Sigma}_{x_o}$ (we identify here $(\pi^{-1}\mathcal{I})_{y_o}$ with $\mathcal{I}_{x_o}$ and $\pi^{-1}\widetilde{\Sigma}_I^\circ(x_o)_{y_o}$ with $\widetilde{\Sigma}_{x_o}$). This proof will be reviewed in the appendix.

(4) By applying now the G-Riemann-Hilbert correspondence of Theorem 2.4 in the reverse direction to $((\Omega, 0), D_\Omega, \widetilde{\Sigma}_{x_o})$, one ends the proof of the theorem. □

3. The irregularity complex

Our aim in this section is to show that, under the goodness assumption as above, the irregularity complex is determined by its restriction to the smooth part of D. More precisely, for every $I \subset J$, and for every connected component of D_I°, we show that there exists a component D_k of D ($k \in I$) such that $\iota_I^{-1} \operatorname{Irr}_D \mathscr{M}$ (on this connected component) is determined by $\iota_k^{-1} \operatorname{Irr}_D \mathscr{M}$.

Let $(\mathscr{L}, \mathscr{L}_\bullet)$ be the Stokes-filtered local system corresponding to a (germ of) good D-meromorphic flat bundle $\mathscr{M}$. We have $\mathscr{L} = \widetilde{\imath}^{-1} \boldsymbol{R}\widetilde{\jmath}_* \operatorname{DR} \mathscr{M}_{|X \smallsetminus D}$, where

$$\widetilde{\imath} : \partial\widetilde{X} \hookrightarrow \widetilde{X} \quad \text{and} \quad \widetilde{\jmath} : X \smallsetminus D \hookrightarrow \widetilde{X}$$

are the natural closed and open inclusions. Let us denote by $\mathscr{A}_{\widetilde{X}}^{\operatorname{mod} D}$ (resp. $\mathscr{A}_{\widetilde{X}}^{\operatorname{rd} D}$) the sheaf on $\widetilde{X}$ of holomorphic functions on $X \smallsetminus D$ having moderate growth (resp. rapid decay) along $\partial\widetilde{X}$. One can then define the moderate (resp. rapidly decaying) de Rham complex $\operatorname{DR}^{\operatorname{mod} D} \mathscr{M}$ (resp. $\operatorname{DR}^{\operatorname{rd} D} \mathscr{M}$) on $\partial\widetilde{X}$. With the goodness assumption, it is known that both have cohomology in degree zero at most. More precisely, the Riemann-Hilbert correspondence recalled in Section 2.e gives

$$\mathscr{L}_{\leqslant 0} = \mathscr{H}^0 \operatorname{DR}^{\operatorname{mod} D} \mathscr{M} \quad \text{and} \quad \mathscr{H}^j \operatorname{DR}^{\operatorname{mod} D} \mathscr{M} = 0 \text{ for } j \neq 0.$$

We set $\mathscr{L}^{>0} := \mathscr{L}/\mathscr{L}_{\leqslant 0}$, and similarly $\operatorname{DR}^{>\operatorname{mod} D} \mathscr{M}$ is defined as the cone of

$$\operatorname{DR}^{\operatorname{mod} D} \mathscr{M} \longrightarrow \widetilde{\imath}^{-1} \boldsymbol{R}\widetilde{\jmath}_* \operatorname{DR} \mathscr{M}_{|X \smallsetminus D},$$

so that $\mathscr{L}^{>0} = \mathscr{H}^0 \operatorname{DR}^{>\operatorname{mod} D} \mathscr{M}$ (and $\mathscr{H}^k \operatorname{DR}^{>\operatorname{mod} D} \mathscr{M} = 0$ for $k \neq 0$).

Proposition 3.1. *We have* $\operatorname{Irr}_D \mathscr{M}[1] = \boldsymbol{R}\varpi_* \mathscr{L}^{>0}$.

Proof. We have

$$\boldsymbol{R}\varpi_* \operatorname{DR}^{\operatorname{mod} D} \mathscr{M} = \operatorname{DR} \mathscr{M}(*D) \quad \text{and} \quad \boldsymbol{R}\varpi_* \boldsymbol{R}\widetilde{\jmath}_* \operatorname{DR} \mathscr{M}_{|X \smallsetminus D} = \boldsymbol{R}j_* \operatorname{DR} \mathscr{M}_{|X \smallsetminus D},$$

where $j : X \smallsetminus D \hookrightarrow X$ is the inclusion. We then apply [Meb04, Def. 3.4-1]. □

Remark 3.2 (The irregularity complex $\operatorname{Irr}_D^* \mathscr{M}$). Recall that Mebkhout also defined the irregularity complex $\operatorname{Irr}_D^* \mathscr{M}$ in [Meb90] (see also [Meb04, Def. 3.4-2]), which is non-canonically isomorphic to the complex $\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_{X|D}}(\mathscr{M}, \mathscr{Q}_D)[-1]$, where $\mathscr{Q}_D = \mathscr{O}_{\widehat{D}}/\mathscr{O}_{X|D}$ (see [Meb04, Cor. 3.4-4]). Let us set $\mathscr{L}_{\prec 0} := \mathscr{H}^0 \operatorname{DR}^{\operatorname{rd} D} \mathscr{M}$. We then have

$$\boldsymbol{R}\varpi_* \mathscr{L}_{\prec 0} \simeq \operatorname{Irr}_D^* \mathscr{M}^\vee, \tag{3.2 $*$}$$

where $\mathscr{M}^\vee$ is the holonomic $\mathscr{D}_X$-module dual to $\mathscr{M}$. Indeed, According to [Kas03, (3.13)] we have

$$\operatorname{DR}(\mathscr{Q}_D \overset{\boldsymbol{L}}{\otimes} \mathscr{M})[-1] \simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_{X|D}}(\mathscr{M}^\vee, \mathscr{Q}_D)[-1] \simeq \operatorname{Irr}_D^* \mathscr{M}^\vee.$$

On the other hand, as $\mathscr{Q}_D$ is flat over $\mathscr{O}_{X|D}$ (because $\mathscr{O}_{\widehat{X|D}}$ is faithfully flat over $\mathscr{O}_{X|D}$) and as $\boldsymbol{R}\varpi_*\mathscr{A}_{\widetilde{X}}^{\mathrm{rd}\,D} \simeq \mathscr{Q}_D[-1]$, we have

$$\mathrm{DR}(\mathscr{Q}_D \otimes \mathscr{M})[-1] \simeq \mathrm{DR}(\mathscr{Q}_D \overset{\boldsymbol{L}}{\otimes} \mathscr{M})[-1] \simeq \boldsymbol{R}\varpi_* \mathrm{DR}^{\mathrm{rd}\,D}\mathscr{M}.$$

We also notice that $\mathrm{Irr}_D^* \mathscr{M}^\vee = \mathrm{Irr}_D^* \mathscr{M}^\vee(*D)$ and $\mathscr{M}^\vee(*D)$ is also a good D-meromorphic flat bundle, which is identified with the dual D-meromorphic flat bundle $\mathscr{H}om_{\mathscr{O}_X(*D)}(\mathscr{M}, \mathscr{O}_X(*D))$.

Let us fix $I \subset J$. Near each point x_o of D_I°, there exists a local ramification

$$\rho : \mathrm{nb}(x_o)_{\boldsymbol{d}_I} \longrightarrow \mathrm{nb}(x_o)$$

along D such that the pullback of $\mathscr{M}$ has a good formal decomposition at each point in $\mathrm{nb}(x_o)_{\boldsymbol{d}_I}$. By the goodness assumption, there exists an index $k(x_o) \in I$ such that each nonzero $\varphi \in \Phi_{x_o}$ has a pole along $D_{k(x_o)}$: indeed, the set $\Phi_{x_o} \cup \{0\}$ is good, so in particular the pole divisors of each of its nonzero elements are totally ordered; the smallest such divisor is nonzero, and we can choose $k(x_o)$ to be the index of a component of this divisor. One can choose this index constant along any connected component of D_I°. For simplicity, we denote by $k(I)$ the locally constant function $x_o \mapsto k(x_o)$ on D_I°.

For every subset $I \subset J$, we have a natural inclusion lifting ι_I:

$$\widetilde{\iota}_I : \partial\widetilde{X}_I^\circ = \varpi^{-1}(D_I^\circ) \hookrightarrow \varpi^{-1}(D) = \partial\widetilde{X}.$$

Proposition 3.3. *Let us fix $I \subset J$ and let us set $k = k(I)$ for simplicity. Then the natural morphism $\widetilde{\iota}_I^{-1}\mathscr{L}^{>0} \to \widetilde{\iota}_I^{-1}\boldsymbol{R}\widetilde{\iota}_{k*}\widetilde{\iota}_k^{-1}\mathscr{L}^{>0}$ is an isomorphism. The same property holds for $\mathscr{L}_{\prec 0}$.*

By applying $\boldsymbol{R}\varpi_*$ and using Proposition 3.1, we obtain:

Corollary 3.4. *With the notation as in Proposition 3.3, the natural morphism $\iota_I^{-1}\mathrm{Irr}_D(\mathscr{M}) \to \iota_I^{-1}\boldsymbol{R}\iota_{k*}\iota_k^{-1}\mathrm{Irr}_D(\mathscr{M})$ is an isomorphism. The same property holds for $\mathrm{Irr}_D^*(\mathscr{M})$.* □

Proof of Proposition 3.3. Since the morphism is globally defined, the proof that it is an isomorphism is a local question. We thus fix $x_o \in D_I^\circ$ and work in some neighbourhood $\mathrm{nb}(x_o)$ of x_o that we may shrink if needed.

Let us first assume that $\mathscr{M} = \mathscr{E}^\varphi$ (see Section 2.a) for some $\varphi \in \mathscr{O}_{X,x_o}(*D)$.

– If $\varphi = 0$ in $\mathscr{O}_{X,x_o}(*D)/\mathscr{O}_{X,x_o}$, then $\mathscr{L}^{>0} = 0$ and there is nothing to prove.

– If $\varphi \neq 0$ in $\mathscr{O}_{X,x_o}(*D)/\mathscr{O}_{X,x_o}$, we set $\varphi(x) = u(x)/x^m$, where $u \in \mathscr{O}_{X,x_o}$ satisfies $u(x_o) \neq 0$, and $m_i \in \mathbb{N}$ for $i \in I$. In particular, $m_{k(I)} \neq 0$. We choose polar coordinates on $\varpi^{-1}(\mathrm{nb}(x_o))$ of the form $(\rho_1, \dots, \rho_\ell, \theta_1, \dots, \theta_\ell, (x_j)_{j \notin I})$ with $\rho_i \in [0, \varepsilon)$. We can assume that, in these coordinates, $m_i \neq 0$ for $i = 1, \dots, p$, $m_i = 0$ for $i = p+1, \dots, \ell$, and that $k(I) = 1$. Then, in these coordinates, $\varpi^{-1}(D \cap \mathrm{nb}(x_o)) = \prod_{i=1}^\ell \rho_i = 0$ and $\mathscr{L}^{>0}$ is the constant sheaf of rank one on the closed subset of $\varpi^{-1}(D \cap \mathrm{nb}(x_o))$ defined by

$$\begin{cases} \sum_{i=1}^p m_i\theta_i \in \arg u(x) + [-\pi/2, \pi/2], \\ \prod_{i=1}^p \rho_i = 0, \end{cases} \tag{3.5}$$

and it is zero outside this closed subset. Let us describe this closed subset. We set

$$x' := (x_j)_{j \notin I} \in \Delta_\varepsilon^{n-\ell}$$

(with $0 < \varepsilon \ll 1$) and $(\rho, e^{\mathrm{i}\,\theta}) \in [0, \varepsilon)^\ell \times (S^1)^\ell$. We can write $u(x) = u(\rho, \theta, x') = u(x_o)e^{g(x)}$ with g holomorphic and $g(0) = 0$ and we set $e^{\mathrm{i}\,\theta_o} := u(x_o)/|u(x_o)|$. A simple computation shows that, if $\varepsilon > 0$ is small enough, the map

$$\begin{aligned}[0,\varepsilon)^\ell \times (S^1)^\ell \times \Delta_\varepsilon^{n-\ell} &\xrightarrow{(F,\rho,x')} S^1 \times [0,\varepsilon)^\ell \times \Delta_\varepsilon^{n-\ell}\\ (\rho,\theta,x') &\longmapsto \Big(\textstyle\prod_{i=1}^p e^{\mathrm{i}\, m_i\theta_i}\cdot e^{-\,\mathrm{i}(\theta_o+\mathrm{im}\, g(\rho,\theta,x'))},\rho,x'\Big)\end{aligned}$$

has everywhere maximal rank (in fact, we have $\partial F/\partial\theta_1(0,\theta,0)\neq 0$ on $(S^1)^\ell$). By Ehresmann's theorem, the map (F,ρ,x') is a C^∞ fibration, which can be trivialized on contractible sets like

$$[-\pi/2,\pi/2]\times[0,\varepsilon)^\ell\times\Delta_\varepsilon^{n-\ell}.$$

For our topological computation, we can thus as well consider the situation where $u(x)$ is constant and replace $u(x)$ with $u(x_o)$ in (3.5).

Each connected component of (3.5) is then homeomorphic to a product

$$\partial[0,\varepsilon)^p\times[a,b]\times(S^1)^{p-1}\times[0,\varepsilon)^{\ell-p}\times(S^1)^{\ell-p}\times\Delta_\varepsilon^{n-\ell}$$

for suitable a,b. The trace of this set on $\varpi^{-1}(D^\circ_{k(I)})$ is the set defined by $\prod_{j=2}^\ell \rho_j\neq 0$. This is the subset

$$\{\rho_1=0\}\times(0,\varepsilon)^{p-1}\times[a,b]\times(S^1)^{p-1}\times[0,\varepsilon)^{\ell-p}\times(S^1)^{\ell-p}\times\Delta_\varepsilon^{n-\ell}. \tag{3.6}$$

Its closure is the subset

$$\{\rho_1=0\}\times[0,\varepsilon)^{p-1}\times[a,b]\times(S^1)^{p-1}\times[0,\varepsilon)^{\ell-p}\times(S^1)^{\ell-p}\times\Delta_\varepsilon^{n-\ell}. \tag{3.7}$$

The ordinary pushforward of the constant sheaf on (3.6) by the open inclusion (3.6) $\hookrightarrow$ (3.7) is the constant sheaf on (3.7) and the higher pushforwards vanish. Since $\varpi^{-1}(D_I)$ is the subset of (3.7) defined by $\rho_i=0$ for $i=2,\dots,\ell$, the restriction of the latter sheaf to $\varpi^{-1}(D_I)$ is the constant sheaf on $\varpi^{-1}(D_I)$, and the morphism $\widetilde{\iota}_I^{-1}\mathscr{L}^{>0}\to\widetilde{\iota}_I^{-1}\boldsymbol{R}\widetilde{\iota}_{k*}\widetilde{\iota}_k^{-1}\mathscr{L}^{>0}$ is nothing but the identity $\mathbb{C}_{\varpi^{-1}(D_I)}\to\mathbb{C}_{\varpi^{-1}(D_I)}$, proving the proposition in this case.

Let us now consider the general case. As already said, the question is local, and we argue now locally on $\partial\widetilde{X}$. One can then reduce the question to the non-ramified case and apply the higher dimensional Hukuhara-Turrittin theorem (see e.g. [Sab13, Th. 12.5]). Let $\mathscr{A}_{\widetilde{X}}$ denote the sheaf of C^∞ functions on $\widetilde{X}$ which are holomorphic on X^* in some neighbourhood of $\widetilde{x}_o$. We can thus assume that $\mathscr{A}_{\widetilde{X}}\otimes\varpi^{-1}\mathscr{M}$ decomposes as the direct sum of terms $\mathscr{A}_{\widetilde{X}}\otimes\varpi^{-1}(\mathscr{E}^\varphi\otimes\mathscr{R}_\varphi)$. By induction on the rank, we can also assume that $\mathscr{R}_\varphi$ has rank one, and locally on $\varpi^{-1}(D_I^\circ)$ the corresponding local system is trivial, so we can finally assume that $\mathscr{M}=\mathscr{E}^\varphi$, a case which was treated above.

The case of $\mathscr{L}_{\prec 0}$ is treated similarly. If we regard all sheaves considered above as external products of constant sheaves of rank one with respect to the product decomposition in (3.6) and (3.7), the case of $\mathscr{L}_{\prec 0}$ is obtained by replacing $[-\pi/2,\pi/2]$ with the complementary open interval in (3.5), and the corresponding sheaf $\mathbb{C}_{[a,b]}$ with the sheaf $\mathbb{C}_{(a',b')}$ for suitable a',b' (i.e., the extension by zero of the constant sheaf on (a',b')). Then the same argument as above applies to this case. ∎

4. Proof of Theorem 1.2

The case $\ell=1$. We first assume that $I=\{i\}$. The transversal slice Ω has dimension one and $D_\Omega=\{0\}$. Let us first prove a statement in dimension one. Let $(\mathscr{L},\mathscr{L}_\bullet)$ be a Stokes-filtered local system on S^1 and let $(\operatorname{gr}\mathscr{L},(\operatorname{gr}\mathscr{L})_\bullet)$ be the associated graded Stokes-filtered local system. We denote by $\mathscr{N}$ resp. $\mathscr{N}'$ the corresponding meromorphic flat bundles on $(\Omega,0)$.

It is well-known that $\mathscr{H}^k\operatorname{Irr}_{D_\Omega}(\mathscr{N})$ and $\mathscr{H}^k\operatorname{Irr}_{D_\Omega}(\mathscr{N}')$ have the same rank for any k, and vanish except for $k=1$, and similarly for $\operatorname{Irr}^*_D\mathscr{N}^\vee$ and $\operatorname{Irr}^*_D\mathscr{N}'^\vee$. They correspond to

$H^0(S^1, \mathscr{L}^{>0})$ and $H^0(S^1, \operatorname{gr}\mathscr{L}^{>0})$ on the one hand, $H^1(S^1, \mathscr{L}_{\prec 0})$ and $H^1(S^1, \operatorname{gr}\mathscr{L}_{\prec 0})$ on the other hand (this is of course a particular case of Proposition 3.1 and Remark 3.2).

Lemma 4.1. *There exists an isomorphism between the vector spaces $H^1(S^1, \mathscr{L}_{\prec 0})$ and $H^1(S^1, \operatorname{gr}\mathscr{L}_{\prec 0})$ such that, for any automorphism λ of $(\mathscr{L}, \mathscr{L}_\bullet)$, the induced automorphism of $H^1(S^1, \mathscr{L}_{\prec 0})$ corresponds, via this isomorphism, to the automorphism induced by $\operatorname{gr}\lambda$ on $H^1(S^1, \operatorname{gr}\mathscr{L}_{\prec 0})$. The same assertion holds for $H^0(S^1, \mathscr{L}^{>0})$ and $H^0(S^1, \operatorname{gr}\mathscr{L}^{>0})$ respectively.*

Proof. We start with $\mathscr{L}_{\prec 0}$. Let us cover S^1 with open intervals $(U_\alpha)_{\alpha=1,\dots,N}$ such that

- every open interval which contains at most one Stokes direction for every pair of distinct exponential factors (see e.g. Example 1.4 in [Sab13]),
- the intersection of two intervals of the covering is an interval not containing any Stokes direction,
- there are no triple intersections of intervals of the covering.

Then this covering is a Leray covering for $\mathscr{L}_{\prec 0}$ (see e.g. the proof of Lemma 3.12 in loc. cit.), and moreover the only nonzero term of the associated Čech complex is the term in degree one. It follows that

$$H^1(S^1, \mathscr{L}_{\prec 0}) = \bigoplus_{\alpha=1,\dots,N} H^0(U_\alpha \cap U_{\alpha+1}, \mathscr{L}_{\prec 0}),$$

if we set $U_{N+1} = U_1$.

Recall that, on each interval U_α, the Stokes-filtered local system $(\mathscr{L}, \mathscr{L}_\bullet)$ is graded, i.e., the Stokes filtration splits (see e.g. Lemma 3.12 in loc. cit.). Let us choose a splitting on $U_\alpha \cap U_{\alpha+1}$. Then Theorem 3.5 (and its proof) in loc. cit. shows that any automorphism λ is graded with respect to the chosen splitting on $U_\alpha \cap U_{\alpha+1}$. It follows that the action of the automorphism on $H^0(U_\alpha \cap U_{\alpha+1}, \mathscr{L}_{\prec 0})$ is the same as the action of the associated graded automorphism on $H^0(U_\alpha \cap U_{\alpha+1}, (\operatorname{gr}\mathscr{L})_{\prec 0})$, so we have found a model where both actions are equal.

For $\mathscr{L}^{>0}$ we argue by duality. Recall that the dual local system $\mathscr{L}^\vee$ is naturally endowed with a Stokes-filtration $\mathscr{L}^\vee_\bullet$ (so that $(\mathscr{L}^\vee, \mathscr{L}^\vee_\bullet)$ RH-corresponds to the dual meromorphic flat bundle), that $\mathscr{L}^{>0} \simeq \mathscr{H}om_{\mathbb{C}}(\mathscr{L}^\vee_{\prec 0}, \mathbb{C})$ (this is similar to [Sab13, Lem. 2.16]), and this isomorphism is compatible with grading. In particular, it induces isomorphisms

$$H^0(S^1, \mathscr{L}^{>0}) \simeq H^1(S^1, \mathscr{L}^\vee_{\prec 0})^\vee \quad \text{and} \quad H^0(S^1, \operatorname{gr}\mathscr{L}^{>0}) \simeq H^1(S^1, \operatorname{gr}\mathscr{L}^\vee_{\prec 0})^\vee,$$

and by the first point applied to $(\mathscr{L}^\vee, \mathscr{L}^\vee_\bullet)$ we obtain a distinguished isomorphism between $H^0(S^1, \mathscr{L}^{>0})$ and $H^0(S^1, \operatorname{gr}\mathscr{L}^{>0})$. Let λ be an automorphism of $(\mathscr{L}, \mathscr{L}_\bullet)$, and let $\lambda^\vee$ be its dual. Then the first point applied to $\lambda^\vee$ gives the desired property for λ. □

End of the proof of Theorem 1.2 in the case $\ell = 1$. We set $I = \{i\}$, $G = \pi_1(D_i^\circ, x_o)$. By Lemma 4.1, given a Stokes-filtered local system $(\mathscr{L}, \mathscr{L}_\bullet)$ endowed with a G-action (i.e., a representation $G \to \operatorname{Aut}(\mathscr{L}, \mathscr{L}_\bullet)$), there exists an isomorphism between $H^0(S^1, \mathscr{L}^{>0})$ and $H^0(S^1, \operatorname{gr}\mathscr{L}^{>0})$, resp. $H^1(S^1, \mathscr{L}_{\prec 0})$ and $H^1(S^1, \operatorname{gr}\mathscr{L}_{\prec 0})$, so that the induced G-action on $H^0(S^1, \mathscr{L}^{>0})$ is transformed into the induced graded G-action on $H^0(S^1, \operatorname{gr}\mathscr{L}^{>0})$, and the induced G-action on $H^1(S^1, \mathscr{L}_{\prec 0})$ into the induced graded G-action on $H^1(S^1, \operatorname{gr}\mathscr{L}_{\prec 0})$.

Recall now that $\operatorname{Irr}_D \mathscr{M}$ is a complex whose cohomology is locally constant on each D_I°. On D_i° it reduces to the local system $\mathscr{H}^1 \operatorname{Irr}_{D_i^\circ} \mathscr{M}$. If we consider the G-Stokes-filtered local system $(\mathscr{L}, \mathscr{L}_\bullet)$ on S^1 corresponding to $\mathscr{M}_{|D_i^\circ}$ by (the proof of) Theorem 2.6, then $\mathscr{H}^1 \operatorname{Irr}_{D_i^\circ} \mathscr{M}$ is the local system corresponding to G-vector space $H^0(S^1, \mathscr{L}^{>0})$ that this G-Stokes-filtered local system defines. We argue similarly with $\mathscr{M}_i^\circ$ and $(\operatorname{gr}\mathscr{L}, \operatorname{gr}\mathscr{L}_\bullet)$, so that the desired isomorphism follows from Lemma 4.1, as explained above. The argument for $\operatorname{Irr}^*_{D_i^\circ} \mathscr{M}$ is identical. □

The case $\ell\geqslant 2$. When $\ell=\#I\geqslant 2$, the structure of a Stokes-filtered local system on $(S^1)^\ell$ is more difficult to analyze, although it shares many properties with the case $\ell=1$ (see e.g. [Sab13, §9.e]). This is why we use another argument. Namely, Proposition 3.1 enables us to deduce the case where $\ell\geqslant 2$ from the case where $\ell=1$.

We set $k=k(I)$ as defined after Proposition 3.1. Let $\mathrm{nb}(D_I^\circ)$ be an open neighbourhood of D_I° in X on which $\mathscr{M}_I^\circ$ is defined. We claim that

$$\iota_k^{-1}\mathscr{M}_I^\circ=\mathscr{M}^\circ_{k\,|\mathrm{nb}(D_I^\circ)}.$$

Indeed, this follows from the uniqueness of $\mathscr{M}_k^\circ$, and from the fact that $\mathscr{M}_I^\circ$ also decomposes after ramification along D at each point of $\mathrm{nb}(D_I^\circ)\cap D_i^\circ$ if this neighbourhood is chosen small enough. We then have

$$\begin{aligned}\mathrm{Irr}_D(\iota_k^{-1}\mathscr{M}_I^\circ)&\simeq \mathrm{Irr}_D(\mathscr{M}_k^\circ)_{|\mathrm{nb}(D_I^\circ)}\\&\simeq \mathrm{Irr}_D(\iota_k^{-1}\mathscr{M})_{|\mathrm{nb}(D_I^\circ)}\quad(\text{case }\ell=1),\end{aligned}$$

and therefore, by applying $\iota_I^{-1}\boldsymbol{R}\iota_{k*}$,

$$\iota_I^{-1}\boldsymbol{R}\iota_{k*}\iota_k^{-1}\,\mathrm{Irr}_D(\mathscr{M}_I^\circ)\simeq\iota_I^{-1}\boldsymbol{R}\iota_{k*}\iota_k^{-1}\,\mathrm{Irr}_D(\mathscr{M}).$$

The assertion of Theorem 1.2 for Irr_D now follows from Corollary 3.4, applied both to $\mathscr{M}$ and $\mathscr{M}_I^\circ$. The case of Irr_D^* is completely similar. $\square$

Appendix. Some properties of Stokes-filtered local systems

In this appendix we keep the setting of Section 3. We review in Proposition A.10 the proof of [Moc11b, Th. 4.13]: by choosing the projection to D_I° of a tubular neighbourhood of D_I° in X and its fibre product over D_I° with a universal covering of D_I°, we are in the situation of loc. cit. except that we do not assume that the C^∞ fibration is topologically trivial. Remark A.11 will then provide the main result used in Step 3 of the proof of Theorem 2.6. We will also review some other essential results which are proved in loc. cit.

A.a. **Grading of a Stokes-filtered local system.** The result in this subsection is local with respect to D, hence we allow a ramification around the components of D. We fix a nonempty subset $I\subset J$. We fix a simply connected open set $U_I^\circ\subset D_I^\circ$.

We assume that $(\mathscr{L},\mathscr{L}_\bullet)$ is non-ramified in the neighbourhood of U_I°. The covering $\widetilde{\Sigma}_I^\circ$ can then be trivialized on $U_I^\circ\times(S^1)^\ell=\varpi^{-1}(U_I^\circ)$, and we set

$$\widetilde{\Sigma}_I^\circ=\Phi\times U_I^\circ\times(S^1)^\ell,$$

where Φ is a finite subset of $\Gamma\big(U_I^\circ,(\mathscr{O}_X(*D)/\mathscr{O}_X)_{|U_I^\circ}\big)$. Moreover, by the goodness assumption on $\widetilde{\Sigma}$, Φ is a good set, namely, for every pair $\varphi\neq\psi$, the divisor of $\varphi-\psi$ is negative. The set $\mathrm{St}(\varphi,\psi)\subset U_I^\circ\times(S^1)^\ell$ of Stokes directions is smooth over U_I° with fibers equal to a union of translated codimension-one subtori

$$\mathrm{St}(\varphi,\psi)_x=\Big\{(\theta_1,\dots,\theta_\ell)\in(S^1)^\ell\mid \textstyle\sum_j m_j\theta_j-\arg c(x)=\pm\pi/2\bmod 2\pi\Big\},\tag{A.1}$$

where $c(x)$ is an invertible holomorphic function on U_I° and $(m_1,\dots,m_\ell)\in\mathbb{N}^\ell\smallsetminus\{0\}$. We denote by $\mathrm{St}(\Phi)$ the union of the subsets $\mathrm{St}(\varphi,\psi)$ for all pairs $\varphi\neq\psi\in\Phi$.

Let us fix

$$\theta_o=(\theta_{o,1},\dots,\theta_{o,\ell})\in(S^1)^\ell\quad\text{and}\quad\alpha_1,\dots,\alpha_\ell\in\mathbb{N}^*$$

such that $\gcd(\alpha_1,\dots,\alpha_\ell)=1$. The map $\theta\mapsto(\alpha_1\theta+\theta_{o,1},\dots,\alpha_\ell\theta+\theta_{o,\ell})$ embeds S^1 in $(S^1)^\ell$. In the following, $S^1_{\boldsymbol{\alpha},\theta_o}$ denotes this circle.

Proposition A.2. *Let A° be an open interval of length $< 2\pi$ in $S^1_{\boldsymbol{\alpha},\theta_o}$ and let A be its closure. Assume that A satisfies the following property.*

- *For every $x \in U_I^\circ$ and every pair $\varphi \neq \psi \in \Phi$,*
$$\#\big(A \cap \mathrm{St}(\varphi,\psi)\big) = \#\big(A^\circ \cap \mathrm{St}(\varphi,\psi)\big) \leqslant 1.$$
If moreover U_I° is contractible, then $(\mathscr{L},\mathscr{L}_\bullet)$ is graded when restricted to a sufficiently small neighbourhood $U_I^\circ \times \mathrm{nb}(A)$ in $U_I^\circ \times (S^1)^\ell$.

Proof. We first prove that, for every $\varphi \in \Phi$, we have $H^k(U_I^\circ \times A, \mathscr{L}_{<\varphi}) = 0$ for $k \geqslant 1$. Note that, since $\varpi : U_I^\circ \times A \to U_I^\circ$ is proper, $R^k\varpi_*\mathscr{L}_{<\varphi|U_I^\circ\times A}$ is compatible with base change, hence its germ at x is equal to $H^k(A, \mathscr{L}_{<\varphi|\{x\}\times A})$. By our assumption on A, this is also equal to $H^k(A^\circ, \mathscr{L}_{<\varphi|\{x\}\times A^\circ})$, and by the proof of [Sab13, Lem. 9.26], this is zero for $k \geqslant 1$. As a consequence, $R^k\varpi_*\mathscr{L}_{<\varphi|U_I^\circ\times A} = 0$ for $k \neq 0$.

We argue as in loc. cit. to obtain that $(\mathscr{L},\mathscr{L}_\bullet)$ is graded in the neighbourhood of $\{x\} \times A$ for every $x \in U_I^\circ$. In particular, it is easy to check that $\varpi_*\mathscr{L}_{<\varphi|U_I^\circ\times A}$ is locally constant, hence constant, on U_I°. Since U_I° is assumed contractible, we obtain the vanishing of $H^k(U_I^\circ \times A, \mathscr{L}_{<\varphi})$ ($k \geqslant 1$). Using once more the argument of loc. cit., we obtain the grading property all over $U_I^\circ \times A$, hence in some open neighbourhood of it. $\square$

By mimicking the proof of [Sab13, Th. 3.5 & Prop. 9.21], we also obtain the following proposition.

Proposition A.3. *Let $\lambda : (\mathscr{L},\mathscr{L}_\bullet) \to (\mathscr{L}',\mathscr{L}'_\bullet)$ between Stokes-filtered local systems as considered in the beginning of this subsection with the same set Φ. For A as in Proposition A.2, there exist gradings of both Stokes-filtered local systems on $U_I^\circ \times \mathrm{nb}(A)$ with respect to which λ is graded.* $\square$

A.b. **Closedness.** Let U_I° be an open subset of D_I° with closure $\overline{U_I^\circ}$ in D_I° and boundary ∂U_I°, and let $j : U_I^\circ \hookrightarrow \overline{U_I^\circ}$ and $\widetilde{\jmath} : \varpi^{-1}(U_I^\circ) \to \varpi^{-1}(\overline{U_I^\circ})$ be the open inclusions. Let $(\mathscr{L},\mathscr{L}_\bullet)$ be a Stokes-filtered local system on $\varpi^{-1}(U_I^\circ)$ with associated covering contained in $\widetilde{\Sigma}^\circ_{I|U_I^\circ}$. Assume that

($*$) any point $x \in \partial U_I^\circ$ has a fundamental system of open neighbourhoods V in D_I° such that $V \cap U_I^\circ$ and $V \cap \overline{U_I^\circ}$ are contractible.

Proposition A.4. *Under this assumption, the functor $\widetilde{\jmath}_*$ induces an equivalence between the category of Stokes-filtered local systems $(\mathscr{L},\mathscr{L}_\bullet)$ on $\varpi^{-1}(U_I^\circ)$ with associated $\mathcal{I}$-covering contained in $\widetilde{\Sigma}^\circ_{I|U_I^\circ}$, and the category of Stokes-filtered local systems on $\varpi^{-1}(\overline{U_I^\circ})$ with associated $\mathcal{I}$-covering contained in $\widetilde{\Sigma}^\circ_{I|\overline{U_I^\circ}}$, a quasi-inverse functor being the restriction $\widetilde{\jmath}^{-1}$.*

Proof. Since the functor is globally defined, the question is local near a point $x_o \in \partial U_I^\circ$. Moreover, as in Section A.a, we can assume that $\widetilde{\Sigma}_I^\circ$ is a trivial covering on some neighbourhood of x_o. It is enough to prove the statement in the non-ramified case since, by uniqueness the construction, it will descend by means of the Galois action of the ramification. We will work with the corresponding set Φ of exponential factors.

Firstly, we note that Assumption ($*$) also holds for $\varpi^{-1}(\overline{U_I^\circ})$, since any point in $\varpi^{-1}(x)$ has a fundamental systems of neighbourhoods of the form of the product of neighbourhoods V with a product of ℓ open intervals. It follows that the local system $\mathscr{L}$ extends in a unique way as a local system on $\varpi^{-1}(\overline{U_I^\circ})$, and the latter is $\widetilde{\jmath}_*\mathscr{L}$. Similarly, a morphism between local systems extends in a unique way by the functor $\widetilde{\jmath}_*$. The same property holds for the local systems $\mathrm{gr}_\psi\mathscr{L}$ for $\psi \in \Phi$.

Let us first show that the functor $\widetilde{\jmath}_*$ takes values in the category of Stokes-filtered local systems. For a pair $\varphi \neq \psi \in \Phi$, we denote by $\beta_{\psi\leqslant\varphi}$ the functor composed of the restriction to the open subset where $\varphi \leqslant \psi$ (i.e., $\operatorname{Re}(\varphi - \psi) < 0$) and the extension by zero to the whole space. The point is to check that every $\widetilde{\jmath}_*\mathscr{L}_{\leqslant\varphi}$ decomposes as $\bigoplus_{\psi\in\Phi} \beta_{\psi\leqslant\varphi}\widetilde{\jmath}_* \operatorname{gr}_\psi \mathscr{L}$ in the neighbourhood of every point (x_o, θ_o) of $\varpi^{-1}(x_o)$. If we fix a small interval A° containing this point as in Proposition A.2, we find that, according to this proposition and Assumption $(*)$,

$$\mathscr{L}_{\leqslant\varphi|(V\cap U_I^\circ)\times\mathrm{nb}(A^\circ)} \simeq \bigoplus_{\psi\in\Phi} \beta_{\psi\leqslant\varphi}(\operatorname{gr}_\psi \mathscr{L})_{|(V\cap U_I^\circ)\times\mathrm{nb}(A^\circ)}. \tag{A.5}$$

We are thus reduced to checking that, for a local system L, the natural morphism

$$\beta_{\psi\leqslant\varphi}L \longrightarrow \widetilde{\jmath}_*\beta_{\psi\leqslant\varphi}\widetilde{\jmath}^{-1}L$$

is an isomorphism: we will apply this to the local system $L = \widetilde{\jmath}_*(\operatorname{gr}_\psi \mathscr{L})_{|(V\cap U_I^\circ)\times\mathrm{nb}(A^\circ)}$ for any ψ. The question is then local, and we can work in the neighbourhood of (x_o, θ_o), with the constant sheaf of rank one as the given local system.

If $(x_o, \theta_o) \notin \operatorname{St}(\varphi, \psi)_{x_o}$, the result is easy. We will thus focus on the case where

$$(x_o, \theta_o) \in \operatorname{St}(\varphi, \psi)_{x_o}.$$

This can be written as $\sum m_j\theta_{o,j} - \arg c(x_o) = \pm\pi/2$. We will consider the case $+\pi/2$, the other one being similar. We need to check that the germ at (x_o, θ_o) of $\widetilde{\jmath}_*\widetilde{\jmath}^{-1}\beta_{\psi\leqslant\varphi}\mathbb{C}$ is zero for any such (x_o, θ_o). For that purpose, it is enough to prove that, for small enough closed neighbourhoods V of x_o and $\mathrm{nb}(\theta_o)$ of θ_o, the cohomology of the sheaf on

$$(V \times \mathrm{nb}(\theta_o)) \cap \bigl\{\textstyle\sum m_j\theta_j - \arg c(x) \in [\pi/2 - \varepsilon, \pi/2]\bigr\} \tag{A.6}$$

which is zero on

$$(V \times \mathrm{nb}(\theta_o)) \cap \bigl\{\textstyle\sum m_j\theta_j - \arg c(x) = \pi/2\bigr\}$$

and constant on the complementary set, is zero for $0 < \varepsilon \ll 1$ and V small enough. We can regard $\sum m_j\theta_j - \arg c(x_o) - \pi/2$ as a coordinate θ' near θ_o vanishing at θ_o, and we can choose the neighbourhood $\mathrm{nb}(\theta_o)$ of the form $[-2\varepsilon, 2\varepsilon] \times [-2\varepsilon, 2\varepsilon]^{\ell-1}$ accordingly. For V small enough, the set (A.6) is a topological fibration above V, and the fiber over $x \in V$ is the product of $[-2\varepsilon, 2\varepsilon]^{\ell-1}$ with the interval

$$\theta' \in \arg c(x) - \arg c(x_o) + [-\varepsilon, 0].$$

Since the projection to V is proper, the base change formula shows that the pushforward to V of this sheaf is identically zero, as the cohomology with compact support of a semi-closed interval is zero. Hence its global cohomology on (A.6) is also zero.

The next step is to show that the extension by $\widetilde{\jmath}_*$ of a morphism λ between Stokes-filtered local systems is compatible with the Stokes filtration. The question is local, and we can assume that the morphism λ is graded on $(V \cap U_I^\circ) \times \mathrm{nb}(A^\circ)$, according to Proposition A.3. Then $\widetilde{\jmath}_*\lambda$ is also graded on this open set with respect to the Stokes filtration constructed above, and is thus also Stokes-filtered.

Once the functor $\widetilde{\jmath}_*$ is defined, that it is essentially surjective is proven similarly, since in the neighbourhood of any point (x_o, θ_o) the sheaves $\mathscr{L}_{\leqslant\varphi}$ are given by a formula like (A.5).

The full faithfulness follows from the full faithfulness for the underlying local systems. $\square$

A.c. **Openness.** We keep the notation as above.

Proposition A.7. *Let $x_o \in D_I^\circ$ and let $(\mathscr{L}, \mathscr{L}_\bullet)_{x_o}$ be a Stokes-filtered local system on*

$$\varpi^{-1}(x_o) \simeq (S^1)^\ell$$

with associated $\mathfrak{I}$*-covering contained in* $\widetilde{\Sigma}^\circ_{I,x_o}$. *Then there exists an open neighbourhood* $\mathrm{nb}(x_o)$ *in* D_I° *such that* $(\mathscr{L},\mathscr{L}_\bullet)_{x_o}$ *extends in a unique way as a Stokes-filtered local system on* $\varpi^{-1}(\mathrm{nb}(x_o)) \simeq \mathrm{nb}(x_o)\times(S^1)^\ell$ *with associated* $\mathfrak{I}$*-covering contained in* $\widetilde{\Sigma}^\circ_{I|\mathrm{nb}(x_o)}$. *Any morphism* $(\mathscr{L},\mathscr{L}_\bullet)_{x_o}\to(\mathscr{L}',\mathscr{L}'_\bullet)_{x_o}$ *between such objects also extends locally in a unique way.*

Proof. The problem is local on D_I° and, by the uniqueness of the extension of morphisms, one can reduce the proof to the non-ramified case. We can therefore assume that $\Sigma_I^\circ = \Phi\times\mathrm{nb}(x_o)$. Moreover, the unique extension of local systems and morphisms between them is clear, so the question reduces to checking that Stokes filtrations extend as well, and that the extended morphism between the extended local systems is compatible with the extended Stokes filtrations.

By Proposition A.2, we can cover $(S^1)^\ell = \varpi^{-1}(x_o)$ by simply connected open sets U_α such that, for every α, there exists a neighbourhood V_α of the compact subset $\overline{U}_\alpha$ and an isomorphism

$$\mathscr{L}_{x_o|V_\alpha} \simeq \bigoplus_{\varphi\in\Phi} \mathrm{gr}_\varphi \mathscr{L}_{x_o|V_\alpha}, \tag{A.8}$$

and the Stokes filtration on V_α is given by

$$\mathscr{L}_{x_o,\leqslant\varphi|V_\alpha} \simeq \bigoplus_{\psi\in\Phi} \beta_{\psi\leqslant\varphi}\,\mathrm{gr}_\psi \mathscr{L}_{x_o|V_\alpha}. \tag{A.9}$$

The transition maps $\lambda_{\alpha\beta}$ for (A.8) on $V_{\alpha\beta} := V_\alpha\cap V_\beta$ satisfy the cocycle condition and are compatible with the Stokes filtration, that is, $\lambda^{\psi,\varphi}_{\alpha\beta} : \mathrm{gr}_\psi\mathscr{L}_{x_o|V_{\alpha\beta}} \to \mathrm{gr}_\varphi\mathscr{L}_{x_o|V_{\alpha\beta}}$ is zero unless $\psi\leqslant\varphi$ on $V_{\alpha\beta}$.

Let us shrink $\mathrm{nb}(x_o)$ to a contractible open neighbourhood such that, for all $\psi\neq\varphi\in\Phi$, $\psi<\varphi$ on $V_{\alpha\beta}$ implies $\psi<\varphi$ on $\mathrm{nb}(x_o)\times U_{\alpha\beta}$. The local system $\mathrm{gr}_\varphi\mathscr{L}_{x_o|U_\alpha}$ extends in a unique way to a local system $\mathrm{gr}_\varphi\mathscr{L}_{|\mathrm{nb}(x_o)\times U_\alpha}$ on $\mathrm{nb}(x_o)\times U_\alpha$, and so do the morphisms $\lambda^{\psi\varphi}_{\alpha\beta}$, which satisfy thus the cocycle condition. In particular, if such an extension $\lambda^{\psi\varphi}_{\alpha\beta}$ is non-zero at one point of $\mathrm{nb}(x_o)\times U_{\alpha\beta}$, it is nonzero everywhere on this open set and we have $\psi<\varphi$ on this open set. Let us set $\mathscr{L}_{|\mathrm{nb}(x_o)\times U_\alpha} := \bigoplus_{\varphi\in\Phi}\mathrm{gr}_\varphi\mathscr{L}_{|\mathrm{nb}(x_o)\times U_\alpha}$, that we equip with the Stokes filtration given by a formula similar to (A.9). It follows that $\lambda_{\alpha\beta}$ is compatible with the Stokes filtrations. We regard now $\lambda_{\alpha\beta}$ as gluing data. The cocycle condition shows that they define a local system $\mathscr{L}$ on $\varpi^{-1}(\mathrm{nb}(x_o))$ whose restriction to $\varpi^{-1}(x_o)$ is isomorphic to $\mathscr{L}$. It is thus uniquely isomorphic to the unique extension of $\mathscr{L}_{x_o}$. Moreover, due to the compatibility with the Stokes filtrations, the latter also glue correspondingly as a Stokes filtration $\mathscr{L}_\bullet$ of this local system, and its restriction to $\varpi^{-1}(x_o)$ is equal to $\mathscr{L}_{x_o\bullet}$.

Let $\mu_{x_o} : (\mathscr{L},\mathscr{L}_\bullet)_{x_o}\to(\mathscr{L}',\mathscr{L}'_\bullet)_{x_o}$ be a morphism. We can choose the covering (U_α) and the decomposition (A.8) so that each $\mu_{x_o,\alpha}$ is graded (see [Sab13, Prop. 9.21]). It extends uniquely as a morphism $\mu : \mathscr{L}_{|\mathrm{nb}(x_o)\times U_\alpha}\to\mathscr{L}'_{|\mathrm{nb}(x_o)\times U_\alpha}$, and it is graded with respect to the corresponding decompositions (A.8). It follows that μ is strictly compatible with the Stokes filtrations $\mathscr{L}_\bullet$ and $\mathscr{L}'_\bullet$, where these Stokes-filtered local systems $(\mathscr{L},\mathscr{L}_\bullet)$ and $(\mathscr{L}',\mathscr{L}'_\bullet)$ are obtained as in the first part.

We can now prove the uniqueness (i.e., up to unique isomorphism) of $(\mathscr{L},\mathscr{L}_\bullet)$ constructed in the first part: the identity automorphism $(\mathscr{L},\mathscr{L}_\bullet)_{x_o}$ extends in a unique way as an isomorphism between two such extensions. ∎

A.d. **An equivalence of categories.** We will use the notation as in Section 2.g. Let

$$\pi : (E_I^\circ(x_o), y_o) \longrightarrow (D_I^\circ(x_o), x_o)$$

be a universal covering of $D_I^\circ(x_o)$ with base point y_o above x_o, and let $\partial\widetilde{Y}_I^\circ(x_o)$ be the pullback of $\partial\widetilde{X}_I^\circ(x_o)$ by π.

Proposition A.10. *The restriction functor*

- *from the category of Stokes-filtered local systems on $\partial\widetilde{Y}_I^\circ(x_o)$ with associated $\pi^{-1}\mathfrak{I}$-covering contained in $\pi^{-1}\widetilde{\Sigma}_I^\circ(x_o)$*
- *to the category of Stokes-filtered local systems on $(\partial\widetilde{\Omega})_0 \simeq (S^1)^\ell$ with associated $\mathfrak{I}_{x_o}$-covering contained in $\widetilde{\Sigma}_{x_o}$*

is an equivalence.

Proof. Let $\Gamma : [0,1]^2 \to E_I^\circ(x_o)$ be a continuous map sending $(0,0)$ to y_o. We pullback by Γ the data from the first item of the proposition. Let us consider the subset of $[0,1]$ consisting of ε's such that the equivalence of the proposition holds with respect to the restriction corresponding to the inclusion $(0,0) \in [0,\varepsilon]^2$. Propositions A.4 and A.7 imply that this set is open and closed, and contains 0, hence it is equal to $[0,1]$. This shows that one can uniquely extend an object in the second category to an object in the first category along paths starting from y_o and that this extension does not depend on the choice of the path. A similar assertion holds for morphisms. $\square$

Remark A.11. The uniqueness of the extension of morphisms enables one to obtain the equivalence between the corresponding G-equivariant categories, and this gives the implication (2) $\Rightarrow$ (3) in the proof of Theorem 2.6.

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(C. Sabbah) CMLS, École polytechnique, CNRS, Université Paris-Saclay, F–91128 Palaiseau cedex, France

E-mail address: Claude.Sabbah@polytechnique.edu

URL: http://www.math.polytechnique.fr/perso/sabbah

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PAIRS OF MORSE FUNCTIONS

OLIVIER THOM

Abstract. The goal of this paper is to classify pairs of Morse functions in general position modulo the action of different groups. In particular, we obtain the classification of generic pairs of Morse functions, with or without target diffeomorphisms, and that of quotients of Morse functions.

We will also present a lemma which gives a sufficient condition for two pairs of functions to be conjugated.

1. Introduction

Throughout this paper, we will denote by $\mathcal{O}_{(\mathbb{C}^n,0)}$ the set of germs at 0 of holomorphic functions on $\mathbb{C}^n$ and by $\mathfrak{m}_{(\mathbb{C}^n,0)}$ its maximal ideal. We will also use the notation $X \cdot f = d_X f$ to mean the derivative of f in the direction given by the holomorphic vector field X.

Following the works of Mather ([Mat]), we will consider the problem of knowing when two objects are diffeomorphic as a problem about group actions. More precisely, for a group $\mathscr{S}$ acting on pairs of functions $f, g \in \mathcal{O}_{(\mathbb{C}^n,0)}$, we will say that two pairs p_1 and p_2 are $\mathscr{S}$-conjugated or $\mathscr{S}$-equivalent if there exists $\varphi \in \mathscr{S}$ such that $\varphi \cdot p_1 = p_2$. In this paper we will consider the groups $\mathscr{R}, \mathscr{A}, \mathscr{F}, \mathscr{Q}$ which follows: $\mathscr{R} = \mathrm{Diff}(\mathbb{C}^n, 0)$ acting by composition at the source, $\mathscr{A} = \mathrm{Diff}(\mathbb{C}^n, 0) \times \mathrm{Diff}(\mathbb{C}^2, 0)$ acting by composition at the source and at the target, $\mathscr{F} = \mathrm{Diff}(\mathbb{C}^n, 0) \times (\mathrm{Diff}(\mathbb{C}, 0))^2$ acting by $(\varphi, \psi_1, \psi_2) \cdot (f, g) = (\psi_1 \circ f, \psi_2 \circ g) \circ \varphi^{-1}$, and $\mathscr{Q} = \mathrm{Diff}(\mathbb{C}^n, 0) \rtimes \mathcal{O}^*_{(\mathbb{C}^n,0)}$ acting by $(\varphi, U) \cdot p = Up \circ \varphi^{-1}$. Classification of pairs of functions up to $\mathscr{F}$-equivalence corresponds to the classification of pairs of foliations up to diffeomorphism at the source; classification of pairs of functions (f, g) up to $\mathscr{Q}$-equivalence corresponds to the classification of meromorphic functions f/g up to diffeomorphism at the source.

For a more complete bibliography about $\mathscr{S}$-equivalence of applications, the reader is referred to [AVG1], [AVG2] and the references therein.

Let f and g be two Morse functions on $(\mathbb{C}^n, 0)$ whith quadratic parts q_f and q_g. Denote by $\mathcal{F}$ and $\mathcal{G}$ the foliations given by the level sets of f and g. Denote also by $I(f, g)$ the tangency ideal between f and g, that is the ideal of $\mathcal{O}_{(\mathbb{C}^n,0)}$ spanned by $(\partial_{x_i} f \partial_{x_j} g - \partial_{x_j} f \partial_{x_i} g)_{i,j}$ for a set of coordinates (x_i) and by $\mathrm{Tang}(f, g) = \mathrm{Tang}(\mathcal{F}, \mathcal{G})$ the set of zeroes of $I(f, g)$, which we will name the tangency locus between f and g.

We will begin by giving the classification up to $\mathscr{R}$-equivalence of pairs of Morse functions, but first, let us recall the well-known classification of pairs of quadratic forms on $\mathbb{C}^n$ (cf [HP]). Seen as matrices, two nondegenerate forms q_f and q_g can be simultaneously diagonalized by blocks with blocks

$$\begin{pmatrix} (0) & & & 1 \\ & & \cdot^{\cdot^{\cdot}} & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & (0) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} (0) & & & \lambda \\ & & \cdot^{\cdot^{\cdot}} & 1 \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ \lambda & 1 & & (0) \end{pmatrix}.$$

As an example, take the quadratic forms given by the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}:$$

$f = 2xy$ and $g = 2xy + y^2$. We see that this pair cannot be simultaneously diagonalized.

Nevertheless, counting the parameters in the diagonalization by blocks we see that a generic (outside a set of codimension 1) pair of quadratic forms (q_f, q_g) can be simultaneously diagonalized.

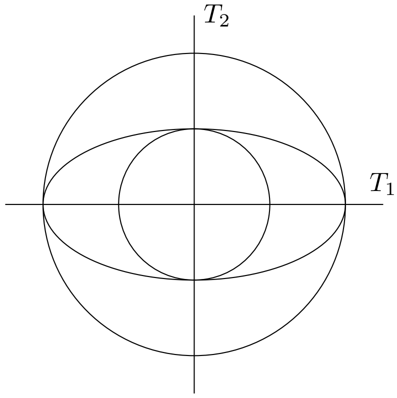
Morse theorem ([Mor]) allows us to assume without loss of generality that $f = \sum x_i^2$. Moreover we suppose that q_f and q_g are in generic position:

$$q_f(x) = \sum x_i^2\ ,\ \ q_g(x) = \sum \lambda_i x_i^2$$

with $\lambda_i \neq \lambda_j \neq 0$ if $i \neq j$ up to a linear change of coordinate.

Next, look at the tangency locus between the foliations $\mathcal{F}$ and $\mathcal{G}$: if f and g were diagonal quadratic forms, this would be the reunion of the coordinate axes. In general, if q_f and q_g are diagonal, it is diffeomorphic and tangent to the reunion of the axes so we can suppose that it is exactly the reunion of the axes; this will be detailed further.

For example, in the case $n = 2$ the functions $f = x^2 + y^2$ and $g = x^2 + 2y^2$ give the following real phase portrait:



If we name the axes T_j as in the picture, we can look at the restriction of each function to each tangency curve, which gives couples $(f|_{T_j}, g|_{T_j})$ for each j. If Φ is a diffeomorphism of $(\mathbb{C}^n, 0)$ stabilizing the T_j's, we have $((f \circ \Phi)|_{T_j}, (g \circ \Phi)|_{T_j}) = (f|_{T_j}, g|_{T_j}) \circ (\Phi|_{T_j})$ so that each couple $(f|_{T_j}, g|_{T_j})$ up to diffeomorphism on the right gives an invariant for the $\mathscr{R}$-equivalence of pairs of functions.

Hence, if C_0 and C_1 are smooth curves, we will say that two couples (u_0, v_0) and (u_1, v_1) with $u_j, v_j \in \mathcal{O}(C_j, 0)$ are conjugated under the action of $\mathrm{Diff}(C_0, C_1)$ on the right if there exists $\psi \in \mathrm{Diff}(C_0, C_1)$ such that $(u_0, v_0) = (u_1, v_1) \circ \psi$.

The use of tangency curves and functions defined on them as invariants for classification problems has already been considered, for example in [ORV]. In our case, these invariants are enough to classify the pairs of Morse functions up to $\mathscr{R}$-equivalence, as stated in the theorem :

Theorem. *Let (f_0, g_0) and (f_1, g_1) be two pairs of Morse functions on $(\mathbb{C}^n, 0)$ with quadratic parts (q_{f_i}, q_{g_i}) in generic position.*

Suppose that we can number the tangency curves T_j^i ($j = 1, \ldots, n$ and $i = 0, 1$) in such a manner that the pairs of Morse functions $(f_i|_{T_j^i}, g_i|_{T_j^i})$ are conjugated under the action of $\mathrm{Diff}(T_j^0, T_j^1)$ on the right. Then (f_0, g_0) and (f_1, g_1) are $\mathscr{R}$-equivalent.

As a consequence, if two pairs of Morse functions with quadratic parts in generic position are topologically conjugated, they are analytically conjugated.

A part of the proof of this theorem is in fact quite general and is expressed as a separate lemma (the key lemma in what follows); Section 2 is devoted to the statement and proof of this lemma. The next section (Section 3) handles the $\mathscr{R}$-classification of pairs of Morse functions.

After the $\mathscr{R}$-classification of pairs of Morse functions, the $\mathscr{A}$-classification and the $\mathscr{F}$-classification are just a matter of rewriting as it will be shown later; these are done in Sections 4 and 5. The $\mathscr{Q}$-classification of pairs of Morse functions is not a straightforward consequence of the former theorem; the main result is that a generic pair (f, g) of Morse functions is determined up to the action of $\mathscr{Q}$ by the 3-jets of f and g, so that a generic quotient of Morse functions is diffeomorphic to an explicit rational function of degree 3. This will be detailed in Section 6.

We will also show that the restriction of a generic Morse function to a quadratic cone (the set of zeroes of a Morse function) is determined up to diffeomorphism by its quadratic part (in section 7).

In the last section, we will show that the key lemma can be applied in a general setting, by rediscovering classical results like the classification of folds, or giving finite determinacy results. As an example, we will give the classification of some special pairs of cusps.

Some of these problems can be restated in terms of diagrams in the sense of Dufour (cf. [D]): the $\mathscr{F}$-classification of pairs of Morse functions corresponds to the classification of divergent diagrams of Morse functions

$$\begin{tikzcd} & (\mathbb{C}, 0) \\ (\mathbb{C}^n, 0) \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & (\mathbb{C}, 0). \end{tikzcd}$$

We should also mention the work of J. Vey about a similar problem: the simultaneous reduction of a Morse function and a volume form (cf. [Vey]).

2. Proof of the key lemma

Recall that two pairs (f_0, g_0) and (f_1, g_1) of functions of $(\mathbb{C}^n, 0)$ are called $\mathscr{R}$-equivalent if there exists a diffeomorphism $\varphi \in \mathrm{Diff}(\mathbb{C}^n, 0)$ such that $(f_0 \circ \varphi, g_0 \circ \varphi) = (f_1, g_1)$. In this section we want to prove the following:

Lemma 1 (Key Lemma)**.** *Let f, g_0 and g_1 be three functions on $(\mathbb{C}^n, 0)$ where f has a singular point at 0. Suppose that the tangency ideals $I(f, g_0)$ and $I(f, g_1)$ are equal and that*

$$g_1 - g_0 \in I(f, g_0).$$

Then (f, g_0) and (f, g_1) are $\mathscr{R}$-conjugated.

The proof of this lemma is based on Moser's path method: we will construct a path (f, g_t) between (f, g_0) and (f, g_1) and show that every (f, g_t) are diffeomorphic. Put $g_t = g_0 + t(g_1 - g_0)$ and $g(t, \cdot) = g_t(\cdot) \in \mathcal{O}(U)$ for a neighborhood U of $[0, 1] \times \{0\}$ in $\mathbb{C}_t \times \mathbb{C}^n$. Introduce also $I = I(f, g)$ (which is an ideal of $\mathcal{O}(U)$) and for each t, $I_t = I(f, g_t)$ (which is an ideal of $\mathcal{O}_{(\mathbb{C}^n,0)}$). Write finally $d_x f \wedge d_x g = \sum_{i<j} h_{ij} dx_i \wedge dx_j$ for a system of coordinates (x_i) on $\mathbb{C}^n$, $J = \langle h_{ij} \rangle_{i<j}$ and note that $I_t = \langle h_{ij}(t, \cdot) \rangle_{i<j}$.

We will first study these ideals to show that $J = I_0 \otimes_{\mathcal{O}_x} \mathcal{O}(U)$ where $\mathcal{O}_x$ denotes the set of germs of holomorphic functions in the variables $x_1, \ldots, x_n$.

Proposition 1. *Suppose $I_0 = I_1$, then $I_0 = I_t$ for t generic.*

Proof. The tangency ideal I_t is spanned by the components of $df \wedge dg_t = t df \wedge dg_1 + (1-t) df \wedge dg_0$ so it is contained in I_0. But I_0/I_t is null for $t = 0$ so the support of I_0/I_t can only consist of finitely many points, hence the result. □

In what follows, we will use the additional hypothesis that I_t is constant along the interval $[0, 1]$. If this is not the case, we could find a point $t_0 \in \mathbb{C}$ such that $I_t = I_0$ for each t in both segments $[0, t_0]$ and $[t_0, 1]$ (thanks to the previous proposition) and use what will follow on these segments to show that $(f, g_0) \simeq (f, g_{t_0}) \simeq (f, g_1)$ so we can indeed suppose without loss of generality that I_t is constant along $[0, 1]$.

Proposition 2. *For each t_0, the localization $J_{(t_0)}$ of J at t_0 satisfies $J_{(t_0)} = I_0 \otimes_{\mathcal{O}_x} \mathbb{C}\{t-t_0, x\}$.*

Proof. It is enough to prove that $J_{(t_0)} = I_{t_0} \otimes \mathbb{C}\{t-t_0, x\}$ because $I_{t_0} = I_0$.

Note first that the h_{ij} are affine in t so that $h_{ij}(t) = h_{ij}(t_0) + \frac{t-t_0}{1-t_0}(h_{ij}(1) - h_{ij}(t_0))$ (we supposed that $t_0 \neq 1$, the case $t_0 = 1$ can be done similarly). Denote by $H(t)$ the vector $(h_{ij}(t))_{i<j}$; the hypothesis that $I_1 = I_{t_0}$ then gives a matrix A with constant coefficients such that $H(1) = AH(t_0)$. Hence the existence of a matrix B satisfying $H(t) = (id + (t-t_0)B)H(t_0)$.

For t near t_0, the matrix $id + (t-t_0)B$ is invertible so the components of the vectors $H(t)$ and $H(t_0)$ span the same germ of ideal around the point t_0. Note finally that the germ of ideal spanned by the components of $H(t_0)$ is $I_0 \otimes \mathbb{C}\{t-t_0, x\}$. □

As a corollary, for each point $p_0 = (t_0, x_0) \in U \subset \mathbb{C}_t \times \mathbb{C}^n$, we have the relation

$$J_{(p_0)} = (I_0)_{(x_0)} \otimes_{\mathbb{C}\{x-x_0\}} \mathbb{C}\{t-t_0, x-x_0\}.$$

Proposition 3. *$J = I_0 \otimes_{\mathcal{O}_x} \mathcal{O}(U)$.*

Proof. We can suppose that the neighborhood U is Stein. The ideal J (resp., $I_0 \otimes \mathcal{O}(U)$) defines a sheaf of ideals $\mathscr{J}$ (resp., $\mathscr{K}$) defined by $\mathscr{J}_{(p_0)} = J_{(p_0)}$ for $p_0 \in U$ (resp.,

$$\mathscr{K}_{(p_0)} = (I_0)_{(x_0)} \otimes \mathbb{C}\{t-t_0, x-x_0\}$$

for $p_0 = (t_0, x_0) \in U$). These sheaves are locally of finite type; if $a_1, \ldots, a_k$ are local sections of $\mathscr{J}$ (resp., $\mathscr{K}$), the sheaf of relations $\mathcal{R}(a_1, \ldots, a_k)$ may be viewed as the relations of the sections a_i of the sheaf $\mathcal{O}$. Hence by Oka's theorem (see, for example, [Hör]), $\mathcal{R}(a_1, \ldots, a_k)$ is locally of finite type and $\mathscr{J}$ and $\mathscr{K}$ are coherent.

Take $a \in I_0 \otimes \mathcal{O}(U)$, then $a_{(p)} \in \mathscr{K}_{(p)} = \mathscr{J}_{(p)}$ for each $p \in U$; since U is Stein and since the global sections h_{ij} span $\mathscr{J}$ locally, there exists holomorphic $r_{ij} \in \mathcal{O}(U)$ such that $a = \sum r_{ij} h_{ij}$, i.e., $a \in J$ (cf. [Hör]).

The converse works in the same way with $h_{ij}(0, \cdot)$ as global sections spanning $\mathscr{K}$ locally. □

Moreover, if $g_1 - g_0 \in I_0$ as in the hypotheses of the lemma, $g_1 - g_0 \in J$ by the former proposition, so J is also equal to I because $df \wedge dg = d_x f \wedge d_x g + (g_1 - g_0) df \wedge dt$.

Now we can prove the key lemma:

Proof of the key lemma. As noted above, the hypothesis $g_1 - g_0 \in I_0$ together with Proposition 3 means that there exists holomorphic $r_{ij}(t, x)$ (for $i < j$) such that $g_1 - g_0 = \sum_{i<j} r_{ij} h_{ij}$.

To use the path method, we need to find a vector field $X = \sum_{i=1}^n X_i \partial_{x_i} + \partial_t$ defined in a neighborhood of $\{0\} \times [0,1] \subset \mathbb{C}^n \times [0,1]$ such that $X \cdot f = X \cdot g = 0$. We also want to have $X(0,t) = \partial_t$ so that the flow $\varphi_s(x,t)$ of X will be defined on a neighborhood of $\{0\} \times [0,1]$. The diffeomorphism $\varphi : x \mapsto \varphi_1(x, 0)$ will then verify $(f \circ \varphi, g_0 \circ \varphi) = (f, g_1)$ on $(\mathbb{C}^n, 0)$.

Remember that

$$X \cdot f = \sum_{i=1}^n X_i \partial_{x_i} f \quad \text{and}$$

$$X \cdot g = \sum_{i=1}^n X_i \partial_{x_i} g_t + (g_1 - g_0).$$

Note that it is enough to find for each $j = 2, \ldots, n$ a vector field X^j satisfying $X^j \cdot f = 0$ and

$$\sum_{i=1}^n X_i^j \partial_{x_i} g_t + \sum_{i=1}^{j-1} r_{ij} h_{ij} = 0$$

because the vector field $X = \sum_{j=2}^n X^j + \partial_t$ would then be as sought.

On $U_j = \{\partial_{x_j} f \neq 0\}$, we may impose

$$X_j^j = \frac{-1}{\partial_{x_j} f} \left(\sum_{i \neq j} (\partial_{x_i} f) X_i^j \right)$$

so that

$$(\partial_{x_j} f)\left(\sum_{i=1}^{n} X_i^j \partial_{x_i} g_t + \sum_{i=1}^{j-1} r_{ij} h_{ij}\right) = \sum_{i\neq j}\left(\partial_{x_j} f \partial_{x_i} g_t - \partial_{x_i} f \partial_{x_j} g_t\right) X_i^j + (\partial x_j f)\left(\sum_{i=1}^{j-1} r_{ij} h_{ij}\right)$$

$$= \sum_{i\neq j} -h_{ij} X_i^j + (\partial_{x_j} f)\left(\sum_{i=1}^{j-1} r_{ij} h_{ij}\right).$$

So we can choose $X_i^j = r_{ij}\partial_{x_j} f$ if $i<j$ and $X_i^j = 0$ for $i>j$ which gives $X_j^j = -\sum_{i<j} r_{ij}\partial_{x_i} f$. We see that every component X_i^j is holomorphic around $\{\partial_{x_j} f = 0\}$ which means that the vector field X^j is defined on $(\mathbb{C}^n, 0)\times[0,1]$. Moreover, since f is singular at 0, every $\partial_{x_i} f$ cancels at 0 so that each X^j cancels on $\{0\}\times[0,1]$.

The vector field $X = \sum_j X^j + \partial_t$ is the one we wanted. □

Remark 1. *The hypothesis "f has a singular point at 0" is only used to show that the vector field $X - \partial_t$ cancels along the t-axis, which is also true if all the r_{ij} cancel on $\{0\}\times[0,1]$. It is also the case if $g_1 - g_0$ cancels at a high enough order at the origin (the exact order depends on the coefficients h_{ij}).*

3. $\mathscr{R}$-classification of pairs of Morse functions

A pair of Morse functions (f,g) is called $\mathscr{R}$-generic if (up to linear isomorphism) the quadratic parts q_f and q_g are diagonal : $q_f(x) = \sum x_i^2$ and $q_g(x) = \sum \lambda_i x_i^2$ with $\lambda_i \neq \lambda_j$ if $i \neq j$.

Let (f,g) be an $\mathscr{R}$-generic pair of Morse functions. Let us first study the tangency loci: if q_f and q_g are diagonal, $\mathrm{Tang}(q_f,q_g)$ is the union of the coordinate axes; in general, we have the following:

Proposition 4. *The sets* $\mathrm{Tang}(f,g)$ *and* $\mathrm{Tang}(q_f,q_g)$ *are diffeomorphic and tangent.*

Proof. We can suppose f quadratic and q_g diagonal. Blow up the origin to get that (recycling the coordinates x_i as coordinates in the blow-up) the transforms of f and g are given by

$$\tilde{f} = x_1^2(1 + x_2^2 + \ldots + x_n^2) \quad \text{and} \quad \tilde{g} = x_1^2(\lambda_1 + \lambda_2 x_2^2 + \ldots) + x_1^3(\ldots).$$

We will simultaneously compute the tangency locuses $\mathrm{Tang}(f,g)$ and $\mathrm{Tang}(q_f,q_g)$ in the blow-up to show this proposition (since we already know $\mathrm{Tang}(q_f,q_g)$, this will help understand $\mathrm{Tang}(f,g)$). Write $\hat{f} = \tilde{f} = \tilde{q_f}$ and $\hat{g} = \tilde{q_g} + x_1^3\varepsilon$ with $\varepsilon = 0$ or $\varepsilon = x_1^{-3}(\tilde{g} - \tilde{q_g})$.

Note that the genericity hypothesis on the n-uple $(\lambda_1,\ldots,\lambda_n)$ implies that q_f and q_g are not tangent near a point of the surface $\{f=0\}$ (except at 0). In the blow-up, put

$$S := \{1 + x_2^2 + \ldots + x_n^2 = 0\}$$

and $E := \{x_1 = 0\}$. The remark above tells that the components of the tangency locus between $\tilde{q_f}$ and $\tilde{q_g}$ which are different from E do not intersect $E \cap S$. So this is the case for $\hat{f}$ and $\hat{g}$ independently of ε. The change of coordinate $x_1 \mapsto \sqrt{\hat{f}}$ is allowed near each point of E far away from the hypersurface S and every component of $\mathrm{Tang}(\hat{f},\hat{g})$ different from E is far away from this hypersurface (note also that this change of coordinate does not depend on ε).

In these new local coordinates,

$$\hat{f} = x_1^2 \quad \text{and} \quad \hat{g} = x_1^2 u = x_1^2(u_0 + x_1\varepsilon')$$

with u_0 not depending on x_1 and ε' holomorphic far from S ($\varepsilon' = 0$ in case $\varepsilon = 0$). The tangency locus is the union of the varieties given by the equations $x_1 = 0$ and $dx_1 \wedge du = 0$. But $dx_1 \wedge du = dx_1 \wedge (du_0 + x_1 d\varepsilon')$ so on the exceptional divisor, the solutions of $dx_1 \wedge du = 0$ are the same as the solutions of $dx_1 \wedge du_0 = 0$. So the solutions of $dx_1 \wedge du = 0$ on E do not depend on ε, thus they are n simple points corresponding to the axes.

Finally, remark that $dx_1 \wedge du = 0$ is given by $n-1$ equations so its solution set is of dimension at least 1. Each point p solution of these equations on E then gives rise to a set T_p of dimension at least 1, but $T_p \cap E = \{p\}$ so that $\dim(T_p) = 1$. The fact that p is a simple point means that T_p is a simple smooth curve intersecting E transversally. Hence, before blowing up, there were n simple smooth tangency curves tangent to the ones between q_f and q_g, which in addition implies that $\mathrm{Tang}(f, g)$ is diffeomorphic to $\mathrm{Tang}(q_f, q_g)$.

□

Even better :

Proposition 5. *There exists a diffeomorphism ϕ which conjugates* $\mathrm{Tang}(f, g)$ *with* $\mathrm{Tang}(q_f, q_g)$ *and f with q_f.*

Proof. If we suppose that f is quadratic and q_g diagonal, it is enough to find ϕ which conjugates $\mathrm{Tang}(f, g)$ with $\mathrm{Tang}(q_f, q_g)$ and preserves f: $f \circ \phi = f$. Call D_n the x_n-axis and T_n the tangency curve tangent to D_n. It is sufficient to find a diffeomorphism ϕ preserving f and fixing the points of $\{x_n = 0\}$ such that $\phi(D_n) = T_n$. Indeed, applying such a ϕ transforms T_n into D_n, but if $\tilde{\phi}$ is a similar diffeomorphism obtained by exchanging the roles of x_n and x_{n-1}, applying $\tilde{\phi}$ transforms (the new) T_{n-1} into D_{n-1} and stabilizes D_n. We can repeat this for each T_j to obtain a diffeomorphism preserving the fibers of f and conjugating the tangency loci.

The curve T_n is tangent to D_n so that it has equations $x_i = x_n^2 \alpha_i(x_n)$ $(i = 1, \dots, n-1)$. We can then search ϕ in the form

$$\phi(x_1, \dots, x_n) = (x_1 - x_n^2 \alpha_1(x_n), \dots, x_{n-1} - x_n^2 \alpha_{n-1}(x_n), (1 + u)x_n)$$

where u is an unknown holomorphic function. The condition that ϕ preserve f can be written

$$\sum_{i \le n} x_i^2 - 2x_n^2 \sum_{i<n} x_i \alpha_i(x_n) + x_n^4 \sum_{i<n} \alpha_i(x_n)^2 + 2x_n^2 u + x_n^2 u^2 = \sum_{i \le n} x_i^2,$$

that is

$$2u + u^2 = 2 \sum_{i<n} x_i \alpha_i - x_n^2 \sum_{i<n} \alpha_i^2.$$

The implicit function theorem then gives a holomorphic solution $u \in \mathfrak{m}_{(\mathbb{C}^n,0)}$ which in turn gives the desired diffeomorphism ϕ (note that $\phi(x_1, \dots, x_{n-1}, 0) = (x_1, \dots, x_{n-1}, 0)$). □

Proposition 6. *If (f, g) is an $\mathscr{R}$-generic pair of Morse functions then the tangency ideal $I(f, g)$ is radical.*

Proof. Suppose that $f = \sum x_i^2$, $q_g = \sum \lambda_i x_i^2$ and that $T := \mathrm{Tang}(f, g)$ is the union of the axes. Write $df \wedge dg = \sum_{i<j} h_{ij} dx_i \wedge dx_j$ with $h_{ij} = 4(\lambda_j - \lambda_i)x_i x_j + O(\mathfrak{m}_{(\mathbb{C}^n,0)}{}^3)$. The ideal of functions vanishing on T is $\langle x_i x_j \rangle$ so $\langle h_{ij} \rangle \subset \langle x_i x_j \rangle$ and we need to show that $\langle h_{ij} \rangle = \langle x_i x_j \rangle$.

Introduce $N = \frac{n(n-1)}{2}$ and the vectors

$$H = (h_{ij})_{i<j} \in (\mathcal{O}_{(\mathbb{C}^n,0)})^N \quad \text{and} \quad X = (x_i x_j)_{i<j} \in (\mathcal{O}_{(\mathbb{C}^n,0)})^N.$$

Note that $h_{ij} - 4(\lambda_j - \lambda_i)x_i x_j \in \mathfrak{m}_{(\mathbb{C}^n,0)} \langle x_i x_j \rangle$ so that there is a matrix A with coefficients in $\mathcal{O}_{(\mathbb{C}^n,0)}$ such that $H = AX$. Note also that $A = \Lambda + B$ where $\Lambda = diag(4(\lambda_j - \lambda_i))$ is invertible and B has coefficients in $\mathfrak{m}_{(\mathbb{C}^n,0)}$. Hence, A is invertible and the coefficients of the vectors H and X span the same ideal. □

With these propositions, we can use the key lemma to conclude the $\mathscr{R}$-classification of pairs of Morse functions:

Theorem 1. *Let (f_0, g_0) and (f_1, g_1) be two $\mathscr{R}$-generic pairs of Morse functions on $(\mathbb{C}^n, 0)$. Suppose that we can number the tangency curves T_j^i ($j = 1, \dots, n$ and $i = 0, 1$) in such a manner that the pairs of Morse functions $(f_i|_{T_j^i}, g_i|_{T_j^i})$ are conjugated under the action of* $\mathrm{Diff}(T_j^0, T_j^1)$ *on the right. Then there is a diffeomorphism φ such that $(f_0 \circ \varphi, g_0 \circ \varphi) = (f_1, g_1)$.*

Proof. By Proposition 5, we can suppose that $f_0 = f_1 = q_f$ and that the tangency loci for both couples are the same. Then, by hypothesis, $(f, g_0) = (f, g_1)$ in restriction to each tangency curve. Since the ideals $I(f, g_0)$ and $I(f, g_1)$ are radical by Proposition 6, this means that

$$I(f, g_0) = I(f, g_1) \text{ and } g_1 - g_0 \in I(f, g_0).$$

The proof is then completed by Lemma 1. □

In particular, we obtain:

Corollary 1. *An $\mathscr{R}$-generic pair of Morse functions (f, g) is $\mathscr{R}$-conjugated to its quadratic parts if and only if f and g are $\mathbb{C}$-proportional on each tangency curve.*

Remark 2. *Given n smooth curves T_j whose tangents at 0 span $\mathbb{C}^n$ and n couples (u_j, v_j) of Morse functions on T_j, there exists a pair of Morse functions having T_j as tangency curves and equal to (u_j, v_j) on T_j. Indeed, we can suppose that T_j is the x_j-axis so that we can take $f(x_1, \ldots, x_n) = \sum u_j(x_j)$ and $g = \sum v_j(x_j)$.*

Hence, since f can be normalized, the moduli space for generic couples of Morse functions is given by the set of generic non-ordered n-uples $(v_1, \ldots, v_n)$ of germs of Morse functions on $(\mathbb{C}, 0)$ modulo the relation $(v_1, \ldots, v_n) \sim (v_1 \circ (\pm id), \ldots, v_n \circ (\pm id))$, the signs $\pm$ being independent.

Note also the corollary:

Corollary 2. *Let (f_0, g_0) and (f_1, g_1) be two $\mathscr{R}$-generic pairs of Morse functions on $(\mathbb{C}^n, 0)$. If these pairs are topologically conjugated, they are analytically conjugated.*

Proof. First, note that the tangency points between f_0 and g_0 are given by the points where the Milnor number of g_0 restricted to a leaf of f_0 is greater or equal to 1. This characterization of the tangency points shows that a topological conjugacy between both couples respects the tangency curves.

As a consequence the restrictions of the couples (f_i, g_i) to each tangency curve are topologically conjugated, and for each tangency curve C there exists an homeomorphism ϕ of C such that $l_0 \circ \phi = l_1$ for $l = f, g$ on C. For coordinates z, w of C such that $f_0(z) = z^2$ and $f_1(w) = w^2$, this equation writes $\phi(z)^2 = w^2$ so that $\phi(z) = \pm w$. This shows that ϕ is holomorphic and each couples $(f_i, g_i)|_{T_j^i}$ are conjugated under the action of $\mathrm{Diff}(T_j^0, T_j^1)$ on the right.

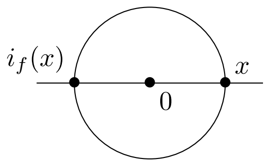
Theorem 1 can then be applied. □

Remark 3. *There is also a link between formal and analytical conjugacy: Artin's approximation theorem shows that if two pairs of germs of Morse functions are formally conjugated, they are also analytically conjugated.*

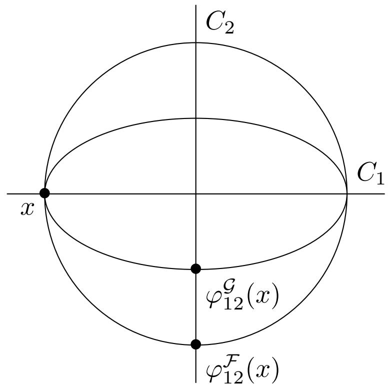
4. Pairs of Morse foliations

As stated in the introduction, two pairs of Morse foliations $(\mathcal{F}_i, \mathcal{G}_i)$ given by the level set of pairs of functions (f_i, g_i) are diffeomorphic if and only if the pairs of Morse functions (f_i, g_i) are $\mathscr{F}$-equivalent. Recall that these pairs are $\mathscr{F}$-equivalent if there exist diffeomorphisms $\varphi \in \mathrm{Diff}(\mathbb{C}^n, 0)$, $\psi_1, \psi_2 \in \mathrm{Diff}(\mathbb{C}, 0)$ such that $(\psi_1 \circ f_0 \circ \varphi, \psi_2 \circ g_0 \circ \varphi) = (f_1, g_1)$. We say that a pair of Morse foliations $(\mathcal{F}, \mathcal{G})$ is $\mathscr{F}$-generic if it has a pair of first integrals (f, g) which is $\mathscr{R}$-generic.

The invariants $(f_i|_{T_j^i}, g_i|_{T_j^i})$ modulo conjugacy on the right are now only defined modulo conjugacy on the right and on the left. First, these new invariants can be re-written in terms of involutions: on $(\mathbb{C}, 0)$, the data of a Morse function modulo conjugacy on the left is equivalent to the data of an involution via $f \mapsto i_f$ where i_f is the function which associates to x the other solution of $f(i_f(x)) = f(x)$.



But some information is lost in the process of considering the invariants modulo conjugacy on the left : for every pair of curves C_1, C_2 transverse to $\mathcal{F}$ and $\mathcal{G}$ and passing through the origin we can consider the holonomy transports $\varphi_{12}^{\mathcal{F}}, \varphi_{12}^{\mathcal{G}}$ from C_1 to C_2 following the leaves of $\mathcal{F}$ or $\mathcal{G}$:



More precisely, we will consider the holonomy transport $\varphi_{ij}^{\mathcal{F}}$ and $\varphi_{ij}^{\mathcal{G}}$ between the tangency curves T_i and T_j. We see on the picture that there are two possible ways to define $\varphi_{nj}^{\mathcal{F}}$ and $\varphi_{nj}^{\mathcal{G}}$, so we have to make a choice (which is equivalent to choosing a local determination of the square root). Put then $\varphi_{njn} = (\varphi_{nj}^{\mathcal{G}})^{-1} \circ \varphi_{nj}^{\mathcal{F}} \in \mathrm{Diff}(T_n)$; this function allows us to recover the pair $(f|_{T_j}, g|_{T_j})$ from $(f|_{T_n}, g|_{T_n})$. Indeed, take two parametrizations $\alpha_j(t)$ and $\alpha_n(t)$ of T_j and T_n such that $\alpha_j = \varphi_{nj}^{\mathcal{F}} \circ \alpha_n$. We want to compute $g \circ \alpha_j$, but $g(\alpha_j(t)) = g((\varphi_{nj}^{\mathcal{G}})^{-1}(\alpha_j(t)))$ and $\alpha_j(t) = \varphi_{nj}^{\mathcal{F}}(\alpha_n(t))$ so $g(\alpha_j(t)) = g(\varphi_{njn}(\alpha_n(t)))$.

Note also that the invariant λ_j/λ_n can be found by taking the linear part of φ_{njn}; hence the following definition:

Definition 1. *Define the invariant of $(\mathcal{F}, \mathcal{G})$ to be $Inv(\mathcal{F}, \mathcal{G}) = ((i_f^n, i_g^n), (\varphi_{njn})_{j<n})$. Two invariants Inv_0, Inv_1 are equivalent if there exists a diffeomorphism $\psi \in \mathrm{Diff}(T_n^0, T_n^1)$ such that $\psi^{-1} \circ Inv_1 \circ \psi = Inv_0$.*

Theorem 2. *Let $(\mathcal{F}_0, \mathcal{G}_0)$ and $(\mathcal{F}_1, \mathcal{G}_1)$ be two $\mathscr{F}$-generic pairs of Morse foliations on $(\mathbb{C}^n, 0)$. Suppose that we can number their tangency curves T_j^i ($j = 1, \ldots, n$ and $i = 0, 1$) such that their invariants $Inv(f, g)$ are equivalent. Then $(\mathcal{F}_0, \mathcal{G}_0)$ and $(\mathcal{F}_1, \mathcal{G}_1)$ are analytically conjugated.*

Proof. Let (f_i, g_i) be first integrals for $(\mathcal{F}_i, \mathcal{G}_i)$; we can suppose that their invariants

$$((i_f^n, i_g^n), (\varphi_{njn})_{j<n})$$

are exactly the same and that $f_0 = f_1 = \sum x_i^2$. We can also compose g_1 with a diffeomorphism on the left in such a manner that $g_0|_{T_n^0} = g_1|_{T_n^1}$ because the involutions i_g^n are the same. Then, as shown above, g_0 and g_1 are equal on each tangency curve because the φ_{njn} are the same.

Hence Theorem 1 can be applied and the pairs (f_i, g_i) are indeed conjugated.

□

Note that for each invariant $((i_1, i_2), (\varphi_{njn})_{j<n})$ there is a pair of Morse foliations having this invariant. Indeed, we can suppose that $i_1 = -id$, $f = \sum x_i^2$ and that T_j is the x_j-axis. Choose g a Morse function on T_n invariant by i_2 and for $p_j = (0, \ldots, 0, x_j, 0, \ldots, 0) \in T_j$ put $g(p_j) = g(\varphi_{njn}(p_n))$ for $p_n = (0, \ldots, 0, x_n)$ with $x_n = x_j$. We thus have for each curve T_j a pair of Morse functions which can be extended to $(\mathbb{C}^n, 0)$ as seen before (in Remark 2).

In order to better understand these invariants, one can find the classification of pairs of involutions in [Vor] or [CM]. In particular, we see that the formal and the analytic classification of pairs of Morse foliations are not the same, because there are some pairs of involutions that are formally but not analytically conjugated.

5. $\mathscr{A}$-classification of pairs of Morse functions

Recall that two couples $\Phi_i = (f_i, g_i)$ are called $\mathscr{A}$-equivalent if there exists two diffeomorphisms $\varphi \in \mathrm{Diff}(\mathbb{C}^n, 0)$ and $\psi \in \mathrm{Diff}(\mathbb{C}^2, 0)$ such that $\psi \circ \Phi_0 \circ \varphi = \Phi_1$. We say that an application $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^2, 0)$ whose components (f, g) are Morse functions is $\mathscr{A}$-generic if the pair (f, g) is $\mathscr{R}$-generic.

Note that the set of such applications Φ is not stable under target diffeomorphisms (for example, the diffeomorphism $(y_1, y_2) \mapsto (y_1, y_2 - \lambda_1 y_1)$ transforms $(\sum x_i^2, \sum \lambda_i x_i^2)$ into $(\sum x_i^2, \sum \mu_i x_i^2)$ with $\mu_1 = 0$). Nevertheless, a pair of functions obtained by a target diffeomorphism from an $\mathscr{R}$-generic pair of Morse functions still has the same tangency locus and is still classified by its values on the tangency locus.

Throughout this section, we will carry on considering pairs of Morse functions to avoid unnecessary notations, but the results extend to pairs $\mathscr{A}$-equivalent to an $\mathscr{R}$-generic pair of Morse functions.

Definition 2. *Let $\Gamma \subset (\mathbb{C}^2, 0)$ be an irreducible curve and $\sigma_1, \sigma_2 : (\mathbb{C}, 0) \to \Gamma$ two parametrizations of Γ. We say that the parametrized curves (Γ, σ_1) and (Γ, σ_2) are σ-equivalent if there is a diffeomorphism $\phi \in \mathrm{Diff}(\mathbb{C}, 0)$ such that $\sigma_1 \circ \phi = \sigma_2$. An equivalence class $[(\Gamma, \sigma)]$ is called a σ-curve; we define its σ-multiplicity to be the integer n such that $\sigma(t) = (at^n + \ldots, bt^n + \ldots)$ with $(a, b) \neq (0, 0)$.*

If the parametrization is clear from the context, we may omit to mention it.

Remark 4. *A σ-curve $[(\Gamma, \sigma)]$ is entirely determined by Γ and its σ-multiplicity.*

A σ-curve $[(\Gamma, \sigma)]$ is of σ-multiplicity 2 in exactly two cases: either Γ is diffeomorphic to a curve $y^2 - x^{2k+1}$ ($k \geq 1$) and σ is a bijection or Γ is smooth and σ is a double cover. The last case happens for example when $\sigma(t) = (t^2, b(t^2))$.

We saw that pairs of Morse functions are classified modulo the action of diffeomorphisms at the source only by the restrictions of $\Phi = (f, g)$ on the tangency curves T_i between f and g, i.e., on the critical set of Φ. Said another way, the classification is given by the functions $\Phi_{|T_i}$ with diffeomorphisms at the source acting as reparametrization, that is by the σ-curves $\Phi(T_i) \subset (\mathbb{C}^2, 0)$.

Each of these σ-curves has σ-multiplicity 2 at the origin and has the line $(t^2, \lambda_i t^2)$ as tangent cone if $f_{|T_i}(t) = t^2 + \ldots$ and $g_{|T_i}(t) = \lambda_i t^2 + \ldots$

Thus the result is the following:

Theorem 3. *Two $\mathscr{A}$-generic pairs of Morse functions Φ_1 and Φ_2 are $\mathscr{A}$-conjugated if and only if the set of σ-curves $\{\Phi_1(T_i^1)\}_{i \leq n}$ and $\{\Phi_2(T_i^2)\}_{i \leq n}$ are conjugated by a diffeomorphism of $(\mathbb{C}^2, 0)$.*

Moreover, for each set of n σ-curves $\{C_i\}$ in $(\mathbb{C}^2, 0)$ with σ-multiplicity 2 and distinct tangent cones, there exists an application $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^2, 0)$ whose components are Morse functions for which $C_i = \Phi(T_i)$.

Remark 5. *A diffeomorphism ψ of $(\mathbb{C}^2, 0)$ conjugates two families of σ-curves $([C_i^1, \sigma_i^1])$ and $([C_i^2, \sigma_i^2])$ if and only if for each i, the σ-curves C_i^1 and C_i^2 have the same multiplicity and ψ conjugates the families of curves (C_i^1) and (C_i^2).*

Proof. Clearly, if two pairs are conjugated by source and target diffeomorphisms, their critical sets are conjugated at the source, so the images of the critical sets are conjugated at the target.

Conversely, suppose that for two generic pairs $\Phi_j = (f_j, g_j)$ there exists a diffeomorphism $\psi \in \mathrm{Diff}(\mathbb{C}^2, 0)$ conjugating the sets of σ-curves $\{\Phi_j(T_i^j)\}_{i \leq n}$. Then we can suppose these sets to be equal, which means that for the right numbering of the tangency curves, the σ-curves $\Phi_1(T_i^1)$ and $\Phi_2(T_i^2)$ are equal for each i. This gives for every i a diffeomorphism $\varphi_i : T_i^1 \to T_i^2$ such that $\Phi_{1|T_i^1} = \Phi_{2|T_i^2} \circ \varphi_i$.

We can then conclude with theorem 1.

For the realization part of the theorem, take n σ-curves C_i in $(\mathbb{C}^2, 0)$ with σ-multiplicity 2 and distinct tangent cones. Note first that we can suppose that no curve has an axe as tangent cone so that these σ-curves can be parametrized by $\sigma_i(t) = (t^2, \lambda_i t^2 + O(t^3)) =: (u_i(t), v_i(t))$ with $\lambda_i \neq 0$. But these curves are the images of the critical locus of the pair $(\sum u_i(x_i), \sum v_i(x_i))$ which is $\mathscr{A}$-generic because $\lambda_i \neq \lambda_j$ if $i \neq j$ and this concludes the proof. □

6. Quotients of Morse functions

Next, consider meromorphic functions $h = g/f$ with $f, g \in \mathcal{O}_{(\mathbb{C}^n,0)}$ Morse functions satisfying the genericity condition. As pointed out in the introduction, two quotients $h_i = g_i/f_i$ are diffeomorphic if and only if the pairs (f_i, g_i) are $\mathscr{Q}$-equivalent, i.e., if there exist a diffeomorphism $\varphi \in \mathrm{Diff}(\mathbb{C}^n, 0)$ and a unity $U \in \mathcal{O}^*_{(\mathbb{C}^n,0)}$ such that $(Uf_0 \circ \varphi, Ug_0 \circ \varphi) = (f_1, g_1)$.

First, consider the critical locus of h : it is given by the zeroes of $\omega = gdf - fdg$, which contain the indeterminacy locus $\{f = 0\} \cap \{g = 0\}$. Note that when $f = \sum x_i^2$ and $g = \sum \lambda_i x_i^2$, the critical locus contains not only $\{f = 0\} \cap \{g = 0\}$ but also the union of the axes. We begin by showing that after a generic perturbation, only the indeterminacy locus remains. Denote by $I(\omega)$ the ideal spanned by the components of ω.

We say that a pair of Morse functions is $\mathscr{Q}$-generic if it is diffeomorphic to

$$\left(\sum x_i^2,\ \sum \lambda_i x_i^2 + \alpha_i x_i^3 + O(\mathfrak{m}^4)\right)$$

with $\lambda_i \neq \lambda_j$ and $\alpha_i \neq 0$; we say that a quotient g/f is $\mathscr{Q}$-generic if the pair (f, g) is $\mathscr{Q}$-generic.

Lemma 2. *For a $\mathscr{Q}$-generic pair of Morse functions (f, g), the ideal $I(\omega)$ contains $\langle f, g\rangle \cdot \mathfrak{m}^4_{(\mathbb{C}^n,0)}$.*

Proof. For simplicity, denote $\mathfrak{m} = \mathfrak{m}_{(\mathbb{C}^n,0)}$. By theorem 1 we can suppose that $f = \sum x_i^2$ and $g = \sum u_i(x_i)$. The genericity hypothesis thus means that $u_i(x_i) = \lambda_i x_i^2 + \alpha_i x_i^3 + O(x_i^4)$ with $\alpha_i \neq 0$. If we write $\omega = \sum \omega_i dx_i$, the coefficient ω_i is

$$\omega_i = 2\sum_{j \neq i} (\lambda_j - \lambda_i) x_i x_j^2 + O(\mathfrak{m}^4)$$

so that $\omega_i = 2x_i(g - \lambda_i f) + O(\mathfrak{m}^4)$. Hence the equalities $x_j\omega_i - x_i\omega_j = 2x_ix_j(\lambda_j - \lambda_i)f + O(\mathfrak{m}^5)$ and $\lambda_j x_j \omega_i - \lambda_i x_i \omega_j = 2x_ix_j(\lambda_j - \lambda_i)g + O(\mathfrak{m}^5)$. As a consequence, for each monomial m of degree 4 except $m = x_k^4$ and each $l = f, g$, we have $ml \in I(\omega) + \mathfrak{m}^7$. Furthermore,

$$\begin{aligned}
\frac{1}{2}\sum_i x_i\omega_i &= \sum_i \frac{1}{2}x_i\left(g\partial_{x_i} f - f\partial_{x_i} g\right) \\
&= g\sum_i \frac{1}{2}x_i\partial_{x_i} f - \frac{1}{2}f\sum_i x_i\partial_{x_i} g \\
&= gf - \frac{1}{2}f\sum_i x_i\partial_{x_i} g \\
&= f\left(g - \sum_i \frac{1}{2}x_i\partial_{x_i} g\right) \\
&= f\left(\frac{-1}{2}\sum_i \alpha_i x_i^3 + O(\mathfrak{m}^4)\right).
\end{aligned}$$

Thus,

$$x_i \sum x_j\omega_j = \beta_i x_i^4 f + \sum_{j\neq i} \beta_j x_j x_i^3 f + O(\mathfrak{m}^7)$$

for some non-zero coefficients β_k, and $x_i^4 f \in I(\omega) + \mathfrak{m}^7$.

A similar computation shows that $x_i^4 g \in I(\omega) + \mathfrak{m}^7$; so for each monomial m of degree 4 and each $l = f, g$, we have $ml \in I(\omega) + \mathfrak{m}^7$. In fact, ml belongs to the ideal $I(\omega) + \langle f, g\rangle \cdot \mathfrak{m}^5$ because $I(\omega)$ is obviously a subset of $\langle f, g\rangle$. It immediately follows that for each index $k \geq 4$, each monomial m of degree k and each $l = f, g$, $ml \in I(\omega) + \langle f, g\rangle \cdot \mathfrak{m}^{k+1}$. This means that ml formally belongs to the ideal $I(\omega)$ hence by flatness, $\langle f, g\rangle \cdot \mathfrak{m}^4 \subset I(\omega)$. □

Remark 6. *Note that the proof is still valid for* 1*-parameter families* $(f_t), (g_t)$ *with fixed* 3*-jets. Indeed, we can show in the exact same way that* $ml \in I(\omega) + \mathfrak{m}^7$ *for each monomial* m *in* x *of degree* 4 *and* $l = f, g$*, the only difference is that* f, g *and* ω *depend on* t *(here* $\mathfrak{m}$ *is still* $\langle x_1, \ldots, x_n\rangle$*).*

Note also that for 1*-parameter families* $(f_t), (g_t)$ *with fixed* 3*-jets, being a* $\mathscr{Q}$*-generic pair of Morse functions for each* $t \in \mathbb{C}$ *is equivalent to being a* $\mathscr{Q}$*-generic pair of Morse functions for* $t = 0$ *because the genericity only depends on the* 3*-jets.*

We thus obtain the following:

Lemma 3. *Consider two functions* $f, g \in \mathcal{O}(t, x_1, \ldots, x_n)$ *defined in a neighborhood of*

$$\mathbb{C}_t \times \{0\} \subset \mathbb{C}_t \times \mathbb{C}_x^n$$

with 3*-jets independent of* t*. Suppose that* $(f(t,\cdot), g(t,\cdot))$ *is a* $\mathscr{Q}$*-generic pair of Morse functions for each* t*. Consider* $\omega_x = g d_x f - f d_x g$ *and* $\mathfrak{m} = \langle x_1, \ldots, x_n\rangle$*, then* $\langle f, g\rangle \mathfrak{m}^4 \subset I(\omega_x)$*.*

Theorem 4. *Let* h_0 *and* h_1 *be* $\mathscr{Q}$*-generic quotients of Morse functions with* $h_i = g_i/f_i$*. Suppose that we have equalities between the* 3*-jets:* $j^3 f_0 = j^3 f_1$ *and* $j^3 g_0 = j^3 g_1$*. Then there exists a diffeomorphism* $\varphi \in \mathrm{Diff}(\mathbb{C}^n, 0)$ *such that* $h_0 \circ \varphi = h_1$*.*

Proof. By Theorem 1, we can suppose that $g_k = \sum_i u_i^k(x_i)$ and $f_k = \sum_i x_i^2$ with

$$u_i^k(x) = \lambda_i x^2 + \alpha_i x^3 + \varepsilon_i^k$$

with $\alpha_i \neq 0$ and $\varepsilon_i^k \in \mathfrak{m}^4_{(\mathbb{C}^n,0)}$. Set for t in a neighborhood of $[0,1]$ in $\mathbb{C}$ $f(t,\cdot) = f_t = f_0 = f_1$, $g(t,\cdot) = g_t = g_0 + t(g_1 - g_0)$, $h(t,\cdot) = h_t = g_t/f_t$ and $\omega = gdf - fdg = \omega_x + rdt$.

Note that $r = -f\partial_t g \in \langle f, g\rangle \mathfrak{m}^4$ and that by Lemma 3, this implies $r \in I(\omega_x)$. We can then find a vector field $X = \sum_i X_i \partial_{x_i} + \partial_t$ such that $\omega(X) = 0$ (note that $X_i \in \langle x_1, \ldots, x_n\rangle$ because there is no linear relation with constant coefficients between the leading terms of the components of ω_x). But this means that h is constant along the trajectories of X so that the flow $\varphi_s(x,t)$ of X (which is defined on a neighborhood of $\{0\} \times [0,1]$) gives a diffeomorphism $\varphi : x \mapsto \varphi_1(x, 0)$ such that $h_0 \circ \varphi = h_1$ on $(\mathbb{C}^n, 0)$. □

Corollary 3. *Let* h *be a* $\mathscr{Q}$*-generic quotient of Morse functions. There exists* $\lambda_i, \alpha_i \in \mathbb{C}^*$ *such that* h *is diffeomorphic to*

$$\frac{\sum_i \lambda_i x_i^2 + \alpha_i x_i^3}{\sum_i x_i^2}.$$

Remark 7. *Since the latter form is stable under homotecies, we can even suppose that* $\alpha_1 = 1$*.*

7. Restriction of a Morse function to a quadratic cone

In this section, we want to study restrictions of Morse functions g to a "quadratic cone" (i.e., a hypersurface $\{f = 0\}$ with f also a Morse function).

Remark 8. *We can see by a cohomological argument that each function and each diffeomorphism defined on a quadratic cone extends to* $(\mathbb{C}^n, 0)$ *(respectively to a function or a diffeomorphism of* $(\mathbb{C}^n, 0)$*). Thus, studying functions on a quadratic cone up to diffeomorphism of the cone is the same as studying functions of* $(\mathbb{C}^n, 0)$ *in restriction to a quadratic cone up to diffeomorphisms of* $(\mathbb{C}^n, 0)$ *fixing the cone.*

Theorem 5. *Let f, g_0 and g_1 be three Morse functions with (f, g_i) $\mathscr{R}$-generic pairs and equalities between the 2-jets $j^2 g_0 = j^2 g_1$. Then there is a diffeomorphism φ such that $f \circ \varphi = f$ and $g_0 \circ \varphi = g_1$ in restriction to $\{f = 0\}$.*

Proof. Let $g_t = g_0 + t(g_1 - g_0)$. We want to find a diffeomorphism φ such that $f \circ \varphi = f$ and $g_0 \circ \varphi - g_1 \in \langle f \rangle$; we will use Moser's path method to find it as the flow of a vector field $X = \sum X_i \partial_{x_i} + \partial_t$ such that $X \cdot g \in \langle f \rangle$ and $X \cdot f = 0$. Note that we can find X verifying $X \cdot g = X \cdot f = 0$ as soon as $\partial_t g \in I(f, g)$, so that we can find X as sought as soon as $\partial_t g \in \langle f \rangle + I(f, g)$. Remark that the components of $X - \partial_t$ will cancel on the t-axis because there is no linear relation with constant coefficients between f and the components of $df \wedge dg$.

We saw in the proof of Proposition 6 that $I(f, g) = \langle x_i x_j + \ldots \rangle$, but x_i^3 is equal to $x_i f$ modulo the ideal $I(f, g) + \mathfrak{m}^4_{(\mathbb{C}^n,0)}$ so that each monomial of degree 3 belongs to $\langle f \rangle + I(f, g) + \mathfrak{m}^4_{(\mathbb{C}^n,0)}$. Thus, the inclusion $\mathfrak{m}^3_{(\mathbb{C}^n,0)} \subset \langle f \rangle + I(f, g)$ holds so that $\partial_t g \in \langle f \rangle + I(f, g)$ and the proof is complete. □

Remark 9. *Note also that g and $g + \lambda f$ represent the same function on $\{f = 0\}$ so that we obtain the following:*

Corollary 4. *Given a Morse function f, each Morse function g such that the pair (f, g) is $\mathscr{R}$-generic is diffeomorphic in restriction to $\{f = 0\}$ to a quadratic function $\sum_{i=1}^{n-1} \lambda_i x_i^2$.*

8. Applications of the Key Lemma

The key lemma can be used in a very general setting for the $\mathscr{R}$-classification of pairs of functions: although the hypotheses might seem strong, they are in fact necessary. For example, it can be applied to rediscover the $\mathscr{R}$-classification of folds.

Definition 3. *Define a fold to be a pair of functions $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ such that f is regular and* $\operatorname{Tang}(f, g)$ *is a simple smooth curve transverse to $\{f = 0\}$.*

Theorem 6. *Let (f, g) be a fold on $(\mathbb{C}^n, 0)$. There exists a unique function $\varphi \in \mathcal{O}(\mathbb{C}, 0)$ and a set of coordinates (x_i) such that $f = x_1$ and $g = \varphi(x_1) + \sum_{i>1} x_i^2$.*

Proof. We can suppose without loss of generality that $f = x_1$ and that $\operatorname{Tang}(f, g)$ is the x_1-axis. This means that $I(f, g) = \langle \partial_{x_i} g \rangle_{i>1} = \langle x_2, \ldots, x_n \rangle$ so $g = \varphi(x_1) + q(x_2, \ldots, x_n) + \varepsilon$ with q a nondegenerate quadratic form and $\varepsilon \in \langle x_2, \ldots, x_n \rangle^2 \mathfrak{m}_{(\mathbb{C}^n,0)}$. Since q is nondegenerate, we can suppose $q = \sum_{i>1} x_i^2$.

We want to use the key lemma in $(\mathbb{C}^n, 0)$ for $f = x_1$, $g_0 = \varphi(x_1) + x_2^2 + \ldots + x_n^2$ and $g_1 = g$. Let us check the hypotheses: first, $g_1 - g_0 = \varepsilon \in \langle x_2, \ldots, x_n \rangle^2 \mathfrak{m}_{(\mathbb{C}^n,0)}$. Then, for each $a \in \langle x_2, \ldots, x_n \rangle^2 \mathfrak{m}_{(\mathbb{C}^n,0)}$, the ideal $I(f, g_0 + a)$ writes $\langle x_2 + \eta_2, \ldots, x_n + \eta_n \rangle$ with

$$\eta_i \in \langle x_2, \ldots, x_n \rangle \mathfrak{m}_{(\mathbb{C}^n,0)},$$

which means that $\operatorname{Tang}(f, g_0 + a)$ is a simple curve and the ideal $I(f, g_0 + a)$ is radical. So the hypotheses $I(f, g_0) = I(f, g_1)$ and $g_1 - g_0 \in I(f, g_0)$ are verified, hence the only hypothesis missing is f having a singular point.

But $g_1 - g_0$ cancels at order 3 at the origin, which will allow us to use the remark 1. Indeed, if we use the same notations, the fact that there is no $\mathbb{C}$-linear relation between the generators of $I(f, g_0)$ implies that the coefficients r_{ij} in the decomposition $g_1 - g_0 = \sum r_{ij} h_{ij}$ cancel on $\{0\} \times [0, 1]$. The lemma can thus be applied and the couples (f, g_0) and (f, g_1) are diffeomorphic.

Last, the function φ is entirely determined by the equality $\varphi \circ f = g$ on $\operatorname{Tang}(f, g)$. □

A first corollary is the classification of regular folds as foliations (i.e., the $\mathscr{F}$-classification):

Corollary 5. *Let $(\mathcal{F}, \mathcal{G})$ be a pair of foliations on $(\mathbb{C}^n, 0)$ given by a fold (f, g) with g regular. Then $(\mathcal{F}, \mathcal{G})$ is diffeomorphic to the pair of foliations given by the first integrals $(x_1, x_1 + \sum_{i>1} x_i^2)$.*

Proof. We can suppose that (f, g) are as in the conclusion of the theorem 6. The hypothesis that g be regular means that φ is a diffeomorphism. In the variables $(\varphi(x_1), x_2, \ldots, x_n)$ the pair $(\mathcal{F}, \mathcal{G})$ is in the right form. □

We can also use this to obtain the $\mathscr{R}$-classification of generic pairs (f, g) with f regular and g a Morse function: this is exactly when φ is a Morse function. In the case of $\mathscr{F}$-equivalence, we obtain the normal form $(x_1, \sum_{i\geq 1} x_i^2)$.

We could also study pairs (f, g) of the form $(x^3 + y^2 + z^2, \lambda x^2 + \mu y^2 + \nu z^2 + \ldots)$, but in this case the tangency ideal $I(f, g)$ will again be radical and this case will be similar to the case of pairs of Morse functions.

The lemma 1 can also be applied for more complicated cases, like for example when the ideal $I(f, g)$ is not radical. To illustrate this, note that if we take $f = x^3 + y^2 + z^2$ and $g = \lambda x^3 + \mu y^2 + \nu z^2$ with $\lambda \neq \mu \neq \nu \neq 0$, the tangency ideal is $I(f, g) = \langle x^2 y, x^2 z, yz\rangle$ and corresponds to $D_x \cup 2D_y \cup 2D_z$ with D_l the l-axis. Let us classify pairs of functions that "look like" this pair. First, recall the following:

Proposition 7. *Let f be a function on $(\mathbb{C}^3, 0)$ having a singular point with Milnor number 2 at the origin; then in a right set of coordinates, $f(x, y, z) = x^3 + y^2 + z^2$.*

Proof. Since the Milnor number of f is 2, the hessian matrix of f at 0 is of rank 2 and in the right set of coordinates, it can be written $diag(0, 2, 2)$. Then $f(x, y, z) = y^2 + z^2 + \varepsilon$ with $\varepsilon \in \mathfrak{m}^3$ and f can be seen as a deformation of $f(0, \cdot, \cdot)$ which has a non-degenerate singular point at 0. By the parametrized Morse lemma, there exists a function φ and a set of coordinates such that $f(x, y, z) = \varphi(x) + y^2 + z^2$.

Since the Milnor number of f is 2, φ is diffeomorphic to x^3 and changing the coordinates once more, we can write $f(x, y, z) = x^3 + y^2 + z^2$. □

So in fact we are interested in pairs (f, g) of functions with Milnor number 2, having hessians $H(f)$, $H(g)$ which can be simultaneously diagonalized with the 0 in the same spot. For such functions, we can then suppose that

$$f = x^3 + y^2 + z^2 \quad \text{and} \quad g = \lambda x^3 + \mu y^2 + \nu z^2 + \varepsilon \tag{1}$$

with $\varepsilon \in \mathfrak{m}^3$ which has no component in x^3.

The tangency locus might not be diffeomorphic to the union of one simple curve and two double curves: the double curves might split. For example, for

$$f = x^3 + y^2 + z^2 \text{ and } g = x^3 + \mu y^2 + \nu z^2 + x^2 y,$$

the y-axis splits into two curves tangent respectively to the y-axis and to the line

$$\{z = 0 = 3(\mu - 1)x - 2y\}.$$

Let us assume the double curves don't split. We will call such a pair (f, g) an exceptional pair of 3-dimensional cusps (or an exceptional pair of cusps because we only deal with the 3-dimensional ones in this example).

Proposition 8. *If (f, g) is an exceptional pair of cusps written as in (1), then* $\operatorname{Tang}(f, g)$ *is tangent and diffeomorphic to the union of the axes. Moreover, the tangency curve tangent to the x-axis is tangent at order 2 with the x-axis.*

The proof is very similar to that of Proposition 4 and is a bit tedious so it is skipped here, but a proof can be found in [T].

Proposition 9. *If (f, g) is an exceptional pair of cusps in the form (1), there exists a diffeomorphism φ preserving f such that* $\operatorname{Tang}(f \circ \varphi, g \circ \varphi)$ *is the union of the axes.*

Once again this proof can be found in [T].

Since the ideal is not radical, the tangency locus is not sufficient to characterize the ideal. The following proposition gives a geometric description of the ideal; it might be interesting in other contexts because it hints at something more general: the characterization of any ideal in terms of cancellation of functions and cancellation of some differential operators on these functions. But I couldn't find mention of such a characterization anywhere, so we only give the following special case:

Proposition 10. *Let (f,g) be an exceptional pair of cusps in the form (1) with* $\mathrm{Tang}(f,g)$ *equal to the union of the axes. Then there is a vector field X such that $X(0)=\partial_x$ and*

$$I(f,g)=\left\{a\in\mathcal{O}_{(\mathbb{C}^3,0)} \text{ such that } a|_{T_x}=a|_{T_y}=a|_{T_z}=0 \text{ and } (X\cdot a)|_{T_y}=(X\cdot a)|_{T_z}=0\right\}.$$

Such a vector field will be said to characterize the tangency ideal.

Proof. The ideal $I(f,g)$ is spanned by the functions $h_1=x^2y+O(\mathfrak{m}^4)$, $h_2=x^2z+O(\mathfrak{m}^4)$ and $h_3=yz+O(\mathfrak{m}^3)$ Note that the tangent cone at 0 of the variety $\{h_3=0\}$ is the union of the planes $\{y=0\}$ and $\{z=0\}$. Moreover, we know by hypothesis that $h_3(T_y)=h_3(T_z)=\{0\}$ so for each z near 0, there is a unique plane tangent to $\{h_3=0\}$ at the point $(0,0,z)$. This plane contains the direction T_z so it is defined by another direction $X(z)$ which we can choose regular in z with $X(0)=\partial_x$. Similarly, the tangent plane to $\{h_3=0\}$ along T_y is defined by a vector field along T_y which we can choose so that both vector fields can be extended to a vector field X on $(\mathbb{C}^3,0)$ with $X(0)=\partial_x$.

Now let

$$J=\{a\in\mathcal{O}_{(\mathbb{C}^3,0)} \text{ such that } a|_{T_x}=a|_{T_y}=a|_{T_z}=0 \text{ and } (X\cdot a)|_{T_y}=(X\cdot a)|_{T_z}=0\}.$$

The set J is an ideal and we first need to show that $I(f,g)\subset J$, i.e., that $(X\cdot h_i)|_{T_l}=0$ for $i=1,2,3$ and $l=y,z$. By construction, $(X\cdot h_3)|_{T_y}$ and $(X\cdot h_3)|_{T_z}$ are null. Next, we know that $h_1\in\langle xy,yz,zx\rangle$ so up to changing h_1 by $h_1-\sum_{i=2,3}\lambda_ih_i$ with $\lambda_i\in\mathfrak{m}_{(\mathbb{C}^3,0)}$, we can suppose that $h_1=ux^2y+x\alpha(y)+x\beta(z)$ with u invertible, α and β in $\mathfrak{m}^3_{(\mathbb{C},0)}$.

The condition that the tangency curves do not split implies that when cutting the curve T_y by a plane $y=y_0$, we obtain a point with multiplicity 2. But if $\alpha\neq 0$, then $\alpha(y_0)$ is generically invertible and $h_1(\cdot,y_0,\cdot)$ is generically regular. The function $h_3(\cdot,y_0,\cdot)$ is also generically regular, so if $\alpha\neq 0$, we obtain a simple point; hence $\alpha=0$. By the same reasons, $\beta=0$ and

$$I(f,g)=\langle x^2y,h_2,h_3\rangle.$$

Similarly, $I(f,g)=\langle x^2y,x^2z,h_3\rangle$ and it is now clear that $I(f,g)\subset J$.

For the converse, we will show that (x^2y,x^2z,h_3) generate J: suppose $a\in J$ and P is his leading homogeneous polynomial (and let $k+1$ be his degree). Since $J\subset\langle xy,yz,zx\rangle$, P has no term in l^{k+1} for $l=x,y$ or z. The only terms that are not spanned by the leading coefficients of x^2y,x^2z or h_3 are the xl^k for $l=y,z$. But if $X=(1+a_1)\partial_x+a_2\partial_y+a_3\partial_z$, then $X\cdot xy^k=(1+a_1)y^k+ka_2xy^{k-1}$ is not nul on T_y: there can't be such a term in P. Therefore (x^2y,x^2z,h_3) generate J and $I(f,g)=J$. □

Proposition 11. *If (f,g) is an exceptional pair of cusps in the form (1), there exists a diffeomorphism φ preserving f such that $I(f\circ\varphi,g\circ\varphi)=\langle x^2y,x^2z,yz\rangle$.*

Proof. By Proposition 9, we can suppose that the tangency locus is the union of the axes. By Proposition 10, we can find a vector field X such that $X(0)=\partial_x$ characterizing the tangency ideal. We want to transform X into ∂_x using a diffeomorphism φ preserving f and the coordinate axes.

As before we will construct φ in two steps by transforming the vector field first on the y-axis and then on the z-axis. We will search the first diffeomorphism in the form

$$\varphi_1(x,y,z)=(x+yxa(y),y+yxb(y),z+yxc(y)),$$

so that

$$\varphi_1^*\partial_x = (1 + ya, yb, yc).$$

We see that for each vector field X tangent to ∂_x at 0, its restriction to the y-axis can be obtained this way. Note that φ_1 fixes $\{y = 0\}$ and preserves the y-axis so that if we do the same construction for the z-axis, the newly constructed diffeomorphism φ_2 will preserve the vector field along the y-axis. Hence $\varphi = \varphi_2\varphi_1$ will conjugate $I(f, g)$ with $\langle x^2y, x^2z, yz\rangle$. □

Theorem 7. *Let (f_0, g_0) and (f_1, g_1) be two exceptional pairs of cusps on $(\mathbb{C}^3, 0)$ with tangency curves T_j^i ($i = 0, 1$, $j = 1, 2, 3$ and T_1^i is the simple one). Suppose that there is a diffeomorphism ψ conjugating the tangency curves and the restrictions $(f_i|_{T_j^i}, g_i|_{T_j^i})$. Then there exists a diffeomorphism φ such that $(f_0 \circ \varphi, g_0 \circ \varphi) = (f_1, g_1)$.*

Proof. After what has been done before, we can suppose that each couple is in the form (1), with tangency ideals $I = \langle x^2y, x^2z, yz\rangle$, with $f_0 = f_1$ everywhere and $g_0 = g_1$ in restriction to the tangency locus T.

Let X be a vector field characterizing the ideal I. If Y is tangent to T, then $\lambda X + \mu Y$ also characterizes I for all $\lambda, \mu \in \mathcal{O}_{(\mathbb{C}^3,0)}$ with λ not vanishing on T so we can suppose that $X \in \mathrm{Ker}(df_0)$ at every point of T (note that $\mathrm{Ker}(df_0)$ is transverse to T at each point different from the origin). By definition of the tangency locus, X then also belongs to the kernel of dg_i for each i on T, hence $g_1 - g_0 \in I$.

The key lemma can then be applied to finish the proof of this theorem. □

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O.Thom, IRMAR, Université de Rennes 1, IRMAR 6625, Campus de Beaulieu, 35042 Rennes Cedex, France

E-mail address: olivier.thom@univ-rennes1.fr

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SMOOTH ARCS ON ALGEBRAIC VARIETIES

DAVID BOURQUI AND JULIEN SEBAG

ABSTRACT. Let k be a field and V be a k-variety. We say that a rational arc $\gamma \in \mathscr{L}_\infty(V)(k)$ is *smooth* if its formal neighborhood $\mathscr{L}_\infty(V)_\gamma$ is an infinite-dimensional formal disk. In this article, we prove that every rational arc $\gamma \in (\mathscr{L}_\infty(V) \setminus \mathscr{L}_\infty(V_{\mathrm{sing}}))(k)$ is smooth if and only if the formal branch containing γ is smooth.

1. Introduction

1.1. The present article is partly motivated by the exegesis of the following statement with respect to singularity theory. This result was obtained by M. Grinberg and D. Kazhdan in case the base field k is contained in $\mathbf{C}$, and by V. Drinfeld for an arbitrary field k (see [8, 6], or [4] for a generalization of such a statement in the context of formal geometry).

Theorem 1.2. *Let k be a field. Let V be a k-variety, and $v \in V(k)$ be a rational point of V. We assume that $\dim_v(V) \geq 1$. Let $\gamma \in \mathscr{L}_\infty(V)(k)$ be a rational point of the associated arc scheme, not contained in $\mathscr{L}_\infty(V_{\mathrm{sing}})$ such that $\gamma(0) = v$. If $\mathscr{L}_\infty(V)_\gamma$ denotes the formal neighborhood of the k-scheme $\mathscr{L}_\infty(V)$ at the point γ, there exists an affine k-scheme S of finite type, with $s \in S(k)$, and an isomorphism of formal k-schemes:*

$$\mathscr{L}_\infty(V)_\gamma \cong S_s \hat{\times}_k \operatorname{Spf}(k[[(T_i)_{i\in\mathbf{N}}]]). \tag{1.1}$$

1.3. Since the work of J. Nash, which introduced the so-called *Nash problem*, one knows that the geometry of $\mathscr{L}_\infty(V)$ is deeply related to the geometry of the singularities of V. As an illustration of this general principle at the level of formal neighborhoods, let us mention the following easy and well-known fact: for every rational arc $\gamma \in \mathscr{L}_\infty(V)(k)$, with origin $v := \gamma(0)$ contained in the smooth locus of V, the formal neighborhood $\mathscr{L}_\infty(V)_\gamma$ is isomorphic to the infinite-dimensional k-formal disk $\mathbf{D}_k^{\mathbf{N}} := \operatorname{Spf}(k[[(T_i)_{i\in\mathbf{N}}]])$. If we translate this remark in the terms of theorem 1.2, it means that, in this case, S can be chosen equal to $\operatorname{Spec}(k)$. In fact, we observe that, in this case, the corresponding algebra $\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}$ is formally smooth over k for the discrete topology. Indeed, one may assume that V is affine and smooth and that there is an étale map $V \to \mathbf{A}^d$. By [14, Lemme 3.3.6] one then has $\mathscr{L}_\infty(V) \cong V \times_{\mathbf{A}_k^d} \mathscr{L}_\infty(\mathbf{A}_k^d)$ thus by [9, Chapter 0, 19.3.3, 19.3.5 (ii)] the k-algebra $\mathcal{O}(\mathscr{L}_\infty(V)) \cong \mathcal{O}(V)[(T_i)_{i\in\mathbf{N}}]$ is formally smooth for the discrete topology. By [9, Chapter 0, 19.3.5 (iv)], this is also the case for $\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}$. In this general context, we address the following natural question:

Question 1.4. *Does the converse property hold true? In other words, if $S = \operatorname{Spec}(k)$ in theorem 1.2, is it true that $\gamma(0)$ is a smooth point of V?*

With respect to theorem 1.2, a positive answer in the direction of question 1.4 clearly indicates that the formal k-scheme S_s in the Drinfeld-Grinberg-Kazhdan theorem would be a measure of the singularities of V at the origin $\gamma(0)$ of the involved arc γ. Since the authors proved in [3] that, in general, theorem 1.2 does not hold if the involved arc γ belongs to $\mathscr{L}_\infty(V_{\mathrm{sing}})$, it seems natural to us, in this perspective, to restrict ourselves to the case of arcs *not* contained in $\mathscr{L}_\infty(V_{\mathrm{sing}})$, that we call *non-degenerate*.

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1.5. In the present paper, we provide a complete answer to question 1.4 for non-degenerate arcs (which are in particular contained in a unique irreducible component of $\mathrm{Spec}(\widehat{\mathcal{O}_{V,\gamma(0)}})$, by proposition 3.6). Precisely we obtain the following statement:

Theorem 1.6. *Let V be a k-variety and $v \in V(k)$ such that $\mathcal{O}_{V,v}$ is reduced and $\dim_v(V) \geq 1$. Let $\gamma \in \mathscr{L}_\infty(V)(k)$ be a non-degenerate rational arc, such that $\gamma(0) = v$. Then, the following conditions are equivalent:*

(1) *The unique formal branch containing γ is smooth.*
(2) *The formal neighborhood $\mathscr{L}_\infty(V)_\gamma$ is isomorphic to $\mathbf{D}_k^{\mathbf{N}}$.*

Let us note that by [9, Chapter 0, 19.3.6, 19.5.4] the second condition in the statement of theorem 1.6 characterizes those non-degenerate rational arcs γ whose local ring $\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}$ is formally smooth over k for the $\mathfrak{m}$-adic topology. In the case of curves, we are able to interpret the above result in terms of a notion of *rigidity* for deformations of arcs (see corollary 4.14). We also obtain analogs of theorem 1.6 in the case of constant arcs (in particular degenerate) and in the context of jet schemes (see proposition 5.2 and theorem 5.4).

1.7. **Conventions, notation.** In this article, k is a field of arbitrary characteristic (unless explicitly stated otherwise); $k[[T]]$ is the ring of power series over the field k. The category of k-schemes is denoted by $\mathfrak{Sch}_k$. The local k-algebra $k[[(T_i)_{i\in\mathbf{N}}]]$ is the completion of $k[(T_i)_{i\in\mathbf{N}}]$ with respect to the maximal ideal $\langle (T_i)_{i\in\mathbf{N}}\rangle$. It is a topological complete k-algebra when we endow it with the projective limit topology. We denote by $\mathbf{D}_k^{\mathbf{N}} := \mathrm{Spf}(k[[(T_i)_{i\in\mathbf{N}}]])$ the associated formal k-scheme. A *k-variety* is a k-scheme of finite type. The singular locus V_{sing} of V is defined as the (unique) reduced closed subscheme associated with the non-smooth locus of V. An arc of V, i.e., a point of the arc scheme $\mathscr{L}_\infty(V)$ associated with V, which is *not* contained in the singular locus V_{sing} of V, is called a *non-degenerate* arc. In other words, the subset $\mathscr{L}_\infty(V) \setminus \mathscr{L}_\infty(V_{\mathrm{sing}})$ is the set of non-degenerate arcs. In this article, by slightly abusing the standard conventions, we introduce the terminology of *smooth rational arcs* on V to designate those arcs $\gamma \in \mathscr{L}_\infty(V)(k)$ such that $\mathscr{L}_\infty(V)_\gamma \cong \mathbf{D}_k^{\mathbf{N}}$ (assuming that the dimension at the origin $\gamma(0)$ of the arc is positive).

2. Arc schemes and arc deformations: recollection

2.1. If V is a k-variety and $n \in \mathbf{N}$, the restriction *à la* Weil of the $k[T]/\langle T^{n+1}\rangle$-scheme

$$V \times_k \mathrm{Spec}(k[T]/\langle T^{n+1}\rangle)$$

with respect to the morphism of k-algebras $k \hookrightarrow k[T]/\langle T^{n+1}\rangle$ exists; it is a k-scheme of finite type which is called the *n-jet scheme* of V and that we denote by $\mathscr{L}_n(V)$. The projective limit $\varprojlim_n(\mathscr{L}_n(V))$ exists in the category of k-schemes; it is the *arc scheme* associated with V and we denote it by $\mathscr{L}_\infty(V)$. For every integer $n \in \mathbf{N}$, the canonical morphism of k-schemes $\pi_n^\infty \colon \mathscr{L}_\infty(V) \to \mathscr{L}_n(V)$ is called the *truncation morphism* of level n. Let A be a k-algebra. As proved in [1], there exists a natural bijection

$$\mathrm{Hom}_{\mathfrak{Sch}_k}(\mathrm{Spec}(A), \mathscr{L}_\infty(V)) \cong \mathrm{Hom}_{\mathfrak{Sch}_k}(\mathrm{Spec}(A[[T]]), V). \tag{2.1}$$

Let us note that in case V is affine or A is local, such a property directly follows from the mere definitions.

2.2. We denote by $\mathfrak{Lacp}$ the following category. The objects are the topological local k-algebras, which are topologically isomorphic to $\mathfrak{m}$-adic completions of local k-algebras and whose residue field are k-algebras isomorphic to k. The morphisms in $\mathfrak{Lacp}$ are the continuous morphisms of local k-algebras. We denote by $\mathfrak{Tes}$ the full subcategory of $\mathfrak{Lacp}$ whose objects are *test-rings*, i.e., local k-algebras in $\mathfrak{Lacp}$ with nilpotent maximal ideal and residue field isomorpic to k. If $\widehat{\mathfrak{Tes}}$ is the category of pre-cosheaves on the category $\mathfrak{Tes}$ (i.e., covariant functors from the category $\mathfrak{Tes}$ to the category of sets), we define the functor

$$\begin{array}{rccc} F\colon & \mathfrak{Lacp} & \longrightarrow & \widehat{\mathfrak{Tes}} \\ & \widehat{\mathcal{O}} & \longmapsto & \mathrm{Hom}_{\mathfrak{Lacp}}(\widehat{\mathcal{O}}, \cdot). \end{array}$$

One has the following seemingly standard observation (see [6]):

Observation 2.3. The functor F is fully faithful.

One will use the following trivial consequence of the observation: let S and S' be k-schemes, let $s \in S(k)$ and $s \in S'(k)$, let S_s and $S'_{s'}$ be the associated formal neighborhoods and let $f_A\colon S_s(A) \to S'_{s'}(A)$ be a natural map defined for every test-ring $(A, \mathfrak{m}_A)$; then there exists a unique morphism of formal k-schemes $f\colon S_s \to S'_{s'}$ inducing f_A for every test-ring A; moreover, f is an isomorphism if and only if f_A is bijective for every A.

2.4. Let V be a k-variety. Let $\gamma \in \mathscr{L}_\infty(V)(k)$. Then, in the sense of observation 2.3, the formal k-scheme $\mathscr{L}_\infty(V)_\gamma$ is *uniquely* determined by the functor $F(\widehat{\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}})$. Let A be a test-ring. Let $\gamma_A \in \mathscr{L}_\infty(V)_\gamma(A)$. The datum of γ_A corresponds to one of the following (equivalent) commutative diagram:

$$\begin{tikzcd} \mathcal{O}_{\mathscr{L}_\infty(V),\gamma} \arrow[r, "\gamma_A"] \arrow[d] & A \arrow[d] \\ k \arrow[r, equal] & k \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}_{V,v} \arrow[r, "\gamma_A"] \arrow[d, "\gamma"'] & A[[T]] \arrow[d, "p_A"] \\ k[[T]] \arrow[r, equal] & k[[T]], \end{tikzcd} \qquad \begin{tikzcd} \widehat{\mathcal{O}_{V,v}} \arrow[r, "\gamma_A"] \arrow[d, "\gamma"'] & A[[T]] \arrow[d, "p_A"] \\ k[[T]] \arrow[r, equal] & k[[T]], \end{tikzcd} \tag{2.2}$$

where we denote by $p_A\colon A[[T]] \to k[[T]]$ the unique local morphism which extends the projection $A \to A/\mathfrak{m}_A \cong k$. The set $\mathscr{L}_\infty(V)_\gamma(A)$ parametrizes the elements $\gamma_A \in V(A[[T]])$ whose reduction modulo $\mathfrak{m}_A$ coincides with γ.

Definition 2.5. *Every morphism $\gamma_A \in \mathscr{L}_\infty(V)_\gamma(A)$ is called an* A-deformation *of γ.*

3. Reduction to formal branches

Definition 3.1. *Let V be a k-variety. Let $\gamma \in \mathscr{L}_\infty(V)(k)$ be a rational arc, viewed as a local morphism $\gamma\colon \widehat{\mathcal{O}_{V,\gamma(0)}} \to k[[T]]$. A* formal branch *(or* formal component*) at $\gamma(0)$ which contains γ is a minimal prime ideal $\mathfrak{p}$ of $\widehat{\mathcal{O}_{V,\gamma(0)}}$ such that $\mathfrak{p} \subset \operatorname{Ker}(\gamma)$.*

In particular, if $\mathfrak{p}$ is such a branch, this definition implies that γ factorizes through the quotient morphism $\widehat{\mathcal{O}_{V,\gamma(0)}} \to \widehat{\mathcal{O}_{V,\gamma(0)}}/\mathfrak{p}$. A classical fact on arc geometry is that every arc on a reduced variety factorizes through the irreducible components of the involved variety which contain the origin of the arc. In the same spirit, the following lemma shows in particular that the formal neighborhood of a given arc contained in a unique formal branch of a reduced variety carries a part of the information on the mere singularities of the formal branch containing the arc.

Proposition 3.2. *Let V be a k-variety. Let $\gamma \in \mathscr{L}_\infty(V)(k)$ be a rational arc contained in a unique formal branch $\mathfrak{p}$. We assume that $\mathcal{O}_{V,\gamma(0)}$ is reduced. Then, for every test-ring $(A, \mathfrak{m}_A)$, for every A-deformation $\gamma_A \in \mathscr{L}_\infty(V)_\gamma(A)$ of γ, the induced morphism of admissible local k-algebras $\gamma_A\colon \widehat{\mathcal{O}_{V,\gamma(0)}} \to A[[T]]$ factorizes through $\widehat{\mathcal{O}_{V,\gamma(0)}} \to \widehat{\mathcal{O}_{V,\gamma(0)}}/\mathfrak{p}$. Besides, the ideal $\mathfrak{p}$ is the only minimal prime ideal with this property.*

In other words, if the arc γ is contained in a unique formal branch at $\gamma(0)$, then every A-deformation of γ is contained in the same branch (and only in this one).

Proof. Let $(A, \mathfrak{m}_A)$ be a test-ring and $\gamma_A \in \mathscr{L}_\infty(V)_\gamma(A)$, corresponding to a diagram of morphisms of complete local k-algebras:

$$\begin{tikzcd} \widehat{\mathcal{O}_{V,\gamma(0)}} \arrow[r, "\gamma_A"] \arrow[dr, "\gamma"'] & A[[T]] \arrow[d] \\ & k[[T]]. \end{tikzcd} \tag{3.1}$$

Then, we have $\mathrm{Ker}(\gamma) = \gamma_A^{-1}(\mathfrak{m}_A[[T]])$. Let $\mathfrak{p}, \mathfrak{q}_1, \dots, \mathfrak{q}_n$ be the minimal prime ideals of the ring $\widehat{\mathcal{O}_{V,\gamma(0)}}$. By assumptions, $\mathrm{Ker}(\gamma)$ contains $\mathfrak{p}$ and does not contain $\mathfrak{q}_i$ for every $i \in \{1, \dots, n\}$. Let us prove that $\mathfrak{p} \subset \mathrm{Ker}(\gamma_A)$.

Let $x \in \mathfrak{p}$. Since the ring $\widehat{\mathcal{O}_{V,\gamma(0)}}$ is reduced, we have $\mathfrak{p} \cap (\cap_{i=1}^n \mathfrak{q}_i) = \langle 0 \rangle$. By assumption, for every integer $i \in \{1, \dots, n\}$, there exists an element $y_i \in \mathfrak{q}_i$ such that $y_i \notin \mathrm{Ker}(\gamma)$. Then, we deduce that $xy_1 \dots y_n = 0$ and that

$$\begin{array}{lcl} \gamma_A(xy_1 \dots y_n) & = & 0 \\ \gamma_A(x) \cdot \gamma_A(y_1) \dots \gamma_A(y_n) & = & 0. \end{array} \tag{3.2}$$

Since, by construction, $y_i \notin \gamma_A^{-1}(\mathfrak{m}_A[[T]])$ for every integer $i \in \{1, \dots, n\}$, we conclude that the element $\gamma_A(y_i)$ does not reduce to zero modulo $\mathfrak{m}_A[[T]]$. In particular (see lemma 3.3), the element $\gamma_A(y_i)$ is not a zero-divisor in the ring $A[[T]]$; hence, by equation (3.2), we have $\gamma_A(x) = 0$, i.e., $x \in \mathrm{Ker}(\gamma_A)$.

In the end, if there exists $i \in \{1, \dots, n\}$ such that $\mathfrak{q}_i \subset \gamma_A^{-1}(0) = \mathrm{Ker}(\gamma_A)$, then we have $\mathfrak{q}_i \subset \gamma_A^{-1}(\mathfrak{m}_A[[T]]) = \mathrm{Ker}(\gamma)$, which contradicts our assumption. It concludes the proof of our statement. ☐

Lemma 3.3. *Let $(A, \mathfrak{m}_A)$ be a test-ring, let $r_A(T) \in A[[T]]$ whose reduction modulo $\mathfrak{m}_A[[T]]$ is a non-zero element of $k[[T]]$. Then, the power series $r_A(T)$ is not a zero-divisor in $A[[T]]$.*

Proof. By the Weierstrass preparation theorem (see [12, Chapter IV, Theorem 9.2]), there is a decompostion $r_A(T) = q_A(T)u_A(T)$ where $q_A(T)$ is a distinguished polynomial and $u_A(T)$ is invertible in $A[[T]]$. By the uniqueness in the Weierstrass division theorem, (see [12, Chapter IV, Theorems 9.1 and 9.2]) a distinguished polynomial is not a zero-divisor in $A[[T]]$. ☐

Remark 3.4. In particular, under the assumptions of proposition 3.2 with V reduced, the arc γ is contained in a unique irreducible component passing through $\gamma(0)$, and every A-deformation of γ is contained in this irreducible component.

Remark 3.5. If one does not assume that the arc γ belongs to a unique formal branch, and $\dim(\mathcal{O}_{V,v}) \geq 2$, it is important to keep in mind that the situation is much more complicated and proposition 3.2 does not hold anymore. Let us consider the example of the affine k-surface

$$V = \mathrm{Spec}(k[X,Y,Z]/\langle Y^2 - X^3 - X \rangle).$$

It is an integral k-variety and $\widehat{\mathcal{O}_{V,\mathfrak{o}}} \simeq k[[U,V,W]]/\langle UV \rangle$, where we denote by $\mathfrak{o}$ the origin of $\mathbf{A}_k^3$. Let $A = k[S]/\langle S^2 \rangle$ and $s := \bar{S}$. We observe that the arc γ, defined by $U \mapsto 0, V \mapsto 0, W \mapsto T$, admits the A-deformation $\gamma_A(T) = (s, s, T)$, which is not contained in any formal branch of V at the origin $\mathfrak{o}$.

Proposition 3.6. *Let V be a k-variety. Let $\gamma \in \mathscr{L}_\infty(V)(k)$ be a rational arc. If the arc γ is non-degenerate, then the arc γ is contained in a unique formal branch.*

Proof. In $\mathrm{Spec}(k[[T]])$, we denote by 0 the closed point, and η the generic point. Let us note that the arc γ is non-degenerate if and only if the point $\gamma(\eta)$ does not belong to V_{sing}. Up to shrinking V, we may assume that the k-variety V is affine and reduced. We also may assume that $\dim(\mathcal{O}_{V,\gamma(0)}) \geq 1$. The arc γ corresponds a morphism of local k-algebras $\gamma \colon \mathcal{O}_{V,\gamma(0)} \to k[[T]]$ which extends to a morphism of local k-algebras $\widehat{\gamma} \colon \widehat{\mathcal{O}_{V,\gamma(0)}} \to k[[T]]$. We denote by $\mathfrak{M}$ the maximal ideal of $\mathcal{O}(V)$ corresponding to $\gamma(0)$. First assume that $\mathrm{Ker}(\gamma)$ contains at least two distinct minimal prime ideals of $\mathcal{O}_{V,\gamma(0)}$; in more geometric terms, that γ lies on at least two distinct irreducible components passing through $\gamma(0)$. Then $(\mathcal{O}_{V,\gamma(0)})_{\mathrm{Ker}(\gamma)} \cong \mathcal{O}_{V,\gamma(\eta)}$ is not a domain, thus $\gamma(\eta)$ is not a smooth point of V and $\gamma \in \mathscr{L}_\infty(V_{\mathrm{sing}})$.

Now consider the general case. Let $\mathcal{O}_{V,\gamma(0)}^h$ be the henselization of $\mathcal{O}_{V,\gamma(0)}$. One has

$$\mathcal{O}_{V,\gamma(0)}^h = \varinjlim B_{\mathfrak{q}},$$

where the limit is taken over all étale $\mathcal{O}(V)$-algebras B localized at a prime $\mathfrak{q}$ such that

$$\mathfrak{q} \cap \mathcal{O}(V) = \mathfrak{M} \quad \text{and} \quad \kappa(\mathfrak{q}) = \kappa(\mathfrak{M}).$$

By [16, Tag 0CB3], one may find such a $(B, \mathfrak{q})$ such that the morphism $B_{\mathfrak{q}} \to \mathcal{O}^h_{V,\gamma(0)}$ induces a bijection on the level of minimal prime ideals. On the other hand, by [16, Tag 0C2E], the morphism $\mathcal{O}^h_{V,\gamma(0)} \to \widehat{\mathcal{O}_{V,\gamma(0)}}$ also induces a bijection on the level of minimal prime ideals. Let $\gamma_B \colon B \to k[[T]]$ (resp. $\gamma_{B_{\mathfrak{q}}} \colon B_{\mathfrak{q}} \to k[[T]]$) be the morphism induced by $\widehat{\gamma}$. Assuming that $\operatorname{Ker}(\widehat{\gamma})$ contains at least two distinct minimal prime ideals, we deduce that the same holds for $\operatorname{Ker}(\gamma_{B_{\mathfrak{q}}})$. By the particular case treated above, one infers that $B_{\operatorname{Ker}(\gamma_B)}$ is not a domain, in particular $\operatorname{Ker}(\gamma_B) = \gamma_B(\eta)$ is not a smooth point of $\operatorname{Spec}(B)$. Since $\operatorname{Spec}(B) \to V$ is étale and maps $\gamma_B(\eta)$ to $\gamma(\eta)$, the point $\gamma(\eta)$ is not a smooth point of V by [10, Chapitre 4, 17.11.1]. $\square$

4. The proof of theorem 1.6

Let $\mathfrak{p}$ be the unique formal branch containing γ, $v := \gamma(0)$ and $\widehat{\mathcal{O}_{\mathfrak{p},v}} := \widehat{\mathcal{O}_{V,v}}/\mathfrak{p}$ be the corresponding local ring.

4.1. Let us show first 1 $\Rightarrow$ 2. This implication is a direct consequence of the following proposition, which is a corollary of proposition 3.2, and of proposition 3.6.

Proposition 4.2. *Let V be a k-variety and $\gamma \in \mathscr{L}_\infty(V)(k)$ be an arc with $v = \gamma(0)$ which is assumed to be contained in a unique formal branch. We assume that $\mathcal{O}_{V,v}$ is reduced and* $\dim_v(V) \geq 1$. *Assume that the formal branch $\mathfrak{p}$ containing γ is smooth. Then the formal k-scheme $\mathscr{L}_\infty(V)_\gamma$ is isomorphic to $\mathbf{D}_k^{\mathbf{N}}$.*

Proof. Let $(A, \mathfrak{m}_A)$ be a test-ring. By assumption, there exists an integer $d \geq 1$ such that

$$\widehat{\mathcal{O}_{\mathfrak{p},v}} \xrightarrow{\sim} k[[S_1, \ldots, S_d]].$$

By proposition 3.2, the A-deformations of γ are in natural bijection with the set of local morphisms $\widehat{\mathcal{O}_{\mathfrak{p},v}} \to A[[T]]$. This set is itself in natural bijection with $\mathfrak{m}_A^{\mathbf{N}}$. By observation 2.3, the k-formal schemes $\mathscr{L}_\infty(V)_\gamma$ and $\mathbf{D}_k^{\mathbf{N}}$ are isomorphic. $\square$

4.3. We prove now 2 $\Rightarrow$ 1. We have to show that the k-algebra $\widehat{\mathcal{O}_{\mathfrak{p},v}}$ is isomorphic (in $\mathfrak{Lacp}$) to a k-algebra of power series in a finite number of variables. Our proof is based on different ingredients which are established in subsections 4.4, 4.6; the main arguments are presented in subsection 4.9.

4.4. Let us start by establishing a basic result. Keep the notation of theorem 1.2.

Lemma 4.5. *Let V be a k-variety and $v \in V(k)$ such that $\mathcal{O}_{V,v}$ is reduced and $\dim_v(V) \geq 1$. Let $\gamma \in \mathscr{L}_\infty(V)(k)$ be a non-degenerate rational arc with $\gamma(0) = v$. Assume that the formal neighborhood $\mathscr{L}_\infty(V)_\gamma$ is isomorphic to $\mathbf{D}_k^{\mathbf{N}}$ and that the minimal prime ideal $\mathfrak{p}$ of $\widehat{\mathcal{O}_{V,v}}$ corresponds to the formal branch containing γ. Let $(B, \mathfrak{m}_B)$ be a local ring. Then, every morphism of local k-algebras $\widehat{\mathcal{O}_{\mathfrak{p},v}} \to (B/\mathfrak{m}_B^2)[[T]]$ lifts to a morphism of local k-algebras $\widehat{\mathcal{O}_{\mathfrak{p},v}} \to \widehat{B}[[T]]$.*

Proof. First, since we have $\mathscr{L}_\infty(V)_\gamma \cong \mathbf{D}_k^{\mathbf{N}}$, we observe that, for every *surjective* morphism of test-rings $f \colon A' \to A$, the natural map

$$\mathfrak{m}_{A'}^{\mathbf{N}} \cong \operatorname{Hom}_{\mathfrak{Lacp}}(\widehat{\mathcal{O}_{\mathfrak{p},v}}, A'[[T]]) \xrightarrow{f\circ \cdot} \operatorname{Hom}_{\mathfrak{Lacp}}(\widehat{\mathcal{O}_{\mathfrak{p},v}}, A[[T]]) \cong \mathfrak{m}_A^{\mathbf{N}}$$

is surjective. Hence, starting from a morphism $\varphi_2 \colon \widehat{\mathcal{O}_{\mathfrak{p},v}} \to B/\mathfrak{m}_B^2[[T]]$, we may construct, by induction, a family of morphisms $\varphi_n \colon \widehat{\mathcal{O}_{\mathfrak{p},v}} \to B/\mathfrak{m}_B^{n+2}[[T]]$, for every integer $n \in \mathbf{N}$, which makes, for every pair $(m,n) \in \mathbf{N}^2$ of integers with $m \geq n$, the following diagram of morphisms in $\mathfrak{Lacp}$ commute

$$\begin{tikzcd} \widehat{\mathcal{O}_{\mathfrak{p},v}} \arrow[r, "\varphi_m"] \arrow[d, equal] & B/\mathfrak{m}_B^{2+m} \arrow[d, "\pi"] \\ \widehat{\mathcal{O}_{\mathfrak{p},v}} \arrow[r, "\varphi_n"'] & B/\mathfrak{m}_B^{2+n}, \end{tikzcd}$$

where we denote by π the canonical projection. By the very definition, we have constructed a morphism $\varphi\colon \widehat{\mathcal{O}_{\mathfrak{p},v}} \to \widehat{B}[[T]]$ lifting φ_2. □

For every noetherian local k-algebra B, we have $B/\mathfrak{m}_B^n \cong \widehat{B}/\widehat{\mathfrak{m}}_B^n$ for every integer $n \geq 1$ by [13, §8]. Under this assumption, the arguments developed in the proof of lemma 4.5 imply in particular that the set of liftings of φ_2 can be identified with $k[[(T_i)_{i\in\mathbf{N}}]]$.

4.6. Using the following lemma, we shall, in some sense, reduce the proof of the theorem 1.6 to the case of a complete intersection. This kind of reduction is a classical "trick" in the construction of motivic measures (see [5] or, e.g., [14]).

Lemma 4.7. *Let V be an affine k-variety, defined by the datum of an ideal I_V of the polynomial ring $k[X_1,\dots,X_N]$ and $\gamma \in \mathscr{L}_\infty(V)(k)$ be a non-degenerate arc. Then, there exist an integer $M \in \{0,\dots,N\}$ and elements $F_1,\dots,F_M \in I_V$, such that:*

(1) *There exists an $(M\times M)$-minor of the jacobian matrix $(\partial_{X_j}F_i)_{i,j}$ which does not vanish at γ.*
(2) *Setting*
$$V' := \operatorname{Spec}(k[X]/\langle F_1,\dots,F_M\rangle),$$
the morphism of formal k-schemes $\mathscr{L}_\infty(V)_\gamma \cong \mathscr{L}_\infty(V')_\gamma$ induced by the closed immersion $V \hookrightarrow V'$ is an isomorphism.

Proof. Let us denote by J_V the ideal generated by the elements $h\delta \in k[X_1,\dots,X_N]$, where δ is an $(M\times M)$-minor of the jacobian matrix of a M-tuple $(F_1,\dots,F_M)$ of elements of I_V, for some integer $M \in \mathbf{N}$, and $h \in (\langle F_1,\dots,F_M\rangle : I_V)$. Using the jacobian criterion, one may show (see [7, §0.2], [17, §4]) that the singular locus V_{sing} of V, i.e., the reduced closed subscheme associated with the non-smooth locus, is the support of the closed subscheme of V associated with the datum of the ideal $I_V + J_V$. Since $\gamma \notin \mathscr{L}_\infty(V_{\text{sing}})(k)$, we obtain all the desired properties, using lemma 4.8 below for the last one. □

Lemma 4.8. *Let V' be an affine k-variety, V be a closed k-subscheme of V' and*
$$h \in (0 : I_V) \subset \mathcal{O}(V').$$
Let $\gamma \in \mathscr{L}_\infty(V)(k)$ such that $h(\gamma) \neq 0$. Then, still denoting by γ the image of γ in V', the natural morphism of formal schemes $\mathscr{L}_\infty(V)_\gamma \to \mathscr{L}_\infty(V')_\gamma$ is an isomorphism of formal k-schemes.

Proof. It suffices to show that for every test-ring A the induced map $\mathscr{L}_\infty(V)_\gamma(A) \to \mathscr{L}_\infty(V')_\gamma(A)$ is bijective. Injectivity is clear; so let us show surjectivity. We pick out $\gamma_A \in \mathscr{L}_\infty(V')_\gamma(A)$ and $G \in I_V$. We have to show that $G(\gamma_A) = 0$. By hypothesis, one has $h(\gamma_A)G(\gamma_A) = 0$. Since $h(\gamma) \neq 0$, the reduction of $h(\gamma_A)$ modulo $\mathfrak{m}_A$ is not zero. By lemma 3.3, one infers that $G(\gamma_A) = 0$. □

4.9. We are ready to complete the proof of theorem 1.6, by proving implication 2 ⇒ 1. We may assume that $V \hookrightarrow \mathbf{A}_k^N$ is affine and, thanks to proposition 3.2 and remark 3.4, irreducible. Let M, $F_1, F_2,\dots,F_M$, δ and h be the elements provided by lemma 4.7 and set $d := N - M$. Up to renumbering, we may assume that δ is the determinant of the matrix $(\partial_{X_{d+j}}(F_i))_{i,j\in\{1,\dots,M\}}$ and that $\operatorname{ord}_T(\delta(\gamma(T))$ is minimal among the T-orders of the evaluation at $\gamma(T)$ of the $(M\times M)$-minors of the jacobian matrix $(\partial_{X_j}(F_i))_{\substack{i\in\{1,\dots,M\}\\ j\in\{1,\dots,N\}}}$. Moreover, up to a translation, we may assume that v is the origin of $\mathbf{A}_k^{d+M}$. For every integer $i \in \{1, \ldots, d+M\}$, we will denote by $\widehat{x_i}$ the image of X_i in $\widehat{\mathcal{O}_{\mathfrak{p},v}}$. Note that, since $h\delta$ does not vanish at γ, the element $h\delta$ does not vanish identically on V; hence,we have $\dim(V) = d$.

We shall identify $\gamma(T)$ with a tuple $(x_j(T))_{i\in\{1,\ldots,N\}} \in k[[T]]^{d+M}$ which satisfies, for every integer $i \in \{1, \ldots, M\}$, the equation

$$F_i((x_j(T))_{j\in\{1,\ldots,d+M\}}) = 0.$$

Using the second property of lemma 4.7, for every test-ring $(A, \mathfrak{m}_A)$, an element of $\mathscr{L}_\infty(V)_\gamma(A)$ may and shall be identified with a tuple $(x_{1,A}(T), \ldots, x_{d+M,A}(T))$ of elements of $\mathfrak{m}_A[[T]]^{d+M}$ such that, for every integer $i \in \{1, \ldots, M\}$,

$$F_i((x_j(T) + x_{j,A}(T))_{j\in\{1,\ldots,d+M\}}) = 0.$$

We denote by $A_{d,2}$ the test-ring $k[S_1, \ldots, S_d]/\langle S_1, \ldots, S_d\rangle^2$ and by s_i the image of S_i in $A_{d,2}$. By lemma 4.11, there exists an element $(x_{1,A_{d,2}}(T), \ldots, x_{d+M,A_{d,2}}(T)) \in \mathscr{L}_\infty(V)_\gamma(A_{d,2})$ such that, for every integer $i \in \{1, \ldots, d\}$,

$$x_{i,A_{d,2}}(T) = s_i.$$

By proposition 3.2, there exists a morphism $\widehat{\mathcal{O}_{\mathfrak{p},v}} \to A_{d,2}[[T]]$ which maps $\hat{x}_i$ to s_i for every integer $i \in \{1, \ldots, d\}$. Since the formal k-scheme $\mathscr{L}_\infty(V)_\gamma$ is isomorphic to $\mathbf{D}_k^{\mathbf{N}}$, by lemma 4.5, there exists a morphism $\widehat{\mathcal{O}_{\mathfrak{p},v}} \to k[[S_1, \ldots, S_d]][[T]]$ which maps, for every integer $i \in \{1, \ldots, d\}$, the element $\hat{x}_i$ to an element of $S_i + \langle S_1, \ldots, S_d\rangle^2[[T]]$. Specializing to $T = 0$, we deduce from lemma 4.10 that the induced morphism $\widehat{\mathcal{O}_{\mathfrak{p},v}} \to k[[S_1, \ldots, S_d]]$ is surjective. Its kernel is a prime ideal of $\widehat{\mathcal{O}_{\mathfrak{p},v}}$. Since $\widehat{\mathcal{O}_{\mathfrak{p},v}}$ is an integral domain of dimension d, this prime ideal is necessarily zero, by the Hauptidealsatz. We deduce the existence of a continuous isomorphism

$$\widehat{\mathcal{O}_{\mathfrak{p},v}} \xrightarrow{\sim} k[[S_1, \ldots, S_d]]$$

of admissible local k-algebras, which shows the desired result by [10, 17.5.3].

For the convenience of the reader, we state and prove the following version of the inverse function theorem for formal power series, probably well-known among the specialists.

Lemma 4.10. *Let $d \geq 1$ be an integer. Let $\mathfrak{m}$ be the maximal ideal of the local k-algebra $k[[S_1, \ldots, S_d]]$. Let $\varphi\colon k[[S_1, \ldots, S_d]] \to k[[S_1, \ldots, S_d]]$ be a morphism of local k-algebras which induces an isomorphism of k-vector spaces $\varphi_1\colon \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2$. Then, the morphism φ is an isomorphism.*

Proof. For every integer $n \geq 1$, we deduce from the assumption a k-linear map

$$\varphi_n\colon \mathfrak{m}^n/\mathfrak{m}^{n+1} \to \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

defined by $\varphi_n(\bar{P}) = \overline{\varphi(P)}$ for every power series $P \in k[[S_1, \ldots, S_d]]$. For every integer $n \geq 1$, the map φ_n is surjective. Indeed, for every $y_1, \ldots, y_n \in \mathfrak{m}$, there exists $x_1, \ldots, x_n \in \mathfrak{m}$ such that $\varphi_2(\bar{x}_i) := \overline{\varphi(x_i)} = \bar{y}_i$ for every integer $i \in \{1, \ldots, n\}$. The element $x := x_1 \ldots x_n$ is a preimage of $y := y_1 \ldots y_n$ by φ_n which concludes the proof of our claim.

Since, for every integer $n \in \mathbf{N}$, the k-vector space $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is finite dimensional, we conclude that the map φ_n are bijective. We deduce the assertion from [2, III/§2/Corollaire 3]. □

Let us recall a convention of subsection 4.9. If a rational arc $\gamma(T)$ is identified with a tuple $(x_j(T))_{i\in\{1,\ldots,N\}} \in k[[T]]^{d+M}$ which satisfies, for every integer $i \in \{1, \ldots, M\}$, the equation

$$F_i((x_j(T))_{j\in\{1,\ldots,d+M\}}) = 0,$$

then, for every test-ring $(A, \mathfrak{m}_A)$, an element of $\mathscr{L}_\infty(V)_\gamma(A)$ may be identified with a tuple

$$(x_{1,A}(T), \ldots, x_{d+M,A}(T))$$

of elements of $\mathfrak{m}_A[[T]]^{d+M}$ such that, for every integer $i \in \{1, \ldots, M\}$,

$$F_i((x_j(T) + x_{j,A}(T))_{j\in\{1,\ldots,d+M\}}) = 0.$$

Lemma 4.11. *Keep the notation and convention of subsection 4.9. Let $(A, \mathfrak{m}_A)$ be a test-ring such that $\mathfrak{m}_A^2 = 0$. Then, the natural application*

$$\begin{array}{ccc} \mathscr{L}_\infty(V)_\gamma(A) & \longrightarrow & (\mathfrak{m}_A[[T]])^d \\ (x_{1,A}(T), \dots, x_{d+M,A}(T)) & \longmapsto & (x_{1,A}(T), \dots, x_{d,A}(T)) \end{array}$$

is bijective.

Proof. We denote by $\mathcal{J}$ the jacobian matrix $[\partial_{X_j} F_i]_{\substack{i \in \{1,\dots,M\} \\ j \in \{1,\dots,d+M\}}}$. Recall that

$$\det \left([\partial_{X_j} F_i]_{\substack{i \in \{1,\dots,M\} \\ j \in \{d+1,\dots,d+M\}}} \right)$$

does not vanish at $\gamma(T)$. Using the Taylor expansion and the fact that $\mathfrak{m}_A^2 = 0$, we observe that, for every tuple $(x_{1,A}(T), \dots, x_{d+M,A}(T)) \in \mathfrak{m}_A[[T]]^{d+M}$, the conditions

$$\forall i \in \{1, \dots, M\} \quad F_i(x_j(T) + x_{j,A}(T))_{j \in \{1,\dots,d+M\}} = 0$$

are equivalent to the condition

$$\mathcal{J}(\gamma(T)) \cdot \begin{pmatrix} x_{1,A}(T) \\ \vdots \\ x_{d+M,A}(T) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Using lemmas 3.3 and 4.12, we deduce that there exist elements $(b_{i,j}(T))_{\substack{i \in \{1,\dots,M\} \\ j \in \{1,\dots,d\}}}$ in $k[[T]]$ such that latter condition is equivalent to the system

$$x_{d+i,A}(T) = \sum_{j=1}^{d} b_{i,j}(T) \cdot x_{j,A}(T), \quad i \in \{1, \dots, M\}.$$

That concludes the proof. □

Lemma 4.12. *Let k be a field, and d, M be positive integers. Let*

$$\mathcal{M} = \left[(\mathcal{M}_{i,j})_{\substack{1 \le i \le M \\ 1 \le j \le d+M}} \right]$$

be a $(M \times (d+M))$ matrix with coefficients in $k[[T]]$. Assume that

$$\mu := \operatorname{ord}_T \left(\det \left[(\mathcal{M}_{i,j})_{\substack{1 \le i \le M \\ d+1 \le j \le d+M}} \right] \right)$$

is an integer, minimal among the orders of the $(M \times M)$-minors of the matrix $\mathcal{M}$. Then there exists an $(M \times M)$ matrix $\mathcal{N}$ with coefficients in $k[[T]]$, whose determinant is not zero, such that

$$\mathcal{N} \cdot \mathcal{M} = \begin{pmatrix} a_{1,1} & \dots & a_{1,d} & T^\mu & 0 & \dots & 0 \\ a_{2,1} & \dots & a_{2,d} & 0 & T^\mu & \dots & 0 \\ \vdots & \dots & \vdots & 0 & 0 & \ddots & 0 \\ a_{M,1} & \dots & a_{M,d} & 0 & 0 & \dots & T^\mu \end{pmatrix} \tag{4.1}$$

$$\forall (i,j) \in \{1, \dots, M\} \times \{1, \dots, d\} \quad \operatorname{ord}_T(a_{i,j}) \ge \mu. \tag{4.2}$$

Proof. This obvious remark was originally made in [5, p. 216]. Write

$$\det \left(\left[(\mathcal{M}_{i,j})_{\substack{1 \le i \le M \\ d+1 \le j \le d+M}} \right] \right) = T^\mu \, u(T)$$

with $u(T) \in k[[T]]^\times$ and set

$$\mathcal{N} = u(T)^{-1} \mathrm{ad} \left(\left[(\mathcal{M}_{i,j})_{\substack{1 \le i \le M \\ d+1 \le j \le d+M}} \right] \right).$$

Clearly equation (4.1) holds. Moreover for $(i,j) \in \{1,\dots,M\} \times \{1,\dots,d\}$ the coefficient $a_{i,j}$ is a linear combination of $(M \times M)$-minors of the matrix $\mathcal{M}$ with coefficients in $k[[T]]$. Hence, formula (4.2) also holds. $\square$

4.13. Let k be a field. Let $(\mathscr{C}, c)$ be an integral k-curve, geometrically unibranch at $c \in \mathscr{C}(k)$. Let $\gamma \in \mathscr{L}_\infty(\mathscr{C})(k)$ be a primitive k-parametrization of $\mathscr{C}$ at c[1]. We say that γ is a *rigid arc*[2] if, for every test-ring $(A, \mathfrak{m}_A)$, for every A-deformation $\gamma_A \in \mathscr{L}_\infty(V)_\gamma(A)$, there exists a unique power series $r_A(T) \in \mathfrak{m}_A[[T]]$ such that $\gamma_A(T) = \gamma(T + r_A(T))$. In the particular case of curves, we may interpret theorem 1.6 as follows.

Corollary 4.14. *Let k be a field. Let $\mathscr{C}$ be an integral k-curve and $c \in \mathscr{C}(k)$. We assume that $(\mathscr{C}, c)$ is geometrically unibranch. Let γ be a primitive k-parametrization at c. Then the following conditions are equivalent:*

(1) *The germ $(\mathscr{C}, c)$ is smooth.*
(2) *The formal neighborhood $\mathscr{L}_\infty(\mathscr{C})_\gamma$ is isomorphic to $\mathbf{D}_k^{\mathbf{N}}$.*
(3) *The arc γ is rigid.*
(4) *Let $\pi\colon \bar{\mathscr{C}} \to \mathscr{C}$ be the normalization of $\mathscr{C}$ and $\bar{\gamma}$ the unique lifting of γ to $\bar{\mathscr{C}}$; then the morphism of formal k-schemes $\mathscr{L}_\infty(\bar{\mathscr{C}})_{\bar{\gamma}} \to \mathscr{L}_\infty(\mathscr{C})_\gamma$ induced by π is an isomorphism.*

Proof. By theorem 1.6 and standard remarks, we only have to show implication 4$\Rightarrow$ 3. Let us assume that γ is a primitive k-parametrization at c such that the morphism of formal k-schemes $\mathscr{L}_\infty(\bar{\mathscr{C}})_{\bar{\gamma}} \to \mathscr{L}_\infty(\mathscr{C})_\gamma$ induced by the normalization $\pi\colon \bar{\mathscr{C}} \to \mathscr{C}$ is an isomorphism, and let us show that γ is rigid. Note that $\bar{\gamma}$ is the unique isomorphism $\widehat{\mathcal{O}_{\bar{\mathscr{C}},\bar{c}}} \xrightarrow{\sim} k[[T]]$ such that $\gamma = \hat{\pi} \circ \bar{\gamma}$. Let $(A, \mathfrak{m}_A)$ be a test-ring. For every power series $r_A \in \mathfrak{m}_A[[T]]$, one has

$$\gamma(r_A(T) + T) \quad = \hat{\pi}(\bar{\gamma}(r_A(T) + T))$$

By assumption, $\bar{\gamma}_A(T) \mapsto \hat{\pi}(\bar{\gamma}_A(T))$ is a natural bijection from $\mathscr{L}_\infty(\bar{\mathscr{C}})_{\bar{\gamma}}(A)$ onto $\mathscr{L}_\infty(\mathscr{C})_\gamma(A)$. Since $\bar{\gamma}$ is rigid, we conclude that γ is rigid too, which concludes the proof of the implication. $\square$

5. Related problems

5.1. A slight variation on an argument of [11, proof of Proposition 1.1] also allows to describe the *constant* arcs whose formal neighborhood is isomorphic to $\mathbf{D}_k^{\mathbf{N}}$ (in arbitrary dimensions), i.e., smooth constant arcs. We denote by σ the canonical section of the projection

$$\pi_0^\infty\colon \mathscr{L}_\infty(V) \to \mathscr{L}_0(V) \cong V.$$

Thus, for every $v \in V$, the point $\sigma(v)$ of $\mathscr{L}_\infty(V)$ is the associated constant arc.

Proposition 5.2. *Let V be a k-variety and $v \in V(k)$ such that $\dim_v(V) \geq 1$. Then the following conditions are equivalent:*

(1) *The k-variety V is smooth at v.*
(2) *The formal neighborhood $\mathscr{L}_\infty(V)_{\sigma(v)}$ is isomorphic to $\mathbf{D}_k^{\mathbf{N}}$.*

In other words, smooth constant arcs on V correspond to smooth points of V.

Proof. We only have to show implication $2 \Rightarrow 1$. By [10, 17.5.1, 17.5.3], it suffices to show that the local k-algebra $\widehat{\mathcal{O}_{V,v}}$ is formally smooth for the $\mathfrak{m}_v$-adic topology (which coincides here with the projective limit topology). By [9, 19.3.3,19.3.6] and the hypothesis, the k-algebra $\widehat{\mathcal{O}_{\mathscr{L}_\infty(V),\sigma(v)}}$ is formally smooth for the projective limit topology. Since the continuous morphism

$$\widehat{\mathcal{O}_{V,v}} \to \widehat{\mathcal{O}_{\mathscr{L}_\infty(V),\sigma(v)}}$$

induced by the projection $\mathscr{L}_\infty(V) \to V$ admits a continuous retraction (induced by σ) we may conclude the proof by the very definition of formal smoothness. $\square$

[1] If k is assumed to be perfect, the assumption that c is geometrically unibranch guarantees the existence of primitive k-parametrizations at c.

[2] An analogous notion has been originally introduced in [15] for constant arcs.

5.3. For non-degenerate arcs centered at a unibranch point, we have an analog of theorem 1.6 with regards to the smoothness of the truncations of the involved arc.

Theorem 5.4. *Let V be a k-variety and $v \in V(k)$. We assume that $\widehat{\mathcal{O}_{V,v}}$ is a domain. Let $\gamma \in \mathscr{L}_\infty(V)(k)$ be a rational non-degenerate arc with $\gamma(0) = v$. Then the following conditions are equivalent:*

1. *The k-variety V is smooth at v.*
2. *There exists an integer $n \in \mathbf{N}$ such that $\gamma_n := \pi_n^\infty(\gamma)$ is a smooth point of the jet scheme $\mathscr{L}_n(V)$.*
3. *For every $n \in \mathbf{N}$, the point γ_n is a smooth point of $\mathscr{L}_n(V)$.*

Implication 1 $\Rightarrow$ 3 is well-kown (e.g., see [14, Lemme 3.4.2]); 3 $\Rightarrow$ 2 is formal. In the end, the proof of implication 2 $\Rightarrow$ 1 is very similar to the proof of theorem 1.6. Indeed, we have to mimick the original proof and replace the use of lemma 4.5 by that of the following lemma, whose proof is completely similar to that of lemma 4.5.

Lemma 5.5. *Let V be a k-variety and $v \in V(k)$ such that $\mathcal{O}_{V,v}$ is reduced and $\dim_v(V) \geq 1$. Let $\gamma \in \mathscr{L}_\infty(V)(k)$ be a non-degenerate rational arc with $\gamma(0) = v$. Let $n \in \mathbf{N}$ be an integer. Assume that the formal neighborhood $\mathscr{L}_n(V)_{\gamma_n}$ is isomorphic to $\mathbf{D}_k^r$ and that the minimal prime ideal $\mathfrak{p}$ of $\widehat{\mathcal{O}_{V,v}}$ corresponds to the formal branch containing γ. Let $(B, \mathfrak{m}_B)$ be a local ring. Then, every morphism of local k-algebras $\widehat{\mathcal{O}_{\mathfrak{p},v}} \to (B/\mathfrak{m}_B^2)[T]/\langle T^{n+1}\rangle$ lifts to a morphism of local k-algebras $\widehat{\mathcal{O}_{\mathfrak{p},v}} \to \widehat{B}[T]/\langle T^{n+1}\rangle$.*

Remark 5.6. This completes in particular a result of [11]. In *loc. cit.*, S. Ishii shows that the jet scheme $\mathscr{L}_n(V)$ is not smooth at any constant jet centered at a non-smooth point of V (see the proof of proposition 1.1 in *op.cit.*). Theorem 5.4 shows that $\mathscr{L}_n(V)$ is not smooth at *any* jet which is the truncation of a non-degenerate arc centered at a non-smooth unibranch point of V.

Remark 5.7. If the reduced germ (V, v) is no longer assumed analytically irreducible, even if the formal branch containing γ is smooth, the truncations γ_n can be *non*-smooth points of the corresponding jet scheme in general. This is already clear for $n = 0$ but this may fail more generally for every n. For example let $V = \operatorname{Spec}(k[X, Y]/\langle X\,Y\rangle)$ and $\gamma(T) = (T, 0)$; then one may check that for every non-negative integer n one has

$$\widehat{\mathcal{O}_{\mathscr{L}_n(V),\gamma_n}} \cong k[[X_0, \ldots, X_n, Y]]/\langle X_0^{n+1} Y\rangle.$$

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Institut de recherche mathématique de Rennes, UMR 6625 du CNRS, Université de Rennes 1, Campus de Beaulieu, 35042 Rennes cedex (France)
E-mail address: david.bourqui@univ-rennes1.fr

Institut de recherche mathématique de Rennes, UMR 6625 du CNRS, Université de Rennes 1, Campus de Beaulieu, 35042 Rennes cedex (France)
E-mail address: julien.sebag@univ-rennes1.fr

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INTERSECTION SPACES, EQUIVARIANT MOORE APPROXIMATION AND THE SIGNATURE

MARKUS BANAGL AND BRYCE CHRIESTENSON

ABSTRACT. We generalize the first author's construction of intersection spaces to the case of stratified pseudomanifolds of stratification depth 1 with twisted link bundles, assuming that each link possesses an equivariant Moore approximation for a suitable choice of structure group. As a by-product, we find new characteristic classes for fiber bundles admitting such approximations. For trivial bundles and flat bundles whose base has finite fundamental group these classes vanish. For oriented closed pseudomanifolds, we prove that the reduced rational cohomology of the intersection spaces satisfies global Poincaré duality across complementary perversities if the characteristic classes vanish. The signature of the intersection spaces agrees with the Novikov signature of the top stratum. As an application, these methods yield new results about the Goresky-MacPherson intersection homology signature of pseudomanifolds. We discuss several nontrivial examples, such as the case of flat bundles and symplectic toric manifolds.

1. Introduction

Classical approaches to Poincaré duality on singular spaces are Cheeger's L^2 cohomology with respect to suitable conical metrics on the regular part of the space ([16], [15], [17]), and Goresky-MacPherson's intersection homology [22], [23], depending on a perversity parameter $\bar{p}$. More recently, the first author has introduced and investigated a different, spatial perspective on Poincaré duality for singular spaces ([3]). This approach associates to certain classes of singular spaces X a cell complex $I^{\bar{p}}X$, which depends on a perversity $\bar{p}$ and is called an *intersection space* of X. Intersection spaces are required to be generalized rational geometric Poincaré complexes in the sense that when X is a closed oriented pseudomanifold, there is a Poincaré duality isomorphism $\widetilde{H}^i(I^{\bar{p}}X;\mathbb{Q}) \cong \widetilde{H}_{n-i}(I^{\bar{q}}X;\mathbb{Q})$, where n is the dimension of X, $\bar{p}$ and $\bar{q}$ are complementary perversities in the sense of intersection homology theory, and $\widetilde{H}^*, \widetilde{H}_*$ denote reduced singular (or cellular) cohomology and homology.

The resulting homology and cohomology theories

$$HI^{\bar{p}}_*(X) = H_*(I^{\bar{p}}X;\mathbb{Q}) \quad \text{and} \quad HI^*_{\bar{p}}(X) = H^*(I^{\bar{p}}X;\mathbb{Q})$$

are *not* isomorphic to intersection (co)homology $I^{\bar{p}}H_*(X;\mathbb{Q})$, $I_{\bar{p}}H^*(X;\mathbb{Q})$. Since its inception, the theory $HI^*_{\bar{p}}$ has so far had applications in areas ranging from fiber bundle theory and computation of equivariant cohomology ([4]), K-theory ([3, Chapter 2.8], [37]), algebraic geometry (smooth deformation of singular varieties ([10], [11]), perverse sheaves [8], mirror symmetry [3, Chapter 3.8]), to theoretical Physics ([3, Chapter 3], [8]). For example, the approach of intersection spaces makes it straightforward to define intersection K-groups by $K^*(I^{\bar{p}}X)$. These techniques are not accessible to classical intersection cohomology. There are applications to L^2-theory as well: In [9], for every perversity $\bar{p}$ a Hodge theoretic description of the theory

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$\widetilde{H}I^*_{\bar{p}}(X;\mathbb{R})$ is found; that is, a Riemannian metric on the top stratum (which is in fact a fiberwise scattering metric and thus very different from Cheeger's class of metrics) and a suitable space of L^2 harmonic forms with respect to this metric (the extended weighted L^2 harmonic forms for suitable weights) which is isomorphic to $\widetilde{H}I^*_{\bar{p}}(X;\mathbb{R})$. A de Rham description of $HI^*_{\bar{p}}(X;\mathbb{R})$ has been given in [5] for two-strata spaces whose link bundle is flat with respect to the isometry group of the link.

At present, intersection spaces have been constructed for isolated singularities and for spaces with stratification depth 1 whose link bundles are a global product, [3]. Constructions of $I^{\bar{p}}X$ in some depth 2 situations have been provided in [7]. The fundamental idea in all of these constructions is to replace singularity links by their Moore approximations, a concept from homotopy theory Eckmann-Hilton dual to the concept of Postnikov approximations. In the present paper, we undertake a systematic treatment of twisted link bundles. Our method is to employ equivariant Moore approximations of links with respect to the action of a suitable structure group for the link bundle.

Equivariant Moore approximations are introduced in Section 3. On the one hand, the existence of such approximations is obstructed and we give a discussion of some obstructions. For instance, if S^{n-1} is the fiber sphere of a linear oriented sphere bundle, then the structure group can be reduced so as to allow for an equivariant Moore approximation to S^{n-1} of degree k, $0 < k < n$, if and only if the Euler class of the sphere bundle vanishes (Proposition 12.1). If the action of a group G on a space X allows for a G-equivariant map $X \to G$, then the existence of a G-equivariant Moore approximation to X of positive degree k implies that the rational homological dimension of G is at most $k-1$. On the other hand, we present geometric situations where equivariant Moore approximations exist. If the group acts trivially on a simply connected CW complex X, then a Moore approximation of X exists. If the group acts cellularly and the cellular boundary operator in degree k vanishes or is injective, then X has an equivariant Moore approximation. Furthermore, equivariant Moore approximations exist often for the effective Hamiltonian torus action of a symplectic toric manifold. For instance, we prove (Proposition 12.3) that 4-dimensional symplectic toric manifolds always possess T^2-equivariant Moore approximations of any degree.

In Section 6, we use equivariant Moore approximations to construct fiberwise homology truncation and cotruncation. Throughout, we use homotopy pushouts and review their properties (universal mapping property, Mayer-Vietoris sequence) in Section 2. Proposition 6.5 relates the homology of fiberwise (co)truncations to the intersection homology of the cone bundle of the given bundle. Of fundamental importance for the later developments is Lemma 6.6, which shows how the homology of the total space of a bundle is built up from the homology of the fiberwise truncation and cotruncation. In order to prove these facts, we employ a notion of precosheaves together with an associated local to global technique explained in Section 4. Proposition 6.7 establishes Poincaré duality between fiberwise truncations and complementary fiberwise cotruncations.

At this point, we discover a new set of *characteristic classes*

$$\mathcal{O}_i(\pi,k,l) \subset H^d(E;\mathbb{Q}),\ d = \dim E,\ i = 0,1,2,\ldots,$$

defined for fiber bundles $\pi : E \to B$ which possess degree k,l fiberwise truncations (Definition 6.8). We show that these characteristic classes vanish if the bundle is a global product (Proposition 6.11). Furthermore, they vanish for flat bundles if the fundamental group of the base is finite (Theorem 7.1). On the other hand, we construct in Example 6.13 a bundle π for which $\mathcal{O}_2(\pi,2,1)$ does not vanish. The example shows also that the characteristic classes $\mathcal{O}_*$ seem to be rather subtle, since the bundle of the example is such that all the differentials of its Serre spectral sequence do vanish.

Now the relevance of these characteristic classes vis-à-vis Poincaré duality is the following: While, as mentioned above, there is always a Poincaré duality isomorphism between truncation and complementary cotruncation, this isomorphism is not determined uniquely and may not commute with Poincaré duality on the given total space E. Proposition 6.9 states that the duality isomorphism in degree r between fiberwise truncation and cotruncation can be chosen to commute with Poincaré duality on E if and only if $\mathcal{O}_r(\pi, k, l)$ vanishes. In this case, the duality isomorphism is uniquely determined by the commutation requirement. Thus, we refer to the classes $\mathcal{O}_*$ as *local duality obstructions*, since in the subsequent application to singular spaces, these classes are localized at the singularities.

The above bundle-theoretic analysis is then applied in Section 9 in constructing intersection spaces $I^{\bar{p}}X$ for stratified pseudomanifolds X of stratification depth 1 such that every connected component of every singular stratum has a closed neighborhood whose boundary is the total space of a fiber bundle, the *link bundle*, while the neighborhood itself is described by the corresponding cone bundle. A large and well-studied class of stratified spaces that have such link bundle structures are the Thom-Mather stratified spaces, which we review in Section 8 with particular emphasis on depth 1. We assume that the link bundles allow for structure groups with equivariant Moore approximations. The central definition is 9.1; the main result here, Theorem 9.5, establishes generalized Poincaré duality

$$\widetilde{H}^r(I^{\bar{p}}X;\mathbb{Q}) \cong \widetilde{H}_{n-r}(I^{\bar{q}}X;\mathbb{Q}) \tag{1.1}$$

for complementary perversity intersection spaces, provided the local duality obstructions of the link bundle vanish.

In the Sections 10, 11, we investigate the signature and Witt element of intersection forms. We show first that if a Witt space allows for middle-degree equivariant Moore approximation, then its intersection form on intersection homology agrees with the intersection form of the top stratum as an element in the Witt group $W(\mathbb{Q})$ of the rationals (Corollary 10.2). Section 11 shows that the duality isomorphism (1.1), where we now use the (lower) middle perversity, can in fact be constructed so that the associated middle-degree intersection form is symmetric when the dimension n is a multiple of 4. Let $\sigma(IX)$ denote the signature of this symmetric form. Theorem 11.3 asserts that $\sigma(IX) = \sigma(M, \partial M)$, where $\sigma(M, \partial M)$ denotes the signature of the top stratum. In particular then, $\sigma(IX)$ agrees with the intersection homology signature. For the rather involved proof of this theorem, we build on the method of Spiegel [37], which in turn is partially based on the methods introduced in the proof of [3, Theorem 2.28]. It follows from all of this that there are interesting global signature obstructions to fiberwise homology truncation in bundles. For instance, viewing the complex projective space $\mathbb{CP}^2$ as a stratified space with bottom stratum $\mathbb{CP}^1 \subset \mathbb{CP}^2$, the signature of $\mathbb{CP}^2$ is 1, whereas the signature of the top stratum D^4 vanishes. Indeed, the normal circle bundle of $\mathbb{CP}^1$, i.e. the Hopf bundle, does not have a degree 1 fiberwise homology truncation, as can of course be verified directly.

On notation: Throughout this paper, all homology and cohomology groups are taken with rational coefficients. Reduced homology and cohomology will be denoted by $\widetilde{H}_*$ and $\widetilde{H}^*$. The linear dual of a K-vector space V is denoted by $V^\dagger = \operatorname{Hom}(V, K)$.

2. Properties of Homotopy Pushouts

This paper uses homotopy pushouts in many constructions. We recall here their definition, as well as the two properties we will need: their universal mapping property and the associated Mayer-Vietoris sequence.

Definition 2.1. Given continuous maps $Y_1 \xleftarrow{f_1} X \xrightarrow{f_2} Y_2$ between topological spaces we define the *homotopy pushout* of f_1 and f_2 to be the topological space $Y_1 \cup_X Y_2$, the quotient of the disjoint union $X \times [0,1] \sqcup Y_1 \sqcup Y_2$ by the smallest equivalence relation generated by

$$\{(x,0) \sim f_1(x) \mid x \in X\} \cup \{(x,1) \sim f_2(x) \mid x \in X\}$$

We denote $\xi_i : Y_i \to Y_1 \cup_X Y_2$, for $i = 1, 2$, and $\xi_0 : X \times I \to Y_1 \cup_X Y_2$, to be the inclusions into the disjoint union followed by the quotient map, where $I = [0,1]$.

Remark 2.2. The homotopy pushout satisfies the following universal mapping property: Given any topological space Z, continuous maps $g_i : Y_i \to Z$, and homotopy $h : X \times I \to Z$ satisfying $h(x,i) = g_{i+1} \circ f_{i+1}(x)$ for $x \in X$, and $i = 0, 1$, then there exists a unique continuous map $g : Y_1 \cup_X Y_2 \to Z$ such that $g_i = g \circ \xi_i$ for $i = 1, 2$, and $h = g \circ \xi_0$.

From the data of a homotopy pushout we get a long exact sequence of homology groups

$$\cdots \longrightarrow H_r(X) \xrightarrow{(f_{1*}, f_{2*})} H_r(Y_1) \oplus H_r(Y_2) \xrightarrow{\xi_{1*} - \xi_{2*}} H_r(Y_1 \cup_X Y_2) \xrightarrow{\delta} \cdots \tag{2.1}$$

This is the usual Mayer-Vietoris sequence applied to $Y_1 \cup_X Y_2$ when it is decomposed into the union of $(Y_1 \cup_X Y_2) \setminus Y_i$ for $i = 1, 2$, whose overlap is X crossed with the open interval. If X is not empty, then there is also a version for reduced homology:

$$\cdots \longrightarrow \widetilde{H}_r(X) \xrightarrow{(f_{1*}, f_{2*})} \widetilde{H}_r(Y_1) \oplus \widetilde{H}_r(Y_2) \xrightarrow{\xi_{1*} - \xi_{2*}} \widetilde{H}_r(Y_1 \cup_X Y_2) \xrightarrow{\delta} \cdots \tag{2.2}$$

3. Equivariant Moore Approximation

Our method to construct intersection spaces for twisted link bundles rests on the concept of an equivariant Moore approximation. The transformation group of the general abstract concept will eventually be a suitable reduction of the structure group of a fiber bundle, which will enable fiberwise truncation and cotruncation. The basic idea behind degree-k Moore approximations of a space X is to find a space $X_{<k}$, whose homology agrees with that of X below degree k, and vanishes in all other degrees. It is well-known that Moore-approximations cannot be made functorial on the category of all topological spaces and continuous maps, as explained in [3]. The equivariant Moore space problem was raised in 1960 by Steenrod, who asked whether given a group G, a right $\mathbb{Z}[G]$-module M and an integer $k > 1$, there exists a topological space X such that $\pi_1(X) = G$, $H_i(\widetilde{X};\mathbb{Z}) = 0$, $i \neq 0, k$, $H_0(\widetilde{X};\mathbb{Z}) = \mathbb{Z}$, and $H_k(\widetilde{X};\mathbb{Z}) = M$, where $\widetilde{X}$ is the universal cover of X, equipped with the G-action by covering translations. The first counterexample was due to Gunnar Carlsson, [14]. Further work on Steenrod's problem has been done by Douglas Anderson [1], James Arnold [2], Peter Kahn [26], [27], Frank Quinn [34], and Justin Smith [36].

Definition 3.1. Let G be a topological group. A *G-space* is a pair (X, ρ_X), where X is a topological space and $\rho_X : G \to \operatorname{Homeo}(X)$ is a continuous group homomorphism. A *morphism* between G-spaces $f : (X, \rho_X) \to (Y, \rho_Y)$ is a continuous map $f : X \to Y$ that satisfies

$$\rho_Y(g) \circ f = f \circ \rho_X(g), \text{ for every } g \in G.$$

We denote the set of morphisms by $\operatorname{Hom}_G(X, Y)$. Morphisms are also called *G-equivariant maps.* We will write $g \cdot x = \rho_X(g)(x)$, $x \in X$, $g \in G$.

Let cX be the closed cone $X \times [0,1]/X \times \{0\}$. If X is a G-space, then the cone cX becomes a G-space in a natural way: the cone point is a fixed point and for $t \in (0,1]$, $g \in G$ acts by $g \cdot (x,t) = (g \cdot x, t)$. More generally, given G-equivariant maps $Y_1 \xleftarrow{f_1} X \xrightarrow{f_2} Y_2$, the homotopy pushout $Y_1 \cup_X Y_2$ is a G-space in a natural way.

Definition 3.2. Given a G-space X and an integer $k \geq 0$, a *G-equivariant Moore approximation* to X of degree k is a G-space $X_{<k}$ together with a continuous G-equivariant map $f_{<k} : X_{<k} \to X$, satisfying the following properties:

- $H_r(f_{<k}) : H_r(X_{<k}) \to H_r(X)$ is an isomorphism for all $r < k$, and
- $H_r(X_{<k}) = 0$ for all $r \geq k$.

Definition 3.3. Let X be a nonempty topological space. The ($\mathbb{Q}$-coefficient) *homological dimension* of X is the number

$$\mathrm{Hdim}(X) = \min\{n \in \mathbb{Z} : \ H_m(X) = 0 \text{ for all } m > n\},$$

if such an n exists. If no such n exists, then we say that X has infinite homological dimension.

Example 3.4. There are two extreme cases, in which equivariant Moore approximations are trivial to construct. For $k = 0$, any Moore approximation must satisfy $H_i(X_{<0}) = 0$, for all $i \geq 0$. This forces $X_{<0} = \emptyset$, and $f_{<0}$ is the empty function. If X has $\mathrm{Hdim}(X) = n$, then for $k \geq n+1$ set $X_{<k} = X$ and $f_{<k} = \mathrm{id}_X$. Hence, any space of homological dimension n has an equivariant Moore approximation of degrees $k \leq 0$ and $k > n$.

Example 3.5. If G acts trivially on a simply connected CW complex X, then Moore approximations of X exist in every degree. For spatial homology truncation in the nonequivariant case, see Chapter 1 of [3], which also contains a discussion of functoriality issues arising in connection with Moore approximations. The simple connectivity condition is sufficient, but far from necessary.

Example 3.6. Let G be a compact Lie group acting smoothly on a smooth manifold X. Then, according to [25], one can arrange a CW structure on X in such a way that G acts cellularly. Now suppose that X is any G-space equipped with a CW structure such that G acts cellularly. If the k-th boundary operator $\partial_k : C_k(X) \to C_{k-1}(X)$ in the cellular chain complex of X vanishes, then the $(k-1)$-skeleton of X, together with its inclusion into X and endowed with the restricted G-action, is an equivariant Moore-approximation $X_{<k} = X^{k-1}$. This condition is for example satisfied for the standard minimal CW structure on complex projective spaces and tori. However, in order to make a given action cellular, one may of course be forced to endow spaces with larger, nonminimal, CW structures. Similarly, if ∂_k is injective, then $X_{<k} = X^k$ is an equivariant Moore-approximation.

The following observation can sometimes be used to show that certain G-spaces and degrees do not allow for an equivariant Moore approximation.

Proposition 3.7. *Let G be a topological group and X a nonempty G-space. Let G_λ be the G-space G with the action by left translation. If*

$$\mathrm{Hom}_G(X, G_\lambda) \neq \emptyset$$

and X has a G-equivariant Moore approximation of degree $k > 0$, then

$$k - 1 \geq \mathrm{Hdim}(G).$$

Proof. Let $f_{<k} : X_{<k} \to X$ be a G-equivariant Moore approximation, $k > 0$. Precomposition with $f_{<k}$ induces a map

$$f^\sharp_{<k} : \mathrm{Hom}_G(X, G_\lambda) \to \mathrm{Hom}_G(X_{<k}, G_\lambda).$$

As $k > 0$ and X is not empty, we have $H_0(X_{<k}) \cong H_0(X) \neq 0$. Thus $X_{<k}$ is not empty. For each $\phi \in \mathrm{Hom}_G(X_{<k}, G_\lambda)$, we note that ϕ is surjective since $X_{<k}$ is not empty, left translation is transitive and ϕ is equivariant. Choose $x \in X_{<k}$ such that $\phi(x) = e$. Define $h_x : G \to X_{<k}$ by $h_x(g) = g \cdot x$. Then $\phi \circ h_x = \mathrm{id}_G$, since

$$\phi(h_x(g)) = \phi(g \cdot x) = g\phi(x) = ge = g.$$

Therefore the map induced by ϕ on homology has a splitting induced by h_x, so there is an isomorphism

$$H_r(X_{<k}) \cong A_r \oplus H_r(G)$$

for some subgroup $A_r \subset H_r(X_{<k})$ and every r. Since by definition $H_r(X_{<k}) = 0$ for $r \geq k$, then if such a ϕ exists we must have $\mathrm{Hdim}(G) \leq k-1$. The condition $\mathrm{Hom}_G(X, G_\lambda) \neq \emptyset$ is sufficient to guarantee the existence of such a ϕ. $\square$

Example 3.8. By Proposition 3.7, the action of S^1 on itself by rotation does not have an equivariant Moore space approximation of degree 1.

Consider S^1 acting on $X = S^1 \times S^2$ by rotation in the first coordinate and trivially in the second coordinate. Example 3.4 shows that for $k \leq 0$ and $k \geq 4$, S^1-equivariant Moore approximations exist trivially. By Proposition 3.7, there is no such approximation for $k = 1$. We shall now construct an approximation for degree $k = 2$. Fix a point $y_0 \in S^2$. Let $i : S^1 \to X$, $\theta \mapsto (\theta, y_0)$, be the inclusion at y_0. Let S^1 act on itself by rotation, then the map i is equivariant. Furthermore, by the Künneth theorem we know that $H_1(X) \cong \mathbb{Q}$ is generated by the class $[S^1 \times y_0]$, and $H_1(i) : H_1(S^1) \to H_1(X)$ is an isomorphism taking $[S^1]$ to $[S^1 \times y_0]$. Thus since both S^1 and X are connected, we have that the map i gives a S^1-equivariant Moore space approximation of degree 2.

Further positive results asserting the existence of Moore approximations in geometric situations such as symplectic toric manifolds are discussed in Section 12.

4. Precosheaves and Local to Global Techniques

The material of this section is fairly standard ([12]); we include it in order to fix terminology and notation. Let B be a topological space and let $VS_{\mathbb{Q}}$ denote the category of rational vector spaces and linear maps.

Definition 4.1. A covariant functor $\mathcal{F} : \tau B \to VS_{\mathbb{Q}}$ from the category τB of open sets on B, with inclusions for morphisms, to the category $VS_{\mathbb{Q}}$, is called a *precosheaf* on B. For open sets $U \subset V \subset B$, we denote the result of applying $\mathcal{F}$ to the inclusion map $U \subset V$ by

$$i^{\mathcal{F}}_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V).$$

A *morphism* $f : \mathcal{F} \to \mathcal{G}$ of precosheaves on B is a natural transformation of functors.

Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ be an open cover of B, and let $\tau\mathcal{U}$ be the category whose objects are unions of finite intersections of open sets in $\mathcal{U}$ and whose morphisms are inclusions. There is a natural inclusion functor $u : \tau\mathcal{U} \to \tau B$, regarding an open set in $\tau\mathcal{U}$ as an object of τB. This realizes $\tau\mathcal{U}$ as a full subcategory of τB.

Definition 4.2. A precosheaf $\mathcal{F}$ on B is $\mathcal{U}$*-locally constant* if for any $U_\alpha \in \mathcal{U}$ and any U which is a finite intersection of elements of $\mathcal{U}$ and intersects U_α nontrivially, the map

$$i^{\mathcal{F}}_{U_\alpha \cap U, U_\alpha} : \mathcal{F}(U_\alpha \cap U) \to \mathcal{F}(U_\alpha)$$

is an isomorphism.

Consider the product category $\tau\mathcal{U} \times \tau\mathcal{U}$ whose objects are pairs (U, V) with $U, V \in \tau\mathcal{U}$, and whose morphism are pairs of inclusions $(U, V) \to (U', V')$ given by $U \subset U'$ and $V \subset V'$. Define the functors $\cap, \cup : \tau\mathcal{U} \times \tau\mathcal{U} \to \tau\mathcal{U}$ that take the object (U, V) to $U \cap V$ and $U \cup V$, respectively, and the morphism $(U, V) \to (U', V')$ to the inclusions $U \cap V \subset U' \cap V'$ and $U \cup V \subset U' \cup V'$. Similarly we have the projection functors $p_i : \tau\mathcal{U} \times \tau\mathcal{U} \to \tau\mathcal{U}$, for $i = 1, 2$ where p_i projects onto the i-th factor. The inclusions $U, V \subset U \cup V$ and $U \cap V \subset U, V$ induce natural transformations of functors $j_i : p_i \to \cup$, and $\iota_i : \cap \to p_i$ for $i = 1, 2$. Applying a precosheaf $\mathcal{F}$ to the $j_i(U, V)$, we obtain linear maps $\mathcal{F}(U) \to \mathcal{F}(U \cup V)$, $\mathcal{F}(V) \to \mathcal{F}(U \cup V)$, which we will again denote by j_1, j_2 (rather than $\mathcal{F}(j_i(U, V))$). Similarly for the ι_i. Thus for any precosheaf $\mathcal{F}$ on B we have the morphisms

$$\mathcal{F}(U \cap V) \xrightarrow{(\iota_1, \iota_2)} \mathcal{F}(U) \oplus \mathcal{F}(V) \xrightarrow{j_1 - j_2} \mathcal{F}(U \cup V)$$

for any object (U, V) in $\tau\mathcal{U} \times \tau\mathcal{U}$. The functoriality of $\mathcal{F}$ implies that $(j_1 - j_2) \circ (\iota_1, \iota_2) = 0$.

Any morphism of precosheaves $f : \mathcal{F} \to \mathcal{G}$ gives a commutative diagram

$$\begin{tikzcd}
\mathcal{F}(U \cap V) \arrow[r, "(\iota_1,\iota_2)"] \arrow[d, "f(U\cap V)"] & \mathcal{F}(U) \oplus \mathcal{F}(V) \arrow[r, "j_1 - j_2"] \arrow[d, "f(U)\oplus f(V)"] & \mathcal{F}(U \cup V) \arrow[d, "f(U\cup V)"] \\
\mathcal{G}(U \cap V) \arrow[r, "(\iota_1,\iota_2)"] & \mathcal{G}(U) \oplus \mathcal{G}(V) \arrow[r, "j_1 - j_2"] & \mathcal{G}(U \cup V).
\end{tikzcd} \tag{4.1}$$

Definition 4.3. Let $\mathcal{F}_r$ be a collection of precosheaves on B, for $r \geq 0$, and let $\mathcal{U}$ be an open cover of B. We say that the sequence $\mathcal{F}_r$ satisfies the *$\mathcal{U}$-Mayer-Vietoris property* if there is a natural transformation of functors on $\tau\mathcal{U} \times \tau\mathcal{U}$,

$$\delta_i^{\mathcal{F}} : \mathcal{F}_i \circ \cup \longrightarrow \mathcal{F}_{i-1} \circ \cap,$$

for each i, such that for every pair of open sets $U, V \in \tau\mathcal{U}$ the following sequence is exact:

$$\longrightarrow \mathcal{F}_{i+1}(U \cup V) \xrightarrow{\delta_{i+1}^{\mathcal{F}}} \mathcal{F}_i(U \cap V) \xrightarrow{(\iota_1^i, \iota_2^i)} \mathcal{F}_i(U) \oplus \mathcal{F}_i(V) \xrightarrow{j_1^i - j_2^i} \mathcal{F}_i(U \cup V) \xrightarrow{\delta_i^{\mathcal{F}}}.$$

A collection of morphisms $f_r : \mathcal{F}_r \to \mathcal{G}_r$, for $r \geq 0$, is called *δ-compatible* if for each pair of open sets $U, V \in \tau\mathcal{U}$ the following diagram commutes for all $i \geq 0$:

$$\begin{tikzcd}
\mathcal{F}_{i+1}(U \cup V) \arrow[r, "\delta_{i+1}^{\mathcal{F}}(U,V)"] \arrow[d, "f_{i+1}(U\cup V)"'] & \mathcal{F}_i(U \cap V) \arrow[d, "f_i(U\cap V)"] \\
\mathcal{G}_{i+1}(U \cup V) \arrow[r, "\delta_{i+1}^{\mathcal{G}}(U,V)"] & \mathcal{G}_i(U \cap V).
\end{tikzcd} \tag{4.2}$$

Proposition 4.4. *Let B be a compact topological space and let $\mathcal{U}$ be an open cover of B. Let $f_i : \mathcal{F}_i \to \mathcal{G}_i$ be a sequence of δ-compatible morphisms between $\mathcal{U}$-locally constant precosheaves on B that satisfy the $\mathcal{U}$-Mayer-Vietoris property. If $f_i(U) : \mathcal{F}_i(U) \to \mathcal{G}_i(U)$ is an isomorphism for every $U \in \mathcal{U}$ and for every $i \geq 0$, then $f_i(B) : \mathcal{F}_i(B) \to \mathcal{G}_i(B)$ is an isomorphism for all $i \geq 0$.*

Proof. We shall prove the following statement by induction on n: For every $U \in \tau\mathcal{U}$ which can be written as a union $U = U_1 \cup \cdots \cup U_n$ of n open sets $U_j \in \tau\mathcal{U}$, each of which is a finite intersection of open sets in $\mathcal{U}$, the map $f_i(U) : \mathcal{F}_i(U) \to \mathcal{G}_i(U)$ is an isomorphism for all $i \geq 0$. The base case $n = 1$ follows from the fact that $\mathcal{F}_i, \mathcal{G}_i$ are $\mathcal{U}$-locally constant together with the assumption on $f_i(U)$ for $U \in \mathcal{U}$. Denote $U^j = U_1 \cup \cdots \cup \hat{U}_j \cup \cdots \cup U_n$ and $V^j = (U_1 \cap U_j) \cdots \cup \hat{U}_j \cup \cdots \cup (U_n \cap U_j)$; then $U = U^j \cup U_j$ and $V^j = U^j \cap U_j$. Since the f_i are δ-compatible, by (4.2) and (4.1) we have the commutative diagram below, whose rows are the $\mathcal{U}$-Mayer-Vietoris sequences associated to the pair U^j and U_j:

$$\begin{tikzcd}
\arrow[r] & \mathcal{F}_i\left(V^j\right) \arrow[r] \arrow[d, "f_i(V^j)"] & \mathcal{F}_i\left(U^j\right)\oplus\mathcal{F}_i\left(U_j\right) \arrow[r] \arrow[d, "f_i(U^j)\oplus f_i(U_j)"] & \mathcal{F}_i\left(U\right) \arrow[r, "\delta"] \arrow[d, "f_i(U)"] & \mathcal{F}_{i-1}\left(V^j\right) \arrow[r] \arrow[d, "f_{i-1}(V^j)"] & {} \\
\arrow[r] & \mathcal{G}_i\left(V^j\right) \arrow[r] & \mathcal{G}_i\left(U^j\right)\oplus\mathcal{G}_i\left(U_j\right) \arrow[r] & \mathcal{G}_i\left(U\right) \arrow[r, "\delta"] & \mathcal{G}_{i-1}\left(V^j\right) \arrow[r] & {} .
\end{tikzcd}$$

Each of V^j, U^j, and U_j is a union of less than n open sets, each of which is a finite intersection of elements of $\mathcal{U}$. Thus by induction hypothesis, $f_i(V^j)$, $f_i(U^j)$ and $f_i(U_j)$ are isomorphisms for all i. By the 5-lemma, $f_i(U)$ is an isomorphism for all i, which concludes the induction step. Since B is compact, there is a finite number of open sets in $\mathcal{U}$ which cover B. Thus the induction yields the desired result. □

5. Examples of Precosheaves

Throughout this section we consider a topological fiber bundle $\pi : E \to B$ with fiber L and topological structure group G. We assume that B, E, and L are compact oriented topological manifolds such that E is compatibly oriented with respect to the orientation of B and L. Set $n = \dim E$, $b = \dim B$ and $c = \dim L = n - b$. We may form the fiberwise cone of this bundle, DE, by defining DE to be the homotopy pushout, Definition 2.1, of the pair of maps $B \xleftarrow{\pi} E \xrightarrow{\mathrm{id}} E$. By Remark 2.2, the map π induces a map $\pi_D : DE \to B$, given by id_B on B and $(x,t) \mapsto \pi(x)$ for $(x,t) \in E \times I$. This makes DE into a fiber bundle whose fiber is cL, the cone on L, and whose structure group is G. We point out, for $U \subset B$ open, that $\pi_D^{-1}U \to U$ is obtained as the homotopy pushout of the pair of maps $U \xleftarrow{\pi|_{\pi^{-1}U}} \pi^{-1}U \xrightarrow{\mathrm{id}} \pi^{-1}U$. One more fact that will be needed is that the pair (DE, E), where E is identified with $E \times \{1\} \subset DE$, along with a stratification of DE given by $B \subset DE$, is a compact $\mathbb{Q}$-oriented ∂-stratified topological pseudomanifold, in the sense of Friedman and McClure [21]. Here we have identified B with the image $\sigma(B)$ of the "zero section" $\sigma : B \to DE$, sending $x \in B$ to the cone point of cL over x. Similarly for any open $U \subset B$, the pair $\left(\pi_D^{-1}U, \pi^{-1}U\right)$ is a $\mathbb{Q}$-oriented ∂-stratified pseudomanifold, though it will not be compact unless U is compact. We write $\partial\pi_D^{-1}U = \pi^{-1}U$.

Example 5.1. For each $r \geq 0$, there are precosheaves $\pi_*\mathcal{H}_r$ on B defined by

$$U \mapsto H_r\left(\pi^{-1}\left(U\right)\right).$$

By the Eilenberg-Steenrod axioms, these are $\mathcal{U}$-locally constant, and satisfy the $\mathcal{U}$-Mayer-Vietoris property for any good open cover $\mathcal{U}$ of B. (An open cover $\mathcal{U}$ of a b-dimensional manifold is *good*, if every nonempty finite intersection of sets in $\mathcal{U}$ is homeomorphic to $\mathbb{R}^b$. Such a cover exists if the manifold is smooth or PL.)

Let $\pi' : E' \to B$ be another fiber bundle, and $f : E \to E'$ a morphism of fiber bundles. Then f induces a morphism of precosheaves $f_* : \pi_*\mathcal{H}_r \to \pi'_*\mathcal{H}_r$, given on any open set $U \subset B$ by

$$f_*(U) := \left(f|_{\pi^{-1}U}\right)_* : H_r\left(\pi^{-1}U\right) \to H_r\left(\pi'^{-1}U\right).$$

Furthermore, for any pair of open sets $U, V \subset B$, we have the following commutative diagram whose rows are exact Mayer-Vietoris sequences:

$$\begin{tikzcd} \arrow[r] & H_r\left(\pi^{-1}(U\cap V)\right) \arrow[r] \arrow[d, "f_r(U\cap V)"] & H_r\left(\pi^{-1}U\right)\oplus H_r\left(\pi^{-1}V\right) \arrow[r] \arrow[d, "f_r(U)\oplus f_r(V)"] & H_r\left(\pi^{-1}(U\cup V)\right) \arrow[r, "\delta"] \arrow[d, "f_r(U\cup V)"] & {} \\ \arrow[r] & H_r\left(\pi'^{-1}(U\cap V)\right) \arrow[r] & H_r\left(\pi'^{-1}U\right)\oplus H_r\left(\pi'^{-1}V\right) \arrow[r] & H_r\left(\pi'^{-1}(U\cup V)\right) \arrow[r, "\delta"] & {} \end{tikzcd} \tag{5.1}$$

Thus, for any good open cover $\mathcal{U}$, the map f induces a δ-compatible sequence of morphisms between precosheaves which satisfy the $\mathcal{U}$-Mayer-Vietoris property, and are $\mathcal{U}$-locally constant.

Example 5.2. Define the precosheaf of intersection homology groups, $\pi_{D*}\mathcal{I}^{\overline{p}}\mathcal{H}_r$ for each $r \geq 0$, and each perversity $\overline{p}$, by assigning to the open set $U \subset B$ the vector space, $I^{\overline{p}}H_r\left(\pi_D^{-1}U\right)$. We use the definition of intersection homology via finite singular chains as in [21]. This is a slightly more general definition than that of King,[28], and Kirwan-Woolf [29]. For our situation the definitions all agree with the exception that the former allows for more general perversities, see the comment after Prop. 2.3 in [21] for more details. In Section 4.6 of Kirwan-Woolf [29] it is shown that each $\pi_{D*}\mathcal{I}^{\overline{p}}\mathcal{H}_r$ is a precosheaf for each $r \geq 0$, and that this sequence satisfies the $\mathcal{U}$-Mayer-Vietoris property for any open cover $\mathcal{U}$ of B. Furthermore, these are all $\mathcal{U}$-locally constant for any good cover $\mathcal{U}$ of B.

Let $f : E \to E'$ be a bundle morphism with $\dim E \geq \dim E'$. Using the levelwise map $E \times I \to E' \times I$, $(e,t) \mapsto (f(e),t)$, and the identity map on B, f induces a bundle morphism $f_D : DE \to DE'$. Recall that a continuous map between stratified spaces is called *stratum-preserving* if the image of every pure stratum of the source is contained in a single pure stratum of the target. A stratum-preserving map g is called *placid* if $\operatorname{codim} g^{-1}(S) \geq \operatorname{codim} S$ for every pure stratum S of the target. Placid maps induce covariantly linear maps on intersection homology (which is not true for arbitrary continuous maps). The map f_D is indeed stratum-preserving and, since $\dim E \geq \dim E'$, placid and thus induces maps

$$(f_D|_{\pi_D^{-1}(U)})_* : I^{\overline{p}}H_r\left(\pi_D^{-1}U\right) \longrightarrow I^{\overline{p}}H_r\left({\pi'_D}^{-1}U\right)$$

for each open set $U \subset B$. This way, we obtain a sequence of δ-compatible morphisms

$$f_{D*} : \pi_{D*}\mathcal{I}^{\overline{p}}\mathcal{H}_r \to \pi'_{D*}\mathcal{I}^{\overline{p}}\mathcal{H}_r.$$

With $I^{\overline{p}}C_*(X)$ the singular rational intersection chain complex as in [21], we define intersection cochains by $I_{\overline{p}}C^*(X) = \operatorname{Hom}(I^{\overline{p}}C_*(X), \mathbb{Q})$ and define intersection cohomology by $I_{\overline{p}}H^*(X) = H^*(I_{\overline{p}}C^*(X))$. Then the universal coefficient theorem

$$I_{\overline{p}}H^*(X) \cong \operatorname{Hom}(I^{\overline{p}}H_*(X), \mathbb{Q})$$

holds. Theorem 7.10 of [21] establishes Poincaré-Lefschetz duality for compact $\mathbb{Q}$-oriented n-dimensional ∂-stratified pseudomanifolds $(X, \partial X)$. Some important facts are established there in the proof:

(1) For complementary perversities $\overline{p} + \overline{q} = \overline{t}$, there is a commutative diagram whose rows are exact:

$$\begin{tikzcd} \arrow[r, "j_\partial^r"] & I_{\overline{p}}H^r(X) \arrow[r, "i_\partial^r"] \arrow[d, "\cong"', "D_X^r"] & I_{\overline{p}}H^r(\partial X) \arrow[r, "\delta_\partial^r"] \arrow[d, "\cong"', "D_{\partial X}^r"] & I_{\overline{p}}H^{r+1}(X,\partial X) \arrow[r] \arrow[d, "\cong"', "D_{n-r-1}^X"] & {} \\ \arrow[r, "j_{n-r}^\partial"] & I^{\overline{q}}H_{n-r}(X,\partial X) \arrow[r, "\delta_{n-r}^\partial"] & I^{\overline{q}}H_{n-r-1}(\partial X) \arrow[r, "i_{n-r-1}^\partial"] & I^{\overline{q}}H_{n-r-1}(X) \arrow[r] & {} \end{tikzcd} \tag{5.2}$$

(2) The inclusion $X \setminus \partial X \to X$ induces an isomorphism

$$I^{\overline{q}} H_{n-r}\left(X \setminus \partial X\right) \cong I^{\overline{q}} H_{n-r}\left(X\right). \tag{5.3}$$

Consider the smooth oriented c-dimensional manifold L. The closed cone cL is a compact $\mathbb{Q}$-oriented $(c+1)$-dimensional ∂-stratified pseudomanifold. Thus the long exact sequence coming from the bottom row of diagram (5.2) gives

$$\longrightarrow I^{\overline{p}} H_{r+1}\left(cL, L\right) \xrightarrow{\delta^{\partial}_{r+1}} I^{\overline{p}} H_r\left(L\right) \xrightarrow{i^{\partial}_r} I^{\overline{p}} H_r\left(cL\right) \xrightarrow{j^{\partial}_r} I^{\overline{p}} H_r\left(cL, L\right) \longrightarrow \ . \tag{5.4}$$

Proposition 5.3. *Let $\overline{p}$ be a perversity and let $k = c - \overline{p}\left(c+1\right)$. Then for the maps in the exact sequence (5.4) we have an isomorphism*

$$i^{\partial}_r : H_r\left(L\right) \to I^{\overline{p}} H_r\left(cL\right),$$

when $r < k$, and an isomorphism

$$\delta^{\partial}_{r+1} : I^{\overline{p}} H_{r+1}\left(cL, L\right) \to H_r\left(L\right),$$

when $r \geq k$.

Proof. The standard cone formula for intersection homology asserts that for a closed c-dimensional manifold L, the inclusion $L \hookrightarrow cL$ as the boundary induces an isomorphism

$$I^{\overline{p}} H_r\left(L\right) \cong I^{\overline{p}} H_r\left(cL\right) \text{ for } r < c - \overline{p}(c+1),$$

whereas $I^{\overline{p}} H_r\left(cL\right) = 0$ for $r \geq c - \overline{p}(c+1)$. (By (5.3) above, this holds both for the closed and the open cone.) This already establishes the first claim. The second one follows from the cone formula together with the exact sequence (5.4). ☐

6. Fiberwise Truncation and Cotruncation

Let $\pi : E \to B$ be a fiber bundle of closed topological manifolds with fiber L and structure group G such that B, E and L are compatibly oriented. Suppose that a G-equivariant Moore approximation $L_{<k}$ of degree k exists for the fiber L. The bundle E has an underlying principal G-bundle $E_P \to B$ such that $E = E_P \times_G L$. Using the G-action on $L_{<k}$, we set

$$\mathrm{ft}_{<k} E = E_P \times_G L_{<k}.$$

Then $\mathrm{ft}_{<k} E$ is the total space of a fiber bundle $\pi_{<k} : \mathrm{ft}_{<k} E \to B$ with fiber $L_{<k}$, structure group G and underlying principal bundle E_P. The equivariant structure map $f_{<k} : L_{<k} \to L$ defines a morphism of bundles

$$F_{<k} : \mathrm{ft}_{<k} E = E_P \times_G L_{<k} \to E_P \times_G L = E.$$

Definition 6.1. The pair $(\mathrm{ft}_{<k} E, F_{<k})$ is called the *fiberwise k-truncation* of the bundle E.

Definition 6.2. The *fiberwise k-cotruncation* $\mathrm{ft}_{\geq k} E$ is the homotopy pushout of the pair of maps

$$B \xleftarrow{\pi_{<k}} \mathrm{ft}_{<k} E \xrightarrow{F_{<k}} E\ .$$

Let $c_{\geq k} : E \to \mathrm{ft}_{\geq k} E$, and $\sigma : B \to \mathrm{ft}_{\geq k} E$ be the maps ξ_2 and ξ_1, respectively, appearing in Definition 2.1.

Since $F_{<k}$ satisfies $\pi_{<k} = \pi \circ F_{<k}$ we have, by the universal property of Remark 2.2, using the constant homotopy, a unique map $\pi_{\geq k} : \mathrm{ft}_{\geq k} E \to B$ satisfying $\pi = \pi_{\geq k} \circ c_{\geq k}$, $\pi_{\geq k} \circ \sigma = \mathrm{id}_B$ and $(\pi_{\geq k} \circ \xi_0)(x, t) = \pi_{<k}(x)$ for all $t \in I$, where $\xi_0 : \mathrm{ft}_{<k} E \times I \to \mathrm{ft}_{\geq k} E$ is induced by the

inclusion (as in Definition 2.1). The map $\pi_{\geq k} : \mathrm{ft}_{\geq k}E \to B$ is a fiber bundle projection with fiber the homotopy pushout of

$$\star \longleftarrow L_{<k} \xrightarrow{f_{<k}} L \ ,$$

i.e. the mapping cone of $f_{<k}$. Note that this mapping cone is a G-space in a natural way (with $\star$ as a fixed point), since $f_{<k}$ is equivariant. The map $c_{\geq k} : E \to \mathrm{ft}_{\geq k}E$ is a morphism of fiber bundles. Furthermore, the bundle $\pi_{\geq k}$ has a canonical section σ, sending $x \in B$ to $\star$ over x.

Definition 6.3. Define the space $Q_{\geq k}E$ to be the homotopy pushout of the pair of maps

$$\star \longleftarrow B \xrightarrow{\sigma} \mathrm{ft}_{\geq k}E \ .$$

This is the mapping cone of σ and hence

$$\widetilde{H}_*(Q_{\geq k}E) \cong H_*(\mathrm{ft}_{\geq k}E, B),$$

where we identified B with its image under the embedding σ. Define the maps

$$\xi_{\geq k} : \mathrm{ft}_{\geq k}E \to Q_{\geq k}E \quad \text{and} \quad [c] : \star \to Q_{\geq k}E$$

to be the maps ξ_2 and ξ_1, respectively (Definition 2.1). Set

$$C_{\geq k} = \xi_{\geq k} \circ c_{\geq k} : E \to Q_{\geq k}E.$$

For each open set $U \subset B$, the space $\pi_{\geq k}^{-1}U$ is the pushout of the pair of maps

$$U \xleftarrow{\pi_{<k}|} \pi_{<k}^{-1}U \xrightarrow{F_{<k}|} \pi^{-1}U$$

and the restrictions of $c_{\geq k}$ induce a morphism of fiber bundles $c_{\geq k}(U) : \pi^{-1}U \to \pi_{\geq k}^{-1}U$. Define the precosheaf $\pi_*^Q\mathcal{H}_r$ by the assignment $U \mapsto H_r(\pi_{\geq k}^{-1}U, U)$ (again identifying U with its image under σ). That this assignment is indeed a precosheaf follows from the functoriality of homology applied to the commutative diagram of inclusions

$$\begin{tikzcd}
(\pi_{\geq k}^{-1}U, U) \arrow[r] \arrow[dr] & (\pi_{\geq k}^{-1}V, V) \arrow[d] \\
& (\pi_{\geq k}^{-1}W, W)
\end{tikzcd}$$

associated to nested open sets $U \subset V \subset W$. The maps $C_r^k(U) : H_r(\pi^{-1}U) \to H_r(\pi_{\geq k}^{-1}U, U)$, given by the composition

$$H_r(\pi^{-1}U) \xrightarrow{c_{\geq k}(U)_*} H_r(\pi_{\geq k}^{-1}U) \longrightarrow H_r(\pi_{\geq k}^{-1}U, U),$$

define a morphism of precosheaves

$$\mathcal{C}_r^k : \pi_*\mathcal{H}_r \to \pi_*^Q\mathcal{H}_r$$

for all $r \geq 0$. The following lemma justifies the terminology "cotruncation".

Lemma 6.4. *For $U \cong \mathbb{R}^b$, the map $C_r^k(U)$ is an isomorphism for $r \geq k$, while $H_r(\pi_{\geq k}^{-1}U, U) = 0$ for $r < k$.*

Proof. Let $L_{\geq k}$ denote the mapping cone of $f_{<k} : L_{<k} \to L$. Since the bundles π and $\pi_{\geq k}$ both (compatibly) trivialize over $U \cong \mathbb{R}^b$, the map $C_r^k(U)$ can be identified with the composition

$$H_r(\mathbb{R}^b \times L) \longrightarrow H_r(\mathbb{R}^b \times L_{\geq k}) \longrightarrow H_r(\mathbb{R}^b \times (L_{\geq k}, \star)),$$

which can further be identified with

$$H_r(L) \longrightarrow \widetilde{H}_r(L_{\geq k}).$$

This map fits into a long exact sequence

$$H_r(L_{<k}) \xrightarrow{f_{<k*}} H_r(L) \longrightarrow \widetilde{H}_r(L_{\geq k}) \longrightarrow H_{r-1}(L_{<k}).$$

The result then follows from the defining properties of the Moore approximation $f_{<k}$. □

As in Example 5.1, the map $F_{<k,r} : H_r(\mathrm{ft}_{<k}E) \to H_r(E)$ is $\mathcal{F}_{<k,r}(B)$ for the morphism of precosheaves $\mathcal{F}_{<k,r} : \pi_{<k*}\mathcal{H}_r \to \pi_*\mathcal{H}_r$ given by $F_{<k}|_* : H_r(\pi_{<k}^{-1}U) \to H_r(\pi^{-1}U)$ for each $r \geq 0$.

For each open set U we have the long exact sequence of perversity $\overline{p}$-intersection homology groups

$$\cdots \longrightarrow I^{\overline{p}}H_{r+1}\left(\pi_D^{-1}U, \partial\pi_D^{-1}U\right) \xrightarrow{\delta^{\partial}_{r+1}(U)} H_r\left(\pi^{-1}U\right) \xrightarrow{i^{\partial}_r(U)} I^{\overline{p}}H_r\left(\pi_D^{-1}U\right) \xrightarrow{j^{\partial}_r(U)} \cdots \tag{6.1}$$

(Recall that $\pi_D : DE \to B$ is the projection of the cone bundle.) When U varies, this exact sequence forms a precosheaf of acyclic chain complexes. In particular the morphisms i^{∂}_r and δ^{∂}_{r+1} are morphisms of precosheaves for every $r \geq 0$. From now on, in order to have good open covers, we assume that B is either smooth or at least PL.

Proposition 6.5. *Fix a perversity $\overline{p}$. Let $n-1 = \dim E$, $b = \dim B$, $c = n-b-1$, and $k = c - \overline{p}(c+1)$. Assume that B is compact and that an equivariant Moore approximation $f_{<k} : L_{<k} \to L$ to L of degree k exists. Then the compositions*

$$i^{\partial}_r(B) \circ F_{<k*} : H_r(\mathrm{ft}_{<k}E) \to I^{\overline{p}}H_r(DE)$$

and

$$C^k_r \circ \delta^{\partial}_{r+1}(B) : I^{\overline{p}}H_{r+1}(DE, E) \to H_r(\mathrm{ft}_{\geq k}E, B) \cong \widetilde{H}_r(Q_{\geq k}E)$$

are isomorphisms for all $r \geq 0$.

Proof. We use our local to global technique. Let $\mathcal{U}$ be a finite good open cover of B which trivializes E. The map $F_{<k}$ induces (by restrictions to preimages of open subsets) a map of precosheaves as demonstrated in Example 5.1. Both i^{∂}_r and $F_{<k,*}$ are sequences of δ-compatible morphisms of $\mathcal{U}$-locally constant precosheaves that satisfy the $\mathcal{U}$-Mayer-Vietoris property. Let $U \in \mathcal{U}$, then $H_r\left(\pi_{<k}^{-1}U\right) \cong H_r(L_{<k})$ and $\mathcal{F}_{<k,r} = f_{<k*}$ is an isomorphism in degrees $r < k$ and 0 in degrees $r \geq k$. Likewise by Proposition 5.3, the map i^{∂}_r induces an isomorphism $H_r(L) \cong I^{\overline{p}}H_r\left(\pi_D^{-1}U\right)$ in degrees $r < k$ and 0 in degrees $r \geq k$, since

$$\pi_D^{-1}U \cong U \times cL \cong \mathbb{R}^b \times cL,$$

$I^{\overline{p}}H_r\left(\mathbb{R}^b \times cL\right) \cong I^{\overline{p}}H_r(cL)$, and we can identify $i^{\partial}_r(U)$ with i^{∂}_r from (5.4). Thus, the composition is an isomorphism in every degree. We can now apply Proposition 4.4 to obtain the desired result.

A analogous argument gives the desired result for the second statement, using Lemma 6.4 in conjunction with Proposition 5.3 to establish the base case. □

It follows from Proposition 6.5 that $i^{\partial}_r(B) : H_r(E) \to I^{\overline{p}}H_r(DE)$ is surjective for all r, $F_{<k*} : H_r(\mathrm{ft}_{<k}E) \to H_r(E)$ is injective for all r, $C^k_r : H_r(E) \to H_r(\mathrm{ft}_{\geq k}E, B)$ is surjective for all r, and $\delta^{\partial}_{r+1}(B) : I^{\overline{p}}H_{r+1}(DE, E) \to H_r(E)$ is injective for all r. We may use the isomorphisms in Proposition 6.5 to identify $H_r(\mathrm{ft}_{<k}E)$ with $I^{\overline{p}}H_r(DE)$ and $\widetilde{H}_r(Q_{\geq k}E)$ with $I^{\overline{p}}H_{r+1}(DE, E)$. In doing so, we may consider the exact sequence

$$\longrightarrow I^{\overline{p}}H_{r+1}(DE, E) \xrightarrow{\delta^{\partial}_{r+1}} H_r(E) \xrightarrow{i^{\partial}_r} I^{\overline{p}}H_r(DE) \xrightarrow{j^{\partial}_r} , \tag{6.2}$$

and identify $F_{<k,r}$ as a section of i_r^∂, and C_r^k as a section of δ_{r+1}^∂. Thus we see that $j_r^\partial = 0$ for every $r \geq 0$, and we have a split short exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & I^{\overline{p}}H_{r+1}(DE,E) \arrow[r, shift left, "\delta_{r+1}^\partial"] & H_r(E) \arrow[l, shift left, "C_r^k"] \arrow[r, shift left, "i_r^\partial"] & I^{\overline{p}}H_r(DE) \arrow[l, shift left, "F_{<k,r}"] \arrow[r] & 0.
\end{tikzcd} \tag{6.3}
$$

Lemma 6.6. *The sequence*

$$0 \to H_r(\mathrm{ft}_{<k}E) \xrightarrow{F_{<k,*}} H_r(E) \xrightarrow{C_{\geq k,*}} \widetilde{H}_r(Q_{\geq k}E) \to 0$$

is exact.

Proof. Only exactness in the middle remains to be shown. The standard sequence

$$\mathrm{ft}_{<k}E \xrightarrow{F_{<k}} E \hookrightarrow \mathrm{cone}(F_{<k})$$

induces an exact sequence

$$H_r(\mathrm{ft}_{<k}E) \xrightarrow{F_{<k,r}} H_r(E) \longrightarrow \widetilde{H}_r(\mathrm{cone}(F_{<k})). \tag{6.4}$$

Collapsing appropriate cones yields homotopy equivalences

$$\mathrm{cone}(F_{<k}) \xrightarrow{\simeq} \mathrm{ft}_{\geq k}E/B \xleftarrow{\simeq} Q_{\geq k}E$$

such that the diagram

$$
\begin{tikzcd}
E \arrow[r, hook] \arrow[d, hook, "c_{\geq k}"'] & \mathrm{cone}(F_{<k}) \arrow[r, "\simeq"] & \mathrm{ft}_{\geq k}E/B \arrow[d, equal] \\
\mathrm{ft}_{\geq k}E \arrow[r, hook, "\xi_{\geq k}"] & Q_{\geq k}E \arrow[r, "\simeq"] & \mathrm{ft}_{\geq k}E/B
\end{tikzcd}
$$

commutes. The induced diagram on homology,

$$
\begin{tikzcd}
H_r(E) \arrow[r] \arrow[d, "c_{\geq k*}"'] & \widetilde{H}_r(\mathrm{cone}(F_{<k})) \arrow[r, "\cong"] & \widetilde{H}_r(\mathrm{ft}_{\geq k}E/B) \arrow[d, equal] \\
H_r(\mathrm{ft}_{\geq k}E) \arrow[r, "\xi_{\geq k*}"] & \widetilde{H}_r(Q_{\geq k}E) \arrow[r, "\cong"] & \widetilde{H}_r(\mathrm{ft}_{\geq k}E/B),
\end{tikzcd}
$$

shows that the homology kernel of $E \to \mathrm{cone}(F_{<k})$ equals the kernel of $\xi_{\geq k*}c_{\geq k*} = C_{\geq k*}$, but it also equals the image of $F_{<k,r}$ by the exactness of (6.4). ☐

Proposition 6.7. *Let $n-1 = \dim E$, $b = \dim B$ and $c = n-b-1$. For complementary perversities $\overline{p}+\overline{q} = \overline{t}$, let $k = c - \overline{p}(c+1)$ and $l = c - \overline{q}(c+1)$. Assume that an equivariant Moore approximation to L exists of degree k and of degree l. Then there is a Poincaré duality isomorphism*

$$D_{k,l} : H^r(\mathrm{ft}_{<k}E) \cong \widetilde{H}_{n-r-1}(Q_{\geq l}E).$$

Proof. We use the isomorphisms in Proposition 6.5 and the Poincaré-Lefschetz duality of [21], as described here in (5.2), applied to the ∂-stratified pseudomanifold (DE,E). By definition, $D_{k,l}$ is the unique isomorphism such that

$$
\begin{tikzcd}
I_{\bar{p}}H^r(DE) \arrow[r, "F^*_{<k}\circ i^*", "\cong"'] \arrow[d, "\cong"', "D_{DE}"] & H^r(\mathrm{ft}_{<k}E) \arrow[d, dotted, "D_{k,l}"] \\
I^{\bar{q}}H_{n-r}(DE,E) \arrow[r, "C^l_{n-r-1}\circ\delta", "\cong"'] & \widetilde{H}_{n-r-1}(Q_{\geq l}E)
\end{tikzcd}
$$

commutes. □

It need not be true, however, that the diagram

$$\begin{tikzcd} H^r(E) \arrow[r, "F^*_{<k}"] \arrow[d, "D_E"', "\cong"] & H^r(\mathrm{ft}_{<k}E) \arrow[d, "\cong"', "D_{k,l}"] \\ H_{n-r-1}(E) \arrow[r, "C_{\geq l*}"] & \widetilde{H}_{n-r-1}(Q_{\geq l}E) \end{tikzcd} \tag{6.5}$$

commutes, see Example 6.13 below. It turns out that there is an obstruction to the existence of any isomorphism $H^r(\mathrm{ft}_{<k}E) \cong \widetilde{H}_{n-r-1}(Q_{\geq l}E)$ such that the diagram (6.5) commutes.

Definition 6.8. Let k, l be two integers. Given G-equivariant Moore approximations

$$f_{<k} : L_{<k} \to L, \quad f_{<l} : L_{<l} \to L,$$

the *local duality obstruction* in degree i is defined to be

$$\mathcal{O}_i(\pi, k, l) = \{C^*_{\geq k}(x) \cup C^*_{\geq l}(y) \mid x \in \widetilde{H}^i(Q_{\geq k}E),\ y \in \widetilde{H}^{n-1-i}(Q_{\geq l}E)\} \subset H^{n-1}(E).$$

Locality of this obstruction refers to the fact that in the context of stratified spaces, the obstruction arises only near the singularities of the space. Clearly, the definition of $\mathcal{O}_i(\pi, k, l)$ does not require any smooth or PL structure on B and thus is available for topological base manifolds. The obstruction set $\mathcal{O}_i(\pi, k, l)$ is a cone: If $z = C^*_{\geq k}(x) \cup C^*_{\geq l}(y)$ is in $\mathcal{O}_i(\pi, k, l)$ then for any $\lambda \in \mathbb{Q}$,

$$\lambda z = C^*_{\geq k}(\lambda x) \cup C^*_{\geq l}(y) \in \mathcal{O}_i(\pi, k, l).$$

If E is connected, then $H^{n-1}(E) \cong \mathbb{Q}$ is one-dimensional, so

$$\text{either } \ \mathcal{O}_i(\pi, k, l) = 0 \ \text{ or } \ \mathcal{O}_i(\pi, k, l) \cong \mathbb{Q}.$$

Proposition 6.9. *There exists an isomorphism $D : H^r(\mathrm{ft}_{<k}E) \cong \widetilde{H}_{n-r-1}(Q_{\geq l}E)$ such that*

$$\begin{tikzcd} H^r(E) \arrow[r, "F^*_{<k}"] \arrow[d, "D_E"', "\cong"] & H^r(\mathrm{ft}_{<k}E) \arrow[d, "\cong"', "D"] \\ H_{n-r-1}(E) \arrow[r, "C_{\geq l*}"] & \widetilde{H}_{n-r-1}(Q_{\geq l}E) \end{tikzcd}$$

commutes if and only if the local duality obstruction $\mathcal{O}_r(\pi, k, l)$ vanishes. In this case, D is uniquely determined by the diagram.

Proof. We have seen that both $F^*_{<k}$ and $C_{\geq l*}$ are surjective and their respective images have equal rank. Thus by linear algebra D exists if and only if $D_E(\ker F^*_{<k}) = \ker C_{\geq l*}$. By Lemma 6.6, $\ker F^*_{<k} = \mathrm{im}\, C^*_{\geq k}$. Thus the condition translates to: For every $x \in \widetilde{H}^r(Q_{\geq k}E)$, $C_{\geq l*}D_E C^*_{\geq k}(x) = 0$. Rewriting this entirely cohomologically using the universal coefficient theorem, this translates further to

$$C^*_{\geq k}(x) \cup C^*_{\geq l}(y) = 0$$

for all x, y.

The uniqueness of D is standard: If $x \in H^r(\mathrm{ft}_{<k}E))$, then $D(x) = C_{\geq l*}D_E(x')$, where $x' \in H^r(E)$ is any element with $F^*_{<k}(x') = x$. By the condition on the kernels, this is independent of the choice of x'. □

Proposition 6.10. *If $\mathcal{O}_i(\pi, k, l) = 0$, then the unique D given by Proposition 6.9 equals the $D_{k,l}$ constructed in Proposition 6.7.*

Proof. This follows from the diagram

$$\begin{tikzcd}
I_{\overline{p}}H^r(DE) \arrow[r, "i^*"] \arrow[d, "D_{DE}"', "\cong"] & H^r(E) \arrow[r, "F^*_{<k}"] \arrow[d, "D_E"', "\cong"] & H^r(\mathrm{ft}_{<k} E) \arrow[d, "\cong"', "D"] \\
I^{\overline{q}}H_{n-r}(DE, E) \arrow[r, "\delta"] & H_{n-r-1}(E) \arrow[r, "C_{\geq l*}"] & \widetilde{H}_{n-r-1}(Q_{\geq l}E).
\end{tikzcd}$$

The left hand square is part of the commutative ladder (5.2). The right hand square commutes by the construction of D. Since the horizontal compositions are isomorphisms, $D = D_{k,l}$. ☐

Although superficially simple, this proposition has rather interesting geometric ramifications: Since $D_{k,l}$ can *always* be defined, even when the duality obstruction is not zero, the proposition implies that in such a case, diagram (6.5) cannot commute. This means that $D_{k,l}$ is not always a geometrically "correct" duality isomorphism, and the duality obstructions govern when it is and when it is not.

It was already shown in [3, Section 2.9] that if the link bundle is a global product, then Poincaré duality holds for the corresponding intersection spaces. This suggests that the duality obstruction vanishes for a global product. We shall now verify this directly:

Proposition 6.11. *For complementary perversities $\overline{p} + \overline{q} = \overline{t}$, let*

$$k = c - \overline{p}\,(c+1) \quad \text{and} \quad l = c - \overline{q}\,(c+1)\,.$$

If $\pi : E = B \times L \to B$ is a global product, then $\mathcal{O}_i(\pi, k, l) = 0$ for all i.

Proof. We have $\mathrm{ft}_{\geq k} E = B \times L_{\geq k}$ and by the Künneth theorem, the reduced cohomology of $Q_{\geq k}E$ is given by

$$\begin{aligned}
\widetilde{H}^*(Q_{\geq k}E) &= H^*(\mathrm{ft}_{\geq k} E, B) = H^*(B \times L_{\geq k}, B \times \star) = H^*(B \times (L_{\geq k}, \star)) \\
&\cong H^*(B) \otimes H^*(L_{\geq k}, \star).
\end{aligned}$$

Let $f_{\geq k} : L \to L_{\geq k}$ be the structural map associated to the cotruncation. By the naturality of the cross product, the square

$$\begin{tikzcd}
H^*(E) & H^*(B) \otimes H^*(L) \arrow[l, "\times"', "\cong"] \\
\widetilde{H}^*(Q_{\geq k}E) \arrow[u, "C^*_{\geq k}"] & H^*(B) \otimes H^*(L_{\geq k}, \star) \arrow[l, "\times"', "\cong"] \arrow[u, "\mathrm{id} \otimes f^*_{\geq k}"']
\end{tikzcd}$$

commutes. Let $x \in \widetilde{H}^i(Q_{\geq k}E)$, $y \in \widetilde{H}^{n-1-i}(Q_{\geq l}E)$. Their images under the Eilenberg-Zilber map are of the form

$$\mathrm{EZ}(x) = \sum_r b_r \otimes e_r^{\geq k},\ b_r \in H^*(B),\ e_r^{\geq k} \in H^*(L_{\geq k}, \star),$$

$$\mathrm{EZ}(y) = \sum_s b'_s \otimes e_s^{\geq l},\ b'_s \in H^*(B),\ e_s^{\geq l} \in H^*(L_{\geq l}, \star),$$

$\deg b_r + \deg e_r^{\geq k} = i$, $\deg b'_s + \deg e_s^{\geq l} = n - 1 - i$. Thus

$$(\mathrm{id} \otimes f^*_{\geq k})\,\mathrm{EZ}(x) \cup (\mathrm{id} \otimes f^*_{\geq l})\,\mathrm{EZ}(y) = \left(\sum_r b_r \otimes f^*_{\geq k}(e_r^{\geq k})\right) \cup \left(\sum_s b'_s \otimes f^*_{\geq l}(e_s^{\geq l})\right)$$

and

$$\begin{aligned} C^*_{\geq k}(x) \cup C^*_{\geq l}(y) &= \times \circ (\mathrm{id} \otimes f^*_{\geq k}) \,\mathrm{EZ}(x) \cup \times \circ (\mathrm{id} \otimes f^*_{\geq l}) \,\mathrm{EZ}(y) \\ &= \left(\sum_r b_r \times f^*_{\geq k}(e_r^{\geq k}) \right) \cup \left(\sum_s b'_s \times f^*_{\geq l}(e_s^{\geq l}) \right) \\ &= \sum_{r,s} \pm (b_r \cup b'_s) \times (f^*_{\geq k}(e_r^{\geq k}) \cup f^*_{\geq l}(e_s^{\geq l})). \end{aligned}$$

If $\deg f^*_{\geq k}(e_r^{\geq k}) + \deg f^*_{\geq l}(e_s^{\geq l}) < \dim L$, then $\deg b_r + \deg b'_s > \dim B$ and thus $b_r \cup b'_s = 0$. If $\deg f^*_{\geq k}(e_r^{\geq k}) + \deg f^*_{\geq l}(e_s^{\geq l}) > \dim L$, then trivially $f^*_{\geq k}(e_r^{\geq k}) \cup f^*_{\geq l}(e_s^{\geq l}) = 0$. Finally, if $\deg f^*_{\geq k}(e_r^{\geq k}) + \deg f^*_{\geq l}(e_s^{\geq l}) = \dim L$, then $f^*_{\geq k}(e_r^{\geq k}) \cup f^*_{\geq l}(e_s^{\geq l}) = 0$ by the defining properties of cotruncation and the fact that k and l are complementary. This shows that

$$C^*_{\geq k}(x) \cup C^*_{\geq l}(y) = 0.$$

□

This result means that, as for other characteristic classes, the duality obstructions of a bundle are a measure of how twisted a bundle is. An important special case is $\overline{p}(c+1) = \overline{q}(c+1)$. Then $k = l$, $Q_{\geq k}E = Q_{\geq l}E$, and for $x \in \widetilde{H}^i(Q_{\geq k}E)$, $y \in \widetilde{H}^{n-1-i}(Q_{\geq k}E)$,

$$C^*_{\geq k}(x) \cup C^*_{\geq l}(y) = C^*_{\geq k}(x \cup y).$$

By the injectivity of $C^*_{\geq k}$, this product vanishes if and only if $x \cup y = 0$. So in the case $k = l$ the local duality obstruction $\mathcal{O}_*(\pi, k, k)$ vanishes if and only complementary cup products in $\widetilde{H}^*(Q_{\geq k}E)$ vanish. For a global product this is indeed always the case, by Proposition 6.11.

Example 6.12. Let $B = S^2$, $L = S^3$ and $E = B \times L = S^2 \times S^3$. Then $c = 3$ and, taking $\overline{p}$ and $\overline{q}$ to be lower and upper middle perversities,

$$k = 3 - \overline{m}(4) = 2 = 3 - \overline{n}(4) = l.$$

The degree 2 Moore approximation is $L_{<2} = \mathrm{pt}$ and the cotruncation is $L_{\geq 2} \simeq S^3 = L$. Thus

$$\mathrm{ft}_{\geq 2} E = B \times L_{\geq 2} \simeq S^2 \times S^3 = E.$$

The reduced cohomology $\widetilde{H}^i(Q_{\geq 2}E) = H^i(S^2 \times (S^3, \mathrm{pt}))$ is isomorphic to $\mathbb{Q}$ for $i = 3, 5$ and zero for all other i. Thus all (and in particular, the complementary) cup products vanish and so the local duality obstruction $\mathcal{O}_*(\pi, 2, 2)$ vanishes.

Here is an example of a fiber bundle whose duality obstruction does not vanish.

Example 6.13. Let Dh be the disc bundle associated to the Hopf bundle $h : S^3 \to S^2$, i.e. Dh is the normal disc bundle of $\mathbb{C}P^1$ in $\mathbb{C}P^2$. Now take two copies $Dh_+ \to S^2_+$ and $Dh_- \to S^2_-$ of this disc bundle and define E as the double

$$E = Dh_+ \cup_{S^3} Dh_-.$$

Then E is the fiberwise suspension of h and so an $L = S^2$-bundle over $B = S^2$, with L the suspension of a circle. Let $\sigma_+, \sigma_- \in L$ be the two suspension points. The bundle E is the sphere bundle of a real 3-plane vector bundle ξ over S^2 with $\xi = \eta \oplus \underline{\mathbb{R}}^1$, where η is the real 2-plane bundle whose circle bundle is the Hopf bundle and $\underline{\mathbb{R}}^1$ is the trivial line bundle. The points $\sigma_\pm$ are fixed points under the action of the structure group on L. Let $\overline{p}$ be the lower, and $\overline{q}$ the upper middle perversity. Here $n = 5$, $b = 2$ and $c = 2$. Therefore, $k = 2$ and $l = 1$. Both structural sequences

$$L_{<1} \xrightarrow{f_{<1}} L \xrightarrow{f_{\geq 1}} L_{\geq 1}$$

and

$$L_{<2} \xrightarrow{f_{<2}} L \xrightarrow{f_{\geq 2}} L_{\geq 2}$$

are given by

$$\{\sigma_+\} \hookrightarrow S^2 \xrightarrow{\mathrm{id}} S^2.$$

The identity map is of course equivariant, but the inclusion of the suspension point is equivariant as well, since this is a fixed point. It follows that the fiberwise (co)truncations

$$\mathrm{ft}_{<1} E \longrightarrow E \longrightarrow \mathrm{ft}_{\geq 1} E$$

and

$$\mathrm{ft}_{<2} E \longrightarrow E \longrightarrow \mathrm{ft}_{\geq 2} E$$

are both given by

$$S^2_+ \overset{s_+}{\hookrightarrow} E \xrightarrow{\mathrm{id}} E,$$

where s_+ is the section of $\pi : E \to S^2$ given by sending a point to the suspension point σ_+ over it. Furthermore,

$$Q_{\geq 1}E = Q_{\geq 2}E = E \cup_{S^2_+} D^3,$$

which is homotopy equivalent to complex projective space $\mathbb{CP}^2$. Indeed, a homotopy equivalence is given by the quotient map

$$Q_{\geq 1}E \xrightarrow{\simeq} \frac{Q_{\geq 1}E}{D^3} \cong \frac{E}{S^2_+} \cong \frac{Dh_+ \cup_{S^3} Dh_-}{S^2_+} \cong D^4 \cup_{S^3} Dh_- = \mathbb{CP}^2.$$

The cohomology ring of $\mathbb{CP}^2$ is the truncated polynomial ring $\mathbb{Q}[x]/(x^3 = 0)$ generated by

$$x \in H^2(\mathbb{CP}^2) \cong \widetilde{H}^2(Q_{\geq 2}E) \cong \widetilde{H}^{n-1-2}(Q_{\geq 1}E).$$

The square x^2 generates $H^4(\mathbb{CP}^2)$, so by the injectivity of $C^*_{\geq 1} = C^*_{\geq 2}$,

$$C^*_{\geq 1}(x) \cup C^*_{\geq 2}(x) = C^*_{\geq 1}(x^2) \in H^4(E)$$

is not zero. Thus the duality obstruction $\mathcal{O}_2(\pi, 2, 1)$ does not vanish.

It follows from Proposition 6.11 that $\pi : E \to S^2$ is in fact a nontrivial bundle, which can here of course also be seen directly. Note that the Serre spectral sequence of any S^2-bundle over S^2 collapses at E_2. Thus the obstructions $\mathcal{O}_*(\pi, k, l)$ are able to detect twisting that is not detected by the differentials of the Serre spectral sequence.

7. Flat Bundles

We have shown that the local duality obstructions vanish for product bundles. We prove here that they also vanish for flat bundles, at least when the fundamental group of the base is finite. The latter assumption can probably be relaxed, but we shall not pursue this further here. A fiber bundle $\pi : E \to B$ with structure group G is *flat* if its G-valued transition functions are locally constant.

Theorem 7.1. *Let $\pi : E \to B$ be a fiber bundle of topological manifolds with structure group G, compact connected base B and compact fiber L, $\dim E = n - 1$, $b = \dim B$, $c = n - b - 1$. For complementary perversities $\bar{p}, \bar{q}$, let $k = c - \bar{p}(c + 1)$, $l = c - \bar{q}(c + 1)$. If*

1. *L possesses G-equivariant Moore approximations of degree k and of degree l,*
2. *π is flat with respect to G, and*
3. *the fundamental group $\pi_1(B)$ of the base is finite,*

then $\mathcal{O}_i(\pi, k, l) = 0$ for all i.

Proof. Let $\widetilde{B}$ be the (compact) universal cover of B and $\pi_1 = \pi_1(B)$ the fundamental group. By the G-flatness of E, there exists a monodromy representation $\pi_1 \to G$ such that

$$E = (\widetilde{B} \times L)/\pi_1,$$

where $\widetilde{B} \times L$ is equipped with the diagonal action of π_1, which is free. If M is any compact space on which a finite group π_1 acts freely, then transfer arguments (using the finiteness of π_1) show that the orbit projection $\rho : M \to M/\pi_1$ induces an isomorphism on rational cohomology,

$$\rho^* : H^*(M/\pi_1) \xrightarrow{\cong} H^*(M)^{\pi_1},$$

where $H^*(M)^{\pi_1}$ denotes the π_1-invariant cohomology classes. Applying this to $M = \widetilde{B} \times L$, we get an isomorphism

$$\rho^* : H^*(E) \xrightarrow{\cong} H^*(\widetilde{B} \times L)^{\pi_1}.$$

Using the monodromy representation, the G-cotruncation $L_{\geq k}$ becomes a π_1-space with

$$\mathrm{ft}_{\geq k} E = (\widetilde{B} \times L_{\geq k})/\pi_1.$$

The closed subspace $\widetilde{B} \times \star \subset \widetilde{B} \times L_{\geq k}$, where $\star \in L_{\geq k}$ is the cone point, is π_1-invariant, since $\star$ is a fixed point of $L_{\geq k}$. Then a relative transfer argument applied to the pair $(\widetilde{B} \times L_{\geq k}, B \times \star)$ yields an isomorphism

$$\rho^* : \widetilde{H}^*(Q_{\geq k} E) = H^*(\mathrm{ft}_{\geq k} E, B) \xrightarrow{\cong} H^*(\widetilde{B} \times L_{\geq k}, \widetilde{B} \times \star)^{\pi_1}.$$

Using the structural map $f_{\geq k} : L \to L_{\geq k}$, we define a map

$$p_{\geq k} = \mathrm{id} \times f_{\geq k} : \widetilde{B} \times L \longrightarrow \widetilde{B} \times L_{\geq k}.$$

Since $f_{\geq k}$ is equivariant, the map $p_{\geq k}$ is π_1-equivariant with respect to the diagonal action. The diagram

$$\begin{tikzcd}
\widetilde{B} \times L \arrow[r, "\rho"] \arrow[d, "p_{\geq k}"'] & E \arrow[d, "c_{\geq k}"] \\
\widetilde{B} \times L_{\geq k} \arrow[r, "\rho"] & \mathrm{ft}_{\geq k} E
\end{tikzcd}$$

commutes and induces on cohomology the commutative diagram

$$\begin{tikzcd}
H^*(E) \arrow[r, "\rho^*", "\cong"'] & H^*(\widetilde{B} \times L)^{\pi_1} \\
H^*(\mathrm{ft}_{\geq k} E) \arrow[u, "c^*_{\geq k}"] \arrow[r, "\rho^*", "\cong"'] & H^*(\widetilde{B} \times L_{\geq k})^{\pi_1} \arrow[u, "p^*_{\geq k}"']
\end{tikzcd} \tag{7.1}$$

as we shall now verify: If $a \in H^*(\widetilde{B} \times L_{\geq k})$ satisfies $g^*(a) = a$ for all $g \in \pi_1$, then the equivariance of $p_{\geq k}$ implies that

$$g^* p^*_{\geq k}(a) = p^*_{\geq k}(g^* a) = p^*_{\geq k}(a),$$

which shows that indeed $p^*_{\geq k}(a) \in H^*(\widetilde{B} \times L)^{\pi_1}$. Similarly, there is a commutative diagram

$$\begin{tikzcd}
H^*(\mathrm{ft}_{\geq k} E) \arrow[r, "\rho^*", "\cong"'] & H^*(\widetilde{B} \times L_{\geq k})^{\pi_1} \\
\widetilde{H}^*(Q_{\geq k} E) \arrow[u, "\xi^*_{\geq k}"] \arrow[r, "\rho^*", "\cong"'] & H^*(\widetilde{B} \times (L_{\geq k}, \star))^{\pi_1}. \arrow[u]
\end{tikzcd} \tag{7.2}$$

Concatenating diagrams (7.1) and (7.2), we obtain the commutative diagram

$$\begin{tikzcd}
H^*(E) \arrow[r, "\rho^*", "\cong"'] & H^*(\widetilde{B} \times L)^{\pi_1} \\
\widetilde{H}^*(Q_{\geq k}E) \arrow[u, "C^*_{\geq k}"] \arrow[r, "\rho^*", "\cong"'] & H^*(\widetilde{B} \times (L_{\geq k}, \star))^{\pi_1}. \arrow[u, "P^*_{\geq k}"']
\end{tikzcd}$$

By the Künneth theorem, the cross product $\times$ is an isomorphism

$$\times : H^*(\widetilde{B}) \otimes H^*(L) \xrightarrow{\cong} H^*(\widetilde{B} \times L)$$

whose inverse is given by the Eilenberg-Zilber map EZ. Define a π_1-action on the tensor product $H^*(\widetilde{B}) \otimes H^*(L)$ by

$$g^*(a) := (\mathrm{EZ} \circ g^* \circ \times)(a), \ g \in \pi_1.$$

This makes the cross-product π_1-equivariant:

$$\times \circ g^*(a) = \times \circ \mathrm{EZ} \circ g^* \circ \times(a) = g^* \circ \times(a).$$

Therefore, the cross-product restricts to a map

$$\times : (H^*\widetilde{B} \otimes H^*L)^{\pi_1} \longrightarrow H^*(\widetilde{B} \times L)^{\pi_1}. \tag{7.3}$$

The Eilenberg-Zilber map is equivariant as well, since

$$g^* \mathrm{EZ}(b) = \mathrm{EZ} \circ g^* \circ \times \circ \mathrm{EZ}(b) = \mathrm{EZ} \circ g^*(b).$$

Consequently, the Eilenberg-Zilber map restricts to a map

$$\mathrm{EZ} : H^*(\widetilde{B} \times L)^{\pi_1} \longrightarrow (H^*\widetilde{B} \otimes H^*L)^{\pi_1}. \tag{7.4}$$

Since $\times$ and EZ are inverse to each other, this shows in particular that the restricted cross-product (7.3) and the restricted Eilenberg-Zilber map (7.4) are isomorphisms. All of these constructions apply just as well to $(L_{\geq k}, \star)$ instead of L. By the naturality of the cross product, the square

$$\begin{tikzcd}
H^*(\widetilde{B} \times L) & H^*\widetilde{B} \otimes H^*L \arrow[l, "\times"', "\cong"] \\
H^*(\widetilde{B} \times (L_{\geq k}, \star)) \arrow[u, "P^*_{\geq k}"] & H^*\widetilde{B} \otimes H^*(L_{\geq k}, \star) \arrow[l, "\times"', "\cong"] \arrow[u, "\mathrm{id} \otimes f^*_{\geq k}"']
\end{tikzcd}$$

commutes. As we have seen, this diagram restricts to the various π_1-invariant subspaces. In summary then, we have constructed a commutative diagram

$$\begin{tikzcd}
H^*(E) \arrow[r, "\rho^*", "\cong"'] & H^*(\widetilde{B} \times L)^{\pi_1} & (H^*\widetilde{B} \otimes H^*L)^{\pi_1} \arrow[l, "\times"', "\cong"] \\
\widetilde{H}^*(Q_{\geq k}E) \arrow[u, "C^*_{\geq k}"] \arrow[r, "\rho^*", "\cong"'] & H^*(\widetilde{B} \times (L_{\geq k}, \star))^{\pi_1} \arrow[u, "P^*_{\geq k}"] & (H^*\widetilde{B} \otimes H^*(L_{\geq k}, \star))^{\pi_1} \arrow[l, "\times"', "\cong"] \arrow[u, "\mathrm{id} \otimes f^*_{\geq k}"']
\end{tikzcd}$$

An analogous diagram is, of course, available for $Q_{\geq l}E$.

Let $x \in H^i(\widetilde{B} \times (L_{\geq k}, \star))^{\pi_1}$, $y \in H^{n-1-i}(\widetilde{B} \times (L_{\geq l}, \star))^{\pi_1}$. Their images under the Eilenberg-Zilber map are of the form

$$\mathrm{EZ}(x) = \sum_r b_r \otimes e_r^{\geq k}, \ b_r \in H^*(\widetilde{B}), \ e_r^{\geq k} \in H^*(L_{\geq k}, \star),$$

$$\mathrm{EZ}(y) = \sum_s b'_s \otimes e_s^{\geq l},\ b'_s \in H^*(\widetilde{B}),\ e_s^{\geq l} \in H^*(L_{\geq l}, \star),$$

$\deg b_r + \deg e_r^{\geq k} = i$, $\deg b'_s + \deg e_s^{\geq l} = n-1-i$. Thus

$$(\mathrm{id} \otimes f^*_{\geq k})\,\mathrm{EZ}(x) \cup (\mathrm{id} \otimes f^*_{\geq l})\,\mathrm{EZ}(y) = \left(\sum_r b_r \otimes f^*_{\geq k}(e_r^{\geq k})\right) \cup \left(\sum_s b'_s \otimes f^*_{\geq l}(e_s^{\geq l})\right)$$

and

$$\begin{aligned} P^*_{\geq k}(x) \cup P^*_{\geq l}(y) &= \times \circ (\mathrm{id} \otimes f^*_{\geq k})\,\mathrm{EZ}(x) \cup \times \circ (\mathrm{id} \otimes f^*_{\geq l})\,\mathrm{EZ}(y) \\ &= \left(\sum_r b_r \times f^*_{\geq k}(e_r^{\geq k})\right) \cup \left(\sum_s b'_s \times f^*_{\geq l}(e_s^{\geq l})\right) \\ &= \sum_{r,s} \pm (b_r \cup b'_s) \times (f^*_{\geq k}(e_r^{\geq k}) \cup f^*_{\geq l}(e_s^{\geq l})). \end{aligned}$$

If $\deg f^*_{\geq k}(e_r^{\geq k}) + \deg f^*_{\geq l}(e_s^{\geq l}) < \dim L$, then $\deg b_r + \deg b'_s > \dim B$ and thus $b_r \cup b'_s = 0$. If $\deg f^*_{\geq k}(e_r^{\geq k}) + \deg f^*_{\geq l}(e_s^{\geq l}) > \dim L$, then trivially $f^*_{\geq k}(e_r^{\geq k}) \cup f^*_{\geq l}(e_s^{\geq l}) = 0$. Finally, if $\deg f^*_{\geq k}(e_r^{\geq k}) + \deg f^*_{\geq l}(e_s^{\geq l}) = \dim L$, then $f^*_{\geq k}(e_r^{\geq k}) \cup f^*_{\geq l}(e_s^{\geq l}) = 0$ by the defining properties of cotruncation and the fact that k and l are complementary. This shows that

$$P^*_{\geq k}(x) \cup P^*_{\geq l}(y) = 0.$$

For $\xi \in \widetilde{H}^i(Q_{\geq k}E)$, $\eta \in \widetilde{H}^{n-1-i}(Q_{\geq l}E)$, we find

$$\rho^*(C^*_{\geq k}(\xi) \cup C^*_{\geq l}(\eta)) = \rho^* C^*_{\geq k}(\xi) \cup \rho^* C^*_{\geq l}(\eta) = P^*_{\geq k}(\rho^*\xi) \cup P^*_{\geq l}(\rho^*\eta) = 0.$$

As ρ^* is an isomorphism,

$$C^*_{\geq k}(\xi) \cup C^*_{\geq l}(\eta) = 0.$$

□

8. Thom-Mather Stratified Spaces

In the present paper, intersection spaces will be constructed for closed topological pseudomanifolds that possess a topological stratification of depth 1 such that every connected component of every singular stratum has a closed neighborhood whose boundary is the total space of a fiber bundle, the *link bundle*, while the neighborhood itself is described by the corresponding cone bundle. A large and well-studied class of stratified spaces that have such link bundle structures are the Thom-Mather stratified spaces, which we shall briefly review with particular emphasis on depth 1. Such spaces are locally compact, second countable Hausdorff spaces X together with a Thom-Mather C^∞-stratification, [30]. We are concerned with *two-strata pseudomanifolds*, which, in more detail, are understood to be pairs (X, Σ), where $\Sigma \subset X$ is a closed subspace and a connected smooth manifold, and $X \setminus \Sigma$ is a smooth manifold which is dense in X. The singular stratum Σ must have codimension at least 2 in X. Furthermore, Σ possesses control data consisting of an open neighborhood $T \subset X$ of Σ, a continuous retraction $\pi : T \to \Sigma$, and a continuous distance function $\rho : T \to [0, \infty)$ such that $\rho^{-1}(0) = \Sigma$. The restriction of π and ρ to $T \setminus \Sigma$ are required to be smooth and $(\pi, \rho) : T \setminus \Sigma \to \Sigma \times (0, \infty)$ is required to be a submersion. (Mather's axioms do *not* require (π, ρ) to be proper.) Without appealing to the method of controlled vector fields required by Thom and Mather for general stratified spaces, we shall prove directly that for two-strata spaces, the bottom stratum Σ possesses a locally trivial link bundle whose projection is induced by π.

Lemma 8.1. *Let $f : M \to N$ be a smooth submersion between smooth manifolds and let $Q \subset N$ be a smooth submanifold. Then $P = f^{-1}(Q) \subset M$ is a smooth submanifold and $f| : P \to Q$ is a submersion.*

Proof. A submersion is transverse to any submanifold. Thus, f is transverse to Q and $P = f^{-1}(Q)$ is a smooth submanifold of M. The differential $f_* : T_xM \to T_{f(x)}N$ at any point $x \in P$ maps T_xP into $T_{f(x)}Q$ and thus induces a map $TM/TP \to TN/TQ$ of normal bundles. This map is a bundle isomorphism (cf. [13, Satz (5.12)]). An application of the four-lemma to the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & T_xP \arrow[r] \arrow[d, "f|_*"'] & T_xM \arrow[r] \arrow[d, "f_*"'] & T_xM/T_xP \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & T_{f(x)}Q \arrow[r] & T_{f(x)}N \arrow[r] & T_{f(x)}N/T_{f(x)}Q \arrow[r] & 0 \end{tikzcd}$$

shows that $f|_* : T_xP \to T_{f(x)}Q$ is surjective for every $x \in P$. □

Proposition 8.2. *Let (X, Σ) be a Thom-Mather C^∞-stratified pseudomanifold with two strata and control data (T, π, ρ). Then there exists a smooth function $\epsilon : \Sigma \to (0, \infty)$ such that the restriction $\pi : E \to \Sigma$ to*

$$E = \{x \in T \mid \rho(x) = \epsilon(\pi(x))\}$$

is a smooth locally trivial fiber bundle with structure group $G = \mathrm{Diff}(L)$, the diffeomorphisms of $L = \pi^{-1}(s) \cap E$, where $s \in \Sigma$.

Proof. If $\epsilon : \Sigma \to (0, \infty)$ is any function, we write

$$T_\epsilon = \{x \in T \mid \rho(x) < \epsilon(\pi(x))\}$$

and

$$\Sigma \times [0, \epsilon) = \{(s, t) \in \Sigma \times [0, \infty) \mid 0 \leq t < \epsilon(s)\}.$$

By [33, Lemma 3.1.2(2)], there exists a smooth ϵ such that $(\pi, \rho) : T_\epsilon \to \Sigma \times [0, \epsilon)$ is proper and surjective (and still a submersion on $T_\epsilon \setminus \Sigma$ because $T_\epsilon \setminus \Sigma$ is open in $T \setminus \Sigma$). (This involves only arguments of a point-set topological nature, but no controlled vector fields. Pflaum's lemma provides only for a continuous ϵ, but it is clear that on a smooth Σ, one may take ϵ to be smooth.) Setting

$$E = \{x \in T \mid \rho(x) = \tfrac{1}{2}\epsilon(\pi(x))\} \subset T_\epsilon \setminus \Sigma,$$

we claim first that $\pi : E \to \Sigma$ is proper. Let $\mathrm{Gr} \subset \Sigma \times [0, \infty)$ be the graph of $\frac{1}{2}\epsilon$. The continuity of ϵ implies that Gr is closed in $\Sigma \times [0, \infty)$ and the smoothness of ϵ implies that Gr is a smooth submanifold. From the description $E = (\pi, \rho)^{-1}(\mathrm{Gr})$ we deduce that E is closed in T_ϵ. The inclusion of a closed subspace is a proper map, and the composition of proper maps is again proper. Hence the restriction of a proper map to a closed subspace is proper. It follows that $(\pi, \rho) : E \to \Sigma \times [0, \infty)$ is proper and then that $(\pi, \rho) : E \to \mathrm{Gr}$ is proper. The first factor projection $\pi_1 : \Sigma \times [0, \infty) \to \Sigma$ restricts to a diffeomorphism $\pi_1 : \mathrm{Gr} \to \Sigma$, which is in particular a proper map. The commutative diagram

(8.1)
$$\begin{tikzcd} E \arrow[r, "{(\pi,\rho)}"] \arrow[dr, "\pi"'] & \mathrm{Gr} \arrow[d, "\pi_1", "\cong"'] \\ & \Sigma \end{tikzcd}$$

shows that $\pi : E \to \Sigma$ is proper.

We prove next that $\pi : E \to \Sigma$ is surjective: Given $s \in \Sigma$, the surjectivity of

$$(\pi, \rho) : T_\epsilon \to \Sigma \times [0, \epsilon)$$

implies that there is a point $x \in T_\epsilon$ such that $(\pi(x), \rho(x)) = (s, \frac{1}{2}\epsilon(s))$, that is, $\rho(x) = \frac{1}{2}\epsilon(\pi(x))$. This means that $x \in E$ and $\pi(x) = s$.

By Lemma 8.1, applied to the smooth map $(\pi, \rho) : T \setminus \Sigma \to \Sigma \times (0, \infty)$ and $Q = \mathrm{Gr}$, $E = (\pi, \rho)^{-1}(\mathrm{Gr})$ is a smooth submanifold and $(\pi, \rho) : E \to \mathrm{Gr}$ is a submersion. Using the diagram (8.1), $\pi : E \to \Sigma$ is a submersion.

Applying Ehresmann's fibration theorem (for a modern exposition see [20]) to the proper, surjective, smooth submersion $\pi : E \to \Sigma$ yields the desired conclusion. $\square$

We call the bundle given by Proposition 8.2 the *link bundle* of Σ in X. The fiber is the *link* of Σ. In this manner, Σ becomes the base space B of a bundle and thus we will also use the notation $\Sigma = B$. More generally, this construction evidently applies to the following class of spaces:

Definition 8.3. A *stratified pseudomanifold of depth* 1 is a tuple $(X, \Sigma_1, \cdots, \Sigma_r)$ such that the Σ_i are mutually disjoint subspaces of X such that $\left(X \setminus \left(\bigcup_{j \neq i} \Sigma_j\right), \Sigma_i\right)$ is a two strata pseudomanifold for every $i = 1, \ldots, r$.

In a depth 1 space, every Σ_i possesses its own link bundle.

Definition 8.4. A stratified pseudomanifold of depth 1, $(X, \Sigma_1, \cdots, \Sigma_r)$, is a *Witt space* if the top stratum $X \setminus \bigcup \Sigma_i$ is oriented and the following condition is satisfied:

- For each $1 \leq i \leq r$ such that Σ_i has odd codimension c_i in X, the middle dimensional homology of the link L_i vanishes:

$$H_{\frac{c_i - 1}{2}}(L_i) = 0.$$

Witt spaces were introduced by P. Siegel in [35]. He assumed them to be endowed with a piecewise linear structure, as PL methods allowed him to compute the bordism groups of Witt spaces. We do not use these computations in the present paper.

9. Intersection Spaces and Poincaré Duality

Let (X, B) be an n-dimensional two strata topological pseudomanifold such that $B \neq \varnothing$ is a b-dimensional manifold that has a good open cover, e.g. B PL or even smooth. We assume furthermore that B has a link bundle $\pi : E \to X$ in X so that a tubular neighborhood of B is the associated cone bundle and the complement of the open tube is a manifold M with boundary $\partial M = E$. This is the case if (X, B) is a Thom-Mather C^∞-stratification: The Thom-Mather control data provide a tubular neighborhood T of B in X and a distance function $\rho : T \to [0, \infty)$. Let $\epsilon : \Sigma = B \to (0, \infty)$ be the smooth function provided by Proposition 8.2 such that $\pi : E \to B$ is a fiber bundle, where $E = \{x \in T \mid \rho(x) = \epsilon(\pi(x))\}$. Let M be the complement in X of $T_\epsilon = \{x \in T \mid \rho(x) < \epsilon(\pi(x))\}$ and let L be the fiber of $\pi : E \to B$. By the surjectivity of π, L is not empty. The space M is a smooth n-dimensional manifold with boundary $\partial M = E$. Let $c = \dim L = n - 1 - b$. Fix a perversity $\bar{p}$ satisfying the Goresky-MacPherson growth conditions $\bar{p}(2) = 0$, $\bar{p}(s) \leq \bar{p}(s+1) \leq \bar{p}(s) + 1$ for all $s \in \{2, 3, \ldots\}$. Set $k = c - \bar{p}\,(c+1)$. The growth conditions ensure that $k > 0$. Let $\bar{q}$ be the dual perversity to $\bar{p}$. The integer $l = c - \bar{q}\,(c+1)$ is positive. Assume that there exist G-equivariant Moore approximations of degree k and l,

$$f_{<k} : L_{<k} \to L \text{ and } f_{<l} : L_{<l} \to L$$

for some choice of structure group G for the bundle $\pi : E \to B$.

We perform the fiberwise truncation and cotruncation of Section 6 on the link bundle

$$\pi : E = \partial M \to B,$$

use these constructions to define two incarnations of intersection spaces, $I^{\overline{p}}X$ and $J^{\overline{p}}X$ associated to X, and show that they are homotopy equivalent. The first, $I^{\overline{p}}X$, agrees with the original definition given by the first author in [3] in all cases where they can be compared, the second $J^{\overline{p}}X$ has not been given before. It is introduced here to facilitate certain computations.

Definition 9.1. Define the map $\tau_{<k} : \mathrm{ft}_{<k}E \to M$ to be the composition

$$\tau_{<k} : \mathrm{ft}_{<k}E \xrightarrow{F_{<k}} E = \partial M \xhookrightarrow{i} M,$$

where i is the canonical inclusion of ∂M as the boundary. Define $I^{\overline{p}}X$ to be the homotopy cofiber of $\tau_{<k}$, i.e. the homotopy pushout of the pair of maps

$$\star \longleftarrow \mathrm{ft}_{<k}E \xrightarrow{\tau_{<k}} M.$$

This is called the $\overline{p}$-*intersection space* for X defined via truncation. If $E \cong B \times L$ is a product bundle, then this agrees with [3, Definition 2.41].

Definition 9.2. In Section 6, we obtained the map $C_{\geq k} : E \to Q_{\geq k}E$. Define the $\overline{p}$-intersection space for X *via cotruncation*, $J^{\overline{p}}X$, to be the space obtained as the homotopy pushout of

$$Q_{\geq k} \xleftarrow{C_{\geq k}} E \xhookrightarrow{i} M.$$

We have the following diagram of topological spaces, commutative up to homotopy, in which every square is a homotopy pushout square:

$$\begin{tikzcd}
\mathrm{ft}_{<k}E \arrow[r, "F_{<k}"] \arrow[d, "\pi_{<k}"] & E \arrow[r, "i"] \arrow[d, "c_{\geq k}"] & M \arrow[dd, "\eta_{\geq k}"] \\
B \arrow[r, "\sigma"] \arrow[d] & \mathrm{ft}_{\geq k}E \arrow[d, "\xi_{\geq k}"] & \\
\star \arrow[r, "{[c]}"] & Q_{\geq k}E \arrow[r, "\nu_{\geq k}"] & J^{\overline{p}}X,
\end{tikzcd}$$

where $\eta_{\geq k}$ and $\nu_{\geq k}$ are defined to be the maps coming from the definition of $J^{\overline{p}}X$ as a homotopy pushout.

Lemma 9.3. *The canonical collapse map $J^{\overline{p}}X \to I^{\overline{p}}X$ is a homotopy equivalence.*

Proof. By construction, the space $J^{\overline{p}}X$ contains the cone on B, cB, as a subspace and $(J^{\overline{p}}X, cB)$ is an NDR-pair. Since cB is contractible, the collapse map $J^{\overline{p}}X \to J^{\overline{p}}X/cB$ is a homotopy equivalence. The quotient $J^{\overline{p}}X/cB$ is homeomorphic to $I^{\overline{p}}X$. ☐

The sequence

$$\mathrm{ft}_{<k}\, E \xrightarrow{\tau_{<k}} M \longrightarrow \mathrm{cone}(\tau_{<k}) = I^{\overline{p}}X$$

induces a long exact sequence

$$\longrightarrow H^{r-1}(\mathrm{ft}_{<k}E) \xrightarrow{\delta^{\overline{p},r}} \widetilde{H}^r\left(I^{\overline{p}}X\right) \xrightarrow{\eta^r_{\geq k}} H^r(M) \xrightarrow{\tau^r_{<k}} H^r(\mathrm{ft}_{<k}E) \longrightarrow . \tag{9.1}$$

Furthermore, we can define $\widehat{M}$ to be the homotopy pushout of the pair of maps

$$\star \longleftarrow \partial M = E \xrightarrow{i} M.$$

This is nothing but the space M with a cone attached to the boundary. Define $J^{-1}X$ to be the homotopy pushout obtained from the pair of maps

$$\star \longleftarrow Q_{\geq k}E \xrightarrow{\nu_{\geq k}} J^{\overline{p}}X.$$

Lemma 9.4. *The canonical collapse map $J^{-1}X \to \widehat{M}$ is a homotopy equivalence.*

Proof. The space $J^{-1}X$ contains the cone $cQ_{\geq k}E$ as a subspace and $(J^{-1}X, cQ_{\geq k}E)$ is an NDR-pair. Thus the collapse map $J^{-1}X \to J^{-1}X/cQ_{\geq k}E$ is a homotopy equivalence. The quotient $J^{-1}X/cQ_{\geq k}E$ is homeomorphic to $\widehat{M}$. ☐

By the lemma, using l and $\overline{q}$ instead of k and $\overline{p}$, we have the long exact sequence (2.2) associated to $J^{-1}X$:

$$\longrightarrow \widetilde{H}_r\left(Q_{\geq l}E\right) \xrightarrow{\nu_{\geq l,r}} \widetilde{H}_r\left(J^{\overline{q}}X\right) \xrightarrow{\zeta_{\geq l,r}} H_r\left(M, \partial M\right) \xrightarrow{\delta_r^{\overline{q}}} \widetilde{H}_{r-1}\left(Q_{\geq l}E\right) \longrightarrow, \tag{9.2}$$

where $\zeta_{\geq l}$ is the composition of the map $J^{\overline{q}}X \to J^{-1}X$, defined by $J^{-1}X$ as a homotopy pushout, with the collapse map $J^{-1}X \xrightarrow{\simeq} \widehat{M}$. In the sequence, we have identified $\widetilde{H}_r(\widehat{M}) \cong H_r\left(M, \partial M\right)$.

Theorem 9.5. *Let (X, B) be a compact, oriented, two strata pseudomanifold of dimension n. Let $\overline{p}$ and $\overline{q}$ be complementary perversities, and $k = c - \overline{p}(c+1)$, $l = c - \overline{q}(c+1)$, where $c = n - 1 - \dim B$. Assume that equivariant Moore approximations to L of degree k and degree l exist. If the local duality obstructions $\mathcal{O}_*(\pi, k, l)$ of the link bundle π vanish, then there is a global Poincaré duality isomorphism*

$$\widetilde{H}^r\left(I^{\overline{p}}X\right) \cong \widetilde{H}_{n-r}\left(I^{\overline{q}}X\right). \tag{9.3}$$

Proof. We achieve this by pairing the sequence (9.1) with the sequence (9.2) (observing Lemma 9.3) and using the five lemma. Consider the following diagram of solid arrows whose rows are exact:

$$\begin{tikzcd}[ampersand replacement=\&] {} \arrow[r] \& H^{r-1}(\mathrm{ft}_{<k}E) \arrow[r, "\delta^{\overline{p},*}"] \arrow[d, "D^{r-1}_{k,l}", "\cong"'] \& \widetilde{H}^r\left(I^{\overline{p}}X\right) \arrow[r, "\eta^*_{\geq k}"] \arrow[d, dashed, "D^r_{IX}"] \& H^r(M) \arrow[r, "\tau^*_{<k}"] \arrow[d, "D^r_M", "\cong"'] \& H^r(\mathrm{ft}_{<k}E) \arrow[d, "D^r_{k,l}", "\cong"'] \\ {} \arrow[r] \& \widetilde{H}_{n-r}\left(Q_{\geq l}E\right) \arrow[r, "\nu_{\geq l,*}"] \& \widetilde{H}_{n-r}\left(I^{\overline{q}}X\right) \arrow[r, "\zeta_{\geq l,*}"] \& H_{n-r}\left(M, \partial M\right) \arrow[r, "\delta^{\overline{q}}_*"] \& \widetilde{H}_{n-r-1}\left(Q_{\geq l}E\right) \end{tikzcd} \tag{9.4}$$

Here $D^r_{k,l}$ comes from Proposition 6.7, and D^r_M comes from the classical Lefschetz duality for manifolds with boundary. The solid arrow square on the right can be written as

$$\begin{tikzcd}[ampersand replacement=\&] H^r(M) \arrow[r, "i^*"] \arrow[d, "D^r_M", "\cong"'] \& H^r(\partial M) \arrow[r, "F^*_{<k}"] \arrow[d, "D^r_{\partial M}", "\cong"'] \& H^r(\mathrm{ft}_{<k}E) \arrow[d, "D^r_{k,l}", "\cong"'] \\ H_{n-r}\left(M, \partial M\right) \arrow[r, "\delta^{M,\partial M}_*"] \& H_{n-r-1}(\partial M) \arrow[r, "C_{\geq l,*}"] \& \widetilde{H}_{n-r-1}\left(Q_{\geq l}E\right) \end{tikzcd}$$

The left square commutes by classical Poincaré-Lefschetz duality, and the right square commutes by Proposition 6.9 and Proposition 6.10, since $\mathcal{O}_*(\pi, k, l) = 0$. Thus diagram (9.4) commutes. By e.g. [3, Lemma 2.46], we may find a map D^r_{IX} to fill in the dotted arrow so that the diagram commutes. By the five lemma, D^r_{IX} is an isomorphism. ☐

It does not follow from this proof that for a $4d$-dimensional Witt space X the associated intersection form $\widetilde{H}_{2d}(IX) \times \widetilde{H}_{2d}(IX) \to \mathbb{Q}$ is symmetric, where $IX = I^{\bar{m}}X = I^{\bar{n}}X$. In Section 11, however, we shall prove that the isomorphism (9.3) can always be constructed so as to yield a symmetric intersection form (cf. Proposition 11.11).

10. Moore Approximations and the Intersection Homology Signature

Assume that (X, B) is a two-strata Witt space with $\dim X = n = 4d$, $d > 0$, and $\dim B = b$, then $c = 4d - 1 - b = \dim L$. If we use the upper-middle perversity $\overline{n}$ and the lower-middle perversity $\overline{m}$, which are complementary, we get the associated pair of integers $k = \lfloor \frac{c+1}{2} \rfloor$ and $l = \lceil \frac{c+1}{2} \rceil$. When c is odd then $k = l = \frac{c+1}{2}$, and when c is even then $k = c/2$ and $l = k + 1$. Notice that the codimension of B in X is $c + 1$. So the Witt condition says that when c is even then $H_{\frac{c}{2}}(L) = 0$. In this case if an equivariant Moore approximation of degree k exists, then so does one of degree $k + 1 = l$ and they can be chosen to be equal. Therefore, when X satisfies the Witt condition and an equivariant Moore approximation to L of degree k exists, we can construct $I^{\overline{m}}X = I^{\overline{n}}X$ and $J^{\overline{m}}X = J^{\overline{n}}X$. We denote the former space IX and the latter JX and call this homotopy type the *intersection space* associated to the Witt space X.

The cone bundle DE is nothing but $\mathrm{ft}_{\geq c+1}E$ with $L_{<c+1} = L$. Note that when $E = \partial M$ as above, then DE is a two strata space with boundary $\partial DE = \partial M$, and we can realize X as the pushout of the pair of maps $M \xleftarrow{i} \partial M \xrightarrow{c_{\geq c+1}} DE$. Thus ∂M is bi-collared in X and by Novikov additivity, Prop. II,3.1 [35], we have that the intersection homology Witt element w_{IH}, defined in I,4.1 [35], is additive over these parts,

$$w_{IH}(X) = w_{IH}(\widehat{M}) + w_{IH}(TE) \in W(\mathbb{Q}), \tag{10.1}$$

where the Thom space TE is DE with a cone attached to its boundary, and $W(\mathbb{Q})$ is the Witt group of $\mathbb{Q}$. When X is Witt, we write $IH_*(X)$ for $I^{\bar{m}}H_*(X) = I^{\bar{n}}H_*(X)$.

Proposition 10.1. *If an equivariant Moore approximation to L of degree $k = \lfloor \frac{1}{2}(\dim L + 1) \rfloor$ exists, then the middle degree, middle perversity intersection homology of the $n = 4d$-dimensional Witt space TE vanishes,*

$$IH_{2d}(TE) = 0.$$

Proof. In this proof we use the notation $\dot{c}E$ and $\dot{D}E$ to mean the open cone on E and the open cone bundle associated to E. According to (5.3),

$$I^{\overline{p}}H_r(\dot{D}E) \cong I^{\overline{p}}H_r(DE), \quad \text{and} \quad I^{\overline{p}}H_r(\dot{c}E) \cong I^{\overline{p}}H_r(cE)$$

for all $r \geq 0$. Hence, as in the proof of Proposition 5.3, we can identify the long exact sequence of intersection homology groups associated to the pair $(\dot{D}E, \dot{D}E \setminus B)$ with the same sequence associated to the ∂-stratified pseudomanifold (DE, E) from (5.2).

Define open subsets U, V of TE by $U = TE \setminus B = \dot{c}E$ and $V = TE \setminus c = \dot{D}E$, where c is the cone point. Then $TE = U \cup V$ and $U \cap V = E \times (-1, 1)$. The Mayer-Vietoris sequence associated to the pair (U, V) gives

$$\longrightarrow H_r(E) \xrightarrow{i_r^{TE}} IH_r(\dot{D}E) \oplus IH_r(\dot{c}E) \xrightarrow{j_r^{TE}} IH_r(TE) \xrightarrow{\delta_r^{TE}} H_{r-1}(E) \longrightarrow \tag{10.2}$$

Here we have identified $IH_r(E \times (-1, 1)) \cong H_r(E)$. After making the identifications as decribed in the previous paragraph, the map $i_r^{TE} = i_r^{DE} \oplus i_r^{cE}$ is identified as the sum of the maps coming from the sequences associated to the pairs (DE, E) and (cE, E) respectively. In degrees $r < 2d$ we know from Proposition 5.3 that i_r^{cE} is an isomorphism $H_r(E) = IH_r(cE)$. Thus i_r^{TE} is injective for $r < 2d$. Consequently, when $r = 2d$, we have an exact sequence

$$\cdots \longrightarrow H_{2d}(E) \longrightarrow IH_{2d}(DE) \oplus IH_{2d}(cE) \longrightarrow IH_{2d}(TE) \longrightarrow 0.$$

By the cone formula for intersection homology, $IH_{2d}(cE) = 0$, since $2d = \dim E - \overline{m}(\dim E + 1)$. Now by Proposition 6.5, the map $H_{2d}(E) \to IH_{2d}(DE)$ is surjective. $\square$

Corollary 10.2. *Let X be a compact, oriented, $n = 4d$-dimensional stratified pseudomanifold of depth 1 which satisfies the Witt condition. If equivariant Moore approximations of degree $k = \lfloor \frac{1}{2}(\dim L + 1) \rfloor$ to the links of the singular set exist, then*

$$w_{IH}(X) = w_{IH}(\widehat{M}) \in W(\mathbb{Q}).$$

In particular, the signature of the intersection form on intersection homology satisfies

$$\sigma_{IH}(X) = \sigma_{IH}(\widehat{M}).$$

Proof. If $IH_{2n}(TE) = 0$, then $w_{IH}(TE) = 0$. The assertion follows from Novikov additivity (10.1). ☐

Example 10.3. Let $X = \mathbb{CP}^2$ be complex projective space with $B = \mathbb{CP}^1 \subset X$ as the bottom stratum, so that the link bundle is the Hopf bundle over B. Then

$$\sigma_{IH}(X) = \sigma(\mathbb{CP}^2) = 1,$$

but

$$\sigma(M, \partial M) = \sigma(D^4, S^3) = 0.$$

Indeed, the link S^1 in the Hopf bundle has no middle-perversity equivariant Moore-approximation because the Hopf bundle has no section.

11. The Signature of Intersection Spaces

Theorem 2.28 in [3] states that for a closed, oriented, $4d$-dimensional Witt space X with only isolated singularities, the signature of the symmetric nondegenerate intersection form

$$\widetilde{H}_{2d}(IX) \times \widetilde{H}_{2d}(IX) \to \mathbb{Q}$$

equals the signature of the Goresky-MacPherson-Siegel intersection form

$$IH_{2d}(X) \times IH_{2d}(X) \to \mathbb{Q}$$

on middle-perversity intersection homology. In fact, both are equal to the Novikov signature of the top stratum. We shall here generalize that theorem to spaces with twisted link bundles that allow for equivariant Moore approximation.

Definition 11.1. Define the signature of a $4d$-dimensional manifold-with-boundary $(M, \partial M)$ to be

$$\sigma(M, \partial M) = \sigma(\beta),$$

where β is the bilinear form

$$\beta : \operatorname{im} j_* \times \operatorname{im} j_* \to \mathbb{Q},\ (j_* v, j_* w) \mapsto (d_M(v))(j_* w),$$

the homomorphism

$$j_* : H_{2d}(M) \longrightarrow H_{2d}(M, \partial M)$$

is induced by the inclusion, and

$$d_M : H_{2d}(M) \longrightarrow H^{2d}(M, \partial M)$$

is Lefschetz duality. This is frequently referred to as the *Novikov signature* of $(M, \partial M)$. It is well-known ([35]) that $\sigma(M, \partial M) = \sigma_{IH}(\widehat{M})$.

Let (X, B) be a two strata Witt space with $\dim X = n = 4d$, $\dim B = b$. We assume that an equivariant Moore approximation of degree $k = 4d - b - 1 - \overline{m}\,(4d - b)$ exists for the link L of B in X, and that the local duality obstruction $\mathcal{O}_*(\pi, k, k)$ vanishes. As discussed in the previous section, this implies that the intersection space IX exists and is well-defined. Theorem 9.5 asserts that IX satisfies Poincaré duality

$$d_{IX} : \widetilde{H}_{2d}(IX) \xrightarrow{\cong} \widetilde{H}^{2d}(IX).$$

We shall show (Proposition 11.11) that d_{IX} can in fact be so constructed that the associated intersection form on the middle-dimensional homology is symmetric. One may then consider its signature:

Definition 11.2. The signature of the space IX,

$$\sigma\,(IX) = \sigma\,(\beta)\,,$$

is defined to be the signature of the symmetric bilinear form

$$\beta : \widetilde{H}_m(IX) \times \widetilde{H}_m(IX) \to \mathbb{Q},$$

with $m = 2d$, defined by

$$\beta(v, w) = d_{IX}(v)(w)$$

for any $v, w \in \widetilde{H}_m(IX)$. Here we have identified $\widetilde{H}^m(IX) \cong \widetilde{H}_m(IX)^\dagger$ via the universal coefficient theorem.

Theorem 11.3. *The signature of IX is supported away from the singular set B, that is,*

$$\sigma\,(IX) = \sigma\,(M, \partial M)\,.$$

Before we prove this theorem, we note that in view of Corollary 10.2, we immediately obtain:

Corollary 11.4. *If a two-strata Witt space (X, B) allows for middle-perversity equivariant Moore-approximation of its link and has vanishing local duality obstruction, then*

$$\sigma_{IH}(X) = \sigma(IX).$$

The rest of this section is devoted to the proof of Theorem 11.3. We build on the method of Spiegel [37], which in turn is partially based on the methods introduced in the proof of [3, Theorem 2.28]. Regarding notation, we caution that the letters i and j will both denote certain inclusion maps and appear as indices. This cannot possibly lead to any confusion.

Let $\{e_1, \ldots, e_r\}$ be any basis for $j_* H_m(M)$, where

$$j_* : H_m(M) \longrightarrow H_m(M, \partial M)$$

is induced by the inclusion. For every $i = 1, \ldots, r$, pick a lift $\overline{e}_i \in H_m(M)$, $j_*(\overline{e}_i) = e_i$. Then $\{\overline{e}_1, \ldots, \overline{e}_r\}$ is a linearly independent set in $H_m(M)$ and

$$\mathbb{Q}\langle \overline{e}_1, \ldots, \overline{e}_r \rangle \cap \ker j_* = \{0\}. \tag{11.1}$$

Let

$$d_M : H_m(M) \xrightarrow{\cong} H^m(M, \partial M) = H_m(M, \partial M)^\dagger$$

be the Lefschetz duality isomorphism, i.e. the inverse of

$$D'_M : H^m(M, \partial M) \xrightarrow{\cong} H_m(M),$$

given by capping with the fundamental class $[M, \partial M] \in H_{2m}(M, \partial M)$. Let

$$d'_M : H_m(M, \partial M) \xrightarrow{\cong} H^m(M)$$

be the inverse of

$$D_M : H^m(M) \xrightarrow{\cong} H_m(M, \partial M),$$

given by capping with the fundamental class. We shall make frequent use of the symmetry identity

$$d_M(v)(w) = d'_M(w)(v),$$

$v \in H_m(M)$, $w \in H_m(M, \partial M)$, which holds since the cup product of m-dimensional cohomology classes commutes as $m = 2d$ is even. The commutative diagram

$$\begin{tikzcd} H_m(M) \arrow[r, "j_*"] \arrow[d, "d_M"'] & H_m(M, \partial M) \arrow[d, "d'_M"] \\ H^m(M, \partial M) \arrow[r, "j^*"'] & H^m(M) \end{tikzcd}$$

implies that the symmetry equation

$$d_M(\overline{e}_i)(e_j) = d_M(\overline{e}_j)(e_i)$$

holds, as the calculation

$$\begin{aligned} d_M(\overline{e}_i)(e_j) &= d_M(\overline{e}_i)(j_*\overline{e}_j) = j^* d_M(\overline{e}_i)(\overline{e}_j) = d'_M(j_*\overline{e}_i)(\overline{e}_j) \\ &= d'_M(e_i)(\overline{e}_j) = d_M(\overline{e_j})(e_i) \end{aligned}$$

shows.

In the proof of [3, Theorem 2.28], the first author introduced the annihilation subspace $Q \subset H_m(M, \partial M)$,

$$Q = \{q \in H_m(M, \partial M) \mid d_M(\overline{e}_i)(q) = 0 \text{ for all } i\}.$$

It is shown on p. 138 of *loc. cit.* that one obtains an internal direct sum decomposition

$$H_m(M, \partial M) = \operatorname{im} j_* \oplus Q.$$

Let $L \subset \widetilde{H}_m(IX)$ be the kernel of the map

$$\zeta_{\geq k*} : \widetilde{H}_m(IX) \longrightarrow H_m(M, \partial M).$$

Once we have completed the construction of a symmetric intersection form, L will eventually be shown to be a Lagrangian subspace of an appropriate subspace of $\widetilde{H}_m(IX)$. Let $\{u_1, \dots u_l\}$ be any basis for L.

We consider the commutative diagram

$$\begin{tikzcd} & H_m(M, \partial M) \arrow[r, equal] & H_m(M, \partial M) & \\ H_m(\operatorname{ft}_{<k} E) \arrow[r, "\tau_{<k*}"] \arrow[d, equal] & H_m(M) \arrow[r, "\eta_{\geq k*}"] \arrow[u, "j_*"] & \widetilde{H}_m(IX) \arrow[r, "\delta_*"] \arrow[u, "\zeta_{\geq k*}"'] & H_{m-1}(\operatorname{ft}_{<k} E) \arrow[d, equal] \\ H_m(\operatorname{ft}_{<k} E) \arrow[r, hook, "F_{<k*}"] & H_m(\partial M) \arrow[r, two heads, "C_{\geq k*}"] \arrow[u, "i_*"] & \widetilde{H}_m(Q_{\geq k} E) \arrow[r, "\delta_* = 0"] \arrow[u, "\nu_{\geq k*}"'] & H_{m-1}(\operatorname{ft}_{<k} E) \end{tikzcd} \tag{11.2}$$

The rows and columns are exact and we have used Lemma 6.6. By exactness of the right hand column, the basis elements u_j can be lifted to $\widetilde{H}_m(Q_{\geq k}E)$, and by the surjectivity of $C_{\geq k*}$, these lifts can be further lifted to $H_m(\partial M)$. In this way, we obtain linearly independent elements $\overline{u}_1, \dots, \overline{u}_l$ in $H_m(\partial M)$ such that

$$\eta_{\geq k*} i_*(\overline{u}_j) = \nu_{\geq k*} C_{\geq k*}(\overline{u}_j) = u_j$$

for all j. Setting

$$w^j = d_M(i_*(\overline{u}_j))$$

yields a linearly independent set $\{w^1, \ldots, w^l\} \subset H^m(M, \partial M)$. From now on, let us briefly write η_*, ζ_*, etc., for $\eta_{\geq k*}, \zeta_{\geq k*}$, etc. Since $\eta_* i_*(\overline{u}_j) = u_j$, we have

$$\mathbb{Q}\langle i_*(\overline{u}_1), \ldots, i_*(\overline{u}_l)\rangle \cap \ker \eta_* = \{0\}.$$

Together with (11.1), and noting $\ker \eta_* \subset \ker j_*$, this shows that there exists a linear subspace $A \subset H_m(M)$ yielding an internal direct sum decomposition

$$H_m(M) = \mathbb{Q}\langle i_*(\overline{u}_1), \ldots, i_*(\overline{u}_l)\rangle \oplus \ker \eta_* \oplus \mathbb{Q}\langle \overline{e}_1, \ldots, \overline{e}_r\rangle \oplus A. \tag{11.3}$$

Setting

$$Z = \ker \eta_* \oplus \mathbb{Q}\langle \overline{e}_1, \ldots, \overline{e}_r\rangle \oplus A,$$

we have

$$H_m(M) = \mathbb{Q}\langle i_*(\overline{u}_1), \ldots, i_*(\overline{u}_l)\rangle \oplus Z,$$

such that

$$\ker \eta_* \subset Z \text{ and } \mathbb{Q}\langle \overline{e}_1, \ldots, \overline{e}_r\rangle \subset Z. \tag{11.4}$$

Choose a basis $\{\widetilde{z}_1, \ldots, \widetilde{z}_s\}$ of Z and put $z^j = d_M(\widetilde{z}_j) \in H^m(M, \partial M)$. Then $\{z^1, \ldots z^s\}$ is a basis for $d_M(Z)$ and

$$H^m(M, \partial M) = \mathbb{Q}\langle w^1, \ldots, w^l\rangle \oplus \mathbb{Q}\langle z^1, \ldots, z^s\rangle.$$

Let

$$\{w_1, \ldots, w_l, z_1, \ldots, z_s\} \subset H_m(M, \partial M)$$

be the dual basis of $\{w^1, \ldots, w^l, z^1, \ldots, z^s\}$, that is,

$$w^i(w_j) = \delta_{ij},\ z^i(z_j) = \delta_{ij},\ w^i(z_j) = 0,\ z^i(w_j) = 0. \tag{11.5}$$

Lemma 11.5. *The set $\{w_1, \ldots, w_l\}$ is contained in the image of ζ_*.*

Proof. In view of the commutative diagram

$$\begin{tikzcd}
\widetilde{H}_m(IX) \arrow[r, "\zeta_*"] & H_m(M, \partial M) \arrow[r, "\delta_*"] \arrow[d, "d'_M"', "\cong"] & \widetilde{H}_{m-1}(Q_{\geq k}E) \\
& H^m(M) \arrow[r, "\tau^*"] & H^m(\operatorname{ft}_{<k} E), \arrow[u, "\cong", "D_{k,k}"']
\end{tikzcd}$$

it suffices to show that $\delta_*(w_j) = 0$, since the top row is exact. Let $x \in H_m(\operatorname{ft}_{<k} E)$ be any element. Then $\tau_* x \in \ker \eta_* \subset Z$, so $d_M(\tau_* x)(w_j) = 0$ by (11.5). Consequently,

$$(\tau^* d'_M(w_j))(x) = d'_M(w_j)(\tau_* x) = d_M(\tau_* x)(w_j) = 0.$$

It follows that $\tau^* d'_M(w_j) = 0$ and in particular

$$\delta_*(w_j) = D_{k,k}\tau^* d'_M(w_j) = 0.$$

$\square$

Suppose that $v \in \ker \zeta_* \cap \eta_*\langle \overline{e}_1, \ldots, \overline{e}_r\rangle$. Then v is a linear combination $v = \eta_* \sum \lambda_i \overline{e}_i$ and

$$0 = \zeta_*(v) = \zeta_*\eta_* \sum \lambda_i \overline{e}_i = \sum \lambda_i j_*(\overline{e}_i) = \sum \lambda_i e_i.$$

Thus $\lambda_i = 0$ for all i by the linear independence of the e_i. This shows that

$$L \cap \eta_*\langle \overline{e}_1, \ldots, \overline{e}_r\rangle = \{0\}.$$

Therefore, it is possible to choose a direct sum complement $W \subset \widetilde{H}_m(IX)$ of $L = \ker \zeta_*$,

$$\widetilde{H}_m(IX) = L \oplus W, \tag{11.6}$$

such that

$$\eta_* \langle \overline{e}_1, \ldots, \overline{e}_r \rangle \subset W. \tag{11.7}$$

The restriction

$$\zeta_*|_W : W \longrightarrow \operatorname{im} \zeta_*$$

is then an isomorphism and thus by Lemma 11.5, we may define

$$\overline{w}_j = (\zeta_*|_W)^{-1}(w_j).$$

We define subspaces $V, L' \subset W$ by

$$V = (\zeta_*|_W)^{-1}(\operatorname{im} j_*), \; L' = (\zeta_*|_W)^{-1}(Q \cap \operatorname{im} \zeta_*).$$

Recall that $\{e_1, \ldots, e_r\}$ is a basis of $\operatorname{im} j_*$. Setting

$$v_j = (\zeta_*|_W)^{-1}(e_j),$$

yields a basis $\{v_1, \ldots, v_r\}$ for V. From

$$\zeta_*(v_i) = e_i = j_*(\overline{e}_i) = \zeta_* \eta_*(\overline{e}_i)$$

it follows that

$$v_i = \eta_*(\overline{e}_i),$$

since both v_i and $\eta_*(\overline{e}_i)$ are in W and ζ_* is injective on W.

The decomposition $H_m(M, \partial M) = \operatorname{im} j_* \oplus Q$ induces a decomposition

$$\operatorname{im} \zeta_* = (\operatorname{im} j_* \oplus Q) \cap \operatorname{im} \zeta_* = \operatorname{im} j_* \oplus (Q \cap \operatorname{im} \zeta_*).$$

Applying the isomorphism $(\zeta_*|_W)^{-1}$, we receive a decomposition

$$W = (\zeta_*|_W)^{-1}(\operatorname{im} j_*) \oplus (\zeta_*|_W)^{-1}(Q \cap \operatorname{im} \zeta_*) = V \oplus L'.$$

By (11.6), we arrive at a decomposition

$$\widetilde{H}_m(IX) = L \oplus V \oplus L'.$$

Lemma 11.6. *The set $\{\overline{w}_1, \ldots, \overline{w}_l\} \subset W$ is contained in L'.*

Proof. By construction of L', we have to show that $\zeta_*(\overline{w}_j) \in Q$ for all j. Now $\zeta_*(\overline{w}_j) = w_j$, so by construction of Q, we need to demonstrate that $d_M(\overline{e}_i)(w_j) = 0$ for all i. By (11.4), $d_M(\overline{e}_i) \in d_M(Z)$, whence the result follows from (11.5). ☐

Lemma 11.7. *The set $\{\overline{w}_1, \ldots, \overline{w}_l\} \subset W$ is a basis for L'.*

Proof. The preimages $\overline{w}_j = (\zeta_*|_W)^{-1}(w_j)$ under the isomorphism $\zeta_*|_W$ are linearly independent since $\{w_1, \ldots, w_l\}$ is a linearly independent set. In particular, $\dim L' \geq l$. It remains to be shown that $\dim L' \leq l$. Standard linear algebra provides the inequality

$$\operatorname{rk} \eta_* \leq \dim \ker \zeta_* + \operatorname{rk}(\zeta_* \eta_*),$$

valid for the composition of any two linear maps. As $\zeta_* \eta_* = j_*$, we may rewrite this as

$$\operatorname{rk} \eta_* \leq l + \operatorname{rk} j_*. \tag{11.8}$$

By Theorem 9.5, there exists some isomorphism $\widetilde{H}^m(IX) \to \widetilde{H}_m(IX)$ such that

$$\begin{tikzcd} \widetilde{H}^m(IX) \arrow[r, "\eta^*"] \arrow[d, "\cong"'] & H^m(M) \arrow[d, "\cong"', "D_M"] \\ \widetilde{H}_m(IX) \arrow[r, "\zeta_*"] & H_m(M, \partial M) \end{tikzcd} \tag{11.9}$$

commutes. Therefore,

$$\operatorname{rk} \zeta_* = \operatorname{rk} \eta^* = \operatorname{rk} \eta_*,$$

and by (11.8),

$$\operatorname{rk} \zeta_* \leq l + \operatorname{rk} j_*.$$

The decomposition (11.6) implies that

$$\dim \widetilde{H}_m(IX) = l + \dim W = l + \operatorname{rk} \zeta_* \leq 2l + \operatorname{rk} j_*.$$

On the other hand, the decomposition $\widetilde{H}_m(IX) = L \oplus V \oplus L'$ implies

$$\dim \widetilde{H}_m(IX) = l + \dim V + \dim L' = l + \operatorname{rk} j_* + \dim L'.$$

It follows that

$$l + \operatorname{rk} j_* + \dim L' \leq 2l + \operatorname{rk} j_*$$

and thus

$$\dim L' \leq l.$$

□

In summary then, we have constructed a certain basis

$$\{u_1, \dots, u_l, v_1, \dots, v_r, \overline{w}_1, \dots, \overline{w}_l\} \tag{11.10}$$

for $\widetilde{H}_m(IX) = L \oplus V \oplus L'$.

Remark 11.8. The above proof shows that $\operatorname{rk} \eta_* \leq l + \operatorname{rk} j_* = l + r$. Thus the restriction of η_* to the subspace $A \subset H_m(M)$ in the decomposition (11.3) is zero, which implies that $A \subset \ker \eta_*$ and so $A = \{0\}$. The decomposition of $H_m(M)$ is thus seen to be

$$H_m(M) = \mathbb{Q}\langle i_*(\overline{u}_1), \dots, i_*(\overline{u}_l)\rangle \oplus \ker \eta_* \oplus \mathbb{Q}\langle \overline{e}_1, \dots, \overline{e}_r\rangle. \tag{11.11}$$

In particular,

$$Z = \ker \eta_* \oplus \mathbb{Q}\langle \overline{e}_1, \dots, \overline{e}_r\rangle.$$

Let

$$\{u^1, \dots, u^l, v^1, \dots, v^r, \overline{w}^1, \dots, \overline{w}^l\}$$

be the dual basis for $\widetilde{H}^m(IX)$. Setting

$$L^\dagger = \mathbb{Q}\langle u^1, \dots, u^l\rangle,\ V^\dagger = \mathbb{Q}\langle v^1, \dots, v^r\rangle,\ (L')^\dagger = \mathbb{Q}\langle \overline{w}^1, \dots, \overline{w}^l\rangle,$$

we get a dual decomposition

$$\widetilde{H}^m(IX) = L^\dagger \oplus V^\dagger \oplus (L')^\dagger.$$

We define the duality map

$$d_{IX} : \widetilde{H}_m(IX) \longrightarrow \widetilde{H}^m(IX)$$

on basis elements to be

$$\begin{aligned} d_{IX}(u_j) &:= \overline{w}^j, \\ d_{IX}(\overline{w}_j) &:= u^j, \\ d_{IX}(v_j) &:= \zeta^* d_M(\overline{e}_j). \end{aligned}$$

We shall now prove that d_{IX} is an isomorphism.

Lemma 11.9. *The image $d_{IX}(V)$ is contained in $V^\dagger$.*

Proof. In terms of the dual basis, $d_{IX}(v_j)$ can be expressed as a linear combination

$$d_{IX}(v_j) = \sum_p \pi_p u^p + \sum_q \epsilon_q v^q + \sum_i \lambda_i \overline{w}^i.$$

The coefficients π_p are

$$\pi_p = \left(\zeta^* d_M(\overline{e}_j)\right)(u_p) = d_M(\overline{e}_j)(\zeta_* u_p) = 0,$$

since $u_p \in L = \ker \zeta_*$. Using (11.5) and $d_M(\overline{e}_j) \in d_M(Z) = \mathbb{Q}\langle z^1, \ldots, z^s \rangle$, we find

$$\lambda_i = \left(\zeta^* d_M(\overline{e}_j)\right)(\overline{w}_i) = d_M(\overline{e}_j)(w_i) = 0.$$

□

Lemma 11.10. *The restriction $d_{IX}| : V \to V^\dagger$ is injective.*

Proof. Suppose that $v = \sum_q \epsilon_q v_q$ is any vector $v \in V$ with $d_{IX}(v) = 0$. Then

$$\begin{aligned} 0 = \eta^* d_{IX}(v) &= \eta^* \sum \epsilon_q d_{IX}(v_q) = \eta^* \sum \epsilon_q \zeta^* d_M(\overline{e}_q) \\ &= j^* d_M \sum \epsilon_q \overline{e}_q = d'_M \sum \epsilon_q j_*(\overline{e}_q) \\ &= d'_M \sum \epsilon_q e_q. \end{aligned}$$

Since d'_M is an isomorphism, $\sum \epsilon_q e_q = 0$ and by the linear independence of the e_q, the coefficients ϵ_q all vanish. This shows that $v = 0$. □

By definition, d_{IX} maps L isomorphically onto $(L')^\dagger$ and L' isomorphically onto $L^\dagger$. Since by Lemma 11.10, $d_{IX}| : V \to V^\dagger$ is an isomorphism, we conclude that the duality map

$$d_{IX} : \widetilde{H}_m(IX) \to \widetilde{H}^m(IX)$$

is an isomorphism.

Proposition 11.11. *The intersection form*

$$\beta : \widetilde{H}_m(IX) \times \widetilde{H}_m(IX) \to \mathbb{Q}$$

given by $\beta(v, w) = d_{IX}(v)(w)$ is symmetric. In fact it is given in terms of the basis (11.10) by the matrix

$$\begin{pmatrix} 0 & 0 & I \\ 0 & S & 0 \\ I & 0 & 0 \end{pmatrix},$$

where I is the $l \times l$-identity matrix and S is a symmetric $r \times r$-matrix, representing the classical intersection form on $\operatorname{im} j_*$ whose signature is the Novikov signature $\sigma(M, \partial M)$.

Proof. On V, we have

$$\begin{aligned} d_{IX}(v_i)(v_j) &= \zeta^* d_M(\overline{e}_i)(v_j) = \zeta^* d_M(\overline{e}_i)(\eta_* \overline{e}_j) \\ &= d_M(\overline{e}_i)(j_* \overline{e}_j) = d_M(\overline{e}_i)(e_j) = d_M(\overline{e}_j)(e_i) \\ &= d_M(\overline{e}_j)(j_* \overline{e}_i) = \zeta^* d_M(\overline{e}_j)(\eta_* \overline{e}_i) \\ &= \zeta^* d_M(\overline{e}_j)(v_i) = d_{IX}(v_j)(v_i). \end{aligned}$$

These are the symmetric entries of S. Between V and L we find

$$d_{IX}(v_i)(u_j) = \zeta^* d_M(\overline{e}_i)(u_j) = d_M(\overline{e}_i)(\zeta_* u_j) = 0,$$

as $u_j \in L = \ker \zeta_*$. This agrees with

$$d_{IX}(u_j)(v_i) = \overline{w}^j(v_i) = 0,$$

by definition of the dual basis. The intersection pairing between V and L' is trivial as well:

$$d_{IX}(v_i)(\overline{w}_j) = \zeta^* d_M(\overline{e}_i)(\overline{w}_j) = d_M(\overline{e}_j)(\zeta_* \overline{w}_j) = d_M(\overline{e}_i)(w_j) = 0,$$

since $d_M(\overline{e}_i) \subset d_M(Z)$. This agrees with

$$d_{IX}(\overline{w}_j)(v_i) = u^j(v_i) = 0,$$

again by definition of the dual basis. On L,

$$d_{IX}(u_i)(u_j) = \overline{w}^i(u_j) = 0$$

and on L',

$$d_{IX}(\overline{w}_i)(\overline{w}_j) = u^i(\overline{w}_j) = 0.$$

Finally, the intersection pairing between L and L' is given by

$$d_{IX}(u_i)(\overline{w}_j) = \overline{w}^i(\overline{w}_j) = \delta_{ij} = u^j(u_i) = d_{IX}(\overline{w}_j)(u_i).$$

□

Theorem 11.3 follows readily from this proposition because

$$\sigma(IX) = \sigma(S) + \sigma \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \sigma(S) = \sigma(M, \partial M).$$

It remains to prove that both

(11.12)
$$\begin{tikzcd} \widetilde{H}_m(IX) \arrow[r, "\zeta_*"] \arrow[d, "d_{IX}"'] & H_m(M, \partial M) \arrow[d, "d'_M"] \\ \widetilde{H}^m(IX) \arrow[r, "\eta^*"'] & H^m(M) \end{tikzcd}$$

and

(11.13)
$$\begin{tikzcd} \widetilde{H}_m(Q_{\geq k} E) \arrow[r, "\nu_*"] & \widetilde{H}_m(IX) \arrow[d, "d_{IX}"] \\ H^{m-1}(\mathrm{ft}_{<k} E) \arrow[u, "D_{k,k}"] \arrow[r, "\delta^*"] & \widetilde{H}^m(IX) \end{tikzcd}$$

commute. We begin with diagram (11.12) and check the commutativity on basis elements.

1. We verify that $\eta^* d_{IX}(u_j) = d'_M \zeta_*(u_j)$ for all j. By exactness, $\zeta_* \eta_* i_* = j_* i_* = 0$ and hence
$$d'_M \zeta_*(u_j) = d'_M \zeta_* \eta_* i_*(\overline{u}_j) = 0.$$

So it remains to show that $\eta^* d_{IX}(u_j) = 0$. We break this into three steps according to the decomposition (11.11). Evaluating on elements of the form $i_*\overline{u}_i$ yields

$$\eta^* d_{IX}(u_j)(i_*\overline{u}_i) = (\eta^*\overline{w}^j)(i_*\overline{u}_i) = \overline{w}^j(\eta_* i_*\overline{u}_i) = \overline{w}^j(u_i) = 0.$$

If a is any element in $\ker \eta_*$, then

$$(\eta^*\overline{w}^j)(a) = \overline{w}^j(\eta_* a) = 0.$$

Before evaluating on elements $\overline{e}_i$, we observe that since $\eta_*\overline{e}_i \in W$ (by (11.7)) and

$$\zeta_*(\eta_*\overline{e}_i) = j_*\overline{e}_i \in \operatorname{im} j_*,$$

we have

$$\eta_*\overline{e}_i \in W \cap \zeta_*^{-1}(\operatorname{im} j_*) = V.$$

It follows that

$$(\eta^*\overline{w}^j)(\overline{e}_i) = \overline{w}^j(\eta_*\overline{e}_i) = 0.$$

Thus $\eta^* d_{IX}(u_j) = 0$ as claimed.

2. On basis elements v_j, the commutativity is demonstrated by the calculation

$$\begin{aligned} \eta^* d_{IX}(v_j) &= \eta^*\zeta^* d_M(\overline{e}_j) = j^* d_M(\overline{e}_j) = d'_M j_*(\overline{e}_j) \\ &= d'_M \zeta_* \eta_*(\overline{e}_j) = d'_M \zeta_*(v_j). \end{aligned}$$

3. We prove that $\eta^* d_{IX}(\overline{w}_j) = d'_M\zeta_*(\overline{w}_j)$ for all j. Again it is necessary to break this into three steps according to the decomposition (11.11). Evaluating on elements of the form $i_*\overline{u}_i$ yields

$$\eta^* d_{IX}(\overline{w}_j)(i_*\overline{u}_i) = \eta^*(u^j)(i_*\overline{u}_i) = u^j(\eta_* i_*\overline{u}_i) = u^j(u_i) = \delta_{ij}$$

and

$$d'_M\zeta_*(\overline{w}_j)(i_*\overline{u}_i) = d'_M(w_j)(i_*\overline{u}_i) = d_M(i_*\overline{u}_i)(w_j) = w^i(w_j) = \delta_{ij}.$$

If a is any element in $\ker \eta_*$, then

$$\eta^*(u^j)(a) = u^j(\eta_* a) = 0 = d_M(a)(w_j) = d'_M(w_j)(a),$$

using (11.5) and $d_M(a) \in d_M(Z)$. Finally, on elements $\overline{e}_i$ we find

$$\eta^*(u^j)(\overline{e}_i) = u^j(\eta_*\overline{e}_i) = u^j(v_i) = 0 = d_M(\overline{e}_i)(w_j) = d'_M(w_j)(\overline{e}_i),$$

using (11.5) and $d_M(\overline{e}_i) \in d_M(Z)$. The commutativity of (11.12) is now established.

If $a \in H_m(M)$ and $b \in \widetilde{H}_m(IX)$ are any elements, then using (11.12),

$$\begin{aligned} \zeta^* d_M(a)(b) &= d_M(a)(\zeta_* b) = d'_M(\zeta_* b)(a) = (\eta^* d_{IX} b)(a) \\ &= d_{IX}(b)(\eta_* a) = d_{IX}(\eta_* a)(b), \end{aligned}$$

where the last equation uses the symmetry of d_{IX}, Proposition 11.11. Hence the diagram

(11.14)
$$\begin{tikzcd} H_m(M) \arrow[r, "\eta_*"] \arrow[d, "d_M"'] & \widetilde{H}_m(IX) \arrow[d, "d_{IX}"] \\ H^m(M, \partial M) \arrow[r, "\zeta^*"] & \widetilde{H}^m(IX) \end{tikzcd}$$

commutes as well. The cohomology braid of the triple

$$\begin{tikzcd} \mathrm{ft}_{<k} E \arrow[r, "F_{<k}"] \arrow[dr, "\tau"'] & \partial M \arrow[d, "i"] \\ & M \end{tikzcd}$$

contains the commutative square

$$\begin{tikzcd} H^{m-1}(\partial M) \arrow[r, "\delta^*"] \arrow[d, "F^*_{<k}"'] & H^m(M, \partial M) \arrow[d, "\zeta^*"] \\ H^{m-1}(\mathrm{ft}_{<k} E) \arrow[r, "\delta^*"] & \widetilde{H}^m(IX). \end{tikzcd} \tag{11.15}$$

We are now in a position to prove the commutativity of (11.13).

Let $a \in H^{m-1}(\mathrm{ft}_{<k} E)$ be any element. We must show that $d_{IX}\nu_* D_{k,k}(a) = \delta^*(a)$. As $F^*_{<k} : H^{m-1}(\partial M) \to H^{m-1}(\mathrm{ft}_{<k} E)$ is surjective (Lemma 6.6), there exists an $\overline{a} \in H^{m-1}(\partial M)$ with $a = F^*_{<k}(\overline{a})$. By Propositions 6.9, 6.10, $D_{k,k}$ is the unique isomorphism such that

$$\begin{tikzcd} H^{m-1}(\partial M) \arrow[r, "F^*_{<k}"] \arrow[d, "D_{\partial M}"', "\cong"] & H^{m-1}(\mathrm{ft}_{<k} E) \arrow[d, "\cong"', "D_{k,k}"] \\ H_m(\partial M) \arrow[r, "C_{\geq k*}"] & \widetilde{H}_m(Q_{\geq k} E) \end{tikzcd}$$

commutes. Therefore,

$$D_{k,k}(a) = D_{k,k} F^*_{<k}(\overline{a}) = C_{\geq k*} D_{\partial M}(\overline{a}).$$

Then, by the lower middle square in Diagram (11.2),

$$\nu_* D_{k,k}(a) = \nu_* C_{\geq k*} D_{\partial M}(\overline{a}) = \eta_* i_* D_{\partial M}(\overline{a}).$$

Applying d_{IX} and using the commutative diagram (11.14), we arrive at

$$d_{IX}\nu_* D_{k,k}(a) = d_{IX}\eta_* i_* D_{\partial M}(\overline{a}) = \zeta^* d_M i_* D_{\partial M}(\overline{a}).$$

Now the commutative diagram

$$\begin{tikzcd} H_m(\partial M) \arrow[r, "i_*"] & H_m(M) \arrow[d, "d_M"] \\ H^{m-1}(\partial M) \arrow[u, "D_{\partial M}"] \arrow[r, "\delta^*"] & H^m(M, \partial M) \end{tikzcd}$$

shows that

$$d_{IX}\nu_* D_{k,k}(a) = \zeta^*\delta^*(\overline{a}),$$

which by Diagram (11.15) equals $\delta^* F^*_{<k}(\overline{a}) = \delta^*(a)$, as was to be shown.

12. Sphere Bundles, Symplectic Toric Manifolds

We discuss equivariant Moore approximations for linear sphere bundles and for symplectic toric manifolds.

Proposition 12.1. *Let $\xi = (E, \pi, B)$ be an oriented real n-plane vector bundle over a closed, oriented, connected, n-dimensional base manifold B. Let $S(\xi)$ be the associated sphere bundle and let $e_\xi \in H^n(B;\mathbb{Z})$ be the Euler class of ξ. Then $S(\xi)$ can be given a structure group which allows for a degree k equivariant Moore approximation, for some $0 < k < n$, if and only if $e_\xi = 0$.*

Proof. Assume that $S(\xi)$ can be given a structure group which allows for a degree k equivariant Moore approximation for some $0 < k < n$. If the fiber dimension n of the vector bundle is odd, then the Euler class has order two. Since $H^n(B;\mathbb{Z}) \cong \mathbb{Z}$ is torsion free, $e_\xi = 0$. Thus we may assume that $n = 2d$ is even. We form the double

$$X^{4d} = DE \cup_{SE} DE,$$

where DE is the total space of the disk bundle of ξ, and $SE = \partial DE$. Then X is a manifold, but we may view it as a 2-strata pseudomanifold (X, B) by taking $B \subset X$ to be the zero section in one of the two copies of DE in X. For this stratified space, $M = DE$, $\partial M = SE$, and $\widehat{M} = TE$, the Thom-space of ξ. Since the double of any manifold with boundary is nullbordant, the signature of X vanishes, $\sigma_{IH}(X) = \sigma(X) = 0$. Note that a degree k equivariant Moore approximation to S^{n-1}, some $0 < k < n$, is in particular an equivariant Moore approximation of degree $\lfloor \frac{1}{2}(\dim S^{n-1} + 1)\rfloor = \lfloor \frac{n}{2} \rfloor$. Thus by Corollary 10.2,

$$\sigma_{IH}(TE) = \sigma_{IH}(X) = 0.$$

The middle intersection homology of the Thom space of a vector bundle is given by

$$IH_n(TE) \cong \operatorname{im}(H_n(DE) \to H_n(DE, SE)),$$

[29, p. 77, Example 5.2.5.3]. By homotopy invariance $H_n(DE) \cong H_n(B) \cong \mathbb{Q}[B]$, and by the Thom isomorphism $H_n(DE, SE) \cong H_0(B) \cong \mathbb{Q}$. The intersection form on the, at most one-dimensional, image is determined by the self-intersection number $[B] \cdot [B]$ of the fundamental class of B, which is precisely the Euler number. Since $\sigma_{IH}(TE) = 0$, this self-intersection number, and thus e_ξ, must vanish. (Note that in this case, the map $H_n(DE) \to H_n(DE, SE)$ is the zero map and $IH_n(TE) = 0$, for $IH_n(TE) \cong \mathbb{Q}$ and $[B] \cdot [B] = 0$ would contradict the nondegeneracy of the intersection pairing.)

Conversely, if $e_\xi = 0$, then [24, Thm. 2.10, p. 137] asserts that ξ has a nowhere vanishing section. This section induces a splitting $\xi \cong \xi' \oplus \underline{\mathbb{R}}^1$, where ξ' is an $(n-1)$-plane bundle and $\underline{\mathbb{R}}^1$ denotes the trivial line bundle over B. This splitting reduces the structure group from $\mathrm{SO}(n)$ to $\mathrm{SO}(1) \times \mathrm{SO}(n-1) = \{1\} \times \mathrm{SO}(n-1)$. The action of this reduced structure group on S^{n-1} has two fixed points; let $p \in S^{n-1}$ be one of them. Then $\{p\} \hookrightarrow S^{n-1}$ is an $\{1\} \times \mathrm{SO}(n-1)$-equivariant Moore approximation for every degree $0 < k < n$. ☐

Example 12.2. A symplectic toric manifold is a quadruple (M, ω, T^n, μ), where M is a $2n$-dimensional, compact, symplectic manifold with non-degenerate closed 2-form ω, there is an effective Hamiltonian action of the n-torus T^n on M, and $\mu : M \to \mathbb{R}^n$ is a choice of moment map for this action. There is a one-to-one correspondence between such $2n$-dimensional symplectic toric manifolds and so-called Delzant polytopes in $\mathbb{R}^n$, [19], given by the assignment

$$(M, \omega, T^n, \mu) \mapsto \Delta_M := \mu(M).$$

Recall that a polytope in $\mathbb{R}^n$ is the convex hull of a finite number of points in $\mathbb{R}^n$. Delzant polytopes in $\mathbb{R}^n$ have the property that each vertex has exactly n edges adjacent to it and for each vertex p, every edge adjacent to p has the form $\{p + tu_i \mid T_i \geq t \geq 0\}$ with $u_i \in \mathbb{Z}^n$, and $u_1, \ldots, u_n$ constitute a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

Section 3.3 of [18] uses the Delzant polytope Δ_M to construct Morse functions on M as follows: Let $X \in \mathbb{R}^n$ be a vector whose components are independent over $\mathbb{Q}$. Then X is not parallel to any facet of Δ_M and the orthogonal projection $\pi_X : \mathbb{R}^n \to \mathbb{R}$ onto the line spanned by X, $\pi_X(Y) = \langle Y, X\rangle$, is injective on the vertices of Δ_M. By composing the moment map μ with the projection π_X, one obtains a Morse function $f_X = \pi_X \circ \mu : M \to \mathbb{R}$, $f_X(q) = \langle \mu(q), X\rangle$, whose critical points are precisely the fixed points of the T^n action. The images of the fixed points under the moment map are the vertices of Δ_M. Since the coadjoint action is trivial on a torus, T^n acts trivially on $\mathbb{R}^n$, and as μ is equivariant, it is thus constant on orbits. Hence the level sets of $\pi_X \circ \mu$ are T^n-invariant. The index of a critical point p is twice the number of edge vectors u_i of Δ_M at $\mu(p)$ whose inner product with X is negative, $\langle u_i, X\rangle < 0$. In particular, the index is always even. For $a \in \mathbb{R}$, we set $M_a = f_X^{-1}(-\infty, a] \subset M$.

Suppose that one can choose X in such a way that the critical points satisfy:

(C) For any two critical points p, q of f_X, if the index of p is larger than the index of q, then $f_X(p) > f_X(q)$.

Then, since f_X is Morse, for each critical value a of f_X the set $M_{a+\epsilon}$ is homotopy equivalent to a CW-complex with one cell attached for each critical point p with $f_X(p) < a+\epsilon$. (Here $\epsilon > 0$ has been chosen so small that there are no critical values of f_X in $(a, a+\epsilon]$.) The dimension of the cell associated to p is the index of f_X at p. Let $2i$ be the index of any critical point $p \in M_{a+\epsilon}$ with $f_X(p) = a$. If $q \in M_{a+\epsilon}$ is an arbitrary critical point of f_X, then $f_X(q) \leq f_X(p) = a$ and thus the index of q is at most $2i$ by condition (C). Thus $M_{a+\epsilon}$ contains all cells of M that have dimension at most $2i$ and no other cells. Since M has only cells in even dimensions, the cellular chain complex of M has zero differentials in all degrees. Thus, since f_X is equivariant, $M_{a+\epsilon} \hookrightarrow M$ is a T^n-equivariant Moore approximation of degree $2i+1$ (and of degree $2i+2$), and is a smooth manifold with boundary.

A particular case of this is the complex projective space $(\mathbb{CP}^n, \omega_{\mathrm{FS}}, T^n, \mu)$, where ω_{FS} is the Fubini-Study symplectic form and T^n acts on $\mathbb{CP}^n$ by

$$(e^{it_1}, \dots, e^{it_n}) \cdot (z_0 : z_1 : \cdots : z_n) = (z_0 : e^{it_1} z_1 : \cdots : e^{it_n} z_n).$$

On page 26 of [31], an equivariant Morse function with $n+1$ critical points is constructed, the i-th one having index $2i$ and critical value i. Using this we obtain equivariant Moore approximations to $\mathbb{CP}^n$ of every degree with respect to the torus action.

In the case that M is 4-dimensional, condition (C) is satisfied. The Delzant polytope $\mu(M)$ associated to a 4-dimensional symplectic toric manifold (M, ω, T^2, μ) is a 2-dimensional polytope in $\mathbb{R}^2$. As M is compact, f_X attains its minimum m and its maximum m' on M. Let $p_{\min} \in M$ be a critical point with $f_X(p_{\min}) = m$ and let $p_{\max} \in M$ be a critical point with $f_X(p_{\max}) = m'$. Suppose that $p \in M$ is any critical point such that $f_X(p) = m$. Then $\pi_X\mu(p) = m = \pi_X\mu(p_{\min})$. The moment images $v = \mu(p)$ and $v_{\min} = \mu(p_{\min})$ are vertices of Δ_M. Since the projection π_X is injective on vertices, we have $v = v_{\min}$. Now as μ maps the fixed points (which are precisely the critical points) bijectively onto the vertices, it follows that $p = p_{\min}$. This shows that $p_{\min}$ is unique and similarly $p_{\max}$ is unique. The index of $p_{\min}$ is 0, while the index of $p_{\max}$ is 4. Thus $\langle u_1, X\rangle \geq 0$ and $\langle u_2, X\rangle \geq 0$ at $v_{\min}$ and $\langle u_1, X\rangle < 0$ and $\langle u_2, X\rangle < 0$ at $v_{\max}$.

Geometrically, this means that the two edges that go out from $v_{\min}$ point in the same half-plane as X, while the outgoing edges at $v_{\max}$ point in the half-plane complementary to the one of X. If v is any vertex of the moment polytope different from $v_{\min}, v_{\max}$, then by the convexity of Δ_M, one of the two outgoing edges must point in X's half-plane, while the other outgoing edge points into the complementary half-plane, yielding an index of 2. If $p \in M$ is a critical point different from $p_{\min}, p_{\max}$, then $\mu(p)$ is a vertex different from $v_{\min}, v_{\max}$ and thus must have index 2. From this, it follows that condition (C) is indeed satisfied: If p, q are critical points such that p has larger index than q, then there are two cases: p has index 4 and q has index in $\{0, 2\}$, or p has index 2 and q has index 0. In the first case, $p = p_{\max}$ and in the second case

$q = p_{\min}$. In both cases it is then clear, using the uniqueness of $p_{\min}, p_{\max}$, that $f_X(p) > f_X(q)$. We have thus shown:

Proposition 12.3. *Every 4-dimensional symplectic toric manifold (M, ω, T^n, μ) has an equivariant Moore approximation $M_{<k}$ of degree k for every $k \in \mathbb{Z}$. Furthermore, the space $M_{<k}$ can be chosen to be a smooth compact codimension 0 submanifold-with-boundary of M.*

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Mathematisches Institut, Ruprecht-Karls-Universität Heidelberg, Im Neuenheimer Feld 205, 69120 Heidelberg, Germany
E-mail address: `banagl@mathi.uni-heidelberg.de`

Department of Mathematics, Western Oregon University, Monmouth OR 97361, USA
E-mail address: `chriestensonb@mail.wou.edu`

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HOROSPHERICAL AND HYPERBOLIC DUAL SURFACES OF SPACELIKE CURVES IN DE SITTER SPACE

SHYUICHI IZUMIYA, ANA CLAUDIA NABARRO, AND ANDREA DE JESUS SACRAMENTO

Abstract. We define two surfaces, the horospherical surface and the hyperbolic dual surface of a spacelike curve in the de Sitter 3-space, in the Lorentzian-Minkowski 4-space. These surfaces are, respectively, in the lightcone 3-space and in the hyperbolic 3-space (other pseudo-spheres). We use techniques from singularity theory to obtain the generic shape of these surfaces and of their singular point sets. Furthermore, we give a relation between these surfaces from the viewpoint of the theory of Legendrian dualities between pseudo-spheres.

1. Introduction

Submanifolds in Lorentz-Minkowski space are investigated from various mathematical viewpoints and are of interest also in relativity theory. In recent years, using singularity theory, very important progress has been made and many investigations have been conducted to classify and characterize the singularities of submanifolds in Euclidean spaces or in semi-Euclidean spaces (see, for example, [1]-[9] and [11]). The first author introduced Legendrian dualities between three kinds of pseudo-spheres in Lorentz-Minkowski space [5, 6]. Curves in the pseudo-spheres and duality relations between the curves and some surfaces in pseudo-spheres are studied. For example, in [3, 4, 8], curves in the hyperbolic space $H^3(-1)$ in $\mathbb{R}^4_1$, in the de Sitter dual surface in S^3_1, and in the horospherical surface in the lightcone LC^*, are investigated. The results in this paper contribute to the study of the extrinsic geometry of curves in the above different ambient spaces.

We use Legendrian duality to investigate spacelike curves in the de Sitter space $S^3_1 \subset \mathbb{R}^4_1$ and two special surfaces related by duality. For a curve $\gamma : I \to S^3_1$ with nowhere vanishing curvature, we define its associated horospherical surface in the lightcone LC^* and its hyperbolic dual surface in the hyperbolic space $H^3(-1)$. For the study of the generic differential geometry of these surfaces and of their singular sets, we use singularity theory techniques, and in particular, classical deformation theory.

Our paper is organized as follows: Section 2 reviews basic definitions for the Minkowski 4-space and introduces a moving frame along γ together with Frenet-Serret type formulae. We also review the definition of the A_k-singularities and their discriminant sets. We define the hyperbolic focal surface and the horospherical surface of γ. In Sections 3 and 5, we define two families of height functions on γ, horospherical height functions and hyperbolic height functions. These functions measure the contact of γ with special hyperplanes. Differentiating these functions yields invariants related to each surface. We show that the horospherical surface of γ is the discriminant set of the family of horospherical height functions (Corollary 3.2) and that its hyperbolic dual surface is the discriminant set of the family of hyperbolic height functions (Corollary 5.3).

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Furthermore, using the theory of deformations, we give a classification and a characterization of the diffeomorphism-type of these surfaces (Theorems 3.4 and 5.5). It is easy to show that the discriminant sets of these families on timelike curves in S^3_1 are empty. For this reason, we consider only spacelike curves in S^3_1.

In Section 4, we investigate the geometric meaning of the invariants discussed in the previous sections. We prove results that give conditions (related to these invariants) for the curve γ to be on a parabolic de Sitter quadric and we give also conditions for γ to be part of a T-horoparabola or an S-horoparabola (Propositions 4.1 and 4.2). In Section 5, we give information about the geometry of the hyperbolic dual surface and of its singular set. We separate the cases where γ has spacelike normal vectors from those where γ has timelike normal vectors. We prove that, if the normal vector is timelike, then the hyperbolic dual surface of γ has no singular points. For this reason, in Section 5, we consider only the case when γ has spacelike normal vectors.

In Section 6, we show that γ can be part of an elliptic de Sitter quadric (Proposition 6.1) by using an invariant of the curve. When γ is not part of an elliptic de Sitter quadric, we characterize the contact of γ with an elliptic de Sitter quadric using the singularity types of the hyperbolic dual surface of γ (Proposition 6.2).

Finally, in Section 7, we recall the concepts of Legendrian dualities between pseudo-spheres in Lorentz-Minkowski space, introduced in [6]. Several duality relationships are presented in Theorem 7.1. These give a dual relation between the horospherical surface and the hyperbolic dual surface of γ.

2. Preliminaries

The *Minkowski space* $\mathbb{R}^4_1$ is the vector space $\mathbb{R}^4$ endowed with the pseudo-scalar product $\langle x, y\rangle = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3$, for any $x = (x_0, x_1, x_2, x_3)$ and $y = (y_0, y_1, y_2, y_3)$ in $\mathbb{R}^4_1$ (see, e.g., [10]). We say that a non-zero vector $x \in \mathbb{R}^4_1$ is *spacelike* if $\langle x, x\rangle > 0$, *lightlike* if $\langle x, x\rangle = 0$ and *timelike* if $\langle x, x\rangle < 0$. We call $\gamma : I \to \mathbb{R}^4_1$, with $I \subset \mathbb{R}$ an open interval, a *spacelike* (resp. *timelike*) *curve* if $\gamma'(t)$ is a spacelike (resp. timelike) vector for any $t \in I$. We define, for $x \in \mathbb{R}^4_1$,

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x \text{ is spacelike,} \\ 0 & \text{if } x \text{ is lightlike,} \\ -1 & \text{if } x \text{ is timelike.} \end{cases}$$

We call sign (x) the *signature* of x. The norm of a vector $x \in \mathbb{R}^4_1$ is defined by $\| x \| = \sqrt{|\langle x, x\rangle|}$. We now consider the pseudo-spheres in $\mathbb{R}^4_1$. The *hyperbolic 3-space* is defined by

$$H^3(-1) = \{x \in \mathbb{R}^4_1 \mid \langle x, x\rangle = -1\},$$

the *de Sitter 3-space* by

$$S^3_1 = \{x \in \mathbb{R}^4_1 \mid \langle x, x\rangle = 1\},$$

and the *lightcone* by

$$LC^* = \{x \in \mathbb{R}^4_1 \setminus \{0\} \mid \langle x, x\rangle = 0\}.$$

For any $x = (x_0, x_1, x_2, x_3)$, $y = (y_0, y_1, y_2, y_3)$, $z = (z_0, z_1, z_2, z_3) \in \mathbb{R}^4_1$, the pseudo-product of x, y and z is defined by:

$$x \wedge y \wedge z = \begin{vmatrix} -e_0 & e_1 & e_2 & e_3 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ z_0 & z_1 & z_2 & z_3 \end{vmatrix},$$

where $\{e_0, e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^4$.

For a non-zero vector $v \in \mathbb{R}^4_1$ and a real number c, a *hyperplane* with *pseudo-normal vector* v is defined by

$$HP(v,c) = \{x \in \mathbb{R}^4_1 \mid \langle x, v\rangle = c\}.$$

We call $HP(v,c)$ a *spacelike*, a *timelike*, or a *lightlike* hyperplane if v is spacelike, timelike, or lightlike, respectively.

We have three types of models of quadric surfaces in S^3_1, which are given by intersections of S^3_1 with hyperplanes in $\mathbb{R}^4_1$, determined by the type of the hyperplane. A surface $S^3_1 \cap HP(v,c)$ is called an *elliptic de Sitter quadric*, a *hyperbolic de Sitter quadric* or a *parabolic de Sitter quadric* if $HP(v,c)$ is spacelike, timelike, or lightlike, respectively. We denote the parabolic de Sitter quadric by $QDP(v,c)$ and the elliptic de Sitter quadric by $QDE(v,c)$.

Let $\gamma : I \to S^3_1$ be a smooth and regular spacelike curve in S^3_1. We can parametrise it by arc length s, and write $t(s) = \gamma'(s)$ for the unit tangent vector. In this case, we call γ a *unit speed spacelike curve*. If $\langle t'(s), t'(s)\rangle \neq 1$, then $\| t'(s) + \gamma(s) \| \neq 0$, and we define the unit vector $n(s) = \dfrac{t'(s) + \gamma(s)}{\| t'(s) + \gamma(s) \|}$. We also define another unit vector by $e(s) = \gamma(s) \wedge t(s) \wedge n(s)$. Then we obtain a pseudo-orthonormal frame $\{\gamma(s), t(s), n(s), e(s)\}$ of $\mathbb{R}^4_1$ along γ. The Frenet-Serret type formulae for that frame are given by

$$\begin{cases} \gamma'(s) = t(s), \\ t'(s) = -\gamma(s) + k_g(s)\, n(s), \\ n'(s) = -\delta(\gamma(s))\, k_g(s)\, t(s) + \tau_g(s)\, e(s), \\ e'(s) = \tau_g(s) n(s), \end{cases}$$

where $\delta(\gamma(s)) = \mathrm{sign}\,(n(s))$ (which we shall write as simply δ), $k_g(s) = \| t'(s) + \gamma(s) \|$ and

$$\tau_g(s) = \frac{\delta(\gamma(s))}{k_g^2(s)} \det(\gamma(s), \gamma'(s), \gamma''(s), \gamma'''(s)).$$

The invariant k_g is called the *geodesic curvature* and τ_g the *geodesic torsion* of γ (see [7]).

Since $\langle t'(s)+\gamma(s), t'(s)+\gamma(s)\rangle = \langle t'(s), t'(s)\rangle - 1$, it follows that $\langle t'(s), t'(s)\rangle \neq 1$ is equivalent to $k_g(s) \neq 0$.

We define the following maps

$$HS^{\pm}_{\gamma} : I \times J \to LC^* \quad \text{and} \quad HD^{\pm}_{\gamma} : I \times J \to H^3(-1)$$

by

$$HS^{\pm}_{\gamma}(s,\mu) = \gamma(s) + \mu n(s) + \lambda e(s) \quad \text{and} \quad HD^{\pm}_{\gamma}(s,\mu) = \mu n(s) + \lambda e(s),$$

respectively, where $\lambda^2 - \mu^2 = \delta(\gamma(s))$.

In other words,

$$HS^{\pm}_{\gamma}(s,\mu) = \gamma(s) + \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))} e(s) \quad \text{and} \quad HD^{\pm}_{\gamma}(s,\mu) = \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))} e(s),$$

with $\mu^2 + \delta(\gamma(s)) \geq 0$, i.e., $\mu \in J = \mathbb{R}$ for $n(s)$ spacelike and $\mu \in J = (-\infty, -1] \cup [1, \infty)$ for $n(s)$ timelike. We call $HS^{\pm}_{\gamma}$ the *horospherical surface of* γ and $HD^{\pm}_{\gamma}$ the *hyperbolic dual surface of* γ. We can suppose that λ and μ are one of cosh and sinh, depending on $\delta(\gamma(s))$.

Definition 2.1. *Let $F : \mathbb{R}^4_1 \to \mathbb{R}$ be a submersion and $\gamma : I \to S^3_1$ be a regular curve. We say that γ and $F^{-1}(0)$ (respectively $F^{-1}(0) \cap S^3_1$) have contact of order k at s_0, if the function $g(s) = F \circ \gamma(s)$ satisfies $g(s_0) = g'(s_0) = \cdots = g^{(k)}(s_0) = 0$ and $g^{(k+1)}(s_0) \neq 0$, i.e., g has an A_k-singularity at s_0.*

Let $G : \mathbb{R} \times \mathbb{R}^r, (s_0, \bar{x}) \to \mathbb{R}$ be a family of germs of functions. We call G an r-parameter deformation of f if $f(s) = G_{\bar{x}}(s)$. Suppose that f has an A_k-singularity ($k \geq 1$) at s_0. If we write

$$j^{(k-1)}\left(\frac{\partial G}{\partial x_i}(s, \bar{x})\right)(s_0) = \sum_{j=0}^{k-1} \alpha_{ji}(s - s_0)^j,$$

for $i = 1, \dots, r$, then G is a *versal deformation* if the $k \times r$ matrix of coefficients (α_{ji}) has rank k ($k \leq r$) (see [2]).

The *discriminant set* of G is the set

$$\mathcal{D}_G = \left\{ x \in (\mathbb{R}^r, \bar{x}) \;\middle|\; G = \frac{\partial G}{\partial s} = 0 \text{ at } (s, x) \text{ for some } s \in (\mathbb{R}, s_0) \right\}.$$

Theorem 2.2. [2] *Let $G : \mathbb{R} \times \mathbb{R}^r, (s_0, \bar{x}) \to \mathbb{R}$ be an r-parameter deformation of f, with f having an A_k-singularity at s_0. Suppose that G is a versal deformation. Then $\mathcal{D}_G$ is locally diffeomorphic to*

(1) *$C \times \mathbb{R}^{r-2}$, if $k = 2$, and*
(2) *$SW \times \mathbb{R}^{r-3}$, if $k = 3$,*

where $C = \{(x_1, x_2) \mid x_1^2 = x_2^3\}$ is the ordinary cusp and

$$SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\}$$

is the swallowtail surface.

We use families of height functions on curves in S^3_1 to study the horospherical surface and the hyperbolic dual surface. In fact, these surfaces are the discriminant sets of these families.

It is easy to show that the discriminant sets of the family of horospherical height functions and family of hyperbolic height functions on timelike curves in S^3_1 are empty. For this reason, we only consider spacelike curves in S^3_1.

3. Horospherical height functions

In this section, we introduce a family of height functions on a curve that is useful for the study of the horospherical surface. We prove that the horospherical surface is the discriminant set of this family.

For a spacelike curve $\gamma : I \to S^3_1$, we define a function $H : I \times LC^* \to \mathbb{R}$ by

$$H(s, v) = \langle \gamma(s), v \rangle - 1.$$

We call H a family of *horospherical height functions on* γ. We denote $h_v(s) = H(s, v)$ for any fixed $v \in LC^*$. The family of horospherical height functions measures the contact of γ with lightlike hyperplanes in $\mathbb{R}^4_1$. Generically, this contact can be of order k, where $k = 1, 2, 3$.

We obtain equivalent conditions for each A_k-singularity, $k = 1, 2, 3$ of h_v by the following result. For example, h_v has an A_2-singularity at s if and only if

$$v = \gamma(s) + \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}e(s), \quad \mu = \frac{1}{k_g(s)\delta(\gamma(s))}, \quad \text{and } \sigma(s) \neq 0.$$

Proposition 3.1. *Let $\gamma : I \to S^3_1$ be a unit speed spacelike curve such that $k_g(s) \neq 0$. Then*

(1) *$h_v(s) = 0$ if and only if there exist real numbers μ, λ, η with*

$$\eta^2 + \delta(\gamma(s))\mu^2 - \delta(\gamma(s))\lambda^2 = -1$$

such that $v = \gamma(s) + \eta t(s) + \mu n(s) + \lambda e(s)$.

(2) $h_v(s) = h'_v(s) = 0$ *if and only if there exist real numbers* μ, λ *such that*

$$v = \gamma(s) + \mu n(s) + \lambda e(s)$$

with $\lambda^2 - \mu^2 = \delta(\gamma(s))$.

(3) $h_v(s) = h'_v(s) = h''_v(s) = 0$ *if and only if* $v = \gamma(s) + \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}e(s)$ *with* $\mu = \dfrac{1}{k_g(s)\delta(\gamma(s))}$.

(4) $h_v(s) = h'_v(s) = h''_v(s) = h_v^{(3)}(s) = 0$ *if and only if* $v = \gamma(s) + \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}e(s)$, $\mu = \dfrac{1}{k_g(s)\delta(\gamma(s))}$ *and* $\sigma(s) = 0$, *where*

$$\sigma(s) = (k'_g \pm k_g\tau_g(-\delta)\sqrt{1 + k_g^2\delta})(s).$$

(5)

(i) If $n(s)$ *is timelike with* $k_g(s) = 1$ *then* $h_v(s) = h'_v(s) = \cdots = h_v^{(4)}(s) = 0$ *if and only if* $v = \gamma(s) + \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}e(s)$, $\mu = \dfrac{1}{k_g(s)\delta(\gamma(s))}$, $\sigma(s) = 0$ *and* $k''_g(s) + \tau_g^2(s) = 0$.

(ii) If $n(s)$ *is timelike with* $k_g(s) \neq 1$ *or if* $n(s)$ *is spacelike, then*

$$h_v(s) = h'_v(s) = \cdots = h_v^{(4)}(s) = 0$$

if and only if

$$v = \gamma(s) + \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}e(s), \quad mu = \frac{1}{k_g(s)\delta(\gamma(s))}, \quad \text{and} \quad \sigma(s) = \sigma'(s) = 0.$$

Proof. Since $h_v(s) = \langle \gamma(s), v\rangle - 1$, by using the Frenet-Serret type formulae, we have

(a) $h'_v(s) = \langle t(s), v\rangle$,
(b) $h''_v(s) = \langle -\gamma(s) + k_g(s)n(s), v\rangle$,
(c) $h_v^{(3)}(s) = \langle (-1 - k_g^2(s)\delta(\gamma(s)))t(s) + k'_g(s)n(s) + k_g(s)\tau_g(s)e(s), v\rangle$, and
(d) $h^{(4)}(s) = \langle (1 + k_g^2(s)\delta(\gamma(s)))\gamma(s) - 3\delta(\gamma(s))k'_g(s)k_g(s)t(s) + (-k_g(s) + k''_g(s) + k_g(s)\tau_g^2(s) - k_g^3(s)\delta(\gamma(s)))n(s) + (2k'_g(s)\tau_g(s) + k_g(s)\tau'_g(s))e(s), v\rangle$.

The proof follows by simple calculations using (a)-(d). □

Corollary 3.2. *The horospherical surface of* γ *is the discriminant set* $\mathcal{D}_H$ *of the family of horospherical height functions* H.

Proof. The proof follows from the definition of the discriminant set given in Section 2 and by Proposition 3.1 (2). □

Following Proposition 3.1, we define the invariant

$$\sigma(s) = \left(k'_g \pm k_g\tau_g(-\delta)\sqrt{1 + k_g^2\delta}\right)(s)$$

of the curve γ. We will study the geometric meaning of this invariant in Section 4.

In the next result, we show that the family of horospherical height functions on a curve in S_1^3 is a versal deformation of an A_k-singularity, $k = 2, 3$, of its members.

Proposition 3.3. *With the same assumptions as in Proposition 3.1, let* $H : I \times LC^* \to \mathbb{R}$ *be the family of horospherical height functions on* γ. *If* h_v *has an* A_2*-singularity at* s_0, *then* H *is a versal deformation of* h_v. *If* h_v *has an* A_3*-singularity at* s_0 *and* $n(s_0)$ *is timelike with* $k_g(s_0) \neq 1$ *(which is a generic condition) or if* $n(s_0)$ *is spacelike, then* H *is a versal deformation of* h_v.

Proof. The family of horospherical height functions is given by

$$H(s,v) = -v_0x_0(s) + v_1x_1(s) + v_2x_2(s) + v_3x_3(s) - 1,$$

where $v = (v_0, v_1, v_2, v_3)$, $\gamma(s) = (x_0(s), x_1(s), x_2(s), x_3(s))$ is the curve parametrised by arc length, $v_0 = \sqrt{v_1^2 + v_2^2 + v_3^2}$ and $x_0(s) = \sqrt{x_1^2(s) + x_2^2(s) + x_3^2(s) - 1}$.

Writing $H(s,v) = H(s, v_1, v_2, v_3)$, we have

$$\frac{\partial H}{\partial v_i} = x_i(s) - \frac{v_i}{v_0}x_0(s),$$

for $i = 1, 2, 3$. Therefore, the 2-jet of $\dfrac{\partial H}{\partial v_i}$ at s_0, is given by

$$x_i(s_0) - \frac{v_i}{v_0}x_0(s_0) + \left(x_i'(s_0) - \frac{v_i}{v_0}x_0'(s_0)\right)(s - s_0) + \frac{1}{2}\left(x_i''(s_0) - \frac{v_i}{v_0}x_0''(s_0)\right)(s - s_0)^2.$$

We assume first that h_v has an A_3-singularity at $s = s_0$, and we show that the determinant of the 3×3 matrix

$$A = \begin{pmatrix} x_1(s_0) - \frac{v_1}{v_0}x_0(s_0) & x_2(s_0) - \frac{v_2}{v_0}x_0(s_0) & x_3(s_0) - \frac{v_3}{v_0}x_0(s_0) \\ x_1'(s_0) - \frac{v_1}{v_0}x_0'(s_0) & x_2'(s_0) - \frac{v_2}{v_0}x_0'(s_0) & x_3'(s_0) - \frac{v_3}{v_0}x_0'(s_0) \\ x_1''(s_0) - \frac{v_1}{v_0}x_0''(s_0) & x_2''(s_0) - \frac{v_2}{v_0}x_0''(s_0) & x_3''(s_0) - \frac{v_3}{v_0}x_0''(s_0) \end{pmatrix}$$

is nonzero. Denote

$$a = \begin{pmatrix} x_0(s_0) \\ x_0'(s_0) \\ x_0''(s_0) \end{pmatrix}, b_i = \begin{pmatrix} x_i(s_0) \\ x_i'(s_0) \\ x_i''(s_0) \end{pmatrix},$$

for $i = 1, 2, 3$. Then

$$\det A = \frac{v_0}{v_0}\det(b_1\ b_2\ b_3) - \frac{v_1}{v_0}\det(a\ b_2\ b_3) - \frac{v_2}{v_0}\det(b_1\ a\ b_3) - \frac{v_3}{v_0}\det(b_1\ b_2\ a).$$

On the other hand,

$$(\gamma \wedge \gamma' \wedge \gamma'')(s_0) = (-\det(b_1\ b_2\ b_3), -\det(a\ b_2\ b_3), -\det(b_1\ a\ b_3), -\det(b_1\ b_2\ a)).$$

Therefore,

$$\begin{aligned} \det A &= \left\langle \left(\frac{v_0}{v_0}, \frac{v_1}{v_0}, \frac{v_2}{v_0}, \frac{v_3}{v_0}\right), (\gamma \wedge \gamma' \wedge \gamma'')(s_0) \right\rangle \\ &= \frac{1}{v_0}\langle \gamma(s_0) + \mu n(s_0) \pm \sqrt{\mu^2 + \delta}e(s_0), k_g(s_0)e(s_0)\rangle \\ &= \pm\frac{1}{v_0}(-\delta)\sqrt{k_g^2(s_0)\delta + 1}. \end{aligned}$$

In the case where $n(s_0)$ is a spacelike vector, we have $\det A = \mp\frac{1}{v_0}\sqrt{k_g^2(s_0) + 1} \neq 0$ and therefore H is a versal deformation of h_v at $s = s_0$. If $n(s_0)$ is a timelike vector, then we have

$$\det A = \pm\frac{1}{v_0}\sqrt{1 - k_g^2(s_0)}$$

and therefore $\det A \neq 0$ under the condition that $k_g(s_0) \neq 1$, so H is a versal deformation of h_v at $s = s_0$.

When $k = 2$, we require the rank of B to equal 2, where B is the matrix

$$B = \begin{pmatrix} x_1(s_0) - \dfrac{v_1}{v_0}x_0(s_0) & x_2(s_0) - \dfrac{v_2}{v_0}x_0(s_0) & x_3(s_0) - \dfrac{v_3}{v_0}x_0(s_0) \\ x_1'(s_0) - \dfrac{v_1}{v_0}x_0'(s_0) & x_2'(s_0) - \dfrac{v_2}{v_0}x_0'(s_0) & x_3'(s_0) - \dfrac{v_3}{v_0}x_0'(s_0) \end{pmatrix}.$$

Since B consists of the first and second lines of A, we have that if $n(s_0)$ is a spacelike vector, then rank of B is 2 because $\det A \neq 0$. If $n(s_0)$ is a timelike vector, the rank of B is 2 if $k_g(s_0) \neq 1$. For the case $k_g(s_0) = 1$, the rank of B is 2 if $\dfrac{2(x_0(s_0) - v_0)}{v_0} \neq 0$. Then it is enough to show that $x_0(s_0) \neq v_0$. As $k_g(s_0) = 1$, we have by Proposition 3.1 (2) that

$$v(s_0) = \gamma(s_0) - n(s_0).$$

Therefore $v_0 = x_0(s_0) - n_0(s_0)$, where $n(s_0) = (n_0(s_0), n_1(s_0), n_2(s_0), n_3(s_0))$. Without loss of generality, we can suppose $n_0(s_0) \neq 0$, so the rank of B is 2. □

Using Theorem 2.2 and Proposition 3.3, we can obtain the diffeomorphism type of the horospherical surface.

Theorem 3.4. *With the same assumptions as in Proposition 3.1, let $HS^{\pm}_{\gamma}$ be the horospherical surface of γ. Then we have the following:*

(1) *The singular values of $HS^{\pm}_{\gamma}$ are given by*

$$h^{\pm}_{\mu}S_{\gamma}(s) = \gamma(s) + \frac{1}{k_g(s)\delta(\gamma(s))}n(s) \pm \sqrt{\frac{1}{k_g^2(s)} + \delta(\gamma(s))}e(s).$$

(2) *$HS^{\pm}_{\gamma}$ is, at (s_0, μ_0), locally diffeomorphic to a cuspidal edge if and only if*

$$\mu_0 = \frac{1}{k_g(s_0)\delta(\gamma(s_0))} \quad \text{and} \quad \sigma(s_0) \neq 0.$$

(3) *$HS^{\pm}_{\gamma}$ is, at (s_0, μ_0), locally diffeomorphic to a swallowtail surface if and only if*

$$\mu_0 = \frac{1}{k_g(s_0)\delta(\gamma(s_0))}, \quad \sigma(s_0) = 0, \quad \text{and} \quad \sigma'(s_0) \neq 0,$$

for $n(s_0)$ timelike with $k_g(s_0) \neq 1$, or for $n(s_0)$ spacelike.

Proof. Consider the horospherical surface given by $HS^{\pm}_{\gamma}(s, \mu) = \gamma(s) + \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}e(s)$. Then

$$\frac{\partial HS^{\pm}_{\gamma}}{\partial s}(s, \mu) = (1 - \mu\delta(\gamma(s))k_g(s))t(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}\tau_g(s)n(s) + \mu\tau_g(s)e(s) \quad \text{and}$$
$$\frac{\partial HS^{\pm}_{\gamma}}{\partial \mu}(s, \mu) = n(s) \pm \frac{\mu}{\sqrt{\mu^2 + \delta(\gamma(s))}}e(s).$$

The vectors

$$\left\{\frac{\partial HS^{\pm}_{\gamma}}{\partial s}(s_0, \mu_0), \frac{\partial HS^{\pm}_{\gamma}}{\partial \mu}(s_0, \mu_0)\right\}$$

are linearly dependent if and only if

$$\mu_0 = \frac{1}{k_g(s_0)\delta(\gamma(s_0))}.$$

Then the singular values of $HS^{\pm}_{\gamma}$ are given by $h^{\pm}_{\mu_0}S_{\gamma}(s_0) = HS^{\pm}_{\gamma}(s_0, \mu_0)$ and assertion (1) follows. By Corollary 3.2, the discriminant set $\mathcal{D}_H$ of the family of horospherical height functions H of

γ is the horospherical surface of γ. It also follows from assertions (3) and (4) of Proposition 3.1 that h_v has an A_2-singularity (respectively, an A_3-singularity) at $s = s_0$ if and only if

$$\mu_0 = \frac{1}{k_g(s_0)\delta(\gamma(s_0))} \quad \text{and} \quad \sigma(s_0) \neq 0$$

(respectively,

$$\mu_0 = \frac{1}{k_g(s_0)\delta(\gamma(s_0))}, \quad \sigma(s_0) = 0, \quad \text{and} \quad \sigma'(s_0) \neq 0).$$

By Theorem 2.2 and Proposition 3.3, we have assertions (2) and (3). We observe that, in (3), if $n(s_0)$ is timelike, it is necessary to suppose that $k_g(s_0) \neq 1$ in order to obtain Proposition 3.3. □

4. Invariants and special geometry of the horosphe-rical surface

We study the geometric meaning of the invariant $\sigma(s)$ defined in the previous section. Let v be a lightlike vector, w be a spacelike vector, and z be a timelike vector. We call the de Sitter space curve, given by the intersections of the parabolic de Sitter quadric $QDP(v, 1)$ with $HP(w, 0)$ (resp. $HP(z, 0)$), T-horoparabolas (resp. S-horoparabolas).

Given a unit speed spacelike curve γ in S^3_1, the unit normal vector n can be a timelike vector or a spacelike vector. We prove the following results that give conditions depending on the invariants, for the curve γ to be in a parabolic de Sitter quadric. In addition, we also give conditions for γ to be part of a T-horoparabola or a S-horoparabola. These facts are related to the invariants $\sigma(s)$ and $\tau_g(s)$. It is convenient to divide the discussion into two cases: $n(s)$ is timelike (Proposition 4.1) and $n(s)$ is spacelike (Proposition 4.2).

We observe that for a curve in hyperbolic 3-space (see [8]), there is only one case because $n(s)$ is always spacelike.

Proposition 4.1. *Let $\gamma : I \to S^3_1$ be a unit speed spacelike curve such that $n(s)$ is a timelike vector field along γ, $k_g(s) \leq 1$, and $k_g(s) \neq 0$. Consider the singular values $h^{\pm}_{\mu} S_{\gamma}(s)$ of the horospherical surface.*

(1) *Suppose that $k_g(s) \equiv 1$. Then the following conditions are equivalent:*
   (a) *$h^{\pm}_{\mu} S_{\gamma}(s)$ is a constant vector,*
   (b) *$\tau_g(s) \equiv 0$,*
   (c) *γ is a part of a T-horoparabola.*

(2) *Suppose that the set $\{s \in I \mid k_g(s) = 1\}$ consists of isolated points. The following conditions are equivalent:*
   (a) *$h^{\pm}_{\mu} S_{\gamma}(s)$ is a constant vector $v_0 \in LC^*$,*
   (b) *$\sigma(s) \equiv 0$,*
   (c) *γ is located on a parabolic de Sitter quadric $QDP(v_0, 1)$.*

Proof. The proof is similar to that for a curve in hyperbolic space in [8]. Consider the singular values $h^{\pm}_{\mu} S_{\gamma}(s)$ of the surface that we denote by

$$v(s) = \gamma(s) + \mu n(s) \pm \sqrt{\mu^2 - 1}\, e(s) \quad \text{with} \quad \mu = -\frac{1}{k_g(s)}.$$

Suppose that $k_g(s) \equiv 1$. Then $v(s) = \gamma(s) - n(s)$, and $v'(s) = -\tau_g(s)e(s)$. Therefore, $v(s)$ is constant if and only if $\tau_g(s) \equiv 0$, so statements (a) and (b) of (1) are equivalent. If $v(s)$ is constant, then $\tau_g(s) \equiv 0$ and, as $e'(s) = \tau_g(s)n(s)$, this means that $e(s)$ is constant. Thus, the hyperplane $HP(e(s), 0)$ generated by $\gamma(s)$, $t(s)$ and $n(s)$, is constant. In this case, the parabolic de Sitter quadric $QDP(v(s), 1)$ is also constant. Thus, the image of γ is a part of a horoparabola given by $QDP(v(s), 1) \cap HP(e(s), 0)$. Therefore, (a) implies (c). If γ is a part of a

T-horoparabola, then it is a de Sitter plane curve and, hence, $\tau_g(s) \equiv 0$; so (c) implies (b). This completes the proof of (1).

Suppose now that $k_g(s) \neq 1$. Since $\mu(s) = -\dfrac{1}{k_g(s)}$, we have

$$v(s) = \gamma(s) - \frac{1}{k_g(s)} n(s) \pm \frac{\sqrt{1-k_g^2(s)}}{k_g(s)} e(s).$$

Thus

$$v'(s) = \left(\frac{k_g' \pm k_g \tau_g \sqrt{1-k_g^2}}{k_g^2} \right)(s) n(s) - \left(\frac{\sqrt{1-k_g^2} k_g \tau_g \pm k_g'}{k_g^2 \sqrt{1-k_g^2}} \right)(s) e(s).$$

Therefore, $v'(s) \equiv 0$ if and only if $\sigma(s) \equiv 0$, so the statements (a) and (b) of (2) are equivalent at any point $s \in I$.

We now consider the family of horospherical height functions $H(s,v)$ on γ. If γ is located on the parabolic de Sitter quadric $QDP(v_0, 1)$, then $H(s, v_0) \equiv 0$. By Proposition 3.1 (4), we have $(k_g' \pm k_g \tau_g \sqrt{1-k_g^2})(s) \equiv 0$. Therefore, (c) implies (b). If v is a constant vector v_0, then $\langle \gamma(s), v_0 \rangle = 1$ for all $s \in I$ and thus $\gamma(s) \in QDP(v_0, 1)$ for all $s \in I$. Therefore, γ is located on a parabolic de Sitter quadric. □

Proposition 4.2. *Let $\gamma : I \to S_1^3$ be a unit speed spacelike curve such that $n(s)$ is a spacelike vector field along γ and $k_g(s) \neq 0$. Consider the singular values $h_\mu^\pm S_\gamma(s)$ of the horospherical surface. The following conditions are equivalent:*

(a) *$h_\mu^\pm S_\gamma(s)$ is a constant vector $v_0 \in LC^*$,*
(b) *$\sigma(s) \equiv 0$,*
(c) *γ is located on a parabolic de Sitter quadric $QDP(v_0, 1)$ for some v_0.*

Furthermore, if $\gamma \subset QDP(v_0, 1)$ and $\tau_g(s) \equiv 0$, then γ is part of a S-horoparabola.

Proof. The proof is analogous to that of Proposition 4.1 (2). □

5. Hyperbolic height functions

We introduce here a family of functions on a curve which is useful to study the singularities of the hyperbolic dual surface of a spacelike unit speed curve γ. First, we explain why we consider only spacelike curves with spacelike normal vector fields. Let $\gamma : I \to S_1^3$ be a unit speed spacelike curve. We suppose, as we did previously, $\langle t'(s), t'(s) \rangle \neq 1$ (generic condition), equivalently $k_g(s) \neq 0$, in order to define $n(s) = \dfrac{t'(s) + \gamma(s)}{\| t'(s) + \gamma(s) \|}$. Then $n(s)$ is a spacelike normal vector field or a timelike normal vector field along γ.

Proposition 5.1. *Let $\gamma : I \to S_1^3$ be a unit speed spacelike curve such that $k_g(s) \neq 0$ for all $s \in I$.*

(1) *Suppose that $n(s)$ is a spacelike normal vector field along γ. Then the hyperbolic dual surface $HD_\gamma^\pm$ of γ is singular at (s_0, μ_0) if and only if $\mu_0 = 0$. That is, the singular values of the hyperbolic dual surface are given by $h_{\mu_0}^\pm D_\gamma(s) = HD_\gamma^\pm(s, 0)$ with $s \in I$ and $\mu_0 = 0$.*
(2) *If $n(s)$ is a timelike normal vector field along γ, then the hyperbolic dual surface $HD_\gamma^\pm$ of γ does not have singular points.*

Proof. Consider the hyperbolic dual surface of γ,

$$HD_\gamma^\pm(s,\mu) = \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}e(s).$$

Then, we have

$$\frac{\partial HD_\gamma^\pm}{\partial s}(s,\mu) = -\delta(\gamma(s))\mu k_g(s)t(s) \pm \sqrt{\mu^2+\delta(\gamma(s))}\tau_g(s)n(s) + \mu\tau_g(s)e(s) \quad \text{and}$$
$$\frac{\partial HD_\gamma^\pm}{\partial \mu}(s,\mu) = n(s) \pm \frac{\mu}{\sqrt{\mu^2+\delta(\gamma(s))}}e(s).$$

If $n(s)$ is a spacelike normal vector field, the proof of (1) is similar to that of Theorem 3.4 (1). However, if $n(s)$ is a timelike normal vector field, the hyperbolic dual surface is not defined for $\mu_0 = 0$. Therefore assertion (2) holds. □

Since we are interested in studying the singularities of the hyperbolic dual surface of a spacelike curve, then it follows from Proposition 5.1 (2) that we need only to consider spacelike curves with spacelike normal vector fields $n(s)$.

We define a family of functions $H : I \times H^3(-1) \to \mathbb{R}$ on γ given by $H(s,v) = \langle \gamma(s), v\rangle$. We call H the family of *hyperbolic height functions on* γ and denote $h_v(s) = H(s,v)$ for any fixed $v \in H^3(-1)$. By Definition 2.1, the hyperbolic height function measures the contact of γ with spacelike hyperplanes. Generically, the order of this contact can be k, $k = 1,2,3$.

We have the following result about the singularities of h_v.

Proposition 5.2. *Let $\gamma : I \to S_1^3$ be a unit speed spacelike curve such that $n(s)$ is a spacelike vector field along γ and $k_g(s) \neq 0$ for all $s \in I$. Then we have the following:*

(1) *$h_v(s) = 0$ if and only if there exist real numbers μ, λ, η with $\eta^2 + \mu^2 - \lambda^2 = -1$ such that $v = \eta t(s) + \mu n(s) + \lambda e(s)$.*
(2) *$h_v(s) = h_v'(s) = 0$ if and only if there exist real numbers μ, λ such that $v = \mu n(s) + \lambda e(s)$ with $\lambda^2 - \mu^2 = 1$.*
(3) *$h_v(s) = h_v'(s) = h_v''(s) = 0$ if and only if $v = \pm e(s)$.*
(4) *$h_v(s) = h_v'(s) = h_v''(s) = h_v^{(3)}(s) = 0$ if and only if $v = \pm e(s)$ and $\tau_g(s) = 0$.*
(5) *$h_v(s) = h_v'(s) = \cdots = h_v^{(4)}(s) = 0$ if and only if $v = \pm e(s)$ and $\tau_g(s) = \tau_g'(s) = 0$.*

Proof. Since $h_v(s) = \langle \gamma(s), v\rangle$, we have

(a) $h_v'(s) = \langle t(s), v\rangle$,
(b) $h_v''(s) = \langle -\gamma(s) + k_g(s)n(s), v\rangle$,
(c) $h_v^{(3)}(s) = \langle (-1 - k_g^2(s))t(s) + k_g'(s)n(s) + k_g(s)\tau_g(s)e(s), v\rangle$,
(d) $h^{(4)}(s) = \langle (1+k_g^2(s))\gamma(s) - 3k_g'(s)k_g(s)t(s) + (-k_g(s) + k_g''(s) + k_g(s)\tau_g^2(s) - k_g^3(s))n(s) + (2k_g'(s)\tau_g(s) + k_g(s)\tau_g'(s))e(s), v\rangle$.

The proof follows by simple calculations using (a)-(d). □

Corollary 5.3. *The hyperbolic dual surface of γ is the discriminant set $\mathcal{D}_H$ of the family of hyperbolic height functions H.*

Proof. The proof follows from the definition of the discriminant set given in Section 2 and Proposition 5.2 (2). □

Proposition 5.4. *Let $\gamma : I \to S_1^3$ be a unit speed spacelike curve such that $n(s)$ is a spacelike vector field along γ, $k_g \neq 0$. Then the family H of hyperbolic height functions on γ is a versal deformation of the A_2 and A_3-singularities of h_v.*

Proof. The method of the proof is similar to that of Proposition 3.3. □

We can now obtain the diffeomorphism-type of the hyperbolic dual surface.

Theorem 5.5. *Let $\gamma : I \to S^3_1$ be a unit speed spacelike curve such that $n(s)$ is a spacelike vector field along γ and $k_g(s) \neq 0$ for all $s \in I$. Consider the hyperbolic dual surface $HD^{\pm}_{\gamma}$ of γ.*

1. *The singular values of $HD^{\pm}_{\gamma}$ are given by $h^{\pm}_{\mu}D_{\gamma}(s) = \pm e(s)$.*
2. *$HD^{\pm}_{\gamma}$ is, at (s_0, μ_0), locally diffeomorphic to a cuspidal edge if and only if $\mu_0 = 0$ and $\tau_g(s_0) \neq 0$.*
3. *$HD^{\pm}_{\gamma}$ is, at (s_0, μ_0), locally diffeomorphic to a swallowtail surface if and only if $\mu_0 = 0$, $\tau_g(s_0) = 0$ and $\tau'_g(s_0) \neq 0$.*

Proof. By Corollary 5.3, the discriminant set $\mathcal{D}_H$ of the family of hyperbolic height functions H on γ is the hyperbolic dual surface of γ. It follows from Proposition 5.2 (3) and (4) that h_v has an A_2-singularity (respectively, an A_3-singularity) at s_0 if and only if $\mu_0 = 0$ and $\tau_g(s_0) \neq 0$ (respectively, $\mu_0 = 0$, $\tau_g(s_0) = 0$ and $\tau'_g(s_0) \neq 0$). By Theorem 2.2 and Proposition 5.4, this completes the proof. □

6. Invariant and special geometry of the hyperbolic dual surface

In this section, we investigate the geometric properties of a hyperbolic dual surface $HD^{\pm}_{\gamma}$ at its singularities by using the invariant τ_g of γ. The de Sitter focal surfaces of hyperbolic space curves are studied in [3].

Proposition 6.1. *Let $\gamma : I \to S^3_1$ be a unit speed spacelike curve such that $n(s)$ is a spacelike vector field along γ and $k_g(s) \neq 0$ for all $s \in I$. Consider the singular values $h^{\pm}_{\mu}D_{\gamma}(s)$ of the hyperbolic dual surface. The following conditions are equivalent:*

(a) *$h^{\pm}_{\mu}D_{\gamma}(s)$ is a constant vector $v_0 \in H^3(-1)$,*
(b) *$\tau_g(s) \equiv 0$,*
(c) *γ is part of the elliptic de Sitter quadric $QDE(v_0, 0)$.*

Proof. If the hyperbolic dual surface is singular at (s, μ), then $\mu = 0$. Therefore,

$$h^{\pm}_{\mu}D_{\gamma}(s) = HD^{\pm}_{\gamma}(s, \mu) = \pm e(s) \quad \text{and} \quad \frac{\partial HD^{\pm}_{\gamma}}{\partial s}(s, \mu) = \pm\tau_g(s)n(s) \equiv 0$$

if and only if $\tau_g(s) \equiv 0$. This means that assertion (a) is equivalent to assertion (b). Suppose that $\tau_g(s) \equiv 0$ then $h^{\pm}_{\mu}D_{\gamma}(s) = \pm e(s) = \pm v_0$ is constant. Since $\langle \gamma(s), \pm e(s) \rangle = 0$, then $\gamma(s) \in S^3_1 \cap HP(e(s), 0)$, where $v_0 = e(s)$ that is a timelike vector. Therefore, assertion (b) implies assertion (c).

On the other hand, suppose that $Im\gamma \subset QDE(v, 0) = S^3_1 \cap HP(v, 0)$, where v is a timelike fixed vector. Then we have $h_v(s) = \langle \gamma(s), v \rangle = 0$ for all $s \in I$. By Proposition 5.2, (4), $\tau_g(s) \equiv 0$. This completes the proof. □

Proposition 6.1 characterizes the case when γ is contained in the elliptic de Sitter quadric: $\tau_g(s) \equiv 0$. If $\tau_g(s) \not\equiv 0$ the result below shows that the degeneracy of the singularities of $HD^{\pm}_{\gamma}$ characterize the contact of the γ with elliptic de Sitter quadrics.

Theorem 6.2. *Let $\gamma : I \to S^3_1$ be a unit speed spacelike curve such that $n(s)$ is a spacelike vector field along γ, $k_g \neq 0$ and $\tau_g \not\equiv 0$. For $v_0 = HD^{\pm}_{\gamma}(s_0, \mu_0)$, we have the following:*

1. *γ has at least 2-point contact with $QDE(v_0, 0)$ at s_0 if and only if $\mu_0 = 0$, equivalently, the hyperbolic dual surface of γ is singular at (s_0, μ_0).*
2. *γ has 2-point contact with $QDE(v_0, 0)$ at s_0 if and only if $\mu_0 = 0$ and $\tau_g(s_0) \neq 0$, equivalently, the hyperbolic dual surface of γ is locally diffeomorphic to a cuspidal edge.*

(3) *γ has 3-point contact with $QDE(v_0, 0)$ at s_0 if and only if $\mu_0 = 0$, $\tau_g(s_0) = 0$ and $\tau_g'(s_0) \neq 0$, equivalently, the hyperbolic dual surface of γ is locally diffeomorphic to a swallowtail surface.*

Proof. For $v_0 = HD_\gamma^\pm(s_0, \mu_0)$, we define a map $\widetilde{h}_{v_0} : S_1^3 \to \mathbb{R}$ by $\widetilde{h}_{v_0}(x) = \langle x, v_0 \rangle$. Thus, we have $\left(\widetilde{h}_{v_0}\right)^{-1}(0) = QDE(v_0, 0)$. In this case, $g(s) = \widetilde{h}_{v_0} \circ \gamma(s) = h_{v_0}(s)$ and then the proof follows from Definition 2.1, Proposition 5.2 and Theorem 5.5.

□

7. Dual relations on horospherical and hyperbolic dual surfaces

We require some properties of contact manifolds and Legendrian submanifolds for the duality results in this section, and we now review these concepts (for more details see, for example, [1]).

Let N be a $(2m+1)$-dimensional smooth manifold and K be a field of tangent hyperplanes on N. Locally, K is defined as the kernel of a 1-form θ. We say that the tangent hyperplane field K is *non-degenerate* if $\theta \wedge (d\theta)^m \neq 0$ at any point on N. The pair (N, K) is called a *contact manifold* if K is a non-degenerate hyperplane field. In this case, we call K a *contact structure* and θ a *contact form*. A submanifold $i : L \subset N$ of a contact manifold (N, K) is *Legendrian* if $\dim L = m$ and $di_x(T_x L) \subset K_{i(x)}$ at any $x \in L$, where i is an immersion. A smooth fibre bundle $\pi : E \to M$ is a *Legendrian fibration* if its total space E is furnished with a contact structure and the fibers of π are Legendrian submanifolds. For a Legendrian submanifold $i : L \subset E$, $\pi \circ i : L \to M$ is called a *Legendrian map*. We call the image of the Legendrian map $\pi \circ i$ a *wavefront* set of i, which is denoted by $W(i)$.

The duality concepts we use here are those introduced in [6] and [5] (the Legendrian dualities between pseudo-spheres in Lorentz-Minkowski space), where five Legendrian double fibrations are considered on the subsets Δ_i, $i = 1, \ldots, 5$ of the product of two of the pseudo-spheres $H^n(-1)$, S_1^n and LC^*. Here we use only $i = 1, 2, 3$. We define one-forms $\langle dv, w \rangle = w_0 dv_0 + \sum_{i=1}^n w_i dv_i$, $\langle v, dw \rangle = v_0 dw_0 + \sum_{i=1}^n v_i dw_i$ on $\mathbb{R}_1^{n+1} \times \mathbb{R}_1^{n+1}$, and consider the following three Legendrian double fibrations.

$$\begin{aligned}
(1)\quad &(a)\ H^n(-1) \times S_1^n \supset \Delta_1 = \{(v, w) \mid \langle v, w \rangle = 0\}, \\
&(b)\ \pi_{11} : \Delta_1 \to H^n(-1),\ \pi_{12} : \Delta_1 \to S_1^n, \\
&(c)\ \theta_{11} = \langle dv, w \rangle |_{\Delta_1},\ \theta_{12} = \langle v, dw \rangle |_{\Delta_1}.
\end{aligned}$$

$$\begin{aligned}
(2)\quad &(a)\ H^n(-1) \times LC^* \supset \Delta_2 = \{(v, w) \mid \langle v, w \rangle = -1\}, \\
&(b)\ \pi_{21} : \Delta_2 \to H^n(-1),\ \pi_{22} : \Delta_2 \to LC^*, \\
&(c)\ \theta_{21} = \langle dv, w \rangle |_{\Delta_2},\ \theta_{22} = \langle v, dw \rangle |_{\Delta_2}.
\end{aligned}$$

$$\begin{aligned}
(3)\quad &(a)\ LC^* \times S_1^n \supset \Delta_3 = \{(v, w) \mid \langle v, w \rangle = 1\}, \\
&(b)\ \pi_{31} : \Delta_3 \to LC^*,\ \pi_{32} : \Delta_3 \to S_1^n, \\
&(c)\ \theta_{31} = \langle dv, w \rangle |_{\Delta_3},\ \theta_{32} = \langle v, dw \rangle |_{\Delta_3}.
\end{aligned}$$

Here, $\pi_{i1}(v, w) = v$, $\pi_{i2}(v, w) = w$ are the canonical projections. We remark that $\theta_{i1}^{-1}(0)$ and $\theta_{i2}^{-1}(0)$ define the same tangent hyperplane field over Δ_i which is denoted by K_i, $(i = 1, 2, 3)$. It has been shown in [6] that each (Δ_i, K_i) $(i = 1, 2, 3)$ is a contact manifold and π_{i1} and π_{i2} $(i = 1, 2, 3)$ are Legendrian fibrations. Moreover, the contact manifolds (Δ_1, K_1), (Δ_2, K_2) and (Δ_3, K_3) are contact-diffeomorphic to each other.

For a given Legendrian embedding $\mathcal{L}_i : U \to \Delta_i$, $i = 1, 2, 3$, we say that $\pi_{i1}(\mathcal{L}_i(U))$ is the Δ_i-*dual* of $\pi_{i2}(\mathcal{L}_i(U))$ and vice-versa (see [4]). In the next result, to show duality, we have to show that the immersion $\mathcal{L}_i : U \to \Delta_i$, $i = 1, 2, 3$ is a Legendrian immersion , i.e., $\dim U = m$ and $(d\mathcal{L}_i)_x(T_x(U)) \subset K_{\mathcal{L}_i(x)}$ for all $x \in L$ (see also [6]). Equivalently, $\mathcal{L}_i$ is a Legendrian immersion if $\dim U = m$ and ${\mathcal{L}_i}^*\theta_{i1} = 0$ (see, e.g., [9]). Therefore, we can show that a submanifold is Legendrian using the second definition.

We have the following relations on horospherical and hyperbolic dual surfaces. We observe that here $n = 3$, $m = 2$ and $\dim \Delta_i = 5$, $i = 1, 2, 3$. (For hyperbolic curves γ, the are duality results in [4] for hyperbolic focal surface and de Sitter focal surface of γ).

Theorem 7.1. *Let $\gamma : I \to S^3_1$ be a unit speed spacelike curve such that $k_g(s) \neq 0$ for all $s \in I$. Then*

(1) *γ is Δ_1-dual of $HD^{\pm}_{\gamma}$.*
(2) *γ is Δ_3-dual of $HS^{\pm}_{\gamma}$.*
(3) *$HD^{\pm}_{\gamma}$ is Δ_2-dual of $HS^{\pm}_{\gamma}$.*

Proof. (1) Define the mapping $\mathcal{L}_1 : I \times J \to \Delta_1$ by $\mathcal{L}_1(s, \mu) = (HD^{\pm}_{\gamma}(s, \mu), \gamma(s))$, where

$$M = \pi_{11}(\mathcal{L}_1(I \times J)) = HD^{\pm}_{\gamma}(s, \mu) = \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}e(s)$$

and

$$M^* = \pi_{12}(\mathcal{L}_1(I \times J)) = \gamma(s).$$

Then $\langle HD^{\pm}_{\gamma}(s, \mu), \gamma(s)\rangle = 0$, so the mapping is well-defined, i.e., $\mathcal{L}_1(s, \mu) \in \Delta_1$. We have

$$\begin{aligned}
\frac{\partial \mathcal{L}_1}{\partial s}(s, \mu) &= (-\delta(\gamma(s))\mu k_g(s)t(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}\tau_g(s)n(s) + \mu\tau_g(s)e(s), t(s)) \\
\frac{\partial \mathcal{L}_1}{\partial \mu}(s, \mu) &= (n(s) \pm \frac{\mu}{\sqrt{\mu^2 + \delta(\gamma(s))}}e(s), 0),
\end{aligned}$$

and so $\mathcal{L}_1$ is an immersion. Since $\mathcal{L}_1^*\theta_{12} = \langle HD^{\pm}_{\gamma}(s, \mu), t(s)\rangle ds = 0$, then, by definition, $\mathcal{L}_1(I \times J)$ is a Legendrian submanifold in Δ_1.

(2) We also define the mapping $\mathcal{L}_3 : I \times J \to \Delta_3$ by $\mathcal{L}_3(s, \mu) = (HS^{\pm}_{\gamma}(s, \mu), \gamma(s))$, where $HS^{\pm}_{\gamma}(s, \mu) = \gamma(s) + \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}e(s)$. Thus, $\langle HS^{\pm}_{\gamma}(s, \mu), \gamma(s)\rangle = 1$, i.e., $\mathcal{L}_3(s, \mu) \in \Delta_3$ and the proof follows as in (1).

(3) We now define the mapping $\mathcal{L}_2 : I \times J \to \Delta_2$ by $\mathcal{L}_2(s, \mu) = (HD^{\pm}_{\gamma}(s, \mu), HS^{\pm}_{\gamma}(s, \mu))$. Then we have

$$\langle HD^{\pm}_{\gamma}(s, \mu), HS^{\pm}_{\gamma}(s, \mu))\rangle = \mu^2\delta(\gamma(s)) + (\mu^2 + \delta(\gamma(s)))(-\delta(\gamma(s))) = -1.$$

Thus, $\mathcal{L}_2(s, \mu) \in \Delta_2$, so the mapping is well-defined. Since

$$\begin{aligned}
\frac{\partial \mathcal{L}_2}{\partial s}(s, \mu) &= (-\delta(\gamma(s))\mu k_g(s)t(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}\tau_g(s)n(s) + \mu\tau_g(s)e(s), \\
&\qquad (1 - \delta(\gamma(s))\mu k_g(s))t(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}\tau_g(s)n(s) + \mu\tau_g(s)e(s)) \\
\frac{\partial \mathcal{L}_2}{\partial \mu}(s, \mu) &= (n(s) \pm \frac{\mu}{\sqrt{\mu^2 + \delta(\gamma(s))}}e(s),\ n(s) \pm \frac{\mu}{\sqrt{\mu^2 + \delta(\gamma(s))}}e(s)),
\end{aligned}$$

$\mathcal{L}_2$ is an immersion, because $-\delta(\gamma(s))\mu k_g(s) \neq 0$ or $1 - \delta(\gamma(s))\mu k_g(s) \neq 0$. Moreover,

$$\begin{aligned}
&\mathcal{L}_2^*\theta_{21} = \left\langle d(HD_\gamma^\pm(s,\mu)), HS_\gamma^\pm(s,\mu)\right\rangle \\
&= \left\langle \frac{\partial HD_\gamma^\pm}{\partial s}(s,\mu)ds + \frac{\partial HD_\gamma^\pm}{\mu}(s,\mu)d\mu, HS_\gamma^\pm(s,\mu)\right\rangle \\
&= \left\langle -\mu\delta(\gamma(s))k_g(s)t(s) \pm \sqrt{\mu^2+\delta(\gamma(s))}\tau_g(s)n(s) + \mu\tau_g(s)e(s), \gamma(s)\right\rangle ds + \\
&\left\langle \tau_g(s)(\mu e(s) \pm \sqrt{\mu^2+\delta(\gamma(s))}\, n(s)) - \mu\delta(\gamma(s))k_g(s)t(s), \mu n(s) \pm \sqrt{\mu^2+\delta(\gamma(s))}e(s)\right\rangle ds \\
&+ \left\langle n(s) \pm \frac{\mu}{\sqrt{\mu^2+\delta(\gamma(s))}}e(s), \gamma(s) + \mu n(s) \pm \sqrt{\mu^2+\delta(\gamma(s))}e(s)\right\rangle d\mu = 0.
\end{aligned}$$

Therefore, $\mathcal{L}_2(I \times J)$ is a Legendrian submanifold in Δ_2.

□

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Shyuichi Izumiya, Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan
E-mail address: izumiya@math.sci.hokudai.ac.jp

Ana Claudia Nabarro, Departamento de Matemática, ICMC Universidade de São Paulo, Campus de São Carlos, Caixa Postal 668, CEP 13560-970, São Carlos-SP, Brazil
E-mail address: anaclana@icmc.usp.br

Andrea de Jesus Sacramento, Departamento de Matemática, ICMC Universidade de São Paulo, Campus de São Carlos, Caixa Postal 668, CEP 13560-970, São Carlos-SP, Brazil
E-mail address: anddyunesp@yahoo.com.br

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ERRATUM TO: A REMARK ON THE IRREGULARITY COMPLEX

CLAUDE SABBAH

Abstract. We correct a wrong statement in [Sab17].

Proposition 3.3 and Corollary 3.4 of [Sab17] should be modified as follows.

Proposition 3.3. *Let us fix $I \subset J$ and let us set $k = k(I)$ for simplicity. Then the natural morphism $\widetilde{\iota}_I^{-1}\mathscr{L}^{>0} \to \widetilde{\iota}_I^{-1}\boldsymbol{R}\widetilde{\iota}_{k*}\widetilde{\iota}_k^{-1}\mathscr{L}^{>0}$ is an isomorphism. The same property holds for $\mathscr{L}_{\prec 0}$ up to replacing $k(I)$ with $k'(I)$.*

Corollary 3.4. *3. With the notation as in Proposition 3.3, the natural morphism*

$$\iota_I^{-1}\operatorname{Irr}_D(\mathscr{M}) \to \iota_I^{-1}\boldsymbol{R}\iota_{k*}\,\iota_k^{-1}\operatorname{Irr}_D(\mathscr{M})$$

is an isomorphism. The same property holds for $\operatorname{Irr}^*_D(\mathscr{M})$ up to replacing $k(I)$ with $k'(I)$. $\square$

Here, the index $k'(I)$ is any index k' such that the following property is satisfied: any $\varphi \in \Phi_{x_o}$ having a pole along $D_{k'}$ has a pole along all the components of D passing through x_o (such a k' exists, due to the goodness condition). Equivalently, the number of $\varphi \in \Phi_{x_o}$ having *no pole* on $D_{k'}$ is maximum (this maximum could be zero).

The last paragraph of the proof of Proposition 3.3 should be replaced with the following.

The case of $\mathscr{L}_{\prec 0}$ is treated similarly by reducing to the case where $\mathscr{M} = \mathscr{E}^{\varphi}$. Assume first that φ has poles along *all* components of D passing through x_o (i.e., $p = \ell$). If we regard all sheaves considered above as external products of constant sheaves of rank one with respect to the product decomposition in (3.6) and (3.7), the case of $\mathscr{L}_{\prec 0}$ is obtained by replacing $[-\pi/2, \pi/2]$ with the complementary open interval in (3.5), and the corresponding sheaf $\mathbb{C}_{[a,b]}$ with the sheaf $\mathbb{C}_{(a',b')}$ for suitable a', b' (i.e., the extension by zero of the constant sheaf on (a', b')). Then the same argument as above applies to this case.

If the assumption on φ does not hold, then φ has no pole along $D_{k'}$, so that $\iota_{k'}^{-1}\mathscr{L}_{\prec 0} = 0$. We also have $\iota_I^{-1}\mathscr{L}_{\prec 0} = 0$ since e^{φ} is not of rapid decay *all along* D, so the statement is obvious in this case.

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(C. Sabbah) CMLS, École polytechnique, CNRS, Université Paris-Saclay, F–91128 Palaiseau cedex, France
E-mail address: Claude.Sabbah@polytechnique.edu
URL: http://www.math.polytechnique.fr/perso/sabbah

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ON THE TOPOLOGY OF A RESOLUTION OF ISOLATED SINGULARITIES

VINCENZO DI GENNARO AND DAVIDE FRANCO

Abstract. Let Y be a complex projective variety of dimension n with isolated singularities, $\pi : X \to Y$ a resolution of singularities, $G := \pi^{-1}\mathrm{Sing}(Y)$ the exceptional locus. From the Decomposition Theorem one knows that the map $H^{k-1}(G) \to H^k(Y, Y\backslash\mathrm{Sing}(Y))$ vanishes for $k > n$. Assuming this vanishing, we give a short proof of the Decomposition Theorem for π. A consequence is a short proof of the Decomposition Theorem for π in all cases where one can prove the vanishing directly. This happens when either Y is a normal surface, or when π is the blowing-up of Y along $\mathrm{Sing}(Y)$ with smooth and connected fibres, or when π admits a natural Gysin morphism. We prove that this last condition is equivalent to saying that the map $H^{k-1}(G) \to H^k(Y, Y\backslash\mathrm{Sing}(Y))$ vanishes for all k, and that the pull-back $\pi_k^* : H^k(Y) \to H^k(X)$ is injective. This provides a relationship between the Decomposition Theorem and Bivariant Theory.

1. Introduction

Consider an n-dimensional complex projective variety Y with *isolated singularities*. Fix a *desingularization* $\pi : X \to Y$ of Y. This paper is addressed at the study of some topological properties of the map π. In a previous paper [14], we already observed that, even though π is never a *local complete intersection* map, in some very special cases it may nonetheless admit a *natural Gysin morphism*. By a natural Gysin morphism, we mean a *topological bivariant class* [20, §7], [7]

$$\theta \in T^0(X \xrightarrow{\pi} Y) := Hom_{D^b(Y)}(R\pi_*\mathbb{Q}_X, \mathbb{Q}_Y),$$

commuting with restrictions to the smooth locus of Y (here and in the following $D^b(Y)$ denotes the *bounded derived category* of sheaves of $\mathbb{Q}$-vector spaces on Y).

In this paper, we give a complete characterization of morphisms like π admitting a natural Gysin morphism by means of the *Decomposition Theorem* [2], [6], [8], [9]. In some sense, what we are going to prove is that π admits a natural Gysin morphism if and only if Y is a $\mathbb{Q}$-*intersection cohomology manifold*, i.e., $IC_Y^\bullet \simeq \mathbb{Q}_Y[n]$ in $D^b(Y)$ ($IC_Y^\bullet$ denotes the *intersection cohomology complex* of Y [17, p. 156], [27]). Furthermore, in this case, there is a unique natural Gysin morphism θ, and it arises from the Decomposition Theorem (compare with Theorem 1.2 below).

The Decomposition Theorem is a beautiful and very deep result about algebraic maps. In the words of MacPherson, "it contains as special cases the deepest homological properties of algebraic maps that we know"[26], [34]. As observed in [34, Remark 2.14], since the proof of the Decomposition Theorem proceeds by induction on the dimension of the strata of the singular locus, a key point is the case of varieties with isolated singularities:

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Theorem 1.1 (The Decomposition Theorem for varieties with isolated singularities)**.** *In $D^b(Y)$, we have a decomposition*

$$R\pi_*\mathbb{Q}_X \cong IC_Y^\bullet[-n] \oplus \mathcal{H}^\bullet,$$

where $\mathcal{H}^\bullet$ is quasi-isomorphic to a skyscraper complex on $\mathrm{Sing}(Y)$*. Furthermore, we have*

1. $\mathcal{H}^k(\mathcal{H}^\bullet) \cong H^k(G)$*, for all* $k \geq n$*,*
2. $\mathcal{H}^k(\mathcal{H}^\bullet) \cong H_{2n-k}(G)$*, for all* $k < n$*,*

where $G := \pi^{-1}(\mathrm{Sing}(Y))$, and $H^k(G)$ and $H_{2n-k}(G)$ have $\mathbb{Q}$-coefficients.

The relationship between the Gysin morphism and the Decomposition Theorem is closely related to an important topological property of the morphism π. Specifically, in [22] and [32] one proves that Theorem 1.1 implies the following vanishing

$$H^{k-1}(G) \to H^k(Y,U) \text{ vanishes for } k > n, \tag{1}$$

where $U = Y \backslash \mathrm{Sing}(Y)$.

One of the main points we would like to stress in this paper (compare with Theorem 3.1) is that

the vanishing (1) is equivalent to the Decomposition Theorem.

More precisely, what we are going to do in this paper is to prove that assuming (1), one can prove Theorem 1.1 in few pages. Actually this equivalence is already implicit in the argument developed by Navarro Aznar in order to prove [30, (6.3) Corollaire, p. 293]. In fact, after proving (1) using Hodge Theory, Navarro Aznar proves the relative Hard Lefschetz Theorem and observes that the Decomposition Theorem follows from Deligne's results on degeneration of spectral sequences. Instead, here we give a simpler and more direct proof, avoiding the use of the relative Hard Lefschetz Theorem. In fact, we deduce the splitting in derived category from a simple result concerning short exact sequences of complexes (compare with Lemma 4.7).

A byproduct of our result is a short proof of the Decomposition Theorem in all cases where one can prove the vanishing (1) directly. This happens when either $2\dim G < n$ (for trivial reasons), or when Y is a *normal surface* in view of *Mumford's theorem* [23], [29], or when $\pi : X \to Y$ is the *blowing-up* of Y along $\mathrm{Sing}(Y)$ with smooth and connected fibres (see Remark 5.1). It is worth remarking that if Y is a locally complete intersection variety, then *Milnor's theorem* on the connectivity of the *link* [16] implies (via Lemma 4.1 below) that the map $H^{k-1}(G) \to H^k(Y,U)$ vanishes for all $k \geq n+2$. Therefore, in this case the question reduces to check that the map $H^n(G) \to H^{n+1}(Y,U)$ vanishes. This in turn is equivalent to require that $H_n(G)$, which is contained in $H_n(X)$ via push-forward, is a nondegenerate subspace of $H_n(X)$ with respect to the natural intersection form $H_n(X) \times H_n(X) \to H_0(X)$ (see Remark 5.1, (i)). Another case is when π admits a natural Gysin morphism. Indeed, in this case it is very easy to prove the stronger property

$$H^{k-1}(G) \to H^k(Y,U) \text{ vanishes for } k > 0.$$

This is the real reason why in our approach the same argument leads to both Theorem 1.1 and the following:

Theorem 1.2. *There exists a natural Gysin morphism for π if and only if Y is a $\mathbb{Q}$-intersection cohomology manifold. In this case, in $D^b(Y)$ we have a decomposition*

$$R\pi_*\mathbb{Q}_X \cong IC_Y^\bullet[-n] \oplus \mathcal{H}^\bullet \cong \mathbb{Q}_Y \oplus \bigoplus_{k\geq 1} R^k\pi_*\mathbb{Q}_X[-k].$$

Moreover, a natural Gysin morphism is unique, and, up to multiplication by a nonzero rational number, it comes from the decomposition above via projection onto $\mathbb{Q}_Y$.

For a more precise and complete statement see Theorem 3.2 and Remark 3.3 below. For instance, from Theorem 3.2, (ix), we deduce that a natural Gysin morphism exists when Y is nodal of even dimension n, or when Y is a cone over a smooth basis M with $H^\bullet(M) \cong H^\bullet(\mathbb{P}^{n-1})$. We stress that the existence of a natural Gysin morphism forces the exceptional locus G to have dimension 0 or $n-1$ (see Remark 6.1).

Last but not least, we have been led to consider the issues addressed in this paper by our previous work on Noether-Lefschetz Theory. We refer to the papers [10], [11], [12], [13] anyone interested in the overlaps between the topological properties investigated here and the Noether-Lefschetz Theorem (specifically, we made an heavy use of the Decomposition Theorem in [12, Remark 3 and Theorem 6, (6.3), p. 169], and in [13, Theorem 2.1, proof of (a), p. 262]).

2. Notations

(i) Let Y be a complex irreducible projective variety of dimension $n \geq 1$, with isolated singularities. Let $\pi : X \to Y$ be a resolution of the singularities of Y. For all $y \in \mathrm{Sing}(Y)$, set $G_y := \pi^{-1}(y)$. Set $G := \bigcup_{y \in \mathrm{Sing}(Y)} G_y = \pi^{-1}(\mathrm{Sing}(Y))$. Let $i : G \hookrightarrow X$ be the inclusion.

(ii) All cohomology and homology groups are with $\mathbb{Q}$-coefficients. For a function $f : A \to B$ we denote by $\Im(f)$ the image of f, i.e., $\Im(f) = f(A)$.

(iii) Set $U := Y \backslash \mathrm{Sing}(Y) \cong X \backslash G$. Denote by $\alpha : U \hookrightarrow Y$ and $\beta : U \hookrightarrow X$ the inclusions. For all k we have the following natural commutative diagram:

$$\begin{tikzcd} H^k(Y) \arrow[rr, "\pi_k^*"] \arrow[dr, "\alpha_k^*"'] & & H^k(X) \arrow[dl, "\beta_k^*"] \\ & H^k(U) & \end{tikzcd} \tag{2}$$

where all the maps denote pull-back.

Remark 2.1. From the commutativity of (2) we deduce $\Im(\alpha_k^*) \subseteq \Im(\beta_k^*)$. Since $H^k(Y) \cong H^k(X)$ for $k \leq 0$ or $k \geq 2n$, we have $\Im(\alpha_k^*) = \Im(\beta_k^*)$ for $k \leq 0$ or $k \geq 2n$. It may happen that $\Im(\alpha_k^*) \neq \Im(\beta_k^*)$. We may interpret the condition $\Im(\alpha_k^*) = \Im(\beta_k^*)$ as follows. Combining the Universal Coefficient Theorem with the Lefschetz Duality Theorem [31, p. 248 and p. 297] we have $H^k(U) \cong H_{2n-k}(Y, \mathrm{Sing}(Y))$ for all k. Since $\mathrm{Sing}(Y)$ is finite, we also have

$$H_{2n-k}(Y) \cong H_{2n-k}(Y, \mathrm{Sing}(Y))$$

for $k \leq 2n-2$, and $H_1(Y) \subseteq H_1(Y, \mathrm{Sing}(Y))$. Therefore, for $k \leq 2n-2$, (2) identifies with the diagram:

$$\begin{tikzcd} H^k(Y) \arrow[rr] \arrow[dr] & & H_{2n-k}(X) \arrow[dl] \\ & H_{2n-k}(Y) & \end{tikzcd}$$

where the map $H^k(Y) \to H_{2n-k}(X)$ is the composite of Poincaré Duality $H^k(X) \cong H_{2n-k}(X)$ with the pull-back π_k^*, the map $H_{2n-k}(X) \to H_{2n-k}(Y)$ is the push-forward, and the map $H^k(Y) \xrightarrow{\cdot \cap [Y]} H_{2n-k}(Y)$ is the *duality morphism*, i.e., the cap-product with the fundamental class $[Y] \in H_{2n}(Y)$ [28]. It follows that $\Im(\alpha_k^*) = \Im(\beta_k^*)$ *if and only if every cycle in* $H_{2n-k}(Y)$ *coming from* $H_{2n-k}(X)$ *via push-forward is the cap-product of a cocycle in* $H^k(Y)$ *with the fundamental class* $[Y]$. This holds true also for $k = 2n-1$ because $H_1(Y) \subseteq H_1(Y, \mathrm{Sing}(Y)) \cong H^{2n-1}(U)$.

(iv) Embed Y in some projective space $\mathbb{P}^N$. For all $y \in \mathrm{Sing}(Y)$ choose a small closed ball $S_y \subset \mathbb{P}^N$ around y, and set $B_y := S_y \cap Y$, $D_y := \pi^{-1}(B_y)$, $B := \bigcup_{y\in \mathrm{Sing}(Y)} B_y$, and $D := \pi^{-1}(B)$. B_y is homeomorphic to the cone over the link ∂B_y of the singularity $y \in Y$, with vertex at y [16, p. 23]. B_y is contractible, by excision we have

$$H^k(Y,U) \cong H^k(B, B\backslash \mathrm{Sing}(Y)) \cong H^k(B, \partial B)$$

for all k, and from the cohomology long exact sequence of the pair $(B, \partial B)$ we get

$$H^k(Y,U) \cong H^{k-1}(\partial B)$$

for all $k \geq 2$. We have $\partial D \cong \partial B$ via π, and by excision we have

$$H^k(X,U) \cong H^k(D, D\backslash G) \cong H^k(D, \partial D)$$

for all k [17, p. 38]. Since G is homotopy equivalent to D, we have $H^k(G) \cong H^k(D)$. Putting everything together, from the cohomology long exact sequence of the pair $(D, \partial D)$ we get the following exact sequence

$$H^k(X,U) \to H^k(G) \to H^{k+1}(Y,U) \stackrel{\gamma^*_{k+1}}{\to} H^{k+1}(X,U) \tag{3}$$

for all $k \geq 1$, where γ^*_{k+1} denotes the pull-back. Observe that since $\mathrm{Sing}(Y)$ is finite, we have $H^k(G) = \oplus_{y\in\mathrm{Sing}(Y)} H^k(G_y)$, $H^k(B) = \oplus_{y\in\mathrm{Sing}(Y)} H^k(B_y)$, $H^k(\partial B) = \oplus_{y\in\mathrm{Sing}(Y)} H^k(\partial B_y)$.

Remark 2.2. Assume that Y is a locally complete intersection variety. From the connectivity of the link [16, Milnor's theorem p. 76, and Hamm's theorem p. 80], it follows that *the duality morphism $H^k(Y) \to H_{2n-k}(Y)$ is an isomorphism for all $k \notin \{n-1, n, n+1\}$, is injective for $k = n-1$, and is surjective for $k = n+1$. In particular $\Im(\alpha^*_k) = \Im(\beta^*_k)$ for all $k \notin \{n-1, n\}$.* In order to prove this property, we argue as follows. We may assume $0 < k < 2n$ and $n \geq 2$. From the cohomology long exact sequence of the pair (Y,U) we have:

$$\ldots \to H^k(Y,U) \to H^k(Y) \to H^k(U) \to H^{k+1}(Y,U) \to \ldots, \tag{4}$$

and by excision $H^k(Y,U) \cong H^k(B, \partial B)$. Taking into account that each B_y is contractible and that ∂B_y is path connected [16, loc. cit.], from the cohomology long exact sequence of the pair $(B, \partial B)$ we get $H^1(B, \partial B) = 0$ and $H^k(B, \partial B) \cong H^{k-1}(\partial B)$ for $k \geq 2$. Since

$$H^k(U) \cong H_{2n-k}(Y, \mathrm{Sing}(Y)),$$

and $H_{2n-k}(Y) \cong H_{2n-k}(Y, \mathrm{Sing}(Y))$ for $k \leq 2n-2$, from (4) we get the exact sequence for $k \notin \{1, 2n-1\}$ (compare with [15, p. 5]):

$$H^{k-1}(\partial B) \to H^k(Y) \to H_{2n-k}(Y) \to H^k(\partial B).$$

Each ∂B_y is $(n-2)$-connected by Milnor's theorem [16, loc. cit.], and it is a compact oriented real manifold of dimension $2n-1$, in particular $h^k(\partial B_y) = h^{2n-1-k}(\partial B_y)$ by Poincaré Duality [16, p. 91]. It follows that the map $H^k(Y) \to H_{2n-k}(Y)$ is an isomorphism for

$$k \notin \{1, n-1, n, n+1, 2n-1\}.$$

As for the case $k = 1 \neq n-1$, this follows from (4) because

$$H^1(Y,U) \cong H^1(B, \partial B) = 0,$$

$H^1(U) \cong H_{2n-1}(Y, \mathrm{Sing}(Y)) \cong H_{2n-1}(Y)$, and $H^2(Y,U) \cong H^2(B, \partial B) \cong H^1(\partial B) = 0$ by connectivity of the link. When $k = 2n-1 \neq n+1$, we have

$$H^{2n-1}(Y,U) \cong H^{2n-1}(B, \partial B) = H^{2n-2}(\partial B) = 0.$$

Thus, $H^{2n-1}(Y) \hookrightarrow H^{2n-1}(U)$. On the other hand $H_1(Y) \hookrightarrow H_1(Y, \mathrm{Sing}(Y)) \cong H^{2n-1}(U)$. It follows that the duality morphism $H^{2n-1}(Y) \to H_1(Y)$ is injective. Then it is an isomorphism because we have just seen, in the case $k=1$, that $h^1(Y)=h_{2n-1}(Y)$. Finally notice that, when $n\geq 3$, from previous analysis and (4) we get the exact sequence:

$$0\to H^{n-1}(Y)\to H_{n+1}(Y)\to H^{n-1}(\partial B)\to H^n(Y)\to H_n(Y)$$
$$\to H^n(\partial B)\to H^{n+1}(Y)\to H_{n-1}(Y)\to 0.$$

Therefore, the duality morphism

$$H^{n-1}(Y)\to H_{n+1}(Y)$$

is injective, and the map $H^{n+1}(Y)\to H_{n-1}(Y)$ is onto. This holds true also when $n=2$. In fact, also in this case we have $H^1(B,\partial B)=0$, which implies that the duality morphism $H^1(Y)\to H_3(Y)$ is injective. Moreover, a similar analysis as before shows that the image of $H^3(Y)$ and $H_1(Y)$ have the same codimension in $H^3(U)$. Thus, they are equal. This concludes the proof of the claim.

(v) By [31, Lemma 14, p. 351] we have $H^k(X,U)\cong H_{2n-k}(G)$. Therefore, from the cohomology long exact sequence of the pair (X,U) we get a long exact sequence:

$$\dots\to H^{k-1}(U)\to H_{2n-k}(G)\to H^k(X)\overset{\beta_k^*}{\to} H^k(U)\to\dots. \tag{5}$$

(vi) For all $y\in \mathrm{Sing}(Y)$ set:

$$H_y^k:=\begin{cases} H^k(G_y) & \text{if } k\geq n\\ H_{2n-k}(G_y) & \text{if } k<n.\end{cases}$$

Let $\mathcal{H}_y^k$ be the skyscraper sheaf on Y with stalk at y given by H_y^k. Set $H^k:=\oplus_{y\in\mathrm{Sing}(Y)}H_y^k$ and $\mathcal{H}^k:=\oplus_{y\in\mathrm{Sing}(Y)}\mathcal{H}_y^k$. We consider $\mathcal{H}^\bullet$ as a complex of sheaves on Y with vanishing differentials $d^k_{\mathcal{H}^\bullet}=0$.

Remark 2.3. From the Universal Coefficient Theorem [31, p. 248] it follows that the $\mathbb{Q}$-vector spaces H^{n-k} and H^{n+k} are isomorphic for all k. This implies that $\mathcal{H}^\bullet[n]$ *is self-dual, i.e., in the bounded derived category $D^b(Y)$ of Y we have $\mathcal{H}^\bullet[n]\cong D(\mathcal{H}^\bullet[n])$*. Taking into account that in $\mathcal{H}^\bullet[n]$ all the differentials vanish, to prove that $\mathcal{H}^\bullet[n]$ is self-dual it suffices to prove that the complexes $\mathcal{H}^\bullet[n]$ and $D(\mathcal{H}^\bullet[n])$ have isomorphic sheaf cohomology. Since $\mathcal{H}^\bullet[n]$ is supported on a finite set, this amounts to prove that $\mathcal{H}^\bullet[n]$ and $D(\mathcal{H}^\bullet[n])$ have isomorphic hypercohomology, i.e., that

$$\mathbb{H}^k(\mathcal{H}^\bullet[n])\cong \mathbb{H}^k(D(\mathcal{H}^\bullet[n]))$$

for all k. But by Poincaré-Verdier Duality [17, p. 69, Theorem 3.3.10] we have:

$$\mathbb{H}^k(D(\mathcal{H}^\bullet[n]))\cong \mathbb{H}^{-k}(\mathcal{H}^\bullet[n])^\vee\cong \mathbb{H}^{n-k}(\mathcal{H}^\bullet)^\vee\cong (H^{n-k})^\vee\cong H^{n+k}\cong \mathbb{H}^k(\mathcal{H}^\bullet[n]).$$

(vii) We say that a graded morphism $\theta_\bullet: H^\bullet(X)\to H^\bullet(Y)$ is *natural* if for all k one has $\theta_k\circ\pi_k^*=\mathrm{id}_{H^k(Y)}$, and the following diagram commutes [14]:

$$\begin{tikzcd} H^k(Y) \arrow[dr, "\alpha_k^*"'] & & H^k(X) \arrow[ll, "\theta_k"'] \arrow[dl, "\beta_k^*"] \\ & H^k(U), & \end{tikzcd}$$

i.e., $\alpha_k^*\circ\theta_k=\beta_k^*$.

Remark 2.4. The existence of a natural graded morphism $\theta_\bullet: H^\bullet(X)\to H^\bullet(Y)$ is equivalent to saying that, for all k, the pull-back $\pi_k^*: H^k(Y)\to H^k(X)$ is injective and $\Im(\alpha_k^*)=\Im(\beta_k^*)$ (compare with the proof of (i) $\implies$ (ii) in Theorem 3.2 below).

(*viii*) We say that a (topological) bivariant class $\theta \in Hom_{D^b(Y)}(R\pi_*\mathbb{Q}_X, \mathbb{Q}_Y)$ is *natural* if the induced graded morphism $\theta_\bullet : H^\bullet(X) \to H^\bullet(Y)$ is natural [14], [20].

Remark 2.5. Fix a bivariant class

$$\theta \in H^0(X \xrightarrow{\pi} Y) \cong Hom_{D^b(Y)}(R\pi_*\mathbb{Q}_X, \mathbb{Q}_Y).$$

Let $\theta_0 : H^0(X) \to H^0(Y)$ be the induced map. Let $q \in \mathbb{Q}$ be such that

$$\theta_0(1_X) = q \cdot 1_Y \in H^0(Y) \cong \mathbb{Q}$$

[31, p. 238]. Put

$$\deg\theta := q.$$

For all k and all $c \in H^k(Y)$, by the projection formula [20, (G_4), (i), p. 26], and [31, 9, p. 251], we have :

$$\theta_k(\pi_k^*(c)) = \theta_k(1_X \cup \pi_k^*(c)) = \theta_0(1_X) \cup c = \deg\theta \cdot (1_Y \cup c) = \deg\theta \cdot c. \tag{6}$$

It follows that for all k one has:

$$\theta_k \circ \pi_k^* = \deg\theta \cdot \mathrm{id}_{H^k(Y)}. \tag{7}$$

Next consider the independent square:

$$\begin{array}{ccc} U & \overset{\beta}{\hookrightarrow} & X \\ \| & & {}^{\pi}\downarrow \\ U & \overset{\alpha}{\hookrightarrow} & Y \end{array}$$

and set $\theta' := \alpha^*(\theta) \in Hom_{D^b(U)}(\mathbb{Q}_U, \mathbb{Q}_U)$ [20, (G_2), p. 26]. Applying [20, (G_2), (ii), p. 26] to the square:

$$\begin{array}{ccc} H^0(U) & \overset{\beta_0^*}{\leftarrow} & H^0(X) \\ {}^{\theta_0'}\downarrow & & {}^{\theta_0}\downarrow \\ H^0(U) & \overset{\alpha_0^*}{\leftarrow} & H^0(Y) \end{array}$$

we get

$$\theta_0'(1_U) = \theta_0'(\beta_0^*(1_X)) = \beta_0^*(\theta_0(1_X)) = \beta_0^*(\deg\theta \cdot 1_Y) = \deg\theta \cdot \beta_0^*(1_Y) = \deg\theta \cdot 1_U.$$

Since $\pi_{|_U} = \mathrm{id}_U$, as in (6) we deduce for all k and all $c \in H^k(U)$:

$$\theta_k'(c) = \theta_k'((\pi|_U)_k^*(c)) = \theta_k'(1_U \cup c) = \theta_0'(1_U) \cup c = \deg\theta \cdot (1_U \cup c) = \deg\theta \cdot c,$$

i.e.,

$$\theta_k' = \deg\theta \cdot \mathrm{id}_{H^k(U)}. \tag{8}$$

From [20, (G_2), (ii), p. 26] it follows that

$$\deg\theta \cdot \beta_k^* = \theta_k' \circ \beta_k^* = \alpha_k^* \circ \theta_k \tag{9}$$

for all k. By (7) and (9) we see that *a bivariant class* θ *is natural if and only if* $\deg\theta = 1$, *and this is equivalent to saying that* $\beta_k^* = \alpha_k^* \circ \theta_k$ *for all* k. Observe that if θ is a bivariant class with $\deg\theta \neq 0$, then $\frac{1}{\deg\theta}\theta$ is natural.

(*ix*) We say that Y is a $\mathbb{Q}$-*cohomology (or homology) manifold* if for all $y \in Y$ and all $k \neq 2n$ one has $H^k(Y, Y\backslash\{y\}) = 0$, and $H^{2n}(Y, Y\backslash\{y\}) \cong \mathbb{Q}$ [27], [28]. Recall that Y is a $\mathbb{Q}$-intersection cohomology manifold if $IC_Y^\bullet \cong \mathbb{Q}_Y[n]$ in $D^b(Y)$, where $IC_Y^\bullet$ denotes the intersection cohomology complex of Y [17, p. 156], [27].

Remark 2.6. By [20, 3.1.4, p. 34] we know that there is a mapping $\phi : X \to \mathbb{R}^m$ such that $(\pi, \phi) : X \to Y \times \mathbb{R}^m$ is a closed imbedding. In this case one has

$$H^0(X \xrightarrow{\pi} Y) \cong H^m(Y \times \mathbb{R}^m, Y \times \mathbb{R}^m \backslash X_\phi),$$

where X_ϕ is the image of X in $Y \times \mathbb{R}^m$. If Y is a $\mathbb{Q}$-cohomology manifold, then by Poincaré-Alexander-Lefschetz Duality [1, Theorem 1.1] we have:

$$H^m(Y \times \mathbb{R}^m, Y \times \mathbb{R}^m \backslash X_\phi) \cong H_{2n}(X).$$

It follows that

$$\dim_{\mathbb{Q}} H^0(X \xrightarrow{\pi} Y) = 1. \tag{10}$$

On the other hand, since U is smooth, we also have [19, Lemma 2 and (26), p. 217]:

$$H^0(U \xrightarrow{\mathrm{id}_U} U) \cong H^m(U \times \mathbb{R}^m, U \times \mathbb{R}^m \backslash U_\phi) \cong H_{2n}^{BM}(U) \cong H^0(U) \cong \mathbb{Q},$$

where $H_{2n}^{BM}(U)$ denotes the Borel-Moore homology. Therefore, the pull-back

$$\alpha^* : H^0(X \xrightarrow{\pi} Y) \to H^0(U \xrightarrow{\mathrm{id}_U} U)$$

for bivariant classes identifies with the restriction in Borel-Moore homology:

$$H_{2n}(X) \cong H_{2n}^{BM}(U).$$

Comparing with (8) and (10), this proves that *if Y is a $\mathbb{Q}$-cohomology manifold, then there is a unique natural bivariant class.*

(x) Let $\mathcal{I}^\bullet$ be an injective resolution of $\mathbb{Q}_X$. Let $\mathcal{J}^\bullet := \pi_*(\mathcal{I}^\bullet)$ be the derived direct image $R\pi_*\mathbb{Q}_X$ of $\mathbb{Q}_X$ in $D^b(Y)$. When $k \geq 1$ the cohomology sheaves $R^k\pi_*\mathbb{Q}_X = H^k(\mathcal{J}^\bullet)$ are supported on $\mathrm{Sing}(Y)$, and for all $y \in \mathrm{Sing}(Y)$ we have $H^k(\mathcal{J}^\bullet)_y = H^k(G_y)$.

Remark 2.7. The complex $\mathcal{J}^\bullet[n]$ is self-dual. In fact, by [17, p. 69, Proposition 3.3.7, (ii)], we have:

$$D(\mathcal{J}^\bullet[n]) = D(R\pi_*\mathbb{Q}_X[n]) = R\pi_*(D(\mathbb{Q}_X[n])) = R\pi_*(\mathbb{Q}_X[n]) = \mathcal{J}^\bullet[n].$$

(xi) Since Y has only isolated singularities, we have [17, Proposition 5.4.4, p. 157]:

$$IH^k(Y) \cong \begin{cases} H^k(Y) & \text{if } k > n \\ \Im(\alpha_n^*) & \text{if } k = n \\ H^k(U) & \text{if } k < n. \end{cases} \tag{11}$$

3. The main results

Theorem 3.1 below is essentially already known. Property (i) implies (ii) by [32, Theorem 1.11, p. 518]. That property (ii) implies (i) is implicit in the argument developed by Navarro in order to prove [30, (6.3) Corollaire, p. 293] using a relative version of the Hard Lefschetz Theorem. Here we give a simpler and more direct proof that (ii) implies (i), avoiding the use of the relative Hard Lefschetz Theorem.

Theorem 3.1. *The following properties are equivalent.*

(i) In the derived category of Y there is an isomorphism:

$$R\pi_*\mathbb{Q}_X \cong IC_Y^\bullet[-n] \oplus \mathcal{H}^\bullet.$$

(ii) The map $H^{k-1}(G) \to H^k(Y, U)$ vanishes for all $k > n$.

The equivalences of properties (v), (vi) and (vii) in the next Theorem 3.2 are already known [4], [28], [27]. We insert them in the claim for Reader's convenience. We refer to [27] for other equivalences concerning a $\mathbb{Q}$-cohomology manifold.

Theorem 3.2. *The following properties are equivalent.*

(i) The map $H^{k-1}(G) \to H^k(Y, U)$ vanishes for all $k > 0$ and the pull-back π_k^* is injective.

(ii) There exists a natural graded morphism $\theta_\bullet : H^\bullet(X) \to H^\bullet(Y)$.

(iii) There exists a natural bivariant class $\theta \in Hom_{D^b(Y)}(R\pi_*\mathbb{Q}_X, \mathbb{Q}_Y)$.

(iv) The natural map $H^\bullet(Y) \to IH^\bullet(Y)$is an isomorphism;

(v) Y is a $\mathbb{Q}$-intersection cohomology manifold.

(vi) Y is a $\mathbb{Q}$-cohomology manifold.

(vii) The duality morphism $H^\bullet(Y) \xrightarrow{\cdot \cap [Y]} H_{2n-\bullet}(Y)$ is an isomorphism (i.e., Y satisfies Poincaré Duality).

(viii) In $D^b(Y)$ there exists a decomposition

$$R\,\pi_*\mathbb{Q}_X \cong \mathbb{Q}_Y \oplus \bigoplus_{k\geq 1} R^k\pi_*\mathbb{Q}_X[-k]. \tag{12}$$

Moreover, if $\pi : X \to Y$ is the blowing-up of Y along $\mathrm{Sing}(Y)$ *with smooth and connected fibres, then previous properties are equivalent to the following property:*

(ix) For all $y \in Sing(Y)$ one has $H^\bullet(G_y) \cong H^\bullet(\mathbb{P}^{n-1})$.

Remark 3.3. (i) Projecting onto $\mathbb{Q}_Y$, from the decomposition (12), we obtain a bivariant class

$$\eta \in Hom_{D^b(Y)}(R\pi_*\mathbb{Q}_X, \mathbb{Q}_Y),$$

whose induced Gysin morphisms $\eta_k : H^k(X) \to H^k(Y)$ are surjective. In particular $\deg \eta \neq 0$. By Remark 2.6 it follows that $\frac{1}{\deg \eta}\eta$ is the unique natural bivariant class.

(ii) The natural morphism $\theta_\bullet : H^\bullet(X) \to H^\bullet(Y)$ is unique and identifies with the push-forward via Poincaré Duality:

$$H^\bullet(X) \cong H_{2n-\bullet}(X) \to H_{2n-\bullet}(Y) \cong H^\bullet(Y).$$

In fact, by Remark 2.1 we know that, for $k < 2n-1$, the restriction map $\alpha_k^* : H^k(Y) \to H^k(U)$ is nothing but the duality (iso)morphism because $H^k(U) \cong H_{2n-k}(Y)$. Therefore, we have $\theta_k = (\alpha_k^*)^{-1} \circ \beta_k^*$. The case $k = 2n-1$ is similar because $H_1(Y) \subseteq H^{2n-1}(U)$ (again compare with Remark 2.1).

4. Preliminaries

Lemma 4.1. *The following sequences are exact:*

$$0 \to H^k(Y) \stackrel{\pi_k^*}{\to} H^k(X) \stackrel{i_k^*}{\to} H^k(G) \to 0 \quad \textit{for all } k > n,$$

$$H^n(Y) \stackrel{\pi_n^*}{\to} H^n(X) \stackrel{i_n^*}{\to} H^n(G) \to 0,$$

$$0 \to H_{2n-k}(G) \to H^k(X) \stackrel{\beta_k^*}{\to} H^k(U) \to 0 \quad \textit{for all } k < n.$$

Proof. By [18, p. 84, 6*] we know that $H^k(Y, \mathrm{Sing}(Y)) \cong H^k(X, G)$ for all k. Since $\mathrm{Sing}(Y)$ is finite, we also have $H^k(Y, \mathrm{Sing}(Y)) \cong H^k(Y)$ for $k \geq 1$. Therefore, the long exact sequence of the pair:

$$\dots \to H^k(X, G) \to H^k(X) \stackrel{i_k^*}{\to} H^k(G) \to H^{k+1}(X, G) \to \dots$$

identifies, when $k \geq 1$, with the long exact sequence:

$$\dots \to H^k(Y) \stackrel{\pi_k^*}{\to} H^k(X) \stackrel{i_k^*}{\to} H^k(G) \to H^{k+1}(Y) \to \dots. \tag{13}$$

In order to prove that the first two sequences are exact, it suffices to prove that i_k^* is surjective for all $k \geq n$. To this purpose, let L be a general hyperplane section of Y, and put $Y_0 := Y \backslash L$, and $X_0 := \pi^{-1}(Y_0)$. As before, we have a long exact sequence:

$$\dots \to H^k(Y_0) \to H^k(X_0) \to H^k(G) \to H^{k+1}(Y_0) \to \dots$$

and by Deligne's theorem [33, Proposition 4.23], we know that the pull-back maps

$$H^k(X) \stackrel{i_k^*}{\to} H^k(G) \quad \text{and} \quad H^k(X_0) \to H^k(G)$$

have the same image. Then we are done. In fact, since Y_0 is affine, we have $H^{k+1}(Y_0) = 0$ for all $k \geq n$ by stratified Morse Theory [21, p. 23-24].

In order to examine the last sequence, assume $k < n$. Then $2n - k > n$, and we just proved that the pull-back $H^{2n-k}(X, G) \cong H^{2n-k}(Y) \to H^{2n-k}(X)$ is injective. Combining the Poincaré Duality Theorem with the Lefschetz Duality Theorem [31, p. 297] we have $H^{2n-k}(X) \cong H_k(X)$ and $H^{2n-k}(X, G) \cong H_k(U)$. Therefore, the push-forward $H_k(U) \to H_k(X)$ is injective. Hence, the restriction $H^k(X) \to H^k(U)$ is onto for all $k < n$. Now our assertion follows from (5). $\square$

Lemma 4.2. *Fix an integer k, and let $\gamma_k^* : H^k(Y, U) \to H^k(X, U)$ be the pull-back. Assume that $\pi_k^* : H^k(Y) \to H^k(X)$ is injective. Then the following properties are equivalent.*

(i) γ_k^* is injective;

(ii) $\Im(\alpha_{k-1}^*) = \Im(\beta_{k-1}^*)$;

(iii) $H^{k-1}(G) \to H^k(Y, U)$ is the zero map.

Proof. Consider the natural commutative diagram with exact rows:

$$\begin{array}{ccccccc}
H^{k-1}(X) & \stackrel{\beta_{k-1}^*}{\longrightarrow} & H^{k-1}(U) & \longrightarrow & H^k(X, U) & \longrightarrow & H^k(X) \\
{\scriptstyle \pi_{k-1}^*}\uparrow & & \| & & {\scriptstyle \gamma_k^*}\uparrow & & {\scriptstyle \pi_k^*}\uparrow \\
H^{k-1}(Y) & \stackrel{\alpha_{k-1}^*}{\longrightarrow} & H^{k-1}(U) & \longrightarrow & H^k(Y, U) & \longrightarrow & H^k(Y).
\end{array}$$

If γ_k^* is injective, then

$$\ker(H^{k-1}(U) \to H^k(X, U)) = \ker(H^{k-1}(U) \to H^k(Y, U)).$$

It follows that $\Im(\alpha_{k-1}^*) = \Im(\beta_{k-1}^*)$ because $\Im(\alpha_{k-1}^*) = \ker(H^{k-1}(U) \to H^k(Y, U))$ and

$$\Im(\beta_{k-1}^*) = \ker(H^{k-1}(U) \to H^k(X, U)).$$

Conversely, assume that $\Im(\alpha_{k-1}^*) = \Im(\beta_{k-1}^*)$, and fix an element $c \in \ker \gamma_k^*$. Since π_k^* is injective, there exists some $c' \in H^{k-1}(U)$ which maps to c via $H^{k-1}(U) \to H^k(Y, U)$. Since $c \in \ker \gamma_k^*$, a fortiori c' belongs to $\Im(\beta_{k-1}^*)$. Hence, $c' \in \Im(\alpha_{k-1}^*)$ and $c = 0$. The equivalence of (i) with (iii) follows from (3). $\square$

Corollary 4.3. *Let $H_k(G) \to H^{2n-k}(G)$ be the map obtained by composing the map $H_k(G) \to H^{2n-k}(X)$ with the pull-back $H^{2n-k}(X) \to H^{2n-k}(G)$. Assume $k \geq n$ and that $\Im(\alpha_k^*) = \Im(\beta_k^*)$. Then the map $H_k(G) \to H^{2n-k}(G)$ is injective.*

Proof. By Lemma 4.1, Lemma 4.2, and (3), we deduce that the map $H^k(X,U) \to H^k(G)$ is onto. Dualizing we get an injective map $H_k(G) \to H_k(X,U)$. We are done because, by excision and the Lefschetz Duality Theorem [31, p. 298], we have

$$H_k(X,U) \cong H_k(D, \partial D) \cong H^{2n-k}(D) \cong H^{2n-k}(G).$$

□

Corollary 4.4. *We have:*

$$H^k(X) \cong \begin{cases} IH^k(Y) \oplus H^k(G) & \text{if } k > n, \\ IH^k(Y) \oplus H_{2n-k}(G) & \text{if } k < n. \end{cases}$$

Moreover, if $\Im(\alpha_n^*) = \Im(\beta_n^*)$*, then*

$$H^n(X) \cong IH^n(Y) \oplus H^n(G).$$

Proof. In view of Lemma 4.1 we only have to examine the case $k = n$. Since $\beta_n^* \circ \pi_n^* = \alpha_n^*$, there exists a subspace $P \subseteq \Im(\pi_n^*) \subseteq H^n(X)$, which is mapped isomorphically to

$$\Im(\beta_n^*) = \Im(\alpha_n^*) = IH^n(Y)$$

via β_n^*. In particular $P \cap \ker \beta_n^* = \{0\}$, and so $H^n(X) = IH^n(Y) \oplus \ker \beta_n^*$. On the other hand $\ker \beta_n^* = \Im(H^n(X,U) \to H^n(X))$. By Corollary 4.3 we know that the map $H^n(X,U) \to H^n(X)$ is injective because so is the composite $H^n(X,U) \cong H_n(G) \to H^n(X) \to H^n(G)$. Therefore, $\ker \beta_n^* = \Im(H^n(X,U) \to H^n(X)) \cong H^n(X,U) \cong H_n(G) \cong H^n(G)$. □

Lemma 4.5. *Assume that* $\Im(\alpha_k^*) = \Im(\beta_k^*)$ *for all* $k \geq n$. *Then there is an injective map of complexes*

$$0 \to \mathcal{H}^\bullet \to \mathcal{J}^\bullet.$$

Proof. It is enough to prove that for all k there is a monomorphism of sheaves

$$\mathcal{H}^k \hookrightarrow \ker(\mathcal{J}^k \to \mathcal{J}^{k+1}).$$

First, we examine the case $k \geq n$.

To this aim, set $\Gamma^\bullet := \Gamma(\mathcal{J}^\bullet)$ and denote by $d^k : \Gamma^k \to \Gamma^{k+1}$ the differential. Then we have $H^k(X) = H^k(\Gamma^\bullet)$. By Lemma 4.1 every element a of $H^k = H^k(G)$ can be lifted to an element $c \in \ker d^k$. We claim that every $a \in H^k(G)$ can be lifted to an element $b \in \ker d^k \subseteq \Gamma(\mathcal{J}^k)$ which is supported on $\mathrm{Sing}(Y)$. Proving this claim amounts to show that every $a \in H^k(G)$ can be lifted to an element $b \in \ker d^k \subset \Gamma(\mathcal{J}^k) = \Gamma(\mathcal{I}^k)$ such that $b\,|_U = 0 \in \Gamma(\mathcal{J}^k\,|_U)$. But $c\,|_U$ projects to a cohomology class living in $\Im(H^k(X) \to H^k(U))$. By our assumption we have

$$\Im(H^k(X) \stackrel{\beta_k^*}{\to} H^k(U)) = \Im(H^k(Y) \stackrel{\alpha_k^*}{\to} H^k(U)).$$

Since

$$H^k(Y) \cong H^k(Y, \mathrm{Sing}(Y)) \cong H^k(X, G)$$

[18, p. 84, 6*], we find

$$\Im(H^k(Y) \stackrel{\alpha_k^*}{\to} H^k(U)) = \Im(H^k(X,G) \to H^k(U)).$$

On the other hand we have

$$H^k(X,G) \cong H^k(X, \beta_! \mathbb{Q}_U)$$

[5, Theorem 12.1], [17, Remark 2.4.5, (ii)]. By definition of direct image with proper support [24, §2.6], [17, Definition 2.3.21], the sheaf $\beta_! \mathbb{Q}_U$ identifies with the subsheaf of $\mathbb{Q}_X$ consisting of sections with support contained in U. It follows that there exists $e_U \in \Gamma(\mathcal{J}^{k-1} \mid_U)$ and $g \in \Gamma(\mathcal{J}^k)$ supported in U such that

$$c \mid_U - d^{k-1}(e_U) = g \mid_U .$$

Moreover, there exists $e \in \Gamma(\mathcal{J}^{k-1})$ with $e \mid_U = e_U$, because $\mathcal{J}^{k-1}$ is injective (hence flabby). We conclude that the section

$$c - g - d^{k-1}(e) \in \Gamma(\mathcal{J}^k)$$

is supported on $\mathrm{Sing}(Y)$. Our claim is proved because $g + d^{k-1}(e) \in \Gamma(\mathcal{J}^k)$ vanishes in $H^k(G)$. To conclude the proof in the case $k \geq n$, fix a basis $a_r \in H^k = H^k(G)$ and lift every a_r to a $b_r \in \ker d^k \subseteq \Gamma(\mathcal{J}^k)$ as in the claim. We get an isomorphism between $H^k(G)$ and a subspace of $\Gamma(\mathcal{J}^k)$ consisting of sections supported on $\mathrm{Sing}(Y)$. We are done because such an isomorphism projects to a monomorphism of sheaves $\mathcal{H}^k \hookrightarrow \ker(J^k \to J^{k+1})$.

Now we assume $k < n$.

By Lemma 4.1 every element a of $H^k = H_{2n-k}(G) \subseteq H^k(X)$ can be lifted to an element $c \in \ker d^k$. Since a restricts to 0 in $H^k(U)$, there exists $e \in \Gamma(\mathcal{J}^{k-1} \mid_U)$ such that $c \mid_U = d_U^{k-1}(e)$. Since $\mathcal{J}^{k-1}$ is flabby, we may assume $e \in \Gamma(\mathcal{J}^{k-1})$. Therefore, $b := c - d^{k-1}(e) \in \Gamma(\mathcal{J}^k)$ represents a and is supported on $\mathrm{Sing}(Y)$. As in the case $k \geq n$, applying this argument to a basis of $H^k = H_{2n-k}(G)$, we define a monomorphism of sheaves $\mathcal{H}^k \hookrightarrow \ker(\mathcal{J}^k \to \mathcal{J}^{k+1})$. $\square$

With the same assumption as in Lemma 4.5, let $\mathcal{K}^\bullet$ be the cokernel of the inclusion $0 \to \mathcal{H}^\bullet \to \mathcal{J}^\bullet$:

$$0 \to \mathcal{H}^\bullet \to \mathcal{J}^\bullet \to \mathcal{K}^\bullet \to 0.$$

All the sheaves of these complexes are injective. Previous sequence gives rise to a long exact sequence of sheaf cohomology:

$$\ldots \to \mathcal{H}^k \to \mathcal{H}^k(\mathcal{J}^\bullet) \to \mathcal{H}^k(\mathcal{K}^\bullet) \to \ldots,$$

and for all $k \geq 1$ these sheaves are supported on $\mathrm{Sing}(Y)$.

Proposition 4.6. *For all k the sequence*

$$0 \to \mathcal{H}^k \to \mathcal{H}^k(\mathcal{J}^\bullet) \to \mathcal{H}^k(\mathcal{K}^\bullet) \to 0$$

is exact.

Proof. It suffices to prove that the map $H_y^k \to \mathcal{H}^k(\mathcal{J}^\bullet)_y$ is injective for all $y \in \mathrm{Sing}(Y)$ and all $k > 0$. If $k \geq n$ this is obvious because $H^k(\mathcal{J}^\bullet)_y = H^k(G_y) = H_y^k$. When $1 \leq k < n$ we have $H_y^k = H_{2n-k}(G_y)$. And the map $H_{2n-k}(G_y) \to H^k(\mathcal{J}^\bullet)_y = H^k(G_y)$ is injective by Corollary 4.3. $\square$

Lemma 4.7. *Let $0 \to \mathcal{H}^\bullet \xrightarrow{f^\bullet} \mathcal{J}^\bullet \xrightarrow{g^\bullet} \mathcal{K}^\bullet \to 0$ be an exact sequence of complexes of sheaves. Assume that $\mathcal{H}^\bullet$ is a complex of injective sheaves with vanishing differential $d^k_{\mathcal{H}^\bullet} = 0$ for all k. The following properties are equivalent.*

(i) The sequence coming from the cohomology long exact sequence:

$$0 \to \mathcal{H}^k(\mathcal{H}^\bullet) \to \mathcal{H}^k(\mathcal{J}^\bullet) \to \mathcal{H}^k(\mathcal{K}^\bullet) \to 0 \tag{14}$$

is exact for all k.

(ii) There is a complex map $s^\bullet : \mathcal{K}^\bullet \to \mathcal{J}^\bullet$ such that $g^\bullet \circ s^\bullet = \mathrm{id}_{\mathcal{K}^\bullet}$.

Proof. We only have to prove that (i) implies (ii).

Since $\mathcal{H}^0$ is injective, the exact sequence sequence $0 \to \mathcal{H}^0 \to \mathcal{J}^0 \to \mathcal{K}^0 \to 0$ admits a section $s^0 : \mathcal{K}^0 \to \mathcal{J}^0$, with $g^0 \circ s^0 = \mathrm{id}_{\mathcal{K}^0}$. Therefore, we may construct $s^\bullet = \{s^i\}_{i\geq 0}$ using induction on i. Assume $i \geq 0$ and that there are sections $s^0, \dots, s^i$, with $s^h : \mathcal{K}^h \to \mathcal{J}^h$, $g^h \circ s^h = \mathrm{id}_{\mathcal{K}^h}$, and $s^h \circ d^{h-1}_{\mathcal{K}^\bullet} = d^{h-1}_{\mathcal{J}^\bullet} \circ s^{h-1}$ for all $0 \leq h \leq i$. As before, since $\mathcal{H}^{i+1}$ is injective and the sequence $0 \to \mathcal{H}^{i+1} \to \mathcal{J}^{i+1} \to \mathcal{K}^{i+1} \to 0$ is exact, there exists a section $\sigma^{i+1} : \mathcal{K}^{i+1} \to \mathcal{J}^{i+1}$, with $g^{i+1} \circ \sigma^{i+1} = \mathrm{id}_{\mathcal{K}^{i+1}}$. A priori it may happen that $\sigma^{i+1} \circ d^i_{\mathcal{K}^\bullet}$ is different from $d^i_{\mathcal{J}^\bullet} \circ s^i$, so we have to modify σ^{i+1}. To this purpose set:

$$\delta := \sigma^{i+1} \circ d^i_{\mathcal{K}^\bullet} - d^i_{\mathcal{J}^\bullet} \circ s^i \in Hom(\mathcal{K}^i, \mathcal{J}^{i+1}).$$

Since

$$g^{i+1} \circ \delta = g^{i+1} \circ \sigma^{i+1} \circ d^i_{\mathcal{K}^\bullet} - g^{i+1} \circ d^i_{\mathcal{J}^\bullet} \circ s^i = d^i_{\mathcal{K}^\bullet} - d^i_{\mathcal{K}^\bullet} = 0,$$

it follows that

$$\Im(\delta) \subseteq \mathcal{H}^{i+1}. \tag{15}$$

Since (14) is exact, the map g^i sends $\ker d^i_{\mathcal{J}^\bullet}$ onto $\ker d^i_{\mathcal{K}^\bullet}$, i.e.,

$$g^i(\ker d^i_{\mathcal{J}^\bullet}) = \ker d^i_{\mathcal{K}^\bullet}. \tag{16}$$

In view of the exactness of the sequence $0 \to \mathcal{H}^\bullet \xrightarrow{f^\bullet} \mathcal{J}^\bullet \xrightarrow{g^\bullet} \mathcal{K}^\bullet \to 0$, and of the assumption $d^i_{\mathcal{H}^\bullet} = 0$, we also have

$$\ker g^i = \Im(f^i) \subseteq \ker d^i_{\mathcal{J}^\bullet}. \tag{17}$$

Combining (16) and (17) we deduce that:

$$\ker d^i_{\mathcal{J}^\bullet} = (g^i)^{-1}(\ker d^i_{\mathcal{K}^\bullet}). \tag{18}$$

In fact, by (16) we have $\ker d^i_{\mathcal{J}^\bullet} \subseteq (g^i)^{-1}(\ker d^i_{\mathcal{K}^\bullet})$. On the other hand, if $x \in (g^i)^{-1}(\ker d^i_{\mathcal{K}^\bullet})$, then $g^i(x) \in \ker d^i_{\mathcal{K}^\bullet}$, and by (16) we may write $g^i(x) = g^i(y)$ for some $y \in \ker d^i_{\mathcal{J}^\bullet}$. Hence, $x - y \in \ker g^i$, and from (17) it follows that $x \in \ker d^i_{\mathcal{J}^\bullet}$. From (18) we get:

$$s^i(\ker d^i_{\mathcal{K}^\bullet}) \subseteq \ker d^i_{\mathcal{J}^\bullet}. \tag{19}$$

To prove this, recall that $g^i \circ s^i = \mathrm{id}_{\mathcal{K}^i}$. Therefore, $g^i(s^i(\ker d^i_{\mathcal{K}^\bullet})) = \ker d^i_{\mathcal{K}^\bullet}$, and so, taking into account (18), we have:

$$s^i(\ker d^i_{\mathcal{K}^\bullet}) \subseteq (g^i)^{-1}(\ker d^i_{\mathcal{K}^\bullet}) = \ker d^i_{\mathcal{J}^\bullet}.$$

By (19) we deduce that:

$$\ker d^i_{\mathcal{K}^\bullet} \subseteq \ker \delta, \tag{20}$$

and from (15) and (20) we get

$$\delta \in Hom(\mathcal{K}^i / \ker d^i_{\mathcal{K}^\bullet}, \mathcal{H}^{i+1}).$$

Since $\mathcal{H}^{i+1}$ is injective, we may extend δ to a map $\tilde{\delta} \in Hom(\mathcal{K}^{i+1}, \mathcal{H}^{i+1})$ such that

$$\tilde{\delta} \circ d^i_{\mathcal{K}^\bullet} = \delta. \tag{21}$$

We have

$$\tilde{\delta} \in Hom(\mathcal{K}^{i+1}, \mathcal{J}^{i+1})$$

because $\mathcal{H}^{i+1}$ maps to $\mathcal{J}^{i+1}$ via f^{i+1}. Now we define:

$$s^{i+1} := \sigma^{i+1} - \tilde{\delta}.$$

From (21) it follows that

$$s^{i+1} \circ d^i_{\mathcal{K}^\bullet} = d^i_{\mathcal{J}^\bullet} \circ s^i,$$

and since $\Im(\tilde{\delta}) \subseteq \mathcal{H}^{i+1}$, we also have

$$g^{i+1} \circ s^{i+1} = \mathrm{id}_{\mathcal{K}^{i+1}}.$$

$\square$

5. Proof of Theorem 3.1

As we have seen in Section 3, by [32, Theorem 1.11, p. 518] one knows that the Decomposition Theorem implies (ii). Therefore, we only have to prove that (ii) implies (i).

In view of Lemma 4.1 and Lemma 4.2 we have $\Im(\alpha_k^*) = \Im(\beta_k^*)$ for all $k \geq n$. From Lemma 4.5, Proposition 4.6, and Lemma 4.7, we get:

$$R\pi_*\mathbb{Q}_X = \mathcal{J}^\bullet = \mathcal{K}^\bullet \oplus \mathcal{H}^\bullet. \tag{22}$$

Hence, we only have to prove that

$$\mathcal{K}^\bullet \cong IC_Y[-n],$$

where $IC_Y^\bullet = IC_Y^{top}[-n]$ denotes the intersection cohomology complex of Y [17, p. 156]. Observe that the restriction $\alpha^{-1}\mathcal{K}^\bullet$ of $\mathcal{K}^\bullet$ to U is $\mathbb{Q}_U$, and that, by (22), we have $\mathcal{K}^\bullet \in D_c^b(Y)$ [17, p. 81-82]. Therefore, $\mathcal{K}^\bullet[n]$ is an extension of $\mathbb{Q}_U[n]$ [17, p. 134]. So to prove that $\mathcal{K}^\bullet \cong IC_Y[-n]$ it suffices to prove that $\mathcal{K}^\bullet[n] \cong \alpha_{!*}\mathbb{Q}_U[n]$, i.e., that $\mathcal{K}^\bullet[n]$ is the intermediary extension of $\mathbb{Q}_U[n]$ [17, p.156 and p.135]. By [17, Proposition 5.2.8, p. 135], this in turn reduces to prove that for all $y \in \mathrm{Sing}(Y)$ the following two conditions hold true ($i_y : \{y\} \to Y$ denotes the inclusion):

(a) $\mathcal{H}^k i_y^{-1}\mathcal{K}^\bullet[n] = 0$ for all $k \geq 0$;

(b) $\mathcal{H}^k i_y^! \mathcal{K}^\bullet[n] = 0$ for all $k \leq 0$.

As for condition (a) we notice that [17, p.130]:

$$\mathcal{H}^k i_y^{-1}\mathcal{K}^\bullet[n] = \mathcal{H}^k(\mathcal{K}^\bullet[n])_y = \mathcal{H}^{k+n}(\mathcal{K}^\bullet)_y,$$

and $\mathcal{H}^{k+n}(\mathcal{K}^\bullet)_y = 0$ because $\mathcal{J}^\bullet = \mathcal{K}^\bullet \oplus \mathcal{H}^\bullet$, and $\mathcal{H}^{k+n}(\mathcal{J}^\bullet)_y = H^{k+n}(G_y) = \mathcal{H}^{k+n}(\mathcal{H}^\bullet)_y$ for $k \geq 0$.

For the condition (b), first notice that combining (22) with Remarks 2.3 and 2.7, we deduce that $\mathcal{K}^\bullet[n]$ is self-dual. Therefore, condition (b) reduces to (a). In fact, we have [17, p. 130, proof of Lemma 5.1.15]:

$$\mathcal{H}^k i_y^! \mathcal{K}^\bullet[n] = \mathcal{H}^{-k}(i_y^{-1}D(\mathcal{K}^\bullet[n]))^\vee = \mathcal{H}^{-k}(i_y^{-1}(\mathcal{K}^\bullet[n]))^\vee = \mathcal{H}^{-k+n}(\mathcal{K}^\bullet)_y^\vee = 0$$

because $k \leq 0$.

Remark 5.1. (i) If $n = 2$, then the map $H^{k-1}(G) \to H^k(Y, U)$ vanishes for all $k \geq n+2$ for trivial reasons. In view of the connectivity of the link, combining Remark 2.2 with Lemma 4.1 and Lemma 4.2, we see that this holds true also when Y is locally complete intersection. Therefore, either when $n = 2$ or when Y is locally complete intersection, in order to deduce the decomposition (i) in Theorem 3.1, we need only check that the map $H^n(G) \to H^{n+1}(Y, U)$ is the zero map. On the other hand, the vanishing of the map $H^n(G) \to H^{n+1}(Y, U)$ is equivalent to require that the natural map $H_n(G) \to H^n(G) \cong H_n(G)^\vee$ is onto (compare with (3), (5), and Corollary 4.3). Since $H_n(G)$ is contained in $H_n(X)$ via push-forward (Lemma 4.1), it follows that the map $H_n(G) \to H^n(G) \cong H_n(G)^\vee$ is onto if and only if $H_n(G)$ is a nondegenerate subspace of $H_n(X)$ with respect to the natural intersection form $H_n(X) \times H_n(X) \to H_0(X) \cong \mathbb{Q}$. By Mumford's theorem [23], [29] we know this holds true when Y is a normal surface. Therefore, *in the case Y is a normal surface (or when $2\dim G < n$), our Theorem 3.1 gives a new and simplified proof of the Decomposition Theorem for $\pi : X \to Y$.*

(ii) Assume that $\pi : X \to Y$ is the blowing-up of Y along $\mathrm{Sing}(Y)$, with smooth and connected fibres. By Poincaré Duality we have $H_{2n-k}(G_y) \cong H^{k-2}(G_y)$ for all $y \in \mathrm{Sing}(Y)$. It follows that $H^k(X,U) \cong H_{2n-k}(G) \cong \oplus_{y\in\mathrm{Sing}(Y)} H_{2n-k}(G_y) \cong \oplus_{y\in\mathrm{Sing}(Y)} H^{k-2}(G_y)$. Hence, the map $H^k(X,U) \to H^k(G)$ identifies with the map $\oplus_{y\in\mathrm{Sing}(Y)} H^{k-2}(G_y) \to \oplus_{y\in\mathrm{Sing}(Y)} H^k(G_y)$ given, on each summand $H^{k-2}(G_y) \to H^k(G_y)$, by the self-intersection formula, i.e., by the cup-product with the first Chern class $c_1(N_y) \in H^2(G_y)$ of the normal bundle N_y of G_y in X. Since π is the blowing-up along the finite set $\mathrm{Sing}(Y)$, the dual normal bundle $N_y^\vee \cong \mathcal{O}_{G_y}(1)$ is ample for all $y \in \mathrm{Sing}(Y)$. From the Hard Lefschetz Theorem it follows that the map $H^{k-2}(G_y) \to H^k(G_y)$ is onto for all $k \geq n$, and so also the map $H^k(X,U) \to H^k(G)$ is. By (3), this implies the vanishing of the map $H^k(G) \to H^{k+1}(Y,U)$. Therefore, also in this case our Theorem 3.1 gives a new and simplified proof of the Decomposition Theorem for π.

(iii) More generally, assume only that the fibres of $\pi : X \to Y$ are smooth and connected, so that π is not necessarily the blowing-up along $\mathrm{Sing}(Y)$. Using the extension of the Hard Lefschetz Theorem to bundles of higher rank due to Bloch and Gieseker [3], [25], with a similar argument as before one proves that *if the dual normal bundle $N_y^\vee$ of G_y in X is ample for all $y \in Sing(Y)$, then the map $H^k(G) \to H^{k+1}(Y,U)$ vanishes for all $k \geq n$.* In fact, set

$$h_y := \dim X - \dim G_y$$

for all $y \in \mathrm{Sing}(Y)$. Now the map $H^k(X,U) \to H^k(G)$ identifies with the map

$$\oplus_{y\in\mathrm{Sing}(Y)} H^{k-2h_y}(G_y) \to \oplus_{y\in\mathrm{Sing}(Y)} H^k(G_y)$$

given, on each summand $H^{k-2h_y}(G_y) \to H^k(G_y)$, by the cup-product with the top Chern class $c_{h_y}(N_y) = (-1)^{h_y} c_{h_y}(N_y^\vee) \in H^{2h_y}(G_y)$ of the normal bundle N_y of G_y in X. And such a map is onto for $k \geq n$ by the quoted extension of the Hard Lefschetz Theorem, because $N_y^\vee$ is ample. We refer to [15, Proposition 2.12 and proof] for examples of resolution of singularities verifying previous assumptions.

6. Proof of Theorem 3.2

(i) $\implies$ (ii) By Lemma 4.1 and Lemma 4.2 we have $\Im(\alpha_k^*) = \Im(\beta_k^*)$ for all k. Let $y_1, \dots, y_a, y_{a+1}, \dots, y_b$ be a basis of $H^k(Y)$ such that $\alpha_k^* y_1, \dots, \alpha_k^* y_a$ is a basis for $\Im(\alpha_k^*) = \Im(\beta_k^*)$, and $y_{a+1}, \dots, y_b$ a basis for $\ker \alpha_k^*$. Since $\pi_k^*(\ker \alpha_k^*) \subseteq \ker \beta_k^*$, we may extend $\pi_k^* y_{a+1}, \dots, \pi_k^* y_b$ to a basis $\pi_k^* y_{a+1}, \dots, \pi_k^* y_b, x_{b+1}, \dots, x_c$ of $\ker \beta_k^*$. Then

$$\pi_k^* y_1, \dots, \pi_k^* y_a, \pi_k^* y_{a+1}, \dots, \pi_k^* y_b, x_{b+1}, \dots, x_c$$

is a basis for $H^k(X)$. Define $\theta_k : H^k(X) \to H^k(Y)$ setting $\theta_k(\pi_k^*(y_i)) := y_i$, and $\theta_k(x_i) := 0$. Then $\theta_\bullet$ is a natural morphism.

(ii) $\implies$ (i) The existence of a natural morphism implies that π_k^* is injective and $\Im(\beta_k^*) \subseteq \Im(\alpha_k^*)$ for all k. Since in general we have $\Im(\alpha_k^*) \subseteq \Im(\beta_k^*)$, it follows that $\Im(\alpha_k^*) = \Im(\beta_k^*)$ for all k. By Lemma 4.1 and Lemma 4.2 we get (i).

(ii) $\implies$ (iv) Since π_k^* is injective for all k, using (13) we get a short exact sequence:

$$0 \to H^k(Y) \overset{\pi_k^*}{\to} H^k(X) \overset{i_k^*}{\to} H^k(G) \to 0$$

for all $k \geq 1$. In particular, for $k \geq 1$, we have

$$H^k(X) \cong H^k(Y) \oplus H^k(G). \tag{23}$$

On the other hand, since $\theta_k \circ \pi_k^* = \mathrm{id}_{H^k(Y)}$, the short exact sequence

$$0 \to \ker\theta_k \to H^k(X) \stackrel{\theta_k}{\to} H^k(Y) \to 0$$

admits π_k^* as a section. It follows another decomposition:

$$H^k(X) = \pi_k^* H^k(Y) \oplus \ker\theta_k. \tag{24}$$

Comparing (23) with (24) we see that

$$\ker\theta_k \cong H^k(G)$$

for all $k \geq 1$. On the other hand, since $\alpha_k^* \circ \theta_k = \beta_k^*$, we have

$$\ker\theta_k \subseteq \ker(H^k(X) \stackrel{\beta_k^*}{\to} H^k(U)) = \Im(H^k(X,U) \to H^k(X)). \tag{25}$$

Since $H^k(X,U) \cong H_{2n-k}(G)$, it follows that

$$\dim H^k(G) \leq \dim H_{2n-k}(G) \tag{26}$$

for all $k \geq 1$. By the Universal-coefficient formula [31, p. 248] we deduce that, for $1 \leq k \leq 2n-1$,

$$\ker\theta_k \cong H^k(G) \cong H_{2n-k}(G). \tag{27}$$

Taking into account that $\Im(\alpha_n^*) = \Im(\beta_n^*)$, combining (23), (27) and Corollary 4.4, it follows that $\dim H^k(Y) = \dim IH^k(Y)$ for all k. Therefore, by (11), it suffices to prove that

$$\alpha_k^* : H^k(Y) \to H^k(U)$$

is surjective for all $k < n$. To this purpose notice that, for $k < n$, β_k^* is surjective by Lemma 4.1. This implies that also α_k^* is by (24) and (25) (compare with diagram (2)).

(iv) $\implies$ (vii) Since intersection cohomology verifies Poincaré Duality [17, p. 158], we have:

$$H^h(Y) = IH^h(Y) = (IH^{2(m+1)-h}(Y))^\vee = (H^{2(m+1)-h}(Y))^\vee = H_{2(m+1)-h}(Y).$$

(vii) $\implies$ (iv) This follows from (11) and Remark 2.1.

(v) $\iff$ (vi) $\iff$ (vii) By [28, Theorem 2, Lemma 2, Lemma 3] we know that the duality morphism is an isomorphism if and only if Y is a $\mathbb{Q}$-cohomology manifold, which is equivalent to saying that Y is a $\mathbb{Q}$-intersection cohomology manifold by [27, Theorem 1.1] (compare also with [4]).

(vii) $\implies$ (ii) Denote by $d_k^Y : H^k(Y) \to H_{2n-k}(Y)$ the duality isomorphism, by

$$d_k^X : H^k(X) \cong H_{2n-k}(X)$$

the Poincaré Duality isomorphism, by $\pi_{*,k} : H_{2n-k}(X) \to H_{2n-k}(Y)$ the push-forward. Set $\theta_k : H^k(X) \to H^k(Y)$ with

$$\theta_k := (d_k^Y)^{-1} \circ \pi_{*,k} \circ d_k^X.$$

Then $\theta_\bullet$ is a natural morphism.

(iii) $\iff$ (ii) We only have to prove that (ii) implies (iii). This follows from Remark 2.6 because Y is a $\mathbb{Q}$-cohomology manifold.

(ii) $\implies$ (viii) Since Y is a $\mathbb{Q}$-intersection cohomology manifold, combining (27) with Theorem 3.1, we get:

$$R\pi_*\mathbb{Q}_X \cong \mathbb{Q}_Y \oplus \mathcal{H}^\bullet \cong \mathbb{Q}_Y \oplus \bigoplus_{k\geq 1} R^k\pi_*\mathbb{Q}_X[-k].$$

(viii) $\implies$ (ii) See Remark 3.3, (i).

(ii) $\iff$ (ix) By [27, Theorem 1.1] we deduce that Y is a $\mathbb{Q}$-intersection cohomology manifold if and only if for all $y \in \mathrm{Sing}(Y)$ the link ∂B_y has the same $\mathbb{Q}$-homology type as a sphere S^{2n-1}. On the other hand, via deformation to the normal cone, we may identify ∂B_y with the link of the vertex of the projective cone over $G_y \subseteq \mathbb{P}^{N-1}$. Restricting the Hopf bundle $S^{2N-1} \to \mathbb{P}^{N-1}$ to G_y, we obtain an S^1-bundle $\partial B_y \to G_y$ inducing the Thom-Gysin sequence [31, p. 260]

$$\dots \to H^k(G_y) \to H^k(\partial B_y) \to H^{k-1}(G_y) \to H^{k+1}(G_y) \to H^{k+1}(\partial B_y) \to \dots$$

And this sequence implies that ∂B_y has the same $\mathbb{Q}$-homology type as a sphere S^{2n-1} if and only if $H^\bullet(G_y) \cong H^\bullet(\mathbb{P}^{n-1})$.

Remark 6.1. By (26) it follows that $h_2(G) \leq h_{2n-2}(G)$. Therefore, if Y is a $\mathbb{Q}$-cohomology manifold, then $\dim G = 0$ or $\dim G = n-1$.

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Vincenzo Di Gennaro, Università di Roma "Tor Vergata", Dipartimento di Matematica, Via della Ricerca Scientifica, 00133 Roma, Italy.

E-mail address: `digennar@axp.mat.uniroma2.it`

Davide Franco , Università di Napoli "Federico II", Dipartimento di Matematica e Applicazioni "R. Caccioppoli", P.le Tecchio 80, 80125 Napoli, Italy.

E-mail address: `davide.franco@unina.it`

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EULER CHARACTERISTIC RECIPROCITY FOR CHROMATIC, FLOW AND ORDER POLYNOMIALS

TAKAHIRO HASEBE, TOSHINORI MIYATANI, AND MASAHIKO YOSHINAGA

Abstract. The Euler characteristic of a semialgebraic set can be considered as a generalization of the cardinality of a finite set. An advantage of semialgebraic sets is that we can define "negative sets" to be the sets with negative Euler characteristics. Applying this idea to posets, we introduce the notion of semialgebraic posets. Using "negative posets", we establish Stanley's reciprocity theorems for order polynomials at the level of Euler characteristics. We also formulate the Euler characteristic reciprocities for chromatic and flow polynomials.

1. Introduction

Let P be a finite poset. The *order polynomial* $\mathcal{O}^{\leq}(P,t)\in\mathbb{Q}[t]$ and the *strict order polynomial* $\mathcal{O}^{<}(P,t)\in\mathbb{Q}[t]$ are polynomials which satisfy

$$\begin{aligned}\mathcal{O}^{\leq}(P,n)&=\#\operatorname{Hom}^{\leq}(P,[n]),\\ \mathcal{O}^{<}(P,n)&=\#\operatorname{Hom}^{<}(P,[n]),\end{aligned}\tag{1}$$

where $[n]=\{1,\dots,n\}$ with the usual ordering and

$$\operatorname{Hom}^{\leq(<)}(P,[n])=\{f:P\longrightarrow[n]\mid x<y\Longrightarrow f(x)\leq(<)f(y)\}$$

is the set of increasing (resp. strictly increasing) maps.

These two polynomials are related to each other by the following reciprocity theorem proved by Stanley ([10, 11], see also [1, 3, 4] for recent surveys).

$$\mathcal{O}^{<}(P,t)=(-1)^{\#P}\cdot\mathcal{O}^{\leq}(P,-t).\tag{2}$$

By putting $t=n$, the formula (2) can be informally presented as follows.

$$\text{“}\ \#\operatorname{Hom}^{<}(P,[n])=(-1)^{\#P}\cdot\#\operatorname{Hom}^{\leq}(P,[-n]).\ \text{”}\tag{3}$$

It is a natural problem to extend the above reciprocity to homomorphisms between arbitrary (finite) posets P and Q. We may expect a formula of the following type.

$$\text{“}\ \#\operatorname{Hom}^{<}(P,Q)=(-1)^{\#P}\cdot\#\operatorname{Hom}^{\leq}(P,-Q).\ \text{”}\tag{4}$$

Of course this is not a mathematically justified formula. In fact, we do not have the notion of a "negative poset $-Q$."

In [9], Schanuel discussed what "negative sets" should be. A possible answer is that a negative set is nothing but a semialgebraic set which has a negative Euler characteristic (Table 1). For

Finite set	Semialgebraic set
Cardinality	Euler characteristic

Table 1. Negative sets

example, the open simplex

$$\mathring{\sigma}_d = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid 0 < x_1 < \cdots < x_d < 1\}$$

has the Euler characteristic $e(\mathring{\sigma}_d) = (-1)^d$, and the closed simplex

$$\sigma_d = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid 0 \leq x_1 \leq \cdots \leq x_d \leq 1\}$$

has $e(\sigma_d) = 1$. Thus we have the following "reciprocity"

$$e(\mathring{\sigma}_d) = (-1)^d \cdot e(\sigma_d). \tag{5}$$

This formula looks like Stanley's reciprocity (2). This analogy would indicate that (2) could be explained via the computations of Euler characteristic of certain semialgebraic sets.

In this paper, by introducing the notion of *semialgebraic posets*, we settle Euler characteristic reciprocity theorems for poset homomorphisms. Semialgebraic posets also provide a rigorous formulation for the reciprocity (4). A similar idea works also for reciprocities of chromatic and flow polynomials.

Briefly, a semialgebraic poset P is a semialgebraic set with poset structure such that the ordering is defined semialgebraically (see Definition 2.2). Finite posets and the open interval $(0, 1) \subset \mathbb{R}$ are examples of semialgebraic posets. A semialgebraic poset P has the Euler characteristic $e(P) \in \mathbb{Z}$ which is an invariant of semialgebraic structure of P (see §2.1). In particular, if P is a finite poset, then $e(P) = \#P$, and if P is the open interval $(0, 1)$, then $e((0, 1)) = -1$.

The philosophy presented in the literature [9] leads one to consider the "moduli space" $\mathrm{Hom}^{\leq(<)}(P, Q)$ of poset homomorphisms from a finite poset P to a semialgebraic poset Q, and then to compute the Euler characteristic of the moduli space instead of counting the number of maps.

Considering the space $\mathrm{Hom}^{\leq(<)}(P, Q)$ itself and its Euler characteristic is not a new idea for the chromatic theory of finite graphs. For example, in [8], the Euler characteristic of the space of colorings is explored, and in [14] the functorial aspects of colorings are studied. The essential reasons why the Euler characteristic works well in these situations are its additivity properties and its consistency with the inclusion-exclusion principle.

The point of the present paper is to introduce the negative of a poset Q in the category of semialgebraic posets. We define $-Q := Q \times (0, 1)$ (See Definition 3.1). Then we have $e(-Q) = -e(Q)$. Furthermore, we have the following result.

Theorem 1.1 (Proposition 2.8 and Theorems 3.3, 3.7)**.** *Let P be a finite poset and Q be a semialgebraic poset.*

(i) $\mathrm{Hom}^{\leq}(P, Q)$ *and* $\mathrm{Hom}^{<}(P, Q)$ *possess the structure of semialgebraic sets.*

(ii) *The following reciprocity of Euler characteristics holds,*

$$e(\mathrm{Hom}^{<}(P, \pm Q)) = (-1)^{\#P} \cdot e(\mathrm{Hom}^{\leq}(P, \mp Q)).$$

(iii) *Let T be a semialgebraic totally ordered set. Then*

$$e(\mathrm{Hom}^{\leq}(P, T)) = \mathcal{O}^{\leq}(P, e(T)),$$
$$e(\mathrm{Hom}^{<}(P, T)) = \mathcal{O}^{<}(P, e(T)).$$

The most important result is the second assertion (ii) which is a rigorous formulation of the reciprocity (4). It should be emphasized that (ii) is a substantially new result since Q need not be a totally ordered set. When we specialize to the totally ordered sets $Q = [n]$ and $T = [n] \times (0, 1)$, our (ii) and (iii) recover Stanley's reciprocity (2) for order polynomials (see §3.3).

Similar Euler characteristic reciprocities are obtained also for Stanley's chromatic polynomials reciprocity [12] and for Breuer and Sanyal's flow polynomials reciprocity [6].

This paper is organized as follows. In §2, we introduce semialgebraic posets, semialgebraic abelian groups and Euler characteristics. In §3, we prove the main result, Theorem 1.1 (ii). The proof is based on topological (cut and paste) arguments. We also deduce Stanley's reciprocity (2) from the main theorem. In §4, we describe other Euler characteristic reciprocities for chromatic polynomials of simple graphs and flow polynomials of oriented graphs.

2. Semialgebraic posets and Euler characteristics

2.1. **Semialgebraic sets.** A subset $X \subset \mathbb{R}^n$ is said to be a semialgebraic set if it is expressed as a Boolean connection (i.e., a set expressed by a finite combination of $\cup, \cap$ and complements) of subsets of the form

$$\{x \in \mathbb{R}^n \mid p(x) > 0\},$$

where $p(x) \in \mathbb{R}[x_1, \dots, x_n]$ is a polynomial. Let $f\colon X \longrightarrow Y$ be a map (not necessarily continuous) between semialgebraic sets $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$. It is called semialgebraic if the graph

$$\Gamma(f) = \{(x, f(x)) \mid x \in X\} \subset \mathbb{R}^{m+n}$$

is a semialgebraic set. If f is semialgebraic then the pull-back $f^{-1}(Y)$ and the image $f(X)$ are also semialgebraic sets (see [2, 5] for details).

Any semialgebraic set X has a finite partition into Nash cells (see [7] for details), namely, a partition $X = \bigsqcup_{\alpha=1}^{k} X_\alpha$ such that X_α is Nash diffeomorphic (that is a semialgebraic analytic diffeomorphism) to the open cell $(0,1)^{d_\alpha}$ for some $d_\alpha \geq 0$. Then the *Euler characteristic*

$$e(X) := \sum_{\alpha=1}^{k} (-1)^{d_\alpha} \tag{6}$$

is independent of the partition [7]. Moreover, the Euler characteristic satisfies

$$\begin{aligned} e(X \sqcup Y) &= e(X) + e(Y), \\ e(X \times Y) &= e(X) \times e(Y). \end{aligned}$$

Example 2.1. As mentioned in §1, the closed simplex σ_d and the open simplex $\mathring{\sigma}_d$ have $e(\sigma_d) = 1$ and $e(\mathring{\sigma}_d) = (-1)^d$.

2.2. **Semialgebraic posets.**

Definition 2.2. $(P, \leq)$ is called a *semialgebraic poset* if

(a) $(P, \leq)$ is a partially ordered set, and
(b) there is an injection $i\colon P \hookrightarrow \mathbb{R}^n$ $(n \geq 0)$ such that the image $i(P)$ is a semialgebraic set and the image of
$$\{(x, y) \in P \times P \mid x \leq y\}$$
by the map $i \times i\colon P \times P \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$, is also a semialgebraic subset of $\mathbb{R}^n \times \mathbb{R}^n$.

Let P and Q be semialgebraic posets. The set of homomorphisms (strict homomorphisms) of semialgebraic posets is defined by

$$\mathrm{Hom}^{\leq(<)}(P, Q) = \left\{ f\colon P \longrightarrow Q \;\middle|\; \begin{array}{l} f \text{ is a semialgebraic map s.t.} \\ x < y \Longrightarrow f(x) \leq (<) f(y) \end{array} \right\}. \tag{7}$$

Example 2.3. (a) A finite poset $(P, \leq)$ admits the structure of a semialgebraic poset, since any finite subset in $\mathbb{R}^n$ is a semialgebraic set. A finite poset has the Euler characteristic $e(P) = \#P$.

(b) The open interval $(0,1)$ and the closed interval $[0,1]$ are semialgebraic posets with respect to the usual ordering induced from $\mathbb{R}$. Their Euler characteristics are $e((0,1)) = -1$ and $e([0,1]) = 1$, respectively.

In this paper, we always consider the following lexicographic ordering on the product $P \times Q$.

Definition 2.4. Let P and Q be posets. Define an ordering on $P \times Q$ by

$$(p_1, q_1) \le (p_2, q_2) \Longleftrightarrow \begin{cases} p_1 < p_2, \text{ or} \\ p_1 = p_2 \text{ and } q_1 \le q_2, \end{cases}$$

for $(p_i, q_i) \in P \times Q$.

Remark 2.5. There are several ways to define poset structures on the product $P \times Q$. However, the lexicographic ordering in Definition 2.4 seems to be the only one that works for our purposes. In particular, the decomposition (18) in §3.2 is crucial.

Proposition 2.6. Let P and Q be semialgebraic posets. Then the product poset $P \times Q$ (with lexicographic ordering) admits the structure of a semialgebraic poset.

Proof. Suppose $P \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^m$. Then

$$\begin{aligned} &\{((p_1, q_1), (p_2, q_2)) \in (P \times Q)^2 \mid (p_1, q_1) \le (p_2, q_2)\} \\ &= \{(p_1, q_1, p_2, q_2) \in (P \times Q)^2 \mid (p_1 < p_2) \text{ or } (p_1 = p_2 \text{ and } q_1 \le q_2)\} \\ &\simeq \big(\{(p_1, p_2) \in P^2 \mid p_1 < p_2\} \times Q^2\big) \sqcup \big(P \times \{(q_1, q_2) \in Q^2 \mid q_1 \le q_2\}\big) \end{aligned}$$

is also semialgebraic since semialgebraicity is preserved by disjoint union, complement and Cartesian products. $\square$

Proposition 2.7. Let P and Q be semialgebraic posets. Then the projection onto the first factor $\pi\colon P \times Q \longrightarrow P$ is a homomorphism of semialgebraic posets.

Proof. This is straightforward from the definition of the lexicographic ordering. $\square$

The next result shows that the "moduli space" of homomorphisms from a finite poset to a semialgebraic poset has the structure of a semialgebraic set.

Proposition 2.8 (Theorem 1.1 (i))**.** Let P be a finite poset and Q be a semialgebraic poset. Then $\mathrm{Hom}^{\le}(P, Q)$ and $\mathrm{Hom}^{<}(P, Q)$ have structures of semialgebraic sets.

Proof. Let us set $P = \{p_1, \dots, p_n\}$ and $\mathcal{L} = \{(i,j) \mid p_i < p_j\}$. Since each element $f \in \mathrm{Hom}^{\le}(P, Q)$ can be identified with the tuple $(f(p_1), \dots, f(p_n)) \in Q^n$, we have the expression

$$\begin{aligned} \mathrm{Hom}^{\le}(P, Q) &\simeq \{(q_1, \dots, q_n) \in Q^n \mid q_i \le q_j \text{ for } (i,j) \in \mathcal{L}\} \\ &= \bigcap_{(i,j) \in \mathcal{L}} \{(q_1, \dots, q_n) \in Q^n \mid q_i \le q_j\}. \end{aligned}$$

Clearly, the right-hand side is a semialgebraic set.

The semialgebraicity of $\mathrm{Hom}^{<}(P, Q)$ is proved similarly. $\square$

2.3. **Semialgebraic abelian groups.** An abelian group $(\mathcal{A}, +)$ is called a semialgebraic abelian group if there exists an injection $i\colon \mathcal{A} \hookrightarrow \mathbb{R}^n$ ($n \geq 0$) such that the image $i(\mathcal{A})$ is a semialgebraic set and the maps

$$+\colon i(\mathcal{A}) \times i(\mathcal{A}) \longrightarrow i(\mathcal{A}),\ (i(x), i(y)) \longmapsto i(x+y)$$
$$(-1)\colon i(\mathcal{A}) \longrightarrow i(\mathcal{A}),\ i(x) \longmapsto i(-x)$$

are semialgebraic maps. Finite abelian groups and the set of all real numbers $\mathbb{R}$ are semialgebraic abelian groups.

It is easy to see that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are semialgebraic abelian groups, then so is the product $\mathcal{A}_1 \times \mathcal{A}_2$.

3. Euler characteristic reciprocity

3.1. The main result.

Definition 3.1. For a semialgebraic poset Q, let us define the negative by $-Q := Q \times (0,1)$. (Recall that we consider the lexicographic ordering on $-Q$.)

Remark 3.2. Note that since $-(-Q) = (Q \times (0,1)) \times (0,1)$, $-(-Q)$ is not equal to Q.

The main theorem of this paper is the following.

Theorem 3.3 (Theorem 1.1 (ii))**.** *Let P be a finite poset and Q be a semialgebraic poset. Then*

$$e(\mathrm{Hom}^{<}(P, \pm Q)) = (-1)^{\#P} \cdot e(\mathrm{Hom}^{\leq}(P, \mp Q)).$$

In other words,

$$e(\mathrm{Hom}^{<}(P, Q)) = (-1)^{\#P} \cdot e(\mathrm{Hom}^{\leq}(P, Q \times (0,1))) \tag{8}$$

and

$$e(\mathrm{Hom}^{<}(P, Q \times (0,1))) = (-1)^{\#P} \cdot e(\mathrm{Hom}^{\leq}(P, Q)) \tag{9}$$

hold.

Note that since $-(-Q) \neq Q$ (Remark 3.2), two formulas (8) and (9) are not equivalent.

Before the proof of Theorem 3.3, we present an example which illustrates the main idea of the proof.

Example 3.4. Let $P = Q = \{1, 2\}$ with $1 < 2$. Clearly we have

$$\mathrm{Hom}^{<}(P, Q) = \{\mathrm{id}\}.$$

Let us describe $\mathrm{Hom}^{\leq}(P, Q \times (0,1))$. Note that $Q \times (0,1)$ is isomorphic to the semialgebraic totally ordered set $(1, \frac{3}{2}) \sqcup (2, \frac{5}{2})$ by the isomorphism

$$\varphi\colon Q \times (0,1) \longrightarrow \left(1, \frac{3}{2}\right) \sqcup \left(2, \frac{5}{2}\right), (a, t) \longmapsto a + \frac{t}{2}.$$

A homomorphism $f \in \mathrm{Hom}^{\leq}(P, Q \times (0,1))$ is described by the two values $f(1) = (a_1, t_1)$ and $f(2) = (a_2, t_2) \in Q \times (0,1)$. The condition imposed on a_1, a_2, t_1 and t_2 (by the inequality $f(1) \leq f(2)$) is

$$(a_1 < a_2),\ \text{or}\ (a_1 = a_2\ \text{and}\ t_1 \leq t_2),$$

which is equivalent to $a_1 + \frac{t_1}{2} \leq a_2 + \frac{t_2}{2}$. Therefore, the semialgebraic set $\mathrm{Hom}^{\leq}(P, Q \times (0,1))$ can be described as in Figure 1. Each diagonal triangle in Figure 1 has a stratification $\mathring{\sigma}_2 \sqcup \mathring{\sigma}_1$. Therefore the Euler characteristic is $e(\mathring{\sigma}_2 \sqcup \mathring{\sigma}_1) = e(\mathring{\sigma}_2) + e(\mathring{\sigma}_1) = (-1)^2 + (-1)^1 = 0$. On the

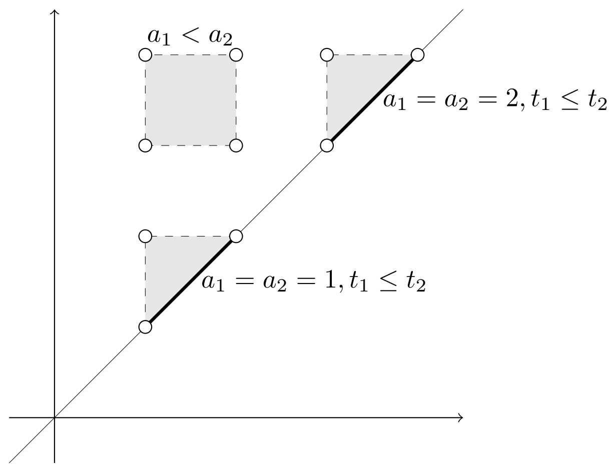

FIGURE 1. $f(1) \leq f(2)$.

other hand, the square region corresponding to $a_1 < a_2$ has the Euler characteristic $(-1)^2 = 1$. Hence we have

$$e(\mathrm{Hom}^{\leq}(P, Q \times (0,1))) = 1 = e(\mathrm{Hom}^{<}(P, Q)).$$

The following lemma will be used in the proof of Theorem 3.3.

Lemma 3.5. *Let $P \subset \mathbb{R}^n$ be a d-dimensional polytope which has a hyperplane description*

$$P = \{\alpha_1 \geq 0\} \cap \cdots \cap \{\alpha_N \geq 0\}$$

of P where α_i are affine maps from $\mathbb{R}^n$ to $\mathbb{R}$ (see [16]). For a given $x_0 \in P$, define the associated locally closed subset P_{x_0} of P (see Figure 2) by

$$P_{x_0} = \bigcap_{\alpha_i(x_0)=0} \{\alpha_i \geq 0\} \cap \bigcap_{\alpha_i(x_0)>0} \{\alpha_i > 0\}.$$

Then the Euler characteristic is

$$e(P_{x_0}) = \begin{cases} (-1)^d, & \text{if } x_0 \in \mathring{P} \\ 0, & \text{otherwise } (x_0 \in \partial P), \end{cases}$$

where $\mathring{P}$ is the relative interior of P and $\partial P = P \smallsetminus \mathring{P}$.

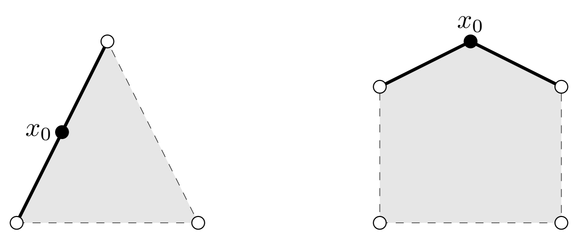

FIGURE 2. P_{x_0}.

Proof. If $x_0 \in \overset{\circ}{P}$, then $P_{x_0} = \overset{\circ}{P}$. The Euler characteristic is $e(\overset{\circ}{P}) = (-1)^d$.

Suppose $x_0 \in \partial P$. Then P_{x_0} can be expressed as

$$P_{x_0} = \bigsqcup_{F \ni x_0} \overset{\circ}{F}, \tag{10}$$

where F runs over the faces of P containing x_0 and $\overset{\circ}{F}$ denotes its relative interior. Then we obtain the decomposition

$$P_{x_0} = \overset{\circ}{P} \sqcup \bigsqcup_{F \ni x_0, F \subset \partial P} \overset{\circ}{F}.$$

We look at the structure of the second component $Z := \bigsqcup_{F \ni x_0, F \subset \partial P} \overset{\circ}{F}$. For any point $y \in Z$, the segment $[x_0, y]$ is contained in Z. Hence Z is contractible open subset of ∂P, which is homeomorphic to the $(d-1)$-dimensional open disk. The Euler characteristic is computed as

$$\begin{aligned} e(P_{x_0}) &= e(\overset{\circ}{P}) + e(Z) \\ &= (-1)^d + (-1)^{d-1} \\ &= 0. \end{aligned}$$

□

3.2. **Proof of the main result.** Now we prove Theorem 3.3. The strategy is to decompose the space $\mathrm{Hom}^{\leq}(P, -Q)$ into appropriate semialgebraic subsets, and then to apply Lemma 3.5 to compute the Euler characteristics.

We first prove (8). Let $\varphi \in \mathrm{Hom}^{<}(P, Q \times (0,1))$. Then φ is a pair of maps

$$\varphi = (f, g),$$

where $f \colon P \longrightarrow Q$ and $g \colon P \longrightarrow (0,1)$. Let $\pi_1 \colon Q \times (0,1) \longrightarrow Q$ be the projection onto the first factor. Since π_1 is order-preserving (Proposition 2.7), so is $f = \pi_1 \circ \varphi$, and hence $f \in \mathrm{Hom}^{\leq}(P, Q)$.

In order to compute the Euler characteristics, we consider the map

$$\pi_{1*} \colon \mathrm{Hom}^{\leq}(P, Q \times (0,1)) \longrightarrow \mathrm{Hom}^{\leq}(P, Q),\ \varphi \longmapsto \pi_1 \circ \varphi = f. \tag{11}$$

Let us set

$$\begin{aligned} M :=& \mathrm{Hom}^{\leq}(P, Q) \smallsetminus \mathrm{Hom}^{<}(P, Q) \\ =& \{f \in \mathrm{Hom}^{\leq}(P, Q) \mid \exists x < y \in P \text{ s.t. } f(x) = f(y)\}. \end{aligned} \tag{12}$$

Then obviously, we have

$$\mathrm{Hom}^{\leq}(P, Q) = \mathrm{Hom}^{<}(P, Q) \sqcup M. \tag{13}$$

This decomposition induces that of $\mathrm{Hom}^{\leq}(P, Q \times (0,1))$,

$$\mathrm{Hom}^{\leq}(P, Q \times (0,1)) = \pi_{1*}^{-1}\left(\mathrm{Hom}^{<}(P, Q)\right) \sqcup \pi_{1*}^{-1}(M). \tag{14}$$

By the additivity of the Euler characteristics, we obtain

$$e\left(\mathrm{Hom}^{\leq}(P, Q \times (0,1))\right) = e\left(\pi_{1*}^{-1}\left(\mathrm{Hom}^{<}(P, Q)\right)\right) + e(\pi_{1*}^{-1}(M)). \tag{15}$$

We claim the following two equalities which are sufficient for the proof of (8).

$$e\left(\pi_{1*}^{-1}\left(\mathrm{Hom}^{<}(P, Q)\right)\right) = (-1)^{\#P} \cdot e\left(\mathrm{Hom}^{<}(P, Q)\right), \tag{16}$$

$$e(\pi_{1*}^{-1}(M)) = 0. \tag{17}$$

We first prove (16). Let $\varphi \in \pi_{1*}^{-1}\left(\mathrm{Hom}^{<}(P,Q)\right)$, that is $\varphi = (f,g)$ with $f \in \mathrm{Hom}^{<}(P,Q)$. By the definition of the ordering of $Q \times (0,1)$, for every $g\colon P \longrightarrow (0,1)$ the pair (f,g) is contained in $\pi_{1*}^{-1}\left(\mathrm{Hom}^{<}(P,Q)\right)$. This implies

$$\pi_{1*}^{-1}\left(\mathrm{Hom}^{<}(P,Q)\right) \simeq \mathrm{Hom}^{<}(P,Q) \times (0,1)^{\#P}, \tag{18}$$

which yields (16).

The proof of (17) requires further stratification of M. Let

$$\mathcal{L}(P) := \{(p_1,p_2) \in P \times P \mid p_1 < p_2\}.$$

For given $f \in M$, consider the set of collapsing pairs,

$$K(f) := \{(p_1,p_2) \in \mathcal{L}(P) \mid f(p_1) = f(p_2)\}.$$

Note that $f \in M$ if and only if $K(f) \neq \emptyset$. We decompose M according to $K(f)$. Namely, for any nonempty subset $X \subset \mathcal{L}(P)$ define a subset $M_X \subset M$ by

$$M_X := \{f \in M \mid K(f) = X\}.$$

Since $\mathcal{L}(P)$ is a finite set,

$$M = \bigsqcup_{\substack{X \subset \mathcal{L}(P) \\ X \neq \emptyset}} M_X \tag{19}$$

is a decomposition of M into finitely many semialgebraic sets. Therefore, we obtain

$$e(\pi_{1*}^{-1}(M)) = \sum_{\substack{X \subset \mathcal{L}(P) \\ X \neq \emptyset}} e(\pi_{1*}^{-1}(M_X)).$$

Thus it is enough to show $e(\pi_{1*}^{-1}(M_X)) = 0$ for all $X \subset \mathcal{L}(P)$ as long as $\pi_{1*}^{-1}(M_X) \neq \emptyset$ (note that $\pi_{1*}^{-1}(M_X) = \emptyset$ can occur for a nonempty X e.g. when $\#Q = 1$).

Now we fix $X \subset \mathcal{L}(P)$ such that $\pi_{1*}^{-1}(M_X) \neq \emptyset$. Then we can show that $\pi_{1*}^{-1}(M_X) \longrightarrow M_X$ is a trivial fibration. Indeed, for any $f \in M_X$, the condition imposed on g by

$$(f,g) \in \mathrm{Hom}^{\leq}(P, Q \times (0,1))$$

is

$$(p_1,p_2) \in X \Longrightarrow g(p_1) \leq g(p_2).$$

Hence the fiber $\pi_{1*}^{-1}(f)$ is independent of $f \in M_X$ and isomorphic to

$$F_X := \{(t_p)_{p \in P} \in (0,1)^P \mid (p_1,p_2) \in X \Longrightarrow t_{p_1} \leq t_{p_2}\}, \tag{20}$$

and we have

$$\pi_{1*}^{-1}(M_X) \simeq M_X \times F_X. \tag{21}$$

The fiber F_X is a locally closed polytope defined by the following inequalities.

$$0 < t_p < 1, t_{p_1} \leq t_{p_2} \text{ for } (p_1,p_2) \in X.$$

The closure $\overline{F_X}$ is defined by

$$\overline{F_X} = \{(t_p)_{p \in P} \in [0,1]^P \mid t_{p_1} \leq t_{p_2} \text{ for } (p_1,p_2) \in X\}.$$

Then F_X is equal to the locally closed polytope $(\overline{F_X})_{x_0}$ associated to the point

$$x_0 = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \in \partial\overline{F_X}.$$

Since $X \neq \emptyset$, x_0 is not contained in the interior of $\overline{F_X}$. By Lemma 3.5, $e(F_X) = 0$. Together with (21), we conclude $e(\pi_{1*}^{-1}(M_X)) = 0$. This completes the proof of (8) of Theorem 3.3.

The proof of the other formula (9) is similar to and actually simpler than that of (8) since we do not need Lemma 3.5. Again the first projection $\pi_1 : Q \times (0,1) \longmapsto Q$ induces the map

$$\pi_{1*} \colon \operatorname{Hom}^{<}(P, Q \times (0,1)) \longrightarrow \operatorname{Hom}^{\leq}(P, Q).$$

We can prove that this map is surjective and each fiber of $\pi_{1*}^{-1}(M_X)$ (now $X = \emptyset$ is allowed) is isomorphic to

$$\mathring{F}_X = \{(t_p)_{p \in P} \in (0,1)^P \mid t_{p_1} < t_{p_2} \text{ for all } (p_1, p_2) \in X\}.$$

This fiber is an open polytope of dimension $\#P$ and hence is isomorphic to $(0,1)^{\#P}$ whose Euler characteristic is $(-1)^{\#P}$. Thus we obtain

$$\begin{aligned} e(\operatorname{Hom}^{<}(P, Q \times (0,1))) &= \sum_{X \subset \mathcal{L}(P)} e(\pi_{1*}^{-1}(M_X)) = \sum_{X \subset \mathcal{L}(P)} e(M_X \times \mathring{F}_X) \\ &= \sum_{X \subset \mathcal{L}(P)} e(M_X) \cdot (-1)^{\#P} = (-1)^{\#P} \cdot e\left(\bigsqcup_{X \subset \mathcal{L}(P)} M_X \right) \\ &= (-1)^{\#P} \cdot e(\operatorname{Hom}^{\leq}(P, Q)). \end{aligned}$$

This completes the proof.

3.3. **Stanley's reciprocity for order polynomials.** In this section, we deduce Stanley's reciprocity (2) from Theorem 3.3. The idea is to take semialgebraic totally ordered posets as the target posets.

Example 3.6. Any semialgebraic set $X \subset \mathbb{R}$ with the induced ordering is a semialgebraic totally ordered set. Furthermore, since $\mathbb{R}^n$ is totally ordered by the lexicographic ordering, any semialgebraic set $X \subset \mathbb{R}^n$ admits the structure of a semialgebraic totally ordered set.

The Euler characteristic of $\operatorname{Hom}^{\leq}(P, T)$, with T a semialgebraic totally ordered set, can be computed by using the order polynomial $\mathcal{O}^{\leq(<)}(P, t)$.

Theorem 3.7 (Theorem 1.1 (iii))**.** *Let P be a finite poset and T be a semialgebraic totally ordered set. Then*

$$e(\operatorname{Hom}^{\leq}(P, T)) = \mathcal{O}^{\leq}(P, e(T)), \tag{22}$$

$$e(\operatorname{Hom}^{<}(P, T)) = \mathcal{O}^{<}(P, e(T)). \tag{23}$$

Before proving Theorem 3.7, we need several lemmas on the Euler characteristics of configuration spaces.

Definition 3.8. Let X be a semialgebraic set. The ordered configuration space of n-points on X, denoted by $C_n(X)$, is defined by

$$C_n(X) = \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ if } i \neq j\}.$$

Lemma 3.9. $e(C_n(X)) = e(X) \cdot (e(X) - 1) \cdots (e(X) - n + 1)$.

Proof. It is proved by induction. When $n = 1$, it is obvious from $C_1(X) = X$. Suppose $n > 1$. Consider the projection

$$\pi \colon C_n(X) \longrightarrow C_{n-1}(X), (x_1, \ldots, x_n) \longmapsto (x_1, \ldots, x_{n-1}).$$

Then the fiber of π at the point $(x_1, \ldots, x_{n-1}) \in C_{n-1}(X)$ is

$$X \smallsetminus \{x_1, \ldots, x_{n-1}\},$$

which has the Euler characteristic

$$e(X \smallsetminus \{x_1, \dots, x_{n-1}\}) = e(X) - (n-1).$$

Therefore, from the inductive assumption, we have

$$\begin{aligned} e(C_n(X)) &= e(C_{n-1}(X)) \cdot (e(X) - n + 1) \\ &= e(X) \cdot (e(X) - 1) \cdots (e(X) - n + 1). \end{aligned}$$

□

Remark 3.10. We will give a stronger result later (Theorem 4.2 and Corollary 4.3).

Lemma 3.11. *Let T be a semialgebraic totally ordered set. Then*

$$e(\mathrm{Hom}^<([n], T)) = \frac{e(T) \cdot (e(T) - 1) \cdots (e(T) - n + 1)}{n!}. \tag{24}$$

Proof. The set

$$\mathrm{Hom}^<([n], T) = \{(x_1, \dots, x_n) \in T^n \mid x_1 < \cdots < x_n\}$$

is obviously a subset of the configuration space $C_n(T)$. Moreover, using the natural action of the symmetric group $\mathfrak{S}_n$ on $C_n(T)$ and the fact that T is totally ordered, we have

$$C_n(T) = \bigsqcup_{\sigma \in \mathfrak{S}_n} \sigma(\mathrm{Hom}^<([n], T)).$$

Since the group action preserves the Euler characteristic, we obtain the following.

$$e(C_n(T)) = n! \cdot e(\mathrm{Hom}^<([n], T)).$$

□

Proof of Theorem 3.7. We fix $\varepsilon \in \{\leq, <\}$. Let $f \in \mathrm{Hom}^\varepsilon(P, T)$. Since P is a finite poset, the image $f(P) \subset T$ is a finite totally ordered set. Suppose $\# f(P) = k$. Then the map f is decomposed as

$$f \colon P \xrightarrow{\alpha} [k] \xrightarrow{\beta} T,$$

where $\alpha \colon P \longrightarrow [k]$ is surjective while $\beta \colon [k] \longrightarrow T$ is injective. Hence β can be considered as an element of $\mathrm{Hom}^<([k], T)$, and we have the following decomposition,

$$\mathrm{Hom}^\varepsilon(P, T) = \bigsqcup_{k \geq 1} \mathrm{Hom}^{\varepsilon,\mathrm{surj}}(P, [k]) \times \mathrm{Hom}^<([k], T), \tag{25}$$

where $\mathrm{Hom}^{\varepsilon,\mathrm{surj}}(P, [k])$ is the set of surjective maps in $\mathrm{Hom}^\varepsilon(P, [k])$. By putting $T = [n]$ and then extending n to real numbers t, we obtain the expression for the (strict) order polynomial,

$$\mathcal{O}^\varepsilon(P, t) = \sum_{k \geq 1} \# \mathrm{Hom}^{\varepsilon,\mathrm{surj}}(P, [k]) \cdot \frac{t(t-1) \cdots (t-k+1)}{k!}, \tag{26}$$

which was already obtained by Stanley [10, Theorem 1]. Using (25), Lemma 3.11 and (26), we have

$$\begin{aligned} e(\mathrm{Hom}^\varepsilon(P, T)) &= \sum_{k \geq 1} e(\mathrm{Hom}^{\varepsilon,\mathrm{surj}}(P, [k])) \cdot e(\mathrm{Hom}^<([k], T)) \\ &= \sum_{k \geq 1} \# \mathrm{Hom}^{\varepsilon,\mathrm{surj}}(P, [k]) \cdot \frac{e(T)(e(T) - 1) \cdots (e(T) - k + 1)}{k!} \\ &= \mathcal{O}^\varepsilon(P, e(T)). \end{aligned}$$

This completes the proof of Theorem 3.7. □

Corollary 3.12. *(Stanley's reciprocity [10]) Let P be a finite poset and $n \in \mathbb{N}$. Then*

$$\# \operatorname{Hom}^{<}(P, [n]) = (-1)^{\# P} \cdot \mathcal{O}^{\leq}(P, -n). \tag{27}$$

Proof. Since $\operatorname{Hom}^{<}(P, [n])$ is a finite poset, the cardinality is equal to the Euler characteristic: $\# \operatorname{Hom}^{<}(P, [n]) = e(\operatorname{Hom}^{<}(P, [n]))$. We apply the Euler characteristic reciprocity (Theorem 3.3),

$$e(\operatorname{Hom}^{<}(P, [n])) = (-1)^{\# P} \cdot e(\operatorname{Hom}^{\leq}(P, [n] \times (0,1))).$$

Note that $[n] \times (0,1)$ is a semialgebraic totally ordered set (with the lexicographic ordering) with the Euler characteristic $e([n] \times (0,1)) = -n$. Applying Theorem 3.7, we have

$$e(\operatorname{Hom}^{\leq}(P, [n] \times (0,1))) = \mathcal{O}^{\leq}(P, -n),$$

which implies (27). □

4. Chromatic and flow polynomials for finite graphs

In this section, we formulate Euler characteristic reciprocities for chromatic polynomials of finite simple graphs and for flow polynomials of finite oriented graphs.

4.1. Chromatic polynomials.

Let $G = (V, E)$ be a finite simple graph with vertex set V and (un-oriented) edge set E. The chromatic polynomial is a polynomial $\chi(G, t) \in \mathbb{Z}[t]$ which satisfies

$$\chi(G, n) = \#\{c \colon V \longrightarrow [n] \mid v_1 v_2 \in E \Longrightarrow c(v_1) \neq c(v_2)\},$$

for all $n > 0$. The chromatic polynomial is also characterized by the following properties:

- if $E = \emptyset$ then $\chi(G, t) = t^{\# V}$;
- if $e \in E$, then $\chi(G, t) = \chi(G - e, t) - \chi(G/e, t)$, where $G - e$ and G/e are the deletion and the contraction with respect to the edge e, respectively.

(See [15] for these terminologies and basic properties of chromatic polynomials.)

Definition 4.1. Given a set X, define the set of vertex coloring with X (or the graph configuration space) by

$$\underline{\chi}(G, X) = \{c \colon V \longrightarrow X \mid v_1 v_2 \in E \Longrightarrow c(v_1) \neq c(v_2)\}. \tag{28}$$

The assignment $X \longmapsto \underline{\chi}(G, X)$ can be considered as a functor [14]. The space $\underline{\chi}(G, X)$ is also called the graph (generalized) configuration space [8].

The chromatic polynomial $\chi(G, t) \in \mathbb{Z}[t]$ satisfies $\chi(G, n) = \# \underline{\chi}(G, [n])$ for all $n \in \mathbb{N}$.

In this section, we investigate the Euler characteristic aspects of the chromatic polynomial for a finite simple graph.

When X is a semialgebraic set, $\underline{\chi}(G, X)$ is also a semialgebraic set. The following result generalizes [8, Theorem 2], where the result is proved when X is a complex projective space.

Theorem 4.2. *Let $G = (V, E)$ be a finite simple graph and X be a semialgebraic set. Then*

$$e(\underline{\chi}(G, X)) = \chi(G, e(X)). \tag{29}$$

Proof. This result is proved by induction on $\# E$. When $E = \emptyset$,

$$e(\underline{\chi}(G, X)) = e(X^{\# V}) = e(X)^{\# V} = \chi(G, e(X)).$$

Suppose $e \in E$. Then we can prove

$$\underline{\chi}(G - e, X) \simeq \underline{\chi}(G, X) \sqcup \underline{\chi}(G/e, X). \tag{30}$$

Using the additivity of the Euler characteristic and the recursive relation for the chromatic polynomial, we obtain (29). □

Note that for the complete graph $G = K_n$, $\underline{\chi}(K_n, X)$ is identical to the configuration space $C_n(X)$ of n-points. Applying Theorem 4.2 to the complete graph K_n (which has the chromatic polynomial $\chi(K_n, t) = t(t-1)\cdots(t-n+1)$), we have the following.

Corollary 4.3. $e(C_n(X)) = e(X)(e(X)-1)\cdots(e(X)-n+1)$.

To formulate the reciprocity for chromatic polynomials, we recall the notion of acyclic orientations on a graph G. (See [3, 12] for details.)

Let $G = (V, E)$ be a finite simple graph. The set of edges E can be considered as a subset of

$$(V \times V \smallsetminus \Delta)/\mathfrak{S}_2,$$

where $\Delta = \{(v, v) \mid v \in V\}$ is the diagonal subset and $\mathfrak{S}_2$ acts on $V \times V$ by transposition. There is a natural projection

$$\pi\colon V \times V \smallsetminus \Delta \longrightarrow (V \times V \smallsetminus \Delta)/\mathfrak{S}_2.$$

An edge orientation on G is a subset $\widetilde{E} \subset V \times V \smallsetminus \Delta$ such that $\pi|_{\widetilde{E}}\colon \widetilde{E} \xrightarrow{\simeq} E$ is a bijection. An orientation $\widetilde{E}$ is said to contain an oriented cycle, if there exists a cyclic sequence $(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1) \in \widetilde{E}$ for some $n > 2$. The orientation $\widetilde{E}$ is called *acyclic* if it does not contain oriented cycles.

Definition 4.4. Let $G = (V, E)$ be a finite simple graph. Fix an acyclic orientation $\widetilde{E} \subset V \times V \smallsetminus \Delta$. Let T be a totally ordered set.

(a) A map $c\colon V \longrightarrow T$ is said to be *compatible* with $\widetilde{E}$ if

$$(v, v') \in \widetilde{E} \Longrightarrow c(v) \leq c(v').$$

(b) A map $c\colon V \longrightarrow T$ is said to be *strictly compatible* with $\widetilde{E}$ if

$$(v, v') \in \widetilde{E} \Longrightarrow c(v) < c(v').$$

We denote the sets of all pairs of an acyclic orientation with a compatible map, and with a strictly compatible map, by

$$\mathcal{AOC}^{\leq}(G, T) := \left\{ (\widetilde{E}, c) \;\middle|\; \begin{array}{l} \widetilde{E} \text{ is an acyclic orientation, and } c\colon V \to T \\ \text{is a map compatible with } \widetilde{E} \end{array} \right\},$$

and

$$\mathcal{AOC}^{<}(G, T) := \left\{ (\widetilde{E}, c) \;\middle|\; \begin{array}{l} \widetilde{E} \text{ is an acyclic orientation, and } c\colon V \to T \\ \text{is a map strictly compatible with } \widetilde{E} \end{array} \right\},$$

respectively.

If T is a semialgebraic totally ordered set, then these spaces possess the structure of semialgebraic sets. We will see a reciprocity between these two spaces from which Stanley's reciprocity for chromatic polynomials is deduced.

It is straightforward that $\mathcal{AOC}^{<}(G, T)$ can be identified with $\underline{\chi}(G, T)$. In particular, we have

$$e(\mathcal{AOC}^{<}(G, T)) = \chi(G, e(T)). \tag{31}$$

We formulate a reciprocity for chromatic polynomials in terms of Euler characteristics.

Theorem 4.5. *Let $G = (V, E)$ be a finite simple graph and T be a semialgebraic totally ordered set. Then*

$$e(\mathcal{AOC}^{\leq}(G, T)) = (-1)^{\#V} \cdot e(\mathcal{AOC}^{<}(G, T \times (0, 1))), \tag{32}$$

$$e(\mathcal{AOC}^{<}(G, T)) = (-1)^{\#V} \cdot e(\mathcal{AOC}^{\leq}(G, T \times (0, 1))). \tag{33}$$

To prove Theorem 4.5, we give alternative descriptions of $\mathcal{AOC}^{\leq(<)}(G,T)$ in terms of poset homomorphisms and graph configuration spaces. Let $\widetilde{E}$ be an acyclic orientation of $G=(V,E)$. Then $\widetilde{E}$ determines an ordering on V, called the transitive closure of $\widetilde{E}$, defined by

$$v < v' \Longleftrightarrow \exists v_0,\dots,v_n \in V \text{ s.t. } \begin{cases} v=v_0, v'=v_n, \text{ and} \\ (v_{i-1},v_i)\in \widetilde{E} \text{ for } 1\leq i\leq n. \end{cases}$$

This ordering defines a poset which we denote by $P(V,\widetilde{E})$.

A map $c\colon V \longrightarrow T$ is compatible with $\widetilde{E}$ if and only if c is an increasing map from $P(V,\widetilde{E})$ to T. Hence the set of maps compatible with $\widetilde{E}$ is identified with $\mathrm{Hom}^{\leq}(P(V,\widetilde{E}),T)$. We have the following decomposition.

$$\mathcal{AOC}^{\leq}(G,T) \simeq \bigsqcup_{\widetilde{E}\text{: acyclic ori.}} \mathrm{Hom}^{\leq}(P(V,\widetilde{E}),T). \tag{34}$$

Similarly, $\mathcal{AOC}^{<}(G,T)$ is decomposed as follows.

$$\mathcal{AOC}^{<}(G,T) \simeq \bigsqcup_{\widetilde{E}\text{: acyclic ori.}} \mathrm{Hom}^{<}(P(V,\widetilde{E}),T). \tag{35}$$

Proof of Theorem 4.5. We prove (32). Using the above decompositions (34) and (35) together with Theorem 3.3, we obtain

$$\begin{aligned} e(\mathcal{AOC}^{\leq}(G,T)) &= e\left(\bigsqcup_{\widetilde{E}\text{: acyclic ori.}} \mathrm{Hom}^{\leq}(P(V,\widetilde{E}),T)\right) \\ &= \sum_{\widetilde{E}\text{: acyclic ori.}} e\left(\mathrm{Hom}^{\leq}(P(V,\widetilde{E}),T)\right) \\ &= (-1)^{\#V}\cdot \sum_{\widetilde{E}\text{: acyclic ori.}} e\left(\mathrm{Hom}^{<}(P(V,\widetilde{E}),T\times(0,1))\right) \\ &= (-1)^{\#V}\cdot e\left(\bigsqcup_{\widetilde{E}\text{: acyclic ori.}} \mathrm{Hom}^{<}(P(V,\widetilde{E}),T\times(0,1))\right) \\ &= (-1)^{\#V}\cdot e(\mathcal{AOC}^{<}(G,T\times(0,1))). \end{aligned}$$

This completes the proof. The second formula (33) is proved similarly. □

We deduce Stanley's reciprocity on chromatic polynomials ([12]). Applying Theorem 4.5 and (31) shows that (note that $T\times(0,1)$ is also a semialgebraic totally ordered set)

$$\begin{aligned} e(\mathcal{AOC}^{\leq}(G,T)) &= (-1)^{\#V}\cdot e(\mathcal{AOC}^{<}(G,T\times(0,1))) \\ &= (-1)^{\#V}\cdot \chi(G,e(T\times(0,1)) \\ &= (-1)^{\#V}\cdot \chi(G,-e(T)). \end{aligned}$$

Putting $T=[n]$, we have the following Stanley's reciprocity.

Corollary 4.6. *Let $G=(V,E)$ be a finite simple graph and $n\in\mathbb{N}$. Then*

$$\#\,\mathcal{AOC}^{\leq}(G,[n]) = (-1)^{\#V}\cdot\chi(G,-n).$$

4.2. **Flow polynomials.** This section treats finite oriented graphs that are allowed to have distinguished multiple edges and loops. Our object is a tuple $G = (V, E, h, t)$ where V and E are finite sets and $h\colon E \longrightarrow V$ and $t\colon E \longrightarrow V$ are maps. An element of V is called a vertex and an element of E is called an edge. For an edge $e \in E$, $h(e) \in V$ is called the head and $t(e) \in V$ is called the tail. An edge $e \in E$ is a loop if $h(e) = t(e)$. In Figure 3, the oriented graph G has five edges $e_1, \dots, e_5$ and their orientations are described by $h(e_1) = h(e_2) = t(e_3) = x$, $t(e_1) = t(e_2) = h(e_3) = h(e_4) = y$ and $t(e_4) = h(e_5) = t(e_5) = z$.

An oriented graph G can also be seen as a 1-dimensional CW-complex. The number of connected components and the 1-st Betti numbers are denoted by $b_0(G)$ and $b_1(G)$, respectively. Note that $b_0(G) - b_1(G) = \#V - \#E$. An edge $e \in E$ is called a coloop if $b_0(G \smallsetminus e) = b_0(G) + 1$. The graph in Figure 3 has the unique coloop e_4.

Let $\mathcal{A}$ be an abelian group. The map $f\colon E \longrightarrow \mathcal{A}$ is called an $\mathcal{A}$-flow if f satisfies

$$\sum_{e:h(e)=v} f(e) = \sum_{e:t(e)=v} f(e) \tag{36}$$

for all $v \in V$ (see [6, 15] more on the notion of flow and flow polynomials). Let f be an $\mathcal{A}$-flow. Denote $\mathrm{Supp}(f) = \{e \in E \mid f(e) \neq 0\}$. An $\mathcal{A}$-flow is called nowhere zero if $\mathrm{Supp}(f) = E$. The set of all $\mathcal{A}$-flows and nowhere zero $\mathcal{A}$-flows are denoted by $\mathcal{F}(G, \mathcal{A})$ and $\mathcal{F}^0(G, \mathcal{A})$, respectively.

Let $\mathcal{A}$ be a semialgebraic abelian group. Then clearly $\mathcal{F}^0(G, \mathcal{A})$ possesses a structure of a semialgebraic set.

The flow polynomial is a polynomial $\phi_G(t) \in \mathbb{Z}[t]$ which satisfies

$$\phi_G(k) = \#\mathcal{F}^0(G, \mathbb{Z}/k\mathbb{Z}),$$

for all $k > 0$. The flow polynomial is also characterized by the following properties:

- if $E = \emptyset$, then $\phi_G(t) = 1$;
- if $e \in E$ is a loop, then $\phi_G(t) = (t-1)\phi_{G \smallsetminus e}(t)$;
- if $e \in E$ is a coloop, then $\phi_G(t) = 0$;
- if $e \in E$ is neither a loop nor a coloop, then $\phi_G(t) = \phi_{G/e}(t) - \phi_{G \smallsetminus e}(t)$.

Proposition 4.7. Let G be a finite oriented graph, and $\mathcal{A}$ be a semialgebraic abelian group.

(a) If $e \in E$ is a loop, then $\mathcal{F}^0(G, \mathcal{A}) \simeq (\mathcal{A} \smallsetminus \{0\}) \times \mathcal{F}^0(G \smallsetminus e, \mathcal{A})$.
(b) If $e \in E$ is a coloop, then $\mathcal{F}^0(G, \mathcal{A}) = \emptyset$.
(c) If $e \in E$ is neither a loop nor a coloop, then $\mathcal{F}^0(G/e, \mathcal{A}) \simeq \mathcal{F}^0(G, \mathcal{A}) \sqcup \mathcal{F}^0(G \smallsetminus e, \mathcal{A})$.

Proof. Straightforward. □

Theorem 4.8. *Let G be a finite oriented graph and $\mathcal{A}$ be a semialgebraic abelian group. Then* $e(\mathcal{F}^0(G, \mathcal{A})) = \phi_G(e(\mathcal{A}))$.

Proof. Using Proposition 4.7, it is proved by induction on the number of edges. (See Theorem 4.2.) □

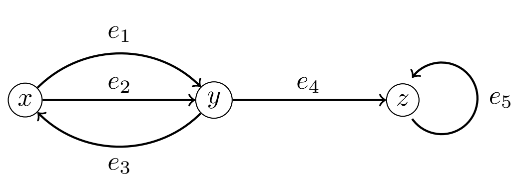

FIGURE 3. An oriented graph.

An oriented graph G is called totally cyclic if every edge is contained in an oriented cycle. Let $\sigma \subset E$ be a subset of edges and denote by ${}_\sigma G$ the reorientation of G along σ. A subset $\sigma \subset E$ is a totally cyclic reorientation if ${}_\sigma G$ is totally cyclic.

Let us denote by $\mathcal{FTC}(G, \mathcal{A})$ the set of all pairs (f, σ) of the flow f and totally cyclic reorientation $\sigma \subset E \smallsetminus \mathrm{Supp}(f)$. Namely,

$$\mathcal{FTC}(G, \mathcal{A}) = \left\{ (f, \sigma) \,\middle|\, \begin{array}{l} f \in \mathcal{F}(G, \mathcal{A}), \text{ and } \sigma \subset E \smallsetminus \mathrm{Supp}(f) \text{ is a} \\ \text{totally cyclic reorientation for } G_{/\,\mathrm{Supp}(f)} \end{array} \right\}.$$

For each subset $\sigma \subset E$, the set of all f with $(f, \sigma) \in \mathcal{FTC}(G, \mathcal{A})$ forms a semialgebraic subset of $\mathcal{F}(G, \mathcal{A})$. Therefore $\mathcal{FTC}(G, \mathcal{A})$ possesses a structure of semialgebraic set. Let us define $-\mathcal{A}$ by

$$-\mathcal{A} := \mathcal{A} \times \mathbb{R}.$$

The following is proved along the same lines of the proof presented in [6, Appendix A], which can be considered as a Breuer-Sanyal's reciprocity at the level of Euler characteristic.

Theorem 4.9. *Let G be a finite oriented graph and $\mathcal{A}$ be a semialgebraic abelian group. Then*

$$e(\mathcal{FTC}(G, \pm\mathcal{A})) = (-1)^{b_1(G)} e(\mathcal{F}^0(G, \mp\mathcal{A})).$$

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Department of Mathematics, Hokkaido University, North 10, West 8, Kita-ku, Sapporo 060-0810, JAPAN

E-mail address: thasebe@math.sci.hokudai.ac.jp

E-mail address: miyatani@math.sci.hokudai.ac.jp

E-mail address: yoshinaga@math.sci.hokudai.ac.jp

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