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SMOOTH ARCS ON ALGEBRAIC VARIETIES

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Abstract. Let k be a field and V be a k-variety. We say that a rational arc $\gamma \in \mathscr{L}_\infty(V)(k)$ is *smooth* if its formal neighborhood $\mathscr{L}_\infty(V)_\gamma$ is an infinite-dimensional formal disk. In this article, we prove that every rational arc $\gamma \in (\mathscr{L}_\infty(V) \setminus \mathscr{L}_\infty(V_{\mathrm{sing}}))(k)$ is smooth if and only if the formal branch containing γ is smooth.

1. Introduction

1.1. The present article is partly motivated by the exegesis of the following statement with respect to singularity theory. This result was obtained by M. Grinberg and D. Kazhdan in case the base field k is contained in $\mathbf{C}$, and by V. Drinfeld for an arbitrary field k (see [8, 6], or [4] for a generalization of such a statement in the context of formal geometry).

Theorem 1.2. *Let k be a field. Let V be a k-variety, and $v \in V(k)$ be a rational point of V. We assume that $\dim_v(V) \geq 1$. Let $\gamma \in \mathscr{L}_\infty(V)(k)$ be a rational point of the associated arc scheme, not contained in $\mathscr{L}_\infty(V_{\mathrm{sing}})$ such that $\gamma(0) = v$. If $\mathscr{L}_\infty(V)_\gamma$ denotes the formal neighborhood of the k-scheme $\mathscr{L}_\infty(V)$ at the point γ, there exists an affine k-scheme S of finite type, with $s \in S(k)$, and an isomorphism of formal k-schemes:*

$$\mathscr{L}_\infty(V)_\gamma \cong S_s \hat{\times}_k \mathrm{Spf}(k[[(T_i)_{i\in\mathbf{N}}]]). \tag{1.1}$$

1.3. Since the work of J. Nash, which introduced the so-called *Nash problem*, one knows that the geometry of $\mathscr{L}_\infty(V)$ is deeply related to the geometry of the singularities of V. As an illustration of this general principle at the level of formal neighborhoods, let us mention the following easy and well-known fact: for every rational arc $\gamma \in \mathscr{L}_\infty(V)(k)$, with origin $v := \gamma(0)$ contained in the smooth locus of V, the formal neighborhood $\mathscr{L}_\infty(V)_\gamma$ is isomorphic to the infinite-dimensional k-formal disk $\mathbf{D}_k^{\mathbf{N}} := \mathrm{Spf}(k[[(T_i)_{i\in\mathbf{N}}]])$. If we translate this remark in the terms of theorem 1.2, it means that, in this case, S can be chosen equal to $\mathrm{Spec}(k)$. In fact, we observe that, in this case, the corresponding algebra $\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}$ is formally smooth over k for the discrete topology. Indeed, one may assume that V is affine and smooth and that there is an étale map $V \to \mathbf{A}^d$. By [14, Lemme 3.3.6] one then has $\mathscr{L}_\infty(V) \cong V \times_{\mathbf{A}^d_k} \mathscr{L}_\infty(\mathbf{A}^d_k)$ thus by [9, Chapter 0, 19.3.3, 19.3.5 (ii)] the k-algebra $\mathcal{O}(\mathscr{L}_\infty(V)) \cong \mathcal{O}(V)[(T_i)_{i\in\mathbf{N}}]$ is formally smooth for the discrete topology. By [9, Chapter 0, 19.3.5 (iv)], this is also the case for $\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}$. In this general context, we address the following natural question:

Question 1.4. *Does the converse property hold true? In other words, if $S = \mathrm{Spec}(k)$ in theorem 1.2, is it true that $\gamma(0)$ is a smooth point of V?*

With respect to theorem 1.2, a positive answer in the direction of question 1.4 clearly indicates that the formal k-scheme S_s in the Drinfeld-Grinberg-Kazhdan theorem would be a measure of the singularities of V at the origin $\gamma(0)$ of the involved arc γ. Since the authors proved in [3] that, in general, theorem 1.2 does not hold if the involved arc γ belongs to $\mathscr{L}_\infty(V_{\mathrm{sing}})$, it seems natural to us, in this perspective, to restrict ourselves to the case of arcs *not* contained in $\mathscr{L}_\infty(V_{\mathrm{sing}})$, that we call *non-degenerate*.

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1.5. In the present paper, we provide a complete answer to question 1.4 for non-degenerate arcs (which are in particular contained in a unique irreducible component of $\mathrm{Spec}(\widehat{\mathcal{O}_{V,\gamma(0)}})$, by proposition 3.6). Precisely we obtain the following statement:

Theorem 1.6. *Let V be a k-variety and $v \in V(k)$ such that $\mathcal{O}_{V,v}$ is reduced and $\dim_v(V) \geq 1$. Let $\gamma \in \mathscr{L}_\infty(V)(k)$ be a non-degenerate rational arc, such that $\gamma(0) = v$. Then, the following conditions are equivalent:*

(1) *The unique formal branch containing γ is smooth.*
(2) *The formal neighborhood $\mathscr{L}_\infty(V)_\gamma$ is isomorphic to $\mathbf{D}_k^{\mathbf{N}}$.*

Let us note that by [9, Chapter 0, 19.3.6, 19.5.4] the second condition in the statement of theorem 1.6 characterizes those non-degenerate rational arcs γ whose local ring $\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}$ is formally smooth over k for the $\mathfrak{m}$-adic topology. In the case of curves, we are able to interpret the above result in terms of a notion of *rigidity* for deformations of arcs (see corollary 4.14). We also obtain analogs of theorem 1.6 in the case of constant arcs (in particular degenerate) and in the context of jet schemes (see proposition 5.2 and theorem 5.4).

1.7. **Conventions, notation.** In this article, k is a field of arbitrary characteristic (unless explicitly stated otherwise); $k[[T]]$ is the ring of power series over the field k. The category of k-schemes is denoted by $\mathfrak{Sch}_k$. The local k-algebra $k[[(T_i)_{i\in\mathbf{N}}]]$ is the completion of $k[(T_i)_{i\in\mathbf{N}}]$ with respect to the maximal ideal $\langle (T_i)_{i\in\mathbf{N}}\rangle$. It is a topological complete k-algebra when we endow it with the projective limit topology. We denote by $\mathbf{D}_k^{\mathbf{N}} := \mathrm{Spf}(k[[(T_i)_{i\in\mathbf{N}}]])$ the associated formal k-scheme. A *k-variety* is a k-scheme of finite type. The singular locus V_{sing} of V is defined as the (unique) reduced closed subscheme associated with the non-smooth locus of V. An arc of V, i.e., a point of the arc scheme $\mathscr{L}_\infty(V)$ associated with V, which is *not* contained in the singular locus V_{sing} of V, is called a *non-degenerate* arc. In other words, the subset $\mathscr{L}_\infty(V)\setminus\mathscr{L}_\infty(V_{\mathrm{sing}})$ is the set of non-degenerate arcs. In this article, by slightly abusing the standard conventions, we introduce the terminology of *smooth rational arcs* on V to designate those arcs $\gamma \in \mathscr{L}_\infty(V)(k)$ such that $\mathscr{L}_\infty(V)_\gamma \cong \mathbf{D}_k^{\mathbf{N}}$ (assuming that the dimension at the origin $\gamma(0)$ of the arc is positive).

2. Arc schemes and arc deformations: recollection

2.1. If V is a k-variety and $n \in \mathbf{N}$, the restriction *à la* Weil of the $k[T]/\langle T^{n+1}\rangle$-scheme

$$V \times_k \mathrm{Spec}(k[T]/\langle T^{n+1}\rangle)$$

with respect to the morphism of k-algebras $k \hookrightarrow k[T]/\langle T^{n+1}\rangle$ exists; it is a k-scheme of finite type which is called the *n-jet scheme* of V and that we denote by $\mathscr{L}_n(V)$. The projective limit $\varprojlim_n(\mathscr{L}_n(V))$ exists in the category of k-schemes; it is the *arc scheme* associated with V and we denote it by $\mathscr{L}_\infty(V)$. For every integer $n \in \mathbf{N}$, the canonical morphism of k-schemes $\pi_n^\infty \colon \mathscr{L}_\infty(V) \to \mathscr{L}_n(V)$ is called the *truncation morphism* of level n. Let A be a k-algebra. As proved in [1], there exists a natural bijection

$$\mathrm{Hom}_{\mathfrak{Sch}_k}(\mathrm{Spec}(A), \mathscr{L}_\infty(V)) \cong \mathrm{Hom}_{\mathfrak{Sch}_k}(\mathrm{Spec}(A[[T]]), V). \tag{2.1}$$

Let us note that in case V is affine or A is local, such a property directly follows from the mere definitions.

2.2. We denote by $\mathfrak{Lacp}$ the following category. The objects are the topological local k-algebras, which are topologically isomorphic to $\mathfrak{m}$-adic completions of local k-algebras and whose residue field are k-algebras isomorphic to k. The morphisms in $\mathfrak{Lacp}$ are the continuous morphisms of local k-algebras. We denote by $\mathfrak{Tes}$ the full subcategory of $\mathfrak{Lacp}$ whose objects are *test-rings*, i.e., local k-algebras in $\mathfrak{Lacp}$ with nilpotent maximal ideal and residue field isomorpic to k. If $\widehat{\mathfrak{Tes}}$ is the category of pre-cosheaves on the category $\mathfrak{Tes}$ (i.e., covariant functors from the category $\mathfrak{Tes}$ to the category of sets), we define the functor

$$\begin{array}{rccc} F\colon & \mathfrak{Lacp} & \longrightarrow & \widehat{\mathfrak{Tes}} \\ & \widehat{\mathcal{O}} & \longmapsto & \mathrm{Hom}_{\mathfrak{Lacp}}(\widehat{\mathcal{O}}, \cdot). \end{array}$$

One has the following seemingly standard observation (see [6]):

Observation 2.3. The functor F is fully faithful.

One will use the following trivial consequence of the observation: let S and S' be k-schemes, let $s \in S(k)$ and $s \in S'(k)$, let S_s and $S'_{s'}$ be the associated formal neighborhoods and let $f_A\colon S_s(A) \to S'_{s'}(A)$ be a natural map defined for every test-ring $(A, \mathfrak{m}_A)$; then there exists a unique morphism of formal k-schemes $f\colon S_s \to S'_{s'}$ inducing f_A for every test-ring A; moreover, f is an isomorphism if and only if f_A is bijective for every A.

2.4. Let V be a k-variety. Let $\gamma \in \mathscr{L}_\infty(V)(k)$. Then, in the sense of observation 2.3, the formal k-scheme $\mathscr{L}_\infty(V)_\gamma$ is *uniquely* determined by the functor $F(\widehat{\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}})$. Let A be a test-ring. Let $\gamma_A \in \mathscr{L}_\infty(V)_\gamma(A)$. The datum of γ_A corresponds to one of the following (equivalent) commutative diagram:

$$\begin{tikzcd} \mathcal{O}_{\mathscr{L}_\infty(V),\gamma} \arrow[r, "\gamma_A"] \arrow[d] & A \arrow[d] \\ k \arrow[r, equal] & k \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}_{V,v} \arrow[r, "\gamma_A"] \arrow[d, "\gamma"'] & A[[T]] \arrow[d, "p_A"] \\ k[[T]] \arrow[r, equal] & k[[T]], \end{tikzcd} \qquad \begin{tikzcd} \widehat{\mathcal{O}_{V,v}} \arrow[r, "\gamma_A"] \arrow[d, "\gamma"'] & A[[T]] \arrow[d, "p_A"] \\ k[[T]] \arrow[r, equal] & k[[T]], \end{tikzcd} \tag{2.2}$$

where we denote by $p_A\colon A[[T]] \to k[[T]]$ the unique local morphism which extends the projection $A \to A/\mathfrak{m}_A \cong k$. The set $\mathscr{L}_\infty(V)_\gamma(A)$ parametrizes the elements $\gamma_A \in V(A[[T]])$ whose reduction modulo $\mathfrak{m}_A$ coincides with γ.

Definition 2.5. *Every morphism $\gamma_A \in \mathscr{L}_\infty(V)_\gamma(A)$ is called an* A-deformation *of γ.*

3. Reduction to formal branches

Definition 3.1. *Let V be a k-variety. Let $\gamma \in \mathscr{L}_\infty(V)(k)$ be a rational arc, viewed as a local morphism $\gamma\colon \widehat{\mathcal{O}_{V,\gamma(0)}} \to k[[T]]$. A* formal branch *(or* formal component*) at $\gamma(0)$ which contains γ is a minimal prime ideal $\mathfrak{p}$ of $\widehat{\mathcal{O}_{V,\gamma(0)}}$ such that $\mathfrak{p} \subset \mathrm{Ker}(\gamma)$.*

In particular, if $\mathfrak{p}$ is such a branch, this definition implies that γ factorizes through the quotient morphism $\widehat{\mathcal{O}_{V,\gamma(0)}} \to \widehat{\mathcal{O}_{V,\gamma(0)}}/\mathfrak{p}$. A classical fact on arc geometry is that every arc on a reduced variety factorizes through the irreducible components of the involved variety which contain the origin of the arc. In the same spirit, the following lemma shows in particular that the formal neighborhood of a given arc contained in a unique formal branch of a reduced variety carries a part of the information on the mere singularities of the formal branch containing the arc.

Proposition 3.2. *Let V be a k-variety. Let $\gamma \in \mathscr{L}_\infty(V)(k)$ be a rational arc contained in a unique formal branch $\mathfrak{p}$. We assume that $\mathcal{O}_{V,\gamma(0)}$ is reduced. Then, for every test-ring $(A, \mathfrak{m}_A)$, for every A-deformation $\gamma_A \in \mathscr{L}_\infty(V)_\gamma(A)$ of γ, the induced morphism of admissible local k-algebras $\gamma_A\colon \widehat{\mathcal{O}_{V,\gamma(0)}} \to A[[T]]$ factorizes through $\widehat{\mathcal{O}_{V,\gamma(0)}} \to \widehat{\mathcal{O}_{V,\gamma(0)}}/\mathfrak{p}$. Besides, the ideal $\mathfrak{p}$ is the only minimal prime ideal with this property.*

In other words, if the arc γ is contained in a unique formal branch at $\gamma(0)$, then every A-deformation of γ is contained in the same branch (and only in this one).

Proof. Let $(A, \mathfrak{m}_A)$ be a test-ring and $\gamma_A \in \mathscr{L}_\infty(V)_\gamma(A)$, corresponding to a diagram of morphisms of complete local k-algebras:

$$\begin{tikzcd} \widehat{\mathcal{O}_{V,\gamma(0)}} \arrow[r, "\gamma_A"] \arrow[dr, "\gamma"'] & A[[T]] \arrow[d] \\ & k[[T]]. \end{tikzcd} \tag{3.1}$$

Then, we have $\mathrm{Ker}(\gamma) = \gamma_A^{-1}(\mathfrak{m}_A[[T]])$. Let $\mathfrak{p}, \mathfrak{q}_1, \dots, \mathfrak{q}_n$ be the minimal prime ideals of the ring $\widehat{\mathcal{O}_{V,\gamma(0)}}$. By assumptions, $\mathrm{Ker}(\gamma)$ contains $\mathfrak{p}$ and does not contain $\mathfrak{q}_i$ for every $i \in \{1, \dots, n\}$. Let us prove that $\mathfrak{p} \subset \mathrm{Ker}(\gamma_A)$.

Let $x \in \mathfrak{p}$. Since the ring $\widehat{\mathcal{O}_{V,\gamma(0)}}$ is reduced, we have $\mathfrak{p} \cap (\cap_{i=1}^n \mathfrak{q}_i) = \langle 0 \rangle$. By assumption, for every integer $i \in \{1, \dots, n\}$, there exists an element $y_i \in \mathfrak{q}_i$ such that $y_i \notin \mathrm{Ker}(\gamma)$. Then, we deduce that $xy_1 \dots y_n = 0$ and that

$$\begin{array}{rcl} \gamma_A(xy_1 \dots y_n) & = & 0 \\ \gamma_A(x) \cdot \gamma_A(y_1) \dots \gamma_A(y_n) & = & 0. \end{array} \tag{3.2}$$

Since, by construction, $y_i \notin \gamma_A^{-1}(\mathfrak{m}_A[[T]])$ for every integer $i \in \{1, \dots, n\}$, we conclude that the element $\gamma_A(y_i)$ does not reduce to zero modulo $\mathfrak{m}_A[[T]]$. In particular (see lemma 3.3), the element $\gamma_A(y_i)$ is not a zero-divisor in the ring $A[[T]]$; hence, by equation (3.2), we have $\gamma_A(x) = 0$, i.e., $x \in \mathrm{Ker}(\gamma_A)$.

In the end, if there exists $i \in \{1, \dots, n\}$ such that $\mathfrak{q}_i \subset \gamma_A^{-1}(0) = \mathrm{Ker}(\gamma_A)$, then we have $\mathfrak{q}_i \subset \gamma_A^{-1}(\mathfrak{m}_A[[T]]) = \mathrm{Ker}(\gamma)$, which contradicts our assumption. It concludes the proof of our statement. ☐

Lemma 3.3. *Let $(A, \mathfrak{m}_A)$ be a test-ring, let $r_A(T) \in A[[T]]$ whose reduction modulo $\mathfrak{m}_A[[T]]$ is a non-zero element of $k[[T]]$. Then, the power series $r_A(T)$ is not a zero-divisor in $A[[T]]$.*

Proof. By the Weierstrass preparation theorem (see [12, Chapter IV, Theorem 9.2]), there is a decompostion $r_A(T) = q_A(T)u_A(T)$ where $q_A(T)$ is a distinguished polynomial and $u_A(T)$ is invertible in $A[[T]]$. By the uniqueness in the Weierstrass division theorem, (see [12, Chapter IV, Theorems 9.1 and 9.2]) a distinguished polynomial is not a zero-divisor in $A[[T]]$. ☐

Remark 3.4. In particular, under the assumptions of proposition 3.2 with V reduced, the arc γ is contained in a unique irreducible component passing through $\gamma(0)$, and every A-deformation of γ is contained in this irreducible component.

Remark 3.5. If one does not assume that the arc γ belongs to a unique formal branch, and $\dim(\mathcal{O}_{V,v}) \geq 2$, it is important to keep in mind that the situation is much more complicated and proposition 3.2 does not hold anymore. Let us consider the example of the affine k-surface

$$V = \mathrm{Spec}(k[X, Y, Z]/\langle Y^2 - X^3 - X \rangle).$$

It is an integral k-variety and $\widehat{\mathcal{O}_{V,\mathfrak{o}}} \simeq k[[U, V, W]]/\langle UV \rangle$, where we denote by $\mathfrak{o}$ the origin of $\mathbf{A}_k^3$. Let $A = k[S]/\langle S^2 \rangle$ and $s := \bar{S}$. We observe that the arc γ, defined by $U \mapsto 0, V \mapsto 0, W \mapsto T$, admits the A-deformation $\gamma_A(T) = (s, s, T)$, which is not contained in any formal branch of V at the origin $\mathfrak{o}$.

Proposition 3.6. *Let V be a k-variety. Let $\gamma \in \mathscr{L}_\infty(V)(k)$ be a rational arc. If the arc γ is non-degenerate, then the arc γ is contained in a unique formal branch.*

Proof. In $\mathrm{Spec}(k[[T]])$, we denote by 0 the closed point, and η the generic point. Let us note that the arc γ is non-degenerate if and only if the point $\gamma(\eta)$ does not belong to V_{sing}. Up to shrinking V, we may assume that the k-variety V is affine and reduced. We also may assume that $\dim(\mathcal{O}_{V,\gamma(0)}) \geq 1$. The arc γ corresponds a morphism of local k-algebras $\gamma\colon \mathcal{O}_{V,\gamma(0)} \to k[[T]]$ which extends to a morphism of local k-algebras $\widehat{\gamma}\colon \widehat{\mathcal{O}_{V,\gamma(0)}} \to k[[T]]$. We denote by $\mathfrak{M}$ the maximal ideal of $\mathcal{O}(V)$ corresponding to $\gamma(0)$. First assume that $\mathrm{Ker}(\gamma)$ contains at least two distinct minimal prime ideals of $\mathcal{O}_{V,\gamma(0)}$; in more geometric terms, that γ lies on at least two distinct irreducible components passing through $\gamma(0)$. Then $(\mathcal{O}_{V,\gamma(0)})_{\mathrm{Ker}(\gamma)} \cong \mathcal{O}_{V,\gamma(\eta)}$ is not a domain, thus $\gamma(\eta)$ is not a smooth point of V and $\gamma \in \mathscr{L}_\infty(V_{\mathrm{sing}})$.

Now consider the general case. Let $\mathcal{O}^h_{V,\gamma(0)}$ be the henselization of $\mathcal{O}_{V,\gamma(0)}$. One has

$$\mathcal{O}^h_{V,\gamma(0)} = \varinjlim B_{\mathfrak{q}},$$

where the limit is taken over all étale $\mathcal{O}(V)$-algebras B localized at a prime $\mathfrak{q}$ such that

$$\mathfrak{q} \cap \mathcal{O}(V) = \mathfrak{M} \quad \text{and} \quad \kappa(\mathfrak{q}) = \kappa(\mathfrak{M}).$$

By [16, Tag 0CB3], one may find such a $(B, \mathfrak{q})$ such that the morphism $B_{\mathfrak{q}} \to \mathcal{O}^h_{V,\gamma(0)}$ induces a bijection on the level of minimal prime ideals. On the other hand, by [16, Tag 0C2E], the morphism $\mathcal{O}^h_{V,\gamma(0)} \to \widehat{\mathcal{O}_{V,\gamma(0)}}$ also induces a bijection on the level of minimal prime ideals. Let $\gamma_B \colon B \to k[[T]]$ (resp. $\gamma_{B_{\mathfrak{q}}} \colon B_{\mathfrak{q}} \to k[[T]]$) be the morphism induced by $\widehat{\gamma}$. Assuming that $\operatorname{Ker}(\widehat{\gamma})$ contains at least two distinct minimal prime ideals, we deduce that the same holds for $\operatorname{Ker}(\gamma_{B_{\mathfrak{q}}})$. By the particular case treated above, one infers that $B_{\operatorname{Ker}(\gamma_B)}$ is not a domain, in particular $\operatorname{Ker}(\gamma_B) = \gamma_B(\eta)$ is not a smooth point of $\operatorname{Spec}(B)$. Since $\operatorname{Spec}(B) \to V$ is étale and maps $\gamma_B(\eta)$ to $\gamma(\eta)$, the point $\gamma(\eta)$ is not a smooth point of V by [10, Chapitre 4, 17.11.1]. □

4. The proof of theorem 1.6

Let $\mathfrak{p}$ be the unique formal branch containing γ, $v := \gamma(0)$ and $\widehat{\mathcal{O}_{\mathfrak{p},v}} := \widehat{\mathcal{O}_{V,v}}/\mathfrak{p}$ be the corresponding local ring.

4.1. Let us show first 1 ⇒ 2. This implication is a direct consequence of the following proposition, which is a corollary of proposition 3.2, and of proposition 3.6.

Proposition 4.2. *Let V be a k-variety and $\gamma \in \mathscr{L}_\infty(V)(k)$ be an arc with $v = \gamma(0)$ which is assumed to be contained in a unique formal branch. We assume that $\mathcal{O}_{V,v}$ is reduced and $\dim_v(V) \geq 1$. Assume that the formal branch $\mathfrak{p}$ containing γ is smooth. Then the formal k-scheme $\mathscr{L}_\infty(V)_\gamma$ is isomorphic to $\mathbf{D}_k^{\mathbf{N}}$.*

Proof. Let $(A, \mathfrak{m}_A)$ be a test-ring. By assumption, there exists an integer $d \geq 1$ such that

$$\widehat{\mathcal{O}_{\mathfrak{p},v}} \xrightarrow{\sim} k[[S_1, \dots, S_d]].$$

By proposition 3.2, the A-deformations of γ are in natural bijection with the set of local morphisms $\widehat{\mathcal{O}_{\mathfrak{p},v}} \to A[[T]]$. This set is itself in natural bijection with $\mathfrak{m}_A^{\mathbf{N}}$. By observation 2.3, the k-formal schemes $\mathscr{L}_\infty(V)_\gamma$ and $\mathbf{D}_k^{\mathbf{N}}$ are isomorphic. □

4.3. We prove now 2 ⇒ 1. We have to show that the k-algebra $\widehat{\mathcal{O}_{\mathfrak{p},v}}$ is isomorphic (in $\mathfrak{Lacp}$) to a k-algebra of power series in a finite number of variables. Our proof is based on different ingredients which are established in subsections 4.4, 4.6; the main arguments are presented in subsection 4.9.

4.4. Let us start by establishing a basic result. Keep the notation of theorem 1.2.

Lemma 4.5. *Let V be a k-variety and $v \in V(k)$ such that $\mathcal{O}_{V,v}$ is reduced and $\dim_v(V) \geq 1$. Let $\gamma \in \mathscr{L}_\infty(V)(k)$ be a non-degenerate rational arc with $\gamma(0) = v$. Assume that the formal neighborhood $\mathscr{L}_\infty(V)_\gamma$ is isomorphic to $\mathbf{D}_k^{\mathbf{N}}$ and that the minimal prime ideal $\mathfrak{p}$ of $\widehat{\mathcal{O}_{V,v}}$ corresponds to the formal branch containing γ. Let $(B, \mathfrak{m}_B)$ be a local ring. Then, every morphism of local k-algebras $\widehat{\mathcal{O}_{\mathfrak{p},v}} \to (B/\mathfrak{m}_B^2)[[T]]$ lifts to a morphism of local k-algebras $\widehat{\mathcal{O}_{\mathfrak{p},v}} \to \widehat{B}[[T]]$.*

Proof. First, since we have $\mathscr{L}_\infty(V)_\gamma \cong \mathbf{D}_k^{\mathbf{N}}$, we observe that, for every *surjective* morphism of test-rings $f \colon A' \to A$, the natural map

$$\mathfrak{m}_{A'}^{\mathbf{N}} \cong \operatorname{Hom}_{\mathfrak{Lacp}}(\widehat{\mathcal{O}_{\mathfrak{p},v}}, A'[[T]]) \xrightarrow{f\circ\cdot} \operatorname{Hom}_{\mathfrak{Lacp}}(\widehat{\mathcal{O}_{\mathfrak{p},v}}, A[[T]]) \cong \mathfrak{m}_A^{\mathbf{N}}$$

is surjective. Hence, starting from a morphism $\varphi_2 \colon \widehat{\mathcal{O}_{\mathfrak{p},v}} \to B/\mathfrak{m}_B^2[[T]]$, we may construct, by induction, a family of morphisms $\varphi_n \colon \widehat{\mathcal{O}_{\mathfrak{p},v}} \to B/\mathfrak{m}_B^{n+2}[[T]]$, for every integer $n \in \mathbf{N}$, which makes, for every pair $(m,n) \in \mathbf{N}^2$ of integers with $m \geq n$, the following diagram of morphisms in $\mathfrak{Lacp}$ commute

$$\begin{tikzcd} \widehat{\mathcal{O}_{\mathfrak{p},v}} \arrow[r, "\varphi_m"] \arrow[d, equal] & B/\mathfrak{m}_B^{2+m} \arrow[d, "\pi"] \\ \widehat{\mathcal{O}_{\mathfrak{p},v}} \arrow[r, "\varphi_n"'] & B/\mathfrak{m}_B^{2+n}, \end{tikzcd}$$

where we denote by π the canonical projection. By the very definition, we have constructed a morphism $\varphi\colon \widehat{\mathcal{O}_{\mathfrak{p},v}} \to \widehat{B}[[T]]$ lifting φ_2. $\square$

For every noetherian local k-algebra B, we have $B/\mathfrak{m}_B^n \cong \widehat{B}/\widehat{\mathfrak{m}}_B^n$ for every integer $n \geq 1$ by [13, §8]. Under this assumption, the arguments developed in the proof of lemma 4.5 imply in particular that the set of liftings of φ_2 can be identified with $k[[(T_i)_{i\in\mathbf{N}}]]$.

4.6. Using the following lemma, we shall, in some sense, reduce the proof of the theorem 1.6 to the case of a complete intersection. This kind of reduction is a classical "trick" in the construction of motivic measures (see [5] or, e.g., [14]).

Lemma 4.7. *Let V be an affine k-variety, defined by the datum of an ideal I_V of the polynomial ring $k[X_1,\dots,X_N]$ and $\gamma \in \mathscr{L}_\infty(V)(k)$ be a non-degenerate arc. Then, there exist an integer $M \in \{0,\dots,N\}$ and elements $F_1,\dots,F_M \in I_V$, such that:*

1. *There exists an $(M\times M)$-minor of the jacobian matrix $(\partial_{X_j}F_i)_{i,j}$ which does not vanish at γ.*
2. *Setting*
$$V' := \operatorname{Spec}(k[X]/\langle F_1,\dots,F_M\rangle),$$
the morphism of formal k-schemes $\mathscr{L}_\infty(V)_\gamma \cong \mathscr{L}_\infty(V')_\gamma$ induced by the closed immersion $V \hookrightarrow V'$ is an isomorphism.

Proof. Let us denote by J_V the ideal generated by the elements $h\delta \in k[X_1,\dots,X_N]$, where δ is an $(M\times M)$-minor of the jacobian matrix of a M-tuple $(F_1,\dots,F_M)$ of elements of I_V, for some integer $M \in \mathbf{N}$, and $h \in (\langle F_1,\dots,F_M\rangle : I_V)$. Using the jacobian criterion, one may show (see [7, §0.2], [17, §4]) that the singular locus V_{sing} of V, i.e., the reduced closed subscheme associated with the non-smooth locus, is the support of the closed subscheme of V associated with the datum of the ideal $I_V + J_V$. Since $\gamma \notin \mathscr{L}_\infty(V_{\mathrm{sing}})(k)$, we obtain all the desired properties, using lemma 4.8 below for the last one. $\square$

Lemma 4.8. *Let V' be an affine k-variety, V be a closed k-subscheme of V' and*
$$h \in (0 : I_V) \subset \mathcal{O}(V').$$
Let $\gamma \in \mathscr{L}_\infty(V)(k)$ such that $h(\gamma) \neq 0$. Then, still denoting by γ the image of γ in V', the natural morphism of formal schemes $\mathscr{L}_\infty(V)_\gamma \to \mathscr{L}_\infty(V')_\gamma$ is an isomorphism of formal k-schemes.

Proof. It suffices to show that for every test-ring A the induced map $\mathscr{L}_\infty(V)_\gamma(A) \to \mathscr{L}_\infty(V')_\gamma(A)$ is bijective. Injectivity is clear; so let us show surjectivity. We pick out $\gamma_A \in \mathscr{L}_\infty(V')_\gamma(A)$ and $G \in I_V$. We have to show that $G(\gamma_A) = 0$. By hypothesis, one has $h(\gamma_A)G(\gamma_A) = 0$. Since $h(\gamma) \neq 0$, the reduction of $h(\gamma_A)$ modulo $\mathfrak{m}_A$ is not zero. By lemma 3.3, one infers that $G(\gamma_A) = 0$. $\square$

4.9. We are ready to complete the proof of theorem 1.6, by proving implication 2 $\Rightarrow$ 1. We may assume that $V \hookrightarrow \mathbf{A}_k^N$ is affine and, thanks to proposition 3.2 and remark 3.4, irreducible. Let M, $F_1, F_2, \dots, F_M$, δ and h be the elements provided by lemma 4.7 and set $d := N - M$. Up to renumbering, we may assume that δ is the determinant of the matrix $(\partial_{X_{d+j}}(F_i))_{i,j\in\{1,\dots,M\}}$ and that $\operatorname{ord}_T(\delta(\gamma(T))$ is minimal among the T-orders of the evaluation at $\gamma(T)$ of the $(M\times M)$-minors of the jacobian matrix $(\partial_{X_j}(F_i))_{\substack{i\in\{1,\dots,M\}\\ j\in\{1,\dots,N\}}}$. Moreover, up to a translation, we may assume that v is the origin of $\mathbf{A}_k^{d+M}$. For every integer $i \in \{1, \ldots, d+M\}$, we will denote by $\widehat{x_i}$ the image of X_i in $\widehat{\mathcal{O}_{\mathfrak{p},v}}$. Note that, since $h\delta$ does not vanish at γ, the element $h\delta$ does not vanish identically on V; hence,we have $\dim(V) = d$.

We shall identify $\gamma(T)$ with a tuple $(x_j(T))_{i\in\{1,\ldots,N\}} \in k[[T]]^{d+M}$ which satisfies, for every integer $i \in \{1, \ldots, M\}$, the equation

$$F_i((x_j(T))_{j\in\{1,\ldots,d+M\}}) = 0.$$

Using the second property of lemma 4.7, for every test-ring $(A, \mathfrak{m}_A)$, an element of $\mathscr{L}_\infty(V)_\gamma(A)$ may and shall be identified with a tuple $(x_{1,A}(T), \ldots, x_{d+M,A}(T))$ of elements of $\mathfrak{m}_A[[T]]^{d+M}$ such that, for every integer $i \in \{1, \ldots, M\}$,

$$F_i((x_j(T) + x_{j,A}(T))_{j\in\{1,\ldots,d+M\}}) = 0.$$

We denote by $A_{d,2}$ the test-ring $k[S_1, \ldots, S_d]/\langle S_1, \ldots, S_d\rangle^2$ and by s_i the image of S_i in $A_{d,2}$. By lemma 4.11, there exists an element $(x_{1,A_{d,2}}(T), \ldots, x_{d+M,A_{d,2}}(T)) \in \mathscr{L}_\infty(V)_\gamma(A_{d,2})$ such that, for every integer $i \in \{1, \ldots, d\}$,

$$x_{i,A_{d,2}}(T) = s_i.$$

By proposition 3.2, there exists a morphism $\widehat{\mathcal{O}_{\mathfrak{p},v}} \to A_{d,2}[[T]]$ which maps $\hat{x}_i$ to s_i for every integer $i \in \{1, \ldots, d\}$. Since the formal k-scheme $\mathscr{L}_\infty(V)_\gamma$ is isomorphic to $\mathbf{D}_k^{\mathbf{N}}$, by lemma 4.5, there exists a morphism $\widehat{\mathcal{O}_{\mathfrak{p},v}} \to k[[S_1, \ldots, S_d]][[T]]$ which maps, for every integer $i \in \{1, \ldots, d\}$, the element $\hat{x}_i$ to an element of $S_i + \langle S_1, \ldots, S_d\rangle^2[[T]]$. Specializing to $T = 0$, we deduce from lemma 4.10 that the induced morphism $\widehat{\mathcal{O}_{\mathfrak{p},v}} \to k[[S_1, \ldots, S_d]]$ is surjective. Its kernel is a prime ideal of $\widehat{\mathcal{O}_{\mathfrak{p},v}}$. Since $\widehat{\mathcal{O}_{\mathfrak{p},v}}$ is an integral domain of dimension d, this prime ideal is necessarily zero, by the Hauptidealsatz. We deduce the existence of a continuous isomorphism

$$\widehat{\mathcal{O}_{\mathfrak{p},v}} \xrightarrow{\sim} k[[S_1, \ldots, S_d]]$$

of admissible local k-algebras, which shows the desired result by [10, 17.5.3].

For the convenience of the reader, we state and prove the following version of the inverse function theorem for formal power series, probably well-known among the specialists.

Lemma 4.10. *Let $d \geq 1$ be an integer. Let $\mathfrak{m}$ be the maximal ideal of the local k-algebra $k[[S_1, \ldots, S_d]]$. Let $\varphi\colon k[[S_1, \ldots, S_d]] \to k[[S_1, \ldots, S_d]]$ be a morphism of local k-algebras which induces an isomorphism of k-vector spaces $\varphi_1\colon \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2$. Then, the morphism φ is an isomorphism.*

Proof. For every integer $n \geq 1$, we deduce from the assumption a k-linear map

$$\varphi_n\colon \mathfrak{m}^n/\mathfrak{m}^{n+1} \to \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

defined by $\varphi_n(\bar{P}) = \overline{\varphi(P)}$ for every power series $P \in k[[S_1, \ldots, S_d]]$. For every integer $n \geq 1$, the map φ_n is surjective. Indeed, for every $y_1, \ldots, y_n \in \mathfrak{m}$, there exists $x_1, \ldots, x_n \in \mathfrak{m}$ such that $\varphi_2(\bar{x}_i) := \overline{\varphi(x_i)} = \bar{y}_i$ for every integer $i \in \{1, \ldots, n\}$. The element $x := x_1 \ldots x_n$ is a preimage of $y := y_1 \ldots y_n$ by φ_n which concludes the proof of our claim.

Since, for every integer $n \in \mathbf{N}$, the k-vector space $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is finite dimensional, we conclude that the map φ_n are bijective. We deduce the assertion from [2, III/§2/Corollaire 3]. □

Let us recall a convention of subsection 4.9. If a rational arc $\gamma(T)$ is identified with a tuple $(x_j(T))_{i\in\{1,\ldots,N\}} \in k[[T]]^{d+M}$ which satisfies, for every integer $i \in \{1, \ldots, M\}$, the equation

$$F_i((x_j(T))_{j\in\{1,\ldots,d+M\}}) = 0,$$

then, for every test-ring $(A, \mathfrak{m}_A)$, an element of $\mathscr{L}_\infty(V)_\gamma(A)$ may be identified with a tuple

$$(x_{1,A}(T), \ldots, x_{d+M,A}(T))$$

of elements of $\mathfrak{m}_A[[T]]^{d+M}$ such that, for every integer $i \in \{1, \ldots, M\}$,

$$F_i((x_j(T) + x_{j,A}(T))_{j\in\{1,\ldots,d+M\}}) = 0.$$

Lemma 4.11. *Keep the notation and convention of subsection 4.9. Let $(A, \mathfrak{m}_A)$ be a test-ring such that $\mathfrak{m}_A^2 = 0$. Then, the natural application*

$$\begin{array}{ccc} \mathscr{L}_\infty(V)_\gamma(A) & \longrightarrow & (\mathfrak{m}_A[[T]])^d \\ (x_{1,A}(T), \dots, x_{d+M,A}(T)) & \longmapsto & (x_{1,A}(T), \dots, x_{d,A}(T)) \end{array}$$

is bijective.

Proof. We denote by $\mathcal{J}$ the jacobian matrix $[\partial_{X_j} F_i]_{\substack{i\in\{1,\dots,M\} \\ j\in\{1,\dots,d+M\}}}$. Recall that

$$\det\left([\partial_{X_j} F_i]_{\substack{i\in\{1,\dots,M\} \\ j\in\{d+1,\dots,d+M\}}} \right)$$

does not vanish at $\gamma(T)$. Using the Taylor expansion and the fact that $\mathfrak{m}_A^2 = 0$, we observe that, for every tuple $(x_{1,A}(T), \dots, x_{d+M,A}(T)) \in \mathfrak{m}_A[[T]]^{d+M}$, the conditions

$$\forall i \in \{1, \dots, M\} \quad F_i(x_j(T) + x_{j,A}(T))_{j\in\{1,\dots,d+M\}} = 0$$

are equivalent to the condition

$$\mathcal{J}(\gamma(T)) \cdot \begin{pmatrix} x_{1,A}(T) \\ \vdots \\ x_{d+M,A}(T) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Using lemmas 3.3 and 4.12, we deduce that there exist elements $(b_{i,j}(T))_{\substack{i\in\{1,\dots,M\} \\ j\in\{1,\dots,d\}}}$ in $k[[T]]$ such that latter condition is equivalent to the system

$$x_{d+i,A}(T) = \sum_{j=1}^{d} b_{i,j}(T) \cdot x_{j,A}(T), \quad i \in \{1, \dots, M\}.$$

That concludes the proof. □

Lemma 4.12. *Let k be a field, and d, M be positive integers. Let*

$$\mathcal{M} = \left[(\mathcal{M}_{i,j})_{\substack{1\le i\le M \\ 1\le j\le d+M}} \right]$$

be a $(M \times (d + M))$ matrix with coefficients in $k[[T]]$. Assume that

$$\mu := \mathrm{ord}_T \left(\det \left[(\mathcal{M}_{i,j})_{\substack{1\le i\le M \\ d+1\le j\le d+M}} \right] \right)$$

is an integer, minimal among the orders of the $(M \times M)$-minors of the matrix $\mathcal{M}$. Then there exists an $(M \times M)$ matrix $\mathcal{N}$ with coefficients in $k[[T]]$, whose determinant is not zero, such that

$$\mathcal{N} \cdot \mathcal{M} = \begin{pmatrix} a_{1,1} & \dots & a_{1,d} & T^\mu & 0 & \dots & 0 \\ a_{2,1} & \dots & a_{2,d} & 0 & T^\mu & \dots & 0 \\ \vdots & \dots & \vdots & 0 & 0 & \ddots & 0 \\ a_{M,1} & \dots & a_{M,d} & 0 & 0 & \dots & T^\mu \end{pmatrix} \tag{4.1}$$

$$\forall (i, j) \in \{1, \dots, M\} \times \{1, \dots, d\} \quad \mathrm{ord}_T(a_{i,j}) \ge \mu. \tag{4.2}$$

Proof. This obvious remark was originally made in [5, p. 216]. Write

$$\det \left(\left[(\mathcal{M}_{i,j})_{\substack{1\le i\le M \\ d+1\le j\le d+M}} \right] \right) = T^\mu\, u(T)$$

with $u(T) \in k[[T]]^\times$ and set

$$\mathcal{N} = u(T)^{-1} \mathrm{ad} \left(\left[(\mathcal{M}_{i,j})_{\substack{1\le i\le M \\ d+1\le j\le d+M}} \right] \right).$$

Clearly equation (4.1) holds. Moreover for $(i,j) \in \{1,\dots,M\} \times \{1,\dots,d\}$ the coefficient $a_{i,j}$ is a linear combination of $(M \times M)$-minors of the matrix $\mathcal{M}$ with coefficients in $k[[T]]$. Hence, formula (4.2) also holds. □

4.13. Let k be a field. Let $(\mathscr{C}, c)$ be an integral k-curve, geometrically unibranch at $c \in \mathscr{C}(k)$. Let $\gamma \in \mathscr{L}_\infty(\mathscr{C})(k)$ be a primitive k-parametrization of $\mathscr{C}$ at c[1]. We say that γ is a *rigid arc*[2] if, for every test-ring $(A, \mathfrak{m}_A)$, for every A-deformation $\gamma_A \in \mathscr{L}_\infty(V)_\gamma(A)$, there exists a unique power series $r_A(T) \in \mathfrak{m}_A[[T]]$ such that $\gamma_A(T) = \gamma(T + r_A(T))$. In the particular case of curves, we may interpret theorem 1.6 as follows.

Corollary 4.14. *Let k be a field. Let $\mathscr{C}$ be an integral k-curve and $c \in \mathscr{C}(k)$. We assume that $(\mathscr{C}, c)$ is geometrically unibranch. Let γ be a primitive k-parametrization at c. Then the following conditions are equivalent:*

(1) *The germ $(\mathscr{C}, c)$ is smooth.*
(2) *The formal neighborhood $\mathscr{L}_\infty(\mathscr{C})_\gamma$ is isomorphic to $\mathbf{D}_k^{\mathbf{N}}$.*
(3) *The arc γ is rigid.*
(4) *Let $\pi\colon \bar{\mathscr{C}} \to \mathscr{C}$ be the normalization of $\mathscr{C}$ and $\bar{\gamma}$ the unique lifting of γ to $\bar{\mathscr{C}}$; then the morphism of formal k-schemes $\mathscr{L}_\infty(\bar{\mathscr{C}})_{\bar{\gamma}} \to \mathscr{L}_\infty(\mathscr{C})_\gamma$ induced by π is an isomorphism.*

Proof. By theorem 1.6 and standard remarks, we only have to show implication 4⇒ 3. Let us assume that γ is a primitive k-parametrization at c such that the morphism of formal k-schemes $\mathscr{L}_\infty(\bar{\mathscr{C}})_{\bar{\gamma}} \to \mathscr{L}_\infty(\mathscr{C})_\gamma$ induced by the normalization $\pi\colon \bar{\mathscr{C}} \to \mathscr{C}$ is an isomorphism, and let us show that γ is rigid. Note that $\bar{\gamma}$ is the unique isomorphism $\widehat{\mathcal{O}_{\bar{\mathscr{C}},\bar{c}}} \xrightarrow{\sim} k[[T]]$ such that $\gamma = \hat{\pi} \circ \bar{\gamma}$. Let $(A, \mathfrak{m}_A)$ be a test-ring. For every power series $r_A \in \mathfrak{m}_A[[T]]$, one has

$$\gamma(r_A(T) + T) \quad = \hat{\pi}(\bar{\gamma}(r_A(T) + T))$$

By assumption, $\bar{\gamma}_A(T) \mapsto \hat{\pi}(\bar{\gamma}_A(T))$ is a natural bijection from $\mathscr{L}_\infty(\bar{\mathscr{C}})_{\bar{\gamma}}(A)$ onto $\mathscr{L}_\infty(\mathscr{C})_\gamma(A)$. Since $\bar{\gamma}$ is rigid, we conclude that γ is rigid too, which concludes the proof of the implication. □

5. Related problems

5.1. A slight variation on an argument of [11, proof of Proposition 1.1] also allows to describe the *constant* arcs whose formal neighborhood is isomorphic to $\mathbf{D}_k^{\mathbf{N}}$ (in arbitrary dimensions), i.e., smooth constant arcs. We denote by σ the canonical section of the projection

$$\pi_0^\infty\colon \mathscr{L}_\infty(V) \to \mathscr{L}_0(V) \cong V.$$

Thus, for every $v \in V$, the point $\sigma(v)$ of $\mathscr{L}_\infty(V)$ is the associated constant arc.

Proposition 5.2. *Let V be a k-variety and $v \in V(k)$ such that $\dim_v(V) \geq 1$. Then the following conditions are equivalent:*

(1) *The k-variety V is smooth at v.*
(2) *The formal neighborhood $\mathscr{L}_\infty(V)_{\sigma(v)}$ is isomorphic to $\mathbf{D}_k^{\mathbf{N}}$.*

In other words, smooth constant arcs on V correspond to smooth points of V.

Proof. We only have to show implication $2 \Rightarrow 1$. By [10, 17.5.1, 17.5.3], it suffices to show that the local k-algebra $\widehat{\mathcal{O}_{V,v}}$ is formally smooth for the $\mathfrak{m}_v$-adic topology (which coincides here with the projective limit topology). By [9, 19.3.3,19.3.6] and the hypothesis, the k-algebra $\widehat{\mathcal{O}_{\mathscr{L}_\infty(V),\sigma(v)}}$ is formally smooth for the projective limit topology. Since the continuous morphism

$$\widehat{\mathcal{O}_{V,v}} \to \widehat{\mathcal{O}_{\mathscr{L}_\infty(V),\sigma(v)}}$$

induced by the projection $\mathscr{L}_\infty(V) \to V$ admits a continuous retraction (induced by σ) we may conclude the proof by the very definition of formal smoothness. □

[1]If k is assumed to be perfect, the assumption that c is geometrically unibranch guarantees the existence of primitive k-parametrizations at c.

[2]An analogous notion has been originally introduced in [15] for constant arcs.

5.3. For non-degenerate arcs centered at a unibranch point, we have an analog of theorem 1.6 with regards to the smoothness of the truncations of the involved arc.

Theorem 5.4. *Let V be a k-variety and $v \in V(k)$. We assume that $\widehat{\mathcal{O}_{V,v}}$ is a domain. Let $\gamma \in \mathscr{L}_\infty(V)(k)$ be a rational non-degenerate arc with $\gamma(0) = v$. Then the following conditions are equivalent:*

(1) *The k-variety V is smooth at v.*
(2) *There exists an integer $n \in \mathbf{N}$ such that $\gamma_n := \pi_n^\infty(\gamma)$ is a smooth point of the jet scheme $\mathscr{L}_n(V)$.*
(3) *For every $n \in \mathbf{N}$, the point γ_n is a smooth point of $\mathscr{L}_n(V)$.*

Implication 1 $\Rightarrow$ 3 is well-kown (e.g., see [14, Lemme 3.4.2]); 3 $\Rightarrow$ 2 is formal. In the end, the proof of implication 2 $\Rightarrow$ 1 is very similar to the proof of theorem 1.6. Indeed, we have to mimick the original proof and replace the use of lemma 4.5 by that of the following lemma, whose proof is completely similar to that of lemma 4.5.

Lemma 5.5. *Let V be a k-variety and $v \in V(k)$ such that $\mathcal{O}_{V,v}$ is reduced and $\dim_v(V) \geq 1$. Let $\gamma \in \mathscr{L}_\infty(V)(k)$ be a non-degenerate rational arc with $\gamma(0) = v$. Let $n \in \mathbf{N}$ be an integer. Assume that the formal neighborhood $\mathscr{L}_n(V)_{\gamma_n}$ is isomorphic to $\mathbf{D}_k^r$ and that the minimal prime ideal $\mathfrak{p}$ of $\widehat{\mathcal{O}_{V,v}}$ corresponds to the formal branch containing γ. Let $(B, \mathfrak{m}_B)$ be a local ring. Then, every morphism of local k-algebras $\widehat{\mathcal{O}_{\mathfrak{p},v}} \to (B/\mathfrak{m}_B^2)[T]/\langle T^{n+1}\rangle$ lifts to a morphism of local k-algebras $\widehat{\mathcal{O}_{\mathfrak{p},v}} \to \widehat{B}[T]/\langle T^{n+1}\rangle$.*

Remark 5.6. This completes in particular a result of [11]. In *loc. cit.*, S. Ishii shows that the jet scheme $\mathscr{L}_n(V)$ is not smooth at any constant jet centered at a non-smooth point of V (see the proof of proposition 1.1 in *op.cit.*). Theorem 5.4 shows that $\mathscr{L}_n(V)$ is not smooth at *any* jet which is the truncation of a non-degenerate arc centered at a non-smooth unibranch point of V.

Remark 5.7. If the reduced germ (V, v) is no longer assumed analytically irreducible, even if the formal branch containing γ is smooth, the truncations γ_n can be *non*-smooth points of the corresponding jet scheme in general. This is already clear for $n = 0$ but this may fail more generally for every n. For example let $V = \mathrm{Spec}(k[X, Y]/\langle X\,Y\rangle)$ and $\gamma(T) = (T, 0)$; then one may check that for every non-negative integer n one has

$$\widehat{\mathcal{O}_{\mathscr{L}_n(V),\gamma_n}} \cong k[[X_0, \ldots, X_n, Y]]/\langle X_0^{n+1} Y\rangle.$$

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