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Table of Contents

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ERRATUM: FREE DIVISORS IN A PENCIL OF CURVES

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In the paper "Free divisors in a pencil of curves", I wrongly said that the Jacobian ideal $J_{\nabla D_k} \subset S$ generated by the partial derivatives of D_k is locally a complete intersection. This is not always true, as can be seen for instance in [2] or in [1, section 1.3].

Because of this error, remark 2.4, Theorem 2.7 and Theorem 2.8 are not true as they were formulated. In remark 2.4, the phrase "the local ideals $(\nabla f \wedge \nabla g)_p$ and $(\nabla f)_p$ coincide" is true when the Jacobian ideal of f is locally a complete intersection at p (for instance when $f = 0$ is a union of lines), but not in general.

Actually, the condition that the Jacobian ideal of a reduced plane curve C is a local complete intersection is equivalent to the claim that any singularity of C is weighted homogeneous (see again [1, section 1.3]).

This hypothesis concerning the nature of the singularities of the curves in the pencil must be added in order to correct theorems 2.7 and 2.8.

The set of all the singularities of all the singular members of the pencil $\mathcal{C}(f,g)$ is denoted by $\mathrm{Sing}(\mathcal{C})$.

A correct statement for Theorem 2.7 is the following one:

Theorem 2.7 *Assume that the base locus of the pencil $\mathcal{C}(f,g)$ is smooth, $n \geq 1$ and $k > 1$. Then, D_k is free with exponents $(2n-2, n(k-2)+1)$ if and only if $D_k \supseteq D^{\mathrm{sg}}$ and $J_{\nabla D_k}$ is locally a complete intersection at every $p \in \mathrm{Sing}(\mathcal{C})$.*

Proof. Let us remark first that D_k is free with exponents $(2n-2, n(k-2)+1)$ if and only if the zero set Z_k of the "canonical section" $s_{\delta,k}$ is empty. Indeed, if $Z_k = \emptyset$, then $\mathrm{H}^1(\mathcal{T}_{D_k}(m)) = 0$ for all $m \in \mathbb{Z}$ and, by Horrocks' criterion, this implies that D_k is free with exponents $(2n-2, n(k-2)+1)$. The other direction is straightforward.

According to lemma 2.6, $Z_k = \emptyset$ if and only if $c_2(\mathcal{J}_{\nabla D_k}) = n^2(k-1)^2 + 3(n-1)^2$. Moreover it is well-known (see [1, section 1.3] for instance) that the length of the Jacobian scheme of D_k is $c_2(\mathcal{J}_{\nabla D_k}) = \sum_{p \in \mathrm{Sing}(D_k)} \tau_p(D_k)$, where $\tau_p(D_k)$ is the Tjurina number of D_k at $p \in D_k$ (this number $\tau_p(D_k)$ is the length of the subscheme of the Jacobian scheme supported by p). Then, to prove the theorem, we show below that $\sum_{p \in \mathrm{Sing}(D_k)} \tau_p(D_k) = n^2(k-1)^2 + 3(n-1)^2$ if and only if $D_k \supseteq D^{\mathrm{sg}}$ and $J_{\nabla D_k}$ is locally a complete intersection at every $p \in \mathrm{Sing}(\mathcal{C})$.

The Jacobian scheme of D_k is supported by the base locus B of the pencil and by the singular points of the k curves forming D_k. The syzygy $\nabla f \wedge \nabla g$ of $J_{\nabla D_k}$ does not vanish at $p \in B$; this implies that $J_{\nabla D_k}$ is locally a complete intersection at $p \in B$; according to [1, section 1.3], this gives $\tau_p(D_k) = \mu_p(D_k)$, where this last number is the Milnor number of D_k at p. Since p is an ordinary singular point of multiplicity k, we obtain $\mu_p(D_k) = (k-1)^2$. Then $\sum_{p \in B} \tau_p(D_k) = n^2(k-1)^2$.

Let us compute now $\sum_{p \in \mathrm{Sing}(D_k) \setminus B} \tau_p(D_k)$.

Let $C_p \subset D_k$ be the unique curve in the pencil singular at $p \in \mathrm{Sing}(D_k) \setminus B$. We can verify without difficulties that their Jacobian ideals coincide locally at $p \in \mathrm{Sing}(D_k) \setminus B$; in particular $\tau_p(D_k) = \tau_p(C_p)$ and $\sum_{p \in \mathrm{Sing}(D_k) \setminus B} \tau_p(D_k) = \sum_{p \in \mathrm{Sing}(D_k) \setminus B} \tau_p(C_p)$.

Let $I = (\nabla f \wedge \nabla g)$ be the ideal generated by the two by two minors of the 3×2 matrix $(\nabla f, \nabla g)$ defining the scheme $\mathrm{sg}(\mathcal{F})$. Let $\mathrm{sg}(\mathcal{F})_p$ be the subscheme of $\mathrm{sg}(\mathcal{F})$ supported by the point p. We have seen in lemma 2.2 that $\mathrm{sg}(\mathcal{F})$ is supported by the whole set of singular points of the pencil and that $l(\mathrm{sg}(\mathcal{F})) = \sum_{p\in \mathrm{Sing}(\mathcal{C})} l(\mathrm{sg}(\mathcal{F})_p) = 3(n-1)^2$.

Let us consider the situation in a fixed point $p \in \mathrm{Sing}(D_k) \setminus B$. To simplify the notation, assume that $f = 0$ is an equation for C_p. Then the other curves of the pencil do not pass through p; in particular, $g(p) \neq 0$. Since $< \nabla f \wedge \nabla g, \nabla g >= 0$, ∇g is a syzygy of I that does not vanish at $p \in \mathrm{Sing}(D_k) \setminus B$. This implies that I is locally a complete intersection at p. Since the ideal I_p is obtained by taking the two by two minors of the matrix $(\nabla f, \nabla g)$ in the local ring S_p, the inclusion $I_p \subset J_{\nabla f,p}$ is straightforward; this inclusion implies $\tau_p(C_p) \leq l(\mathrm{sg}(\mathcal{F})_p)$ because $l(\mathrm{sg}(\mathcal{F})_p) = l(S_p/I_p)$.

Then $Z_k = \emptyset$ if and only if $\sum_{p\in \mathrm{Sing}(D_k)\setminus B} \tau_p(C_p) = \sum_{p\in \mathrm{Sing}(\mathcal{C})} l(\mathrm{sg}(\mathcal{F})_p)$. And this equality is verified if and only if $\mathrm{Sing}(D_k) \setminus B = \mathrm{Sing}(\mathcal{C})$ and $l(\mathrm{sg}(\mathcal{F})_p) = \tau_p(C_p)$ for all $p \in \mathrm{Sing}(\mathcal{C})$. The second equality is equivalent to the equality $I_p = J_{\nabla f,p}$, which implies that the Jacobian ideal of C_p is locally a complete intersection at p.

Since the Jacobian ideals of D_k and C_p coincide locally at $p \in \mathrm{Sing}(D_k) \setminus B$, this proves that $Z_k = \emptyset$ if and only if D_k contains all the singular members of the pencil ($\mathrm{Sing}(D_k)\setminus B = \mathrm{Sing}(\mathcal{C})$) and the Jacobian ideal of D_k is locally a complete intersection in every singular point of the pencil ($\mu_p(C_p) = \tau_p(C_p)$ for all $p \in \mathrm{Sing}(\mathcal{C})$). $\square$

Remark. If the Jacobian ideal of D_k is not locally a complete intersection at p, then Z_k is not empty because $p \in \mathrm{supp}(Z_k)$, even if D_k contains all the singular curves.

This new hypothesis on the singularities must be added also in theorem 2.8 and in proposition 2.10. The proofs remain the same.

Theorem 2.8 *Assume that the base locus of the pencil $\mathcal{C}(f,g)$ is smooth and that the Jacobian ideal of D^{sg} is locally a complete intersection. Assume also that D_k contains all the singular members of the pencil except the singular curves C_{α_i,β_i} for $i = 1,\dots,r$. Then,*

$$\mathcal{J}_{Z_k} = \mathcal{J}_{\nabla C_{\alpha_1,\beta_1}} \otimes \cdots \otimes \mathcal{J}_{\nabla C_{\alpha_r,\beta_r}}.$$

Proposition 2.10 *We assume that the base locus of the pencil $\mathcal{C}(f,g)$ is smooth, that the Jacobian ideal of D^{sg} is locally a complete intersection and that D_k contains D^{sg}. Let C be a singular member in $\mathcal{C}(f,g)$ and Z its scheme of singular points. Then there is an exact sequence*

$$0 \longrightarrow \mathcal{T}_{D_k} \longrightarrow \mathcal{T}_{D_k\setminus C} \longrightarrow \mathcal{J}_{Z/C}(n(3-k)-1) \longrightarrow 0,$$

where $\mathcal{J}_{Z/C} \subset \mathcal{O}_C$ defines Z into C.

References

[1] Schenck, H. and Tohăneanu, Ş. O. *Freeness of conic-line arrangements in* $\mathbb{P}^2$. Comment. Math. Helv. 84 (2009), no. 2, 235–258.

[2] Simis, A. and Tohăneanu, Ş. O. *Homology of homogeneous divisors.* Israel J. Math. 200 (2014), no. 1, 449–487

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MODULI SPACES FOR TOPOLOGICALLY QUASI-HOMOGENEOUS FUNCTIONS

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Abstract. We consider the topological class of a germ of 2-variables quasi-homogeneous complex analytic function. Each element f in this class induces a germ of foliation $(\mathrm{d}f = 0)$ and a germ of curve $(f = 0)$. We first describe the moduli space of the foliations in this class and we give analytic normal forms. The classification of curves induces a distribution on this moduli space. By studying the infinitesimal generators of this distribution, we can compute the generic dimension of the moduli space for the curves, and we obtain the corresponding generic normal forms.

Introduction

From any convergent series f in $\mathbb{C}\{x, y\}$, we can consider three different associated mathematical objects: a germ of holomorphic function defined by the sum of this series, a germ of foliation whose leaves are the connected components of the level curves $f =$ constants, and an embedded curve $f = 0$. Composing f on the left side by a diffeomorphism of $(\mathbb{C}, 0)$ may change the function but nor the foliation or the curve. Multiplying f by an invertible function u may change the function and the foliation but not the related curve. Therefore, there are three different analytic equivalence relations:

- The classification of functions (or right equivalence):
$$f_0 \sim_r f_1 \Leftrightarrow \exists \phi \in \text{Diff } (\mathbb{C}^2, 0),\ f_1 = f_0 \circ \phi.$$
- The classification of foliations (or left-right equivalence):
$$f_0 \sim f_1 \Leftrightarrow \exists \phi \in \text{Diff } (\mathbb{C}^2, 0),\ \psi \in \text{Diff } (\mathbb{C}, 0),\ \psi \circ f_1 = f_0 \circ \phi.$$
- The classification of curves:
$$f_0 \sim_c f_1 \Leftrightarrow \exists \phi \in \text{Diff } (\mathbb{C}^2, 0),\ \exists u \in \mathcal{O}_2, u(0) \neq 0, u f_1 = f_0 \circ \phi.$$

In the same way, one can define topological classifications requiring only topological changes of coordinates. In what follows, we are going to consider mostly the two last equivalence relations for foliations and curves, since the comparison between the two first analytic classifications has been studied in [1].

Finally, we emphasize that in our work, we will always require that the conjugacies that appear above will respect a fixed numbering of the branches of $f = 0$.

A germ of holomorphic function $f_{\mathrm{qh}} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is *quasi-homogeneous* if and only if f_{qh} belongs to its jacobian ideal $J(f_{\mathrm{qh}}) = (\frac{\partial f_{\mathrm{qh}}}{\partial x}, \frac{\partial f_{\mathrm{qh}}}{\partial y})$. If f_{qh} is quasi-homogeneous, there exist

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coordinates (x, y) and positive coprime integers k and l such that the quasi-radial vector field $R = kx\frac{\partial}{\partial x} + ly\frac{\partial}{\partial y}$ satisfies

$$R(f_{\mathrm{qh}}) = d \cdot f_{\mathrm{qh}},$$

where the integer d is *the quasi-homogeneous* (k, l)*-degree* of f_{qh} [15]. In these coordinates, f_{qh} has some cuspidal branches and maybe axial branches, that is to say, f_{qh} is written

$$f_{\mathrm{qh}} = cx^{n_\infty} y^{n_0} \prod_{b=1}^{p} (y^k + a_b x^l)^{n_b} \tag{1}$$

where c is a non vanishing complex number and the multiplicities satisfy $n_0 \geq 0$, $n_\infty \geq 0$ and $n_b > 0$. The complex numbers a_b are non vanishing numbers such that $a_b \neq a_{b'}$. Using a convenient analytic change of coordinates, we may suppose that $a_1 = 1$.
A germ of holomorphic function f is *topologically quasi-homogeneous* if the function f is topologically conjugated to a quasi-homogeneous function f_{qh}, that is to say there is a continuous right-equivalence between f and f_{qh}.

For any couple of coprime positive integers (k, l) with $k < l$ and $(p + 2)$-uple (n) of integers in $\mathbb{N}^2 \times (\mathbb{N}^*)^p$, $(n) = (n_\infty, n_0, n_1, n_2, \cdots, n_p)$ we consider the topological class $\mathcal{T}_{(k,l),(n)}$ of f_{qh} defined in (1), that is the set of all functions topologically conjugated to f_{qh}. The first aim of this paper is to describe the moduli space defined by the quotient

$$\mathcal{M}_{(k,l),(n)} = \mathcal{T}_{(k,l),(n)} /_{\sim}$$

where $\sim$ refers to the left-right analytical equivalence. We give the infinitesimal description of this moduli space by making use of the cohomological tools considered by J.F. Mattei in [13]: the tangent space to the moduli space is given by the first Cěch cohomology group $H^1(D, \Theta_{\mathcal{F}})$, where D is the exceptional divisor of the desingularization of f_{qh}, and $\Theta_{\mathcal{F}}$ is the sheaf of germs of vector fields tangent to the desingularized foliation $\widetilde{\mathcal{F}}$ induced by $\mathrm{d}f_{\mathrm{qh}} = 0$. Using a particular covering of D, we give a triangular presentation of the $\mathbb{C}$-space $H^1(D, \Theta_{\mathcal{F}})$ in Theorem (1.3). This description leads us to consider *triangular analytic normal forms*

$$N_a = x^{n_\infty} y^{n_0} \prod_{b=1}^{p} (y^k + \sum_{\{(b,d),\Phi(b,d)\in\mathbb{T}\}\cup\{(1,kl)\}} a_{b,d} m^d)^{n_b} \tag{2}$$

by perturbing the topological normal form (1) with some monomials m^d following an algorithm described in the subsection (1.2), in which the precise meaning of the indexation $\Phi(b, d)$ is defined. This family of analytic normal forms turns out to be semi-universal as established in Theorem (1.10). In this way, we obtain a local description of $\mathcal{M}_{(k,l),(n)}$. We finally give a global description of this moduli space in Theorem (1.15) and Theorem (1.16) by proving that any function in $\mathcal{T}_{(k,l),(n)}$ is actually conjugated to some normal form N_a , and that the parameter a is unique up to some weighted projective action of $\mathbb{C}^*$. All the results of this first part can be extended to the generic Darboux function:

$$f^{(\lambda)} = f_1^{\lambda_1} \cdots f_p^{\lambda_p}$$

with complex multiplicities λ_i. Nevertheless, we do not insert this extension here, since we have previously explain in [8] how to perform it in the topologically homogeneous case.

The second part of our work is dedicated to the study of the moduli space of curves in the quasi-homogeneous topological class. This problem is a particular case of an open problem known as the *Zariski problem*. It has only a very few answers: Zariski [17] for the very first treatment of some particular cases, Hefez and Hernandes [5, 6] for the irreducible curves, Granger [9] in the homogeneous topological class and [2] for some results which are particular cases of our present

result. Our strategy that we already introduced in a previous work [8], differs from all this works: we consider the integrable distribution $\mathcal{C}$ on the moduli space of foliations $\mathcal{M}_{(k,l),(n)}$ induced by the equivalence relation $\sim_c$: two foliations represented by two points in $\mathcal{M}_{(k,l),(n)}$ are in a same orbit of this distribution if and only if they induce the same curve up to analytic conjugacy. Studying the family of vector fields that induce the distribution $\mathcal{C}$ on $\mathcal{M}_{(k,l),(n)}$, we compute the dimension of the generic strata of the moduli space of curves $\mathcal{M}_{(k,l),(n)}/_{\mathcal{C}}$ in Theorem (2.7). We also give an algorithm in order to construct the corresponding generic normal forms in Theorem (2.8).
Since the cohomological description of the moduli space of foliations is known for a general one-form, we may expect that this strategy can be develop in a general topological class.

In order to keep a sufficiently readable text, we have postponed a lot of technical computations in appendix A.

We thank Jean-François Mattei for helpful discussions, and for suggesting improvements on a first version of this work.

1. The moduli space of foliations

In this section, we will consider a function f in the class $\mathcal{T}_{(k,l),(n)}$. The (k,l)-degree of a monomial $x^m y^n$ is $km+ln$. It induces a valuation on $\mathbb{C}\{x,y\}$ denoted by $\nu_{k,l}$.

Let f be a function in the topological class $\mathcal{T}_{(k,l),(n)}$. We know, from a theorem of Lejeune-Jalabert [10] that the desingularization process of f is identical to that of f_{qh}, that is to say: after a sequence of blowing-ups E, the exceptional divisor D is a chain of components isomorphic to $P^1(\mathbb{C})$, the strict transform of the cuspidal branches intersect the same component D_c, *the principal component*, and the strict transform of the axes, if they appear, intersect the end components of this chain : see Appendix A, and figure (2).

Lemma 1.1 (Prenormalization)**.** *There exists some coordinates (x,y) such that f is written*

$$f(x,y)=cx^{n_\infty}y^{n_0}\left(y^k+x^l+\cdots\right)^{n_1}\left(y^k+a_2x^l+\cdots\right)^{n_2}\cdots\left(y^k+a_px^l+\cdots\right)^{n_p}$$

where c is a non-vanishing complex number, $a_b,\ b=2,\ldots,p$ are non-vanishing complex numbers with $a_b\neq a_{b'}\neq 1$, and the dots are terms of (k,l)-degree greater than kl.

Proof. Let f be a function topologically conjugated to f_{qh}. The number of branches, and their multiplicities are topological invariants. Therefore, we consider the following irreducible decomposition of f:

$$f=f_\infty^{n_\infty}f_0^{n_0}f_1^{n_1}\cdots f_p^{n_p}.$$

Since f has the same desingularization process as f_{qh}, if $n_0>0$ or $n_\infty>0$, the strict transform of the corresponding branches appear on the end components. Therefore, their blowing-down are smooth transverse branches at 0, and we can choose coordinates (x,y) such that

$$f=x^{n_\infty}y^{n_0}f_1^{n_1}\cdots f_p^{n_p}.$$

Now, the strict transform of the other branches meet the principal component D_c. Using the blown-down formulas of proposition (3.2) in Appendix A, we obtain that:

$$f_i=\alpha_i y^k+\beta_i x^l+\cdots$$

with $\alpha_i\neq 0$ and $\beta_i\neq 0$. By factorizing α_i in each component f_i we obtain the existence of a non-vanishing constant c and a family of p non-vanishing complex number a_b, $b=1,\ldots,p$ such that

$$f = cx^{n_\infty} y^{n_0} \left(y^k + a_1 x^l + \cdots\right)^{n_1} \cdots \left(y^k + a_p x^l + \cdots\right)^{n_p}$$

where the dots are terms of (k, l) −degree greater than kl. Finally, by applying a final change of coordinates of the form $(x, y) \to (\alpha x, y)$, we can suppose that $a_1 = 1$. □

Unless any precision is given, from now on, we will only consider system of coordinates (x, y) such that the function $f \in \mathcal{T}_{(k,l),(n)}$ has an expression as in the above lemma.

1.1. **The infinitesimal description.** Since the transverse structure of a foliation defined by a function is rigid, i.e. completely given by the discrete data of the multiplicities, any topologically trivial deformation is an unfolding as defined in [13]. We know from the same reference that the tangent space to the moduli space of unfoldings of a germ of analytic foliation $\mathcal{F}$ is the vector space: $H^1(D, \Theta_{\mathcal{F}})$, where $\Theta_{\mathcal{F}}$ is the sheaf on D of germs of holomorphic vector fields tangent to the desingularized foliation $\widetilde{\mathcal{F}}$. Furthermore, this vector space is a finite dimensional one, whose dimension δ is obtained by a formula involving the multiplicities of the foliation at the singular points appearing at each step of the blowing up process. In the present topological class, we will give an alternative description of this tangent space which will allow us to construct normal forms.

Let f be in $\mathcal{T}_{(k,l),(n)}$. We consider the saturated foliations $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ induced by df and E^*df, where E is the desingularization morphism of f.

Notation 1.2.

(1) *We define two integers ε_0 and ε_∞ in $\{0, 1\}$ as follows: if $n_0 > 0$ then we set $\varepsilon_0 = 1$, else we set $\varepsilon_0 = 0$. We define ε_∞ the same way but relative to n_∞.*

(2) *Let (u, v) be the unique couple of integers defined by the Bézout identity*

$$uk - vl = 1 \text{ with } 0 \leq u < l,\ 0 \leq v < k.$$

(3) *We denote by ν_c the multiplicity of the desingularized foliation on the principal component D_c of the exceptional divisor. According to Proposition (3.4) in Appendix A, we have*

$$\nu_c = klp - k - l + k\varepsilon_\infty + l\varepsilon_0.$$

(4) *Let $\mathbb{T}$ be the triangle in the real half plane (X, Y), $Y \geq 0$, delimited by*

$$\begin{array}{rcl} kX - (k-v)(Y - \nu_c) & > & 0 \\ lX - (l-u)(Y - \nu_c) & < & 0 \end{array}$$

The summit of this triangle is $(0, \nu_c)$. The directions of the non horizontal edges are given by the vectors

$$\vec{x} = (k - v, k) \quad \text{and } \vec{y} = (l - u, l).$$

Theorem 1.3. *There is an explicit linear isomorphism Ψ between $H^1(D, \Theta_{\mathcal{F}})$ and the $\mathbb{C}$-vector space freely generated by the set of integer points $e_{i,j} = (i, j)$ in the triangle $\mathbb{T}$.*

The expression of Ψ is given in the proof below. We give a presentation of the tangent space to the moduli space of a function in the topological class: $(k, l) = (3, 5)$, $p = 4$, $n_0 = n_\infty = 0$, $n_1, \ldots, n_4$ arbitrary, obtained by this theorem in Appendix B, Figure (3).

Proof. Let us consider θ_f the vector field with an isolated singularity defined by

$$\theta_f = \frac{1}{\text{g.c.d.}\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)} \cdot \left(\frac{-\partial f}{\partial x}\frac{\partial}{\partial y} + \frac{\partial f}{\partial y}\frac{\partial}{\partial x}\right). \tag{3}$$

Let $\{U_0, U_\infty\}$ be the covering of the exceptional divisor introduced in the Appendix A. From proposition 3.6, we know that this covering is acyclic with respect to the sheaf $\Theta_{\mathcal{F}}$. Therefore we have

$$H^1(D, \Theta_{\mathcal{F}}) = \frac{\Theta_{\mathcal{F}}(U_0 \cap U_\infty)}{\Theta_{\mathcal{F}}(U_0) \oplus \Theta_{\mathcal{F}}(U_\infty)}.$$

In order to compute each term of this quotient, we consider the principal chart (x_c, y_c) defined near the central component D_c defined in Appendix A. The domain of this chart contains $U_0 \cap U_\infty$. The vector field

$$\theta_{\text{is}} = \frac{E^*\theta_f}{y_c^{\nu_c}}$$

has isolated singularities and defines $\mathcal{F}$ on $U_0 \cap U_\infty$. Therefore we have

$$\Theta_{\mathcal{F}}(U_0 \cap U_\infty) = \mathcal{O}\left(U_0 \cap U_\infty\right) \cdot \theta_{\text{is}},$$

and each θ in $\Theta_{\mathcal{F}}(U_0 \cap U_\infty)$ can be written

$$\theta = \left(\sum_{i \in \mathbb{Z}, j \in \mathbb{N}} \lambda_{i,j}\ x_c^i y_c^j\right) \cdot \theta_{is}.$$

By the local monomial expression of E given by proposition 3.2 in Appendix A, these vector fields θ blow down on meromorphic vector fields with poles on the axes:

$$E_*\theta = \sum_{i \in \mathbb{Z},\ j \in \mathbb{N}} \lambda_{i,j}\ x^{li-(l-u)(j-\nu_c)} y^{-ki+(k-v)(j-\nu_c)} \cdot \theta_f.$$

Let us prove that θ has an holomorphic extension on U_0 if and only if

$$-ki + (k-v)(j-\nu_c) < 0 \implies \lambda_{i,j} = 0. \qquad (\star)$$

If such an extension is possible, then θ has no pole along the curve $y = 0$ whose strict transform belongs to U_0, thus the property $(\star)$ holds. On the converse, if the property $(\star)$ is satisfied, then the multiplicity $\nu_{D_1}(\theta)$ of θ along the component D_1 which meets the strict transform of the x-axis is positive. Indeed, after a standard blow-up, we find

$$\nu_{D_1}(\theta) \geq \min_{\lambda_{i,j} \neq 0} \{(l-k)i + (j-\nu_c)(k-v-l+u)\} \geq 0$$

Now, the intermediate multiplicities $\nu_{D_i}(\theta)$, $1 < i < c$ are also positive. This is a consequence of the relations

$$\nu_{D_2}(\theta) = e_1 \nu_{D_1}(\theta),\ \nu_{D_{i+1}}(\theta) = e_i \nu_{D_i}(\theta) - \nu_{D_{i-1}}(\theta),\ i = 2, \ldots, c-1$$

which can be obtained by a similar argument as in proposition 3.4. Here, $-e_i$ is the self-intersection of the component D_i. Since $e_i \geq 2$ for $i = 1, \ldots, c-1$, we have

$$\nu_{D_2}(\theta) \geq \nu_{D_1}(\theta),\ \nu_{D_{i+1}}(\theta) - \nu_{D_i}(\theta) \geq \nu_{D_i}(\theta) - \nu_{D_{i-1}}(\theta),\ i = 2, \ldots, c-1$$

which proves that $\nu_{D_i}(\theta)$ is positive for any $i = 1, \ldots, c$. In the same way, an element θ in $\Theta_{\mathcal{F}}(U_0 \cap U_\infty)$ belongs to $\Theta_{\mathcal{F}}(U_\infty)$ if and only if

$$li - (l-u)(j-\nu_c) < 0 \implies \lambda_{i,j} = 0.$$

Therefore, there is a linear isomorphism Ψ between the $\mathbb{C}$-space freely generated by the integer points $e_{i,j}$ in $\mathbb{T}$ and $H^1(D, \Theta_{\mathcal{F}})$ defined by:

$$\Psi: \sum_{(i,j)\in\mathbb{T}} \lambda_{i,j} e_{i,j} \longmapsto \left[\left(\sum_{(i,j)\in\mathbb{T}} \lambda_{i,j} x_c^i y_c^j\right)\cdot \theta_{is}\right]. \tag{4}$$

□

This representation gives us a direct formula for the dimension δ of $H^1(D,\Theta_{\mathcal{F}})$, by counting the integers points in the above triangle. In order to give an explicit formula, we need the following fact:

Lemma 1.4 (**and notations**). *The number of integer points in an open interval* $]a,b[$ *is given by* $]b]-[a[$*, where* $[a[$ *stands for the usual integer part* n *of* a*:* $n\leq a<n+1$*, and* $]b]$ *is the "strict" integer part* m *of* b *defined by* $m<b\leq m+1$*.*

Since the intersections of the horizontal levels j with the boundary of $\mathbb{T}$ are respectively given by $\frac{k-v}{k}(j-\nu_c)$ and $\frac{l-u}{l}(j-\nu_c)$, we obtain

Proposition 1.5. *The dimension of* $H^1(D,\Theta_{\mathcal{F}})$ *is*

$$\delta=\sum_{j=0}^{\nu_c}\left(\left]\frac{l-u}{l}(j-\nu_c)\right]-\left[\frac{k-v}{k}(j-\nu_c)\right[\right).$$

Example. For the topological class given by $(k,l)=(3,5)$, $p=4$, without axis, by counting the integers points in figure (3) in Appendix B, or applying the previous formula, we obtain that $\delta=78$.

1.2. **Construction of the local normal forms.** We will construct here analytic models for topologically quasi-homogeneous functions starting from the topological normal form (1). Since it already appears $(p-1)$ analytic invariants that are the values a_b (the cross ratios between branches on the principal component), we have to add $\delta-(p-1)$ monomial terms of higher degrees. The construction to come is *a priori* based upon some algorithmic but arbitrary choices. It will be justified by Theorem (1.10) in the next section.

In our previous work in [8], for the *homogeneous* topological class, in which the topological representative was p transverse lines, we straightened the fourth first lines on $xy(y+x)(y+a_{4,1}x)$, we added the monomials $a_{5,2}x^2$ to the fifth line, $a_{6,2}x^2+a_{6,3}x^3$ to the sixth, and so on. We generalize this triangular construction here by making use of the quasi-homogeneous (k,l)-degree. Nevertheless, the choice of the monomials and their distribution between the branches is not so obvious here.

The following algorithm will associate an analytic normal form starting from the previous triangular presentation of the infinitesimal moduli space.
The figure (3) in Appendix B shows the procedure in order to construct the normal forms associated to the topological class of

$$\left(y^3+x^5\right)^{n_1}\left(y^3+a_2x^5\right)^{n_2}\left(y^3+a_3x^5\right)^{n_3}\left(y^3+a_4x^5\right)^{n_4}.$$

The meaning of all the datas that appear on the figure will be detailed below.
The construction consists in two successive steps.

Step 1. Choice of the monomials.

Notation 1.6. *For any* $d\geq kl$*, there exists a unique monomial* x^iy^j *with quasi-homogeneous* (k,l)*-degree* d*, such that* $j<k$*. We denote it the following way*

$$m^d:=x^iy^j \qquad ik+jl=d,\ j<k.$$

For $(k, l) = (3, 5)$, we find $m^{15} = x^5$, $m^{16} = x^2y^2$, $m^{17} = x^4y$, $m^{18} = x^6$,...

Therefore, to each horizontal line of index j in the triangle $\mathbb{T}$, one can associate the monomial m^d, $d = kl + j$. We put them on a column on the right side in Figure (3).

Step 2. Distribution of the monomials between the cuspidal branches. The link between the monomial terms m^d and m^{d+1} is the multiplication by the meromorphic monomial term m^{d+1}/m^d. We encode this multiplication by a translation in $\mathbb{T}$. We associate to the multiplication by x (resp. y) the translation by $\vec{x} = (k - v, k)$ (resp. $\vec{y} = (l - u, l)$). This choice is suggested by the formulas of Proposition (3.2) in Appendix A. Thus to a degree d we associate the translation in $\mathbb{Z}^2$ by the vector $\vec{t_d}$ defined by

$$\vec{t_d} = i\vec{x} + j\vec{y}$$

where $x^iy^j = m^{d+1}/m^d$.

Lemma 1.7. *For any d, $\vec{t_d}$ is either $(1, 1)$ or $(0, 1)$.*

Proof. Let $m^d = x^iy^j$ and thus $ik + jl = d$ with $0 \leq j < k$. Suppose first that $j - v \geq 0$. Then $m^{d+1} = x^{i+u}y^{j-v}$. Hence, in the the canonical basis, the components of $\vec{t_d}$ are

$$(i + u - i)(k - v, k) + (j - v - j)(l - u, l) = (1, 1).$$

If $j - v < 0$ then $m^{d+1} = x^{i+u-l}y^{j+k-v}$. Indeed, we have $0 \leq j + k - v < k$ and $i + u - l \geq 0$ since from

$$(i + u)k = kl + 1 - (j - v)l > kl.$$

In this case, the components of $\vec{t_d}$ are

$$(u - l)(k - v, k) + (k - v)(l - u, l) = (0, 1).$$

□

For $(k, l) = (3, 5)$, the meromorphic monomials form a periodic sequence of lenght 3 generated by: y^2/x^3, x^2/y, x^2/y. The successive translations are $\vec{t_{15}}$, $\vec{t_{16}}$, $\vec{t_{17}}$, $\vec{t_{18}} = \vec{t_{15}}$ etc..., whose components are $(0, 1)$, $(1, 1)$, $(1, 1)$. We put the translations on a column on the right side of Figure (3).

Now we consider all the parallel paths issued from the integer points $(i, 0)$ on the horizontal axe, under the action of the successive translations $\vec{t_d}$. The point $\left(-\nu_c\frac{k-v}{k}, 0\right)$ is the intersection of the left edge of the triangle with this horizontal axe. We consider the p integer points:

$$M_1 := \left([-\nu_c\frac{k-v}{k}[+p, 0\right),\ M_2 := \left([-\nu_c\frac{k-v}{k}[+p-1, 0\right), \ldots,\ M_p := \left([-\nu_c\frac{k-v}{k}[+1, 0\right).$$

Notice that the $(p - 1)$ last ones are inside the triangle, while the first one is outside.

Proposition 1.8. *The p paths issued from the initial points M_i, $i = 1, \ldots, p$, obtained by the action of the successive translations $\vec{t_d}$ pass through all the integer points inside the triangle $\mathbb{T}$.*

Proof. Let i_n and j_n such that $m^{kl+n} = x^{i_n}y^{j_n}$. Following the arguments in the proof of Lemma (1.7), the sequence (i_n, j_n) is explicitely defined by the following system

$$\begin{cases} i_n = l + ua_n - (l - u)\,b_n \\ j_n = -va_n + (k - v)\,b_n \\ i_nk + j_nl = d_0 + n \\ j_n < k \end{cases}$$

where $d_0 = kl$ and (a_n, b_n) is defined by $(a_0, b_0) = (0, 0)$ and

$$\begin{aligned} \begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} &= \begin{pmatrix} a_n \\ b_n \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ if } j_n - v \geq 0 \\ &= \begin{pmatrix} a_n \\ b_n \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ if } j_n - v < 0. \end{aligned}$$

Notice that a_n is the number of translations of type $(1, 1)$ occuring in a path of lenght n, and corresponds to the horizontal component of the sum of the n first translations. We consider the left side of the triangle given by the equation

$$ki - (k - v)j + \nu_c(k - v) = 0$$

and its intersections (x_n, n) with the horizontal levels $j = n$. We have

$$x_n = \frac{k - v}{k}(n - \nu_c).$$

We consider the path starting from the last integer point $([x_0[+1, 0)$. The successive integer points of this path are given by the sequence $(p_n, n) = ([x_0[+1 + a_n, n)$. We claim that the moving point along this path does not go too far away from the left side of the triangle. More precisely, we have:

$$(p_n - x_n) \in]-1, 1].$$

Indeed, by solving the above system, we obtain $a_n = \frac{-j_n}{k} + n\frac{k-v}{k}$. Therefore we have:

$$p_n - x_n = ([-\nu_c\frac{k - v}{k}[+\nu_c\frac{k - v}{k} + 1) + (a_n - n\frac{k - v}{k}).$$

Clearly, the first part of the sum belongs to $]0, 1]$, and the second one, which equals to $\frac{-j_n}{k}$ belongs to $]-1, 0]$. Therefore this path will catch all the first integer points of the triangle on each level starting from the left side. If we consider the p parallel paths starting from M_i, $i = 1, \ldots, p$, they will catch all the integers points of the triangle, since on each level there is at most p points. □

These p paths give us a unique way to distribute the monomials $a_{b,d}m^d$ on each branch, putting the monomials encountered on the first path (starting from the right hand side) on the first branch, and so on. With this *path game*, we do not miss any point of the triangle according to the previous proposition. Each integer point of the triangle can be represented by the new coordinates (b, d) where b is the index of a path or branch and d the index of a level, or degree. From our construction, they are related to (i, j) by the change of indexation

$$(i, j) = \Phi(b, d) = \left(\left[-\nu_c\frac{k - v}{k} \right[+ p + 1 - b + \sum_{i=kl-1}^{d-1} \alpha_i \ , \ d - kl \right), \tag{5}$$

where $\alpha_{kl-1} = 0$, and for $i \geq kl$, α_i is the horizontal component of $\vec{t_i}$.

To conclude, the general writing of the analytic normal forms for foliations defined by a function in $\mathcal{T}_{(k,l),(n)}$ obtained by our construction is the following definition

Definition 1.9. *Let $\mathcal{A}$ be the following open set of $\mathbb{C}^\delta$*

$$\mathcal{A} = \{(a_{b,d}),\ \Phi(b, d) \in \mathbb{T},\ a_{b,kl} \neq 0,\ a_{b,kl} \neq 1,\ a_{b,kl} \neq a_{b',kl} \text{ for } b \neq b'\}.$$

Furthermore, we set $a_{1,kl} = 1$. For $a = (a_{b,d}) \in \mathcal{A}$ we define the analytic normal form N_a by

$$N_a = x^{n_\infty} y^{n_0} \prod_{b=1}^{p} \left(y^k + \sum_{\{(b,\,d),\,\Phi(b,\,d)\,\in\,\mathbb{T}\}\,\cup\,\{(1,\,kl)\}} a_{b,d} m^d \right)^{n_b} \tag{6}$$

Example. From the figure (3) in the Appendix B, the analytic normal form N_a of the foliation defined by a function f in the topological class $(k,l) = (3,5)$, $p = 4$, $n = (n_1, n_2, n_3, n_4)$ are given in the same Appendix: we add 2 monomials on the first branch, 16 on the second, 31 on the third and 29 on the last one.

1.3. **Local universality.** The construction described in the previous section is justified, *a posteriori*, by the following result. For any $a \in \mathcal{A}$, we consider the saturated foliation $\mathcal{F}_a$ defined by the one-form $\mathrm{d}N_a$.

Theorem 1.10. *For any a^0 in $\mathcal{A}$, the germ of deformation $\{\mathcal{F}_a,\ a \in (\mathcal{A}, a^0)\}$ is an equireducible semi-universal unfolding among the equireducible unfoldings of $\mathcal{F}_{a^0}$.*

This means that for any equireducible unfolding $\{\mathcal{F}_t,\ t \in (T, t^0)\}$ where (T, t^0) is a germ of some space of parameters $t = (t_1, \ldots, t_s)$, such that $\mathcal{F}_{t^0} = \mathcal{F}_{a^0}$, there exists a map $\lambda : T \to A$ with $\lambda(t^0) = a^0$ such that the family $\mathcal{F}_t$ is analytically equivalent to $\mathrm{d}N_{\lambda(t)}$. Furthermore, the universality means that the map λ is unique and the semi-universality only requires that the first derivative of λ at t^0 is unique.

Proof. Let E be the common desingularization map for each foliation $\mathcal{F}_a$ and $\widetilde{\mathcal{F}}_a$ the pull-back of $\mathcal{F}_a$ by E. $\widetilde{\mathcal{F}}_a$ is also the saturated foliation defined by the one-form $\mathrm{d}\widetilde{N}_a$ where $\widetilde{N}_a = N_a \circ E$. Let Θ_0 be the sheaf on $D = E^{-1}(0)$ of germs of holomorphic vector fields tangent to the foliation $\widetilde{\mathcal{F}}_{a^0}$.

Lemma 1.11. *Let $\mathcal{U} = \{U_0, U_\infty\}$ be the covering of the exceptional divisor of E introduced in the Appendix A (notations 3.1). Any unfolding of $\widetilde{\mathcal{F}}_{a^0}$ is locally analytically trivial on each open set U_0, U_∞.*

Proof. Suppose that the unfolding is given by a one-form

$$dF = \frac{\partial F_t}{\partial x} + \frac{\partial F_t}{\partial y} + \sum_{r=1}^{s} \frac{\partial F_t}{\partial t_r},$$

such that dF_{t^0} defines $\widetilde{\mathcal{F}}_{a^0}$. Let m be a point of D, in some local chart (x_i, y_i) of D. For each parameter t_r, we can find a local vector field in some neighborhood U_m of m

$$X_r = \theta_r - \frac{\partial}{\partial t_r} = (\alpha_r(x_i, y_i, t)\frac{\partial}{\partial x_i} + \beta_r(x_i, y_i, t)\frac{\partial}{\partial y_i}) - \frac{\partial}{\partial t_r}$$

such that $d(F \circ E)(X_r) = 0$, which can also be written

$$\theta_r(F \circ E) = \frac{\partial}{\partial t_r}(F \circ E).$$

The existence of X_r is clear around a regular point of the foliation, and still true around a reduced singular point: see [13]. The difference between two such local vector fields is a tangent vector field to the foliation $\widetilde{\mathcal{F}}_{a^0}$. Now, from Proposition (3.6) in Appendix A, we have

$$H^1(U_0, \Theta_0) = H^1(U_\infty, \Theta_0) = 0.$$

Therefore we can glue together these vector fields on U_0 or on U_∞. The trivialization of the unfolding on U_0 or U_∞ in the direction $\frac{\partial}{\partial t_r}$ is obtained by integration of these vector fields X_r. □

For each parameter $a_{b,d}$ of the unfolding defined by dN_a, a in $(\mathcal{A}, a^0)$, the previous lemma proves that there exist two vector fields $\theta^0_{b,d}$ in $\Theta_0(U_0)$ and $\theta^\infty_{b,d}$ in $\Theta_0(U_\infty)$ such that

$$\theta^0_{b,d}(\widetilde{N_{a^0}}) = \left.\frac{\partial \widetilde{N_a}}{\partial a_{b,d}}\right|_{a=a^0} \quad \text{and} \quad \theta^\infty_{b,d}(\widetilde{N_{a^0}}) = \left.\frac{\partial \widetilde{N_a}}{\partial a_{b,d}}\right|_{a=a^0}. \tag{7}$$

We call them "*trivializing vector fields in the direction $a_{b,d}$*". We denote by $\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}$ the difference $\theta^0_{b,d} - \theta^\infty_{b,d}$ in $\Theta_0(U_0 \cap U_\infty)$ and by $\left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0}$ its image in $H^1(D, \Theta_0)$, which does not depend on the choice of the trivializing vector fields. We define a map from the tangent space to $\mathcal{A}$ at a^0 into $H^1(D, \Theta_0)$ by

$$\left\{ \begin{array}{ccc} T_{a^0}\mathcal{A} & \longrightarrow & H^1(D, \Theta_0) \\ \sum_{(b,d)} \lambda_{b,d}(a) \frac{\partial}{\partial a_{b,d}} & \longmapsto & \sum_{(b,d)} \lambda_{b,d}(a) \left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0} \end{array} \right. \tag{8}$$

According to a theorem of J.F. Mattei ([13], Theorem (3.2.1)), the unfolding $\{\mathcal{F}_a,\ a \in (\mathcal{A}, a^0)\}$ is semi-universal among the equireducible unfoldings if and only if this map is a bijective one. By our construction, the dimension of $T_{a^0}\mathcal{A}$ is equal to the one of $H^1(D, \Theta_0)$. Therefore it suffices to prove that the cocycles $\left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0}$ are independent. We denote by

$$\left\langle \left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0}\ ,\ e_{i,j} \right\rangle$$

the component of $\left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0}$ on the element of the basis $\{e_{i,j}\}$ given by Theorem (1.3). These numbers define a square matrix M of size $\delta = \dim H^1(D, \Theta_0)$, and we have to prove that it is an invertible one, that will be done in two steps.

Step 1. *Components of the cocycles on the first level $d = kl$.*
According to our construction of the normal forms, the coefficient $a_{1,kl}$ is constant equal to 1. Nevertheless, in order to perform calculus in a more symmetric way, we first consider here the parameter $a_{1,kl}$ as a free parameter.

Proposition 1.12. *The square matrix of size p defined by*

$$\left(\left\langle \left[\frac{\partial \mathcal{F}_a}{\partial a_{b,kl}}\right]_{a^0}, e_{\Phi(b',kl)} \right\rangle \right)_{b,b'=1,\ldots,p}$$

is an invertible Vandermonde matrix.

Proof. We first compute the p components of degree kl of the trivializing vector fields $\theta^0_{b,kl}$ and $\theta^\infty_{b,kl}$ in the two charts (x_{c-1}, y_{c-1}) and (x_c, y_c) around $(D_c, 0)$ and (D_c, ∞), covering the principal component D_c (see notations (3.1) in Appendix A). Notice that, from Proposition (3.2), we have

$$E^*R = x_{c-1}\frac{\partial}{\partial x_{c-1}} = y_c \frac{\partial}{\partial y_c}.$$

Therefore the R-degree is also the x_{c-1}-degree or the y_c-degree. In what follows, the dots stand for terms of higher R-degree. We set $n_c := \sum_{b=1}^{p} n_b$ where the n_b's are the multiplicities of N_a on the cuspidal branches. We have

$$\widetilde{N_a}(x_{c-1}, y_{c-1}) = x_{c-1}^{kn_\infty+ln_0+kln_c} y_{c-1}^{vn_\infty+un_0+vln_c} \prod_{b=1}^{p}(a_{b,kl} + y_{c-1} + \cdots)^{n_b} \tag{9}$$

$$:= x_{c-1}^{m} P(y_{c-1}) + \cdots \tag{10}$$

where $m = kn_\infty + ln_0 + kln_c$, and P is a one variable polynomial. Now we have

$$\frac{\partial \widetilde{N_a}}{\partial x_{c-1}} = m x_{c-1}^{m-1} P(y_{c-1}) + \cdots, \quad \frac{\partial \widetilde{N_a}}{\partial y_{c-1}} = x_{c-1}^{m} P'(y_{c-1}) + \cdots, \tag{11}$$

$$\frac{\partial \widetilde{N_a}}{\partial a_{b,kl}} = \frac{n_b x_{c-1}^{m} P(y_{c-1})}{a_{b,kl} + y_{c-1}} + \cdots \tag{12}$$

Let us write

$$\theta_{b,kl}^{0} = \frac{x_{c-1}}{m} \alpha_{b,kl}^{0}(y_{c-1}) \frac{\partial}{\partial x_{c-1}} + \beta_{b,kl}^{0}(y_{c-1}) \frac{\partial}{\partial y_{c-1}} + \cdots.$$

Identifying the terms of lower x_{c-1}-degree in equation (7) on U_0, we obtain

$$\alpha_{b,kl}^{0} P + \beta_{b,kl}^{0} P' = \frac{n_b P}{a_{b,kl} + y_{c-1}}. \tag{13}$$

From the solution (A_0, B_0) of the following Bézout identity in $\mathbb{C}[y_{c-1}]$:

$$A_0 P + B_0 P' = P \wedge P',\ \deg(A_0) < \deg(P'/P \wedge P'),\ \deg(B_0) < \deg(P/P \wedge P'),$$

where $P \wedge P'$ stands for $\gcd(P, P')$, we obtain an holomorphic solution of equation (13) by setting

$$\left(\frac{n_b A_0 P}{(P \wedge P')(a_{b,kl} + y_{c-1})},\ \ \frac{n_b B_0 P}{(P \wedge P')(a_{b,kl} + y_{c-1})} \right).$$

We may suppose that the solution $(\alpha_{b,kl}^{0},\ \beta_{b,kl}^{0})$ coincides with this one. Indeed, one can check that another choice for the solution of the Bézout identity differs from this one by a vector field tangent (at the first order kl) to the foliation, holomorphic on U_0. We can perform a similar computation in the other chart (x_c, y_c) on (D_c, ∞). We have:

$$\widetilde{N_a}(x_c, y_c) = y_c^{kn_\infty+ln_0+kln_c} x_c^{(k-v)n_\infty+(l-u)n_0+(kl-ku)n_c} \prod_{b=1}^{p}(1 + a_{b,kl} x_c + \cdots)^{n_b}$$

$$:= y_c^{m} Q(x_c) + \cdots$$

Setting $\theta_{b,kl}^{\infty} = \alpha_{b,kl}^{\infty}(x_c) \frac{\partial}{\partial x_c} + \frac{y_c}{m} \beta_{b,kl}^{\infty}(x_c) \frac{\partial}{\partial y_c} + \cdots$, we have

$$\alpha_{b,kl}^{\infty} Q' + \beta_{b,kl}^{\infty} Q = \frac{n_b x_c Q}{1 + a_{b,kl} x_c}. \tag{14}$$

By considering the solution (A_∞, B_∞) of the following Bézout identity:

$$A_\infty Q + B_\infty Q' = Q \wedge Q',\ \deg(A_\infty) < \deg(\frac{Q'}{Q \wedge Q'}),\ \deg(B_\infty) < \deg(\frac{Q}{Q \wedge Q'})$$

we obtain an holomorphic solution of (14) on U_∞ by setting:

$$\alpha_{b,kl}^{\infty} = \frac{n_b x_c Q B_\infty}{(1 + a_{b,kl} x_c)(Q \wedge Q')},\ \beta_{b,kl}^{\infty} = \frac{n_b x_c Q A_\infty}{(1 + a_{b,kl} x_c)(Q \wedge Q')}.$$

In order to compute the cocycles, we give the expression of $\theta^0_{b,kl}$ in the chart (x_c, y_c). Since we have $x_{c-1} = x_c y_c, \quad y_{c-1} = x_c^{-1}$, we obtain

$$\frac{\partial}{\partial x_{c-1}} = x_c^{-1}\frac{\partial}{\partial y_c}, \quad \frac{\partial}{\partial y_{c-1}} = -x_c^2\frac{\partial}{\partial x_c} + x_c y_c\frac{\partial}{\partial y_c}.$$

Furthermore, by considering the reduced polynomials related to P and Q, we also have

$$\frac{P}{P\wedge P'}(y_{c-1}) = \frac{1}{x_c^{p+2}}\frac{Q}{Q\wedge Q'}(x_c).$$

We obtain:

$$\theta^0_{b,kl} = \frac{n_b x_c^{-(p+2)}Q/Q\wedge Q'(x_c)}{(a_{b,kl}+x_c^{-1})}\left[m^{-1}A_0(x_c^{-1})y_c\frac{\partial}{\partial y_c} + B_0(x_c^{-1})(-x_c^2\frac{\partial}{\partial x_c} + x_c y_c\frac{\partial}{\partial y_c})\right] + \cdots$$

We consider now a vector field θ_{is} on $U_0\cap U_\infty$ tangent to the saturated foliation defined by $\mathrm{d}\widetilde{N_a}$, with isolated singularities. Since

$$-\frac{\partial\widetilde{N_a}}{\partial y_c}\frac{\partial}{\partial x_c} + \frac{\partial\widetilde{N_a}}{\partial x_c}\frac{\partial}{\partial y_c} = \left(-my_c^{m-1}Q(x_c)+\cdots\right)\frac{\partial}{\partial x_c} + \left(y_c^m Q'(x_c)+\cdots\right)\frac{\partial}{\partial y_c}$$

we can choose

$$\theta_{is} := \left(-\frac{Q}{Q\wedge Q'}+\cdots\right)\frac{\partial}{\partial x_c} + \left(y_c\frac{Q'}{mQ\wedge Q'}+\cdots\right)\frac{\partial}{\partial y_c}.$$

Let $\Phi^{0,\infty}_{b,kl}$ be the function such that $\theta^0_{b,kl} - \theta^\infty_{b,kl} = \Phi^{0,\infty}_{b,kl}\cdot\theta_{is}$. By computing the coefficient of $\theta^0_{b,kl} - \theta^\infty_{b,kl}$ on $\partial/\partial x_c$, we have:

$$\Phi^{0,\infty}_{b,kl} = \frac{n_b x_c}{1+a_{b,kl}x_c}[x_c^{-p}B_0(x_c^{-1}) - B_\infty(x_c)].$$

The value of $\left\langle\left[\frac{\partial\mathcal{F}_a}{\partial a_{b,kl}}\right]_{a^0}, e_{\Phi(b',kl)}\right\rangle$ is by construction the coefficients on x_c^i of the Laurent series of $\Phi^{0,\infty}_{b,kl}$ where i is defined by $\Phi(b',kl) = (i,0)$ (i.e., from (5), $i = [-\nu_c\frac{k-v}{k}[+p+1-b')$. Thus we only have to consider the meromorphic part of $\Phi^{0,\infty}_{b,kl}$, i.e.:

$$\frac{n_b x_c}{1+a_{b,kl}x_c}\times\frac{\overline{B_0}(x_c)}{x_c^{2p}},$$

where $\overline{B_0}(x) = \sum_{n=0}^p v_n x^n$ is the polynomial function $x^p B_0(x^{-1})$. We have

$$\begin{aligned}\frac{x_c}{1+a_{b,kl}x_c} &= \sum_{m=0}^{+\infty}(-a_{b,kl})^m x_c^{m+1},\\ \frac{\overline{B_0}(x_c)}{x_c^{2p}} &= \sum_{n=0}^{p} v_n x_c^{n-2p}.\end{aligned}$$

Therefore, the coefficient of the Laurent series of $\Phi^{0,\infty}_{b,kl}$ in x_c^i is

$$\begin{aligned}\sum_{\substack{(m+1)+(n-2p)=i\\ 0\le n\le p}} n_b v_n(-a_{b,kl})^m &= n_b\sum_{n=0}^{p} v_n(-a_{b,kl})^{2p-1-n+i}\\ &= n_b\overline{B_0}(-a_{b,kl}^{-1})\times(-a_{b,kl})^{2p-1+i}\end{aligned}$$

Finally we obtain

$$\left\langle \left[\frac{\partial \mathcal{F}_a}{\partial a_{b,kl}}\right]_{a^0}, e_{\Phi(b',kl)} \right\rangle = C_b(-a_{b,kl}^{-1})^{b'} \tag{15}$$

where $C_b = n_b\overline{B_0}(-a_{b,kl}^{-1}) \times (-a_{b,kl})^{\left(3p+\left[-\nu_c \frac{k-v}{k}\right[\right)}$. This defines a Vandermonde matrix. Furthermore, $C_b = 0$ if and only if $B_0(-a_{b,kl}) = 0$, which cannot happen: evaluating the Bézout identity

$$A_0\frac{P}{P\wedge P'} + B_0\frac{P'}{P\wedge P'} = 1$$

at $y_0 = -a_{b,kl}$, we would obtain a contradiction, since $-a_{b,kl}$ is a root of P.

Moreover, $a_{b,kl} \neq a_{b',kl}$ for $b \neq b'$, thus the Vandermonde matrix is invertible. □

Step 2. *Relationship between the components of the cocycles on different levels.*

Lemma 1.13. *If $\theta^0_{b,kl}$, $\theta^\infty_{b,kl}$ are trivializing vector fields on U_0 (resp. on U_∞) for the direction $\frac{\partial}{\partial a_{b,kl}}$, then for any $d > kl$, the vector fields*

$$\frac{\tilde{m}^d}{\tilde{m}^{kl}}\ \theta^0_{b,kl},\quad \frac{\tilde{m}^d}{\tilde{m}^{kl}}\ \theta^\infty_{b,kl}$$

are trivializing vector fields on U_0 and U_∞ for the direction $\frac{\partial}{\partial a_{b,d}}$ where $\tilde{m} = m \circ E$.

Proof. Let $B_b := y^k + \sum_{(b,d),\Phi(b,d)\in\mathbb{T}} a_{b,d}m^d$ be the branch of index b, and $\widetilde{B_b} := B_b \circ E$. Since we have:

$$\frac{\partial \widetilde{N_a}}{\partial a_{b,d}} = \tilde{m}^d n_b \frac{\widetilde{N_a}}{\widetilde{B_b}}$$

if $\theta^0_{b,kl}$ satisfies equation (7) for $d = kl$, then, given $d > kl$, $\frac{\tilde{m}^d}{\tilde{m}^{kl}}\theta^0_{b,kl}$ satisfies the trivializing equation for the level d. Furthermore, this vector field is still holomorphic on U_0. Indeed, from the trivializing equation (7), we deduce that the multiplicity of the trivializing vector field $\theta^0_{b,kl}$ on a component D_i of $D \cap U_0$ is given by

$$\nu_i(\theta^0_{b,kl}) = \nu_i(\tilde{m}^{kl}) - \nu_i(\widetilde{B_b}) + 1.$$

The multiplicity of $\frac{\tilde{m}^d}{\tilde{m}^{kl}}\theta^0_{b,kl}$ on D_i is thus equal to

$$\nu_i\left(\frac{\tilde{m}^d}{\tilde{m}^{kl}}\theta^0_{b,kl}\right) = \nu_i(\tilde{m}^d) - \nu_i(\tilde{m}^{kl}) + \nu_i(\tilde{m}^{kl}) - \nu_i(\widetilde{B_b}) + 1$$

and therefore is still a positive number. The argument is similar for $\theta^\infty_{b,kl}$. □

We consider the linear operator

$$T_d := \frac{\tilde{m}^d}{\tilde{m}^{kl}}\times\ :\ \Theta_0(U_0\cap U_\infty) \longrightarrow \Theta_0(U_0\cap U_\infty).$$

induced by the previous Lemma. We can remark that when d runs over $\{kl,\ kl+1,\cdots\}$ the points $T_d \cdot e_{\Phi(b,kl)}$ are exactly the paths introduced in the previous section, and the indexation $(i,j) = \Phi(b,d)$ has been introduced such that

$$T_d \cdot e_{\Phi(b,kl)} = e_{\Phi(b,d)}.$$

Proposition 1.14. *Let $d > kl$ be the index of an horizontal level in the half plane representing $\Theta_0(U_0 \cap U_\infty)$. For each $b,\ b'$ in $1,\ldots,p$, we have that*

(1) *for any d' such that $kl \leq d' < d$, one has*

$$\left\langle \left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0}, e_{\Phi(b',d')} \right\rangle = 0.$$

(2) *the coefficient* $\left\langle \left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0}, e_{\Phi(b',d)} \right\rangle$ *is constant with respect to d (i.e. constant along the paths introduced in the previous section).*

(3) *the coefficient* $\left\langle \left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0}, e_{\Phi(b',d')} \right\rangle$ *only depends on the variables $a_{b'',d''}$ with*

$$kl \leq d^{''} \leq kl + d^{'} - d.$$

Proof. For $b = 1, \ldots, p$, we have:

$$\left[\frac{\partial \mathcal{F}_a}{\partial a_{b,kl}}\right]_{a^0} = \sum_{b'} \left\langle \left[\frac{\partial \mathcal{F}_a}{\partial a_{b,kl}}\right]_{a^0}, e_{\Phi(b',kl)} \right\rangle e_{\Phi(b',kl)} + \cdots$$

where the dots correspond to components of higher level. Applying the linear operator T_d, we obtain:

$$\left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0} = \sum_{b'} \left\langle \left[\frac{\partial \mathcal{F}_a}{\partial a_{b,kl}}\right]_{a^0}, e_{\Phi(b',kl)} \right\rangle e_{\Phi(b',d)} + \cdots$$

which proves the statements (1) and (2). For the third point, we consider the meromorphic function $\Phi_{b,d}^{0,\infty}$ defined by

$$\theta_{b,d}^{0} - \theta_{b,d}^{\infty} = \Phi_{b,d}^{0,\infty} \theta_{is} \tag{16}$$

where θ_{is} is the vector field with isolated singularities which generates the foliation on $U_0 \cap U_\infty$, introduced in step 1. Recall that the coefficient

$$\left\langle \left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0}, \ e_{i,j} \right\rangle$$

is nothing but the coefficient of $x_c^i y_c^j$ in the Laurent development of $\Phi_{b,d}^{0,\infty}$ (see the proof of proposition (1.12)). On the first level ($d = kl$), setting

$$\theta_{b,kl}^{\infty} = \left(\theta_{b,kl}^{\infty}\right)_{x_c} \frac{\partial}{\partial x_c} + \left(\theta_{b,kl}^{\infty}\right)_{y_c} y_c \frac{\partial}{\partial y_c},$$

the second relation in (7) can be written

$$\left(\theta_{b,kl}^{\infty}\right)_{x_c} \frac{\partial \widetilde{N_a}}{\partial x_c} + \left(\theta_{b,kl}^{\infty}\right)_{y_c} y_c \frac{\partial \widetilde{N_a}}{\partial y_c} = \frac{\partial \widetilde{N_a}}{\partial a_{b,kl}}. \tag{17}$$

Now, extending the expression of the partial derivatives of $\widetilde{N_a}$ filtered by y_c variable as in (11) leads to expressions of the following form

$$\begin{aligned}
y_c \frac{\partial \widetilde{N_a}}{\partial y_c} &= y_c^m A_0(x_c) + y_c^{m+1} A_1(x_c) + y_c^{m+2} A_2(x_c) \cdots \\
\frac{\partial \widetilde{N_a}}{\partial x_c} &= y_c^m B_0(x_c) + y_c^{m+1} B_1(x_c) + y_c^{m+2} B_2(x_c) \cdots \\
\frac{\partial \widetilde{N_a}}{\partial a_{b,kl}} &= y_c^m C_0(x_c) + y_c^{m+1} C_1(x_c) + y_c^{m+2} C_2(x_c) \cdots
\end{aligned}$$

where m is defined in (9). From the construction of the normal form N_a, it can be seen that for any i, A_i, B_i and C_i depend only on the variable $a_{b',d}$ with $kl \leq d \leq kl + i$. Thus, if one filters the equation (17) with respect to the y_c variable, one can see that the solution $\theta^{\infty}_{b,kl}$ shares the same property of filtration, namely, if one writes

$$\theta^{\infty}_{b,kl} = (D_0(x_c) + D_1(x_c)y_c + \cdots)\frac{\partial}{\partial x_c} + (E_0(x_c) + E_1(x_c)y_c + \cdots)y_c\frac{\partial}{\partial y_c},$$

then D_i and E_i depend only on the variables $a_{b',d}$ with $kl \leq d \leq kl+i$. The same remark can be done for $\theta^{0}_{b,kl}$ using a filtration with respect to the x_{c-1} variable. Finally, since θ_{is} has also the same property of filtration, the relation (16) implies that the jet of order i with respect to the y_c variable of $\Phi^{0,\infty}_{b,kl}$ depends only on the variables $a_{b',d}$ with $kl \leq d \leq kl+i$. Now, since the vertical component of T_d is just a translation induced by $\times y_c^{d-kl}$, this property propagates as in the statement of (3). $\square$

End of the proof of Theorem (1.10).
Notice first that the operator T_d is "quite well defined" in the cohomology group $H^1(D,\Theta_0)$: let $M_2 = e_{\Phi(2,kl)}, \ldots, M_p = e_{\Phi(p,kl)}$ be the roots of the paths indexed by the branches $b = 2,\ldots,p$ and M_1 the point under the root of the first branch (recall that this root is on the second level since we set $a_{1,kl} = 1$). If we pick a point on the first level outside $M_1, M_2, \ldots, M_p$, the action of T_d preserves the half planes corresponding to $\Theta_0(U_0)$ and $\Theta_0(U_\infty)$. This is clear on figure (3) and it is a consequence of Proposition (1.8): these paths cannot go back inside the triangle. Therefore the operator T_d is well defined on $H^1(D,\Theta_0)$ excepted on the line generated by $M_1 = e_{\Phi(1,kl)}$. We add this point to the triangle and now we can write the $(\delta+1)\times(\delta+1)$-matrix of the cocycles $\frac{\partial \mathcal{F}_a}{\partial a_{1,kl}},\ldots\ \frac{\partial \mathcal{F}_a}{\partial a_{p,kl}},\frac{\partial \mathcal{F}_a}{\partial a_{1,kl+1}},\ldots,\frac{\partial \mathcal{F}_a}{\partial a_{p,kl+1}},\ldots$ on $\{e_{\Phi(b,d)}\}_{b,d}$, ordered by the lexicographic order. According to the previous Proposition (1.12) and Proposition (1.14) this matrix is a block triangular matrix:

$$V := \begin{pmatrix} (V_1) & 0 & 0 & \cdots \\ \times & (V_2) & 0 & \cdots \\ \times & \times & (V_3) & \cdots \\ \times & \times & \times & \ddots \end{pmatrix}$$

where V_1 is the invertible Vandermonde matrix obtained in step 1, V_2, V_3... are sub-matrices of consecutive lines and columns of V_1 defined by the paths from the first level to the following levels. Clearly since $\det(V) = \prod \det(V_i)$ this matrix is an invertible one. Finally, since $a_{1,kl} = 1$, $e_{1,kl} \notin \mathbb{T}$, the matrix M is obtained by deleting the first line and first column of V, and is still an invertible one. $\square$

1.4. **The global moduli space of foliations.**

Proposition 1.15 (Existence of normal forms)**.** *For any f in $\mathcal{T}_{(k,l),(n)}$, there exists a in $\mathcal{A}$ such that $f \sim N_a$, where $\sim$ denotes the classification of foliations.*

Proof. We can suppose that f is given under its prenormalization form (1.1). Therefore the deformation defined by

$$f_\lambda := \frac{1}{\lambda^r} f\left(\lambda^k x, \lambda^l y\right)$$

where $r = kn_\infty + ln_0 + kln_c$, $(n_c = \sum_b n_b)$ is an equireducible unfolding of $f_0 = N_{a_0}$,

$$a_0 = (1, a_2, \ldots, a_p, 0, \ldots, 0) \in \mathcal{A}.$$

Using Theorem (1.10), we can ensure that for λ small enough, there exists $a \in \mathcal{A}$ such that $f_\lambda \sim N_a$. Furthermore, this deformation is analytically trivial for $\lambda \neq 0$, since we construct it by a conjugacy. Therefore, $f = f_1 \sim f_\lambda$, for λ small, and the proposition is proved. $\square$

Let us consider the diffeomorphism: $h_\lambda(x, y) = (\lambda^k x, \lambda^l y)$. We have:

$$N_a \circ h_\lambda = \lambda^{n_c} N_{\lambda \cdot a}, \quad \text{with } \lambda \cdot a = \lambda \cdot (a_{b,d}) := (\lambda^{d-kl} a_{b,d}).$$

As above, we have thus $N_a \sim N_{\lambda \cdot a}$. Actually, this action of $\mathbb{C}^*$ the only obstruction to the unicity of normal forms:

Theorem 1.16 (Unicity of normal forms)**.** *$N_a \sim N_{a'}$ if and only if there exists a complex number $\lambda \neq 0$ such that $a^{'} = \lambda \cdot a$.*

Proof - Suppose that there exists a conjugacy relation

$$\psi \circ N_{a'} = N_a \circ \phi. \tag{18}$$

Following [1], we can suppose that ψ is an homothetie $\gamma \mathrm{Id}$. We are going to reduce the proof to the case where ϕ is tangent to the identity. Since the conjugacy preserves the numbering of the branches, looking at the relation induced by (18) on the jet of smaller (k, l)-order, we can ensure that the linear part of ϕ is written $h_\lambda = (\lambda^k x, \lambda^l y)$ for some $\lambda \neq 0$. Then

$$N_a \circ \phi \circ h_\lambda^{-1} = \gamma N_{a'} \circ h_\lambda^{-1} = c N_{\lambda^{-1} \cdot a'}$$

where c stands for some non vanishing number. Since $\phi \circ h_\lambda^{-1}$ is tangent to the identity, it appears that $c = 1$. Thus, setting for the sake of simplicity $a^{'} = \lambda^{-1} \cdot a^{'}$ and $\phi = \phi \circ h_\lambda^{-1}$ we are led to a relation

$$N_{a'} = N_a \circ \phi \tag{19}$$

where ϕ is tangent to the identity. The proof reduces to show that in the situation (19), we have $a = a'$. Let X be a germ of formal vector field such that $\phi = e^X$. The vector field X can be decomposed in the sum of its quasi-homogeneous components

$$X = X_\nu + X_{\nu+1} + \cdots$$

Lemma 1.17. *If $N_a \circ e^{X_\nu + \cdots} = N_{a'}$ then for all b from 1 to p and all $d \leq kl + \nu - 1$, $a_{b,d} = a'_{b,d}$.*

Proof. We set:

$$N_a = N_a^{(N)} + \cdots + N_a^{(N+p-1)} + \ N_a^{(N+p)} + \cdots$$

where $N = kn_\infty + ln_0 + kln_c$ is the degree of the first quasi-homogeneous component of N_a. Since we have

$$e^{X_\nu + \cdots} N_a = N_a + X_\nu \cdot N_a + \cdots$$

we obtain $N_a^{(N+i)} = N_{a'}^{(N+i)}$ for i from 0 to $\nu - 1$. The expression of $N_a^{(N+i)}$ only depends on the variables $a_{b,d}$ for $d \leq kl + i$. Finally we claim that $N_a^{(N+i)} = N_{a'}^{(N+i)}$ if and only if $a_{b,d} = a'_{b,d}$ for $d \leq kl + i$. This fact can be proved by induction on $d \leq kl + i$. It is obvious for $d = kl$. Suppose that $a_{b,d} = a'_{b,d}$ is true for $d \leq kl + j - 1$ with $j - 1 < i$. Then we have:

$$\sum_b \frac{N_a^{(N)}}{y^k + a_{b,kl} x^l} a_{b,kl+j} m^{kl+j} = \sum_b \frac{N_a^{(N)}}{y^k + a_{b,kl} x^l} a^{'}_{b,kl+j} m^{kl+j}.$$

which implies that $a_{b,kl+j} = a'_{b,kl+j}$. □

Now if $\nu \geq pkl + l\epsilon_0 + k\epsilon_\infty$ and $N_a \circ e^{X_\nu + \cdots} = N_{a'}$ then according to the previous lemma, for all b and $d \leq kl + pkl + l\epsilon_0 + k\epsilon_\infty - 1$, $a_{b,d} = a^{'}_{b,d}$. Since $\nu_c = pkl + k\epsilon_\infty + l\epsilon_0 - k - l$ then

$$\underbrace{\nu_c + kl - 1}_{\substack{\text{bigger value of } d \\ (b,\, d) \in \mathbb{T}}} < pkl + kl + l\epsilon_0 + k\epsilon_\infty - 1.$$

Therefore we have $a = a^{'}$. Thus, it remains to prove the following lemma:

Lemma 1.18. *If* $N_a \circ e^{X_\nu+\cdots} = N_{a'}$ *then* $\nu \geq pkl + l\epsilon_0 + k\epsilon_\infty$.

Proof. It suffices to prove that $\nu < pkl + l\epsilon_0 + k\epsilon_\infty$ leads to a contradiction. Since the conjugacy ϕ does not modify the parameter $a_{b,d}$ for $d \leq kl + \nu - 1$ the first non trivial relation of the smallest (k,l)-degree induced by (19) is written

$$X_\nu \cdot N_a^{(N)} = -N_a^{(N+\nu)} + N_{a'}^{(N+\nu)}.$$

Dividing by $N_a^{(N)}$ leads to

$$\sum_{b=1}^{p} n_b \frac{X_\nu \cdot (y^k + a_{b,kl}x^l)}{y^k + a_{b,kl}x^l} + n_\infty \frac{X_\nu \cdot x}{x} + n_0 \frac{X_\nu \cdot y}{y} = m^{kl+\nu} \sum_{b=1}^{p} \frac{-a_{b,kl+\nu} + a'_{b,kl+\nu}}{y^k + a_{b,kl}x^l}.$$

We take the pull-back of the previous equality with respect to the map E and write it in the coordinates (x_c, y_c). Since we are going to look at residus at $x_c = -a_{b,kl}^{-1}$, we only make appear the terms having poles at these points:

$$\cdots + \sum_{b=1}^{p} n_b \frac{a_{b,kl}\widetilde{X_\nu} \cdot x_c}{1 + a_{b,kl}x_c} = x_c^{\nu - iv - ju + ku} \sum_{b=1}^{p} \frac{\delta_{b,kl+\nu}}{1 + a_{b,kl}x_c}$$

where $\widetilde{X_\nu}$ stands for the vector field $\frac{E^* X_\nu}{y_c^\nu}$, $\delta_{b,kl+\nu}$ for the difference $-a_{b,kl+\nu} + a'_{b,kl+\nu}$ and i, j for the couple of integers such that $m^{kl+\nu} = x^i y^j$.
Since the integer $\nu - iv - ju + ku$ is non negative, evaluating the residue at $-a_{b,kl}^{-1}$ yields the relation

$$n_b a_{b,kl} \widetilde{X_\nu} \cdot x_c \left(-a_{b,kl}^{-1}\right) = \left(-a_{b,kl}^{-1}\right)^{\nu - iv - ju + ku} \delta_{b,kl+\nu}. \tag{20}$$

A straightforward computation shows that $\widetilde{X_\nu} \cdot x_c$ is a polynomial function in x_c that is written the following way

(1) if $\epsilon_0 = 1$ –that is if the curve $y = 0$ is invariant– or if $\frac{\nu+l}{k}$ is not an integer

$$\widetilde{X_\nu} \cdot x_c = \sum_{\nu(1-\frac{u}{l}) \leq w \leq \nu(1-\frac{v}{k}),\ w \in \mathbb{N}} p_w x_c^w = x_c^{\left[\nu(1-\frac{u}{l})\right]+1} \left(\sum_{w=0}^{\left[\nu(1-\frac{v}{k})\right] - \left[\nu(1-\frac{u}{l})\right] - 1} q_w x_c^w \right)$$

(2) else

$$\widetilde{X_\nu} \cdot x_c = \sum_{\nu(1-\frac{u}{l}) \leq w \leq \nu(1-\frac{v}{k}) + \frac{1}{k},\ w \in \mathbb{N}} p_w x_c^w = x_c^{\left[\nu(1-\frac{u}{l})\right]+1} \left(\sum_{w=0}^{\left[\nu(1-\frac{v}{k}) + \frac{1}{k}\right] - \left[\nu(1-\frac{u}{l})\right] - 1} q_w x_c^w \right)$$

Now, in view of the construction of the normal form, the coefficient $\delta_{b,kl+\nu}$ has to be zero for

$$p - \sharp\mathbb{Z} \cap \left] \frac{k-v}{k}(\nu - \nu_c), \frac{l-u}{l}(\nu - \nu_c) \right[$$

values of the parameter b. Thus, according to (20), the polynomial function $\widetilde{X_\nu} \cdot x_c$ has the same number of non-vanishing roots among the values $-a_{b,kl}^{-1}, b = 1, \ldots, p$. This number is strictly greater than the degree of the polynomial functions factorized in the above expressions of $\widetilde{X_\nu} \cdot x_c$. Thus, the latter has to be the zero polynomial function. Therefore, looking again at the relation (20) yields

$$\forall b,\ \delta_{b,kl+\nu} = 0.$$

Hence, the vector field X_ν has to be tangent to $N_a^{(N)}$ which is a contradiction with the hypothesis $\nu < pkl + l\epsilon_0 + k\epsilon_\infty$. $\square$

Finally, we can summarize the previous results by

Theorem 1.19. *The moduli space $\mathcal{M}_{(k,l),(n)}$ is isomorphic to ${}^{\mathcal{A}}/_{\mathbb{C}^*}$ where the action of $\mathbb{C}^*$ is defined by*

$$\lambda \cdot a = \lambda \cdot (a_{b,d}) = (\lambda^{d-kl} \cdot a_{b,d})$$

2. The moduli space of curves

Let $\mathcal{C}$ be the partition of $\mathcal{M} = \mathcal{M}_{(k,l),(n)}$ induced by the classification of curves $\sim_c$.

2.1. **The infinitesimal generators of $\mathcal{C}$.** We first recall general facts proved in [8], which are valid in every topological class. Let $\mathcal{F}$ be a foliation defined by an holomorphic function f (or more generally by any generic non dicritical differential form ω), and let S be the curve defined by $f = 0$ (or by the separatrix set of ω). Let $E: M \to (\mathbb{C}^2, 0)$ be the desingularization map of the foliation, and D its exceptional divisor. We denote by $\widetilde{f}$, $\widetilde{\mathcal{F}}$, $\widetilde{S}$ the pull back by E on M of f, $\mathcal{F}$ or S. The tangent space to the point $[S]$ in the moduli space of curves (for $\sim_c$) is the cohomological group $H^1(D, \Theta_S)$ where Θ_S is the sheaf on D of germs of vector fields tangent to $\widetilde{S}$. The inclusion of $\Theta_{\mathcal{F}}$ into Θ_S induces a map i:

$$H^1(D, \Theta_{\mathcal{F}}) \xrightarrow{i} H^1(D, \Theta_S)$$

whose kernel represents the directions of unfolding of foliations with trivial associate unfolding of curves.

Definition 2.1. *An open set U of M is a* quasi-homogeneous open set *(relatively to f) if there exists an holomorphic vector field R_U on U such that $R_U(\widetilde{f}) = \widetilde{f}$.*
We can always cover D by two quasi-homogeneous open sets U and V. The cocycle of quasi-homogeneity $[R_{U,V}]$ of $\mathcal{F}$ is the element of $H^1(D, \Theta_{\mathcal{F}})$ induced by $R_U - R_V$.

Recall that $H^1(D, \Theta_{\mathcal{F}})$ has a natural structure of $\mathcal{O}_2$-module. We have:

Theorem 2.2. [8] *The kernel of the map i is generated by the cocycle of quasi-homogeneity, i.e.:*

$$\ker(i) = \{h \cdot [R_{U,V}],\ h \in \mathcal{O}_2\}.$$

Notice that the distribution induced by these directions is integrable and defines a singular foliation $\mathcal{C}$ on $\mathcal{A}$. The point corresponding to the topological model is a singular one: indeed, this model is quasihomogeneous. Therefore the whole open set $U = M$ is quasi-homogeneous, and the cocycle $[R_{U,V}]$ is trivial for this foliation.

Let $X_{m,n}$ be the vector fields on $\mathcal{A}$ generated by $x^m y^n \cdot [R_{U,V}]$. Below, we describe some properties of the distribution induced by the vector fields $X_{m,n}$.

Proposition 2.3.

(1) *The $\mathcal{O}_2$-generator of $\mathcal{C}$ is given by:*

$$X_{0,0} = -\frac{1}{r} \sum_{\Phi(b,d) \in \mathbb{T} \cup \{(1,kl)\}} (d - kl) a_{b,d} \left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0}$$

where $r = kn_0 + ln_\infty + kl \sum_{b=1}^{p} n_b$

(2) *For any level d we denote by $X^d_{m,n}$ the components of the vector field $X_{m,n}$ on the subspace Vect$\{e_{\phi(b,d)},\ b=1,\dots,p\}$. For any $m,\ n,\ X_{m,n}$ is quasihomogeneous with respect to the degree induced by $rX_{0,0}$. Indeed, we have*

$$[rX_{0,0}, X_{m,n}] = (km+ln)\,X_{m,n}.$$

The coefficients of $X^{\nu}_{m,n}$ are quasi-homogeneous with respect to the weight $rX_{0,0}$ of degree $\nu-km-ln-kl$. In particular, they only depend on the variables $a_{b,d}$ with $d\le \nu-km-ln$.

(3) *If we decompose the vector field $X_{0,0}$*

$$\begin{aligned} X_{0,0} &= -\frac{1}{r}\sum_d\sum_b (d-kl)\,a_{b,d}\left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0} \\ &= -\frac{1}{r}\sum_{i\in\mathbb{Z},j\ge 1}\underbrace{\left(\sum_{0\le d-kl\le j}\sum_b (d-kl)\,a_{b,d}\left\langle\left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0}\ ,\ e_{i,j}\right\rangle\right)}_{\Gamma_{i,j}(a)} e_{i,j}. \end{aligned}$$

then the functions $\Gamma_{i,j}(a)$ are algebraically independent.

(4) *The vector fields defined by*

$$\widetilde{X_{m,n}} = a^{km+ln}_{2,kl+1}X_{m,n},$$

commute with $X_{0,0}$. Therefore, they induce the distribution $\mathcal{C}$ on $\mathcal{M}$.

Proof. 1. The proof is the same as the one of proposition (5.5) of [8] with a very slight change where we replace $(\lambda x,\lambda y)$ with $(\lambda^k x,\lambda^l y)$.
2. The proof is also a slight generalization of the proof of Proposition (5.9) in [8].
3. Let us decompose the coefficient $\Gamma_{i,j}(a)$

$$\Gamma_{i,j}(a) = \underbrace{\sum_b j a_{b,kl+j}\left\langle\left[\frac{\partial \mathcal{F}_a}{\partial a_{b,kl+j}}\right]_{a^0}\ ,\ e_{i,j}\right\rangle}_{L_{i,j}} + \underbrace{\sum_{0\le d-kl<j}\sum_b (d-kl)\,a_{b,d}\left\langle\left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0}\ ,\ e_{i,j}\right\rangle}_{R_{i,j}}.$$

Following, the proposition (1.14) the function $\left\langle\left[\frac{\partial \mathcal{F}_a}{\partial a_{b,kl+j}}\right]_{a^0}\ ,\ e_{i,j}\right\rangle$ depends only on the variables $a_{b',kl}$ with $\Phi(b',d)=(i,j)$. The expression $\left\langle\left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0}\ ,\ e_{i,j}\right\rangle$ in $R_{i,j}$ depends only on the variables $a_{b,d'}$ where d' satisfies

$$0\le d^{'}-kl\le j-(d-kl) \implies d^{'}\le j+kl-(d-kl)<j+kl.$$

In view of the proposition (1.12), for a fixed value of $j=J$, the functions $L_{i,J}$ considered as linear functions of the variables $a_{b,kl+j}$ are linearly independent because their matrix is an extraction of consecutive rows and columns in the Vandermonde matrix of 1.12. Thus, they are also algebraically independent as a whole. Now, let us consider an algebraic relation between the functions $\Gamma_{i,j}(a)$ given by a polynomial function $P\left(\{X_{i,j}\}_{(i,j)\in\mathbb{T}}\right)$ where the $X_{i,j}$'s are some independent variables

$$P\left(\Gamma_{i,j}(a)\right)=0.$$

Let J be the greatest integer such that there exists a point (i,J) in $\mathbb{T}$ and denote by

$$\{(i_0,J),(i_1,J),\dots,(i_q,J)\}$$

the family of points in $\mathbb{T}$ at the level J. The relation P is written

$$P\left(\{\Gamma_{i,j}(a)\}_{j<J}, L_{i_0,J}(a)+R_{i_0,J}(a),\dots,L_{i_q,J}(a)+R_{i_q,J}(a)\right)=0.$$

We fix all the variables $a_{b,d}$ with $d-kl<J$ at a generic value. Then, the above relation becomes an algebraic relation between the affine forms $L_{i,J}(a)+R_{i,J}(a)$. Let us decompose the relation P as follows

$$P=\sum_{I\subset\{(i_k,J)\}_{k=0..q}} Q_I(X_{i,j})\,X_I$$

where $X_I=\prod_{(i,J)\in I} X_{i,J}$. Here, Q_I depends only on the variables $X_{i,j}$ with $j<J$. Since, the affine form $L_{i,J}(a)+R_{i,J}(a)$ are algebraically independent, for any I, we have

$$Q_I(\Gamma_{i,j}(a))=0,$$

which are algebraic relations between the functions $\Gamma_{i,j}(a)$ with $j<J$. Therefore, an inductive argument ensures that P has to be the trivial relation, which proves the property.
4. We recall that the global moduli space of foliations is obtained from the local one by considering the weighted action of $\mathbb{C}^*$ on $\mathcal{A}$ which is also the flow of $X_{0,0}$. Since the $X_{0,0}$-degree of the variable $a_{1,kl+1}$ is equal to 1, we have

$$\begin{aligned}\left[rX_{0,0},a_{2,kl+1}^{km+ln}X_{m,n}\right] &= rX_{0,0}(a_{2,kl+1}^{km+ln})X_{m,n}+a_{2,kl+1}^{km+ln}\left[rX_{0,0},X_{m,n}\right]\\ &= -(km+ln)a_{2,kl+1}^{km+ln}X_{m,n}+a_{2,kl+1}^{km+ln}\left[rX_{0,0},X_{m,n}\right]=0.\end{aligned}$$

□

2.2. **The dimension of the generic strata.** The dimension τ of the generic strata of the local moduli space of curves corresponds to the codimension of the distribution $\mathcal{C}$ at a generic point of $\mathcal{M}$. According to proposition 2.3, the family of coefficients $\{\Gamma_{ij}\}_{i,j}$ of $X_{0,0}$ is functionally independent: thus, any family of r vector fields in dimension r whose coefficients are chosen among the Γ_{ij}'s is generically free: indeed, their determinant cannot identically vanish since it would produce a functional relation between the Γ_{ij}'s. Thus, to compute the dimension of the generic strata, we just have to browse the triangle of moduli and to compute at each level d how many moduli can actually be reached by the vector fields $X_{m,n}$. For the following computations, we recommend to refer at each step to the example presented in Appendix B, figure 3.

Let us denote by $\nu(X_{m,n})=km+ln+kl+1$ the order of $X_{m,n}$. By construction, $\nu(X_{m,n})-kl$ is the first level of the triangle of moduli on which $X_{m,n}$ may have an action: indeed, since $X_{m,n}=x^my^nX_{0,0}$, its projections on the previous levels vanish. In most cases, $X_{m,n}$ can be used to kill a modulus which is exactly at its first level $\nu(X_{m,n})-kl$. However, in some cases, $X_{m,n}$ cannot be used this way because, for instance, the triangle of moduli has no modulus on this particular level: therefore, we use $X_{m,n}$ to kill a modulus on some level above. To take care of all this possibilities, we introduce a decomposition by blocks of the triangle of moduli and we prove some related arithmetical properties:

A block B_i in the triangle of moduli is a union of kl consecutive horizontal lines from the line of index $d_i=ikl+1$, see Figure 3. We denote by

- n_d the "dimension" of the line of index d which means the number of integer points on this line.
- $N_i=\sum_{d=ikl+1}^{(i+1)kl} n_d$ the dimension of the block B_i which is also the number of integer points in the whole block.
- $n_i^{\max}=\max\{n_d,\ d=ikl+1,\ldots,(i+1)kl\}$ which is the greatest dimension of a line in the block B_i.

One can easily prove, by using the equations of the edges of the triangle, the following lemma –see also figure 3– :

Lemma 2.4.

(1) *We have:* $N_{i+1} = N_i - kl, \quad n_i^{\max} = p - i$.
(2) *For each line of level* d *of the block* B_i, $n_d = n_i^{\max}$ *or* $n_i^{\max} - 1$.
(3) *On the first line* d_i *of the block* B_i, *the number* n_{d_i} *reaches the maximum* $n_i^{\max}$.

We denote by:

- q_d the number of vector fields $X_{m,n}$ such that $\nu(X_{m,n}) = d$
- $Q_i = \sum_{d=ikl+1}^{(i+1)kl} q_d$
- $q_i^{\max} = \max\{q_d,\ d = ikl+1, \ldots, (i+1)kl\}$.

One can check a similar result to (2.4):

Lemma 2.5.

(1) *We have:* $Q_{i+1} = Q_i + kl, \quad q_i^{\max} = i$.
(2) *For each line of level* d *of the block* B_i, $q_d = q_i^{\max}$ *or* $q_i^{\max} - 1$.
(3) *On the first line* d_i *of the block* B_i, *the number* q_{d_i} *reaches the maximum* $q_i^{\max}$.

We consider the maximal sequence of blocks B_i such that $q_i^{\max} = i < n_i^{\max} = p - i$, i.e. the sequence $B_1, \ldots, B_{]p/2]}$, where $]p/2]$ is the strict integer part of $p/2$. We call *critical block*, the block $B_{\frac{p}{2}}$ when p is even or the unique block B_1 that appears when $p = 1$. This block is going to be analyzed independently. In figure 3, this block is the second one, and in figure 1, since $p = 1$, this block is the sole block B_1.
Consider a block B_i such that $q_i^{max} > n_i^{max}$. For each line of index d of this block, since $q_d = q_i^{max}$ or $q_i^{max} - 1$, we have: $q_d \geq n_i^{max} \geq n_d$. According to the previous functional independence of the vector fields $X_{m,n}$, we can conclude that in this case, their action is transitive on such a block and the block above.
In the critical block $B_{\frac{p}{2}}$ or B_1, the integers $n_d - q_d$ for $d = d_i, \ldots, d_i + kl - 1$ can only take the values +1, 0 or -1, starting from the value 0 on the first level of the block. On the latter level, the action of the $X_{m,n}$ is thus transitive. We consider the first line of this block on which $n_d - q_d \neq 0$:

- If we have $n_d - q_d = +1$, there remains one dimension which cannot be reached by the action of the $X_{m,n}$. We have to count it in the codimension of the generic leaves of $\mathcal{C}$.
- If $n_d - q_d = -1$, the action of the vector fields $X_{m,n}$ is transitive on this level. Furthermore we have an extra vector field $X_{m,n}$ such that $\nu(X_{m,n}) = d$ whose higher components will act on the higher levels. Suppose that there exists a level $d' > d$ such that $n_{d'} - q_{d'} = +1$.

Therefore, in order to compute the generic dimension of the distribution $\mathcal{C}$ on the critical block, we have to introduce the following non commutative sum :

Definition 2.6. *Let* r_d *be a sequence taking its values in* $\{-1, 0, +1\}$. *The notation* $\widetilde{\sum}_d r_d$ *denotes the value obtained by the following operations:*

(1) *delete the values 0;*
(2) *delete recursively the consecutive values* $(-1, +1)$ *(but not the consecutive values* $(+1, -1)$*);*
(3) *after the two first steps, remains a sequence of* n *consecutive terms with value +1, followed by* m *consecutive terms with value -1. We set:* $\widetilde{\sum}_d r_d = n$.

Example. In the critical block of Figure 3, the sequence of values $n_d - q_d$ is:

$$\{0, +1, +1, 0, +1, 0, -1, +1, 0, -1, 0, -1, -1, 0, -1\}.$$

The extra vector field appearing on the 7^{th} position acts on the next level. The next extra vector fields are unuseful. Therefore, the number of free dimensions under the action of these vector fields is

$$\widetilde{\sum}\{0,+1,+1,0,+1,0,-1,+1,0,-1,0,-1,-1,0,-1\}=3.$$

From all the considerations above, we deduce the following:

Theorem 2.7. *The dimension of the generic strata of the moduli space for curves is*

$$\tau=\sum_{d=d_0}^{d_0+]p/2]+kl-1}(n_d-q_d)+\widetilde{\sum}_{d=d_{p/2}}^{d_{p/2}+kl-1}(n_d-q_d),$$

where $n_d=]\frac{v-k}{k}(\nu_c-d+kl)]-[\frac{u-l}{l}(\nu_c-d+kl)[$, q_d *is the number of positive integer solutions* (m,n) *of the equation* $km+ln+kl+1=d$*, and the second sum* $\widetilde{\sum}$ *is defined above and only appears if* p *is even or if* $p=1$ *(in this case, we set* $d_{1/2}=kl$*).*

Example. In the topological class $(k,l)=(3,5)$ and $p=4$ of figure 3, we obtain $\tau=35$.

2.3. Normal forms for curves.

Theorem 2.8. *We consider the reduced normal form*

$$N_a=x^{\epsilon_\infty}y^{\epsilon_0}\prod_{b=1}^{p}\left(y^k+\sum_{\{(b,\,d),\,\Phi(b,\,d)\,\in\,\mathbb{T}\}\,\cup\,\{(1,\,kl)\}}a_{b,d}m^d\right)$$

obtained for the classification of foliations defined by topologically quasi-homogeneous functions. We obtain a generic unique normal form $N_b,\ b\in\mathbb{C}^\tau$ *for the classification of curves by performing the following operations on* N_a*:*

1. *we set:* $a_{1,kl+1}=1$*;*
2. *for each level* d *in a block* B_i*,* $i\leq]p/2]$*, we set* $a_{b,d}=0$ *for the first* q_d *coefficients starting from the rightside of the line* d*;*
3. *for each level in the critical block* $B_{p/2}$ *(which appears if* p *is even), we consider the sequence of number* n_d-q_d *(recall that in this block we have* $n_d-q_d\in\{-1,0,+1\}$*).*
   - *if* $n_d-q_d=0$*, we vanish all the coefficients of the line;*
   - *if* $n_d-q_d=+1$*, we set* $a_{b,d}=0$ *for the first coefficient starting from the right side of the line* d*;*
   - *for the first lines such that* $n_d-q_d=-1$ *and encountered in the sequence on some line* d*, we set* $a_{b,d}=0$ *for the unique coefficient on this line. Furthermore, we set* $a_{b,d'}=0$ *for the second coefficient on the next line* $d'>d$ *such that* $n_{d'}-q_{d'}=+1$*, if such line exists.*
   - *for the last line such that* $n_d-q_d=-1$ *without upper line* d' *such that* $n_{d'}-q_{d'}=+1$ *we set* $a_{b,d}=0$ *for the unique coefficient on this line.*
4. *for each level* d *in a block* B_i*,* $i>]p/2]$*, and every index* b*, we set* $a_{b,d}=0$*.*

Proof. Since the projection $X_{0,0}^{(d_1)}$ of $X_{0,0}$ on the first line of the block B_1 is the radial vector field in the variables a_{b,d_1}, its flow acts by homothety on this level and we can make use of its action to normalize one coefficient to the value 1. We choose the first one starting from the right side.

On all the higher levels of index $d>d_1$ and for the q_d vector fields $X_{m,n}$ such that $\nu(X_{m,n})=d$, we have

$$X_{m,n}^{(d)}=\sum_b\Gamma_{m,n}(a_{d_0},a_{d_1})\frac{\partial}{\partial a_{b,d}}$$

in which $\Gamma_{m,n}(a_{d_0},a_{d_1})$ only depends on the variables a_{b,d_0} and a_{b,d_1}. This is a consequence of the relation $X_{m,n}=x^my^n\cdot X_{0,0}$ and of the proposition 1.14. Therefore this vector field is

constant with respect to the variables of the level $d > d_1$. Its flow acts by translation and we make use of this flow (and the independence property) to vanish q_d coefficients.
In the critical block, if there is an extra vector field $X_{m,n}$ on a line d such that $n_d - q_d = -1$, we make use of the component $X^{(d')}_{m,n}$ to act on the next level d' such that $n_{d'} - q_{d'} = +1$. Suppose that this level is the next one ($d' = d+1$). This means that we have to consider the action of the second non vanishing component of $X_{m,n}$. According to the proposition 1.14, this one will depend on the variables a_{b,d_0}, a_{b,d_1} and a_{b,d_1+1}. If we have to skip two lines it will depend on the variables a_{b,d_0}, a_{b,d_1}, a_{b,d_1+1} and a_{b,d_1+2}, and so on. Therefore, it turns out that the components of $X^{(d')}_{m,n}$ will only depend on variables $a_{b,d}$ with $d < d'$. Its flow still acts by translation and we make use of it to vanish the second coefficient of this line. $\square$

We give in Appendix B the generic normal form obtained in the topological class $(k,l) = (3,5)$ and $p = 4$.

2.4. **An example: the case $y^n + x^{n+1}$.** In [17], O. Zariski computes the dimension of the generic stratum of the moduli space of the curve

$$y^n + x^{n+1} = 0$$

for $n \geq 2$. We are going to apply our strategy to recover this dimension.
Let us consider $k = n$ and $l = n+1$. In this situation, the fundamental Bezout relation is written

$$n \cdot n - (n-1) \cdot (n+1) = 1.$$

Thus, $u = n$, $v = n-1$, $\nu_c = n^2 - n - 1$, and the triangle $\mathbb{T}$ is delimited by the two lines

$$j - ni = n^2 - n - 1$$
$$j - (n+1)i = n^2 - n - 1.$$

On a level j, this triangle bounds an interval

$$]l(j), r(j)[=] - \frac{n^2 - n - 1 - j}{n}, -\frac{n^2 - n - 1 - j}{n+1}[.$$

For $j \geq 0$, all these intervals have length less than 1, and we have:

$$l(j) \in \mathbb{Z} \quad \Leftrightarrow \quad \exists \alpha \in \mathbb{N}, j = -1 + (\alpha+1)n$$
$$r(j) \in \mathbb{Z} \quad \Leftrightarrow \quad \exists \alpha \in \mathbb{N}, j = 1 + \alpha(n+1).$$

Therefore, the interval $]l(j), r(j)[$ contains an integer if and only if there exists α in $\mathbb{N}$ such that

$$1 + \alpha(n+1) < j < -1 + (\alpha+1)n.$$

Thus we have $n_j = 1$ for the above values of the index j, and $n_j = 0$ else. Now we have for each $k \geq 0$

$$\nu(X_{k,0}) - d_0 = kn + 1,\ \nu(X_{k-1,1}) - d_0 = kn + 2, \cdots, \nu(X_{0,k}) - d_0 = kn + k + 1$$

where $d_0 = n(n+1)$. This gives $q_j = 1$ for the above values of the index j and 0 else. We summarize these results in figure 1.

From the previous remarks the sequence $n_j - q_j$, $j \geq 0$ takes the following values:

$$0, -1, \underbrace{1, 1, \ldots, 1}_{j=2,\ldots,n-2}, 0, 0, -1, -1, \underbrace{1, 1, \ldots, 1}_{j=n+3,\ldots,2n-2}, 0, 0, -1, -1, -1, \underbrace{1, 1, \ldots, 1}_{j=2n+4,\ldots,3n-2} \cdots$$

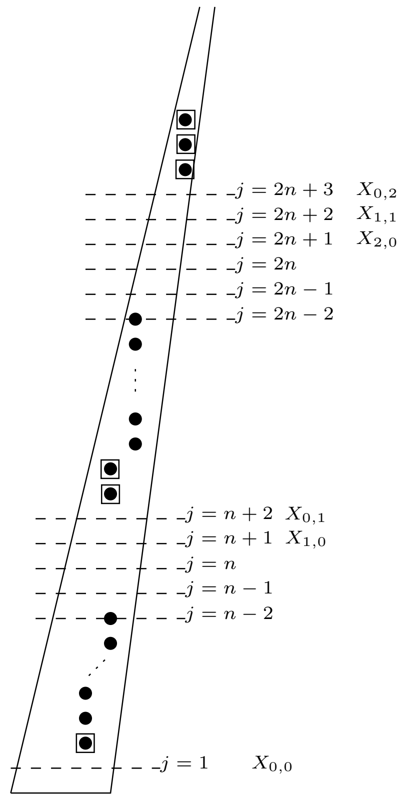


FIGURE 1. The case $y^n + x^{n+1}$

Since there is only one branch, there is only one block and it is a critical block. Therefore we have

$$\begin{array}{rcl}\tau = \widetilde{\Sigma}_{j\geq 0}(n_j - q_j) & = & (n-4)+(n-6)+(n-8)+\cdots+(0 \textit{ or } 1) \\ & = & \displaystyle\sum_{\alpha\geq 0}\sup(n-4-2\alpha,0) \\ & = & \dfrac{(n-4)(n-2)}{4} \text{ if } n \text{ is even} \\ & = & \dfrac{(n-3)^2}{4} \text{ if } n \text{ is odd}\end{array}$$

which are the formulas given in [17].

3. Appendix A: reduction of singularities of a topologically quasi-homogeneous function

Let f be a topologically quasihomogeneous function of weight (k,l) with p cuspidal branches, and multiplicities $(n_\infty, n_0, n_1, \ldots, n_p)$. From Lemma (1.1), we can consider a system of coordinates (x,y) such that f is written

$$f(x,y) = cx^{n_\infty}y^{n_0}\left(y^k + x^l + \cdots\right)^{n_1}\left(y^k + a_{2,kl}x^l + \cdots\right)^{n_2}\cdots\left(y^k + a_{p,kl}x^l + \cdots\right)^{n_p}$$

where the dots contains terms of (k,l)-degree bigger than kl.

Let θ_f be the vector field with an isolated singularity defined by

$$\theta_f = \frac{1}{\text{g.c.d.}\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)} \cdot \left(\frac{-\partial f}{\partial x}\frac{\partial}{\partial y} + \frac{\partial f}{\partial y}\frac{\partial}{\partial x}\right) \tag{21}$$

The vector field θ_f can be also defined as a dual of the 1-form $f^{\text{red}}\frac{df}{f}$ for the standard volume form $dx \wedge dy$

$$dx \wedge dy\,(\theta_f, \cdot) = f^{\text{red}}\frac{df}{f} \tag{22}$$

where

$$f^{\text{red}}(x,y) = cxy\left(y^k + x^l + \cdots\right)\left(y^k + a_{2,kl}x^l + \cdots\right)\cdots\left(y^k + a_{p,kl}x^l + \cdots\right).$$

3.1. **The desingularization.** The desingularization of f is exactly the same as its topological quasihomogeneous model f_{qh}

$$f_{\text{qh}}(x,y) = cx^{n_\infty}y^{n_0}\left(y^k + x^l\right)^{n_1}\left(y^k + a_{2,kl}x^l\right)^{n_2}\cdots\left(y^k + a_{p,kl}x^l\right)^{n_p}.$$

The process of desingularization $E : M \to (\mathbb{C}^2, 0)$ can be inductively described as follows: the map E is written $E_1 \circ \tilde{E}$ where

- E_1 is the standard blow-up of $(0,0)$ in $\mathbb{C}^2$.
- $\tilde{E}$ is the process of reduction of $E_1^* f_{\text{qh}}$ which is a quasi-homogeneous function of degree $(k, l-k)$.

Therefore, the process of desingularization will follow the Euclide algorithm for the couple (k,l). In particular, the exceptional divisor is a chain of compact components $\mathbb{CP}^1$ such that each of them is linked exactly with two others except the extremal components. There is exactly one component called the central component along which is attached the strict transform of the cuspidal branches of f_{qh}.

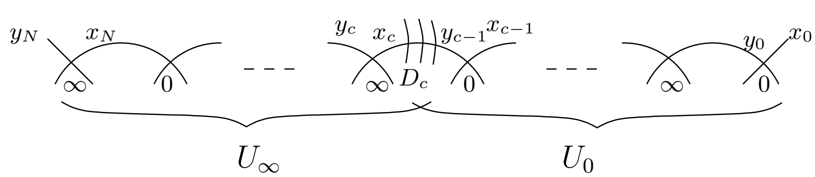


FIGURE 2. Desingularization of a topologically quasi-homogeneous function.

In what follows we will keep the following notations:

Notation 3.1.

- *The integers u and v are defined by:*

$$uk - vl = 1,\ 0 \le u < l,\ 0 \le v < k.$$

- *The numbering $D_1, \ldots, D_N$ of the components of D is a geometric order of the chain, from the one which contains the strict transform of $y = 0$, to the one which contains the strict transform of $x = 0$. It is not the "historical" order of the process.*

- *On each D_i, we denote by 0 the intersection point with D_{i-1}, or with the strict transform of $y = 0$ for D_1, and by ∞ the intersection with D_{i+1} or with the strict transform of $x = 0$ for D_N.*
- *Each component D_i is covered by two charts (x_{i-1}, y_{i-1}) and (x_i, y_i) whose domains V_{i-1} and V_i contains $(D_i, 0)$ and (D_i, ∞). The change of coordinates are given by*

$$x_{i-1} = x_i^{e_i} y_i, \ y_{i-1} = x_i^{-1}$$

where $-e_i$ is the self intersection of the component D_i.
- *On the principal component D_c, we choose the two charts such that each domain V_{c-1}, V_c, contains all the strict transforms of the cuspidal branches.*
- *We define the covering of D by the two open sets:*

$$U_0 = \cup_{i=0}^{c-1} V_i, \quad U_\infty = \cup_{i=c}^{N} V_i.$$

Proposition 3.2. *The desingularization map E is given in the chart (x_c, y_c) by*

$$(x, y) = (x_c^{k-v} y_c^k \ , \ x_c^{l-u} y_c^l).$$

The blowing down is given in this chart by:

$$x_c = \frac{x^l}{y^k}, \quad y_c = \frac{x^{u-l}}{y^{v-k}}.$$

Proof. We prove this result by an induction on the number of blowing up's of the minimal desingularization of f. For one blow-up, we have: $k = l = 1$, $u = 1$, $v = 0$ therefore, the formula is valid in this case. After one blow-up E_1, the germ of $E_1^* f$ at its singular point along the exceptional divisor is a quasi-homogeneous function in the class $(k, l-k)$. Notice that if $uk - vl = 1$ is the Bézout identity of (k, l), the corresponding Bézout identity for the new pair is $(u - v)k - v(l - k) = 1$. Let us suppose that the formula of Proposition (3.2) is valid for the pair $(k, l - k)$. Therefore, after one blowing-up we have in the first chart

$$x_1 = x_c^{k-v} y_c^k, \quad y_1 = x_c^{l-k-u+v} y_c^{l-k}. \tag{23}$$

Thus, we obtain:

$$x = x_1 = x_c^{k-v} y_c^k, \quad y = x_1 y_1 = x_c^{l-u} y_c^l.$$

From this, we easily obtain the inverse formulas defining the blowing-down. □

3.2. **Computing multiplicities.** We first recall the classical result which allows us to compute the multiplicities of a function along the components D_i of the exceptional divisor D of its desingularization [4]: we consider the matrix of intersections J defined for $i \neq j$ by $J_{i,j} = 1$ if the two components D_i and D_j meet together, $J_{i,j} = 0$ otherwise, and $J_{i,i} = -e_i$, where $-e_i$ is the self intersection of each component. For any component D_i, let n_i be the number of strict branches of $f \circ E$ meeting D_i, counted with their multiplicities, and let B be the column matrix induced by these numbers.

Proposition 3.3. *The multiplicities m_i of $(f \circ E)$ along each D_i define a column matrix M which satisfy*

$$JM + B = 0.$$

In the quasi-homogeneous case, since $D = D_1 \cup \cdots \cup D_{c-1} \cup D_c \cup D_{c+1} \cup \cdots \cup D_N$, the column matrix B is here: $(n_0, 0, \ldots, 0, n_c, 0, \ldots, 0, n_\infty)^t$, where $n_c = \sum_{b=1}^{p} n_b$ is on index c. The intersection matrix is given by:

$$J = \begin{pmatrix} -e_1 & 1 & 0 & \cdots & & & & 0 \\ 1 & -e_2 & 1 & 0 & \cdots & & & 0 \\ 0 & 1 & -e_3 & 1 & 0 & \cdots & & 0 \\ \vdots & & & & & & & \\ 0 & \cdots & & & 0 & 1 & -e_{N-1} & 1 \\ 0 & \cdots & & & & 0 & 1 & -e_N \end{pmatrix}$$

Therefore we obtain the multiplicities of f by $M = -J^{-1}B$ (see example below).

We compute now the multiplicities of the *desingularized foliation*, i.e. of the vector field $E^*\theta_f$, where θ_f is the vector field (21) with isolated singularity, defining the foliation $\mathcal{F}$.

Proposition 3.4.

(1) *The multiplicities ν_i of $E^*\theta_f$ along each component D_i define a column matrix N which satisfy*
$$JN + C = 0,$$
where $C = (\varepsilon_0 - 1, 0, \ldots, 0, p, 0, \ldots, 0, \varepsilon_\infty - 1)^t$, with p on index c.

(2) *The multiplicity of $E^*\theta_f$ on the principal component D_c of D is*
$$\nu_c = klp - k - l + k\varepsilon_\infty + l\varepsilon_0.$$

(3) *The multiplicities of $E^*\theta_f$ on the $(y_0 = 0)$ (strict transform of the x-axis) and on $(x_N = 0)$ (strict transform of the y-axis) are $\varepsilon_0 - 1$ and $\varepsilon_\infty - 1$.*

Proof. Let $V = (v_i)$ be the multiplicities of $E^*dx \wedge dy$ along each D_i. From
$$E^*(dx \wedge dy)(E^*\theta_f, \cdot) = (f^{red} \circ E)d(f \circ E)/(f \circ E)$$
we obtain:
$$v_i + \nu_i = r_i + (m_i - 1) - m_i = r_i - 1$$
where $r_i = \nu(f^{red} \circ E, D_i)$. We consider the "axis function": $a = xy$. Let $A = (a_i)$ be the column matrix of multiplicities of $a \circ E$ along each D_i. We claim that $v_i = a_i - 1$. Indeed, let (x_i, y_i) be the chart induced by (x, y) and E around the origin of D_i. Since E is here monomial in these coordinates, there exist positive integers p, q, r, s, such that:
$$E^*dx \wedge dy = a \circ E \cdot E^*\Big(\frac{dx}{x} \wedge \frac{dy}{y}\Big) = a \circ E \cdot (ps - qr)\frac{dx_i}{x_i} \wedge \frac{dy_i}{y_i}$$
from which we deduce $v_i = a_i - 1$. Therefore we obtain $A + N = R$, where R, A are the matrices of multiplities of $(f^{red} \circ E)$ and $a \circ E$ along each D_i. Now, from the previous proposition applied to the functions f^{red} and a we have: $JR + B^{red} = 0$, with $B^{red} = (\varepsilon_0, 0, \ldots, 0, p, 0, \ldots, 0, \varepsilon_\infty)^t$ and $JA + B' = 0$ where B' is the column matrix such that $b'_i = 1$ for $i = 1$ or $i = N$ and $b'_i = 0$ otherwise. We obtain:
$$JN = J(R - A) = -B^{red} + B' = -C.$$
For the principal component, by making use of the formulas of proposition (3.2), we obtain:
$$\begin{aligned} \nu_c &= \nu_{y_c}(E^*\theta_f) = r_c - 1 - v_c = (klp + k\varepsilon_\infty + l\varepsilon_0 - 1) - (k + l - 1) \\ &= klp + k\varepsilon_\infty + l\varepsilon_0 - k - l. \end{aligned}$$
On the branch $(y_0 = 0)$, we have $\nu_{y_0}(E^*f^{red}) = \varepsilon_0$ and $\nu_{y_0}(a \circ E) = 1$. Therefore,
$$\nu_{y_0}(E^*\theta_f) = \nu_{y_0}\left(E^*f^{red}\frac{df}{f}\right) - \nu_{y_0}(E^*dx \wedge dy) = \varepsilon_0 - 1.$$
We obtain the multiplicity on $(x_N = 0)$ by a similar computation. $\square$

Example. For $(k,l)=(3,5)$, the matrix of intersections is:

$$J=\begin{pmatrix} -3 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -3 \end{pmatrix}$$

and we have $B=(n_0,n_c,0,n_\infty)^t$, where $n_c=\sum_{b=1}^p n_b$, and $C=(\varepsilon_0-1,p,0,\varepsilon_\infty-1)$. Therefore we obtain:

$$M=\begin{pmatrix} 2n_0+n_\infty+5n_c \\ 5n_0+3n_\infty+15n_c \\ 3n_0+2n_\infty+9n_c \\ n_0+n_\infty+3n_c \end{pmatrix}\ ;\ N=\begin{pmatrix} 2\varepsilon_0+\varepsilon_\infty+5p-3 \\ 5\varepsilon_0+3\varepsilon_\infty+15p-8 \\ 3\varepsilon_0+2\varepsilon_\infty+9p-5 \\ \varepsilon_0+\varepsilon_\infty+3p-2 \end{pmatrix}.$$

The multiplicity of the foliation on the principal component D_2 is

$$\nu_c=5\varepsilon_0+3\varepsilon_\infty+15p-8.$$

3.3. **Acyclic covering of D for the sheaf $\Theta_{\mathcal{F}}$.** We consider the covering $\{U_0,U_\infty\}$ defined in (3.1).

Lemma 3.5. *There exists a global section T_0 (resp. T_∞) of the sheaf $\Theta_{\mathcal{F}}$ of germs of vector fields tangent to $E^*\mathcal{F}$ on U_0 (resp. U_∞) which admits only isolated singularities.*

Proof. From Proposition (3.4), the following holomorphic vector fields

$$\theta_0=\frac{E^*\theta_f}{x_0^{\nu_1}y_0^{\varepsilon_0-1}},\ \theta_i=\frac{E^*\theta_f}{x_i^{\nu_{i+1}}y_i^{\nu_i}},\ i=1,\ldots,c-1,$$

have isolated singularities. We claim that they glue together on their common domains, defining a global section T_0 of $\Theta_{\mathcal{F}}$ on U_0. Indeed, from the previous relation $JN+C=0$, we have :

$$\begin{aligned} -e_1\nu_1+\nu_2+\varepsilon_0-1 &= 0 \\ \nu_{i-1}-e_i\nu_i+\nu_{i+1} &= 0,\quad i=2,\ldots,c-1 \end{aligned}$$

Therefore, using the change of coordinates between two consecutive charts, we have

$$\begin{aligned} x_1^{\nu_2}y_1^{\nu_1} &= y_0^{-\nu_2}x_0^{\nu_1}y_0^{e_1\nu_1}=x_0^{\nu_1}y_0^{\varepsilon_0-1} \\ x_i^{\nu_{i+1}}y_i^{\nu_i} &= y_{i-1}^{-\nu_{i+1}}x_{i-1}^{\nu_i}y_{i-1}^{e_i\nu_i}=x_{i-1}^{\nu_i}y_{i-1}^{\nu_{i-1}},\ i=1,\ldots,c-1. \end{aligned}$$

The proof is similar for constructing T_∞ on U_∞. □

Proposition 3.6. *We have $H^1(U_0,\Theta_{\mathcal{F}})=H^1(U_\infty,\Theta_{\mathcal{F}})=0$.*

Proof. The previous section T_0 with isolated singularities allows us to identify the sheaf $\Theta_{\mathcal{F}}|_{U_0}$ to $\mathcal{O}_M|_{U_0}$. Since the Chern class of each branch is negative, a direct computation with the change of charts shows that $H^1(U_0,\mathcal{O}_M)=0$. The proof is similar for U_∞. □

4. Appendix B: normal forms for $(k,l)=(3,5)$ and $p=4$

According to the figure draw below, the analytical normal form for the topological class of

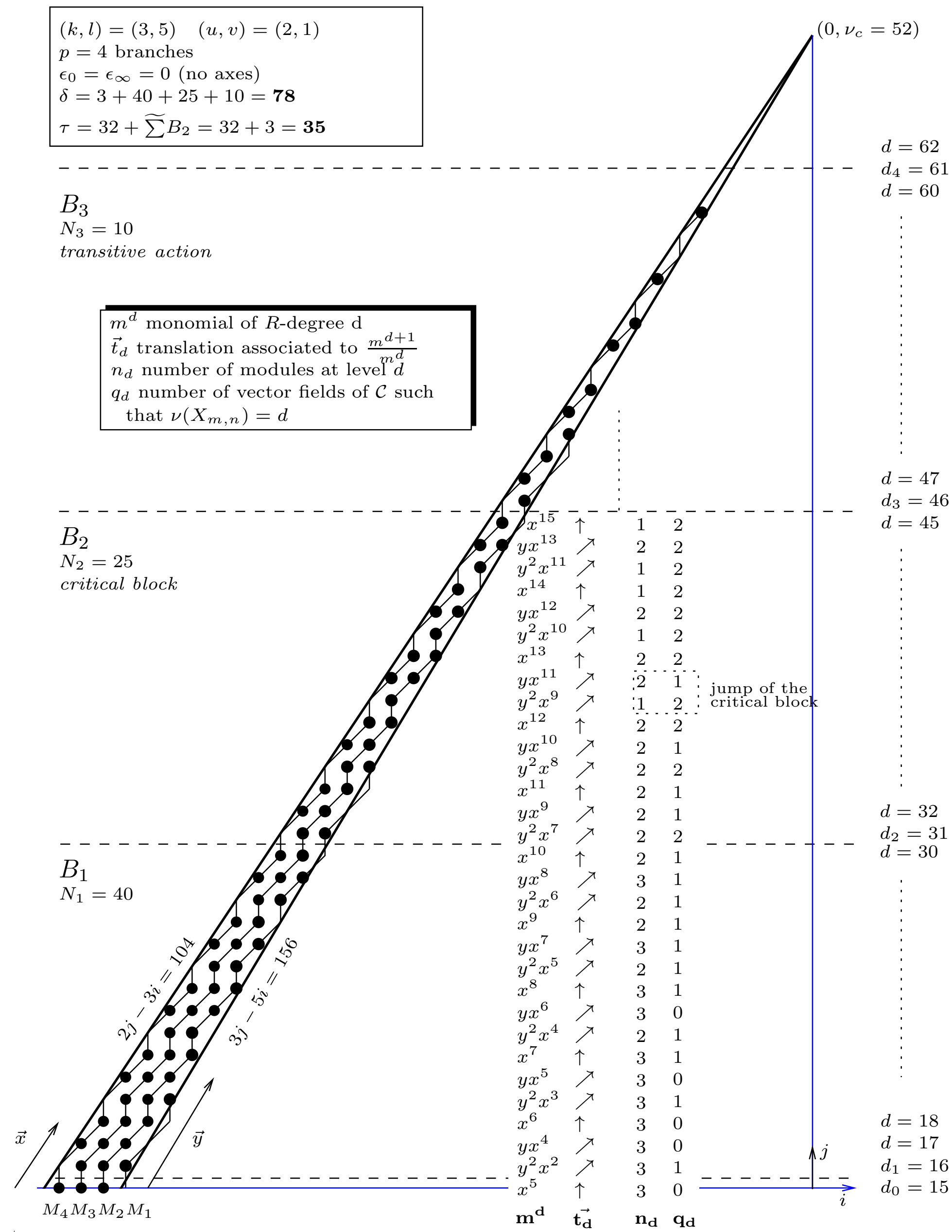
$$\left(y^3+x^5\right)^{n_1}\left(y^3+a_2x^5\right)^{n_2}\left(y^3+a_3x^5\right)^{n_3}\left(y^3+a_4x^5\right)^{n_4}$$

is given by the following family of functions with 78 parameters

$$
\begin{aligned}
N_a \;=\; & \big(y^3+x^5+a_{1,16}x^2y^2+a_{1,19}x^3y^2\big)^{n_1}\times\\
& \big(y^3+a_{2,15}x^5+a_{2,16}x^2y^2+a_{2,17}x^4y+a_{2,18}x^6+a_{2,19}x^3y^2+a_{2,20}x^5y\\
& \quad +a_{2,21}x^7+a_{2,22}x^4y^2+a_{2,23}x^6y+a_{2,24}x^8+a_{2,25}x^5y^2+a_{2,26}x^7y\\
& \quad +a_{2,28}x^6y^2+a_{2,29}x^8y+a_{2,31}x^7y^2+a_{2,34}x^8y^2\big)^{n_2}\times\\
& \big(y^3+a_{3,15}x^5+a_{3,16}x^2y^2+a_{3,17}x^4y+a_{3,18}x^6+a_{3,19}x^3y^2+a_{3,20}x^5y\\
& \quad +a_{3,21}x^7+a_{3,22}x^4y^2+a_{3,23}x^6y+a_{3,24}x^8+a_{3,25}x^5y^2+a_{3,26}x^7y\\
& \quad +a_{3,27}x^9+a_{3,28}x^6y^2+a_{3,29}x^8y+a_{3,30}x^{10}+a_{3,31}x^7y^2+a_{3,32}x^9y\\
& \quad +a_{3,33}x^{11}+a_{3,34}x^8y^2+a_{3,35}x^{10}y+a_{3,36}x^{12}+a_{3,37}x^8y^2+a_{3,38}x^{11}y\\
& \quad +a_{3,39}x^{13}+a_{3,40}x^9y^2+a_{3,41}x^{12}y+a_{3,43}x^{10}y^2+a_{3,44}x^{13}y+a_{3,46}x^{12}y^2\\
& \quad +a_{3,49}x^{13}y^2\big)^{n_3}\times\\
& \big(y^3+a_{4,15}x^5+a_{4,17}x^4y+a_{4,18}x^6+a_{4,20}x^5y+a_{4,21}x^7+a_{4,23}x^6y\\
& \quad +a_{4,24}x^8+a_{4,26}x^7y+a_{4,27}x^9+a_{4,29}x^8y+a_{4,30}x^{10}+a_{4,32}x^9y\\
& \quad +a_{4,33}x^{11}+a_{4,35}x^{10}y+a_{4,36}x^{12}+a_{4,38}x^{11}y+a_{4,39}x^{13}+a_{4,41}x^{12}y\\
& \quad +a_{4,42}x^{14}+a_{4,44}x^{13}y+a_{4,45}x^{15}+a_{4,47}x^{14}y+a_{4,48}x^{16}+a_{4,50}x^{15}y\\
& \quad +a_{4,51}x^{17}+a_{4,53}x^{16}y+a_{4,54}x^{18}+a_{4,56}x^{17}y+a_{4,59}x^{18}y\big)^{n_4}.
\end{aligned}
$$

Moreover, the normal forms for the generic curve are given by the 35-parameters family

$$
\begin{aligned}
N_a \;=\; & \big(y^3+x^5+x^2y^2\big)\times\\
& \big(y^3+a_{2,15}x^5+a_{2,16}x^2y^2+a_{2,17}x^4y+a_{2,18}x^6+a_{2,19}x^3y^2+a_{2,20}x^5y\\
& \quad +a_{2,23}x^6y\big)\times\\
& \big(y^3+a_{3,15}x^5+a_{3,16}x^2y^2+a_{3,17}x^4y+a_{3,18}x^6+a_{3,19}x^3y^2+a_{3,20}x^5y\\
& \quad +a_{3,21}x^7+a_{3,22}x^4y^2+a_{3,23}x^6y+a_{3,24}x^8+a_{3,25}x^5y^2+a_{3,26}x^7y\\
& \quad +a_{3,28}x^6y^2+a_{3,29}x^8y\big)\times\\
& \big(y^3+a_{4,15}x^5+a_{4,17}x^4y+a_{4,18}x^6+a_{4,20}x^5y+a_{4,21}x^7+a_{4,23}x^6y\\
& \quad +a_{4,24}x^8+a_{4,26}x^7y+a_{4,27}x^9+a_{4,29}x^8y+a_{4,30}x^{10}+a_{4,32}x^9y\\
& \quad +a_{4,33}x^{11}+a_{4,35}x^{10}y\big).
\end{aligned}
$$



$\mathbf{m^d}$	$\vec{\mathbf{t}}_{\mathbf{d}}$	$\mathbf{n_d}$	$\mathbf{q_d}$
x^{15}	↑	1	2
yx^{13}	↗	2	2
y^2x^{11}	↗	1	2
x^{14}	↑	1	2
yx^{12}	↗	2	2
y^2x^{10}	↗	1	2
x^{13}	↑	2	2
yx^{11}	↗	2	1
y^2x^9	↗	1	2
x^{12}	↑	2	2
yx^{10}	↗	2	1
y^2x^8	↗	2	2
x^{11}	↑	2	1
yx^9	↗	2	1
y^2x^7	↗	2	2
x^{10}	↑	2	1
yx^8	↗	3	1
y^2x^6	↗	2	1
x^9	↑	2	1
yx^7	↗	3	1
y^2x^5	↗	2	1
x^8	↑	3	1
yx^6	↗	3	0
y^2x^4	↗	2	1
x^7	↑	3	1
yx^5	↗	3	0
y^2x^3	↗	3	1
x^6	↑	3	0
yx^4	↗	3	0
y^2x^2	↗	3	1
x^5	↑	3	0

FIGURE 3. Moduli triangle of the topological class $(k,l) = (3,5)$ and $p = 4$

References

[1] M. Berthier, D. Cerveau and R. Meziani, *Transformations isotropes des germes de feuilletages holomorphes* J. Math. Pures Appl. 9-78 (1999) DOI: 10.1016/s0021-7824(99)00013-6

[2] J. Briançon, M. Granger, and Ph. Maisonobe, *Le nombre de modules du germe de courbe plane* $x^a + y^b = 0$, Math. Ann. 279-3 (1988)

[3] C. Camacho, A. Lins Neto and P. Sad, *Topological invariants and equidesingularization for holomorphic vector fields* J. Differential Geom. 20-1 (1984)

[4] D. Eisenbud and W.D. Neumann, *Three-dimensional link theory and invariants of plane curve singularities,* Ann. Math. Stud. 101 (1985)

[5] A. Hefez and M. Hernandes, *Analytic classification of plane branches up to multiplicity 4*, J. Symbolic Comput. 44-6 (2009)

[6] A. Hefez and M. Hernandes, *The analytic classification of plane branches*, Bull. Lond. Math. Soc. 43-2 (2011) DOI: 10.1112/blms/bdq113

[7] A. Hefez and Hernandes M. *Standard bases for local rings of branches and their modules of differentials,* J. Symbolic Comput. 42 (2007)

[8] Y. Genzmer and E. Paul - *Normal forms of foliations and curves defined by a function with a generic tangent cone,* Moscow math. jour. 11-1 (2011)

[9] J.M. Granger - *Sur un espace de modules de germe de courbe plane,* Bull. Sc. math. 103-2 (1979)

[10] M. Lejeune-Jalabert, *Sur l'équivalence des courbes algébroïdes planes. Coefficients de Newton. Contribution à l'étude des singularités du point de vue du polygone de Newton*, Université de Paris VII, Paris, (1973)

[11] J. N. Mather and S. S. T. Yau. *Classication of isolated hypersurface singularities by their moduli algebras.* Invent. Math., 69-2 (1982)

[12] J.F. Mattei - *Quasihomogénité et équiréductibilité de feuilletages holomorphes en dimension 2,* Astérisque S.M.F. 261 (2000)

[13] J.F. Mattei - *Modules de feuilletages holomorphes singuliers: I équisingularité,* Invent. Math. 103 (1991)

[14] J.F. Mattei and R. Moussu - *Holonomie et intégrales premières,* Ann. Sci. Ecole Norm. Sup. 13 (1980)

[15] K. Saito, *Quasihomogene isolierte Singularitäten von Hyperflächen* Invent. Math. 14 (1971)

[16] A. Seidenberg - *Reduction of singularities of the differential equation* $Ady = Bdx$, Amer. J. Math. 90 (1968)DOI: 10.2307/2373435

[17] O. Zariski - *Le problème des modules pour les branches planes,* Hermann, Paris, seconde édition (1986), course given at the Centre de Mathématiques de l'Ecole Polytechnique (1973).

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STRANGELY DUAL ORBIFOLD EQUIVALENCE I

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Abstract. In this brief note we prove orbifold equivalence between two potentials described by strangely dual exceptional unimodular singularities of type E_{14} and Q_{10} in two different ways. The matrix factorizations proving the orbifold equivalence give rise to equations whose solutions are permuted by Galois groups which differ for different expressions of the same singularity.

1. Introduction

In this paper, we present two ways of proving an orbifold equivalence between two potentials describing two strangely dual unimodular exceptional singularities, namely Q_{10} and E_{14}. In addition, we observe that each matrix factorization proving this orbifold equivalence depends on a different Galois orbit. First, we will recall the notion of orbifold equivalence and motivate this research direction, leaving computations for Sections 3 and 4. We also include an appendix, written by the second author with Federico Zerbini, which discusses the Kreuzer–Skarke theorem and gives a way to count invertible potentials for any number of variables.

1.1. Orbifold equivalence.

We will work in the graded ring of polynomials over the complex numbers, $\mathbb{C}[x_1, \ldots, x_n]$, with degrees $|x_i| \in \mathbb{Q}_{\geq 0}$ associated to each variable x_i.

Definition 1.1. A *potential* is a polynomial $W \in \mathbb{C}[x_1, \ldots, x_n]$ satisfying

$$\dim_{\mathbb{C}} \left(\frac{\mathbb{C}[x_1, \ldots, x_n]}{\langle \partial_1 W, \ldots, \partial_n W \rangle} \right) < \infty.$$

We say that a potential is *homogeneous of degree* $d \in \mathbb{Q}_{\geq 0}$ if in addition it satisfies

$$W\left(\lambda^{|x_1|} x_1, \ldots, \lambda^{|x_n|} x_n\right) = \lambda^d W(x_1, \ldots, x_n)$$

for all $\lambda \in \mathbb{C}^\times$.

From now on, the word potential will be used to mean 'homogeneous potential of degree 2'.

We will denote the set of all possible potentials with complex coefficients, and any number of variables, by $\mathcal{P}_{\mathbb{C}}$. To a potential $W \in \mathcal{P}_{\mathbb{C}}$ with n variables, we can associate a number called the *central charge*, which is defined as:

$$c_W = \sum_{i=1}^{n} (1 - |x_i|).$$

Definition 1.2.

- A *matrix factorization* of W consists of a pair (M, d^M) where
  - M is a $\mathbb{Z}_2$-graded free module over $\mathbb{C}[x_1, \ldots, x_n]$;
  - $d^M : M \to M$ is a degree 1 $\mathbb{C}[x_1, \ldots, x_n]$–linear endomorphism (the *twisted differential*) such that:

$$d^M \circ d^M = W.\mathrm{id}_M. \tag{1}$$

We may display the $\mathbb{Z}_2$-grading explicitly as $M = M_0 \oplus M_1$ and

$$d^M = \begin{pmatrix} 0 & d_1^M \\ d_0^M & 0 \end{pmatrix}.$$

If there is no risk of confusion, we will denote $\left(M, d^M\right)$ simply by M.

- We call M a *graded matrix factorization* if, in addition, M_0 and M_1 are $\mathbb{Q}$-graded, acting with x_i is an endomorphism of degree $|x_i|$ with respect to the $\mathbb{Q}$-grading on M, and the twisted differential has degree 1 with respect to the $\mathbb{Q}$–grading on M. Note that these conditions imply that W has degree 2 (as desired).

We will denote by $\mathrm{hmf}^{\mathrm{gr}}(W)$ the idempotent complete full subcategory of graded finite–rank matrix factorizations: its objects are homotopy equivalent to direct summands of finite–rank matrix factorizations. The morphisms are homogeneous even linear maps up to homotopy with respect to the twisted differential. This category is indeed monoidal and has duals and adjunctions which can be described in a very explicit way. This leads to the following result which gives precise formulas for the left and right quantum dimensions of a matrix factorization.

Proposition 1.3. [CM, CR1] Let $V(x_1, \ldots, x_m)$ and $W(y_1, \ldots, y_n) \in \mathcal{P}_{\mathbb{C}}$ be two potentials and M a matrix factorization of $W - V$. Then the left quantum dimension of M is:

$$\mathrm{qdim}_l(M) = (-1)^{\binom{m+1}{2}} \operatorname{Res}\left[\frac{\operatorname{str}\left(\partial_{x_1} d^M \ldots \partial_{x_m} d^M \partial_{y_1} d^M \ldots \partial_{y_n} d^M\right) dy_1 \ldots dy_n}{\partial_{y_1} W, \ldots, \partial_{y_n} W}\right]$$

and the right quantum dimension is:

$$\mathrm{qdim}_r(M) = (-1)^{\binom{n+1}{2}} \operatorname{Res}\left[\frac{\operatorname{str}\left(\partial_{x_1} d^M \ldots \partial_{x_m} d^M \partial_{y_1} d^M \ldots \partial_{y_n} d^M\right) dx_1 \ldots dx_m}{\partial_{x_1} V, \ldots, \partial_{x_m} V}\right],$$

where by str we mean the supertrace of the corresponding supermatrix.

Quantum dimensions allow us to define the following equivalence relation:

Definition and Theorem 1.4. [CR2, CRCR] Let V, W and M be as in the previous proposition. We say that V and W are *orbifold equivalent* ($V \sim_{\mathrm{orb}} W$) if there exists a finite–rank matrix factorization of $V - W$ for which the left and the right quantum dimensions are non-zero. Orbifold equivalence is an equivalence relation in $\mathcal{P}_{\mathbb{C}}$.

Remark 1.5. [CR2, Proposition 6.4] (or [CRCR, Proposition 1.3]) If two potentials V and W are orbifold equivalent, then their associated central charges are equal: $c_V = c_W$.

Let us give some comments on quantum dimensions and orbifold equivalences [CRCR, CR2]:

- [CRCR, Lemma 2.5] The quantum dimensions of graded matrix factorizations take values in $\mathbb{C}$. One can see this by counting degrees in the formulas given in Proposition 1.3.
- The definitions of the quantum dimensions are also valid for ungraded matrix factorizations (in which case they will take values in $\mathbb{C}[x_1, \ldots, x_n]$ instead of in $\mathbb{C}$). Furthermore, the quantum dimensions are independent of the $\mathbb{Q}$-grading on a graded matrix factorization.
- So far, the difficulty of establishing an orbifold equivalence lies in constructing the explicit matrix factorization which proves it.

1.2. **Motivation: an interlude on Arnold's strange duality.** From now on, we fix the number of variables of our polynomial ring to be $n = 3$.

The aim of this work was to discover more orbifold equivalent potentials as in [CRCR]. In that paper, orbifold equivalence between simple singularities was proven. These singularities have modality zero and fall into an ADE classification. A natural next step for finding new orbifold equivalences is to focus on potentials described by singularities of modality one. Thanks to the classification performed by Arnold in the late 60's, we know that such singularities fall into 3 families of parabolic singularities, a three-suffix series of hyperbolic singularities, and 14 families of exceptional singularities. For more details on this classification, we refer to [Ar, AGV].

A singularity can be described with a regular weight system [Sai], that is, a quadruple of positive integers $(a_1, a_2, a_3; h)$ with:

- $a_1, a_2, a_3 < h$,
- $\gcd(a_1, a_2, a_3) = 1$, and
- There exists a polynomial $W \in \mathbb{C}[x_1, x_2, x_3]$ that has an isolated singularity at the origin (with the degrees of the variables x_i being $|x_i| = \frac{2a_i}{h}$, $i \in \{1, 2, 3\}$) which is invariant under the Euler field E, that is,

$$E.W = \left(\frac{a_1}{h} x_1 \frac{\partial}{\partial x_1} + \frac{a_2}{h} x_2 \frac{\partial}{\partial x_2} + \frac{a_3}{h} x_3 \frac{\partial}{\partial x_3} \right) W = W.$$

  In other words, the polynomial associated to a regular weight system must be a potential invariant under the Euler field.

With the assignment of degrees made, this is the same as requesting homogeneity of degree 2 for the potentials [1]. The integer h is called the *Coxeter number*.

From now on, we write $x_1 = x$, $x_2 = y$ and $x_3 = z$. Some examples of regular weight systems, those corresponding to each of the 14 unimodular exceptional singularities are shown in Table 1. The associated potentials are also described. For most of the exceptional unimodular singularities, there is only one way to write the associated potential, whereas there are two expressions for each of Q_{12}, Z_{13}, W_{12}, W_{13} and E_{14}. Exceptionally, there are 4 potentials which can describe the singularity U_{12}. In order to find these potentials, combine invariance under the Euler field (or homogeneity of order 2) with the Kreuzer–Skarke theorem [KS] to see that any variable x_i shows up in a potential only as a power of itself, x_i^a (for some $a > 2$) or as $x_i^a x_j$ (with $i \neq j$). [2].

Let us illustrate this with an example: take E_{14}. The degrees assigned to the variables are: $|x| = \frac{6}{24} = \frac{1}{4}$, $|y| = \frac{16}{24} = \frac{2}{3}$ and $|z| = \frac{24}{24} = 1$. Imposing homogeneity of degree 2, we need to find monomials of the shape $x^{k_1} y^{k_2} z^{k_3}$ where $k_i \in \mathbb{Z}_+$, $i \in \{1, 2, 3\}$ must satisfy $\frac{2}{3}k_1 + \frac{1}{4}k_2 + k_3 = 2$. The only solutions are four tuples: $(8, 0, 0)$, $(4, 0, 1)$, $(0, 3, 0)$, $(0, 0, 2)$, i.e., the monomials $x^4 z$, x^8, y^3 and z^2. Combining them and taking into account the Kreuzer–Skarke theorem, we get the two potentials appearing in Table 1.

[1]This argument goes as follows: a potential in three variables can only have seven possible shapes, which are specified in a graphical way in Table 4 in the Appendix A, or in [AGV, Chapter 13]. Imposing invariance under the Euler field boils down to some conditions on the powers of the monomials in the potential. With the assignment of degrees made, one can easily see that these conditions are exactly the same as those we should impose if we want homogeneity of degree 2.

[2]A complete statement of this theorem, as well as a discussion of it, is presented in the Appendix A.

Type	Potential (1)	Potential (2)	$(a_1, a_2, a_3; h)$
Q_{10}	$x^4 + y^3 + xz^2$	–	$(9, 8, 6; 24)$
Q_{11}	$x^3y + y^3 + xz^2$	–	$(7, 6, 4; 18)$
Q_{12}	$x^3z + y^3 + xz^2$	$x^5 + y^3 + xz^2$	$(6, 5, 3; 15)$
S_{11}	$x^4 + y^2z + xz^2$	–	$(5, 4, 6; 16)$
S_{12}	$x^3y + y^2z + xz^2$	–	$(4, 3, 5; 13)$
U_{12}	$x^4 + y^3 + z^3$	$x^4 + y^3 + z^2y$	$(4, 4, 3; 12)$
Z_{11}	$x^5 + xy^3 + z^2$	–	$(8, 6, 15; 30)$
Z_{12}	$yx^4 + xy^3 + z^2$	–	$(6, 4, 11; 22)$
Z_{13}	$x^3z + xy^3 + z^2$	$x^6 + y^3x + z^2$	$(5, 3, 9; 18)$
W_{12}	$x^5 + y^2z + z^2$	$x^5 + y^4 + z^2$	$(5, 4, 10; 20)$
W_{13}	$yx^4 + y^2z + z^2$	$x^4y + y^4 + z^2$	$(4, 3, 8; 16)$
E_{12}	$x^7 + y^3 + z^2$	–	$(14, 6, 21; 42)$
E_{13}	$y^3 + yx^5 + z^2$	–	$(10, 4, 15; 30)$
E_{14}	$x^4z + y^3 + z^2$	$x^8 + y^3 + z^2$	$(8, 3, 12; 24)$

TABLE 1. Unimodular singularities of exceptional type (note that U_{12} can also be described in two additional different ways: $x^4 + y^2z + z^3$ and $x^4 + y^2z + z^2y$).

As discovered by Kobayashi [Kob], there is some duality between these weight systems – which corresponds to what is known as *Arnold's strange duality*[3]. Four pairs of these exceptional singularities share the same Coxeter number: Q_{10} and E_{14} ($h = 24$), Q_{11} and Z_{13} ($h = 18$), S_{11} and W_{13} ($h = 16$) and Z_{11} and E_{13} ($h = 30$).

In addition, one notices the following phenomenon. For potentials described by strange dual pairs, the associated central charges have a close relationship with the Coxeter number h [Ma2],

$$c_W = \frac{h+2}{h}$$

which implies that the potentials related to strange dual singularities have the same central charge. As mentioned in Remark 1.5, equality of central charges is one consequence of orbifold equivalence between two potentials. Hence, it makes sense to conjecture from the mathematics point of view that strangely dual exceptional unimodular singularities are orbifold equivalent.

Another consequence of orbifold equivalence between strangely dual exceptional unimodular singularities would be that the Ginzburg algebras [Gin] for these singularities with Dynkin diagrams [Gab] sharing the same Coxeter number are orbifold equivalent in the bicategory whose objects are smooth dg algebras with finite dimensional cohomology and whose morphism categories are the respective perfect derived categories. We refer to the recent paper [CQ] for a complete exposition and details of this statement.

Furthermore, from the physics point of view, we have known for some time that for each of these exceptional singularities there is a uniform construction of a $K3$ surface obtained by compactifying the singularity [Sai, Pin]. Landau-Ginzburg models with potentials described by strangely dual singularities correspond to the same $K3$ surface [Ma1, Ma2].

[3]This duality roughly states that, given two singularities, the Dolgachev numbers associated to the first singularity are the same as the Gabrielov numbers of the second one (and vice versa). We refer to the bibliography for further details, e.g. [Ar, Dol, Eb].

This can also be regarded as well as a prediction of orbifold equivalence between these singularities. In addition, it would be interesting to see the implications of orbifold equivalence for $N = 2$ superconformal four–dimensional gauge theories [CDZ].

A further motivation for this work (if not the primary for the second author) is given by the so-called Landau-Ginzburg/conformal field theory correspondence [HW, LVW, VW, RC], which predicts a certain relation between categories of matrix factorizations of the potential of the Landau-Ginzburg model and categories of representations of the vertex operator algebra associated to some conformal field theory. An immediate consequence of orbifold equivalence between two potentials is the following result:

Proposition 1.6. [CR2] Let V, $W \in \mathcal{P}_{\mathbb{C}}$ be two potentials which are orbifold equivalent and let $M \in \mathrm{hmf}^{\mathrm{gr}}(W - V)$ have non-zero quantum dimensions. Then,

$$\mathrm{hmf}^{\mathrm{gr}}(W) \simeq \mathrm{mod}\left(X^{\dagger} \otimes X\right)$$

where by $X^{\dagger}$ we mean the right adjoint of X and $\mathrm{mod}\left(X^{\dagger} \otimes X\right)$ is the category of modules over $X^{\dagger} \otimes X$.

$X^{\dagger} \otimes X$ is a separable symmetric Frobenius algebra [CR2] (see e.g. [BCP] for a review on Frobenius algebras). These algebras are related to full CFTs [FRS1]. Hence, proving orbifold equivalences is a way to match together both sides of the Landau-Ginzburg/conformal field theory correspondence, providing a better understanding of a mathematical conjecture for it. Due to the need for computational software improvements, we postpone the analysis of the results of this paper from the point of view of the Landau-Ginzburg/conformal field theory correspondence to later works [RCN].

Proving more orbifold equivalences requires at this point some strong computational tool which for the moment we lack [4]. For this reason we focus on a first example – that of $E_{14} - Q_{10}$ – and analyze it in detail.

This paper is organized as follows. In Section 2, we explain orbifold equivalence as well as some basics on matrix factorizations. In Section 3, we describe the method followed to find the matrix factorizations of $E_{14} - Q_{10}$ which prove orbifold equivalence in two different fashions. In Section 4, we describe the Galois orbits on which the matrix factorizations obtained in Section 3 depend. We wrap up with some conclusions and an appendix by the second author and Federico Zerbini on the Kreuzer–Skarke theorem.

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2. $Q_{10} \sim_{\mathrm{orb}} E_{14}$ in two fashions

Our method to find matrix factorizations of finite rank consists of a variation of the perturbation method used in [CRCR]. The starting point is the paper [KST], where we find the

[4] Upon the writing of this manuscript, the second author became aware of a project by Andreas Recknagel et al. to create a computer algorithm to prove orbifold equivalences. We do not know any further details about this project, but it seems that this algorithm was able to reproduce the orbifold equivalences of [CRCR] and the one in this paper as well - apparently via a different method but nonetheless pretty simultaneously.

full strongly exceptional collection of objects of the category of matrix factorizations of each potential described by unimodular singularities. Our recipe proceeds as follows:

(1) Consider the difference between two potentials. Set to zero one of the variables (typically the one with the smallest degree associated). Factorize the resulting potential.
(2) Pick one of the exceptional objects from the [KST] collection for the potential which doesn't contain the variable set to zero in the previous step. The entries of these matrices are factorizations of each of the monomials of the corresponding potential. We change these factorizations in order to obtain entries in the matrix similar to the factors in the factorization of Step 1, being careful to ensure that the result is still a matrix factorization.
(3) Perturb à la [CRCR] all possible entries of the matrix factorization (not necessarily only with respect to the variable set to zero), except for the zero entries.
(4) Impose Equation 1 and reduce the system of equations obtained from the perturbation constants as much as possible. We obtain a matrix factorization depending on a small number of parameters satisfying some equations.

In an attempt to elucidate this recipe, we will explain in detail how to prove $Q_{10} \sim_{\text{orb}} E_{14}$ in two ways.

2.1. $\mathbf{Q_{10}} \sim_{\text{orb}} \mathbf{E_{14}}$, version 1.

(1) Consider the potentials:

$$Q_{10} = x^4 + y^3 + xz^2$$

$$E_{14} = u^4 w + v^3 + w^2$$

whose variables have the following associated degrees:

$$|x| = \frac{6}{12} \qquad |y| = \frac{8}{12} \qquad |z| = \frac{9}{12} \qquad |u| = \frac{3}{12} \qquad |v| = \frac{8}{12} \qquad |w| = \frac{12}{12}.$$

It is easy to check that both potentials have a central charge of $c_{Q_{10}} = \frac{13}{12} = c_{E_{14}}$. The variable with the smallest degree is u and we will perturb with respect to it. Set u equal to zero; the resulting potential is then:

$$\overline{Q_{10} - E_{14}} = x^4 + y^3 + xz^2 - v^3 - w^2$$

We can factorize this potential as:

$$\overline{Q_{10} - E_{14}} = \left(x^2 + w\right)\left(x^2 - w\right) + (y - v)\left(y^2 + yv + v^2\right) + (xz)(z). \tag{2}$$

(2) First, we will start from the indecomposables of Q_{10}. The matrix factorization associated to the vertex V_0 of the Auslander-Reiten quiver associated to this singularity is given by ([KST]):

$$[!h]d_0 = \begin{pmatrix} xz & y^2 & x^3 & 0 \\ y & -z & 0 & x^3 \\ x & 0 & -z & -y^2 \\ 0 & x & -y & xz \end{pmatrix} \qquad d_1 = \begin{pmatrix} z & y^2 & x^3 & 0 \\ y & -xz & 0 & x^3 \\ x & 0 & -xz & -y^2 \\ 0 & x & -y & z \end{pmatrix}$$

Note that the determinant of d_1 is precisely Q_{10}^2. Then, similarly to the procedure followed to prove the orbifold equivalence $A_{29} \sim_{\text{orb}} E_8$ in [CRCR], we make the ansatz that it is possible to recover d_0 as $Q_{10} d_1^{-1}$. Hence we will only need to work with d_1.

Modify d_1 as follows:

$$\widetilde{d_1} = \begin{pmatrix} z & y^2 & x^2 & 0 \\ y & -xz & 0 & x^2 \\ x^2 & 0 & -xz & -y^2 \\ 0 & x^2 & -y & z \end{pmatrix}$$

The determinant of this matrix is still equal to Q_{10}^2. Then, using the factorization in Eq. 2, we can construct a similar d_1 whose determinant is precisely $\left(\overline{Q_{10} - E_{14}}\right)^2$:

$$[!h]\widetilde{\widetilde{d_1}} = \begin{pmatrix} z & v^2 + vy + y^2 & x^2 + w & 0 \\ y - v & -xz & 0 & x^2 + w \\ x^2 - w & 0 & -xz & -\left(v^2 + yv + y^2\right) \\ 0 & x^2 - w & -y + v & z \end{pmatrix}$$

which has a degree distribution (in units of 1/12) specified in Table 2.

9	16	12	0
8	15	0	12
12	0	15	16
0	12	8	9

TABLE 2. Degree distribution of the entries of $\tilde{\tilde{d}}_1$

From this matrix, construct $\widetilde{\widetilde{d_0}}$:

$$[!h]\widetilde{\widetilde{d_0}} = \begin{pmatrix} -xz & -\left(v^2 + vy + y^2\right) & -\left(x^2 + w\right) & 0 \\ v - y & z & 0 & -\left(x^2 + w\right) \\ -\left(x^2 - w\right) & 0 & z & v^2 + yv + y^2 \\ 0 & -\left(x^2 - w\right) & -v + y & -xz \end{pmatrix}$$

which has a degree distribution (in units of 1/12) specified in Table 3.

15	16	12	0
8	9	0	12
12	0	9	16
0	12	8	15

TABLE 3. Degree distribution of the entries of $\tilde{\tilde{d}}_0$

Now form the whole matrix factorization (which we will denote by d_X). Indeed, we see that $d_X \circ d_X = \overline{Q_{10} - E_{14}}$.

(3) Perturb all possible entries with terms (at least) linear in u. Note that, in contrast to [CRCR], the zero entries are not perturbed. Those which can be perturbed in this way are those of degree:
   - 9: u^3, ux.
   - 12: uz, u^4, xu^2.
   - 15: ux^2, u^2z, uw, u^5.

Implement the perturbation in $d_X = \tilde{\tilde{d}}_0 \oplus \tilde{\tilde{d}}_1 = (x_{ij})$ $(i,j = 1,\dots,8)$; the entries of this matrix will be

[!h]

$$\begin{aligned}
x_{15} &= z + p_{111}u^3 + p_{112}ux \\
x_{16} &= v^2 + vy + y^2 \\
x_{17} &= x^2 + w + p_{131}uz + p_{132}u^4 + p_{133}xu^2 \\
x_{25} &= y - v \\
x_{26} &= -xz + p_{221}ux^2 + p_{222}u^2z + p_{223}uw + p_{224}u^5 \\
x_{28} &= w + x^2 + p_{241}uz + p_{242}u^4 + p_{243}xu^2 \\
x_{35} &= -w + x^2 + p_{311}uz + p_{312}u^4 + p_{313}xu^2 \\
x_{37} &= -xz + p_{331}ux^2 + p_{332}u^2z + p_{333}uw + p_{334}u^5 \\
x_{38} &= -v^2 - vy - y^2 \\
x_{46} &= -w + x^2 + p_{421}uz + p_{422}u^4 + p_{423}xu^2 \\
x_{47} &= v - y \\
x_{48} &= z + p_{441}u^3 + p_{442}ux
\end{aligned}$$

for d_1, and similarly for d_0, with the rest of entries of the matrix zeros and where $p_{lmn} \in \mathbb{C}$ $(l = 1,\dots,8;\ m,n = 1,\dots,4)$. Imposing Equation 1 and linear conditions on the p_{ijk}'s, we finally recover a diagonal matrix where in order to recover the original potential $Q_{10} - E_{14}$ we need to solve a system of 11 equations with 12 variables, which can indeed be further reduced. Changing $p_{112} \rightsquigarrow a$, $p_{131} \rightsquigarrow b$ and $p_{221} \rightsquigarrow c$, we are left with only two equations and three variables:

(3)
$$-\frac{1}{64}\left(-4 + 3a^4 + 8a^3b + 8a^2b^2 - 4a^3c - 8a^2bc\right)\cdot\left(4 + 3a^4 + 8a^3b + 8a^2b^2 - 4a^3c - 8a^2bc\right) = 0$$
and
$$-\frac{1}{8}a^2\left(a^4 - 8a^2b^2 - 16ab^3 - 8b^4 + 8a^2bc + 24ab^2c + 16b^3c - 2a^2c^2 - 8abc^2 - 8b^2c^2\right) = 0.$$

For the sake of simplification, introduce the following notation:

$$\kappa_1 := \left(\frac{a^3}{2} + a^2b + ab^2 - \frac{a^2c}{2} - abc\right),$$

$$\kappa_2 := 1 + \frac{3a^4}{4} + 3a^3b + 4a^2b^2 + 2ab^3 - a^3c - 3a^2bc - 2ab^2c.$$

The entries of d_X finally look like:

$$\begin{aligned}
x_{15} &= \kappa_1 u^3 + aux + z,\\
x_{16} &= v^2 + vy + y^2,\\
x_{17} &= \frac{1}{2}\kappa_2 u^4 + w - \frac{1}{2}a\left(-a-2b\right)u^2x + x^2 + buz,\\
x_{25} &= y - v,\\
x_{26} &= \left(-b - b^2\kappa_1 + \frac{1}{2}\left(c-a\right)\kappa_2\right)u^5 + \left(-a-2b+c\right)uw\\
&\quad + cux^2 + b\left(-a-b+c\right)u^2z - xz,\\
x_{35} &= \left(-1 + \left(-a-2b+c\right)\kappa_1 + \frac{\kappa_2}{2}\right)u^4\\
&\quad - w + \frac{1}{2}a\left(-a-2b+2c\right)u^2x + x^2 + \left(-a-b+c\right)uz,
\end{aligned}$$

with

$$\begin{aligned}
x_{15} &= x_{48} = x_{62} = x_{73}\\
x_{16} &= -x_{38} = -x_{52} = x_{74}\\
x_{17} &= x_{28} = -x_{53} = -x_{64}\\
x_{25} &= -x_{47} = -x_{61} = x_{83}\\
x_{26} &= x_{37} = x_{84} = x_{51}\\
x_{35} &= x_{46} = -x_{71} = -x_{82}
\end{aligned}$$

and with all other entries of the matrix zero.

The quantum dimensions of our matrix factorization are

$$\begin{aligned}
\mathrm{qdim}_l\left(d_X\right) &= \frac{1}{2}a^2\left(a+2b-c\right)\\
\mathrm{qdim}_r\left(d_X\right) &= -2\left(a-c\right)
\end{aligned}$$

which are not zero for any values of a, b, c satisfying Eqs. 3.

2.2. $\mathbf{Q_{10}} \sim_{\mathrm{orb}} \mathbf{E_{14}}$, version 2.

(1) This time we consider the potentials:

$$Q_{10} = x^4 + y^3 + xz^2$$

$$E_{14} = u^3 + v^8 + w^2$$

that is, the same Q_{10} but a different E_{14}. The variables of the potential Q_{10} have the same associated degree, while u and v of E_{14} switch theirs. This time, we will perturb with respect to w (the variable with the biggest degree). Set it equal to zero, and the resulting potential is:

$$\overline{Q_{10} - E_{14}} = x^4 + y^3 + xz^2 - u^3 - v^8$$

which has again a factorization similar to that of Eq. 2:

$$\overline{Q_{10} - E_{14}} = \left(x^2 + v^4\right)\left(x^2 - v^4\right) + \left(y-u\right)\left(y^2 + yu + u^2\right) + \left(xz\right)\left(z\right)$$

(2) Proceeding analogously to 2.1, we get:

$$\tilde{\tilde{d}}_1 = \begin{pmatrix} z & u^2+uy+y^2 & v^4+x^2 & 0 \\ -u+y & -xz & 0 & v^4+x^2 \\ -v^4+x^2 & 0 & -xz & -u^2-uy-y^2 \\ 0 & -v^4+x^2 & u-y & z \end{pmatrix}$$

whose determinant is precisely $\overline{Q_{10}-E_{14}}^2$. The degrees are distributed in the matrix in the same way as in Table 2. Again, $\tilde{\tilde{d}}_0$ is given by $\overline{Q_{10}-E_{14}}\tilde{\tilde{d}}_1^{-1}$.

(3) In this case, we will allow all possible perturbations – not only those linear in w. The perturbations associated to each degree are then:
- 9: v^3, vx.
- 12: vz, v^2x, w.
- 15: vw, v^5, v^2z, v^3x, vx^2.

We proceed as in the previous example. We obtain a matrix factorization with entries:

$$\begin{aligned} x_{15} &= bv^3 + cvx + z, \\ x_{16} &= u^2+uy+y^2, \\ x_{17} &= v^4 + aw + \frac{1}{2}\left(c^2+2cd\right)v^2x + x^2 + dvz, \\ x_{25} &= -u+y, \\ x_{26} &= -\frac{2avw}{b} + \left(b + \frac{2c}{b^2} - \frac{2cd}{b} + c^2d + 2cd^2 - \frac{c^2+2cd}{b}\right)v^3x \\ &\quad + \left(-\frac{2}{b}+c+2d\right)vx^2 - \frac{2v^2z}{b^2} - xz, \\ x_{35} &= -v^4 - aw + \left(c\left(-\frac{2}{b}+c+2d\right) + \frac{1}{2}\left(-c^2-2cd\right)\right)v^2x \\ &\quad + x^2 + \left(-\frac{2}{b}+d\right)vz, \end{aligned}$$

and

$$\begin{aligned} x_{15} &= x_{48} = x_{62} = x_{73} \\ x_{16} &= -x_{38} = -x_{52} = x_{74} \\ x_{17} &= x_{28} = x_{53} = -x_{64} = x_{82} \\ x_{25} &= -x_{47} = -x_{61} = x_{83} \\ x_{26} &= x_{37} = x_{26} = x_{51} = x_{84} \\ x_{35} &= x_{46} = -x_{71} \end{aligned}$$

with the rest of the entries of the matrix factorization being zero.

a, b, c and d must satisfy:

$$\begin{aligned} a^2 &= 1 \\ b^2 + \frac{4c}{b} - c^2 - 4cd + bc^2d + 2bcd^2 &= 0 \\ -2 + 2bc + \frac{2c^2}{b^2} - \frac{c^4}{4} + 2bd - \frac{2c^2d}{b} + c^2d^2 &= 0 \\ \frac{-2}{b^2} + \frac{2d}{b} - d^2 &= 0 \end{aligned} \tag{4}$$

The quantum dimensions of this matrix factorization are:

$$\mathrm{qdim}_l\left(d_X\right) = \frac{24a\left(-1+bc+bd\right)}{b}$$
$$\mathrm{qdim}_r\left(d_X\right) = \frac{6a}{b^2}\left(-3b^3-12c+7bc^2+3b^4d+24bcd-6b^2c^2d-18b^2cd^2+3b^3c^2d^2+6b^3cd^3\right)$$

which are not zero for any values of a, b, c, d which satisfy Eqs. 4.

3. Galois theory

In this section, we analyze in detail the solutions of Eqs. 3 and 4. These solutions lie in Galois orbits, which are described in the following two propositions.

Proposition 3.1. The solutions of Eqs. 3 are permuted by a Galois group isomorphic to $D_8 \times C_2$. Moreover, the solutions comprise three distinct orbits for the Galois action.

Proof. Define

$$\begin{aligned} f_1 &= 4+3a^4+8a^3b+8a^2b^2-4a^3c-8a^2bc \\ f_2 &= f_1-8=-4+3a^4+8a^3b+8a^2b^2-4a^3c-8a^2bc \\ g &= a^4-8a^2b^2-16ab^3-8b^4+8a^2bc+24ab^2c+16b^3c-2a^2c^2-8abc^2-8b^2c^2. \end{aligned}$$

Eqns. 3 reduce to $f_1f_2 = g = 0$. Thus, the solutions to Eqns. 3 come in two disjoint families. Family 1 consists of solutions to $f_1 = g = 0$, and Family 2 consists of solutions to $f_2 = g = 0$.

Solving the equations shows that the solutions in Family 1 have $a = i^k\sqrt[4]{-12\pm 8\sqrt{2}}$ for some $k \in \mathbb{Z}/4\mathbb{Z}$, and all eight possibilities for a occur. In other words, a is a root of x^8+24x^4+16, which is irreducible over $\mathbb{Q}$.

Solutions in Family 2 have $a = i^k\sqrt[4]{12\pm 8\sqrt{2}} = i^k\sqrt{2\pm 2\sqrt{2}}$, for some $k \in \mathbb{Z}/4\mathbb{Z}$, and all eight possibilities for a occur. in other words a is a root of

$$x^8-24x^4+16 = (x^4-4x^2-4)(x^4+4x^2-4) = 0.$$

The family of solutions with a a root of the irreducible polynomial x^4-4x^2-4 will be called Family 2A. The solutions with a a root of the irreducible polynomial x^4+4x^2-4 will be called Family 2B.

Every solution (a,b,c) to Eqs. 3 has a defined over $L = \mathbb{Q}(\sqrt[4]{-3+2\sqrt{2}}, \sqrt{1+\sqrt{2}})$ and, moreover, the values of a for all solutions of Eqs. 3 generate $L/\mathbb{Q}$. The field L is a degree 16 Galois extension of $\mathbb{Q}$ whose Galois group is isomorphic to $D_8 \times C_2$ and has generators ρ, σ, τ with the following actions on $m = \sqrt[4]{-3+2\sqrt{2}}$ and $n = \sqrt{1+\sqrt{2}}$:

$$\begin{aligned} \rho: &\quad m \mapsto im^{-1}, \quad n \mapsto in^{-1} \\ \sigma: &\quad m \mapsto m^{-1}, \quad n \mapsto in^{-1} \\ \tau: &\quad m \mapsto m, \quad n \mapsto -n. \end{aligned}$$

Note that $i = (m^2+m^{-2})/2$, so ρ has order 4, whereas σ and τ have order 2.

The a-values of solutions in Family 1 generate $\mathbb{Q}(m)/\mathbb{Q}$, the fixed field of τ. The a-values of solutions in Family 2 generate $\mathbb{Q}(i,n)/\mathbb{Q}$, the fixed field of $\tau\rho^2$. Both $\mathbb{Q}(m)/\mathbb{Q}$ and $\mathbb{Q}(i,n)/\mathbb{Q}$ are Galois extensions with Galois groups isomorphic to D_8.

The solutions (a,b,c) in Family 1 satisfy the equations $a^8+24a^4+16=0$ and

$$16(a+2b)c = 32ab+32b^2-12a^2-a^6.$$

They make up one Galois orbit.

The solutions (a,b,c) in Family 2A satisfy $a^4-4a^2-4=0$ and $2(a+2b)c=(a+2b)^2+2$. They make up one Galois orbit. The solutions (a,b,c) in Family 2B satisfy $a^4+4a^2-4=0$ and $2(a+2b)c=(a+2b)^2-2$. They make up one Galois orbit. $\square$

Proposition 3.2. The solutions of Eqs. 4 are permuted by a Galois group isomorphic to $V_4=C_2\times C_2$. The solutions comprise eight orbits for the Galois action, with each orbit having 4 elements.

Proof. The solutions of Eqs. 4 consist of two families: solutions in Family(+1) have $a=1$, whereas solutions in Family(-1) have $a=-1$. We define a new variable t by $t=bd$. The last equation in Eqs. 4 becomes

$$t^2-2t+2=0 \tag{5}$$

and hence $t=1\pm i$. Substituting (5) into the second and third equations in Eqs. 4 and simplifying gives the following equivalent system of equations.

$$\begin{aligned} a^2&=1\\ \left(\frac{b}{c}\right)^2&=1-t\\ c^4-8\left(\frac{b}{c}\right)c^2+8\left(\frac{b}{c}\right)^2&=0\\ t^2-2t+2&=0. \end{aligned} \tag{6}$$

Hence, the solutions only depend on a,b and c, and b/c is a primitive 8th root of unity. The solutions for c are the roots of $f(x)=x^{16}+2^7.17x^8+2^{12}$, which decomposes into four quartic polynomials over $\mathbb{Q}$, and splits completely into linear factors over $\mathbb{Q}(\zeta_8)$. Therefore, all values of c are defined over $\mathbb{Q}(\zeta_8)$, which has Galois group V_4. For each value of c, there is a unique primitive 8th root of unity β such that $c^4-8\beta c^2+8\beta^2=0$. In other words, each value of c determines a value of b/c, and hence also a value of t.

Each family of solutions, Family(+1) and Family(-1), breaks down into four Galois orbits, one for each quartic factor in the decomposition of f over $\mathbb{Q}$. So, in total we have eight Galois orbits, each with four elements corresponding to the four roots of a quartic factor of f. $\square$

Remark 3.3. Note the marked differences between the solutions of Eqs. 3 and those of Eqs. 4. In particular, there are infinitely many solutions to Eqs. 3, whereas Eqs. 4 admit precisely 32 solutions.

The elements in the Galois group interfere with our matrix factorizations in the following way. Let $W\in\mathbb{Q}[x,y,z]$ be a potential and let M be a finite–rank matrix factorization of W given by $\left(\mathbb{C}[x,y,z]^{\oplus 2r},d^M\right)$ $(r\in\mathbb{N})$. Let σ be an element of the Galois group and denote by $\sigma(d_M)$ the twisted differential obtained by applying σ to each entry. Since σ leaves the potential invariant, i.e., $\sigma(W)=W$, $\sigma(d_M)$ is still a factorization of W, $\sigma(M)=\left(\mathbb{C}[x,y,z]^{\oplus 2r},\sigma(d_M)\right)$. Therefore, we obtain not only one matrix factorization proving orbifold equivalence between Q_{10} and E_{14}, but infinitely many for Eqs. 3 and 32 for Eqs. 4 – one for each solution.

Remark 3.4. Note that the two Galois groups we obtain are quite different. V_4 is abelian and order 4, whereas $D_8\times C_2$ is non-abelian and order 16. In fact, V_4 is a subgroup of $D_8\times C_2$ – and actually also of D_8 alone. Both matrix factorizations prove the same orbifold equivalence, but the second version has the advantage that the resulting equations are much easier to handle.

It would certainly be interesting to further explore the connection between Galois groups and matrix factorizations proving orbifold equivalence between potentials described by singularities. That is the aim of the second part of this paper, [RCN]. Some ideas we would like to explore are the following.

We intend to investigate whether it is possible to predict from the outset whether a given expression of a singularity will lead to a Galois group which is easy to handle (e.g. abelian). In the particular example we have dealt with in this paper, in Version 1 the potential for E_{14} had a cross term, whereas in Version 2 (the easier one), the potential for E_{14} had only pure power monomials. But as we have seen in Table 1, not all the candidates for orbifold equivalence which at the same time are strangely dual have an associated potential which only has pure power monomials. In our case indeed a simpler shape of the potentials led to a simpler Galois group, but further analysis of other cases may give us some hints about how the Galois groups vary for each expression of the potentials.

While proving orbifold equivalence, in both [CRCR] and this paper we observe the repeated appearance of C_2 in the resulting Galois groups. We would like to investigate whether this is a coincidence or there is some intrinsic relationship with the structure of matrix factorizations.

Altogether, we look for(ward to) a better understanding of the orbifold equivalence, and we hope to provide further insights very soon.

Appendix A. Counting invertible potentials – by Ana Ros Camacho and Federico Zerbini

Besides the Arnold classification, one may ask the following question: given a polynomial ring with n variables over the complex numbers, how many kinds of potentials can we have and what do they look like?

A partial answer is provided by the Kreuzer–Skarke theorem [KS, HK]. In these papers they provide a graphical algorithm to generate potentials that we recall here.

Fix a regular set of weights. We call a *configuration* the set of polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ with this regular set of weights. A classification of potentials is encoded in certain graphs representing configurations. Every variable is represented by a dot, and a term of the form $x_i^a x_j$ is represented by an arrow from x_i to x_j ("x_i *points at* x_j").

Definition A.1. We call a variable x_i a *root* if the polynomial W contains a term x_i^a. A monomial $x_j^a x_k$ is called a *pointer* at x_k. The number a is called the *exponent* of x_i or x_j, respectively. We recursively define a link between two expressions, which may themselves be variables or links, as a monomial depending only on the variables occurring in these expressions. A link may further be linear in additional variables, which don't count as variables of the link. In this case we say that the link points at x_k, extending the definition of a pointer. It is possible that a specific monomial could have more than one interpretation as a link or a pointer. Given a potential W, any graph whose lines allow the above interpretation in terms of monomials in W is a graphic representation of W.

The following result is taken verbatim from [KS].

Theorem A.2. [5] *For a configuration a necessary and sufficient condition for a polynomial to be a potential is that it has a member which can be represented by a graph where:*

(1) *Each variable is either a root or points at another variable.*

[5] This theorem has been reformulated in a slightly more general setting in [HK], but we keep here the original formulation from [KS] as the graphical language proves intuitive and useful for explanations.

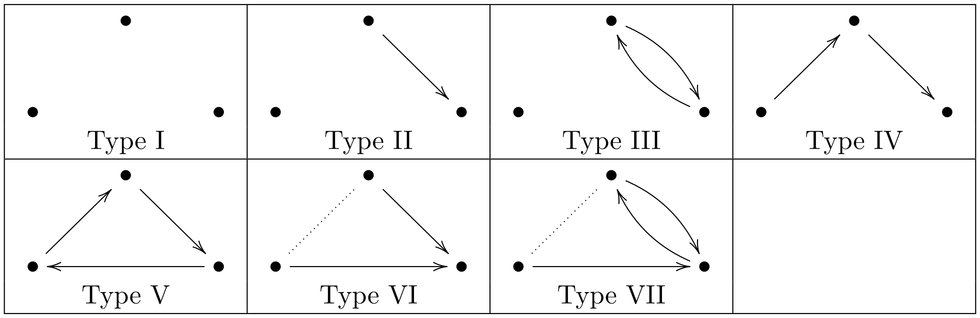
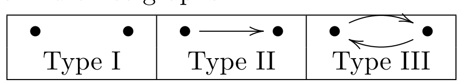


TABLE 4. Types of potentials for $n = 3$.

(2) *For any pair of variables and/or links pointing at the same variable x_i there is a link joining the two pointers and not pointing at x_i or any of the targets of the sublinks which are joined*[6].

Let us explain how this theorem works presenting a couple of examples for a small number of variables:

- [Ar, AGV] For $n = 2$, we find three graphs:



- [Ar, AGV] For $n = 3$, we find seven graphs as specified in Table 4.

Remark A.3. Notice that for $n = 3$ the second condition of Theorem A.2 is only relevant for Types VI and VII. Actually, one can reformulate this second condition for Types VI and VII as follows [AGV]. Every potential of Type VI contains a monomial in $\{x^a, y^b x, z^c x\}$, and those of Type VII contain a monomial in $\{x^a y, y^b x, z^c x\}$ (up to suitable changes of variables). The exponents of these potentials must satisfy the following conditions:

- Type VI: the least common multiple of b and c must be divisible by $a - 1$.
- Type VII: $(b - 1)\, c$ must be divisible by the product of $a - 1$ and the greatest common divisor of b and c.

The potentials generated via this theorem can be divided into two classes:

Definition A.4.

- Let W be a potential. We say W is *invertible* when the following conditions are satisfied:
  - The number of variables n coincides with the number of monomials in W,
  $$W\left(x_1, \ldots, x_n\right) = \sum_{i=1}^{n} a_i \prod_{j=1}^{n} x_j^{E_{ij}}$$
  for some coefficients $a_i \in \mathbb{C}^*$ and $E_{ij} \in \mathbb{Z}_{\geq 0}$.
  - The matrix $E := (E_{ij})$ is invertible over $\mathbb{Q}$.
  - [BH] The *Berglund-Hübsch transpose* of W, written W^T and defined by
  $$W^T\left(x_1, \ldots, x_n\right) = \sum_{i=1}^{n} a_i \prod_{j=1}^{n} x_j^{E_{ji}},$$

[6] We will draw these links as dotted arrows to distinguish them from those coming from the first condition of the theorem.

is also a potential.

- If a potential is not invertible, we call it a *beserker*.

As an example, notice that a potential in two variables is always invertible. In three variables, it is invertible if it is of type I–V, and it is a beserker if it is of type VI or VII.

Remark A.5.

- The Berglund-Hübsch transposition is closely related to mirror symmetry and the Landau-Ginzburg/Calabi-Yau correspondence, see for example [Chi] or [ET].
- For the potentials associated to the singularities Q_{10} and E_{14}, notice that in Version 2.1 they are Berglund-Hübsch transposes of each other, while this is not the case in Version 2.2. Actually, whenever we take the Berglund-Hübsch transpose of a potential (from the first column of Table 1) described by an exceptional unimodular singularity, we either obtain the same potential or the corresponding strange dual.
- In addition, notice that the Berglund-Hübsch transposition preserves the central charge for invertible potentials [RC], which suggests that Berglund-Hübsch may be a source of orbifold equivalences (see Remark 1.5).
- For invertible potentials, the Berglund-Hübsch transposition corresponds graphically to reversing the directions of the arrows.

Remark A.6. Invertible potentials can only be of three types (or combinations of them) [KS]:

- *Fermat*: $x_1^{a_1} + x_2^{a_2} + \ldots + x_n^{a_n}$
- *Chain*: $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \ldots + x_{n-1}^{a_{n-1}} + x_n^{a_n}$
- *Loop*: $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \ldots + x_{n-1}^{a_{n-1}} + x_n^{a_n} x_1$

Translating this in terms of dots and arrows, the Fermat part of the potentials is represented by isolated dots (see Type I in Table 4), the chain part by the union of all chains, i.e., the sequences of arrows leading from one dot to another distinct dot (see Type IV), and the loop part by the union of all loops, i.e., the sequences of arrows leading from one dot to itself (see Type V). This means that the invertible potentials are in one-to-one correspondence with mappings of n points to themselves that never involve two points mapping to a third distinct point.

A question that may arise at this point is, using this description in terms of mappings of points, how many invertible potentials do we get for a given number of variables?

We denote by $P(n)$ the number of partitions of n, and we denote by $P_1(n)$ the number of partitions of n where we exclude parts with cardinality 1. For example $P(3) = 3$, because we have the partitions $\{3\}$, $\{2,1\}$ and $\{1,1,1\}$, but $P_1(3) = 1$, since only $\{3\}$ is allowed.
It is easy to see that the two sequences are related by $P_1(n) = P(n) - P(n-1)$.

Proposition A.7. The number of invertible potentials (or, for brevity, invertibles) is given by

$$\mathrm{Inv}(n) = 1 + 2\sum_{k=2}^{n} P_1(k) + \sum_{k=4}^{n}\sum_{i=2}^{k-2} P_1(i)P_1(k-i). \tag{7}$$

Proof. First, notice that the number of invertibles given by Equation 7 matches our computation by hand:

$$\begin{aligned}\mathrm{Inv}\,(2) &= 3\\ \mathrm{Inv}\,(3) &= 5\\ \mathrm{Inv}\,(4) &= 10\\ \mathrm{Inv}\,(5) &= 16\\ \mathrm{Inv}\,(6) &= 29\\ \mathrm{Inv}\,(7) &= 45\\ \mathrm{Inv}\,(8) &= 75\\ \mathrm{Inv}\,(9) &= 115\end{aligned}$$

The proof relies on the fact that one can easily compute $\mathrm{Inv}(n)$ if $\mathrm{Inv}(n-1)$ is given. This is because every invertible with n dots that has at least one isolated dot can be thought as an invertible with $n-1$ dots plus the mentioned isolated one. This means that counting the invertibles without isolated dots is the same as computing $\mathrm{Inv}(n) - \mathrm{Inv}(n-1)$.
One can think of an invertible without isolated dots as divided into 2 blocks: one constituted by chains and one constituted by loops. Note that the number of dots in any chain or loop is at least 2, so one gets the following intuitive formula:

$$\mathrm{Inv}(n) - \mathrm{Inv}(n-1) = 2P_1(n) + \sum_{i=2}^{n-2} P_1(i)P_1(n-i),$$

where $2P_1(n)$ counts the invertibles constituted either only by chains or only by loops (this is why there is a factor 2!), and the sum counts the invertibles with a mix of chains and loops. Now the proof of Equation 7 is trivial, because we already now that it works for the first values of n, so we just need to check that Equation 7 also gives the difference predicted above, which is straightforward. □

Remark A.8. (Courtesy of G. Sanna) This formula can be rewritten as

$$\mathrm{Inv}(n) = \sum_{k=0}^{n} P(n-k)[P(k) - P(k-1)], \tag{8}$$

with $P(0) := 1$, $P(-1) := 0$. One can easily prove that the two formulas give the same result by induction, rewriting $P(n-k)$ as $P_1(n-k) + P(n-1-k)$ in Equation 8 and using the fact that $P_1(1) = 0$ and that $P_1(k) = P(k) - P(k-1)$.

Thanks to Remark A.8[7], one can immediately see what is the generating function for the numbers $\mathrm{Inv}(n)$:

Corollary A.9. Setting $\mathrm{Inv}(0) := 1$, we have

$$\sum_{n\geq 0} \mathrm{Inv}(n)q^n = (1-q)\prod_{m\geq 1}(1-q^m)^{-2} \tag{9}$$

Proof. Expanding every term in the product on the right as a power series in q shows that

$$\sum_{n\geq 0} P(n)q^n = \prod_{m\geq 1}(1-q^m)^{-1}.$$

[7] Actually, the generating function was found originally using Equation 7 and without Equation 8, but the proof was less elegant.

So the left hand side of Equation 9 can be rewritten as

$$(1-q)\Big(\sum_{n\geq 0} P(n)q^n\Big)^2.$$

The result now follows from the observation that

$$\Big(\sum_{n\geq 0} P(n)q^n\Big)^2 = \sum_{n\geq 0}\Big(\sum_{k=0}^{n} P(n-k)P(k)\Big)q^n$$

and

$$q\Big(\sum_{n\geq 0} P(n)q^n\Big)^2 = \sum_{n\geq 0}\Big(\sum_{k=0}^{n} P(n-k)P(k-1)\Big)q^n.$$

$\square$

Note that this is the same generating function as the one generating the sequence A000990 in the Encyclopedia of Integer Sequences ([Slo]), which counts the number of plane partitions of n with at most two rows.

References

[Ar] V. I. Arnol'd, *Critical points of smooth functions, and their normal forms.* Uspehi Mat. Nauk **30** (1975), no. 5(185), 3–65 (English translation in Russ. Math. Surv. **30:5** (1975), 1–75).

[AGV] V. I. Arnol'd, S. M. Guseĭn-Zade and A. N. Varchenko, *Singularities of differentiable maps. Volume I. The classification of critical points, caustics and wave fronts.* Translated from the Russian by Ian Porteous and Mark Reynolds. Monographs in Mathematics, **82**. Birkhäuser Boston, Inc., Boston, MA, 1985. xi+382 pp.

[BCP] I. Brunner, N. Carqueville and D. Plencner, *A quick guide to defect orbifolds.* String-Math 2013, 231–241, Proc. Sympos. Pure Math., **88**, Amer. Math. Soc., Providence, RI, 2014.

[BH] P. Berglund and T. Hübsch, *A generalized construction of mirror manifolds.* Nucl. Phys. B **393** (1993), no. 1-2, 377–391.

[CRCR] N. Carqueville, A. Ros Camacho and I. Runkel, *Orbifold equivalent potentials.* J. Pure Appl. Algebra **220** (2016), no. 2, 759–781.

[CR1] N. Carqueville and I. Runkel, *Rigidity and defect actions in Landau-Ginzburg models.* Comm. Math. Phys. **310** (2012), no. 1, 135–179.

[CR2] N. Carqueville and I. Runkel, *Orbifold completion of defect bicategories.* Quantum Topol. **7** (2016), no. 2, 203–279.

[CM] N. Carqueville and D. Murfet, *Adjunctions and defects in Landau-Ginzburg models.* Adv. Math. **289** (2016), 480–566.

[CQ] N. Carqueville and A. Quintero Vélez, *Calabi–Yau completions and orbifold equivalences.* arXiv:1509.00880 [math.RT].

[CDZ] S. Cecotti and M. Del Zotto, *On Arnold's 14 'exceptional' $\mathcal{N}=2$ superconformal gauge theories.* J. High Energy Phys. **99** (2011), no. 10, 53 pp. .

[Chi] A. Chiodo and Y. Ruan, *LG/CY correspondence: the state space isomorphism.* Adv. Math. **227** (2011), no. 6, 2157–2188.

[Dol] I. V. Dolgačev, *Conic quotient singularities of complex surfaces.* (Russian) Funkcional. Anal. i Priložen. **8** (1974), no. 2, 75–76. (Engl. translation in Funct. Anal. Appl. **8** (1974), 160–161); *Automorphic forms, and quasihomogeneous singularities.* (Russian) Funkcional. Anal. i Priložen. **9** (1975), no. 2, 67–68 (Engl. translation in Funct. Anal. Appl. **9** (1975), 149–151).

[Eb] W. Ebeling, *Strange duality, mirror symmetry, and the Leech lattice.* Singularity theory (Liverpool, 1996), xv–xvi, 55–77, London Math. Soc. Lecture Note Ser., **263**, Cambridge Univ. Press, Cambridge, 1999.

[ET] W. Ebeling and A. Takahashi, *Strange duality between hypersurface and complete intersection singularities.* arXiv:1508.02226 [math.AG].

[FRS1] J. Fuchs, I. Runkel and C. Schweigert, *TFT construction of RCFT correlators. I. Partition functions.* Nuclear Phys. B **646** (2002), no. 3, 353–497.

[Gab] A. M. Gabrièlov, *Dynkin diagrams of unimodal singularities.* (Russian) Funkcional. Anal. i Priložen. **8** (1974), no. 3, 1–6.

[Gin] V. Ginzburg, *Calabi–Yau algebras.* arXiv:math/0612139 [math.AG].

[HK] C. Hertling and R. Kurbel, *On the classification of quasihomogeneous singularities.* J. Singul. **4** (2012), 131–153.

[HW] P. S. Howe and P. C. West, *$N = 2$ superconformal models, Landau-Ginsburg Hamiltonians and the ϵ expansion*, Phys. Lett. B **223** (1989), 377–385; *Chiral correlators in Landau-Ginsburg [Ginzburg] theories and $N = 2$ superconformal models.* Phys. Lett. B **227** (1989), no. 3-4, 397–405; *Fixed points in multi-field Landau-Ginzburg models.* Phys. Lett. B **244** (1990), no. 2, 270–274.

[KST] H. Kajiura, K. Saito and A. Takahashi, *Triangulated categories of matrix factorizations for regular systems of weights with $\epsilon = -1$.* Adv. Math. **220** (2009), no. 5, 1602–1654.

[Kob] M. Kobayashi, *Duality of weights, mirror symmetry and Arnold's strange duality.* Tokyo J. Math. **31** (2008), no. 1, 225–251.

[KS] M. Kreuzer and H. Skarke, *On the classification of quasihomogeneous functions.* Comm. Math. Phys. **150** (1992), no. 1, 137–147.

[LVW] W. Lerche, C. Vafa and N. P. Warner, *Chiral rings in N=2 superconformal theories.* Nucl. Phys. B **324** (1989), 427–474.

[Ma1] E. J. Martinec, *Algebraic geometry and effective Lagrangians.* Phys. Lett. B **217** (1989), no. 4, 431–437.

[Ma2] E. J. Martinec, *Criticality, catastrophes and compactifications. Physics and mathematics of strings*, 389–433, World Sci. Publ., Teaneck, NJ, 1990.

[Pin] H. Pinkham, *Singularités exceptionelles, la dualité étrange d'Arnold et les surfaces $K-3$.* C. R. Acad. Sci. Paris Sér. A-B **284** (1977), no. 11, A615–A618.; E. Looijenga, *The smoothing components of a triangle singularity. I. Singularities, Part 2* (Arcata, Calif., 1981), 173–183, Proc. Sympos. Pure Math., **40**, Amer. Math. Soc., Providence, RI, 1983; E. Brieskorn, *The unfolding of exceptional singularities.* Leopoldina Symposium: Singularities (Thüringen, 1978). Nova Acta Leopoldina (N.F.) **52** (1981), no. 240, 65–93.

[RC] A. Ros Camacho, *Matrix factorizations and the Landau-Ginzburg/conformal field theory correspondence.* PhD thesis, available at E-Dissertationen der Universität Hamburg.

[RCN] A. Ros Camacho and R. Newton, *Strangely dual orbifold equivalence II.* Work in progress.

[Sai] K. Saito, *Regular systems of weights and associated singularities.* Complex analytic singularities, 479–526, Adv. Stud. Pure Math. **8** (1987).

[Slo] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, sequence A000990.

[VW] C. Vafa and N. Warner, *Catastrophes and the classification of conformal theories.* Phys. Lett. B **218** (1989), 51–58.

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CLASSIFICATION OF FOLIATIONS ON $\mathbb{CP}^2$ OF DEGREE 3 WITH DEGENERATE SINGULARITIES

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Abstract. The aim of this work is to classify foliations on $\mathbb{CP}^2$ of degree 3 with degenerate singular points. For that we construct a stratification of the space of holomorphic foliations by locally closed, irreducible, non-singular algebraic subvarieties which parametrize foliations with a special degenerate singularity. We also prove that there are only two foliations with isolated singularities with automorphism group of dimension two, the maximum possible dimension. Finally we obtain the unstable foliations with only one singular point, that is, a singular point with Milnor number 13.

1. Introduction

The aim of this work is to classify holomorphic foliations on $\mathbb{CP}^2$ of degree 3 with certain degenerate singular point using Geometric Invariant Theory (GIT). This theory was developed principally by David Hilbert and David Mumford (see [6]). We obtain locally closed, irreducible, non-singular algebraic subvarieties which parametrize foliations of degree 3 with a special degenerate singularity. We also get the dimension and explicit generators for each stratum. Similar results for degree 2 are given in [2] and in [3], we have some general results for degree d.

Geometric Invariant Theory gives a method for constructing quotients for group actions on algebraic varieties. More specifically, we have a linear action by a reductive group on a projective variety and we can construct a good quotient if we remove the closed set of unstable points. When the projective variety parametrizes geometric objects, the unstable points are in some sense degenerate objects. For example, the unstable plane algebraic curves with respect to the action by projective transformations are curves with non-ordinary singularities with order greater than 2.

In this article the projective variety $\mathcal{F}_3$ is the space of holomorphic foliations on $\mathbb{CP}^2$ of degree 3 and the action is given by change of coordinates. For this action we obtain the closed set of unstable foliations. We will prove that a foliation is unstable if and only if it has a special degenerate singular point (see Theorem 8). In this closed set we construct the stratification studied by Kirwan (in [12]), Hesselink (in [9]) and Kempf (in [11]). The strata are locally closed, non-singular, irreducible algebraic subvarieties of $\mathcal{F}_3$. We characterize the generic foliation on every stratum according to the Milnor number and multiplicity of their singularities. We also obtain the dimension of the strata (see Theorem 7).

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As a corollary we describe the irreducible components of the closed set of unstable foliations. We find, up to change of coordinates, the only two foliations with isolated singularities with automorphism group of dimension 2 (see Theorem 10). Finally we classify unstable foliations on $\mathbb{CP}^2$ of degree 3 with only one singular point, that is with Milnor number 13 (see Theorem 11). This result is important because the classification of foliations on $\mathbb{CP}^2$ with only one singular point is known only for degree 2 (see [5] and [2]).

In sections 2 and 3 we recall the basic results about Geometric Invariant Theory and foliations that we need in the sequel. We compute in section 4 the unstable foliations of degree 3 using the numerical criterion of one parameter subgroups. The construction of the stratification of the space of foliations and the characterization of the generic foliation on every stratum is included in section 5. The last section is devoted to give some important corollaries of the construction.

2. Geometric Invariant Theory

In this section we recall basic facts about Geometric Invariant Theory. All the definitions and results can be found in [14] and [11].

Let V be a projective variety in $\mathbb{CP}^n$, and consider a reductive group G acting linearly on V.

Definition 1. *Let $x \in V \subset \mathbb{CP}^n$, and consider $\overline{x} \in \mathbb{C}^{n+1}$ such that $\overline{x} \in x$. Denote by $O(\overline{x})$ the orbit of $\overline{x}$ in the affine cone of V and by $O(x)$ the orbit of x. Then*
*(i) x is **unstable** if $0 \in \overline{O(\overline{x})}$.*
*(ii) x is **semi-stable** if $0 \notin \overline{O(\overline{x})}$. The set of semi-stable points will be denoted by V^{ss}.*
*(iii) x is **stable** if it is semi-stable, $O(x)$ is closed in V^{ss} and $\dim O(x) = \dim G$. The set of stable points will be denoted by V^s.*

The main result in GIT is the following:

Theorem 1. *(see page 74 in [14])*
(i) There exists a projective variety Y and a morphism $\phi : V^{ss} \to Y$, which is a good quotient.

(ii) There exists an open set $Y^s \subset Y$ such that $\phi^{-1}(Y^s) = V^s$ and the morphism $\phi_| : V^s \to Y^s$ is a good quotient and an orbit space.

It is very often difficult to find the unstable points for a given action, but there exists a very useful criterion due to Hilbert and Mumford. Let us describe it.

A 1-parameter subgroup (1-PS) of the group G is an algebraic morphism $\lambda : \mathbb{C}^* \to G$. Since the action on V is linear, this induces a diagonal representation of $\mathbb{C}^*$:

$$\begin{aligned} \mathbb{C}^* &\to GL(n+1, \mathbb{C}) \\ t &\mapsto \lambda(t) : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1} \\ &\qquad\qquad\quad v \mapsto \lambda(t)v. \end{aligned}$$

Therefore there exists a basis $\{v_0, ..., v_n\}$ of $\mathbb{C}^{n+1}$ such that $\lambda(t)v_i = t^{r_i}v_i$, where $r_i \in \mathbb{Z}$.

Definition 2. *Let $x \in X$ and let $\lambda : \mathbb{C}^* \to G$ be a 1-PS of G. If $\bar{x} \in x$ and $\bar{x} = \sum_{i=0}^n a_i v_i$, then $\lambda(t)\bar{x} = \sum_{i=0}^n t^{r_i} a_i v_i$. We define the following function*

$$\mu(x, \lambda) := min\{r_i : a_i \neq 0\}.$$

The numerical criterion can now be stated.

Theorem 2. *(see Theorem 4.9 of [14])*
(i) x is stable if and only if $\mu(x,\lambda) < 0$ for every 1-PS, λ, of G.
(ii) x is unstable if and only if there exists a 1-PS, λ, of G such that $\mu(x,\lambda) > 0$.

Definition 3. *If $\mu(x,\lambda) > 0$ we will say that x is λ-**unstable**.*

The following is a useful tool for applying the criterion of 1-PS when $G = SL(n,\mathbb{C})$. We formulate the result for the case $n = 3$.

Lemma 1. *(see [14]) Every 1-parameter subgroup of $SL(3,\mathbb{C})$ has the form*

$$\lambda(t) = g\begin{pmatrix} t^{k_1} & 0 & 0 \\ 0 & t^{k_2} & 0 \\ 0 & 0 & t^{k_3} \end{pmatrix} g^{-1},$$

for some $g \in SL(3,\mathbb{C})$ and some integers k_1, k_2, k_3 such that $k_1 \geq k_2 \geq k_3$ and $k_1 + k_2 + k_3 = 0$.

3. Foliations on $\mathbb{CP}^2$ of degree d

This section provides the definitions and results that we need to know about holomorphic foliations on $\mathbb{CP}^2$ for the development of the paper.

Definition 4. *A holomorphic foliation X of $\mathbb{CP}^2$ of degree d is a non-trivial morphism of vector bundles:*

$$X : \mathcal{O}(1-d) \to \mathcal{T}\mathbb{CP}^2,$$

modulo multiplication by a nonzero scalar. The space of foliations of degree d is

$$\mathcal{F}_d := \mathbb{P}H^0(\mathbb{CP}^2, \mathcal{T}\mathbb{CP}^2(d-1)),$$

where $d \geq 0$.

Take homogeneous coordinates $(x : y : z)$ on $\mathbb{CP}^2$. Up to multiplication by a nonzero scalar there are two equivalent ways to describe a foliation of degree d (see [8]):

(1) By a homogeneous vector field:

$$X = P(x,y,z)\frac{\partial}{\partial x} + Q(x,y,z)\frac{\partial}{\partial y} + R(x,y,z)\frac{\partial}{\partial z} = \begin{pmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{pmatrix}$$

where $P, Q, R \in \mathbb{C}[x,y,z]$ are homogeneous of degree d. And if we consider the radial foliation

$$E = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z},$$

then X and $X + F(x,y,z)E$ represent the same foliation for all $F \in \mathbb{C}[x,y,z]$ homogeneous of degree $d-1$.

(2) By a homogeneous 1-form: $\Omega = L(x,y,z)dx + M(x,y,z)dy + N(x,y,z)dz$, such that $L, M, N \in \mathbb{C}[x,y,z]$ are homogeneous of degree $d+1$ and these satisfy the Euler's condition $xL + yM + zN = 0$.

With this we can see that the space of foliations on $\mathbb{CP}^2$ of degree d is a projective space of dimension $d^2 + 4d + 2$. We will use the description 1 for the rest of the paper.

We now define the notion of singular point for a foliation and two important invariants for this.

Definition 5. *A point $p = (a : b : c) \in \mathbb{CP}^2$ is **singular** for the above foliation X if*

$$(P(a,b,c), Q(a,b,c), R(a,b,c)) = (ka, kb, kc)$$

for some $k \in \mathbb{C}$. The set of singular points of X will be denoted by $Sing(X)$.

Definition 6. *Let*

$$\begin{pmatrix} f(y,z) \\ g(y,z) \end{pmatrix}$$

be a local generator of X in $p = (1 : b : c)$. Then

***the Milnor number of** p is $\mu_p(X) := \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^2,p}}{<f,g>}$,*
***the multiplicity of** p is $m_p(X) := min\{ord_p(f), ord_p(g)\}$.*

Proposition 1. *(see [4]) Let X be a foliation of degree d with isolated singularities then*

$$d^2 + d + 1 = \sum_{p \in \mathbb{CP}^2} \mu_p(X).$$

From Lemma 1.2 in [7] we can deduce that

$$\{X \in \mathcal{F}_d : \text{there exists } p \in \mathbb{CP}^2 \text{ such that } \mu_p(X) \geq 2\}$$

is a divisor in $\mathcal{F}_d$, therefore we have the following:

Theorem 3. *The set $\{X \in \mathcal{F}_d :$ every singular point for X has Milnor number $1\}$ is open and non-empty in $\mathcal{F}_d$.*

Finally we give the definition of algebraic leaf for a foliation.

Definition 7. *A plane curve defined by a polynomial $F(x,y,z)$ is an **algebraic leaf for** X **or invariant by** X if and only if there exists a polynomial $H(x,y,z)$ such that:*

$$P(x,y,z)\frac{\partial F(x,y,z)}{\partial x} + Q(x,y,z)\frac{\partial F(x,y,z)}{\partial y} + R(x,y,z)\frac{\partial F(x,y,z)}{\partial z} = FH.$$

Theorem 4. *(see Theorem 1.1, p.158 in [10] and [13]) The set*

$$\{X \in \mathcal{F}_d : X \text{ has no algebraic leaves}\}$$

is open and non-empty in $\mathcal{F}_d$.

Generically a foliation on $\mathbb{CP}^2$ of degree d does not have degenerate singularities and does not have algebraic leaves. So it is important to classify foliations in the complement of these sets. In this article we say something about that for degree 3.

The group $PGL(3,\mathbb{C})$ of automorphisms of $\mathbb{CP}^2$ is a reductive group that acts linearly on $\mathcal{F}_d$ by change of coordinates:

$$\begin{aligned} PGL(3,\mathbb{C}) \times \mathcal{F}_d &\to \mathcal{F}_d \\ (g, X) &\mapsto gX = DgX \circ (g^{-1}). \end{aligned}$$

In the computations we will use $SL(3,\mathbb{C})$ instead of $PGL(3,\mathbb{C})$, we will get the same results.

4. Unstable Foliations on $\mathbb{CP}^2$ of degree 3

As we saw before the space of foliations $\mathcal{F}_3$ is a projective space of dimension 23. In this section we apply the numerical criterion of one parameter subgroups to obtain the closed set of unstable foliations of degree 3. Remember that $X \in \mathcal{F}_d$ is unstable with respect to the action by change of coordinates if and only if there exists λ a 1-PS of $SL(3,\mathbb{C})$ such that $\mu(X,\lambda) > 0$ (see Theorem 2). For all λ a 1-PS of $SL(3,\mathbb{C})$ there exists $g \in SL(3,\mathbb{C})$ such that $D(t) := g\lambda(t)g^{-1}$ is a diagonal 1-PS, with the form:

$$D : \mathbb{C}^* \to SL(3,\mathbb{C}), \quad t \mapsto \begin{pmatrix} t^{k_1} & 0 & 0 \\ 0 & t^{k_2} & 0 \\ 0 & 0 & t^{k_3} \end{pmatrix},$$

for some intergers k_1, k_2, k_3 such that $k_1 \geq k_2 \geq k_3$ and $k_1 + k_2 + k_3 = 0$.

Since $\mu(gX, D) = \mu(X, g^{-1}Dg) = \mu(X,\lambda)$ (see remark 4.10 of [14]), every unstable foliation is in the orbit of an unstable point with respect to a diagonal 1-PS. Therefore, we will find the unstable foliations with respect to a diagonal one parameter subgroup and then we will take the set of orbits of these points.

Let us consider the basis for the vector space $H^0(\mathbb{CP}^2, \mathcal{T}\mathbb{CP}^2(2))$ given by

$$\{M\frac{\partial}{\partial x}, M\frac{\partial}{\partial y}, x^3\frac{\partial}{\partial z}, x^2y\frac{\partial}{\partial z}, xy^2\frac{\partial}{\partial z}, y^3\frac{\partial}{\partial z} : M \in \mathbb{C}[x,y,z] \text{ is a monic monomial of degree 3}\}.$$

This basis diagonalizes the action of $SL(3,\mathbb{C})$. Let $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}$ be a foliation on $\mathbb{CP}^2$ of degree 3 where

$$\begin{aligned} P(x,y,z) &= \sum a_{\alpha,\beta} x^\alpha y^\beta z^{3-\alpha-\beta} \\ Q(x,y,z) &= \sum b_{\alpha,\beta} x^\alpha y^\beta z^{3-\alpha-\beta} \\ R(x,y,z) &= \sum c_{\alpha,\beta} x^\alpha y^\beta z^{3-\alpha-\beta}. \end{aligned}$$

Then we are looking for the points $X \in \mathcal{F}_3$ such that there exist $k_1, k_2, k_3 \in \mathbb{Z}$ with $k_1 \geq k_2 \geq k_3$ and $k_1 + k_2 + k_3 = 0$ and such that $\max\{-E_P, -E_Q, -E_R\} < 0$, where

$$\begin{aligned} E_P &= \min\{-k_1(\alpha-1) - k_2\beta - k_3\gamma : a_{\alpha,\beta} \neq 0\} \\ E_Q &= \min\{-k_1\alpha - k_2(\beta-1) - k_3\gamma : b_{\alpha,\beta} \neq 0\} \\ E_R &= \min\{-k_1\alpha - k_2\beta - k_3(\gamma-1) : c_{\alpha,\beta} \neq 0\}. \end{aligned}$$

From definition 2, $\mu(X,D) = -\max\{-E_P, -E_Q, -E_R\}$, where D is the diagonal 1-PS defined above.

Since $k_1 > 0$ and $k_3 < 0$, then we can define $q_i := \frac{k_i}{k_1}$. Therefore $q_1 + q_2 + q_3 = 0$, $1 \geq q_2 \geq q_3$ and $q_2 \in [-\frac{1}{2}, 1] \cap \mathbb{Q}$. We must find the conditions in the rational numbers q_i to have non-zero coefficients for the monomials of P, Q and R. It is easy to obtain the following conclusion.

Coefficients $a_{\alpha,\beta}$ $(\alpha-1)k_1+\beta k_2+\gamma k_3<0$
$\alpha=2,\beta=0,\gamma=1,\ q_2\in(0,1]$
$\alpha=1,\beta=2,\gamma=0,\ q_2\in[-\frac{1}{2},0)$
$\alpha=1,\beta=1,\gamma=1,\ q_2\in[-\frac{1}{2},1]$
$\alpha=1,\beta=0,\gamma=2,\ q_2\in[-\frac{1}{2},1]$
$\alpha=0,\beta=3,\gamma=0,\ q_2\in[-\frac{1}{2},\frac{1}{3})$
$\alpha=0,\beta=2,\gamma=1,\ q_2\in[-\frac{1}{2},1]$
$\alpha=0,\beta=1,\gamma=2,\ q_2\in[-\frac{1}{2},1]$
$\alpha=0,\beta=0,\gamma=3,\ q_2\in[-\frac{1}{2},1]$

Coefficients $b_{\alpha,\beta}$ $\alpha k_1+(\beta-1)k_2+\gamma k_3<0$
$\alpha=2,\beta=0,\gamma=1,\ q_2\in(\frac{1}{2},1]$
$\alpha=1,\beta=1,\gamma=1,\ q_2\in(0,1]$
$\alpha=1,\beta=0,\gamma=2,\ q_2\in(-\frac{1}{3},1]$
$\alpha=0,\beta=3,\gamma=0,\ q_2\in[-\frac{1}{2},0)$
$\alpha=0,\beta=2,\gamma=1,\ q_2\in[-\frac{1}{2},1]$
$\alpha=0,\beta=1,\gamma=2,\ q_2\in[-\frac{1}{2},1]$
$\alpha=0,\beta=0,\gamma=3,\ q_2\in[-\frac{1}{2},1]$

Coefficients $c_{\alpha,\beta}$ $\alpha k_1+\beta k_2+(\gamma-1)k_3<0$
$\alpha=0,\beta=3,\gamma=0,\ q_2\in[-\frac{1}{2},-\frac{1}{4})$

From this, we see that $a_{3,0}$, $a_{2,1}$, $b_{3,0}$, $b_{2,1}$, $b_{1,2}$, $c_{3,0}$, $c_{2,1}$, $c_{1,2}=0$ and we can have $a_{1,1}$, $a_{1,0}$, $a_{0,2}$, $a_{0,1}$, $a_{0,0}$, $b_{0,2}$, $b_{0,1}$, $b_{0,0}\neq 0$. Now we do a partition of $\left[-\frac{1}{2},1\right]$ to have the subspaces of unstable foliations with respect to a diagonal 1-PS.

$$
\begin{aligned}
q_2\in\left[-\frac{1}{2},-\frac{1}{3}\right] &\Rightarrow a_{2,0},b_{2,0},b_{1,1},b_{1,0}=0\\
q_2\in\left(-\frac{1}{3},-\frac{1}{4}\right) &\Rightarrow a_{2,0},b_{2,0},b_{1,1}=0\\
q_2\in\left[-\frac{1}{4},0\right) &\Rightarrow a_{2,0},b_{2,0},b_{1,1},c_{0,3}=0\\
q_2=0 &\Rightarrow a_{2,0},a_{1,2},b_{2,0},b_{1,1},b_{0,3},c_{0,3}=0\\
q_2\in\left(0,\frac{1}{3}\right) &\Rightarrow a_{1,2},b_{2,0},b_{0,3},c_{0,3}=0\\
q_2\in\left[\frac{1}{3},\frac{1}{2}\right] &\Rightarrow a_{1,2},a_{0,3},b_{2,0},b_{0,3},c_{0,3}=0\\
q_2\in\left(\frac{1}{2},1\right] &\Rightarrow a_{1,2},a_{0,3},b_{0,3},c_{0,3}=0.
\end{aligned}
$$

Consider the seven subspaces with the corresponding coefficients equal to zero. In these sets we have 3 maximal subspaces of $H^0(\mathbb{CP}^2,\mathcal{T}\mathbb{CP}^2(2))$. That we describe below:

$$
\begin{aligned}
V_1:=\Big\langle & xy^2\frac{\partial}{\partial x},xyz\frac{\partial}{\partial x},xz^2\frac{\partial}{\partial x},y^3\frac{\partial}{\partial x},y^2z\frac{\partial}{\partial x},yz^2\frac{\partial}{\partial x},z^3\frac{\partial}{\partial x},\\
& xz^2\frac{\partial}{\partial y},y^3\frac{\partial}{\partial y},y^2z\frac{\partial}{\partial y},yz^2\frac{\partial}{\partial y},z^3\frac{\partial}{\partial y},y^3\frac{\partial}{\partial z}\Big\rangle_{\mathbb{C}}
\end{aligned}
$$

$$
\begin{aligned}
V_2:=\Big\langle & x^2z\frac{\partial}{\partial x},xyz\frac{\partial}{\partial x},xz^2\frac{\partial}{\partial x},y^2z\frac{\partial}{\partial x},yz^2\frac{\partial}{\partial x},z^3\frac{\partial}{\partial x},x^2z\frac{\partial}{\partial y},xyz\frac{\partial}{\partial y},xz^2\frac{\partial}{\partial y},\\
& y^2z\frac{\partial}{\partial y},yz^2\frac{\partial}{\partial y},z^3\frac{\partial}{\partial y}\Big\rangle_{\mathbb{C}}
\end{aligned}
$$

$$V_3 := \Big\langle x^2z\frac{\partial}{\partial x}, xyz\frac{\partial}{\partial x}, xz^2\frac{\partial}{\partial x}, y^3\frac{\partial}{\partial x}, y^2z\frac{\partial}{\partial x}, yz^2\frac{\partial}{\partial x}, z^3\frac{\partial}{\partial x}, xyz\frac{\partial}{\partial y}, xz^2\frac{\partial}{\partial y}, y^2z\frac{\partial}{\partial y}, yz^2\frac{\partial}{\partial y}, z^3\frac{\partial}{\partial y}\Big\rangle_{\mathbb{C}},$$

Then,

$\{X \in H^0(\mathbb{CP}^2, \mathcal{T}\mathbb{CP}^2(2)) :$ there exists D a 1-PS diagonal such that $\mu(X,D) > 0\} = V_1 \cup V_2 \cup V_3$.

Therefore we can state:

Theorem 5. *The closed set of unstable foliations on $\mathbb{CP}^2$ of degree 3 is*

$$\mathcal{F}_3^{un} = SL(3,\mathbb{C})\mathbb{P}V_1 \cup SL(3,\mathbb{C})\mathbb{P}V_2 \cup SL(3,\mathbb{C})\mathbb{P}V_3.$$

5. The Stratification of $\mathcal{F}_3$

In the previous section we exhibit the closed set of unstable foliations on $\mathbb{CP}^2$ of degree 3. In this section we will use properties of the singularities of the foliations to construct locally closed, non-singular subvarieties of $\mathcal{F}_3^{un}$. Firstly we will explain the stratification described in the following Theorem by F. Kirwan and then apply it to $\mathcal{F}_3$.

Theorem 6. *(see Theorem 13.5 in [12]) Let V be a non-singular projective variety with a linear action by a reductive group G. Then there exists a stratification*

$$\{S_\beta : \beta \in \mathcal{B}\}$$

of V such that the unique open stratum is V^{ss} and every stratum S_β in the set of unstable points is non-singular, locally closed and isomorphic to $G \times_{P_\beta} Y_\beta^{ss}$, where Y_β^{ss} is a non-singular locally-closed subvariety of V and P_β is a parabolic subgroup of G.

Throughout the text we will use the same notation as in §12 of [12].

Definition 8. *Let $Y(G)$ be the set of one parameter subgroups $\lambda : \mathbb{C}^* \to G$. Define in $Y(G) \times \mathbb{N}$ the equivalence relation: (λ_1, n_1) is related with (λ_2, n_2) if and only if $\lambda_1(t^{n_2}) = \lambda_2(t^{n_1})$ for all $t \in \mathbb{C}^*$.* ***A virtual one parameter subgroups of*** *G is an equivalence class of this relation, the set of these classes will be denoted by $M(G)$.*

The indexing set $\mathcal{B}$ of the stratification is a finite subset of $M(G)$ and this may be described in terms of the weights of the representation of G which defines the action. For the construction we must consider on $M(G)$ a norm q which is the square of an inner product $\langle\ ,\ \rangle$. This norm gives the partial order $>$ on $\mathcal{B}$.

On the other hand, the representation of $\mathbf{D}$ on $\mathbb{C}^{n+1}$, where $\mathbf{D}$ is a maximal torus of G, splits as a sum of scalar representations given by characters $\alpha_0, ..., \alpha_n$. These characters are elements of the dual of $M(\mathbf{D})$ but we can identify them with elements of $M(\mathbf{D})$ using $\langle\ ,\ \rangle$.

Definition 9. *Once we have the indexing set $\mathcal{B}$ we can describe the objects that appear in Theorem 6. Let $\beta \in \mathcal{B}$, we define:*

$$\begin{aligned} Z_\beta &= \{(x_0 : ... : x_n) \in V : x_j = 0 \text{ if } \langle \alpha_j, \beta\rangle \neq q(\beta)\}, \\ Y_\beta &= \{(x_0 : ... : x_n) \in V : x_j = 0 \text{ if } \langle \alpha_j, \beta\rangle < q(\beta) \\ &\qquad \text{and } x_j \neq 0 \text{ for some } j \text{ with } \langle \alpha_j, \beta\rangle = q(\beta)\}, \end{aligned}$$

the map $p_\beta : Y_\beta \to Z_\beta, (x_0, ..., x_n) \mapsto (x'_0, ..., x'_n)$ *as* $x'_j = x_j$ *if* $\langle \alpha_j, \beta \rangle = q(\beta)$ *and* $x'_j = 0$ *otherwise.*

Consider $Stab(\beta)$, the stabilizer of β under the adjoint action of G. There exists a unique connected reductive subgroup G_β of $Stab_\beta$ such that $M(G_\beta) = \{\lambda \in M(Stab_\beta) : \langle \lambda, \beta \rangle = 0\}$ (see 12.21 in [12]). With this group we can define

$$Z_\beta^{ss} = \{x \in Z_\beta : x \text{ is semistable under the action of } G_\beta \text{ on } Z_\beta\}$$

and $Y_\beta^{ss} = p_\beta^{-1}(Z_\beta^{ss})$.

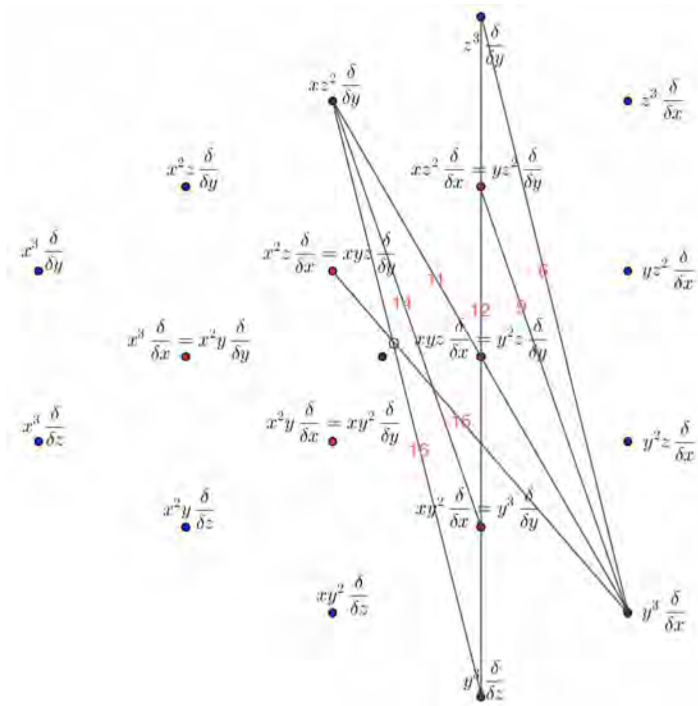
Finally the parabolic group of β is: if $x \in Y_\beta^{ss}$ then $P_\beta = \{g \in G : gx \in Y_\beta^{ss}\}$.

Remark 1. *Since* S_β *is isomorphic to* $G \times_{P_\beta} Y_\beta^{ss}$*, it has dimension* $\dim Y_\beta^{ss} + \dim G - \dim P_\beta$.

5.1. **The representation of $\mathcal{F}_3$.** Norbert A'Campo and Vladimir Popov give in [15] a computer program such that given a reductive group and one of its representation, the output is the finite subset $\mathcal{B}$ of virtual 1-parameter subgroups for the above stratification. For a more detailed construction of the virtual 1-parameter subgroups in the case of the action by change of coordinates of $SL(3, \mathbb{C})$ in $\mathcal{F}_d$ we refer to section 3 of [3]. For $\mathcal{F}_3$ the virtual 1-parameter subgroups for the stratification are:

$$\begin{gathered}
\beta_1 := \left(\frac{5}{3}, \frac{2}{3}, -\frac{7}{3}\right), \beta_2 := \left(\frac{5}{3}, -\frac{1}{3}, -\frac{4}{3}\right), \beta_3 := \left(\frac{3}{2}, 0, -\frac{3}{2}\right), \beta_4 := \left(\frac{5}{3}, -\frac{5}{6}, -\frac{5}{6}\right), \\
\beta_5 := \left(\frac{55}{42}, -\frac{11}{42}, -\frac{22}{21}\right), \beta_6 := \left(\frac{7}{6}, -\frac{1}{3}, -\frac{5}{6}\right), \beta_7 := \left(\frac{2}{3}, \frac{2}{3}, -\frac{4}{3}\right), \beta_8 := (1, 0, -1), \\
\beta_9 := \left(\frac{20}{21}, -\frac{4}{21}, -\frac{16}{21}\right), \beta_{10} := \left(\frac{2}{3}, \frac{1}{6}, -\frac{5}{6}\right), \beta_{11} := \left(\frac{1}{2}, 0, -\frac{1}{2}\right), \\
\beta_{12} := \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right), \beta_{13} := \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right), \beta_{14} := \left(\frac{5}{21}, -\frac{1}{21}, -\frac{4}{21}\right), \\
\beta_{15} := \left(\frac{2}{21}, \frac{1}{42}, -\frac{5}{42}\right), \beta_{16} := \left(\frac{7}{78}, -\frac{1}{39}, -\frac{5}{78}\right).
\end{gathered}$$

Now we consider the induced representation $H^0(\mathbb{CP}^2, T\mathbb{CP}^2(2))$ of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$. The weight diagrama for this irreducible representation is the following (the number i denoted the virtual 1-PS β_i):



From this information we can easily obtain the sets Z_i and Y_i described in definition 9. In Y_i with $i \in \{1, 2, 3, 4, 5, 7, 8, 10, 13\}$ every foliation has a curve of singularities, we can study these foliations as foliations of degree 2, so we are going to discard these strata.

To obtain the strata with foliations with isolated singularities we must find

$$Z_i^{ss} := \{x \in Z_i : \mu(x, \lambda) \leq \langle \lambda, \beta_i \rangle \text{ for all } \lambda \in M(\text{Stab}(\beta_i))\}.$$

See definition 12.10 of [12]. For this we will use the following results.

Lemma 2. *(see [2, p. 430]) Let $X \in Z_i$ such that the virtual one parameter subgroup (n_0, n_1, n_2) corresponding to β_i satisfies $n_0 > n_1 > n_2$. Then $X \in Z_i^{ss}$ if and only if β_i is the closest point to zero in C_X with respect to $\mathbf{D}$, where C_X is the convex hull formed with the weights of X.*

The only virtual 1-PS where $n_1 = n_2$ in β_{12}, for finding Z_{12}^{ss} we need further analysis. We must recall that $\text{Stab}(\beta_{12})$ is the stabilizer of β_{12} under the adjoint action of $SL(3, \mathbb{C})$ on $M(SL(3, \mathbb{C}))$ (see 12.21 in [12]), i.e.,

$$\begin{aligned}\text{Stab}(\beta_{12}) &= \left\{ g \in SL(3,\mathbb{C}) : g \begin{pmatrix} \frac{2}{3} & & \\ & -\frac{1}{3} & \\ & & -\frac{1}{3} \end{pmatrix} g^{-1} = \begin{pmatrix} \frac{2}{3} & & \\ & -\frac{1}{3} & \\ & & -\frac{1}{3} \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} \in SL(3,\mathbb{C}) \right\}.\end{aligned}$$

We know that if $\lambda \in M(\text{Stab}(\beta_{12}))$ then there exists $g \in \text{Stab}(\beta_{12})$ such that $g\lambda g^{-1}$ has the form $Diag(t^{k_1}, t^{k_2}, t^{k_3})$, where $k_1 \geq k_2 \geq k_3$; therefore:

$Z_{12}^{ss} = \{x \in Z_{12} : \mu(gx, \lambda) \leq \langle \lambda, \beta_{12} \rangle,$ for all $\lambda = Diag(t^{k_1}, t^{k_2}, t^{k_3})$, where $k_1 \geq k_2 \geq k_3$ and $g \in \mathrm{Stab}(\beta_{12})\}$.

Where, from the weight diagram:

$$Z_{12} = \mathbb{P}\Big\langle xy^2\frac{\partial}{\partial x}, xyz\frac{\partial}{\partial x}, xz^2\frac{\partial}{\partial x}, y^3\frac{\partial}{\partial y}, y^2z\frac{\partial}{\partial y}, yz^2\frac{\partial}{\partial y}, z^3\frac{\partial}{\partial y}, y^3\frac{\partial}{\partial z}\Big\rangle_{\mathbb{C}},$$

and we have $\langle \lambda(t) = Diag(t^{k_1}, t^{k_2}, t^{k_3}), \beta_{12} \rangle = \frac{2}{3}k_1 - \frac{1}{3}k_2 - \frac{1}{3}k_3$. For $X \in Z_{12}$ we obtain

$$\lambda(t) \cdot X = \begin{pmatrix} a_{1,2}t^{-2k_2}xy^2 + a_{1,1}t^{-k_2-k_3}xyz + a_{1,0}t^{-2k_3}xz^2 \\ b_{0,3}t^{-2k_2}y^3 + b_{0,2}t^{-k_2-k_3}y^2z + b_{0,1}t^{-2k_3}yz^2 + b_{0,0}t^{k_2-3k_3}z^3 \\ c_{0,3}t^{k_3-3k_2}y^3 \end{pmatrix},$$

therefore $\mu(X, \lambda) = \min\{-2k_2, -k_2 - k_3, -2k_3, k_2 - 3k_3, k_3 - 3k_2\}$. With the conditions

$$\begin{array}{rcc} -2k_2 \leq \frac{2}{3}k_1 - \frac{1}{3}k_2 - \frac{1}{3}k_3 & \Leftrightarrow & k_2 \geq k_3 \\ -k_2 - k_3 \leq \frac{2}{3}k_1 - \frac{1}{3}k_2 - \frac{1}{3}k_3 & \Leftrightarrow & 0 \geq 0 \\ -2k_3 \leq \frac{2}{3}k_1 - \frac{1}{3}k_2 - \frac{1}{3}k_3 & \Leftrightarrow & k_2 \leq k_3 \\ k_2 - 3k_3 \leq \frac{2}{3}k_1 - \frac{1}{3}k_2 - \frac{1}{3}k_3 & \Leftrightarrow & k_2 \leq k_3 \\ k_3 - 3k_2 \leq \frac{2}{3}k_1 - \frac{1}{3}k_2 - \frac{1}{3}k_3 & \Leftrightarrow & k_2 \geq k_3. \end{array}$$

we conclude that $Z_{12}^{ss} = \{X \in Z_{12} : (a_{1,2}, a_{1,1}, b_{0,3}, b_{0,2}, c_{0,3}) \neq 0, (a_{1,1}, a_{1,0}, b_{0,2}, b_{0,1}, b_{0,0}) \neq 0\}$. Now we can give the full list of linear subspaces of $\mathcal{F}_3$ for the construction of the strata.

$$Z_6^{ss} = \mathbb{P}\left\{\begin{pmatrix} a_{0,3}y^3 \\ b_{0,0}z^3 \\ 0 \end{pmatrix} \in Z_6 : a_{0,3} \neq 0, b_{0,0} \neq 0\right\}$$

$$Z_9^{ss} = \mathbb{P}\left\{\begin{pmatrix} a_{1,0}xz^2 + a_{0,3}y^3 \\ b_{0,1}yz^2 \\ 0 \end{pmatrix} : a_{0,3} \neq 0, (a_{1,0}, b_{0,1}) \neq 0\right\}$$

$$Z_{11}^{ss} = \mathbb{P}\left\{\begin{pmatrix} a_{1,1}xyz + a_{0,3}y^3 \\ b_{1,0}xz^2 + b_{0,2}y^2z \\ 0 \end{pmatrix} : b_{1,0} \neq 0, (a_{1,1}, a_{0,3}, b_{0,2}) \neq 0\right\}$$

$$Z_{12}^{ss} = \mathbb{P}\left\{\begin{pmatrix} a_{1,2}xy^2 + a_{1,1}xyz + a_{1,0}xz^2 \\ b_{0,3}y^3 + b_{0,2}y^2z + b_{0,1}yz^2 + b_{0,0}z^3 \\ c_{0,3}y^3 \end{pmatrix} : (a_{1,2}, a_{1,1}, b_{0,3}, b_{0,2}, c_{0,3}) \neq 0, (a_{1,1}, a_{1,0}, b_{0,2}, b_{0,1}, b_{0,0}) \neq 0\right\}$$

$$Z_{14}^{ss} = \mathbb{P}\left\{\begin{pmatrix} a_{1,2}xy^2 \\ b_{1,0}xz^2 + b_{0,3}y^3 \\ 0 \end{pmatrix} : b_{1,0} \neq 0, (a_{1,2}, b_{0,3}) \neq 0\right\}$$

$$Z_{15}^{ss} = \mathbb{P}\left\{\begin{pmatrix} a_{2,0}x^2z + a_{0,3}y^3 \\ b_{1,1}xyz \\ 0 \end{pmatrix} : a_{0,3} \neq 0, (a_{2,0}, b_{1,1}) \neq 0\right\}$$

$$Z_{16}^{ss} = \mathbb{P}\left\{ \begin{pmatrix} 0 \\ b_{1,0}xz^2 \\ c_{0,3}y^3 \end{pmatrix} : b_{1,0} \neq 0, c_{0,3} \neq 0 \right\},$$

and

$$Y_6^{ss} = \mathbb{P}\left\{ \begin{pmatrix} a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3 \\ b_{0,0}z^3 \\ 0 \end{pmatrix} : a_{0,3} \neq 0, b_{0,0} \neq 0 \right\}$$

$$Y_9^{ss} = \mathbb{P}\left\{ \begin{pmatrix} a_{1,0}xz^2 + a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3 \\ b_{0,1}yz^2 + b_{0,0}z^3 \\ 0 \end{pmatrix} : a_{0,3} \neq 0, (a_{1,0}, b_{0,1}) \neq 0 \right\}$$

$$Y_{11}^{ss} = \mathbb{P}\left\{ \begin{pmatrix} a_{1,1}xyz + a_{1,0}xz^2 + a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3 \\ b_{1,0}xz^2 + b_{0,2}y^2z + b_{0,1}yz^2 + b_{0,0}z^3 \\ 0 \end{pmatrix} : \right.$$
$$\left. b_{1,0} \neq 0, (a_{1,1}, a_{0,3}, b_{0,2}) \neq 0 \right\}$$

$$Y_{12}^{ss} = \mathbb{P}\left\{ \begin{pmatrix} \sum_{j=0}^{2} a_{1,j}xy^jz^{2-j} + \sum_{j=0}^{3} a_{0,j}y^jz^{3-j} \\ \sum_{j=0}^{3} b_{0,j}y^jz^{3-j} \\ c_{0,3}y^3 \end{pmatrix} : \right.$$
$$\left. (a_{1,2}, a_{1,1}, b_{0,3}, b_{0,2}, c_{0,3}) \neq 0, (a_{1,1}, a_{1,0}, b_{0,2}, b_{0,1}, b_{0,0}) \neq 0 \right\}$$

$$Y_{14}^{ss} = \mathbb{P}\left\{ \begin{pmatrix} a_{1,2}xy^2 + a_{1,1}xyz + a_{1,0}xz^2 + a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3 \\ b_{1,0}xz^2 + b_{0,3}y^3 + b_{0,2}y^2z + b_{0,1}yz^2 + b_{0,0}z^3 \\ 0 \end{pmatrix} : \right.$$
$$\left. b_{1,0} \neq 0, (a_{1,2}, b_{0,3}) \neq 0 \right\}$$

$$Y_{15}^{ss} = \mathbb{P}\left\{ \begin{pmatrix} a_{2,0}x^2z + a_{1,1}xyz + a_{1,0}xz^2 + a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3 \\ b_{1,1}xyz + b_{1,0}xz^2 + b_{0,2}y^2z + b_{0,1}yz^2 + b_{0,0}z^3 \\ 0 \end{pmatrix} : \right.$$
$$\left. a_{0,3} \neq 0, (a_{2,0}, b_{1,1}) \neq 0 \right\}$$

$$Y_{16}^{ss} = \mathbb{P}\left\{ \begin{pmatrix} a_{1,2}xy^2 + a_{1,1}xyz + a_{1,0}xz^2 + a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3 \\ b_{1,0}xz^2 + b_{0,3}y^3 + b_{0,2}y^2z + b_{0,1}yz^2 + b_{0,0}z^3 \\ c_{0,3}y^3 \end{pmatrix} : \right.$$
$$\left. b_{1,0} \neq 0, c_{0,3} \neq 0 \right\}.$$

6. Strata of the space of foliations of degree 3 and its singularities

In this section we calculate the Milnor number and the multiplicity of a common singularity in the generic foliation in every stratum. We also obtain the dimension of the strata.

Note that the point $p = (1 : 0 : 0)$ is a singularity for every foliation in Y_i^{ss} for all $i = 6$, 9, 11, 12, 14, 15, 16. Along this section we use the following notation: given a foliation,

$$X = \begin{pmatrix} P(x, y, z) \\ Q(x, y, z) \\ R(x, y, z) \end{pmatrix} \in Y_i^{ss},$$

we consider the corresponding local polynomial vector field around $(0, 0)$:

$$X_0 = (Q(1, y, z) - yP(1, y, z))\frac{\partial}{\partial y} + (R(1, y, z) - zP(1, y, z))\frac{\partial}{\partial z}.$$

We define $f_i(y, z) := Q(1, y, z) - yP(1, y, z)$, $g_i(y, z) := R(1, y, z) - zP(1, y, z)$ and $I_0(f, g)$ will be the intersection index of f and g at $(0, 0)$.

6.1. **Stratum 6.** As we saw before if $X \in Y_6^{ss}$ then $a_{0,3}$ and $b_{0,0}$ are different from zero and

$$\begin{aligned} f_6(y, z) = Q(1, y, z) - yP(1, y, z) &= b_{0,0}z^3 - a_{0,3}y^4 - a_{0,2}y^3z - a_{0,1}y^2z^2 - a_{0,0}yz^3 \\ g_6(y, z) = -zP(1, y, z) &= -a_{0,3}y^3z - a_{0,2}y^2z^2 - a_{0,1}yz^3 - a_{0,0}z^4. \end{aligned}$$

Note that $b_{0,0}z^3$ and $P(1, y, z)$ does not have common tangent lines, therefore

$$\begin{aligned} \mu_p(X) &= I_0(f_6(y, z), g_6(y, z)) = I_0(f_6(y, z), z) + I_0(f_6(y, z), P(1, y, z)) \\ &= I_0(-a_{0,3}y^4, z) + 9 = 13 \\ m_p(X) &= 3. \end{aligned}$$

Finally, the 2-jet of $\begin{pmatrix} f_6 \\ g_6 \end{pmatrix}$ is trivial and the 3-jet is $\begin{pmatrix} z^3 \\ 0 \end{pmatrix}$, if we suppose $b_{0,0} = 1$. On the other hand, if X is a foliation of degree 3 with $m_{(1:0:0)}(X) = 3$, $\mu_{(1:0:0)}(X) = 13$ and with 3-jet $\begin{pmatrix} z^3 \\ 0 \end{pmatrix}$, it is easy to see that $X \in Y_6^{ss}$. In this case the corresponding parabolic subgroup P_6 is the subgroup of upper triangular matrices, therefore $\dim S_6 = \dim Y_6^{ss} + \dim SL(3, \mathbb{C}) - \dim P_6 = 7$ (see Remark 1).

6.2. **Stratum 9.** If $X \in Y_9^{ss}$ then $a_{0,3} \neq 0$ and $(a_{1,0}, b_{0,1}) \neq (0, 0)$; therefore

$$\begin{aligned} f_9(y, z) = Q(1, y, z) - yP(1, y, z) &= (b_{0,1} - a_{1,0})yz^2 + b_{0,0}z^3 \\ &\quad - a_{0,3}y^4 - a_{0,2}y^3z - a_{0,1}y^2z^2 - a_{0,0}yz^3 \\ g_9(y, z) = -zP(1, y, z) &= -a_{1,0}z^3 - a_{0,3}y^3z - a_{0,2}y^2z^2 - a_{0,1}yz^3 - a_{0,0}z^4, \end{aligned}$$

and

$$\begin{aligned} \mu_p(X) &= I_0(f_9(y, z), g_9(y, z)) = I_0(f_9(y, z), z) + I_0(f_9(y, z), P(1, y, z)) \\ &= I_0(a_{0,3}y^4, z) + 2I_0(z, a_{0,3}y^3) + I_0(b_{0,1}y + b_{0,0}z, P(1, y, z)) \\ &= 10 + I_0(b_{0,1}y + b_{0,0}z, P(1, y, z)). \end{aligned}$$

Note that $I_0(b_{0,1}y + b_{0,0}z, P(1, y, z))$ is

$$\begin{cases} 2 & a_{1,0}\neq 0, b_{0,1}\neq 0 \\ 3 & (b_{0,1}=0, b_{0,0}\neq 0) \text{ or } (a_{1,0}=0 \text{ and } b_{0,1}y+b_{0,0}z \text{ is not tangent for } P(1,y,z)) \\ \infty & (b_{1,0}, b_{0,0})=0 \text{ or } (a_{1,0}=0 \text{ and } b_{0,1}y+b_{0,0}z \text{ is tangent for } P(1,y,z)) \end{cases}$$

If $a_{1,0}\neq 0$ it is clear that the multiplicity of the singular point is 3. If $a_{1,0}=0$ then $b_{0,1}\neq 0$ and also the multiplicity is 3. Finally, the 3-jet is $\begin{pmatrix}(b_{0,1}-a_{1,0})yz^2+b_{0,0}z^3\\ -a_{1,0}z^3\end{pmatrix}$. Then in the open set where $a_{1,0}\neq 0, b_{0,1}\neq 0$ every foliation has a singular point with multiplicity 3 and Milnor number 12. In this case the corresponding parabolic subgroup is the subgroup of upper triangular matrices, therefore $\dim S_9=9$.

6.3. **Stratum 11.** Remember that if $X\in Y_{11}^{ss}$ then $b_{1,0}$ and $(a_{1,1}, a_{0,3}, b_{0,2})$ are different from zero, and

$$\begin{aligned} f_{11}(y,z) &= Q(1,y,z)-yP(1,y,z)\\ &= b_{1,0}z^2+(b_{0,2}-a_{1,1})y^2z+(b_{0,1}-a_{1,0})yz^2+b_{0,0}z^3\\ &\quad -a_{0,3}y^4-a_{0,2}y^3z-a_{0,1}y^2z^2-a_{0,0}yz^3,\\ g_{11}(y,z) &= -zP(1,y,z)=-a_{1,1}yz^2-a_{1,0}z^3-a_{0,3}y^3z-a_{0,2}y^2z^2-a_{0,1}yz^3-a_{0,0}z^4. \end{aligned}$$

If $a_{0,3}=0$, then $z=0$ is a curve of singularities. Suppose $a_{0,3}\neq 0$. Note that

$$\begin{aligned} I_0(f_{11},g_{11}) &= I_0(-a_{0,3}y^4,z)+I_0(z,-a_{0,3}y^3)+\\ &\qquad I_0(b_{1,0}z+b_{0,2}y^2+b_{0,1}yz+b_{0,0}z^2, P(1,y,z))\\ &= 7+I_0(b_{1,0}z+b_{0,2}y^2+b_{0,1}yz+b_{0,0}z^2, P(1,y,z)). \end{aligned}$$

And

$$\begin{aligned} &I_0(b_{1,0}z+b_{0,2}y^2+b_{0,1}yz+b_{0,0}z^2, a_{1,1}yz+a_{1,0}z^2+a_{0,3}y^3+a_{0,2}y^2z+a_{0,1}yz^2+a_{0,0}z^3)\\ &= I_0\Bigg(b_{1,0}z+b_{0,2}y^2+b_{0,1}yz+b_{0,0}z^2, \left(a_{0,3}-\frac{a_{1,1}}{b_{1,0}}b_{0,2}\right)y^3\\ &\quad+\left(a_{0,2}-\frac{a_{1,0}}{b_{1,0}}b_{0,2}-\frac{a_{1,1}}{b_{1,0}}b_{0,1}\right)y^2z+\left(a_{0,1}-\frac{a_{1,0}}{b_{1,0}}b_{0,1}-\frac{a_{1,1}}{b_{1,0}}b_{0,0}\right)yz^2\\ &\quad+\left(a_{0,0}-\frac{a_{1,0}}{b_{1,0}}b_{0,0}\right)z^3\Bigg)\\ &= \begin{cases} 3 & \text{if } a_{0,3}b_{1,0}\neq a_{1,1}b_{0,2}\\ 4 & \text{if } a_{0,3}b_{1,0}=a_{1,1}b_{0,2} \text{ and } a_{0,2}b_{1,0}\neq a_{1,1}b_{0,1}+a_{1,0}b_{0,2}\\ 5 & \text{if } [\ldots] \text{ and } a_{0,1}b_{1,0}\neq a_{1,1}b_{0,0}+a_{1,0}b_{0,1}\\ 6 & \text{if } [\ldots], a_{0,1}b_{1,0}=a_{1,1}b_{0,0}+a_{1,0}b_{0,1} \text{ and } a_{0,0}b_{1,0}\neq a_{1,0}b_{0,0}\\ \infty & \text{if } [\ldots], a_{0,1}b_{1,0}=a_{1,1}b_{0,0}+a_{1,0}b_{0,1} \text{ and } a_{0,0}b_{1,0}=a_{1,0}b_{0,0} \end{cases} \end{aligned}$$

where $[\ldots]$ is $a_{0,3}b_{1,0}=a_{1,1}b_{0,2}, a_{0,2}b_{1,0}=a_{1,1}b_{0,1}+a_{1,0}b_{0,2}$.

We conclude that in the open set of Y_{11} where $a_{0,3}b_{1,0}\neq a_{1,1}b_{0,2}$ every foliation has a singularity with Milnor number 10. Since $b_{1,0}\neq 0$ then the multiplicity for the singular point $(1:0:0)$ is equal to 2 and the 2-jet is $\begin{pmatrix}z^2\\0\end{pmatrix}$. The corresponding parabolic subgroup is the subgroup of upper triangular matrices, therefore $\dim S_{11}=12$.

6.4. **Stratum 12.** If $X \in Y_{12}^{ss}$ then we have that

$$(a_{1,2}, a_{1,1}, b_{0,2}, b_{0,3}, c_{0,3}) \qquad \text{and} \qquad (a_{1,1}, a_{1,0}, b_{0,0}, b_{0,1}, b_{0,2})$$

are different from zero and

$$\begin{aligned} f_{12}(y,z) &= Q(1,y,z) - yP(1,y,z) \\ &= (b_{0,3} - a_{1,2})y^3 + (b_{0,2} - a_{1,1})y^2z + (b_{0,1} - a_{1,0})yz^2 + b_{0,0}z^3 \\ &\qquad - a_{0,3}y^4 - a_{0,2}y^3z - a_{0,1}y^2z^2 - a_{0,0}yz^3 \\ g_{12}(y,z) &= c_{0,3}y^3 - zP(1,y,z) \\ &= c_{0,3}y^3 - a_{1,2}y^2z - a_{1,1}yz^2 - a_{1,0}z^3 \\ &\qquad - a_{0,3}y^3z - a_{0,2}y^2z^2 - a_{0,1}yz^3 - a_{0,0}z^4. \end{aligned}$$

These polynomials are homogenous in two variables, then generically we have

$$I_p(f_{12}, g_{12}) = 9.$$

If $(a_{1,2}, a_{1,1}, c_{0,3}) \neq 0$ then $g_{12} \neq 0$ and $m_p(X) = 3$. If $(a_{1,2}, a_{1,1}, c_{0,3}) = 0$ then $(b_{0,3}, b_{0,2}) \neq 0$ and we have also $m_p(X) = 3$. In this case the parabolic subgroup is

$$P_{12} = \left\{ \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ 0 & \alpha_{22} & \alpha_{23} \\ 0 & \alpha_{32} & \alpha_{33} \end{pmatrix} \in SL(3,\mathbb{C}) \right\},$$

therefore $\dim S_{12} = 13$.
Moreover, the set

$$\{X \in \mathcal{F}_3 : \text{there exists } p \text{ such that } m_p(X) = 3,\ \mu_p(X) = 9\},$$

is an open set in S_{12} because a foliation with these properties for the point $(1:0:0)$ is unstable and it does not be in another stratum.

6.5. **Stratum 14.** If $X \in Y_{14}^{ss}$ then $b_{1,0}$ and $(a_{1,2}, b_{0,3})$ are different from zero, and

$$\begin{aligned} f_{14}(y,z) &= Q(1,y,z) - yP(1,y,z) \\ &= b_{1,0}z^2 + (b_{0,3} - a_{1,2})y^3 + (b_{0,2} - a_{1,1})y^2z + (b_{0,1} - a_{1,0})yz^2 + b_{0,0}z^3 \\ &\qquad - a_{0,3}y^4 - a_{0,2}y^3z - a_{0,1}y^2z^2 - a_{0,0}yz^3 \\ g_{14}(y,z) &= -zP(1,y,z) = -a_{1,2}y^2z - a_{1,1}yz^2 - a_{1,0}z^3 \\ &\qquad - a_{0,3}y^3z - a_{0,2}y^2z^2 - a_{0,1}yz^3 - a_{0,0}z^4. \end{aligned}$$

Note that

$$\begin{aligned} I_0(f_{14}, g_{14}) = I_0(&(b_{0,3} - a_{1,2})y^3 - a_{0,3}y^4, z) \\ &+ I_0(f_{14}, -a_{1,2}y^2 - a_{1,1}yz - a_{1,0}z^2 - a_{0,3}y^3 - a_{0,2}y^2z - a_{0,1}yz^2 - a_{0,0}z^3). \end{aligned}$$

If we suppose that $a_{1,2} \neq 0$ and $b_{0,3} \neq a_{1,2}$ then the Milnor number of $(1:0:0)$ is 7. If $a_{1,2} \neq 0, b_{0,3} = a_{1,2}$ and $a_{0,3} \neq 0$ we have $\mu_{(1:0:0)}(X) = 8$. On the other hand, $a_{1,2} \neq 0, b_{0,3} = a_{1,2}$

and $a_{0,3} = 0$ implies that we have a curve of singularities. Supposing $a_{1,2} = 0$, we obtain $b_{0,3} \neq a_{1,2}, 0$, and with this we have

$$\begin{aligned}
&I_0(f_{14}, a_{1,1}yz + a_{1,0}z^2 + a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3) \\
&\quad = I_0\Bigg(b_{1,0}z^2 + b_{0,3}y^3 + b_{0,2}y^2z + b_{0,1}yz^2 + b_{0,0}z^3, a_{1,1}yz + \left(a_{1,0} - \frac{a_{0,3}}{b_{0,3}}b_{1,0}\right)z^2 \\
&\qquad\quad + \left(a_{0,2} - \frac{a_{0,3}}{b_{0,3}}b_{0,2}\right)y^2z + \left(a_{0,1} - \frac{a_{0,3}}{b_{0,3}}b_{0,1}\right)yz^2 + \left(a_{0,0} - \frac{a_{0,3}}{b_{0,3}}b_{0,0}\right)z^3\Bigg) \\
&\quad = 3 + I_0\Bigg(b_{1,0}z^2 + b_{0,3}y^3 + b_{0,2}y^2z + b_{0,1}yz^2 + b_{0,0}z^3, a_{1,1}y + \left(a_{1,0} - \frac{a_{0,3}}{b_{0,3}}b_{1,0}\right)z \\
&\qquad\qquad + \left(a_{0,2} - \frac{a_{0,3}}{b_{0,3}}b_{0,2}\right)y^2 + \left(a_{0,1} - \frac{a_{0,3}}{b_{0,3}}b_{0,1}\right)yz + \left(a_{0,0} - \frac{a_{0,3}}{b_{0,3}}b_{0,0}\right)z^2\Bigg).
\end{aligned}$$

We can verify that if $a_{1,1} \neq 0$ then the last expression is equal to 5. When $a_{1,1} = 0$ we can see that

$$\begin{aligned}
&I_0(f_{14}, a_{1,1}yz + a_{1,0}z^2 + a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3) \\
&\quad = I_0\Big(b_{1,0}z^2 + b_{0,3}y^3 + b_{0,2}y^2z + b_{0,1}yz^2 + b_{0,0}z^3, a'_{0,3}y^3 + a'_{0,2}y^2z + a'_{0,1}yz^2 + a'_{0,0}z^3\Big),
\end{aligned}$$

where $a'_{i,j}$ denotes $a_{i,j} - \frac{a_{1,0}}{b_{1,0}}b_{i,j}$.

In conclusion,

$$\begin{aligned}
&I_0(b_{1,0}z^2 + b_{0,3}y^3 + b_{0,2}y^2z + b_{0,1}yz^2 + b_{0,0}z^3, a'_{0,3}y^3 + a'_{0,2}y^2 + a'_{0,1}yz + a'_{0,0}z^2) \\
&\qquad = \begin{cases} 6 & \text{if } a'_{0,3} \neq 0 \\ 7 & \text{if } a'_{0,3} = 0 \text{ and } a'_{0,2} \neq 0 \\ 8 & \text{if } (a'_{0,3}, a'_{0,2}) = 0 \text{ and } a'_{0,1} \neq 0 \\ 9 & \text{if } (a'_{0,3}, a'_{0,2}, a'_{0,1}) = 0 \text{ and } a'_{0,0} \neq 0 \\ \infty & \text{if } (a'_{0,3}, a'_{0,2}, a'_{0,1}, a'_{0,0}) = 0. \end{cases}
\end{aligned}$$

Therefore,

$$\mu_{(1:0:0)}(X) = \begin{cases} 7 & \text{if } b_{0,3} \neq a_{1,2} \neq 0 \\ 8 & \text{if } (b_{0,3} = a_{1,2} \neq 0 \text{ and } a_{0,3} \neq 0) \text{ or if } (a_{1,2} = 0 \text{ and } a_{1,1} \neq 0) \\ 9 & \text{if } (a_{1,2}, a_{1,1}) = 0 \text{ and } a'_{0,3} \neq 0 \\ 10 & \text{if } (a_{1,2}, a_{1,1}, a'_{0,3}) = 0 \text{ and } a'_{0,2} \neq 0 \\ 11 & \text{if } (a_{1,2}, a_{1,1}, a'_{0,3}, a'_{0,2}) = 0 \text{ and } a'_{0,1} \neq 0 \\ 12 & \text{if } (a_{1,2}, a_{1,1}, a'_{0,3}, a'_{0,2}, a'_{0,1}) = 0 \text{ and } a'_{0,0} \neq 0 \\ \infty & \text{if } (a_{1,2}, a_{1,1}, a'_{0,3}, a'_{0,2}, a'_{0,1}, a'_{0,0}) = 0 \end{cases}$$

Since $b_{1,0} \neq 0$ we get $m_{(1:0:0)}(X) = 2$ with 2-jet $\begin{pmatrix} z^2 \\ 0 \end{pmatrix}$. Since the parabolic subgroup is the subgroup of upper triangular matrices we obtain $\dim S_{14} = 14$. Moreover, the set

$$\{X \in \mathcal{F}_3 : \text{there exists } p \text{ such that } m_p(X) = 2,\ \mu_p(X) = 7 \text{ and 2-jet linearly equivalent to } \begin{pmatrix} z^2 \\ 0 \end{pmatrix}\},$$

is an open set in S_{14}, because a foliation with these properties for the point $(1:0:0)$ is unstable and it does not be in another stratum.

6.6. **Stratum 15.** If $X \in Y_{15}^{ss}$ we have that $a_{0,3}$ and $(a_{2,0}, b_{1,1})$ are different from zero, and

$$\begin{aligned}
f_{15}(y,z) &= Q(1,y,z) - yP(1,y,z)\\
&= (b_{1,1}-a_{2,0})yz + b_{1,0}z^2 + (b_{0,2}-a_{1,1})y^2z + (b_{0,1}-a_{1,0})yz^2 + b_{0,0}z^3\\
&\quad - a_{0,3}y^4 - a_{0,2}y^3z - a_{0,1}y^2z^2 - a_{0,0}yz^3\\
g_{15}(y,z) &= -zP(1,y,z)\\
&= -a_{2,0}z^2 - a_{1,1}yz^2 - a_{1,0}z^3 - a_{0,3}y^3z - a_{0,2}y^2z^2 - a_{0,1}yz^3 - a_{0,0}z^4.
\end{aligned}$$

Note that

$$\begin{aligned}
I_0(f_{15}, g_{15}) &= I_0(-a_{0,3}y^4, z)\\
&\quad + I_0(f_{15}, -a_{1,2}y - a_{1,1}z - a_{1,0}z^2 - a_{0,3}y^3 - a_{0,2}y^2z - a_{0,1}yz^2 - a_{0,0}z^3)\\
&= 4 + I_0(b_{1,1}yz + b_{1,0}z^2 + b_{0,2}y^2z + b_{0,1}yz^2 + b_{0,0}z^3,\\
&\qquad a_{2,0}z + a_{1,1}yz + a_{1,0}z^2 + a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3)\\
&= 4 + I_0(z, a_{0,3}y^3) + I_0(b_{1,1}y + b_{1,0}z + b_{0,2}y^2 + b_{0,1}yz + b_{0,0}z^2,\\
&\qquad a_{2,0}z + a_{1,1}yz + a_{1,0}z^2 + a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3).
\end{aligned}$$

It is clear that if $b_{1,1}$ and $a_{2,0}$ are different from zero then the intersection index of f_{15} and g_{15} is 8. Suppose that $a_{2,0} \neq 0$ and $b_{1,1} = 0$, then

$$\begin{aligned}
I_0(b_{1,0}z + b_{0,2}y^2 + b_{0,1}yz + b_{0,0}z^2, &\, a_{2,0}z + a_{1,1}yz + a_{1,0}z^2 +\\
a_{0,3}y^3 + a_{0,2}y^2z &+ a_{0,1}yz^2 + a_{0,0}z^3)\\
&= \begin{cases} 2 & \text{if } b_{0,2} \neq 0\\ 3 & \text{if } b_{0,2} = 0 \text{ and } b_{1,0} \neq 0\\ 4 & \text{if } (b_{0,2}, b_{1,0}) = 0 \text{ and } b_{0,1} \neq 0\\ 6 & \text{if } (b_{0,2}, b_{1,0}, b_{0,1}) = 0 \text{ and } b_{0,0} \neq 0\\ \infty & \text{if } (b_{0,2}, b_{1,0}, b_{0,1}, b_{0,0}) = 0 \end{cases}
\end{aligned}$$

Now suppose $a_{2,0} = 0$ and $b_{1,1} \neq 0$. We define

$$\begin{aligned}
&L(y,z) = b_{1,1}y + b_{1,0}z \quad M(y,z) = a_{1,1}yz + a_{1,0}z^2\\
&N(y,z) = b_{0,2}y^2 + b_{0,1}yz + b_{0,0}z^2 \quad F(y,z) = a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3.
\end{aligned}$$

We have

$$I_0(L+N, M+F) = 2 \qquad \text{if } L \nmid M.$$

Now suppose $L \mid M$, we have two cases: if $L \nmid N$,

$$\begin{aligned} I_0(L+N, M+F) &= I_0\left(L+N, F-\frac{M}{L}N\right) \\ &= \begin{cases} 3 & \text{if } L^2 \nmid (LF-MN) \\ 4 & \text{if } L^2 \mid (LF-MN) \text{ and } L^3 \nmid (LF-MN) \\ 5 & \text{if } L^3 \mid (LF-MN) \text{ and } L^4 \nmid (LF-MN) \\ 6 & \text{if } L^4 \mid (LF-MN) \end{cases}. \end{aligned}$$

On the other hand, if $L \mid N$:

$$\begin{aligned} I_0\left(L+N, F-\frac{M}{L}N\right) &= I_0\left(L, F-\frac{M}{L}N\right) + I_0\left(1+\frac{N}{L}, F-\frac{M}{L}N\right) \\ = I_0\left(L, F-\frac{M}{L}N\right) &= \begin{cases} 3 & \text{if } L \nmid (LF-MN) \\ \infty & \text{if } L \mid (LF-MN) \end{cases} \end{aligned}$$

We conclude that in S_{15} we have a nonempty open set which consists of foliations with a singularity with multiplicity 2 and Milnor number 8. But we can have in this stratum foliations with a singularity with multiplicity 2 and Milnor number 9, 10, 11, 12 and 13 or with a curve of singularities. The 2-jet around the singular point $(1:0:0)$ of X is $\begin{pmatrix} (b_{1,1}-a_{2,0})yz + b_{1,0}z^2 \\ -a_{2,0}z^2 \end{pmatrix}$. Since the parabolic subgroup is the subgroup of upper triangular matrices we obtain $\dim S_{15} = 14$.

6.7. **Stratum 16.** If $X \in Y_{16}^{ss}$ then $b_{1,0} \neq 0$ and $c_{0,3} \neq 0$. We have

$$\begin{aligned} f_{15}(y,z) &= Q(1,y,z) - yP(1,y,z) \\ &= b_{1,0}z^2 + (b_{0,3}-a_{1,2})y^3 + (b_{0,2}-a_{1,1})y^2z + (b_{0,1}-a_{1,0})yz^2 + b_{0,0}z^3 \\ &\quad - a_{0,3}y^4 - a_{0,2}y^3z - a_{0,1}y^2z^2 - a_{0,0}yz^3 \\ g_{15}(y,z) &= c_{0,3}y^3 - zP(1,y,z) \\ &= c_{0,3}y^3 - a_{1,2}y^2z - a_{1,1}yz^2 - a_{1,0}z^3 - a_{0,3}y^3z - a_{0,2}y^2z^2 - a_{0,1}yz^3 - a_{0,0}z^4, \end{aligned}$$

and the polynomials $c_{0,3}y^3 - a_{1,2}y^2z - a_{1,1}yz^2 - a_{1,0}z^3$ and $b_{1,0}z^2$ do not have common factors. As a result the Milnor number of $(1:0:0)$ is 6. And the multiplicity of this point is 2 with 2-jet $\begin{pmatrix} z^2 \\ 0 \end{pmatrix}$. The corresponding parabolic subgroup is the subgroup of upper triangular matrices, therefore $\dim S_{16} = 15$.
In the following theorem we summarize the above.

Theorem 7. *The spaces $S_i = SL(3,\mathbb{C})Y_i^{ss}$ for $i \in \{1,\dots,16\}$, are locally closed, irreducible non-singular algebraic subvarieties of $\mathcal{F}_3$. They form a stratification of the closed set of unstable foliations $\mathcal{F}_3^{un}$, and $\overline{S_i} \subset \bigcup_{j \leq i} S_j$. Moreover, these varieties satisfy the following:*

Stratum	*Characterization of the generic foliation*
S_1, S_2, S_3, S_4, S_5, S_7, S_8, S_{10}, S_{13}	Every foliation has a curve of singularities
S_6 $\dim S_6 = 7$	$\{X \in \mathcal{F}_3 : \exists p$ with $m_p(X) = 3$, $\mu_p(X) = 13$ and 3-jet linearly equivalent to $\begin{pmatrix} z^3 \\ 0 \end{pmatrix} \} = S_6$
S_9 $\dim S_9 = 9$	$\{X \in \mathcal{F}_3 : \exists p$ with $m_p(X) = 3$, $\mu_p(X) = 12\} \cap S_9$ is open in S_9
S_{11} $\dim S_{11} = 12$	$\{X \in \mathcal{F}_3 : \exists p$ with $m_p(X) = 2$, $\mu_p(X) = 10$ and 2-jet linearly equivalent to $\begin{pmatrix} z^2 \\ 0 \end{pmatrix}\} \cap S_{11}$ is open in S_{11}
S_{12} $\dim S_{12} = 13$	Contains $\{X \in \mathcal{F}_3 : \exists p$ with $m_p(X) = 3$, $\mu_p(X) = 9\}$ as an open set
S_{14} $\dim S_{14} = 14$	*Contains* $\{X \in \mathcal{F}_3 : \exists p$ with $m_p(X) = 2$, $\mu_p(X) = 7$ and 2-jet linearly equivalent to $\begin{pmatrix} z^2 \\ 0 \end{pmatrix} \}$ *as an open set*
S_{15} $\dim S_{15} = 14$	$\{X \in \mathcal{F}_3 : \exists p$ with $m_p(X) = 2$, $\mu_p(X) = 8\} \cap S_{15}$ *is open in* S_{15}
S_{16} $\dim S_{16} = 15$	$\{X \in \mathcal{F}_3 : \exists p$ with $m_p(X) = 2$, $\mu_p(X) = 6$ and 2-jet linearly equivalent to $\begin{pmatrix} z^2 \\ 0 \end{pmatrix}\} = S_{16}$

In [3] we also have studied the strata S_6, S_9 and S_{16}. As a consequence of the above we can mention the following general result.

Theorem 8. *Let $X \in \mathcal{F}_3$ with isolated singularities. Then X is unstable if and only if:*

(1) *X has a singular point with multiplicity* 3 *or*
(2) *X has a singular point with multiplicity* 2 *and 2-jet linearly equivalent to $z^2 \frac{\partial}{\partial y}$ or*
(3) *$X \in S_{15}$.*

Moreover, the irreducible components of $\mathcal{F}_3^{un}$ are the closure of the locally closed subvarieties S_{15} and S_{16}. The first one has dimension 14 *and the second one has dimension* 15.

Proof. The first affirmation is consequence of the results in the table. For the second one we use proposition 4.2 of [9], it says that S_j is irreducible and $\overline{S_j} = SL(3, \mathbb{C})\overline{Y_j}$ for all j. Since $\overline{Y_{16}} \subset \bigcup_{j \neq 15} Y_j$ and $S_{15} \not\subset S_{16}$, we have that $\mathcal{F}_3^{un} = \overline{S_{15}} \cup \overline{S_{16}}$ is the decomposition in irreducible components.

□

Remark 2. *Note that $V_3 = Y_{15}$ and $V_1 = Y_{16}$ in theorem 5.*

6.8. **Semistable non-stable foliations on $\mathbb{CP}^2$ of degree** 3**.** In this subsection we describe the semistable non-stable foliations on $\mathbb{CP}^2$ of degree 3.

Theorem 9. *The set of semistable non-stable foliations on $\mathbb{CP}^2$ of degree* 3 *with isolated singularities is*

$$SL(3, \mathbb{C})\mathbb{P}\left\{ \begin{pmatrix} P(y,z) \\ b_{1,1}xyz + b_{1,0}xz^2 + \sum_{j=0}^{3} b_{0,j} y^j z^{3-j} \\ c_{1,0}xz^2 + c_{0,2}y^2z + c_{0,1}yz^2 + b_{0,0}z^3 \end{pmatrix} : \right.$$
$$\left. P(y,z) \in \mathbb{C}_3[y,z], (b_{0,3}, c_{0,2}) \neq 0, (b_{1,1}, c_{1,0}) \neq 0 \right\}.$$

Proof. Let $X \in \mathcal{F}_3^{ss} - \mathcal{F}_3^{s}$ then X is not in any strata and it satisfies one of the following properties:

(1) $\dim O(X) < 8$: in this case, by Theorem 1.2 of [1], the foliation is λ-invariant for some 1-PS λ and it is not in any Z_j. Then the foliation is, up to change of coordinates, such that the line with its weights pass through zero, if we see the representation we conclude that X is:
$$\begin{pmatrix} 0 \\ b_{1,1}xyz + b_{0,3}y^3 \\ c_{1,0}xz^2 + c_{0,2}y^2z \end{pmatrix},$$
where $(b_{0,3}, c_{0,2}) \neq 0, (b_{1,1}, c_{1,0}) \neq 0$, since X has isolated singularities we can suppose $c_{1,0} = 1$.

(2) $O(X)$ is not closed in $\mathcal{F}_3^{ss}$: then there exists $Y \in \mathcal{F}_3^{ss} \cap (\overline{O(X)} - O(X))$. Since $\dim O(Y) < 8$, we conclude that Y is the above foliation, therefore X has its weights in one hyperplane given by the weights of Y, therefore X is, up to change of coordinates, in

$$\mathbb{P}\left\{ \begin{pmatrix} P(y,z) \\ b_{1,1}xyz + b_{1,0}xz^2 + \sum_{j=0}^{3} b_{0,j}y^j z^{3-j} \\ c_{1,0}xz^2 + c_{0,2}y^2z + c_{0,1}yz^2 + b_{0,0}z^3 \end{pmatrix} : P(y,z) \in \mathbb{C}_3[y,z], (b_{0,3}, c_{0,2}) \neq 0, (b_{1,1}, c_{1,0}) \neq 0 \right\}.$$

□

7. Corollaries

7.1. **The dimension of the orbits.** Generically the orbit of a foliation on $\mathbb{CP}^2$ has dimension 8, for example, a stable foliation satisfies this property. It theorem 1.2 of [1] we classify foliations with isolated singular points such that the dimension of the orbit is less than or equal to 7. We can see in proposition 2.3 of [5] that the dimension of an orbit of a foliation with isolated singularities of degree d is greater than or equal to 6. In the same paper the authors describe the two unique foliations of degree 2, up to change of coordinates, such that the orbit has dimension 6. For the case of foliations of degree 3 we have the same situation.

Theorem 10. *There are, up to change of coordinates, two foliations on $\mathbb{CP}^2$ of degree 3 with isolated singularities with automorphism group of dimension 2: $y^3\frac{\partial}{\partial x} + z^3\frac{\partial}{\partial y}$ and $y^3\frac{\partial}{\partial x} + z^3\frac{\partial}{\partial z}$.*

Proof. In proposition 2.5 of [5] we can see that if $Aut(X)$ has dimension 2 then it is isomorphic to the group of affine transformations of the line. Therefore by theorem 1.2 of [1], X is λ-invariant for some 1-PS λ, and X is also invariant by $(\mathbb{C}, +)$. This last affirmation implies, by the same theorem, that X is unstable with a singular point with Milnor number ≥ 12. An unstable foliation invariant by a 1-PS is, up to change of coordinates, in $Z_{15} \cup Z_{14} \cup Z_{12} \cup Z_{11} \cup Z_9 \cup Z_6$. It is easy to see that the unique foliations with a singular point with Milnor number ≥ 12 are in Z_9 and Z_6, they are $y^3\frac{\partial}{\partial x} + z^3\frac{\partial}{\partial y}$ and $y^3\frac{\partial}{\partial x} + z^3\frac{\partial}{\partial z}$.

□

7.2. **Foliations on $\mathbb{CP}^2$ of degree 3 with one singular point.** The classification of foliations on $\mathbb{CP}^2$ with one singular point is known only for degree 2 (see [5] and [2]). In this section we describe all the unstable foliations on $\mathbb{CP}^2$ of degree 3 with one singular point, that means with a singular point with Milnor number 13. To obtain the result we need the following lemma.

Lemma 3. *Let X be a foliation on $\mathbb{CP}^2$ of degree d. If X has a singular point p with multiplicity d and Milnor number greater than d^2, then X has an invariant line that passes through p.*

Proof. We can suppose that X is a foliation on $\mathbb{CP}^2$ of degree d such that $m_{(1:0:0)}(X) = d$ and $\mu_{(1:0:0)}(X) > d^2$. Then

$$X = \begin{pmatrix} xP_{d-1} + P_d \\ Q_d \\ R_d \end{pmatrix}$$

where $P_k, Q_k, R_k \in \mathbb{C}[y, z]$ are homogeneous of degree k. In the chart U_0, the foliation is

$$(Q_d - yP_{d-1})\frac{\partial}{\partial y} + (R_d - zP_{d-1})\frac{\partial}{\partial z},$$

since $\mu_{(1:0:0)}(X) > d^2$ then there exists a line $L = \alpha y - \beta z$ such that $Q_d - yP_{d-1} = LF$ and $R_d - zP_{d-1} = LG$ for some $F, G \in \mathbb{C}[y, z]$. Therefore $\alpha Q_d - \beta R_d = L(P_{d-1} + \alpha F - \beta G)$, and this means that L is invariant for X and it passes through $(1 : 0 : 0)$. □

Theorem 11. *The unstable foliations on $\mathbb{CP}^2$ of degree 3 with one singular point are:*

(1) *The stratum S_6, which has dimension 7.*

(2) *The subspace of S_9:*

$$SL(3,\mathbb{C})\left\{ \begin{pmatrix} a_{1,0}xz^2 + y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3 \\ b_{0,1}yz^2 + b_{0,0}z^3 \\ 0 \end{pmatrix} : b_{0,1} = 0, a_{1,0}b_{0,0} \neq 0 \textit{ or} \right.$$

$$\left. a_{1,0} = 0, b_{01} \neq 0 \textit{ and } b_{0,1}y + b_{0,0}z \nmid y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3 \right\},$$

of dimension 8.

(3) *The subspace of S_{11}:*

$$SL(3,\mathbb{C})\left\{ \begin{pmatrix} a_{1,1}xyz + a_{1,0}xz^2 + a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3 \\ xz^2 + b_{0,2}y^2z + b_{0,1}yz^2 + b_{0,0}z^3 \\ 0 \end{pmatrix} : (a_{1,1}, a_{0,3}, b_{0,2}) \neq 0, \right.$$

$$\left. a_{0,0} \neq a_{1,0}b_{0,0}; a_{1,0} = a_{1,1}b_{0,0} + a_{1,0}b_{0,1}; a_{0,2} = a_{1,1}b_{0,1} + a_{1,0}b_{0,2}; a_{0,3} = a_{1,1}b_{0,2} \right\},$$

of dimension 9.

(4) *The subspace of S_{12}:*

$$SL(3,\mathbb{C})\left\{ \begin{pmatrix} x(\alpha_1 y - \beta_1 z)(\alpha_2 y - \beta_2 z) + y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3 \\ \alpha(\alpha_1 y - \beta_1 z)^2(\alpha_2 y - \beta_2 z) \\ 0 \end{pmatrix} : \right.$$

$$\left. \textit{It has isolated singularities, } (\alpha_1, \alpha_2) \neq 0 \textit{ and } \alpha_1\alpha_2 = 0\textit{; or } \alpha\alpha_1 = 1 \textit{ and } \beta_1, \beta_2 \in \mathbb{C}^* \right\},$$

of dimension 10.

(5) *The subspace of S_{15}:*

$$SL(3,\mathbb{C})\left\{ \begin{pmatrix} x^2z + a_{1,1}xyz + a_{1,0}xz^2 + a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3 \\ b_{0,0}z^3 \\ 0 \end{pmatrix} : a_{0,3} \neq 0 \right\},$$

of dimension 10.

In S_6, S_9 and S_{12} the singular point has multiplicity 3 *and in S_{11}, S_{15} has multiplicity* 2.

Proof. It remains only to find the foliations with one singular point in S_{12}, the other cases were studied in the construction of the strata.

To obtain the dimension of these spaces, we must observe that if X is a foliation in any of the described linear subspaces of Y_9^{ss}, Y_{11}^{ss}, or Y_{15}^{ss} then for $g \in SL(3,\mathbb{C})$ we have that gX is in the same linear subspace if and only if g is in the corresponding parabolic subgroup. Therefore the dimension of the space is the dimension of the linear subspace plus 3.

Now let $X \in Y_{12}^{ss}$ such that $(1:0:0)$ is the unique singularity. By the above lemma, X has an invariant line $\alpha y - \beta z$. There exists $g \in P_{12}$ such that z is invariant for $gX \in Y_{12}^{ss}$. For that we can suppose:

$$X = \begin{pmatrix} xL_1(y,z)L_2(y,z) + a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3 \\ L_3(y,z)L_4(y,z)L_5(y,z) \\ 0 \end{pmatrix}$$

where $L_k(y,z) = \alpha_k y - \beta_k z$ and $\alpha_k, \beta_k, a_{0,j}, \in \mathbb{C}$ for $k = 1,...,5$ and $j = 0,1,2,3$. The Milnor number of $(1:0:0)$ is

$$\begin{aligned} &I\big(L_3L_4L_5 - y(L_1L_2 + a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3), \\ &\qquad\qquad z(L_1L_2 + a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3)\big) \\ &= I(z, L_3L_4L_5 - y(L_1L_2 + a_{0,3}y^3)) + I(L_3L_4L_5, L_1L_2 + a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3), \end{aligned}$$

and this is 13 if and only if

$$\begin{aligned} &I(z, L_3L_4L_5 - y(L_1L_2 + a_{0,3}y^3)) = 4, \\ &I(L_3L_4L_5, L_1L_2 + a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3) = 9, \end{aligned}$$

and this happens if and only if $a_{0,3} \neq 0$, $z|(L_3L_4L_5 - yL_1L_2)$, $L_3L_4L_5 = \alpha L_1^2L_2$ for some $\alpha \neq 0$, and $L_1^2L_2 \nmid (a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3)$. Since $z|L_1L_2(\alpha L_1 - y)$ then we have the following cases: $\alpha_1 = 0$, $\alpha_2 = 0$ or $\alpha\alpha_1 = 1$. The condition for X to be in Y_{12}^{ss} says that if $\alpha_1 = 0$ then $\alpha_2 \neq 0$ and if $\alpha_2 = 0$ then $\alpha_1 \neq 0$. If $\alpha\alpha_1 = 1$ then $\beta_1, \beta_2 \in \mathbb{C}^*$. We get

$$X = \begin{pmatrix} x(\alpha_1 y - \beta_1 z)(\alpha_2 y - \beta_2 z) + y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3 \\ \alpha(\alpha_1 y - \beta_1 z)^2(\alpha_2 y - \beta_2 z) \\ 0 \end{pmatrix}.$$

Then the dimension of the projectivization of the linear space where X lives is 7. When we move the invariant line through $(1:0:0)$ we obtain a family of foliations of dimension 8 and when we take the action by $SL(3,\mathbb{C})$ module the parabolic subgroup P_{12} we obtain a space of dimension 10. □

To finish the classification of foliations on $\mathbb{CP}^2$ of degree 3 with one singularity with Milnor number 13 remains to find the semistable foliations with this property. For foliations on $\mathbb{CP}^2$ of degree 2 we know that there exists only one semistable foliation, up to change of coordinates (see Theorem 5.9 of [2]), with only one singular point. In this case the singularity is a saddle-node, that means multiplicity 1 non-nilpotent. For degree 3 the situation is different, for example, we have the following foliations:

$$X_1 = z^3\frac{\partial}{\partial x} + (x^2z + xy^2)\frac{\partial}{\partial y} - (xyz + y^3)\frac{\partial}{\partial z}$$

$$X_2 = y^2z\frac{\partial}{\partial x} + (xyz + z^3)\frac{\partial}{\partial y} - y^3\frac{\partial}{\partial z}.$$

Both are semistable foliations of degree 3 with only one singularity, the first one has a nilpotent singularity and in second one the singularity has multiplicity 2. In general is very difficult to find foliations on $\mathbb{CP}^2$ of degree d with one singular point. It is clear that using this stratification we can get all the unstable foliations. We think that using recursively this construction it is possible to find also the semi-stable foliations of degree d with a singularity with Milnor number d^2+d+1.

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References

[1] Alcántara, C. R. *Singular points and automorphisms of unstable foliations of CP2* Bol. Soc. Mat. Mexicana (3) 16 (2010), no. 1, 39-51.

[2] Alcántara, C. R. *Foliations on $\mathbb{CP}^2$ of degree 2 with degenerate singularities.* Bull. Braz. Math. Soc. (N.S.) 44 (2013), no. 3, 421-454.

[3] Alcántara, C. R. *Stratification of the space of foliations on CP2.* J. Symbolic Comput. 72 (2016), 147-160. DOI: 10.1016/j.jsc.2015.02.003

[4] Brunella, M. *Birational Geometry of Foliations.* IMPA Monographs, 1. Springer, Cham, 2015.

[5] Cerveau, D.; Déserti, J.; Garba Belko, D.; Meziani, R. *Géométrie classique de certains feuilletages de degré deux.* Bull Braz Math Soc, New Series 41(2), (2010), 161-198.

[6] Fogarty, J.; Kirwan, F.; Mumford, D. *Geometric Invariant Theory.* Springer-Verlag, 1994.

[7] Gómez-Mont, X.; Kempf, G. *Stability of Meromorphic Vector Fields in Projective Spaces.* Comment. Math. Helvetici 64 (1989) 462-473. DOI: 10.1007/BF02564687

[8] Gómez-Mont, X.; Ortiz-Bobadilla, L. *Sistemas dinámicos holomorfos en superficies.* Aportaciones Matemáticas: Notas de Investigación [Mathematical Contributions: Research Notes], 3. Sociedad Matemática Mexicana, México, 1989.

[9] Hesselink, W. H. *Desingularizations of varieties of nullforms.* Invent. Math. 55 (1979), no. 2, 141-163.

[10] Jouanolou, J.P. *Equations de Pfaff Algébriques.* Springer-Verlag, 1979.

[11] Kempf, G. *Instability in Invariant Theory.* Annals of Mathematics 108 (1978) 299-316. DOI: 10.2307/1971168

[12] Kirwan, F. *Cohomology of Quotients in Symplectic and Algebraic Geometry.* Mathematical Notes 31 (1984), 299-316.

[13] Lins Neto, A.; Soares, M.G. *Algebraic Solutions of one-dimensional foliations.* J. Differ. Geom. 43 (1996), 652-673.

[14] Newstead, P. *Introduction to Moduli Problems and Orbit Spaces.* Springer-Verlag, 1978.

[15] Popov, V. *The Cone of Hilbert Nullforms.* arχiv: 1009.6107v1

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VANISHING RESULTS FOR THE AOMOTO COMPLEX OF REAL HYPERPLANE ARRANGEMENTS VIA MINIMALITY

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Abstract. We prove vanishing results for the cohomology groups of the Aomoto complex over an arbitrary coefficient ring for real hyperplane arrangements. The proof uses the minimality of arrangements and descriptions of the Aomoto complex in terms of chambers.

Our methods are used to present a new proof for the vanishing theorem of local system cohomology groups, a result first proved by Cohen, Dimca, and Orlik.

1. Introduction

The theory of hypergeometric integrals originated with Gauss, and has been generalized to higher dimensions for applications in various areas of mathematics and physics ([1, 9, 17]). In this generalization, the notion of local system cohomology groups of the complement of a hyperplane arrangement plays a crucial role.

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of affine hyperplanes in $\mathbb{C}^\ell$ and let

$$M(\mathcal{A}) = \mathbb{C}^\ell \smallsetminus \bigcup_{H \in \mathcal{A}} H$$

be its complement. Let us fix a defining equation α_i of H_i. An arrangement $\mathcal{A}$ is called essential if the normal vectors of hyperplanes generate $\mathbb{C}^\ell$. The first homology group $H_1(M(\mathcal{A}), \mathbb{Z})$ is a free abelian group generated by the meridians $\gamma_1, \dots, \gamma_n$ of hyperplanes. We denote their dual basis by $e_1, \dots, e_n \in H^1(M(\mathcal{A}), \mathbb{Z})$. The element e_i can be identified with $\frac{1}{2\pi\sqrt{-1}} d\log \alpha_i$ via the de Rham isomorphism.

The isomorphism class of a rank one complex local system $\mathcal{L}$ is determined by a homomorphism $\rho : H_1(M(\mathcal{A}), \mathbb{Z}) \longrightarrow \mathbb{C}^\times$, which is also determined by an n-tuple $q = (q_1, \dots, q_n) \in (\mathbb{C}^\times)^n$, where $q_i = \rho(\gamma_i)$.

For a generic parameter $(q_1, \dots, q_n)$, it is known that the following vanishing result holds.

$$\dim H^k(M(\mathcal{A}), \mathcal{L}) = \begin{cases} 0, & \text{if } k \neq \ell, \\ |\chi(M(\mathcal{A}))|, & \text{if } k = \ell. \end{cases} \tag{1}$$

Several sufficient conditions for the vanishing of (1) are known ([1, 8]). Cohen, Dimca and Orlik ([4]) proved the following.

Theorem 1.1. *(CDO-type vanishing theorem) Suppose that $q_X \neq 1$ for each dense edge X contained in the hyperplane at infinity. Then, the vanishing (1) holds. (See §2.1 for description of the notation.)*

The above result is stronger than many other vanishing results. Indeed, for the case $\ell = 2$, it has been proved ([19]) that the vanishing (1) with an additional property holds if and only if the assumption of Theorem 1.1 holds.

The local system cohomology group $H^k(M(\mathcal{A}), \mathcal{L})$ is computed using the twisted de Rham complex $(\Omega^\bullet_{M(\mathcal{A})}, d + \omega\wedge)$ with $\omega = \sum \lambda_i d\log \alpha_i$, where λ is a complex number such that

$\exp(-2\pi\sqrt{-1}\lambda_i) = q_i$ (we denote $\mathcal{L} = \exp(\omega)$). The algebra of rational differential forms $\Omega^\bullet_{M(\mathcal{A})}$ has a natural $\mathbb{C}$-subalgebra $A^\bullet_{\mathbb{C}}(\mathcal{A})$ generated by $e_i = \frac{1}{2\pi\sqrt{-1}} d\log\alpha_i$. This subalgebra is known to be isomorphic to the cohomology ring $H^\bullet(M(\mathcal{A}), \mathbb{C})$ of $M(\mathcal{A})$ ([3]) and to have a combinatorial description, the so-called Orlik-Solomon algebra [11] (see §2.1 for details). The Orlik-Solomon algebra provides a subcomplex $(A^\bullet_{\mathbb{C}}(\mathcal{A}), \omega\wedge)$ of the twisted de Rham complex, which is called the Aomoto complex. There exists a natural morphism

$$(A^\bullet_{\mathbb{C}}(\mathcal{A}), \omega\wedge) \hookrightarrow (\Omega^\bullet_{M(\mathcal{A})}, d + \omega\wedge) \tag{2}$$

of complexes. The Aomoto complex $(A^\bullet_{\mathbb{C}}(\mathcal{A}), \omega\wedge)$ has a purely combinatorial description. Furthermore, it can be considered as a linearization of the twisted de Rham complex $(\Omega^\bullet_{M(\mathcal{A})}, d+\omega\wedge)$. Indeed, there exists a Zariski open subset $U \subset (\mathbb{C}^\times)^n$ that contains $(1, 1, \dots, 1) \in (\mathbb{C}^\times)^n$ such that (2) is a quasi-isomorphism for $q \in U$ ([7, 16, 10]). However, this is not an isomorphism in general.

The following vanishing result for the cohomology of the Aomoto complex has been obtained by Yuzvinsky.

Theorem 1.2. *([21, 22]) Let $\omega = \sum_{i=1}^n 2\pi\sqrt{-1}\lambda_i e_i \in A^1_{\mathbb{C}}(\mathcal{A})$. Suppose $\lambda_X \neq 0$ for all dense edges X in $L(\mathcal{A})$. Then, we have*

$$\dim H^k(A^\bullet_{\mathbb{C}}(\mathcal{A}), \omega\wedge) = \begin{cases} 0, & \text{if } k \neq \ell, \\ |\chi(M(\mathcal{A}))|, & \text{if } k = \ell. \end{cases} \tag{3}$$

Note that the assumptions in Theorem 1.1 and Theorem 1.2 are somewhat complementary: the first one requires nonresonant conditions along the hyperplane at infinity, whereas the second imposes nonresonant conditions on all dense edges in the affine space.

Recently, Papadima and Suciu proved that the dimension of the local system cohomology group for a torsion local system is bounded by that of the Aomoto complex with finite field coefficients.

Theorem 1.3. *([14]) Let $p \in \mathbb{Z}$ be a prime. Suppose $\omega = \sum_{i=1}^n \lambda_i e_i \in A^1_{\mathbb{F}_p}(\mathcal{A})$ and $\mathcal{L}$ is the local system determined by $q_i = \exp(\frac{2\pi\sqrt{-1}}{p}\lambda_i)$. Then,*

$$\dim_{\mathbb{C}} H^k(M(\mathcal{A}), \mathcal{L}) \leq \dim_{\mathbb{F}_p} H^k(A^\bullet_{\mathbb{F}_p}(\mathcal{A}), \omega\wedge), \tag{4}$$

for all $k \geq 0$.

In view of the Papadima-Suciu inequality (4), it is natural to expect that a CDO-type vanishing theorem for a p-torsion local system may be deduced from that of the Aomoto complex with finite field coefficients. Furthermore, Papadima and Suciu ([15]) clarified the relationship between multinet structures and $H^1(A^\bullet_{\mathbb{F}_p}(\mathcal{A}), \omega\wedge)$. These results motivate the study of the Aomoto complex with coefficients in an arbitrary commutative ring R. The main result of this paper is the following CDO-type vanishing theorem.

Theorem 1.4. *(Theorem 3.1) Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an essential affine hyperplane arrangement in $\mathbb{R}^\ell$. Let R be a commutative ring with multiplicative unit 1. Let $\omega = \sum_{i=1}^n \lambda_i e_i \in A^1_R(\mathcal{A})$. Suppose that $\lambda_X \in R^\times$ for any dense edge X contained in the hyperplane at infinity. Then, the following holds.*

$$H^k(A^\bullet_R(\mathcal{A}), \omega\wedge) \simeq \begin{cases} 0, & \text{if } k \neq \ell, \\ R^{|\chi(M(\mathcal{A}))|}, & \text{if } k = \ell. \end{cases} \tag{5}$$

Our proof relies on several results ([18, 19, 20]) concerning the minimality of arrangements. We also provide an alternative proof of Theorem 1.1 for real arrangements.

Remark 1.5. If $R = \mathbb{C}$, one can deduce Theorem 1.4 from Theorem 1.1. We present a sketch of the argument. If $\omega = \sum_{i=1}^{n} \lambda_i e_i \in A^1_R(\mathcal{A})$ satisfies the assumption of Theorem 1.4, then so does $t\omega$ for $t \in \mathbb{C}^\times$. Clearly, we have $H^k(A^\bullet_{\mathbb{C}}(\mathcal{A}), \omega\wedge) \simeq H^k(A^\bullet_{\mathbb{C}}(\mathcal{A}), t\omega\wedge)$. However, the tangent-cone theorem ([6, 7]) gives, for $0 < |t| \ll 1$, an isomorphism $H^k(A^\bullet_{\mathbb{C}}(\mathcal{A}), t\omega\wedge) \simeq H^k(M(\mathcal{A}), \mathcal{L}_t)$, where $\mathcal{L}_t = \exp(t\omega)$. Then, Theorem 1.1 gives $H^k(M(\mathcal{A}), \mathcal{L}_t) = 0$.

The remainder of this paper is organized as follows.

In §2, we give some basic terminology and a description of the Aomoto complex in terms of chambers developed in [18, 19, 20]. We also recall the description of a twisted minimal complex in terms of chambers. Two cochain complexes $(R[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_{\omega_\lambda})$ and $(\mathbb{C}[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_{\mathcal{L}})$ are constructed using the real structures of $\mathcal{A}$ (adjacency relations of chambers). These cochain complexes provide a parallel description of the cohomology of the Aomoto complex and the local system cohomology group. Indeed, using these complexes, we can simultaneously prove CDO-type vanishing results for both cases.

In §3, we state the main result and describe the strategy for the proof. The proof consists of an easy part and a hard part. The easy part of the proof mainly uses elementary arguments relating to cochain complexes, which are also stated in this section. The hard part is tackled in §4.

§4 is devoted to an analysis of the polyhedral structures of chambers that are required for matrix presentations of the coboundary map of $(R[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_{\omega_\lambda})$.

2. Notation and Preliminaries

2.1. **Orlik-Solomon algebra and Aomoto complex.** Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an affine hyperplane arrangement in $V = \mathbb{R}^\ell$. Denote the complement of the complexified hyperplanes by $M(\mathcal{A}) = \mathbb{C}^\ell \smallsetminus \cup_{i=1}^{n} H_i \otimes \mathbb{C}$. By identifying $\mathbb{R}^\ell$ with $\mathbb{P}^\ell_{\mathbb{R}} \smallsetminus \overline{H}_\infty$, define the projective closure by $\overline{\mathcal{A}} = \{\overline{H}_1, \dots, \overline{H}_n, \overline{H}_\infty\}$, where $\overline{H}_i \subset \mathbb{P}^\ell_{\mathbb{R}}$ is the closure of H_i in the projective space. We denote the intersection posets of $\mathcal{A}$ and $\overline{\mathcal{A}}$ as $L(\mathcal{A})$ and $L(\overline{\mathcal{A}})$, respectively; these are the posets of subspaces obtained as intersections of some hyperplanes with reverse inclusion order. An element of $L(\mathcal{A})$ (and $L(\overline{\mathcal{A}})$) is also called an edge. We denote the set of all k-dimensional edges by $L_k(\mathcal{A})$. For example, $L_\ell(\mathcal{A}) = \{V\}$ and $L_{\ell-1}(\mathcal{A}) = \mathcal{A}$. Then, $\mathcal{A}$ is essential if and only if $L_0(\mathcal{A}) \neq \emptyset$.

Let R be a commutative ring. Orlik and Solomon gave a simple combinatorial description of the algebra $H^*(M(\mathcal{A}), R)$, which is the quotient of the exterior algebra on classes dual to the meridians, modulo a certain ideal determined by $L(\mathcal{A})$ (see [11]). More precisely, by associating a generator $e_i \simeq \frac{1}{2\pi\sqrt{-1}} d\log\alpha_i$ to any hyperplane H_i, the Orlik-Solomon algebra $A^\bullet_R(\mathcal{A})$ of $\mathcal{A}$ is the quotient of the exterior algebra generated by the elements e_i, $1 \leq i \leq n$, modulo the ideal $I(\mathcal{A})$ generated by:

- elements of the form $\{e_{i_1} \wedge \cdots \wedge e_{i_s} \mid H_{i_1} \cap \cdots \cap H_{i_s} = \emptyset\}$,
- elements of the form $\{\partial(e_{i_1} \wedge \cdots \wedge e_{i_s}) \mid H_{i_1} \cap \cdots \cap H_{i_s} \neq \emptyset$ and $\mathrm{codim}(H_{i_1} \cap \cdots \cap H_{i_s}) < s\}$, where $\partial(e_{i_1} \wedge \cdots \wedge e_{i_s}) = \sum_{\alpha=1}^{s} (-1)^{\alpha-1} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_\alpha}} \wedge \cdots \wedge e_{i_s}$.

Let $\lambda = (\lambda_1, \dots, \lambda_n) \in R^n$ and $\omega_\lambda = \sum_{i=1}^{n} \lambda_i e_i \in A^1_R(\mathcal{A})$. The cochain complex

$$(A^\bullet_R(\mathcal{A}), \omega_\lambda\wedge) = \{A^\bullet_R(\mathcal{A}) \xrightarrow{\omega_\lambda\wedge} A^{\bullet+1}_R(\mathcal{A})\}$$

is called *the Aomoto complex*.

We say that an edge $X \in L(\overline{\mathcal{A}})$ is *dense* if the localization $\overline{\mathcal{A}}_X = \{\overline{H} \in \overline{\mathcal{A}} \mid X \subseteq \overline{H}\}$ is indecomposable (see [13] for more details). Each hyperplane $\overline{H} \in \overline{\mathcal{A}}$ is considered to be a dense edge. In this paper, the set of dense edges of $\overline{\mathcal{A}}$ contained in $\overline{H}_\infty$ plays an important role. We denote this set by $\mathsf{D}_\infty(\overline{\mathcal{A}})$. We will characterize $X \in \mathsf{D}_\infty(\overline{\mathcal{A}})$ in terms of chambers in Proposition 2.6.

Set $\lambda_\infty := -\sum_{i=1}^n \lambda_i$. For any $X \in L(\overline{\mathcal{A}})$, $\lambda_X := \sum_{\overline{H}_i \supset X} \lambda_i$, where the index i-runs through the set $\{1, 2, \dots, n, \infty\}$.

The isomorphism class of a rank one local system $\mathcal{L}$ on the complexified complement $M(\mathcal{A})$ is determined by the monodromy $q_i \in \mathbb{C}^\times$ around each hyperplane H_i. As in the case of the Aomoto complex, we denote $q_\infty = (q_1 q_2 \cdots q_n)^{-1}$ and $q_X = \prod_{\overline{H}_i \supset X} q_i$ for an edge $X \in L(\overline{\mathcal{A}})$.

2.2. **Chambers and minimal complexes.** In this section, we recall the description of the minimal complex in terms of real structures from [18, 19, 20]. Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an essential hyperplane arrangement in $\mathbb{R}^\ell$. A connected component of $\mathbb{R}^\ell \smallsetminus \bigcup_{i=1}^n H_i$ is called a chamber. The set of all chambers of $\mathcal{A}$ is denoted by $\mathrm{ch}(\mathcal{A})$. A chamber $C \in \mathrm{ch}(\mathcal{A})$ is called a bounded chamber if C is bounded. The set of all bounded chambers of $\mathcal{A}$ is denoted by $\mathrm{bch}(\mathcal{A})$. For a chamber $C \in \mathrm{ch}(\mathcal{A})$, denote the closure of C in $\mathbb{P}^\ell_\mathbb{R}$ by $\overline{C}$. It is easily seen that a chamber C is bounded if and only if $\overline{C} \cap \overline{H}_\infty = \emptyset$.

Given two chambers $C, C' \in \mathrm{ch}(\mathcal{A})$, we denote the set of separating hyperplanes of C and C' by

$$\mathrm{Sep}(C, C') := \{H_i \in \mathcal{A} \mid \ H_i \text{ separates } C \text{ and } C'\}.$$

To describe the minimal complex, we must fix a generic flag. Let

$$\mathcal{F} : \emptyset = F^{-1} \subset F^0 \subset F^1 \subset \cdots \subset F^\ell = \mathbb{R}^\ell$$

be a generic flag (i.e., F^k is a generic k-dimensional affine subspace, that is,

$$\dim(\overline{X} \cap \overline{F}^k) = \dim \overline{X} + k - \ell$$

for any $\overline{X} \in L(\overline{\mathcal{A}})$). The genericity of $\mathcal{F}$ is equivalent to

$$F^k \cap L_i(\mathcal{A}) = L_{k+i-\ell}(\mathcal{A} \cap F^k)$$

for $k + i \geq \ell$.

Definition 2.1. We say that the hyperplane $F^{\ell-1}$ is near to $\overline{H}_\infty$ when $F^{\ell-1}$ does not separate 0-dimensional edges $L_0(\mathcal{A}) \subset \mathbb{R}^\ell$. Similarly, we say the flag $\mathcal{F}$ is near to $\overline{H}_\infty$ when F^{k-1} does not separate $L_0(\mathcal{A} \cap F^k)$ for all $k = 1, \dots, \ell$.

From this point, we assume that the flag $\mathcal{F}$ is near to $\overline{H}_\infty$. For a generic flag $\mathcal{F}$ near to $\overline{H}_\infty$, we define

$$\begin{aligned}
\mathrm{ch}^k(\mathcal{A}) &= \{C \in \mathrm{ch}(\mathcal{A}) \mid C \cap F^k \neq \emptyset, C \cap F^{k-1} = \emptyset\} \\
\mathrm{bch}^k(\mathcal{A}) &= \{C \in \mathrm{ch}^k(\mathcal{A}) \mid C \cap F^k \text{ is bounded}\} \\
\mathrm{uch}^k(\mathcal{A}) &= \{C \in \mathrm{ch}^k(\mathcal{A}) \mid C \cap F^k \text{ is unbounded}\}.
\end{aligned}$$

It is then clear that

$$\mathrm{ch}^k(\mathcal{A}) = \mathrm{bch}^k(\mathcal{A}) \sqcup \mathrm{uch}^k(\mathcal{A})$$

$$\mathrm{ch}(\mathcal{A}) = \bigsqcup_{k=0}^{\ell} \mathrm{ch}^k(\mathcal{A}).$$

Note that $\mathrm{bch}^\ell(\mathcal{A}) = \mathrm{bch}(\mathcal{A})$. However, for $k < \ell$, $C \in \mathrm{bch}^k(\mathcal{A})$ is an unbounded chamber.

Definition 2.2. ([19, Definition 2.1]) Let $C \in \mathrm{bch}(\mathcal{A})$. There exists a unique chamber, denoted by $C^\vee \in \mathrm{uch}(\mathcal{A})$, which is the opposite with respect to $\overline{C} \cap \overline{H}_\infty$, where $\overline{C}$ is the closure of C in the projective space $\mathbb{P}^\ell_\mathbb{R}$.

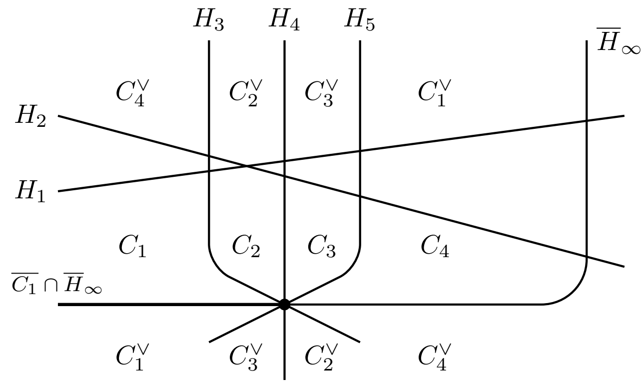

FIGURE 1. Opposite chambers

Let us denote the projective subspace generated by $\overline{C} \cap \overline{H}_\infty$ as $X(C) = \langle \overline{C} \cap \overline{H}_\infty \rangle$.

Proposition 2.3. *Let* $C \in \mathrm{bch}(\mathcal{A})$. *Then*

$$\mathrm{Sep}(C, C^\vee) = \{H \in \mathcal{A} \mid \overline{H} \not\supset X(C)\} = \overline{\mathcal{A}} \smallsetminus \overline{\mathcal{A}}_{X(C)}. \tag{6}$$

Proof. Let $p \in C$ and p' be a point in the relative interior of $\overline{C} \cap \overline{H}_\infty$. Take the line

$$L = \langle p, p' \rangle \subset \mathbb{P}^\ell_{\mathbb{R}},$$

and choose a point $p'' \in C^\vee \cap L$. Then, consider the segment $[p, p''] \subset \mathbb{R}^\ell = \mathbb{P}^\ell_{\mathbb{R}} \smallsetminus \overline{H}_\infty$ (see Figure 2). On the projective space $\mathbb{P}^\ell_{\mathbb{R}}$, the line $L = \langle p, p' \rangle$ must intersect every hyperplane $\overline{H} \in \overline{\mathcal{A}}$ exactly once. Furthermore, L intersects $\overline{H} \in \overline{\mathcal{A}}_{X(C)}$ at p'. Additionally, the segment $[p, p'']$ intersects $H \in \mathrm{Sep}(C, C^\vee)$. Hence, we have (6). $\square$

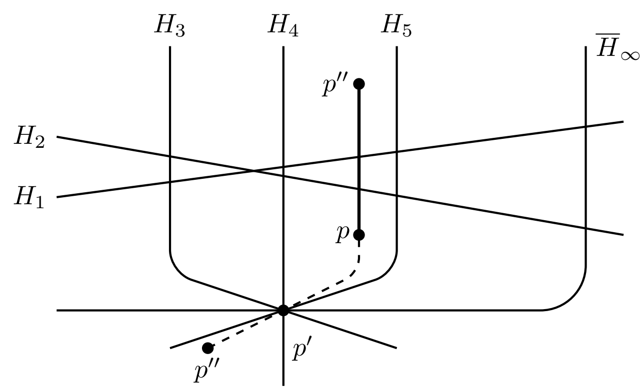

FIGURE 2. The segment $[p, p'']$ (thick segment).

Corollary 2.4. *If* $\dim X(C) = \ell - 1$, *then* $\mathrm{Sep}(C, C^\vee) = \mathcal{A}$.

Proof. In this case, $\overline{\mathcal{A}}_{X(C)} = \{\overline{H}_\infty\}$. Proposition 2.3 implies that $\mathrm{Sep}(C, C^\vee) = \mathcal{A}$. $\square$

Proposition 2.5. *(*[18, 19]*)*

(*i*) $\#\,\mathrm{ch}^k(\mathcal{A}) = b_k$, *where* $b_k = b_k(M(\mathcal{A}))$.

(*ii*) $\#\,\mathrm{bch}^k(\mathcal{A}) = \#\,\mathrm{uch}^{k+1}(\mathcal{A})$.
(*iii*) $\#\,\mathrm{bch}^k(\mathcal{A}) = b_k - b_{k-1} + \cdots + (-1)^k b_0$.

Concerning Proposition 2.5 (*ii*), an explicit bijection is given by the map to the opposite chamber,

$$\iota : \mathrm{bch}^k(\mathcal{A}) \xrightarrow{\simeq} \mathrm{uch}^{k+1}(\mathcal{A}), C \longmapsto C^\vee.$$

The next result characterizes the dense edges contained in $\overline{H}_\infty$.

Proposition 2.6. *([19, Proposition 2.4]) Let $\mathcal{A}$ be an affine arrangement in $\mathbb{R}^\ell$. An edge $X \in L(\overline{\mathcal{A}})$ with $X \subseteq \overline{H}_\infty$ is dense if and only if $X = X(C)$ for some chamber $C \in \mathrm{uch}(\mathcal{A})$. In particular, we have*

$$\mathsf{D}_\infty(\mathcal{A}) = \{X(C) \mid C \in \mathrm{uch}(\mathcal{A})\}. \tag{7}$$

Next, we define the degree map

$$\deg : \mathrm{ch}^k(\mathcal{A}) \times \mathrm{ch}^{k+1}(\mathcal{A}) \longrightarrow \mathbb{Z}.$$

Let $B = B^k \subset F^k$ be a k-dimensional ball of sufficiently large radius so that every 0-dimensional edge $X \in L_0(\mathcal{A} \cap F^k) \simeq L_{\ell-k}(\mathcal{A})$ is contained in the interior of B^k. Let $C \in \mathrm{ch}^k(\mathcal{A})$ and $C' \in \mathrm{ch}^{k+1}(\mathcal{A})$. Then, there exists a vector field $U^{C'}$ on F^k ([18]) that satisfies the following conditions.

- $U^{C'}(x) \neq 0$ for $x \in \partial\overline{C} \cap B^k$.
- Let $x \in \partial(B^k) \cap \overline{C}$. Then, $T_x(\partial B^k)$ can be considered as a hyperplane of T_xF^k. We impose the condition that $U^{C'}(x) \in T_xF^k$ is contained in the half-space corresponding to the inside of B^k.
- If $x \in H \cap F^k$ for a hyperplane $H \in \mathcal{A}$, then $U^{C'}(x) \notin T_x(H \cap F^k)$ and is directed to the side in which C' is lying with respect to H.

When the vector field $U^{C'}$ satisfies the above conditions, we say that *the vector field $U^{C'}$ is directed to the chamber C'*. The above conditions imply that if either $x \in H \cap F^k$ or $x \in \partial B^k$, then $U^{C'}(x) \neq 0$. Thus, for $C \in \mathrm{ch}^k(\mathcal{A})$, U is not vanishing on $\partial(\overline{C} \cap B^k)$. Hence, we can consider the following Gauss map.

$$\frac{U^{C'}}{|U^{C'}|} : \partial(\overline{C} \cap B^k) \longrightarrow S^{k-1}.$$

Fixing an orientation on F^k induces an orientation on $\partial(\overline{C} \cap B^k)$.

Definition 2.7. Define the degree $\deg(C, C')$ between $C \in \mathrm{ch}^k(\mathcal{A})$ and $C' \in \mathrm{ch}^{k+1}(\mathcal{A})$ by

$$\deg(C, C') := \deg\left(\left.\frac{U^{C'}}{|U^{C'}|}\right|_{\partial(\overline{C} \cap B^k)} : \partial(\overline{C} \cap B^k) \longrightarrow S^{k-1} \right) \in \mathbb{Z}.$$

This is independent of the choice of $U^{C'}$ ([18]).

If the vector field $U^{C'}$ does not have zeros on $\overline{C} \cap B^k$, then the Gauss map can be extended to the map $\overline{C} \cap B^k \longrightarrow S^{k-1}$. Hence, $\frac{U^{C'}}{|U^{C'}|} : \partial(\overline{C} \cap B^k) \longrightarrow S^{k-1}$ is homotopic to a constant map, and we have the following result.

Proposition 2.8. *If the vector field $U^{C'}$ is nowhere zero on $\overline{C} \cap B^k$, then $\deg(C, C') = 0$.*

Example 2.9. Let $p_0 \in F^k$ be such that $p_0 \notin \bigcup_{H \in \mathcal{A}} H \cup \partial B^k$. Define the pointing vector field U^{p_0} by

$$U^{p_0}(x) = \overrightarrow{x; p_0} \in T_x F^k, \tag{8}$$

where $\overrightarrow{x; p_0}$ is a tangent vector at x pointing toward p_0 (see Figure 3). The vector field U^{p_0} is directed to the chamber containing p_0. Note that $U^{p_0}(x) = 0$ if and only if $x = p_0$. Hence, if $p_0 \notin C \cap B^k$, the Gauss map $\frac{U^{p_0}}{|U^{p_0}|} : \partial(\overline{C} \cap B^k) \longrightarrow S^{k-1}$ satisfies $\deg\left(\frac{U^{p_0}}{|U^{p_0}|}\right) = 0$. Otherwise, if $p_0 \in C \cap B^k$, then $\deg\left(\frac{U^{p_0}}{|U^{p_0}|}\right) = (-1)^k$.

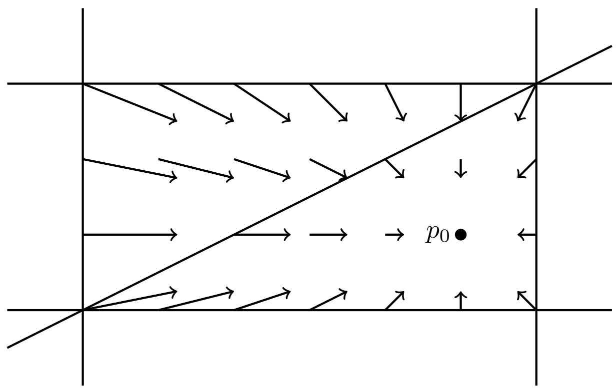

FIGURE 3. Pointing vector field $\frac{1}{4}U^{p_0}$

Consider the Orlik-Solomon algebra $A^{\bullet}_R(\mathcal{A})$ over the commutative ring R. Let

$$\omega_\lambda = \sum_{i=1}^{n} \lambda_i e_i \in A^1_R(\mathcal{A}) \qquad (\lambda_i \in R).$$

We will describe the Aomoto complex $(A^{\bullet}_R(\mathcal{A}), \omega_\lambda \wedge)$ in terms of chambers. For two chambers $C, C' \in \mathrm{ch}(\mathcal{A})$, define $\lambda_{\mathrm{Sep}(C,C')}$ by

$$\lambda_{\mathrm{Sep}(C,C')} := \sum_{H_i \in \mathrm{Sep}(C,C')} \lambda_i.$$

Proposition 2.10. *Let C be an unbounded chamber. Then,*

$$\lambda_{\mathrm{Sep}(C,C^\vee)} = -\lambda_{X(C)}.$$

Proof. By Proposition 2.3, we have $\overline{\mathcal{A}} = \overline{\mathcal{A}}_{X(C)} \sqcup \mathrm{Sep}(C, C^\vee)$. Hence, from the definition of $\lambda_\infty = -\sum_{i=1}^{n} \lambda_i$, we obtain $\lambda_{\mathrm{Sep}(C,C^\vee)} + \lambda_{X(C)} = 0$. □

Let $R[\mathrm{ch}^k(\mathcal{A})] = \bigoplus_{C \in \mathrm{ch}^k(\mathcal{A})} R \cdot [C]$ be the free R-module generated by $\mathrm{ch}^k(\mathcal{A})$. Let

$$\nabla_{\omega_\lambda} : R[\mathrm{ch}^k(\mathcal{A})] \longrightarrow R[\mathrm{ch}^{k+1}(\mathcal{A})]$$

be the R-homomorphism defined by

$$\nabla_{\omega_\lambda}([C]) = \sum_{C' \in \mathrm{ch}^{k+1}} \deg(C, C') \cdot \lambda_{\mathrm{Sep}(C,C')} \cdot [C']. \tag{9}$$

Proposition 2.11. *([20]) $(R[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_{\omega_\lambda})$ is a cochain complex. Furthermore, there is a natural isomorphism of cochain complexes,*

$$(R[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_{\omega_\lambda}) \simeq (A^\bullet_R(\mathcal{A}), \omega_\lambda \wedge).$$

In particular,

$$H^k(R[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_{\omega_\lambda}) \simeq H^k(A^\bullet_R(\mathcal{A}), \omega_\lambda \wedge).$$

Let $\mathcal{L}$ be a rank one local system on $M(\mathcal{A})$ with monodromy $q_i \in \mathbb{C}^\times$ around H_i $(i = 1, \dots, n)$. Fix $q_i^{1/2} = \sqrt{q_i}$ and define $q_\infty^{1/2}$ and $\Delta(C, C')$ by $q_\infty^{1/2} := \left(q_1^{1/2} \cdots q_n^{1/2}\right)^{-1}$ and

$$\Delta(C, C') := \prod_{H_i \in \mathrm{Sep}(C,C')} q_i^{1/2} - \prod_{H_i \in \mathrm{Sep}(C,C')} q_i^{-1/2}.$$

The local system cohomology groups can then be computed in a similar way to the cohomology groups of the Aomoto complex. Indeed, let us define the linear map

$$\nabla_{\mathcal{L}} : \mathbb{C}[\mathrm{ch}^k(\mathcal{A})] \longrightarrow \mathbb{C}[\mathrm{ch}^{k+1}(\mathcal{A})]$$

by

$$\nabla_{\mathcal{L}}([C]) = \sum_{C' \in \mathrm{ch}^{k+1}} \deg(C, C') \cdot \Delta(C, C') \cdot [C'].$$

Then, we have the following.

Proposition 2.12. *([18]) $(\mathbb{C}[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_{\mathcal{L}})$ is a cochain complex. Furthermore, there is a natural isomorphism of cohomology groups:*

$$H^k(\mathbb{C}[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_{\mathcal{L}}) \simeq H^k(M(\mathcal{A}), \mathcal{L}).$$

3. Main results and strategy

3.1. **Main theorems.** In this section, let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a hyperplane arrangement in $\mathbb{R}^\ell$ and R be a commutative ring with 1.

Theorem 3.1. *If $\lambda_X \in R^\times$ for all $X \in \mathsf{D}_\infty(\overline{\mathcal{A}})$, then*

$$H^k(\mathbb{C}[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_{\omega_\lambda}) \simeq \begin{cases} 0, & \text{if } k < \ell, \\ R[\mathrm{bch}(\mathcal{A})], & \text{if } k = \ell. \end{cases}$$

More generally, we can prove the following.

Corollary 3.2. *Let $0 \le p < \ell$. If $\lambda_X \in R^\times$ for all $X \in \mathsf{D}_\infty(\overline{\mathcal{A}})$ with $\dim(X) \ge p$, then*

$$H^k(\mathbb{C}[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_{\omega_\lambda}) = 0, \text{ for all } 0 \le k < \ell - p.$$

Proof. We prove Corollary 3.2 based on Theorem 3.1 with $\mathcal{A} \cap F^{\ell-p}$. The Orlik-Solomon algebra $A^\bullet_R(\mathcal{A} \cap F^{\ell-p})$ is isomorphic to $A_R^{\le \ell-p}(\mathcal{A})$. Hence, we have an isomorphism

$$H^k(A^\bullet_R(\mathcal{A} \cap F^{\ell-p}), \omega_\lambda \wedge) \simeq H^k(A^\bullet_R(\mathcal{A}), \omega_\lambda \wedge), \tag{10}$$

for $k < \ell - p$. Note that $L(\mathcal{A} \cap F^{\ell-p}) \simeq L^{\ge p}(\mathcal{A})$. By assumption, we have that $\lambda_X \in R^\times$ for any $X \in \mathsf{D}_\infty(\mathcal{A} \cap F^{\ell-q})$. Hence, by Theorem 3.1, the left-hand side of (10) vanishes. □

The following vanishing result for the Aomoto complex follows from Proposition 2.11.

Corollary 3.3. *Let $0 \le p < \ell$. If $\lambda_X \in R^\times$ for all $X \in \mathsf{D}_\infty(\overline{\mathcal{A}})$ with $\dim(X) \ge p$, then*

$$H^k(A^\bullet_R(\mathcal{A}), \omega_\lambda \wedge) = 0, \text{ for all } 0 \le k < \ell - p.$$

Remark 3.4. A very similar proof can be used for the case of local systems. Namely, if the local system $\mathcal{L}$ satisfies $q_X \neq 1$ for all $X \in \mathsf{D}_\infty(\overline{\mathcal{A}})$ with $\dim(X) \geq p$, then

$$H^k(\mathbb{C}[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_{\mathcal{L}}) = 0, \text{ for all } k < \ell - p.$$

Using Proposition 2.12, this implies

$$H^k(M(\mathcal{A}), \mathcal{L}) = 0, \text{ for all } k < \ell - p,$$

which gives an alternative proof of Theorem 1.1 given by Cohen, Dimca, and Orlik.

3.2. **Strategy for the proof of Theorem 3.1.** To analyze the cohomology group

$$H^k(R[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_\omega) = \frac{\ker\left(\nabla_\omega : R[\mathrm{ch}^k(\mathcal{A})] \longrightarrow R[\mathrm{ch}^{k+1}(\mathcal{A})]\right)}{\mathrm{im}\left(\nabla_\omega : R[\mathrm{ch}^{k-1}(\mathcal{A})] \longrightarrow R[\mathrm{ch}^k(\mathcal{A})]\right)},$$

we will use the direct decomposition $R[\mathrm{ch}^k(\mathcal{A})] = R[\mathrm{bch}^k(\mathcal{A})] \oplus R[\mathrm{uch}^k(\mathcal{A})]$, and then consider the map

$$\overline{\nabla}_{\omega_\lambda} : R[\mathrm{bch}^k(\mathcal{A})] \hookrightarrow R[\mathrm{ch}^k(\mathcal{A})] \xrightarrow{\nabla_\omega} R[\mathrm{ch}^{k+1}(\mathcal{A})] \twoheadrightarrow R[\mathrm{uch}^{k+1}(\mathcal{A})]. \tag{11}$$

We will study the map $\overline{\nabla}_{\omega_\lambda} : R[\mathrm{bch}^k(\mathcal{A})] \longrightarrow R[\mathrm{uch}^{k+1}(\mathcal{A})]$ in detail below. Recall that there is a natural bijection $\iota : \mathrm{bch}^k(\mathcal{A}) \xrightarrow{\simeq} \mathrm{uch}^{k+1}(\mathcal{A})$ (see Proposition 2.5 and subsequent remarks). Once we fix an ordering $C_1, \ldots, C_b$ of $\mathrm{bch}^k(\mathcal{A})$, we obtain a matrix expression of the map $\overline{\nabla}_{\omega_\lambda}$. We will prove the following.

(i) Let $C \in \mathrm{bch}^k(\mathcal{A})$. Then, $\deg(C, C^\vee) = (-1)^{\ell-1-\dim X(C)}$.
(ii) For an appropriate ordering of $\mathrm{bch}^k(\mathcal{A}) = \{C_1, \ldots, C_b\}$, the matrix expression of

$$\overline{\nabla}_{\omega_\lambda} : R[\mathrm{bch}^k(\mathcal{A})] \longrightarrow R[\mathrm{uch}^{k+1}(\mathcal{A})]$$

is upper-triangular.
(iii) $\det \overline{\nabla}_\omega \in R^\times$
(iv) These imply Theorem 3.1.

(i) and (ii) will be proved in §4.

Here, we prove (iii) and (iv) based on (i) and (ii). First, note that Proposition 2.10, along with the definition (9) of the coboundary map of the complex $(R[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_\omega)$ and the upper-triangularity in (ii) above implies that

$$\det \overline{\nabla}_\omega = \pm \prod_{C \in \mathrm{bch}^k(\mathcal{A})} \deg(C, C^\vee) \lambda_{X(C)}.$$

Then, from the assumption that $\lambda_X \in R^\times$ for $X \in \mathsf{D}_\infty(\mathcal{A})$ (see also Proposition 2.6), we have (iii) directly. Because $\overline{\nabla}_\omega : R[\mathrm{bch}^k(\mathcal{A})] \xrightarrow{\simeq} R[\mathrm{uch}^{k+1}(\mathcal{A})]$, which are diagonals of the following diagram, are isomorphisms of free R-modules, we have $H^k(R[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_\omega) = 0$ for $k < \ell$ and $H^\ell(R[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_\omega) \simeq R[\mathrm{bch}^\ell(\mathcal{A})]$.

$$\begin{array}{ccccccccccc}
R[\mathrm{ch}^0] & \xrightarrow{\nabla_\omega} & R[\mathrm{ch}^1] & \xrightarrow{\nabla_\omega} & \cdots & \xrightarrow{\nabla_\omega} & R[\mathrm{ch}^k] & \xrightarrow{\nabla_\omega} & R[\mathrm{ch}^{k+1}] & \xrightarrow{\nabla_\omega} & \cdots \\
\| & & \| & & & & \| & & \| & & \\
R[\mathrm{bch}^0] & & R[\mathrm{bch}^1] & & \cdots & & R[\mathrm{bch}^k] & & R[\mathrm{bch}^{k+1}] & & \\
& \searrow & \oplus & \searrow & & \searrow & \oplus & \searrow & \oplus & & \\
& & R[\mathrm{uch}^1] & & \cdots & & R[\mathrm{uch}^k] & & R[\mathrm{uch}^{k+1}] & &
\end{array}$$

4. Proofs

In this section, we prove (i) and (ii) stated in §3.2 for $k = \ell - 1$. Namely:

(i') For a chamber $C \in \mathrm{bch}^{\ell-1}(\mathcal{A})$, $\deg(C, C^{\vee}) = (-1)^{\ell-1-\dim X(C)}$.

(ii') For an appropriate ordering of $\{C_1, \dots, C_b\} = \mathrm{bch}^{\ell-1}(\mathcal{A})$, the matrix expression of $\overline{\nabla}_{\omega_\lambda} : R[\mathrm{bch}^{\ell-1}(\mathcal{A})] \longrightarrow R[\mathrm{uch}^{\ell}(\mathcal{A})]$ is upper-triangular.

For other $k < \ell$, note that the assertions can be proved in a similar way to that for $k = l - 1$ by using the generic section F^{k+1} (see the arguments in the proof of Corollary 3.2).

4.1. **Structure of Walls.** For simplicity, we will set $F = F^{\ell-1}$. Recall that

$$\mathrm{bch}^{\ell-1}(\mathcal{A}) = \{C \in \mathrm{ch}(\mathcal{A}) \mid C \cap F \text{ is a bounded chamber of } F \cap \mathcal{A}\}.$$

Let $C \in \mathrm{bch}^{\ell-1}(\mathcal{A})$. A hyperplane $H \in \mathcal{A}$ is said to be a wall of C if $H \cap F$ is a supporting hyperplane of a facet of $\overline{C} \cap F$. For any $C \in \mathrm{bch}^{\ell-1}(\mathcal{A})$, we denote the set of all walls of C by $\mathrm{Wall}(C)$.

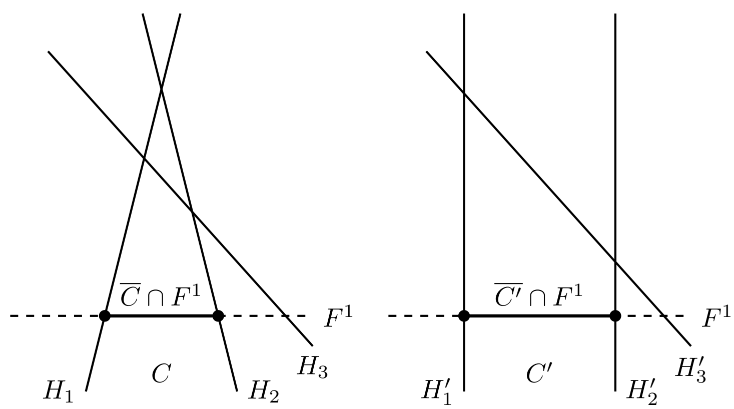

Figure 4. $\mathrm{Wall}(C) = \mathrm{Wall}_2(C) = \{H_1, H_2\}$, $\mathrm{Wall}(C') = \mathrm{Wall}_1(C') = \{H'_1, H'_2\}$

We divide the set of walls into two types.

Definition 4.1. A wall $H \in \mathrm{Wall}(C)$ is said to be of the first kind if $\overline{H} \supset X(C)$. Otherwise, we say that H is a wall of the second kind. The sets of the first and the second kind of walls are denoted by $\mathrm{Wall}_1(C)$ and $\mathrm{Wall}_2(C)$, respectively. We have $\mathrm{Wall}(C) = \mathrm{Wall}_1(C) \sqcup \mathrm{Wall}_2(C)$ (see Figure 4 and 5).

Let $C \in \mathrm{bch}^{\ell-1}(\mathcal{A})$ and $\mathrm{Wall}_1(C) = \{H_{i_1}, \dots, H_{i_k}\}$ be walls of the first kind. We choose defining equations $\alpha_{i_1}, \dots, \alpha_{i_k}$ for the walls in $\mathrm{Wall}_1(C)$ so that

$$C \subset \{\alpha_{i_1} > 0\} \cap \cdots \cap \{\alpha_{i_k} > 0\}.$$

Note that $\widetilde{C} := \{\alpha_{i_1} > 0\} \cap \cdots \cap \{\alpha_{i_k} > 0\}$ is a chamber of $\mathrm{Wall}_1(\mathcal{A})$. Let $D \in \mathrm{uch}(\mathcal{A})$ be another unbounded chamber of $\mathcal{A}$. Then, D is said to be inside $\mathrm{Wall}_1(C)$ if

$$D \subset \widetilde{C} = \{\alpha_{i_1} > 0\} \cap \cdots \cap \{\alpha_{i_k} > 0\}.$$

This condition is equivalent to $\mathrm{Sep}(C, D) \cap \mathrm{Wall}_1(C) = \emptyset$.

Recall that the opposite chamber of $C \in \mathrm{bch}^{\ell-1}(\mathcal{A})$ is defined as the opposite chamber with respect to $X(C) \subset \overline{H}_\infty$. Using (6), we have the following.

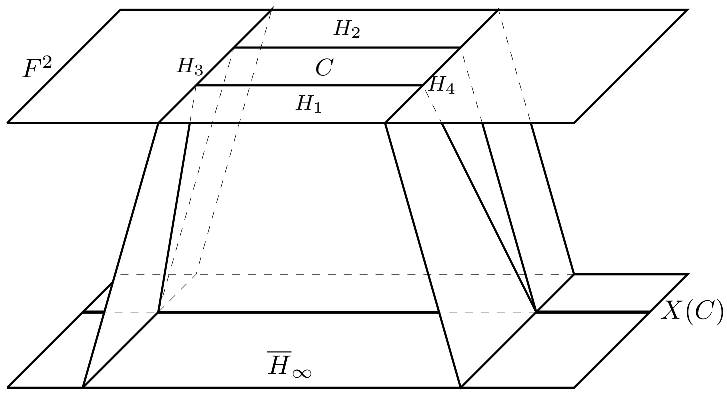

FIGURE 5. $\mathrm{Wall}_1(C) = \{H_1, H_2\}, \mathrm{Wall}_2(C) = \{H_3, H_4\}$.

Proposition 4.2. *Let* $C \in \mathrm{bch}^{\ell-1}(\mathcal{A})$. *Then,* $\mathrm{Sep}(C, C^\vee) \cap \mathrm{Wall}(C) = \mathrm{Wall}_2(C)$.

Remark 4.3. Let $C \in \mathrm{bch}^{\ell-1}(\mathcal{A})$. If D is inside $\mathrm{Wall}_1(C)$, then

$$X(D) \subset X(C) \quad \text{and} \quad \dim X(D) \leq \dim X(C).$$

4.2. **Fibered structure of chambers.** Let $C \in \mathrm{bch}^{\ell-1}(\mathcal{A})$ and $d = \dim X(C)$. As above, we let $\widetilde{C} \in \mathrm{ch}(\mathrm{Wall}_1(C))$ be the unique chamber such that $C \subset \widetilde{C}$.

For each point $p \in \overline{\widetilde{C}}$, denote $G_1(p) := \langle X(C), p\rangle \cap F$ (see Figure 6). Then, $G_1(p)$ is a d-dimensional affine subspace that is parallel to each $H \in \mathrm{Wall}_1(C)$. Fix a base point $p_0 \in \widetilde{C}$. We also fix an $(\ell - 1 - d)$-dimensional subspace $G_2(p_0) \subset F$ that passes through p_0 and is transversal to $G_1(p_0)$ (see Figure 6). Let us call $Q_0 := G_2(p_0) \cap \overline{\widetilde{C}}$ the base polytope.

Consider the map $\pi_C : \overline{C} \cap F \longrightarrow Q_0, p \longmapsto G_1(p) \cap Q_0$. For each $q \in Q_0$, the fiber $\pi_C^{-1}(q) = G_1(q) \cap \overline{C}$ is a d-dimensional polytope. This is a conclusion of the assumption that F is generic and near to $\overline{H}_\infty$ and the following elementary proposition.

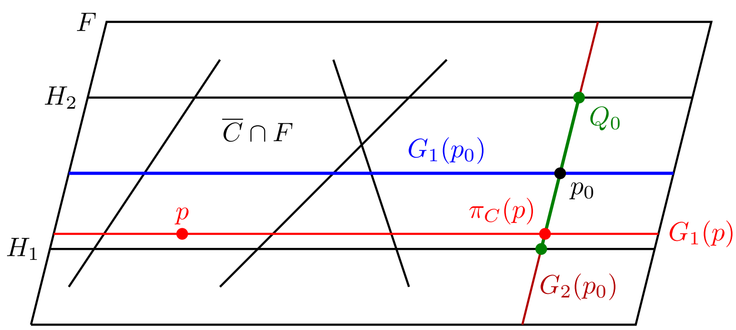

FIGURE 6. Base polytope Q_0 ($\mathrm{Wall}_1(C) = \{H_1, H_2\}$)

Proposition 4.4. *Let* $P \subset \mathbb{R}^\ell$ *be an* ℓ*-dimensional polytope. Let* $X \subset P$ *be a* d*-dimensional face* ($0 \leq d \leq \ell$)*. We denote by* $\langle X\rangle$ *the* d*-dimensional affine subspace spanned by* X*. Then, for* $\varepsilon \in \mathbb{R}^\ell$ *with sufficiently small* $0 \leq |\varepsilon| \ll 1$, $(\langle X\rangle + \varepsilon) \cap P$ *is either an empty set or a* d*-dimensional polytope.*

Remark 4.5. As $\pi_C : \overline{C} \cap F \longrightarrow Q_0$ is a fibration with contractible fibers, there exists a continuous section $\sigma_C : Q_0 \longrightarrow \overline{C} \cap F$ such that $\pi_C \circ \sigma_C = \mathrm{id}_{Q_0}$.

4.3. **Upper-triangularity.** Let us fix an ordering of the chambers of $\mathrm{bch}^{\ell-1}(\mathcal{A}) = \{C_1, \dots, C_b\}$ in such a way that

$$\dim X(C_1) \geq \dim X(C_2) \geq \cdots \geq \dim X(C_b).$$

The main result of this section is the following.

Theorem 4.6. *The matrix* $(\deg(C_i, C_j^\vee))_{i,j=1,\dots,b}$ *is upper-triangular. In other words, if* $i > j$, $\deg(C_i, C_j^\vee) = 0$.

Proof. Let $C, D \in \mathrm{bch}^{\ell-1}(\mathcal{A})$. Suppose $\dim X(D) \geq \dim X(C)$ and $C \neq D$. Then, we will show that $\deg(C, D^\vee) = 0$. The idea of the proof is to construct a vector field $U^{D^\vee}$ directed to $D^\vee$ on F that is nowhere vanishing on a neighbourhood of $\overline{C} \cap F \subset F$. Then, by Proposition 2.8, we have $\deg(C, D^\vee) = 0$.

Let us study the following three cases separately.

(a) $\dim X(C) = \ell - 1$.
(b) $\dim X(C) < \ell - 1$ and D is not inside $\mathrm{Wall}_1(C)$.
(c) $\dim X(C) < \ell - 1$ and D is inside $\mathrm{Wall}_1(C)$.

First, we consider case (a). In this situation, because $\dim X(D) \geq \dim X(C)$, we have $\dim X(D) = \ell - 1$. Choose a point $p \in D \cap F$, and define the vector field U on F by

$$U(x) = \overrightarrow{x; p} \in T_x F.$$

Then, the vector field is directed to p and nowhere vanishing on $\overline{C} \cap F$ (because $p \notin \overline{C}$). By Corollary 2.4, $-U$ is a vector field directed to $D^\vee$ that is also nowhere vanishing on $\overline{C} \cap F$. Hence, $\deg(C, D^\vee) = 0$.

From this point, we assume $\dim X(C) < \ell - 1$. If D is inside $\mathrm{Wall}_1(C)$, then $X(D) \subset X(C)$ by Remark 4.3, and we have $\overline{\mathcal{A}}_{X(D)} \supset \overline{\mathcal{A}}_{X(C)}$. Proposition 4.2 indicates $\mathrm{Sep}(D, D^\vee) \cap \overline{\mathcal{A}}_{X(C)} = \emptyset$. We can conclude that $D^\vee$ is also inside $\mathrm{Wall}_1(C)$. Conversely, if D is not inside $\mathrm{Wall}_1(C)$, then $D^\vee$ is also not inside $\mathrm{Wall}_1(C)$.

Second, we consider case (b). In this situation, $\mathrm{Sep}(C, D^\vee) \cap \mathrm{Wall}_1(C) \neq \emptyset$. Choose a hyperplane $H_{i_0} \in \mathrm{Sep}(C, D^\vee) \cap \mathrm{Wall}_1(C)$ and let α_{i_0} be the defining equation of H_{i_0}. Without loss of generality, we may assume that

$$H_{i_0}^+ = \{\alpha_{i_0} > 0\} \supset D^\vee$$
$$H_{i_0}^- = \{\alpha_{i_0} < 0\} \supset C.$$

We will construct a vector field $U^{D^\vee}$ on F that is directed to $D^\vee$ and satisfies

$$U^{D^\vee}(x)\alpha_{i_0} > 0 \tag{12}$$

for $x \in \overline{C} \cap F$, where the left hand side of (12) is the derivative of α_{i_0} with respect to the vector field. In particular, we obtain a vector field directed to $D^\vee$ that is nowhere vanishing on $\overline{C} \cap F$. It is sufficient to show that, at any point $x_0 \in \overline{C}$, there exists a local vector field around x_0 that satisfies (12). Then, we will obtain a global vector field that satisfies (12) using a partition of unity.

It is sufficient to show the existence of such a vector field around each vertex x_0 of $\overline{C} \cap F$. By the genericity of F, $Z := \bigcap \mathcal{A}_{x_0} = \bigcap_{x_0 \in H \in \mathcal{A}} H$ is a 1-dimensional flat of $\mathcal{A}$, which is transversal to F. By the assumption that F does not separate 0-dimensional flats of $\mathcal{A}$, we have

$$\overline{Z} \cap \overline{H}_\infty \subset \overline{C} \cap \overline{H}_\infty. \tag{13}$$

(See Figure 7.)

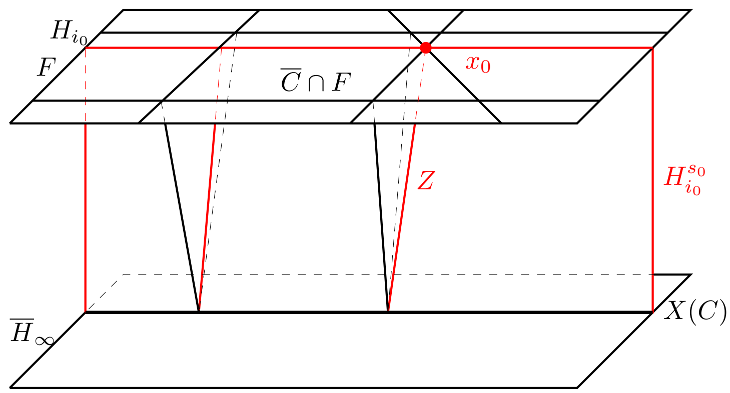

FIGURE 7. Z and $H_{i_0}^{s_0}$.

Set $s_0 := \alpha_{i_0}(x_0)$ and let $H_{i_0}^{s_0} = \{\alpha_{i_0} = s_0\}$ be the hyperplane passing through x_0 that is parallel to H_{i_0}. Then, we have $Z \subset H_{i_0}^{s_0}$, as otherwise we have a contradiction with (13). The hyperplanes $\mathcal{A}_{x_0} = \mathcal{A}_Z$ determine chambers (cones), one of which (denoted by Γ) contains $D^\vee$ (see Figure 8). Hence, the tangent vector $U^{D^\vee}(x_0)$ must be contained in Γ. Furthermore,

$$D \subset \Gamma \cap H_{i_0}^{+} \subset \Gamma \cap H_{i_0}^{>s_0}. \tag{14}$$

In particular, we have $\Gamma \cap H_{i_0}^{>s_0} \neq \emptyset$. Thus, we can construct a vector field $U^{D^\vee}$ around x_0 so that $U^{D^\vee}(x_0) \in \Gamma \cap H_{i_0}^{>s_0}$ and (12) is satisfied around x_0. Hence, we have $\deg(C, D^\vee) = 0$ in case (b).

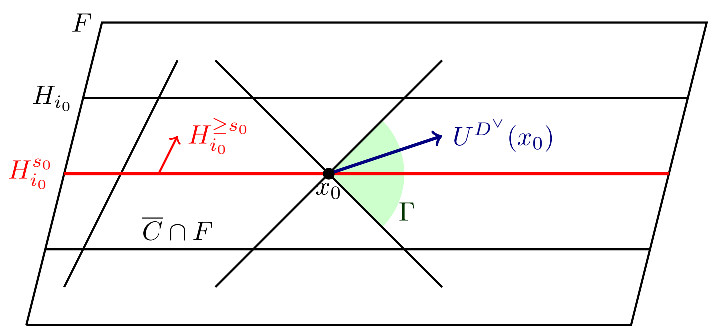

FIGURE 8. Construction of the vector field $U^{D^\vee}$

Third, suppose D is inside $\mathrm{Wall}_1(C)$ (equivalently, $D \subset \widetilde{C}$), and let us handle case (c). As $X(D) \subset X(C)$ and $\dim X(D) \geq \dim X(C)$, we have $X(D) = X(C)$. In this case,

$$\mathrm{Wall}_1(C) = \mathrm{Wall}_1(D) \quad \text{and} \quad \widetilde{C} = \widetilde{D}.$$

We consider the fibration $\pi_D : \overline{D} \cap F \longmapsto Q_0$ that also has d-dimensional polytopes as fibers. Because the fibers are contractible, there exists a continuous section $\sigma_D : Q_0 \longmapsto \overline{D} \cap F$ such that $\pi_D \circ \sigma_D = \mathrm{id}_{Q_0}$.

We now move to the construction of a vector field. For each $p \in \overline{C} \cap F$, we denote the $(\ell - 1 - d)$-dimensional subspace that passes through p and is parallel to $G_2(p_0)$ (see Figure 9).

Let $\{p'\} = G_2(p) \cap G_1(p_0)$. The tangent space decomposes into the direct sum

$$T_pF = T_pG_1(p) \oplus T_pG_2(p).$$

Let us first construct a vector field on the second component. For this, define the tangent vector $V_2(p) \in T_pG_2(p) \subset T_pF$ by

$$V_2(p) = \overrightarrow{p;p'}. \tag{15}$$

The vector field V_2 is obviously inward with respect to $\mathrm{Wall}_1(C)$, and vanishes on the reference fiber $G_1(p_0) \cap \overline{C}$.

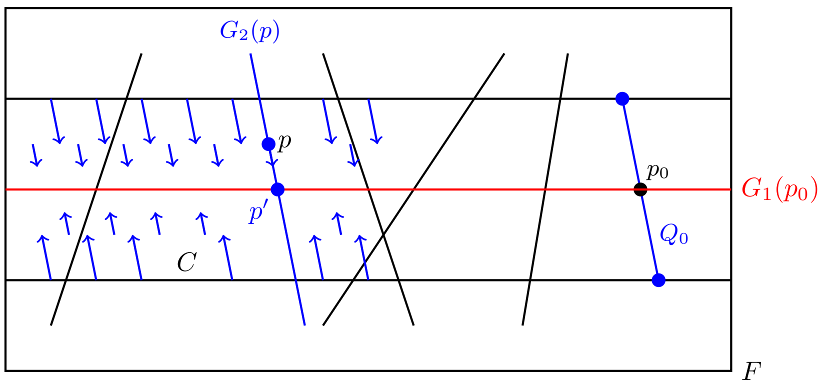

FIGURE 9. V_2.

Let us now construct a vector field V_1 along the fibers $G_1(p)$. Using the section

$$\sigma_D : Q_0 \longrightarrow \overline{C} \cap F \qquad \text{(Remark 4.5)},$$

define V_1 by

$$V_1(p) = \overrightarrow{p;\sigma_D(\pi_C(p))}. \tag{16}$$

(See Figure 10.)

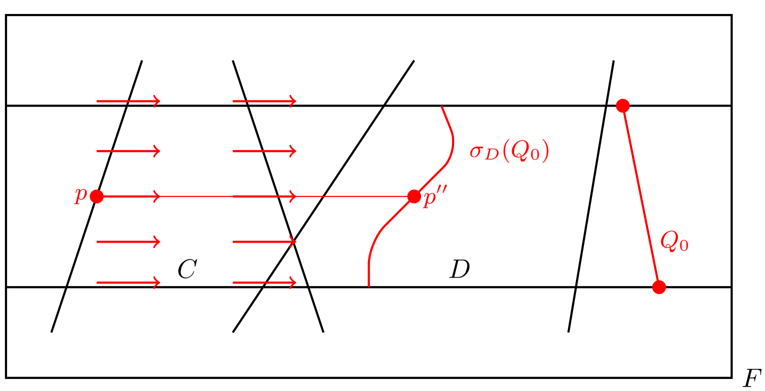

FIGURE 10. V_1, $p'' = \sigma_D(\pi_C(p))$.

Proposition 4.7. *For sufficiently large $t \gg 0$, the vector field tV_1+V_2 is directed to D. Similarly, $-tV_1 + V_2$ is a vector field directed to $D^\vee$.*

Proof. Let $p \in H \in \mathrm{Wall}_1(C)$. Recall that D is inside $\mathrm{Wall}_1(C)$. As V_2 is inward and V_1 is tangent to H, the vector field $\pm tV_1 + V_2$ is also inward. Let $H \in \mathrm{Wall}_2(C)$ and $p \in H \cap F$. Then, V_1 (resp. $-V_1$) is directed to D (resp. $D^\vee$) with respect to H. Hence, for sufficiently large t, $tV_1 + V_2$ (resp. $-tV_1 + V_2$) is directed to D (resp. $D^\vee$). □

Because V_1 is a nowhere vanishing vector field on $\overline{C} \cap F$, $-tV_1 + V_2$ is a nowhere vanishing vector field around $\overline{C} \cap F$ that is directed to $D^\vee$. Hence, $\deg(C, D^\vee) = 0$. This completes the proof of Theorem 4.6. □

4.4. **The degree formula.** This section is devoted to a prove the following theorem.

Theorem 4.8. *Let $C \in \mathrm{bch}^{\ell-1}(\mathcal{A})$ and $d = \dim X(C)$. Then,*

$$\deg(C, C^\vee) = (-1)^{\ell-1-d}. \tag{17}$$

We will construct a vector field around $\overline{C} \cap F$ that is directed to $C^\vee$. The vector field V_2 is the same as in the previous section (§4.3). Define the vector field V_1 along the fibers of π_C by

$$V_1(p) = \overrightarrow{p; \sigma_C(\pi_C(p))}. \tag{18}$$

(See Figure 11.)

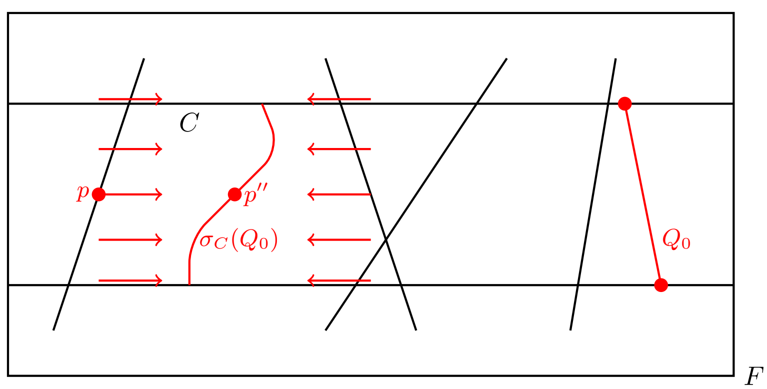

FIGURE 11. V_1, $p'' = \sigma_C(\pi_C(p))$.

Then, $tV_1 + V_2$ is a vector field directed to C (for $t \gg 0$). Since C and $C^\vee$ are separated by $H \in \mathcal{A} \smallsetminus \mathrm{Wall}_1(C)$, the vector field $-tV_1 + V_2$ is directed to $C^\vee$. We can compute the degree $\deg(C, C^\vee)$ using the vector field $-tV_1 + V_2$. Note that $-tV_1(p)$ is an outward vector field along a d-dimensional space $G_1(p)$, and $V_2(p)$ is an inward vector field that is tangent to an $(\ell-1-d)$-dimensional space $G_2(p)$. Hence, $\deg(C, C^\vee)$ is equal to the index of the following vector field in $\mathbb{R}^{\ell-1}$ at the origin:

$$V = \sum_{i=1}^{d} x_i \frac{\partial}{\partial x_i} - \sum_{i=d+1}^{\ell-1} x_i \frac{\partial}{\partial x_i}. \tag{19}$$

Finally, recall that the de Rham cohomology group $H^{\ell-1}(S^{\ell-2})$ is generated by the differential form ([2])

$$\sum_{i=1}^{\ell-1}(-1)^{i-1}x_i dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_{\ell-1}.$$

It is easily seen that the self-map of $H^{\ell-1}(S^{\ell-2})$ induced by the Gauss map of the vector field (19) is equal to multiplication by $(-1)^{\ell-1-d}$. This completes the proof of Theorem 4.8.

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References

[1] K. Aomoto, M. Kita, Theory of hypergeometric functions. With an appendix by Toshitake Kohno. Translated from the Japanese by Kenji Iohara. Springer Monographs in Mathematics. Springer-Verlag, Tokyo, 2011. xvi+317 pp. DOI: 10.1007/978-4-431-53938-4

[2] R. Bott, L. Tu: Differential forms in algebraic topology. Graduate Texts in Mathematics, Springer-Verlag, 1982. DOI: 10.1007/978-1-4757-3951-0

[3] E. Brieskorn, Sur les groupes de tresses [d'après V. I. Arnol'd]. (French) Séminaire Bourbaki, 24ème année (1971/1972), Exp. No. 401, pp. 21-44. Lecture Notes in Math., Vol. 317, Springer, Berlin, 1973.

[4] D. Cohen, A. Dimca, P. Orlik, Nonresonance conditions for arrangements. *Ann. Inst. Fourier* **53** (2003), 1883–1896. DOI: 10.5802/aif.1994

[5] A. Dimca, S. Papadima, Hypersurface complements, Milnor fibers and higher homotopy groups of arrangements. *Ann. of Math.* (2) **158**(2003), no. 2, 473–507.

[6] A. Dimca, S. Papadima, A. I. Suciu, Topology and geometry of cohomology jump loci. *Duke Math. J.* **148** (2009), no. 3, 405-457.

[7] H. Esnault, V. Schechtman, and E. Viehweg, Cohomology of local systems of the complement of hyperplanes. *Invent. Math.* **109** (1992), 557-561. DOI: 10.1007/BF01232040

[8] T. Kohno, Homology of a local system on the complement of hyperplanes. *Proc. Japan Acad. Ser. A Math. Sci.* **62** (1986), no. 4, 144-147.

[9] T. Kohno, Conformal field theory and topology. Translations of Mathematical Monographs, 210. Iwanami Series in Modern Mathematics. American Mathematical Society, Providence, RI, 2002. x+172 pp.

[10] S. Nazir, M. Torielli, M. Yoshinaga, On the admissibility of certain local systems. *Topology Appl.* **178** (2014), 288-299. DOI: 10.1016/j.topol.2014.10.001

[11] P. Orlik, L. Solomon, Combinatorics and topology of complements of hyperplanes. *Invent. Math.* **56** (1980), 167-189. DOI: 10.1007/BF01392549

[12] P. Orlik,H. Terao, Arrangements of hyperplanes. Grundlehren der Mathematischen Wissenschaften, 300. Springer-Verlag, Berlin, 1992. xviii+325 pp. DOI: 10.1007/978-3-662-02772-1

[13] P. Orlik, H. Terao, Arrangements and hypergeometric integrals. Second edition. MSJ Memoirs, 9. Mathematical Society of Japan, Tokyo, 2007. x+112 pp.

[14] S. Papadima, A. Suciu, The spectral sequence of an equivariant chain complex and homology with local coefficients. *Trans. A. M. S.*, **362** (2010), no. 5, 2685-2721.

[15] S. Papadima, A. Suciu, The Milnor fibration of a hyperplane arrangement: from modular resonance to algebraic monodromy. arχiv: 1401.0868

[16] V. Schechtman, H. Terao, and A. Varchenko, Local systems over complements of hyperplanes and the Kac–Kazhdan conditions for singular vectors. *J. Pure Appl. Algebra* **100** (1995), 93-102. DOI: 10.1016/0022-4049(95)00014-N

[17] A. N. Varchenko, Multidimensional hypergeometric functions the representation theory of Lie Algebras and quantum groups, Advanced Series in Mathematical Physics, **2**, World Scientific, 1995. DOI: 10.1142/2467

[18] M. Yoshinaga, Hyperplane arrangements and Lefschetz's hyperplane section theorem. *Kodai Math. J.* **30**, no. 2 (2007), 157–194. DOI: 10.2996/kmj/1183475510

[19] M. Yoshinaga, Minimality of hyperplane arrangements and basis of local system cohomology. *Singularities in geometry and topology*, 345-362, IRMA Lect. Math. Theor. Phys., 20, *Eur. Math. Soc., Zürich*, 2012.

[20] M. Yoshinaga, The chamber basis of the Orlik-Solomon algebra and Aomoto complex. *Ark. Mat.* **47** (2009), no. 2, 393-407. DOI: 10.1007/978-4-431-53938-4

[21] S. Yuzvinsky, Orlik-Solomon algebras in algebra and topology. *Russian Math. Surveys* **56** (2001), no. 2, 293-364.

[22] S. Yuzvinsky, Cohomology of the Brieskorn-Orlik-Solomon algebras. *Comm. Algebra* **23** (1995), no. 14, 5339-5354.

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FINE POLAR INVARIANTS OF MINIMAL SINGULARITIES OF SURFACES

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ABSTRACT. We consider the polar curves $P_{S,0}$ arising from generic projections of a germ $(S, 0)$ of a complex surface singularity onto $\mathbb{C}^2$. Taking $(S, 0)$ to be a minimal singularity of normal surface (i.e., a rational singularity with reduced tangent cone), we give the δ-invariant of these polar curves, as well as the equisingularity-type of their generic plane projections, which are also the discriminants of generic projections of $(S, 0)$.

These two pieces of equisingularity data for $P_{S,0}$ are described on the one hand by the geometry of the tangent cone of $(S, 0)$, and on the other hand by the limit-trees introduced by T. de Jong and D. van Straten for the deformation theory of these minimal singularities. These trees give a combinatorial device for the description of the polar curve which makes it much clearer than in our previous note on the subject. This previous work mainly relied on a result of M. Spivakovsky. Here, we give a geometrical proof via deformations (on the tangent cone, and what we call Scott deformations) and blow-ups, although we need Spivakovsky's result at some point, extracting some other consequences of it along the way.

INTRODUCTION

The local polar varieties of any germ $(X, 0)$ of a reduced complex analytic space were introduced by Lê D.T. and B. Teissier in [17]. In particular, the multiplicities of the *general* polar varieties are important analytic invariants of the germ $(X, 0)$.

However, as also emphasised by these authors (see also [23] p. 430–431 and [24]), there is more information to be gained on the geometry of $(X, 0)$ by considering not only the multiplicity but the (e.g., Whitney-) equisingularity class of these general polar varieties, which can also be shown to be an analytic invariant.

In this work, we will focus on the polar curves of a two-dimensional germ $(S, 0)$.

Our reference on equisingularity theory for space curves will be the mémoire [8]. Of course, as opposed to the case of plane curves, there is no complete set of invariants attached to a germ of a space curve describing its equisingularity class. As a general rule, results on equisingularity beyond the case of plane curves only make sense by considering the constancy of invariants in given families. Here we look at the family of polar curves and will consider the following invariants:

Definition 0.1. Our equisingularity data for a germ of space curve consists of both:
(i) the value of the delta invariant of the curve, and
(ii) the equisingularity class of its generic plane projection.

We recall the definitions of these notions in the text (see Def. 6.1 and Def. 1.2). The constancy of these two invariants in a family of space curves ensures Whitney conditions and actually the stronger *equisaturation condition* (cf. [8]).

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In general, this is still partial information; for example, another interesting invariant for space curves, namely the semi-group of each branch, is completely independent of this equisaturation condition.

The purpose of this paper is to describe the equisingularity data in 0.1 for the *the general polar curve* of a class of normal surface singularities called *minimal*.

These *minimal singularities* were studied in any dimension by J. Kollár in [19]. In the case of normal surfaces, these are also the *rational singularities with reduced fundamental cycle* and were studied by M. Spivakovsky in [21] and T. de Jong and D. van Straten in [15].

For these surfaces, we prove the following:

Proposition $(*)$ **(cf. 5.5 for a more precise statement)**: *the general polar curve is a union of A_{n_i}-plane curves singularities*[1]*, where the n_i's and the contacts between these curves can be deduced from the resolution graph of the surface.*

This information gives in particular a complete description of part (ii) of the data in 0.1, i.e., of the general plane projection of the polar curve, which is also the discriminant of the general projection (the coincidence of these two concepts is a theorem, cf. section 1).

The information on the discriminant was already given in the note [5] as a consequence of a result of Spivakovsky, but the statement there was clumsy.

Here we give a much nicer device that allows us to read directly the information about of the discriminant (or the polar curve as well) from both the information contained in the tangent cone of these singularities and the information given by a graph deduced from the resolution graph, which is precisely the *limit tree* introduced by T. de Jong and D. van Straten in their study of the deformation theory for these minimal singularities (see [15]).

We also provide an inductive proof relying much more on the geometrical properties of these minimal singularities. This proof makes up the core of the paper. It still uses Spivakovsky's theorem, however, mainly through a characterisation of generic polar curves on the resolution which we deduce along the way.

The several plane branches of the polar curve lie in distinct planes in a bigger linear space, and the value of the delta invariant (part (i) in 0.1) gives some (partial) information on the configuration of these planes in the space. We explain how this delta invariant is easily computed from what we call the Scott deformation of the surface, which turns out to give a delta-constant deformation of the polar curve onto bunches of generic configurations of lines.

Organisation of the paper:

In Section 1, we recall the definition of the general polar curve $P_{S,0}$ of a germ of surface $(S,0)$, of the discriminant $\Delta_{S,0}$ of a generic projection of $(S,0)$ onto $\mathbb{C}^2$ and the important result that $\Delta_{S,0}$ is a *generic* projection of the curve $P_{S,0}$.

Section 2 gives the definition of minimal singularities in general, the particular case of normal surfaces, and their characterisation by their dual resolution graph. We then define, in Section 3, a notion of *height* on the vertices of this resolution graph, which was used in other places such as [21] and [15], and corresponds to the number of point blow-ups necessary to let the corresponding exceptional component appear. We also give there our convention in representing dual graphs with $\bullet$ and $*$ and define *reduced dual graphs* to be the ones in which the self-intersections for components of the tangent cone have the minimum absolute value.

In Section 4, we give the description of generic polar curves on a resolution of a minimal singularity as proved by M. Spivakovsky (Thm. 4.2). This result will play the following somehow different roles in the sections following it:

[1] Hence the information about the semi-group of the branches is obvious.

(i) Section 5 explains how, using the full strength of this result, one may derive quite quickly a description of the generic discriminant $\Delta_{S,0}$ (more precisely, of Proposition 5.5 for the polar curve). This sums-up the note [5] in an improved way, and a mistake in an example there is corrected.
(ii) In Section 6, we mention how, using a result of J. Giraud, Theorem 4.2 also permits one, at least theoretically, to deduce the δ-invariant for the general polar curve from the shape of its transform on the minimal resolution. This result is however not useful for concrete computations, for which we use another approach in Section 11.
(iii) In Section 7, we get, as a purely qualitative consequence of (i) and (ii), a characterisation of generic polar curves on the minimal resolution of the singularity $(S, 0)$. This will be the application of Spivakovsky's result we will use in the proof of our main result.

Sections 8 to 11 form the core of the text:

• in Section 8, the polar curve for the tangent cone of a minimal singularity is made geometrically explicit and through the process of deformation onto the tangent cone is also seen as "part" of the polar curve of the singularity.
• in Section 9, we recall what we need from the limit tree construction of de Jong and van Straten. With this,
• in Section 10, we give, and prove, our main theorem giving more details about the information in Proposition $(*)$ page 92 using the limit tree construction and the contribution of the tangent cone.
• in Section 11, we show how a special deformation of minimal singularities has a nice interpretation in our description of polar curves and also gives a nice method for computing the delta invariant of these, completing the information in Def. 0.1 (i).

This leads us to ask: can (part of) the deformation theory of these minimal singularities of surfaces be recovered from their discriminants?

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Several years have passed since I wrote the first version of this paper, and as it turned out, it happened to be useful to other people: I am very grateful to A. Pichon for her interest in this work, for inviting me to submit the paper to this Journal, and to the referee for his/her remarks.

1. Polar invariants of a surface singularity

1.1. **The general polar curve as an analytic invariant.** We recall here the definition of the local polar variety of a germ of surface following [17]:

Let $(S, 0)$ be a complex surface singularity $(S, 0)$, embedded in $(\mathbb{C}^N, 0)$: for any $(N-2)$-dimensional vector subspace D of $\mathbb{C}^N$, we consider a linear projection $\mathbb{C}^N \to \mathbb{C}^2$ with kernel D and denote by $p_D : (S, 0) \to (\mathbb{C}^2, 0)$ the restriction of this projection to $(S, 0)$.

Restricting ourselves to the D such that p_D is finite, and considering a small representative S of the germ $(S, 0)$, we define, as in [17] (2.2.2), the *polar curve* $C(D)$ of the germ $(S, 0)$ for the direction D, as the closure in S of the critical locus of the restriction of p_D to $S \setminus \mathrm{Sing}(S)$. It is a reduced analytic curve.

As explained in loc. cit., it makes sense to say that for an open dense subset of the Grassmann manifold $G(N-2, N)$ of $(N-2)$-planes in $\mathbb{C}^N$, the space curves $C(D)$ are *equisingular*, e.g., in terms of Whitney-equisingularity (or strong simultaneous resolution, but this is the same for families of space curves, cf. [8]). We call this equisingularity class the *general polar curve* for $(S,0)$ embedded in $\mathbb{C}^N$.

One may then compare the general polar curves obtained by two distinct embeddings of the surface into a $(\mathbb{C}^N,0)$ and it turns out that they are still Whitney-equisingular; this is essentially proved in [23] (see p. 430) in a much more general setting (arbitrary dimension and "relative" polar varieties). Summing up, we have:

Theorem 1.1. *The Whitney equisingularity-type of the general polar curve $C(D)$ depends only on the analytic type of the germ $(S,0)$.*

In this paper, following, in a sense, the program in [24], we want to study this invariant $C(D)$ for a special class of surface singularities.

1.2. **The generic discriminant as a derived invariant.** With the same notation as before, we define the *discriminant* Δ_{p_D} as (the germ at 0 of) the reduced analytic curve of $(\mathbb{C}^2,0)$ which is the image of the polar curve $C(D)$ by the finite morphism p_D.

Again, one may show that, for a generic choice of D, the discriminants obtained are *equisingular germs of plane curves*, and that this in turn defines an analytic invariant of $(S,0)$.

We will denote $\Delta_{S,0}$ the equisingularity class of the discriminant of a generic projection of $(S,0)$.

A first advantage of $\Delta_{S,0}$, as a germ of a plane curve, is that its equisingularity class is well-defined in terms of classical invariants such as the Puiseux pairs of the branches and the intersection numbers between branches (cf. e.g., the introduction of [8] for references on this subject).

As it turns out, there is a very nice relationship between the general polar curve and $\Delta_{S,0}$. For this we recall the following:

Definition 1.2. Let $(X,0)\subset(\mathbb{C}^N,0)$ be a germ of reduced curve. Then a linear projection $p:\mathbb{C}^N\to\mathbb{C}^2$ is said to be *generic* with respect to $(X,0)$ if the kernel of p does not contain any limit of bisecants to X (cf. [8] for an explicit description of the cone $C_5(X,0)$ formed by the limits of bisecants to $(X,0)$). For future reference, we will write $BS(X,0)$ for this cone denoted $C_5(X,0)$ in [8]).

Then the equisingularity type of the germ of plane curve $(p(X),0)$ image of $(X,0)$ by such a generic projection is uniquely defined by the saturation of the ring $\mathcal{O}_{X,0}$ (cf. [8]).

We now state the following transversality result (proved for curves on surfaces of $\mathbb{C}^3$ in [9] Theorem 3.12 and in general as the "lemme-clé" in [23] V (1.2.2)) relating polar curves and discriminants:

Theorem 1.3. *Let $p_D:(S,0)\to(\mathbb{C}^2,0)$ be as above, and $C(D)\subset(S,0)\subset(\mathbb{C}^N,0)$ be the corresponding polar curve. Then there is an open dense subset U of $G(N-2,N)$ such that for $D\in U$ the restriction of p to $C(D)$ is generic in the sense of Definition 1.2.*

Definition 1.4. Let us define $P_{S,0}$ to be not the *Whitney-equisingularity* class of the general polar curve as in Thm. 1.1, but the *equisaturation class* of the general polar curve (which may be a smaller class). As we recalled after Definition 0.1, this class is precisely given by the constancy of the invariants there. Then, the foregoing Theorem 1.3 states that $\Delta_{S,0}$ is *the generic plane projection* of $P_{S,0}$.

As stated in the introduction, the goal of this work is to determine $P_{S,0}$ completely.

2. Definition of minimal singularities

We begin with a definition valid in any dimension (following [19] § 3.4):

Definition 2.1. We call a singularity $(X, 0)$ *minimal* if it is reduced, Cohen-Macaulay, and the multiplicity and embedding dimension of $(X, 0)$ fulfill:
i) $\mathrm{mult}_0 X = \mathrm{emdim}_0 X - \dim_0 X + 1$, and
ii) the tangent cone $C_{X,0}$ of X at 0 is reduced.

Considering normal surfaces, one has the following characterisation:

Theorem 2.2. *Minimal singularities of normal surfaces are precisely the rational surface singularities with reduced fundamental cycles (with the terminology of* [2]*).*

Condition (i) follows for any rational surface singularity from Artin's formulas for multiplicities and embedding dimension (cf. [2]). Condition (ii) follows from the fact that the fundamental cycle of rational singularities is also the cycle defined by the maximal ideal. Conversely, the fact that minimal normal singularities are rational is proved in [19] 3.4.9. The proof that "reduced tangent cone" implies "reduced fundamental cycle" is easy (after our Thm. 3.2 or see, e.g., [26] p. 245).

Taking $(S, 0)$ to be a normal surface singularity and $\pi : (X, E) \to (S, 0)$ to be the *minimal* resolution of the singularity, one associates as usual to the exceptional curve configuration $E = \pi^{-1}(0)$ a dual graph Γ where each irreducible component L_i in E is represented by a vertex and two vertices are connected by an edge if, and only if, the corresponding components intersect.

Each vertex x of Γ (we will frequently abuse notation and write $x \in \Gamma$) is given a weight $w(x)$ defined as:

$$w(x) := -L_x^2,$$

where L_x^2 is the self-intersection of the corresponding component L_x on X.

For any rational surface singularity, it is well-known that all the L_i are smooth rational curves and that Γ is a tree. But in general, it takes some computation to check whether a given tree is the dual tree of a rational singularity (cf. [2]).

On the other hand, one reads at first sight from the dual graph that a surface singularity is minimal (cf. [21] II 2.3):

Remark 2.3. Let Γ be any weighted graph. Then, it is the dual graph of resolution of a minimal singularity if, and only if, Γ is a tree and, for each vertex $x \in \Gamma$, one has the following inequality:

$$w(x) \geq v(x),$$

where $v(x)$ denotes the valence of x, i.e., the number of edges attached to x.

3. More about the dual graphs

In the representation of the dual graph Γ of a minimal singularities, we will distinguish between the vertices with $w(x) = v(x)$ and the others.

Notation 3.1. In representing the dual graphs of minimal singularities, we chose to represent with a $\bullet$ the vertices with $w(x) = v(x)$, so that there is no need to mention the weight above them.

On the other hand, we enumerate as $x_1, \ldots, x_k$ the vertices with $w(x_i) > v(x_i)$, and let them be denoted by $*$ on the graph. One should then mention the weights of the (x_i) to define the graph.

In this work, we will pay much attention to the minimal singularities with the property that, for all vertices x_i with $w(x_i) > v(x_i)$, one has, in fact, the equality $w(x_i) = v(x_i) + 1$.

Let us call *reduced* the graphs with this property; it is then clear that in representing these dual graphs, there is no longer any need to mention the weights.

For example, saying that the graph in Figure 1 is *reduced* amounts to saying that

$$w(x_1) = w(x_n) = 2 \text{ and } w(x_i) = 3 \text{ for } 1 < i < n$$

(and the vertices with $\bullet$'s all have weight two here).

FIGURE 1.

The geometrical meaning of this distinction between vertices comes from:

Theorem 3.2 (Tyurina, cf. [25])**.** *Let $(S, 0)$ be a rational surface singularity and let $\pi : (X, E) \to (S, 0)$ be its minimal resolution. Let $b : S_1 \to S$ be the blow-up of 0 in S.*

Then there is a morphism $r : X \to S_1$ such that $\pi = b \circ r$ and a component L_i of $E = \pi^{-1}(0)$ is contracted to a point by r if, and only if, the intersection $(L_i \cdot Z) = 0$, where Z is the fundamental cycle.

Of course, the components of E which are not contracted by r are the strict transform by r of the components of the $\mathbb{P}(C_{S,0})$ appearing on S_1.

When $(S, 0)$ is a minimal singularity, the fundamental cycle is $Z = \sum_{x \in \Gamma} L_x$ and hence, for a given vertex $y \in \Gamma$, the intersection $(L_y \cdot Z)$ is just $v(y) - w(y)$.

This should justify:

Definition 3.3. Let $(S, 0)$ be a minimal normal surface singularity and Γ be the dual graph of its minimal resolution.

We will say that a vertex x in Γ has *height one* if $w(x) > v(x)$, which from the foregoing remarks means that the corresponding component L_x corresponds to a component of (the proj of) the tangent cone $C_{S,0}$. Hence we will denote by $\Gamma_{TC} = \{x_1, \dots, , x_n\}$ the set of these vertices.

Then, we define the height of any vertex x in Γ as the number s_x defined by:

$$s_x := \operatorname{dist}(x, \Gamma_{TC}) + 1,$$

where dist is the distance on the graph (number of edges on the geodesic between two vertices).

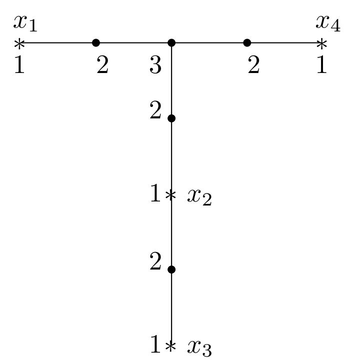
The reader should check that this height corresponds to the number of blow-ups necessary to make the corresponding component "appear".[2] The notation s_x here comes from [21] II 5.1 and was the one used in the previous work [5].

Example 3.4. As an example, we indicate the heights on the graph in Figure 2, where the (x_i) are, as before, the vertices of height one (with $*$'s):

We will also need the following:

[2]This latter notion is studied more systematically for any rational singularity as "desingularisation depth" in [18]; of course in this general case, it is not given directly from a distance!

FIGURE 2. Minimal graph with the heights for the vertices.



Definition 3.5. Let Γ be a minimal graph. The connected components Γ_i (for $i = 1, \ldots, r$) of $\Gamma \setminus \Gamma_{TC}$ are called the *Tyurina components* of Γ.

Hence, Theorem 3.2 states that the blow-up S_1 of $(S, 0)$ has exactly r singularities (S_1, O_i) which are minimal singularities with dual resolution graph Γ_i.

4. A result of Spivakovsky

To state this result, we introduce further terminology:

Let $\pi : (X, E) \to (S, 0)$ be the minimal resolution of the singularity $(S, 0)$, where $E = \pi^{-1}(0)$ is the exceptional divisor, with components L_i. A *cycle* will be by definition a divisor with support on E, i.e., a linear combination $\sum a_i L_i$ with $a_i \in \mathbb{Z}$ (or $a_i \in \mathbb{Q}$ for a $\mathbb{Q}$-cycle).

Let Γ be the dual graph of the minimal resolution π and, for each vertex x, let s_x denote the *height* defined in Def. 3.3.

Definition 4.1. Let then x, y be two adjacent vertices on Γ; the edge (x, y) in Γ is called a *central arc* if $s_x = s_y$. A vertex x is called a *central vertex* if there are at least two vertices y adjacent to x such that $s_y = s_x - 1$ (cf. [21]).

We then define a $\mathbb{Q}$-cycle Z_Ω on the minimal resolution X of $(S, 0)$ by:

$$Z_\Omega = \sum_{x \in \Gamma} s_x L_x - Z_K, \qquad (1)$$

where Γ is the dual graph, and Z_K is the numerically canonical $\mathbb{Q}$-cycle [3].

The theorem from [21] (Theorem 5.4) is now:

Theorem 4.2. *Let $(S, 0)$ be a* minimal *normal surface singularity. There is a open dense subset U' of the open set U of Theorem 1.3, such that, for all $D \in U'$, the strict transform $C'(D)$ of $C(D)$ on X:*
a) is a multi-germ of smooth curves intersecting each component L_x of E transversally in exactly $-Z_\Omega.L_x$ points,

[3]Uniquely defined by the condition that, for all $x \in \Gamma$, $Z_K.L_x = -2 - L_x^2$, since the intersection product on E is negative-definite.

b) goes through the point of intersection of L_x and L_y if and only if $s_x = s_y$ (point corresponding to a central arc *of the graph). Furthermore, the curves $C'(D)$, with $D \in U'$ do not share other common points (base points) and these base points are simple, i.e., the curves $C'(D)$ are separated when one blows up these points once.*

Referring to loc. cit. for unexplained terminology, let us make the following observation:

Remark 4.3. Blowing-up once the base points referred to in b) above, one gets a resolution X_N of the Nash blow-up of the germ $(S, 0)$. The map from X_N to the normalized Nash blow-up $N(S)$ is simply the contraction of the exceptional components which are not intersected by a branch of the generic polar curve.

5. First description of the polar curve and the discriminant

This section essentially describes the results obtained in [5] in an improved form. We refer to this note (Section 3) for the proofs of the following lemmas:

Lemma 5.1. *Let $(S, 0)$ be a* minimal *normal surface singularity and $\pi : X \to (S, 0)$ its minimal resolution. It is known that π is (the restriction to S of) a composition $\pi_1 \circ \cdots \circ \pi_r$ of point blow-ups. Then, this composition of blow-ups is also the minimal resolution of the generic polar curve $C(D)$ for $D \in U'$ as in Theorem 4.2.*

The following is a slightly more precise version of loc. cit. Lem. 3.2:

Lemma 5.2. *For $D \in U'$ as in Theorem 4.2, the polar curve $(C(D), 0)$ on $(S, 0)$ is a union of germs of curves of multiplicity two. In particular, it has only smooth branches and branches of multiplicity two, the latter being exactly those for which the strict transform goes through a central arc as in b) of Theorem 4.2.*

Let us now make a perhaps not so standard definition:

Definition 5.3. Let $(C_1, 0)$ and $(C_2, 0)$ be two analytically irreducible curve germs in $(\mathbb{C}^N, 0)$. We will *hereafter* call *contact* between the C_i simply the number of point blow-ups necessary to separate these two branches.

For the description of the polar curve, just recall that *an A_n-curve* is a curve analytically isomorphic to the plane curve defined by $x^2 + y^{n+1} = 0$:

Proposition 5.4. *Let $(S, 0)$ be a minimal surface singularity and $C = C(D)$ be a generic polar curve corresponding to D in the open set U' of thm. 4.2.*

Then, if $C = \cup_i C_i$ is the decomposition of C into analytic branches, denote by L_{C_i} the irreducible exceptional component on the minimal resolution of S which intersects the strict transform of C_i. It is unique except in the case of central arcs. In this case, just choose one between the two intersecting components. Then:

(i) *The contact between C_i and C_j in the sense of Def. 5.3 above is the minimum height in the chain between L_{C_i} and L_{C_j} (cf. Def. 3.3).*

(ii) *We may write rather C as a union of $C = \bigcup C_i$ of curves of multiplicity two by taking by pairs branches intersecting the same exceptional component on X that we will now denote L_{C_i}.*

Then, each C_i is a A_{n_i}-curve, where the number n_i equals $2.s(L_{C_i})$ if C_i goes through a central arc, and $2.s(L_{C_i}) - 1$ otherwise (which comprises the case of central vertices and components of height s equal to one).

We may obviously define the contact between these A_{n_i}-curves just by taking one branch in each, so that it is still given by (i).

Proof. The statement about the contact in (i) follows from Lemma 5.1. The first statement in (ii) is Lemma 5.2.

Any curve of multiplicity two is a A_n-curve, see, e.g., [4] p. 62. The statement about the n_i follows from (i) just like the statement about the contacts. □

The result in Prop. 5.4 gives a complete description of the equisingularity class of the discriminant plane curve in $(\mathbb{C}^2, 0)$ using Theorem 1.3 [4]:

Proposition 5.5. *The discriminant $\Delta_{p_D} = p_D(C(D))$ has exactly the same properties as the polar curve $C(D)$ in Prop. 5.4. This describes the generic discriminant $\Delta_{S,0}$ as a union of A_{n_i}-curves with the n_i and the contacts described in 5.4.*

Proof. The curves C_i in Prop. 5.4, being plane curves, are their own generic plane projections. Hence by Thm. 1.3, the image Δ_{p_D} of $C(D)$ by the *generic projection* p_D decomposes as the same union of A_{n_i}-curves.

We give here a direct argument to prove that the contact (in sense of Def. 5.3) between the branches in Δ_{p_D} is the same as the one in $C(D)$ (in [5], we invoked a bilipschitz invariance which is perhaps not obvious with our definition of contact).

Let us write down equations for a special case: considering a pair C_1, C_2 of branches in $C(D)$, we embed $C_1 \cup C_2$ into a $(\mathbb{C}^3, 0)$ and suppose there are coordinates so that C_1 is parameterised by $(x = t^{\varepsilon_1}, y = t^n, z = 0)$ and C_2 is parameterised by $(x = t^{\varepsilon_2}, y = 0, z = t^m)$, with $\varepsilon_i = 1$ or 2.

We then leave it to the reader to verify that the projection defined by $(x, y + z)$ is transverse to the cone of bisecants $BS(C, 0)$ of Def. 1.2 and that the contact in our sense is preserved.

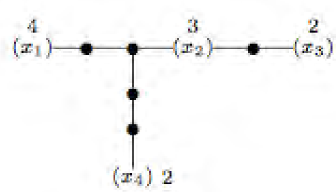
In the general case, the contact between C_1 and C_2 may be smaller, but the results remain valid with another parameterisation of C_2.

□

The foregoing description of the discriminant still involves the computation of the number of branches on each central vertex by Spivakovsky's formula. We will describe a much better and condensed one in Section 10, which does not involve any computation and is geometrically more significant. Before, the author would like to make amends to the readers of [5] for a mistake in the following:

Example 5.6 (Correct version of Example 1 in [5]). Consider $(S, 0)$ with dual graph Γ as in Figure 3, where, following the convention of Section 3, the $\bullet$ denote vertices with $w(x) = v(x)$, and the others form $\Gamma_{TC} = \{x_1, \ldots, x_4\}$ with the weights indicated on the graph.

FIGURE 3.



[4] This is an equivalent, but more simply expressed, version of the statement in [5], Cor. 4.3.

The branches of the polar curve going through the components of Γ_{TC} are just four branches going through L_{x_1}, which gives in the equisingularity class $\Delta_{S,0}$ four distinct lines through the origin with contact one with any other branch of $\Delta_{S,0}$.

Then, we have two central vertices (of heights 3 and 2) and a central arc (with boundaries of height 2), which give, respectively, an A_5, an A_3, and an A_4-curve from Prop. 5.4 and 5.5 above.

The contact between the A_5 and the A_4 is *two* (and not 3 as claimed in loc. cit.) and their contact with the other A_3 is one.

Hence, using coordinates, a representative of the equisingularity class of $\Delta_{S,0}$ can be chosen to be:

$$\underbrace{(x^4+y^4)}_{\text{The two } A_1\text{'s}}(x^2+y^5)\underbrace{(x+y^2+iy^3)(x+y^2-iy^3)}_{\text{The } A_5}(y^2+x^4)=0.$$

6. The delta invariant of the polar curve

Definition 6.1. Let $(C,0)$ be a germ of a reduced complex curve. Let $n:\overline{C}\to(C,0)$ be its normalisation map, which provides a finite inclusion of the local ring $\mathcal{O}_{C,0}$ into the semi-local ring $\mathcal{O}_{\overline{C}}$.

The δ-invariant of $(C,0)$ is by definition $\delta(C,0):=\dim_{\mathbb{C}}\mathcal{O}_{\overline{C}}/\mathcal{O}_{C,0}$.

In the paper [14], J. Giraud gives a way to compute $\delta(C,0)$ for any curve on a *rational* surface singularity $(S,0)$ if one knows a resolution of the surface singularity where the strict transform C' of C is a multi-germ of smooth curves.

To quote this result, we need the following lemma, proved in loc. cit. 3.6.2:

Lemma 6.2. *Let $p:(X,E)\to(S,0)$ be a resolution of a normal surface singularity $(S,0)$, with $E=\pi^{-1}(0)=\cup_i E_i$. Let $D=\sum_i a_iE_i$ be a $\mathbb{Q}$-cycle on X.*

There exists a unique $\mathbb{Z}$-cycle $V=\sum_i\alpha_iE_i$ with the property that the intersection $(V\cdot E_i)$ is less than or equal to $(D\cdot E_i)$ for all i, and is a minimum among cycles with this property.

This $\mathbb{Z}$-cycle will be denoted as $\lfloor D\rfloor$.

(In the previous lemma, "minimum" means that any other $\mathbb{Z}$-cycle with this property has the form $\lfloor D\rfloor+W$ with W a cycle with non-negative coefficients.)

In the situation of Lemma 6.2, let us associate to any curve $(C,0)\subset(S,0)$ a $\mathbb{Q}$-cycle Z_C uniquely defined by the condition that, for all irreducible component E_i of E, the intersection number $(E_i\cdot Z_C)$ equals $(E_i\cdot C')$, where C' denotes the strict transform of C on X. We may then quote loc. cit. Cor. (3.7.2):

Theorem 6.3. *Let $p:(X,E)\to(S,0)$ be a resolution of a* rational *surface singularity. Let $(C,0)$ be a germ of a reduced curve on $(S,0)$, such that, denoting by C' the strict transform of C on X, C' is a multi-germ of smooth curves on X.*

Then, using the $\mathbb{Q}$-cycle Z_C associated C in the way defined above, and letting

$$D_C:=Z_C+\lfloor -Z_C\rfloor,$$

one has the following formula[5]*:*

$$\delta(C,0)=-\frac{1}{2}(Z_C\cdot(Z_C+Z_K))+\frac{1}{2}(D_C\cdot(D_C+Z_K)).$$

[5] Beware that, in loc. cit., the + before the second term in the right hand-side of the corresponding formula (5) is not properly printed, yet it *is* a plus. One should also read formula (3) there as $\underline{D}:=e(D_s)-\lceil e(D_s)\rceil=e(D_s)+\lfloor -e(D_s)\rfloor$, which agrees with my definition for D_C.

Thanks to Spivakovsky's theorem 4.2, we may apply the foregoing to a general polar curve of a minimal singularity $(C, 0) \subset (S, 0)$, X the minimal resolution of $(S, 0)$, and $Z_C = -Z_\Omega$. As a corollary to these two theorems, we have:

Corollary 6.4. *Let $(S, 0)$ be a minimal singularity of a normal surface (hence, rational by Thm. 2.2). The δ-invariant of the generic polar curve is a topological invariant of $(S, 0)$, i.e., depends only on the data of the weighted resolution graph.*

Applying the formula in 6.3 to get δ for the polar curve in concrete cases leads to huge computations, except in some simple examples:

Example 6.5. Let $(S, 0)$ be the singularity at the vertex of the cone over a rational normal curve of degree n. It is the minimal singularity whose (dual) resolution graph has only one vertex of weight n. Assume that $n \geq 3$. Check that, using E to denote the irreducible exceptional divisor, one has $Z_\Omega = (2n-2)/nE$, $Z_K = -(n-2)/nE$, $\lfloor Z_\Omega \rfloor = 2E$ and hence $\delta(C, 0) = 3n - 6$.

In Section 8, we will obtain the result of the foregoing example (and more) from a geometric argument, with no use of the theorems above. The problem of computing δ for the general polar curve of any minimal singularity is solved in 11.4.

7. A characterisation of the generic polar curve in a resolution

As a consequence of the results of Sections 5 and 6, we get the following characterisation for generic polar curves on the minimal resolution of the surface:

Theorem 7.1. *Let $(S, 0)$ be a minimal normal surface singularity, and X the minimal resolution of $(S, 0)$. Let $C(D)$ be* any polar curve *of $(S, 0)$ with the property that its strict transform $C'(D)$ on X is exactly as depicted in Thm. 4.2.*

Then $C(D)$ is a generic polar curve $P_{S,0}$ *as defined in Def. 1.4, i.e., has the generic invariants defined in 0.1 of the introduction.*

Proof. The description of Prop. 5.4 rests only on the the shape of the polar curve in the resolution X, and gives in particular the datum (ii) in 0.1 (cf. Def 1.4 and Prop. 5.5). Giraud's theorem 6.3 gives the value of the delta invariant also from the data of the resolution. Considering the linear system of polar curves, our special polar curve is now equisingular in the sense of Def. 0.1 to the generic polar curve. $\square$

Remark 7.2. We explained in [6] how such characterisations of "general" curves on a resolution may be useful; here, it will be used in Rem. 10.6.

We also need the following inductive property for which we will use[6] the explicit form of the cycle Z_Ω in (1) before Spivakovsky's Thm. 4.2:

Proposition 7.3. *Let $(S, 0)$ be a minimal singularity of a normal surface, with dual resolution graph Γ. Let S_1 be the blow-up of $(S, 0)$ at 0 and O_i a singular point of S_1. Let $\Gamma_i \subset \Gamma$ be the Tyurina component corresponding to O_i as in Def. 3.5. Let Z_{Ω_i} be the cycle associated to Γ_i as Z_Ω is associated to Γ in Thm. 4.2.*

Then, for every vertex $x \in \Gamma_i$, the corresponding component L_x on X satisfies the following intersection property:

$$(Z_\Omega \cdot L_x) = (Z_{\Omega_i} \cdot L_x). \tag{2}$$

This means that the corresponding component L_x is intersected by exactly the same number of branches of the generic polar curve for $(S, 0)$ or for (S_1, O_i), and the central arcs in Γ_i are obviously also central arcs in Γ.

[6] Ideally, we would have liked not to do so; see precisely (a) of the proof of this proposition.

Proof. Although the assertion in (2) follows easily from the explicit form of the cycles Z_Ω and Z_{Ω_i} (cf. (1) p. 97), we distinguish between:

(a) the components L_x with $w(x) - v_{\Gamma_i}(x) \geq 2$. Since $x \in \Gamma_i$, $w(x) = v_\Gamma(x)$; hence the property in Γ_i implies that x is a central vertex in Γ. Hence L_x bears components of the strict transform of the general polar curve of $(S,0)$, and here we know no method other than computing to prove (2).

(b) the central components L_x in Γ_i (central vertex or boundary of a central arc). Then, it is also central in Γ, and we believe (2) should be understood without any reference to the cited formula, using the following remark in [21], p. 459 (first lines): "in the neighbourhood of L_x, $\tilde{\Omega}$ is generated by sections whose zero set is contained in the exceptional divisor near L_x". □

8. The contribution of the tangent cone in the polar curve

In Section 5, we said that $P_{S,0}$ was formed by A_n-curves. Here we explain how the bunches of A_1-curves arise, and will be more precise about their geometry.

8.1. Discriminant and polar curve for cones over Veronese curves.

Remark 8.1. Let $(S,0)$ be the singularity of the cone over the rational normal curve of degree m in $\mathbb{P}^m_\mathbb{C}$, whose dual graph has just one vertex, with weight m.

Denoting by P_m the polar curve for a generic projection of $(S,0)$ onto $(\mathbb{C}^2,0)$, it is just the cone over the critical set of the projection of the rational normal curve onto $\mathbb{P}^1_\mathbb{C}$, which is a set of $2m-2$ distinct points by the Hurwitz formula.

Hence we know that here P_m is given by $(2m-2)$ lines in $(\mathbb{C}^{m+1},0)$ with:
i) δ-invariant $3m-6$ as computed in Example 6.5, from Giraud's formula.
ii) obviously a set of $2m-2$ distinct lines in $(\mathbb{C}^2,0)$ as generic plane projection, denoted δ_m.

We can say more on the geometry of P_m in this case, and re-find the value of δ:

Lemma 8.2. *The general polar curve P_m of the singularity of a cone over a Veronese curve of degree $m \geq 3$ is a set of $(2m-2)$ lines in $(\mathbb{C}^{m+1},0)$, which has the generic (i.e, minimum) value of the δ-invariant for* any set *of $2m-2$ lines in $(\mathbb{C}^{m+1},0)$, and this value is $3m-6$.*

Proof. (a) We will denote by $V = v_m(\mathbb{P}^1)$ the rational normal curve of degree m in $\mathbb{P}^m_\mathbb{C}$ and by $G_p(m-2,m)$ the Grassmann manifold of subspaces of codimension two in this $\mathbb{P}^m_\mathbb{C}$, and consider the map

$$G_p(m-2,m) \to \mathrm{Hilb}_V^{2m-2}$$

onto the Hilbert scheme parameterising the set of $2m-2$-points in V, which assign to each Λ the critical subscheme of the projection along Λ.

Using a result of H. Flenner and M. Manaresi (in [11] 3.3-3.5), this map is generically finite, and since both spaces have dimension $2m-2$ and the target space is irreducible, the image of this map is dense.

(b) Now, from a result of G.M. Greuel in [13] (3.3), a set of r-lines through the origin in $\mathbb{C}^{m+1}$, corresponding to a set $p_1,\dots,p_r$ of points in $\mathbb{P}^m_\mathbb{C}$, has the generic δ invariant, if for all d in some bounded set of integers, their images $v_d(p_1),\dots,v_d(p_r)$ by the corresponding Veronese embedding $v_d : \mathbb{P}^m_\mathbb{C} \to \mathbb{P}^{N_d}_\mathbb{C}$ span a projective space of maximal dimension.

If we take $V \subset \mathbb{P}^m_\mathbb{C}$ to be a Veronese curve, one may always find such generic sets of points on V since by composing the Veronese embeddings in Greuel's condition with the Veronese embedding defining V, this amounts to a genericity condition for points in $\mathbb{P}^1_\mathbb{C}$.

Hence, there is a non-trivial open subset $U \subset \mathrm{Hilb}_V^{2m-2}$ with the property that the cone over this set of points has the minimum delta invariant. Applying (a) gives that these points actually occur as critical locus.

(c) A formula for the delta invariant for such a generic configuration of r lines in $\mathbb{C}^n$ is given by Greuel in loc. cit. We leave it to the reader to check that it gives $3m-6$ in our situation. □

8.2. **Geometry of the tangent cone of a minimal singularity.**

Remark 8.3. Let $(S,0)$ be a minimal normal surface singularity with embedding dimension N, and $C_{S,0}$ be its tangent cone in $(\mathbb{C}^N,0)$.

Then, if $\mathbb{P}:\mathbb{C}^N\setminus\{0\}\to\mathbb{P}^{N-1}_{\mathbb{C}}$ denotes the standard projection, the projective curve $\mathbb{P}(C_{S,0})$ is a connected, non-degenerate[7] curve of minimal degree in $\mathbb{P}^{N-1}_{\mathbb{C}}$. Indeed, condition (i) in Def. 2.1 immediately passes to $\mathbb{P}(C_{S,0})$.

It then follows by a standard argument (cf., e.g., [3], p. 67–68) that each of its irreducible components is a rational normal curve of a linear subspace of $\mathbb{P}^{N-1}_{\mathbb{C}}$.

Let Γ be the dual graph of the minimal resolution of $(S,0)$. From Tyurina's Thm. 3.2 and the remarks following it, an irreducible component L_{x_i} of $\mathbb{P}(C_{S,0})$ corresponds to a vertex x_i with $w(x_i)>v(x_i)$ in Γ and it is easy to compute that the degree $m(x_i)$ of the rational normal curve L_{x_i} is precisely $w(x_i)-v(x_i)$.

Conclusion 8.4. Hence the tangent cone $C_{S,0}$ is embedded in $(\mathbb{C}^N,0)$ as a union of cones over rational normal curves of degree m_i intersecting along singular lines.

8.3. **Scheme-theoretic critical spaces and discriminants.** To study deformations of polar curves and discriminants, we need a scheme-theoretic definition for these objects, introduced by B. Teissier in [22] through the use of Fitting ideals.

Further, when non-isolated singularities occur, the right objects for deformations are not polar curves but critical spaces, which also contain the singular locus of the surface.

Definition 8.5. We call $C^F(S,0)$ the *critical space* of a generic projection p of a surface $(S,0)$ onto $(\mathbb{C}^2,0)$ as defined by the Fitting ideal $F_0(\Omega_p)$ in $\mathcal{O}_{S,O}$ and $\Delta^F_{S,0}$ its image as defined by $F_0(p_*(\mathcal{O}_{C^F(S,0)}))$ in $\mathcal{O}_{\mathbb{C}^2,0}$.

Beware, $C^F(S,0)$ always contains $\mathrm{Sing}(S)$, which, if $\mathrm{Sing}(S)$ is not reduced to $\{0\}$, makes $C^F(S,0)$ even set-theoretically bigger than the polar curve $P_{S,0}$ defined in Section 1. But by a Bertini type theorem, one gets that:

Remark 8.6. (i) For a generic projection p of any reduced surface $(S,0)$, the intersection of the $C^F(S,0)$ with $S\setminus\mathrm{Sing}(S)$ is reduced, and hence the divisorial part $\operatorname{div} C^F(S,0)$ is formed of the generic polar curve $P_{S,0}$ and of (possibly non reduced) components of $\mathrm{Sing}(S)$. The same is true for $\operatorname{div}\Delta^F_{S,0}$.

(ii) In particular, if $(S,0)$ is an isolated singularity $\operatorname{div} C^F(S,0)$ and $\operatorname{div}\Delta^F_{S,0}$ coincide with the $P_{S,0}$ and $\Delta_{S,0}$ defined in Section 1.[8]

Lemma 8.7. *Let $(S,0)$ be a minimal normal surface singularity, with tangent cone $C_{S,0}$, and Γ the dual graph of the minimal resolution of S. Recall that we then denote Γ_{TC} the set of vertices x_i in Γ with $w(x_i)>v(x_i)$.*

[7]This means not contained in a hyperplane of $\mathbb{P}^{N-1}_{\mathbb{C}}$.

[8]But as explicitly proved in [7] 3.5.2, the Fitting critical curves and discriminants for minimal singularities do have embedded components as soon as the multiplicity is bigger than 3.

Here, we denote by $m(x_i)$ the difference $w(x_i)-v(x_i)$, and we have just seen that $C_{S,0}$ is made of cones over rational normal curves of degree $m(x_i)$ intersecting along singular lines. Hence, considering a projection of $\mathbb{C}^N$ onto $\mathbb{C}^2$ restricted to $C_{S,0}$ we get that:

$$\operatorname{div} C^F_{C_{S,0}} = \bigcup_{x_i \in \Gamma_{TC}} P_{m(x_i)} \cup \textit{the singular lines in } C_{S,0} \textit{ with some multiplicity,}$$

and

$$\operatorname{div} \Delta^F_{C_{S,0}} = \bigcup_{x_i \in \Gamma_{TC}} \delta_{m(x_i)} \cup \textit{non reduced lines,}$$

where the P_m and δ_m were defined in Rem. 8.1 and Lem. 8.2.

8.4. **Deformations of polar curves and discriminants.** We first recall what we need from the construction of the deformation of $(S,0)$ onto its tangent cone $C_{S,0}$, as described in [12], Chap. 5: let M be the blow-up of $(0,0)$ in $S \times \mathbb{C}$, and $\rho : M \to \mathbb{C}$ the flat map induced by composing the blow-up map with the second projection. One then shows that: for all $t \neq 0$, the fiber $M_t := \rho^{-1}(t)$ is isomorphic to S and M_0 is the sum of the two divisors on M, namely $\mathbb{P}(C_{S,0} \oplus 1) + S_1$, where S_1 stands for the blow-up of S in 0. To this deformation, we will apply the following:

Proposition 8.8. *Let $\rho : X \to \mathbb{D}$ be a flat map, with a section σ so that the germs $(X_t, \sigma(t))$ are isolated singularities for $t \neq 0$, X_0 is a reduced possibly non-isolated singularity, and $\dim X_t$ is two for all t.*

Then, reducing the disk $\mathbb{D}$, one may find a projection $p : X \to \mathbb{C}^2 \times \mathbb{D}$ compatible with ρ so that, for all $t \in \mathbb{D}$, the polar curve of $p_t : X_t \to \mathbb{C}^2 \times \{t\}$ is generic, and its image is also the generic discriminant $\Delta_{X_t,0}$ as defined in Section 1.

The proposition above is well-known to specialists and may be deduced from more general results (see also [1], Th. 3.1).

Applying the proposition to the foregoing deformation $\rho : M \to \mathbb{D}$ gives that $P^F_{S,0}$ deforms onto $P^F_{C_{S,0}}$ and the same statement for the Fitting discriminants. The description of the generically reduced branches of $P^F_{C_{S,0}}$ in Lem. 8.7 now implies:

Corollary 8.9. *Let $(S,0)$ be a minimal singularity, with notation as in Lem. 8.7, and let us denote by L_{x_i} the component of $\mathbb{P}(C_{S,O})$ corresponding to $x_i \in \Gamma_{TC}$. Then:*
(i) *The generic polar curve $P_{S,0}$ of $(S,0)$ contains a union:*

$$P_{TC} = \bigcup_{x_i \in \Gamma_{TC}} P_{m(x_i)}$$

of generic configuration of lines $P_{m(i)}$ as described in Lem. 8.2. The bunch $P_{m(x_i)}$ in $P_{TC} \subset P_{S,0}$ is by definition the set of branches of $P_{S,0}$ which are deformed onto the (scheme-theoretically) smooth branches $P_{m(x_i)} \subset P^F_{C_{S,O}}$ of Lem. 8.7.
(ii) *The same statement is true for the generic discriminant $\Delta_{S,0}$ of $(S,0)$:*

writing $\Delta_{TC} = \cup_{x_i \in \Gamma_{TC}} \delta_{m(x_i)}$, with $\delta_{m(x_i)}$ standing for $2m(x_i) - 2$ lines in $(\mathbb{C}^2, 0)$, we may just as well say that these smooth branches with pairwise distinct tangents just form a Δ_{TC} part in $\Delta_{S,0}$.
(iii) *Denote by S_1 the blow-up of $(S,0)$. The strict transforms on S_1 of the smooth curves in $P_{m(x_i)} \subset P_{S,0}$ intersect the exceptional divisor only in L_{x_i} and this intersection is transverse.*

Proof. (i) A curve deforming onto a smooth curve is certainly smooth, hence locally a line. In Lem. 8.2, we said the P_m-curves are characterised by the minimality of δ. By semi-continuity of this δ applied to the family deforming onto $P_{m(x_i)} \subset P_{C_{S,O}}$, we get the full conclusion for the curves in $P_{S,0}$. (ii) is follows directly from (i).

(iii) Let us denote by ρ the deformation onto the tangent cone as recalled at the beginning of Section 8.4. The fiber $\rho^{-1}(0)$ contains the blow-up S_1 of $(S,0)$ intersecting $\mathbb{P}(C_{S,O} \oplus 1)$ in $\mathbb{P}(C_{S,O})$. Since the lines $P_{m(x_i)}$ in $C_{S,O}$ are transverse to the Veronese curve L_{x_i} in the $\mathbb{P}(C_{S,O})$ at infinity, it also follows that the strict transforms of the curves in $P_{m(x_i)} \subset P_{S,0}$ are transverse to the corresponding exceptional component $L_{x_i} \subset \mathbb{P}(C_{S,O})$ on the blow-up S_1. ☐

9. Limit trees

We proceed to identify the remaining part in $P_{S,0}$ besides the P_{TC}-part just exhibited. The following *limit tree* construction introduced by T. de Jong and D. van Straten in [15] will turn out to be very relevant to this description. Precisely, using the height function we defined in 3.3, one finds in loc. cit. (1.13):

Definition 9.1. Let Γ be the dual graph of a minimal resolution of a minimal singularity of a normal surface. A *limit equivalence relation* $\sim$ is an equivalence relation on the vertices of Γ satisfying the following two conditions:

(a) Vertices x with height $s_x = 1$, i.e., with $w(x) > v(x)$, belong to different equivalence classes,

(b) for every vertex x in Γ with height $s_x = k+1$, $k \geq 1$, there is exactly *one vertex* y connected to x with height $s_y = k$ and $y \sim x$.

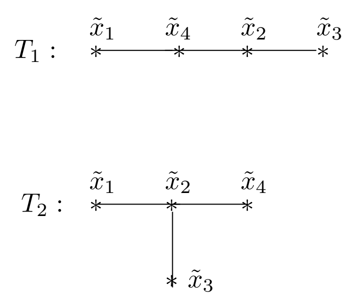
Then, the tree $T = \Gamma / \sim$ is a called a *limit tree* associated to Γ.

It is clear that any equivalence class contains exactly one vertex x_i of height one, so that we denote these equivalences classes as vertices $\widetilde{x_i}$ in T.

In fact, we only make this construction in the particular case of minimal singularities with *reduced graphs* in the sense of notation 3.1, so that the definition above really correspond to the definition in loc. cit.[9]

Starting with Γ as in Example 3.4 , one may associate non-isomorphic limit trees to the same reduced graph Γ, depending on the equivalence classes chosen, namely:

FIGURE 4. Two distinct limit trees for the dual graph in Figure 2.



Notation 9.2. For any pair x, y of vertices on the dual graph Γ, we denote by $C(x,y)$ the (minimal) chain on Γ joigning them (including the end points). This is unique since Γ is a tree.

[9] For the non-reduced case, one has to use the extended resolution graph of loc. cit. to build the limit tree, to really get a bijection between vertices of T and element of the set $\mathcal{H}$ considered in loc. cit. But, again, we will not use this.

We define the length $l(x,y)$ to be the number of vertices on $C(x,y)$ and the overlap $\rho(x,y;z)$ as the number of vertices on $C(x,z)\cap C(y,z)$.

As in [15], we attach to a limit tree T the following data:

- for any edge $(\tilde{x},\tilde{y})$ of T, the length $l(x,y)$, where x,y are the corresponding vertices of height one in the resolution graph Γ,
- for any pair of adjacent edges $(\tilde{x},\tilde{z})$ and $(\tilde{z},\tilde{y})$ in T, the overlap $\rho(x,y;z)$.

We use the notation (T,l,ρ) for the data above. In loc. cit. Lemma (1.16), it is shown that these data determine uniquely the resolution graph Γ.

10. Description of the polar curve using the limit tree

The following is our main result; we formulate it for the polar curve $P_{S,0}$, reminding the reader that this implies the analogous statements for the discriminant $\Delta_{S,0}$:

Theorem 10.1. *Let $(S,0)$ be a minimal singularity of a normal surface. Let Γ be the dual graph of the minimal resolution of S.*

Let Γ^r be the reduced graph associated to Γ in the sense of notation 3.1, i.e., the same graph with the weights of the x_i of height one reduced to $v(x_i)+1$, and let $(S^r,0)$ be a minimal singularity with dual resolution graph Γ^r.

Then the generic polar curve $P_{S,0}$ decomposes into:

$$P_{S,0}=P_{TC}\cup P_{S^r}$$

where the contact between any line in P_{TC} and any branch in $P_{S^r,0}$ is one and P_{TC} was described in Cor. 8.9 as the "contribution of the tangent cone".

Let T be the limit tree for Γ^r, as defined in Section 9 and (T,l,ρ) the set of data (length and overlap) associated to it at the end of that section.

These data give the following easy description of P_{S^r} (as a union of A_n-curves):

- *each edge $(\tilde{x_i},\tilde{x_j})$ in the limit tree T defines exactly one $A_{l_{i,j}}$-curve in P_{S^r}, where $l_{i,j}$ stands for $l(\tilde{x_i},\tilde{x_j})$.*
- *For each pair of adjacent edges $(\tilde{x_i},\tilde{x_j})$ and $(\tilde{x_j},\tilde{x_k})$, the contact (Def. 5.3) between the corresponding $A_{l_{i,j}}$ and $A_{l_{j,k}}$-curves in P_{S^r} is exactly the overlap $\rho(i,k;j)$.*
- *For non adjacent edges $(\tilde{x_i},\tilde{x_j})$ and $(\tilde{x_k},\tilde{x_l})$, the contact between the corresponding $A_{l_{i,j}}$ and $A_{l_{k,l}}$-curves in P_{S^r} is the* minimum *of the contacts between adjacent edges on the chain joining them.*

Let us first illustrate this on the following:

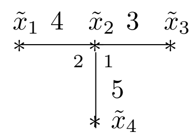
Example 10.2. (i) For a minimal singularity $(S,0)$ with dual graph as in Figure 2, p. 97, using any of the limit trees in Figure 4, we get: $P_{S,0}=A_5\cup A_5'\cup A_3$, with contact three between the two A_5 and contact one between the A_5's and the A_3.
(ii) For Example 5.6, the description of the discriminant was already given there. It is now more directly seen from the limit tree in Figure 5 given below together with the data (l,ρ), where the lengths l are put above the edges and the ρ as smaller numbers in-between a pair of edges (following the same convention as in [15] (1.19)).

The rest of this section is devoted to the proof of Thm. 10.1 above.

First we recall the following well-known:

Lemma 10.3. *Let $(S,0)$ be a minimal singularity of a normal surface and m be the multiplicity of $(S,0)$. Then the multiplicity of the generic polar curve (resp. discriminant) is $2m-2$.*

FIGURE 5. Limit tree for the reduced graph associated to the graph in Example 5.6



Proof. This is easily deduced from the following two facts (we refer, e.g., to [7] (3.9) and § 5):
(a) for any normal surface $(S, 0)$ and any projection $p : S \to \mathbb{C}^2$ whose degree equals the multiplicity m of the surface, the multiplicity of the discriminant Δ_p is $m + \mu - 1$, where μ is the Milnor number of a generic hyperplane section of $(S, 0)$.
(b) When $(S, 0)$ is minimal, $\mu = m - 1$. ☐

The proof of Thm. 10.1 is by induction on the maximal height of the vertices in Γ:

A) Initial step – The maximal height in Γ is one. We prove the result by a direct argument (independent from Spivakovsky's theorem). Now, all the vertices x_i in Γ are in Γ_{TC}, and the minimal resolution X of $(S, 0)$ is the first blow-up.
(a)We know from the deformation onto the tangent cone that each exceptional component E_{x_i} bears the strict transform of $(2n_i - 2)$ smooth branches of the polar curve, cutting E_{x_i} transversely at general points, with $n_i = w_i - v_i$ (cf. 8.9 (iii)).
(b) A general theorem of J. Snoussi ([20], Thm. 6.6), valid for any normal surface singularity, describes the base points of the linear system of polar curves on the first normalized blow-up of $(S, 0)$. In our situation, the blow-up is already normal and even smooth, and hence Snoussi's theorem implies that here the bases points are exactly the singular points of the exceptional divisor, i.e., the intersection points of two components E_{x_i} and E_{x_j}.

Let N be the number of vertices in Γ; then Γ has $N - 1$ edges (it is a tree), which represent the intersections points of exceptional components.

By Bertini's Theorem, the part of the generic polar curve $P_{S,0}$ whose strict transform goes through a base point is singular, i.e., has multiplicity at least two.

Hence, adding the contributions of the smooth branches in (a) and the singular curves in (b), the multiplicity $m(P_{S,0}, 0)$ of the polar curve satisfies the inequality:

$$m(P_{S,0}, 0) \geq \sum_{i=1}^{N} (2n_i - 2) + 2(N - 1). \tag{3}$$

Comparing this to the equality $m(P_{S,0}, 0) = 2m - 2$ of Lemma 10.3 above, where the multiplicity m of $(S, 0)$ equals the $\sum_{i=1}^{N} n_i$, proves that (3) is in fact an equality.

Hence, each point of intersection of two exceptional components bears the strict transform of a curve of multiplicity exactly two on $(S, 0)$. We now claim that the curve in question is a A_2-curve singularity on $(S, 0)$. Let C be such a curve.

Then, the multiplicity $m(C, 0) = 2$ is the intersection number of C with a generic hyperplane section of $(S, 0)$. This intersection number may be computed on X as the intersection number of the strict transform C' with the reduced exceptional divisor (which is the cycle defined by the maximal ideal of $(S, 0)$). Since we know C' intersects two exceptional components, the intersection of C' with each one should be transverse.

Hence C is a branch of multiplicity two resolved in one blow-up, i.e., an A_2-curve. This completes the proof of the initial step A.

B) The induction step – We use first the following general lemma in [7], 6.1:

Lemma 10.4. *Let $(S,0)$ be any normal surface singularity and $p:(S,0)\to(\mathbb{C}^2,0)$ any projection with degree equal to the multiplicity $\nu = m(S,0)$.*

Then, denoting by $b_0:\widetilde{\mathbb{C}^2}\to(\mathbb{C}^2,0)$ the blow-up of the origin, and by Σ_1 the analytic fiber product of b_0 and p above $(\mathbb{C}^2,0)$, one proves that the normalisation of Σ_1 coincides with the normalised blow-up $\overline{S_1}$ of $(S,0)$, which yields the following commutative diagram:

$$\begin{tikzcd}
\overline{S_1} \arrow[dr, "n"] \arrow[drr, bend left, "\overline{b_1}"] \arrow[ddr, bend right, "\varphi_1"'] & & \\
& \Sigma_1 \arrow[r] \arrow[d, "p_1"] & (S,0) \arrow[d, "p"] \\
& \widetilde{\mathbb{C}^2} \arrow[r, "b_0"] & (\mathbb{C}^2,0)
\end{tikzcd}$$

where $\varphi_1:\overline{S_1}\to\widetilde{\mathbb{C}^2}$ is the composition of the pulled-back projection p_1 with the normalisation n.

The following formula can then be obtained for the discriminant Δ_{φ_1}:

$$\Delta_{\varphi_1} = (\Delta_p)' + (\nu - r)E, \tag{4}$$

where $(\Delta_p)'$ is the strict transform of the discriminant of p, E denotes the reduced exceptional divisor, ν is the multiplicity of the germ $(S,0)$ and r the number of branches of a general hyperplane section of $(S,0)$.

We refer to loc. cit. for the proof, we just make precise that the discriminants in the equality of the lemma are the divisorial parts of Fitting discriminants as defined in Section 8.3, which are hence allowed to have non-reduced components.

Here, $(S,0)$ being a minimal singularity, the blow-up S_1 is already normal (cf. e.g., [7], Thm. 5.9), so that $\overline{S_1}$ is just S_1. The generic projection that we consider certainly fulfills the property $\deg p = m(S,0)$ as a necessary condition. Since the general hyperplane section of a minimal singularity of multiplicity ν is just ν lines (cf., e.g., loc. cit., lem. 5.4), formula (4) in the above lemma simply reads:

$$\Delta_{\varphi_1} = (\Delta_p)',$$

and similarly, denoting by $C(D)$ the polar curve of the projection p, $C'(D)$ its strict transform on S_1 and C_{φ_1} the polar curve for the projection φ_1, we get:

$$C'(D) = C_{\varphi_1}. \tag{5}$$

From Thm. 3.2 (see also Def. 3.5), we know that the singularities O_i of S_1 are minimal singularities whose resolution graphs are the Tyurina components Γ_i.

Localising the result of (5) in O_i yields the following:

Conclusion 10.5. Let $C(D)$ be a generic polar curve for $(S,0)$ and $C'(D)$ its strict transform on the blow-up S_1 of S at 0. Let O_i be a singular point of S_1. We proved that the part of $C(D)'$ going through O_i is the polar curve for the projection φ_1 obtained of the germ (S_1,O_i) onto a plane, as in the lemma above.

Remark 10.6. To apply induction, we need to know that the projection

$$\varphi_1:(S_1,O_i)\to(\mathbb{C}^2,0),$$

in question is generic, i.e., has the generic polar curve.

Counting multiplicity as in A) gives that this projection has degree equal to the multiplicity of (S_1, O_i), but this is not enough to prove that the polar curve is generic. This will be proved thanks to the results of Section 7.

Indeed, once we know from Conclusion 10.5 that $C'(D)$ is a polar curve for (S_1, O_i), we may use Prop. 7.3 to see that the strict transform of $C'(D)$ on X, which is also part of the strict transform of $C(D)$, actually fulfills the conditions of the characterisation in Thm. 7.1. Then:

Conclusion 10.7. With the same notation as in Conclusion 10.5, the part of $C'(D)$ going through O_i is the *generic polar curve* P_{S_1,O_i} for the germ (S_1, O_i).

Now, the induction hypothesis applied to each (S_1, O_i) yields that P_{S_1,O_i} is a union of A_n-curves described by a limit tree T_i for Γ_i as stated in Theorem 10.1.

C) Reconstructing $P_{S,0}$ from its strict transform

Let $(S, 0)$ be a minimal surface singularity and let S_1 be its blow-up, and E the exceptional divisor with components $E_1, \ldots, E_r$. We will denote by $O_1, \ldots, O_s$ the singular points of S_1 and by $Q_1, \ldots, Q_t$ the points of intersection of components of E which are not singular points of S_1.

We already know that the generic polar curve $P_{S,0}$ of $(S, 0)$ is precisely made of:

(1) A_1-curves in number $\sum_{i=1}^{r}(m_i-1)$, whose strict transforms intersect each E_i as $(2m_i-2)$ lines going through generic points of E_i, for $i = 1, \ldots, r$,
(2) A_2-curves singularities in number t, each one having its strict transform on S_1 intersecting a different point Q_i defined above,
(3) curves whose strict transforms go through the singular points O_i of S_1.

The first two points are proved by the same reasoning as in step A). Step B) applied to curves in (3) for each O_i gives the description of the strict transforms of these curves as A_n-curves described by the limit tree T_i associated to (S_1, O_i).

The corresponding description, for all the curves in (3) whose strict transforms go through the same O_i, on $(S, 0)$ itself, is then obtained by adding 2 to all the n's and one to the ρ by elementary properties of these A_n-curves and our Def. 5.3 of the contact.

But now from [15] (1.18), we know that the data associated to limit trees T_i of Γ_i are related to T exactly the same way (length:= length-2, overlap := overlap -1).

This completes the proof by induction for the first two points of Theorem 10.1, the last point follows by definition of the contact.

11. Scott deformations and polar invariants

The following was first proved by de Jong and van Straten in [15] Thm. 2.13:

Theorem 11.1. *Let $(S, 0)$ be a minimal singularity of a normal surface with multiplicity m. Let S_1 be the blow-up of 0 in S, with singular points $O_1, \ldots, O_r$. Then there exists a one-parameter deformation $\rho : X \to \mathbb{D}$ of $(S, 0)$ on the Artin component such that X_s for $s \neq 0$ has $r + 1$ singular points isomorphic respectively to the (S_1, O_i) for $i = 1, \ldots, r$ and to the cone over the rational normal curve of degree m.*

This has to be compared to a standard result for plane curves, attributed to C. A. Scott in [16], where a proof is also given (see p. 460):

Lemma 11.2. *Let $(C, 0) \in (\mathbb{C}^2, 0)$ be a plane curve singularity of multiplicity m. Let O_i for $i = 1, \ldots, r$ be the singularities of the first blow-up C_1 of $(C, 0)$. Then there exists a one-parameter δ-constant deformation Γ of $(C, 0)$ such that Γ_s for $s \neq 0$ is a plane curve which has $r + 1$ singular points isomorphic respectively to the (C_1, O_i) for $i = 1, \ldots, r$ and to an ordinary m-tuple point.*

Beyond the formal analogy between Thm. 11.1 and Lem. 11.2, de Jong and van Straten prove the result in Thm. 11.1 for the more general class of *sandwiched singularities* as a consequence of their theory of *decorated curves*: all the deformations of these surface singularities can be obtained from deformations of *decorated curves* associated to the singularity. In particular, the Scott deformation of a decorated curve (conveniently adjusted) gives rise to the deformation in Thm. 11.1.

As an application of our description for generic discriminants in Thm. 10.1, however, we get first a new relation between these two deformations:

Corollary 11.3. *Let the notation be the same as in Thm. 11.1. We will also call the deformation X the* Scott deformation *of the surface* $(S, 0)$.

Considering a projection p of X in $\mathbb{D} \times C^2$ as in Prop. 8.8, i.e., compatible with ρ and such that the discriminant $\Delta(p_t : X_t \to \mathbb{C}^2)$ is the generic discriminant Δ_t for all the singularities in X_t, for all t, one gets a deformation $\rho' : \Delta \to \mathbb{D}$ of the generic discriminant $\Delta_{S,0}$ of $(S, 0)$, which is exactly the Scott deformation of this curve as defined in Lem. 11.2.

Proof. In our proof in Section 10, it is proved that the discriminant of (S_1, O_i) is the part of the strict transform of the discriminant of $(S, 0)$ going through the image of O_i in the plane (it is of course also obvious from the result there).

The discriminant of the cone over the m-th Veronese curve is a $2m-2$-tuple ordinary point in the plane (cf. Rem. 8.1). This is indeed the last singularity occurring in the Scott deformation of $\Delta_{S,0}$, since, by Lem. 10.3, the multiplicity of the $\Delta_{S,0}$ is $2m - 2$. □

Considering polar curves in this Scott deformation, we get the more interesting:

Theorem 11.4. *Let the notation be as in Cor. 11.3. Then, the polar curve for $p_t : X_t \to (\mathbb{C}^2, 0)$ is also the generic polar curve P_{X_t} (which is a multi-germ of space curves for $t \neq 0$). Further, P_{X_t} is a δ-constant deformation of the generic polar curve $P_{S,0}$. Hence iterating Scott deformations, one may compute the δ-invariant of $P_{S,0}$ as sum of δ-invariants for sets of generic lines P_m as in Lem. 8.2.*

Proof. In Theorem 11.1, the deformation X_t is said to belong to the Artin-component of $(S, 0)$. This means that it has a simultaneous resolution, in which (cf. Lem 5.1) the P_{X_t} are also resolved. One then has a *normalisation in family* for the family P_{X_t}, which is equivalent to "δ-constant" (cf. [22], p. 609). □

We illustrate the second statement in Thm 11.4 by giving:

Example 11.5. Taking a singularity with graph as in Figure 6, and applying twice the Scott deformation of the surface, one gets two cones over a cubic and two cones over a conic. Hence the polar curve deforms onto two P_3's and two P_2's (in the notation of Lem. 8.2), which gives 8 for the δ-invariant.[10]

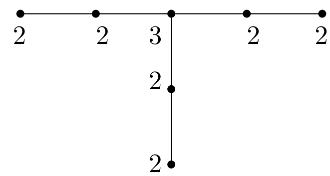
Let us end this with the following:

Remark 11.6. The information on (the resolution graph of) $(S, 0)$ given by the generic discriminant $\Delta_{S,0}$ is of course partial: e.g., one may permute the Tyurina components in the resolution graph of $(S, 0)$ or the weights on the tangent cone without changing $\Delta_{S,0}$. However, when one looks at deformations on $(S, 0)$, we believe the information on the discriminant is most valuable:

(a) As a very basic occurence of this: a family of normal surfaces S_t with constant generic discriminant $\Delta_{S_t,0}$ is Whitney-equisingular and in particular has constant topological type (encoded by the minimal resolution graph). As a consequence of our result, these three equisingularity

[10] Beware that $\delta(P_2) = 1$ is not given by the formula $\delta(P_n) = 3n - 6$, valid for $n \geq 3$.

FIGURE 6. Graph with weights on the vertices for example 11.5



conditions are in fact equivalent for minimal singularities of surfaces (see also [1], Th. 3.6. and Cor. 4.3).

(b) Much more generally, one can deform the discriminant $\Delta_{S,0}$ and ask which deformation of $(S,0)$ "lies above" the curve-deformation: for example, can one deduce the existence of the Scott deformation of the surface $(S,0)$ in the sense of Thm. 11.1 as deformation "lying above" the Scott deformation of $\Delta_{S,0}$?

This would give a description of some deformation theory of the surface through an invariant which, as opposed to the birational join construction of Spivakovsky or the decorated tree construction of De Jong and Van Straten, is uniquely defined from $(S,0)$.

References

[1] E. D. Akeke, Equisingular generic discriminants and Whitney conditions, Ann. Fac. Sci. Toulouse 17, 661-671, (2008). DOI: 10.5802/afst.1197

[2] M. Artin, On isolated rational singularities of surfaces, Amer. J. Math 88, 129–136, (1966). DOI: 10.2307/2373050

[3] M. Artin, Deformations of singularities, Tata Institute of Fund. Research, Bombay, (1976).

[4] W. Barth, C. Peters, A. Van de Ven, Compact complex surfaces, Ergeb. der Math. 4, Springer Verlag, (1984). DOI: 10.1007/978-3-642-96754-2

[5] R. Bondil, Discriminant of a generic projection of a minimal normal surface singularity, C.R. Acad. Sci. Paris, Ser. I 337, 195–200, (2003).

[6] R. Bondil, General elements of an m-primary ideal on a normal surface singularity, proc. of the second french-japanese congress on singularities (C.I.R.M. 2002), ed. by J.-P. Brasselet and T. Suwa, S.M.F. Séminaire et Congrès 10, 11–20, (2005).

[7] R. Bondil, Lê D.T., Résolution des singularités de surfaces par éclatements normalisés, in Trends in Singularities, 31–81, ed. by. A. Libgober and M. Tibăr, Birkhäuser Verlag, (2002).

[8] J. Briançon, A. Galligo, M. Granger, Déformations équisingulières des germes de courbes gauches réduites, Mém. Soc. Math. France, no. 1, 69 pp., (1980/81).

[9] J. Briançon, J.P. Henry, Equisingularité générique des familles de surfaces à singularité isolée, Bull. Soc. math. France 108, 259–281, (1980).

[10] J. Briançon, J.P. Speder, Familles équisingulières de surfaces à singularité isolée, C.R. Acad. Sci. Paris, t. 280, Série A, 1013–1016, (1975).

[11] H. Flenner, M. Manaresi, Variation of ramification loci of generic projections, Math. Nachr. 194, 79–92, (1998). DOI: 10.1002/mana.19981940107

[12] W. Fulton, Intersection theory. Second edition. Ergeb. Math. u. Grenzgebiete. 3, Springer-Verlag, (1998). DOI: 10.1007/978-1-4612-1700-8

[13] G.M. Greuel, On deformation of curves and a formula of Deligne, Algebraic Geometry (La Rábida, 1981), 141–168, Lecture Notes in Math. 961, Springer Verlag, (1982).

[14] J. Giraud, Intersections sur les surfaces normales, in Introduction à la théorie des singularités, ed. by Lê D.T., Travaux en cours 37, Hermann, (1988).

[15] T. de Jong, D. van Straten, On the deformation theory of rational surface singularities with reduced fondamental cycle, J. Algebraic Geometry 3,117–172, (1994).

[16] T. de Jong, D. van Straten, Deformation theory of sandwiched singularities, Duke Math. J. 95, no. 3, 451–522, (1998). DOI: 10.1215/S0012-7094-98-09513-8
[17] Lê D.T., Teissier B., Variétés polaires locales et classes de Chern des variétés singulières, Ann. of Math. 114, 457–491, (1981). DOI: 10.2307/1971299
[18] D.T. Lê, M. Tosun, Combinatorics of rational surface singularities, Comment. Math. Helvetici. 79, 582–604, (2004).
[19] J. Kollár, Toward moduli of singular varieties, Comp. Math. 56, 369–398, (1985).
[20] J. Snoussi, Limites d'espaces tangents à une surface normale, Comment. Math. Helv. 76, 61-88, (2001). DOI: 10.1007/s000140050150
[21] M. Spivakovsky, Sandwiched singularities and desingularisation of surfaces by normalized Nash transformations, Ann. math. 131, 411–491, (1990). DOI: 10.2307/1971467
[22] B. Teissier, The hunting of invariants in the geometry of discriminants, Real and complex singularities (Proc. Ninth Nordic Summer School, Oslo, 1976) 565–678, Sijthoff and Noordhoff, (1977).
[23] B. Teissier, Variétés polaires II, Multiplicités polaires, sections planes et conditions de Whitney, in Algebraic Geometry (La Rábida 1981), 314–491, Lectures Notes in Math. 961, Springer Verlag, (1982).
[24] B. Teissier, On B. Segre and the theory of polar varieties, in Geometry and complex variables (Bologna, 1988/1990), 357–367, Lecture Notes in Pure and Appl. Math. 132, Dekker, New York, (1991).
[25] G.N. Tyurina, Absolute isolatedness of rational singularities and triple rational points, Func. Anal. Appl. 2, 324–332, (1968). DOI: 10.1007/BF01075685
[26] J. Wahl, Equations defining rational singularities, Ann. scient. Ec. Norm. Sup. 10, 231–264, (1977).

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ON CHARACTERISTIC CLASSES OF SINGULAR HYPERSURFACES AND INVOLUTIVE SYMMETRIES OF THE CHOW GROUP

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Abstract. For every choice of an integer and a line bundle on an algebraic scheme we construct an associated involution on its Chow group, and show that various notions of characteristic class for singular hypersurfaces are interchanged via such involutions. As an application, we apply our formulas to effectively compute some non-trivial characteristic classes associated with a graph hypersurface. In the case of projective space we show that such involutions are induced by involutive correspondences.

1. Introduction

Fix an algebraically closed field $\mathfrak{K}$ of characteristic zero, let M be a smooth $\mathfrak{K}$-variety and let $X \subset M$ be a hypersurface. For singular X there exists a generalization of the notion of 'Milnor number' to arbitrary singularities which is a characteristic class supported on the singular locus of X referred to in the literature as the *Milnor class* of X, which we denote by $\mathcal{M}(X)$. Milnor classes have received significant interest in the recent literature [17][8][21][18][11][10], and –for a general closed subscheme $Y \hookrightarrow M$– are defined (up to sign) as the difference between the *Fulton class* $c_{\mathrm{F}}(Y)$ and its *Chern-Schwartz-MacPherson* class $c_{\mathrm{SM}}(Y)$. Both the Fulton class and CSM class are elements of the Chow group which are generalizations of Chern classes to the realm of singular varieties in the sense that the classes both agree with the total homology Chern class in the smooth case[1]. Another characteristic class supported on the singular locus of a hypersurface X is the *Lê-class* of X, denoted $\Lambda(X) \in A_*X$, which was first defined in [10] and named as such as the *Lê-class* is closely related to the so-called *Lê-cycles* of X, which were initially defined and studied independently of Milnor classes [16]. The main result announced in [10] was that if $\mathscr{O}(X)$ is very ample then both $\mathcal{M}(X)$ and $\Lambda(X)$ determine each other in a completely symmetric way, i.e.,

$$\mathcal{M}_k(X) = \sum_{j=0}^{d-k} (-1)^{j+k} \binom{j+k}{k} c_1(\mathscr{O}(X))^j \cap \Lambda_{j+k}, \tag{1.1}$$

and

$$\Lambda_k(X) = \sum_{j=0}^{d-k} (-1)^{j+k} \binom{j+k}{k} c_1(\mathscr{O}(X))^j \cap \mathcal{M}(X)_{j+k}, \tag{1.2}$$

where d is the dimension of the singular locus of X and an ith subscript on a class denotes its component of dimension i.

However, it was soon discovered that formulas (1.1) and (1.2) did *not* in fact hold, as an erratum appeared stating that there had been a subtle error which lead to a misidentification of the global *Lê-class* $\Lambda(X)$ with the Segre class $s(X_s, M)$ of the singular scheme X_s of X in M [9]. In any case, a direct corollary of Theorem 4.3 which we prove in §4 is that formulas (1.1)

[1] We give a more in-depth discussion of all classes mentioned here in §2.

and (1.2) *do* in fact hold once the components of $\Lambda(X)$ in formulas (1.1) and (1.2) are replaced by components of a class $\widetilde{\Lambda}(X)$ closely related to the Segre class $s(X_s, M)$, namely

$$\widetilde{\Lambda}(X) = c(\mathscr{O}(X))c(T^*M \otimes \mathscr{O}(X)) \cap s(X_s, M) \in A_*X_s, \tag{1.3}$$

where again X_s denotes the *singular scheme* of X, i.e., the subscheme of X whose ideal sheaf is locally generated by all partial derivatives of a defining equation for X. Moreover, we require no assumption that $\mathscr{O}(X)$ be very ample.

In [3], the class $c(T^*M \otimes \mathscr{O}(X)) \cap s(X_s, M)$ was taken as the definition of a class referred to as the μ-*class* of the singular scheme X_s of X (as it generalized Parusiǹski's 'μ-number' [19]), denoted $\mu(X_s)$, thus the class $\widetilde{\Lambda}(X)$ properly realizing formulas (1.1) and (1.2) is precisely given by

$$\widetilde{\Lambda}(X) = c(\mathscr{O}(X)) \cap \mu(X_s) \in A_*X_s.$$

Moreover, if we define $\widetilde{\Lambda}^{(k)}(X)$ for $k \in \mathbb{Z}$ as

$$\widetilde{\Lambda}^{(k)}(X) = c(\mathscr{O}(X))^k \cap \mu(X_s) \in A_*X_s,$$

we show that symmetric formulas analogous to (1.1) and (1.2) hold between the Milnor class $\mathcal{M}(X)$ and $\widetilde{\Lambda}^{(k)}(X)$ for all $k \in \mathbb{Z}$. As such, it is essentially the μ-class which is at the heart of this duality with the Milnor class. Applications of μ-classes to the study of dual varieties varieties and contact schemes of hypersurfaces were also considered in [3].

The symmetry of formulas (1.1) and (1.2) seem to suggest the existence of some non-trivial involutive symmetry of A_*X which exchanges $\mathcal{M}(X)$ and $\widetilde{\Lambda}(X)$, which we show in §4 is in fact the case. Furthermore, we show in §3 that for every integer $n \in \mathbb{Z}$ and line bundle $\mathscr{L} \to X$ there exists an associated involution

$$i_{n,\mathscr{L}} : A_*X \to A_*X,$$

and that other notions of characteristic class for singular varieties are interchanged via such involutions as well.

In what follows we give a brief review of the characteristic classes under consideration in §2. In §3 we define the maps $i_{n,\mathscr{L}}$ and show they are in fact involutions. In §4 we prove involutive formulas which relate different characteristic classes, and we give an application of our formulas by computing the Segre class and μ-class of a highly non-reduced scheme which is the singular scheme of a graph hypersurface. Such classes would be extremely difficult to compute solely from their definitions. We then close in §5 with an interpretation of the involutions $i_{n,\mathscr{L}}$ for X projective in terms of involutive correspondences on projective spaces.

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2. Characteristic classes of singular hypersurfaces

The total Chern class $c(X)$ of a smooth $\mathfrak{K}$-variety X is the most basic characteristic class for $\mathfrak{K}$-varieties in the sense that all other reasonable notions of characteristic class are linear combinations of Chern classes over a suitable ring. For those interested in singularities, it is then only natural that one would want to generalize the notion of Chern class to the realm of singular varieties (and schemes) in such a way that that they agree with the usual Chern class for smooth varieties. The CSM class $c_{\mathrm{SM}}(X)$ of a possibly singular variety X is in some sense the most direct generalization, since for $\mathfrak{K} = \mathbb{C}$ it generalizes the Poincaré-Hopf (or Gauß-Bonnet) theorem to the realm of singular varieties, i.e.,

$$\int_X c_{\mathrm{SM}}(X) = \chi(X),$$

where $\chi(X)$ denotes the topological Euler characteristic with compact support, and the integral sign is notation for taking the dimension zero component of a class[2]. For arbitrary $\mathfrak{K}$ (algebraically closed of characteristic zero) we simply *define* the Euler characteristic of a $\mathfrak{K}$-variety as the 'integral' of its CSM class. Moreover, CSM classes are a generalization of counting in the sense that they obey inclusion-exclusion (which of course is very useful for computations). In [4], Aluffi obtained a very nice formula for the CSM class of a hypersurface in terms of the Segre class (see Definition 2.1) of its singular scheme, and since we are only concerned with hypersurfaces in this note we may use his formula as a working definition (we recall Aluffi's formula in §4, after introducing some useful notations).

Another class generalizing the Chern class to the realm of singular varieties and schemes is the Fulton class, which is defined for any subscheme of a smooth $\mathfrak{K}$-variety M. From here on we will refer to such schemes as *embeddable schemes*. For X a (possibly singular) local complete intersection, its Fulton class $c_{\mathrm{F}}(X)$ agrees (after pushforward to M) with the total Chern class of a smooth variety in the same rational equivalence class as X, and so $c_{\mathrm{SM}}(X)$ differs from $c_{\mathrm{F}}(X)$ only in terms of dimension less than or equal to the dimension of its singular locus. The difference $c_{\mathrm{SM}}(X) - c_{\mathrm{F}}(X)$ then measures the discrepancy of $c_{\mathrm{SM}}(X)$ from the Chern class of a smooth deformation of X (parametrized by $\mathbb{P}^1$), and is an invariant precisely of the *singularities* of X. For X with only isolated singularities (over $\mathbb{C}$) the integral of $c_{\mathrm{SM}}(X) - c_{\mathrm{F}}(X)$ agrees (up to sign) precisely with the sum of the Milnor numbers of each singular point of X, thus it seemed natural to refer to this class generalization of global Milnor number as the 'Milnor class' of X, which we denote by $\mathcal{M}(X) := c_{\mathrm{SM}}(X) - c_{\mathrm{F}}(X)$[3].

To define the Fulton class of an arbitrary embeddable scheme, we first need the following

Definition 2.1. Let M be a smooth $\mathfrak{K}$-variety and $Y \hookrightarrow M$ a subscheme. For Y regularly embedded (so that its normal cone is in fact a vector bundle, which we denote by $N_Y M$), the *Segre class* of Y relative to M is denoted $s(Y, M)$, and is defined as

$$s(Y, M) := c(N_Y M)^{-1} \cap [Y] \in A_* Y.$$

For Y 'irregularly' embedded, let $f : \widetilde{M} \to M$ be the blowup of M along Y and denote the exceptional divisor of f by E. The Segre class of Y relative to M is then defined as

$$s(Y, M) := f|_{E*} s(E, \widetilde{M}) \in A_* Y,$$

where $f|_{E*}$ denotes the proper pushforward of f restricted to E. As E is always regularly embedded, this is enough to define the Segre class of Y (relative to M) in any case.

The Fulton class is then given by the following

Definition 2.2. Let Y be a subscheme of some smooth variety M. It's *Fulton class* is denoted $c_{\mathrm{F}}(Y)$, and is defined as

$$c_{\mathrm{F}}(Y) := c(TM) \cap s(Y, M) \in A_* Y.$$

Remark 2.1. As shown in [12] (Example 4.2.6), $c_{\mathrm{F}}(Y)$ is intrinsic to Y, i.e., it is independent of an embedding into some smooth variety (thus justifying the absence of an ambient M anywhere in its notation).

[2] We note that while CSM classes were first defined over $\mathbb{C}$ [15], their definition was later generalized to an arbitrary algebraically closed field of characteristic zero in [13].

[3] We blindly ignore any sign conventions some may associate with this class in the literature.

Remark 2.2. While the Fulton class is sensitive to scheme structure, the CSM class of a scheme by definition coincides with that of its support with natural reduced structure, and thus is *not* sensitive to any non-trivial scheme structure. As for Milnor classes, since they are defined as the difference between the CSM and Fulton classes, they are scheme-theoretic invariants as well. More precisely, in the case of a possibly singular/non-reduced hypersurface X, $\mathcal{M}(X)$ is an invariant of the *singular scheme* of X, i.e., the subscheme of X whose ideal sheaf is locally generated by the partial derivatives of a local defining equation for X. We note that at present it is not clear what scheme structure on the singular locus of an arbitrary local complete intersection determines its Milnor class, though for a large class of global complete intersections it was shown in [11] that the Milnor class is determined by a direct generalization of the notion of singular scheme of a hypersurface to complete intersections.

As noted in Remark 2.2, while Fulton classes are sensitive to scheme structure, in some sense they are not sensitive to the singularities of a hypersurface (or more generally a local complete intersection), since (as mentioned earlier) the Fulton class of a local complete intersection coincides with that of a smooth representative of its rational equivalence class (e.g, the Fulton class of two distinct lines in the plane is the same as the Fulton class of a smooth conic). A scheme-theoretic characteristic class which is also sensitive to the singularities of an embeddable scheme Y is the *Aluffi class* of Y, denoted by $c_{\mathrm{A}}(Y)$, which may be integrated to yield the Donaldson-Thomas type invariant of Y. Aluffi classes were first defined by Aluffi in [5], where he referred to them as *weighted Chern-Mather classes*. Behrend then later coined the term 'Aluffi class' in [7], where he makes the first connection between Aluffi's weighted Chern-Mather classes (albeit with a different sign convention) and Donaldson-Thomas invariants of Calabi-Yau threefolds. For Y the singular scheme of a hypersurface X it was shown in [5] that (up to sign) $c_{\mathrm{A}}(Y) = c(\mathscr{O}(X)) \cap \mathcal{M}(X)$, and since this is the only context in which we consider Aluffi classes we refer the reader to both [7][5] for precise definitions and further discussion.

3. The involutions $i_{n,\mathscr{L}}$

Let X be an algebraic $\mathfrak{K}$-scheme. For every $(n, \mathscr{L}) \in \mathbb{Z} \times \mathrm{Pic}(X)$ we now define a map $i_{n,\mathscr{L}} : A_*X \to A_*X$, and show that it is an involutive automorphism of A_*X (these will be precisely the involutions which relate various characteristic classes alluded to above). But before doing so, we first introduce two intersection theoretic operations, which will not only provide an efficient way for defining the involutions $i_{n,\mathscr{L}}$, but will also be of computational utility.

So let $\alpha \in A_*X$ be written as $\alpha = \alpha^0 + \cdots + \alpha^n$, where α^i is the component of α of *codimension* i (in X). We denote by $\alpha^\vee$ the class

$$\alpha^\vee := \sum (-1)^i \alpha^i,$$

and refer to it as the 'dual' of α.

We now define an action of $\mathrm{Pic}(X)$ on A_*X. Given a line bundle $\mathscr{L} \to X$ we denote its action on $\alpha = \sum \alpha^i \in A_*X$ by $\alpha \otimes_X \mathscr{L}$ [4], which we define as

$$\alpha \otimes_X \mathscr{L} := \sum \frac{\alpha^i}{c(\mathscr{L})^i}.$$

It is straightforward to show that this defines an honest action (i.e.,

$$(\alpha \otimes_X \mathscr{L}) \otimes_X \mathscr{M} = \alpha \otimes_X (\mathscr{L} \otimes \mathscr{M})$$

[4]The notation '$\otimes_X$' is not to be confused with a similar notation used in a different context in [14] §8.1

for any line bundles $\mathscr{L}$ and $\mathscr{M}$), and we refer to this action as 'tensoring by a line bundle'. For $\mathscr{E}$ a rank r class in the Grothendieck group of vector bundles on X (note that r may be non-positive), the formulas

$$(c(\mathscr{E}) \cap \alpha)^{\vee} = c(\mathscr{E}^{\vee}) \cap \alpha^{\vee} \tag{3.1}$$

$$(c(\mathscr{E}) \cap \alpha) \otimes_X \mathscr{L} = \frac{c(\mathscr{E} \otimes \mathscr{L})}{c(\mathscr{L})^r} \cap (\alpha \otimes_X \mathscr{L}) \tag{3.2}$$

were proven in [2] (along with the first appearance of the 'tensor' and 'dual' operations), and will be indispensable throughout the remainder of this note[5]. We now arrive at the following

Proposition 3.1. *Let X be an algebraic $\mathfrak{K}$-scheme, $n \in \mathbb{Z}$ and $\mathscr{L} \to X$ be a line bundle. Then the map $i_{n,\mathscr{L}} : A_*X \to A_*X$ given by*

$$\alpha \mapsto c(\mathscr{L})^n \cap (\alpha^{\vee} \otimes_X \mathscr{L})$$

is an involutive automorphism of A_*X (i.e., $i_{n,\mathscr{L}} \circ i_{n,\mathscr{L}} = \mathrm{id}_{A_*X}$).

Proof. Let $\alpha \in A_*X$ and denote $i_{n,\mathscr{L}}(\alpha)$ by β, i.e.,

$$\beta = c(\mathscr{L})^n \cap (\alpha^{\vee} \otimes_X \mathscr{L})\,. \tag{3.3}$$

We will show that $i_{n,\mathscr{L}}(\beta) = \alpha$, which implies the conclusion of the proposition. Capping both sides of the equation 3.3 by $c(\mathscr{L})^{-n}$ we get

$$c(\mathscr{L})^{-n} \cap \beta = \alpha^{\vee} \otimes_X \mathscr{L}. \tag{3.4}$$

By formula 3.2, for any line bundle $\mathscr{M} \to X$ we have

$$\big(c(\mathscr{L})^{-n} \cap \beta\big) \otimes_X \mathscr{M} = \frac{c(\mathscr{M})^n}{c(\mathscr{L} \otimes \mathscr{M})^n} \cap (\beta \otimes_X \mathscr{M}),$$

thus tensoring both sides of equation 3.4 by $\mathscr{L}^{\vee}$ yields

$$c(\mathscr{L}^{\vee})^n \cap (\beta \otimes_X \mathscr{L}^{\vee}) = \alpha^{\vee}. \tag{3.5}$$

Finally, taking the 'dual' (i.e. applying formula 3.1) to both sides of equation 3.5 we have

$$\alpha = c(\mathscr{L})^n \cap (\beta^{\vee} \otimes_X \mathscr{L}) = i_{n,\mathscr{L}}(\beta),$$

as desired.

The fact that $i_{n,\mathscr{L}}$ is a homomorphism (i.e. $\mathbb{Z}$-linear) follows from the fact that dualizing, tensoring by a line bundle and capping with Chern classes are all linear operations. □

Remark 3.1. The map $\alpha \mapsto \alpha^{\vee}$ sending a class to its dual coincides with $i_{n,\mathscr{O}}$ for every $n \in \mathbb{Z}$.

4. Symmetric formulas abound

We now assume M is a smooth proper $\mathfrak{K}$-variety and $X \subset M$ is an arbitrary hypersurface (i.e., the zero-*scheme* associated with a non-trivial section of line bundle on M). We denote the singular scheme of X by X_s, which is the subscheme of X whose ideal sheaf is the restriction to X of the ideal sheaf on M which is locally generated by a defining equation for X and each of its partial derivatives. In what follows, as we prefer to work mostly in M, we will not distinguish between classes in A_*X and their pushforwards (via the natural inclusion) to A_*M. We will call two classes *k -$\mathscr{L}$ dual* if one is the image of the other (and so vice-versa) under the map $i_{k,\mathscr{L}}$. In this section, we show formulas (1.1) and (1.2) both hold when $\Lambda(X)$ is replaced by $\widetilde{\Lambda}(X)$ as defined via 1.3, and that these symmetric relations are consequences of the fact that $\mathcal{M}(X)$

[5]The tensor and dual operations, along with formulas 3.1 and 3.2 are what we refer to as Aluffi's 'intersection-theoretic calculus' in §1.

and $\widetilde{\Lambda}(X)$ are simply $\dim(M)$-$\mathscr{O}(X)$ dual. Similar relations are then derived for other notions of characteristic class for singular varieties.

We now recall Aluffi's formula for the CSM class of X, which as mentioned earlier we will take as a working definition.

Theorem 4.1 (Aluffi, [4])**.**

$$c_{\mathrm{SM}}(X) = \frac{c(TM)}{c(\mathscr{O}(X))} \cap \left([X] + s(X_s, M)^{\vee} \otimes_M \mathscr{O}(X)\right).$$

We then immediately arrive at the following

Corollary 4.2.

$$\mathcal{M}(X) = \frac{c(TM)}{c(\mathscr{O}(X))} \cap \left(s(X_s, M)^{\vee} \otimes_M \mathscr{O}(X)\right).$$

Proof. This follows directly from definitions of Fulton class and Milnor class, as

$$\mathcal{M}(X) = c_{\mathrm{SM}}(X) - c_{\mathrm{F}}(X) \quad \text{and} \quad c_{\mathrm{F}}(X) = c(TM) \cap s(X, M) = \frac{c(TM)}{c(\mathscr{O}(X))} \cap [X].$$

□

The fact that formulas (1.1) and (1.2) hold after replacing Λ by $\widetilde{\Lambda}$ are a special case of the following

Theorem 4.3. *Let n be an integer. Then*

$$\mathcal{M}(X) = i_{n,\mathscr{O}(X)}(\alpha_X(n)) \quad \textit{and} \quad \alpha_X(n) = i_{n,\mathscr{O}(X)}(\mathcal{M}(X)),$$

where

$$\alpha_X(n) := c(T^*M \otimes \mathscr{O}(X)) c(\mathscr{O}(X))^{n+1-\dim(M)} \cap s(X_s, M).$$

Proof. By Corollary 4.2 we have

$$\begin{aligned}
\mathcal{M}(X) &= \frac{c(TM)}{c(\mathscr{O}(X))} \cap \left(s(X_s, M)^{\vee} \otimes_M \mathscr{O}(X)\right) \\
&= c(\mathscr{O}(X))^n \cap \left(\frac{c(TM)c(\mathscr{O})^{n+1-\dim(M)}}{c(\mathscr{O}(X))^{n+1}} \cap \left(s(X_s, M)^{\vee} \otimes_M \mathscr{O}(X)\right)\right) \\
&\overset{3.2}{=} c(\mathscr{O}(X))^n \cap \left(\left(c(TM \otimes \mathscr{O}(-X))c(\mathscr{O}(-X))^{n+1-\dim(M)} \cap s(X_s, M)^{\vee}\right) \otimes_M \mathscr{O}(X)\right) \\
&\overset{3.1}{=} c(\mathscr{O}(X))^n \cap \left(\left(c(TM^* \otimes \mathscr{O}(X))c(\mathscr{O}(X))^{n+1-\dim(M)} \cap s(X_s, M)\right)^{\vee} \otimes_M \mathscr{O}(X)\right) \\
&= i_{n,\mathscr{O}(X)}(\alpha_X(n)).
\end{aligned}$$

The formula $\alpha_X(n) = i_{n,\mathscr{O}(X)}(\mathcal{M}(X))$ then follows as $i_{n,\mathscr{O}(X)}$ is an involution by Proposition 3.1. □

Remark 4.1. The most natural case of Theorem 4.3 is when $n = \dim(X)$, in which case we have the formulas

$$\mathcal{M}(X) = i_{\dim(X),\mathscr{O}(X)}(\mu(X_s)) \quad \text{and} \quad \mu(X_s) = i_{\dim(X),\mathscr{O}(X)}(\mathcal{M}(X)),$$

where we recall $\mu(X_s)$ denotes the μ-class of the singular scheme X_s of X, which is defined via the formula

$$\mu(X_s) = c(T^*M \otimes \mathscr{O}(X)) \cap s(X_s, M) \in A_* X_s. \tag{4.1}$$

The μ-class was first defined by Aluffi [3], and is an intrinsic invariant of the singularities of X. Such classes arise often in the study of projective duality [20] (though they are actually referred to as 'Milnor classses' in that text!), have applications to the study of contact schemes of hypersurfaces [3], and are closely related to the Donaldson-Thomas type invariant of X_s [7].

Remark 4.2. As n varies over $\mathbb{Z}$, writing out the formula for the kth dimensional piece $\mathcal{M}_k(X)$ of the Milnor class of X via Theorem 4.3 yields infinitely many symmetric formulas similar to (1.1) and (1.2). In particular, for $n = \dim(M)$ we have $\alpha_X(\dim(M)) = \widetilde{\Lambda}(X)$ as defined in (1.3), a fact which implies formulas (1.1) and (1.2) indeed hold after $\Lambda(X)$ is replaced by $\widetilde{\Lambda}(X)$, which we now state and prove via

Corollary 4.4. *Formulas* (1.1) *and* (1.2) *hold after* Λ *is replaced by* $\widetilde{\Lambda}$.

Proof. Denote the dimension of M by d. By Theorem 4.3,

$$\begin{aligned}
\mathcal{M}(X) &= i_{d,\mathscr{O}(X)}(\alpha_X(d)) \\
&= i_{d,\mathscr{O}(X)}(\widetilde{\Lambda}(X)) \\
&= c(\mathscr{O}(X))^d \cap \left(\widetilde{\Lambda}(X)^\vee \otimes_M \mathscr{O}(X)\right) \\
&= c(\mathscr{O}(X))^d \cap \left(\sum_{i=0}^{d} \frac{(-1)^i \widetilde{\Lambda}_{d-i}(X)}{c(\mathscr{O}(X))^i}\right) \\
&= \sum_{i=0}^{d} (-1)^i c(\mathscr{O}(X))^{d-i} \cap \widetilde{\Lambda}_{d-i}(X) \\
&= \sum_{i=0}^{d} (-1)^i (1 + c_1(\mathscr{O}(X)))^{d-i} \cap \widetilde{\Lambda}_{d-i}(X) \\
&= \sum_{i=0}^{d} \sum_{j\geq 0} (-1)^i \binom{d-i}{j} c_1(\mathscr{O}(X))^j \cap \widetilde{\Lambda}_{d-i}(X).
\end{aligned}$$

In the last equality the term $c_1(\mathscr{O}(X))^j \cap \widetilde{\Lambda}_{d-i}(X)$ is of dimension $d-i-j$, and so $\mathcal{M}_k(X)$ corresponds to setting $i = d-k-j$, which yields

$$\mathcal{M}_k(X) = \sum_{j\geq 0} (-1)^{d-k-j} \binom{j+k}{j} c_1(\mathscr{O}(X))^j \cap \widetilde{\Lambda}_{j+k}(X),$$

which is equivalent (up to sign) to formula (1.1) with Λ replaced by $\widetilde{\Lambda}$ via the identity

$$\binom{a+b}{a} = \binom{a+b}{b}.$$

The (possible) disparity in sign comes from the fact that in [10] their definition of Milnor class differs from ours by a factor of $(-1)^d$. The analogue of formula (1.2) then immediately follows as $\mathcal{M}(X)$ and $\widetilde{\Lambda}(X)$ are d-$\mathscr{O}(X)$ dual. □

Remark 4.3. We note that it was much more work to write out formulas for the individual components $\mathcal{M}_k(X)$ than that of the total Milnor class $\mathcal{M}(X)$ (as in Theorem 4.3). And this is a general principle when computing characteristic classes, i.e., it is often simpler to compute a *total* class rather than its individual components.

Remark 4.4. As mentioned in §2, in [5] Aluffi defined a scheme-theoretic characteristic class for arbitrary embeddable $\mathfrak{K}$-schemes which Behrend refers to as the 'Aluffi class' in his theory of Donaldson-Thomas type invariants [7]. The analogue of the Gauß-Bonnet theorem in this theory is the formula

$$\int_Y c_{\mathrm{A}}(Y) = \chi_{\mathrm{DT}}(Y),$$

where Y is an embeddable scheme with Aluffi class $c_{\mathrm{A}}(Y)$, and $\chi_{\mathrm{DT}}(Y)$ denotes the Donaldson-Thomas type invariant of Y. If Y is the singular scheme of a hypersurface X it was shown in [5] that

$$c_{\mathrm{A}}(Y) = c(\mathscr{O}(X)) \cap \mathcal{M}(X).$$

Thus capping both sides of the formulas constituting Theorem 4.3 with $c(\mathscr{O}(X))$ then yields

Corollary 4.5. *Let n be an integer, Y be the singular scheme of a hypersurface X and let $\alpha_X(n)$ be defined as in Theorem 4.3. Then*

$$c_{\mathrm{A}}(Y) = i_{n+1,\mathscr{O}(X)}(\alpha_X(n)) \quad \textit{and} \quad \alpha_X(n) = i_{n+1,\mathscr{O}(X)}(c_{\mathrm{A}}(Y)).$$

We now give an application of such formulas by computing classes that would be considerably difficult using only their definitions.

Example 4.6. Let X be the hypersurface in $\mathbb{P}^4$ given by

$$X : (t_1t_2t_3t_4 + t_1t_2t_3t_5 + t_1t_2t_4t_5 + t_1t_3t_4t_5 + t_2t_3t_4t_5 = 0) \subset \mathbb{P}^4.$$

Such a hypersurface is the graph hypersurface associated with the 'banana graph' with 5 edges [6]. The homogeneous ideal associated with its singular scheme X_s is then

$$(t_2t_3t_4 + t_2t_3t_5 + t_2t_4t_5 + t_3t_4t_5, \ldots, t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4).$$

In [6], the Milnor class of X was computed as

$$\mathcal{M}(X) = 60H^4 - 10H^3,$$

where H denotes the class of a hyperplane in $\mathbb{P}^4$. By Theorem 4.3 we have

$$\begin{aligned}
\mu(X_s) &= c(\mathscr{O}(X))^3 \cap (\mathcal{M}(X)^\vee \otimes_{\mathbb{P}^4} \mathscr{O}(X)) \\
&= (1+4H)^3 \cdot \left(\frac{60H^4}{(1+4H)^4} + \frac{10H^3}{(1+4H)^3}\right) \\
&= \frac{60H^4}{(1+4H)} + 10H^3 \\
&= 60H^4(1-4H) + 10H^3 \\
&= 60H^4 + 10H^3,
\end{aligned}$$

so that the μ-class of the singular scheme of X is in fact the dual of the Milnor class. The Aluffi class of X_s is then given by

$$c_{\mathrm{A}}(X_s) = c(\mathscr{O}(X)) \cap \mathcal{M}(X) = (1+4H)(60H^4 - 10H^3) = 20H^4 - 10H^3,$$

so that the Donaldson-Thomas type invariant of X_s is 20. By definition of the μ-class (4.1) we may compute the Segre class of X_s in $\mathbb{P}^4$ via the formula

$$s(X_s, \mathbb{P}^4) = c(T^*\mathbb{P}^4 \otimes \mathscr{O}(X))^{-1} \cap \mu(X_s),$$

thus

$$s(X_s, \mathbb{P}^4) = \frac{(1+4H)}{(1+3H)^5} \cdot (60H^4 + 10H^3) = -50H^4 + 10H^3.$$

We conclude this section by identifying the 'n-$\mathscr{O}(X)$ dual partners' of the CSM class of X, which we state via the following

Theorem 4.7. *Let n be an integer. Then*

$$c_{\mathrm{SM}}(X)=i_{n,\mathscr{O}(X)}(\nu_X(n)+\alpha_X(n)) \quad \textit{and} \quad \nu_X(n)+\alpha_X(n)=i_{n,\mathscr{O}(X)}(c_{\mathrm{SM}}(X)),$$

where

$$\nu_X(n)=c(T^*M\otimes\mathscr{O}(X))c(\mathscr{O}(X))^{n-\dim(M)}\cap -[X]$$

and $\alpha_X(n)$ is as defined in Theorem 4.3.

Proof. By Proposition 3.1 and Theorem 4.3, the proof amounts to showing

$$c_{\mathrm{F}}(X)=i_{n,\mathscr{O}(X)}(\nu_X(n)),$$

as $c_{\mathrm{SM}}(X)=c_{\mathrm{F}}(X)+\mathcal{M}(X)$. Thus

$$\begin{aligned}
c_{\mathrm{F}}(X) &= c(TM)\cap s(X,M)\\
&= c(TM)\cap\left(c(N_XM)^{-1}\cap[X]\right)\\
&= c(TM)\cap([X]\otimes_M\mathscr{O}(X))\\
&= c(\mathscr{O}(X))^n\cap\left(\frac{c(TM)c(\mathscr{O})^{n-\dim(M)}}{c(\mathscr{O}(X))^n}\cap([X]\otimes_M\mathscr{O}(X))\right)\\
&\overset{3.2}{=} c(\mathscr{O}(X))^n\cap\left(\left(c(TM\otimes\mathscr{O}(-X))c(\mathscr{O}(-X))^{n-\dim(M)}\cap[X]\right)\otimes_M\mathscr{O}(X)\right)\\
&\overset{3.1}{=} c(\mathscr{O}(X))^n\cap\left(\left(c(T^*M\otimes\mathscr{O}(X))c(\mathscr{O}(X))^{n-\dim(M)}\cap-[X]\right)^\vee\otimes_M\mathscr{O}(X)\right)\\
&= i_{n,\mathscr{O}(X)}(\nu_X(n)),
\end{aligned}$$

as desired. □

5. $i_{n,\mathscr{L}}$ VIA INVOLUTIVE CORRESPONDENCES

Let M and N be smooth proper $\mathfrak{K}$-varieties. A *correspondence* from M to N is a class $\alpha\in A_*(M\times N)$, and such an α induces homomorphisms

$$\alpha_*\in\mathrm{Hom}(A_*M,A_*N) \qquad \text{and} \qquad \alpha^*\in\mathrm{Hom}(A_*N,A_*M)$$

given by

$$\beta\overset{\alpha_*}{\longmapsto}q_*(\alpha\cdot p^*\beta), \quad \gamma\overset{\alpha^*}{\longmapsto}p_*(\alpha\cdot q^*\gamma),$$

where p is the projection $M\times N\to M$, q is the projection $M\times N\to N$ and '$\cdot$' denotes the intersection product in $A_*(M\times N)$ (which is well defined via the smoothness assumption on $M$and N). Correspondences are at the heart of Grothendieck's theory of motives, and generalize algebraic morphisms in the sense that we think of an arbitrary class $\alpha\in A_*(M\times N)$ as a generalization of the graph Γ_f of a (proper) morphism $f\in\mathrm{Hom}(M,N)$. Just as a morphism $f\in\mathrm{Hom}(M,N)$ induces morphisms on the corresponding Chow groups via proper pushforward (f_*) and flat pullback (f^*), the morphisms α_* and α^* are direct generalizations of proper pushforward and flat pullback as $f_*=(\Gamma_f)_*$ and $f^*=(\Gamma_f)^*$. Moreover, correspondences may be composed in such a way that the functorial properties of proper pushforward and flat pullback still hold, i.e., $(\alpha\circ\vartheta)_*=\alpha_*\circ\vartheta_*$ and $(\alpha\circ\vartheta)^*=\vartheta^*\circ\alpha^*$ for composable correspondences α and ϑ. From this perspective we were naturally led to the question of whether or not for an algebraic scheme X the involutions $i_{n,\mathscr{L}}$ defined in §3 are induced by involutive correspondences in $A_*(X\times X)$. We answer this question for $X=\mathbb{P}^N$ via the following

Theorem 5.1. *Let N be a positive integer and $(n,m) \in \mathbb{Z}\times\mathbb{Z}$. Then there exists a unique $\alpha = \sum_{i+j\leq N} a_{i,j}x^iy^j \in \mathbb{Z}[x,y]/(x^{N+1},y^{N+1}) \cong A_*(\mathbb{P}^N \times \mathbb{P}^N)$ such that $i_{n,\mathscr{O}(m)} = \alpha_*$*[6]*, and the coefficients of α are given by*

$$a_{N-j,i} = (-1)^j \binom{n-j}{i-j} m^{i-j}.$$

Proof. Consider $\mathbb{P}^N \times \mathbb{P}^N$ with the natural projections onto its first and second factors, which we denote by p and q respectively. Denote by x the hyperplane class in the first factor and by y the hyperplane class in the second factor (we use the same notations for their pullbacks via the natural projections). Let $\beta = \sum_{i=0}^N \beta_i x^i \in A_*\mathbb{P}^N$. It follows directly from the definition of $i_{n,\mathscr{O}(m)}$ and induction that

$$i_{n,\mathscr{O}(m)}(\beta) = \sum_{i=0}^N \left(\sum_{j=0}^N (-1)^j \binom{n-j}{i-j} m^{i-j}\beta_j \right) y^i.$$

We now let $\alpha = \sum_{i+j\leq N} a_{i,j}x^iy^j \in A_*(\mathbb{P}^N \times \mathbb{P}^N)$ be arbitrary, compute $\alpha_*(\beta) = q_*(\alpha\cdot p^*\beta)$, set its coefficients equal to those of $i_{n,\mathscr{O}(m)}(\beta)$, and then observe that this determines the $a_{i,j}$ uniquely. Since we are not using a notational distinction for x and its pullback p^*x, $p^*\beta$ retains exactly the same form as β in its expansion with respect to x. Now $\alpha\cdot p^*\beta$ is just usual multiplication in the ring $\mathbb{Z}[x,y]/(x^{N+1},y^{N+1})$, and $q_*(\alpha\cdot p^*\beta)$ is just the coefficient of x^N in the expansion of $\alpha\cdot p^*\beta$ with respect to x, which yields

$$\alpha_*(\beta) = \sum_{i=0}^N \left(\sum_{j=0}^N a_{N-j,i}\beta_j \right) y^i.$$

By setting $\alpha_*(\beta) = i_{n,\mathscr{O}(m)}(\beta)$ the $a_{i,j}$ are then uniquely determined to be as stated in the conclusion of the theorem.

To see that $q_*(\gamma)$ for arbitrary $\gamma \in A_*(\mathbb{P}^N\times\mathbb{P}^N)$ is indeed the coefficient of x^N in the expansion of γ with respect to x, one may first view q as the natural projection of the projective bundle $\mathbb{P}(\mathscr{E})$ with $\mathscr{E}$ the trivial rank $N+1$ bundle over $\mathbb{P}^N$ and $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) = x$. Then by the projection formula, to compute $q_*(\gamma)$ we need only to compute $q_*(x^i)$ in the expansion of γ with respect to x, which we do using the notion of *Segre class* of a vector bundle[7]. By definition of the *Segre class* of $\mathscr{E}$, denoted $s(\mathscr{E})$, we have

$$s(\mathscr{E}) := q_*(1 + x + x^2 + \cdots).$$

And since $s(\mathscr{E}) = c(\mathscr{E})^{-1} = 1$, matching terms of like dimension we see that all powers of x map to 0 except for x^N which maps to 1. $\square$

It would be interesting to determine objects of the bounded derived category of $\mathbb{P}^N \times \mathbb{P}^N$ whose Chern characters coincide with α as given in Theorem 5.1. And certainly there must be a larger class of varieties (other than projective spaces) for which an analogue of Theorem 5.1 holds.

[6] Note that $\alpha_* = \alpha^*$ in this case.

[7] We note that the notion of Segre class of a vector bundle is different than the *relative* Segre class we define in §2 (see [12], Chapter 3 for a precise definition).

REFERENCES

[1] P. Aluffi. Remarks on Lê cycles and Milnor classes. *Unpublished note.*
[2] P. Aluffi. MacPherson's and Fulton's Chern classes of hypersurfaces. *Internat. Math. Res. Notices*, (11):455ff., approx. 11 pp. (electronic), 1994. doi:10.1155/S1073792894000498.
[3] P. Aluffi. Singular schemes of hypersurfaces. *Duke Math. J.*, 80(2):325–351, 1995. doi:10.1215/S0012-7094-95-08014-4.
[4] P. Aluffi. Chern classes for singular hypersurfaces. *Trans. Amer. Math. Soc.*, 351(10):3989–4026, 1999. doi:10.1090/S0002-9947-99-02256-4.
[5] P. Aluffi. Weighted Chern-Mather classes and Milnor classes of hypersurfaces. In *Singularities—Sapporo 1998*, volume 29 of *Adv. Stud. Pure Math.*, pages 1–20. Kinokuniya, Tokyo, 2000.
[6] Paolo Aluffi and Matilde Marcolli. Feynman motives of banana graphs. *Commun. Number Theory Phys.*, 3(1):1–57, 2009. doi:10.4310/CNTP.2009.v3.n1.a1.
[7] Kai Behrend. Donaldson-Thomas type invariants via microlocal geometry. *Ann. of Math. (2)*, 170(3):1307–1338, 2009. doi:10.4007/annals.2009.170.1307.
[8] J.-P. Brasselet, D. Lehmann, J. Seade, and T. Suwa. Milnor classes of local complete intersections. *Trans. Amer. Math. Soc.*, 354(4):1351–1371 (electronic), 2002. doi:10.1090/S0002-9947-01-02846-X.
[9] R. Callejas-Bedregal, M. F. Z. Morgado, and J. Seade. Erratum to: Lê cycles and Milnor classes. *Invent. Math.*, 197(2):483–489, 2014. doi:10.1007/s00222-014-0529-9.
[10] R. Callejas-Bedregal, M. F. Z. Morgado, and J. Seade. Lê cycles and Milnor classes. *Invent. Math.*, 197(2):453–482, 2014. doi:10.1007/s00222-013-0450-7.
[11] J. Fullwood. On Milnor classes via invariants of singular schemes. *Journal of singularities*, 8:1–10, 2014.
[12] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge.* Springer-Verlag, Berlin, second edition, 1998. doi:10.1007/978-1-4612-1700-8.
[13] Gary Kennedy. MacPherson's Chern classes of singular algebraic varieties. *Comm. Algebra*, 18(9):2821–2839, 1990. doi:10.1080/00927879008824054.
[14] Robert Lazarsfeld. *Positivity in algebraic geometry. II*, volume 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals. doi:10.1007/978-3-642-18808-4.
[15] R. D. MacPherson. Chern classes for singular algebraic varieties. *Ann. of Math. (2)*, 100:423–432, 1974.
[16] D. B. Massey. The Lê varieties. I. *Invent. Math.*, 99(2):357–376, 1990. doi:10.1007/BF01234423.
[17] Laurentiu Maxim. On Milnor classes of complex hypersurfaces. In *Topology of stratified spaces*, volume 58 of *Math. Sci. Res. Inst. Publ.*, pages 161–175. Cambridge Univ. Press, Cambridge, 2011.
[18] Laurentiu Maxim, Morihiko Saito, and Jörg Schürmann. Hirzebruch-Milnor classes of complete intersections. *Adv. Math.*, 241:220–245, 2013. doi:10.1016/j.aim.2013.04.001.
[19] Adam Parusiński. A generalization of the Milnor number. *Math. Ann.*, 281(2):247–254, 1988. doi:10.1007/BF01458431.
[20] E. A. Tevelev. *Projective duality and homogeneous spaces*, volume 133 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2005. Invariant Theory and Algebraic Transformation Groups, IV.
[21] Shoji Yokura. Verdier-Riemann-Roch for Chern class and Milnor class. *Asian J. Math.*, 6(1):1–22, 2002.

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ON SAITO'S NORMAL CROSSING CONDITION

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Abstract. Kyoji Saito defined a residue map from the logarithmic differential 1-forms along a reduced complex analytic hypersurface to the meromorphic functions on the hypersurface. He studied the condition that the image of this map coincides with the weakly holomorphic functions, that is, with the functions on the normalization. With Michel Granger, the author proved that this condition is equivalent to the hypersurface being normal crossing in codimension one. In this article, the condition is given a natural interpretation in terms of regular differential forms beyond the hypersurface case. For reduced equidimensional complex analytic spaces which are free in codimension one, the geometric interpretation of being normal crossing in codimension one is shown to persist.

Introduction

Saito [29] introduced the complex of logarithmic differential forms along a reduced hypersurface D in a smooth complex manifold S. It is defined as

$$\Omega^\bullet(\log D) = \{\omega \in \Omega_S^\bullet(D) \mid d\mathscr{I}_D \wedge \omega \subseteq \Omega_S^{\bullet+1}\}$$

where $\mathscr{I}_D$ is the ideal sheaf of D. Locally, if $\mathscr{I}_D = \langle h \rangle$, such forms are characterized by having a presentation as

$$g\omega = \frac{dh}{h} \wedge \xi + \eta$$

where $\xi \in \Omega_S^{\bullet-1}$ and $\eta \in \Omega_S^\bullet$ have no pole and $g \in \mathscr{O}_S$ maps to a non-zero divisor in $\mathscr{O}_D$. He defined a logarithmic residue map

$$\rho_D \colon \Omega^\bullet(\log D) \to \mathscr{M}_D \otimes_{\mathscr{O}_D} \Omega_D^{\bullet-1}, \quad \omega \mapsto \frac{\xi}{g}|_D \tag{0.1}$$

where $\mathscr{M}_D = Q(\mathscr{O}_D)$ denotes the meromorphic functions on D. This residue map gives rise to an exact sequence

$$0 \longrightarrow \Omega_S^\bullet \longrightarrow \Omega^\bullet(\log D) \xrightarrow{\rho_D} \sigma_D^{\bullet-1} \longrightarrow 0 \tag{0.2}$$

where $\sigma_D^{\bullet-1}$ denotes the image of ρ_D. Let $\nu_D \colon \tilde{D} \to D$ be a normalization and note that $\mathscr{M}_D = \mathscr{M}_{\tilde{D}}$. Saito [29, (2.8),(2.11)] showed that

$$(\nu_D)_* \mathscr{O}_{\tilde{D}} \subseteq \sigma_D^0 \tag{0.3}$$

and that, if D is a plane curve, equality holds if and only D is normal crossing. Generalizing this result to reduced hypersurfaces D, Granger and the author [13] showed that equality in (0.3)

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is equivalent to D being normal crossing in codimension one. The purpose of this article is to further generalize this preceding result.

In §2, we suggest a more general point of view for the equality in (0.3). It is based on Aleksandrov's result [2, §4, Cor. 2] that $\sigma_D^\bullet = \omega_D^\bullet$ where the latter denotes the regular differential forms on D. With Tsikh [4, Thm. 2.4] (or [5, Thm. 3.1]) and later in [3, Thm. 2] he generalized this result to complete intersections using (different versions of) multilogarithmic differential forms and their residues. We relate it to Aleksandrov's multilogarithmic residue map and we comment on some claims made in [3]. Regular differential forms are defined under more general hypotheses. More specifically let X be a reduced equidimensional complex analytic singularity with normalization $\nu_X\colon \tilde{X} \to X$. Due to normality of $\tilde{X}$, we have $\mathscr{O}_{\tilde{X}} = \omega_{\tilde{X}}^0$ (see Corollary 2.3). We shall therefore refer to the equality

$$(\nu_X)_*\omega_{\tilde{X}}^0 = \omega_X^0$$

resulting from (0.3) as *Saito's normal crossing condition*. Our approach is independent of an embedding and does not require a generalization of logarithmic differential forms such as multilogarithmic differential forms in the complete intersection case. While Aleksandrov and Tsikh use Barlet's description of regular differential forms in the complex analytic context (see [7]) we prefer to rely on a general algebraic approach due to Kersken that is reviewed in §1. In §4 and §5, we study Saito's normal crossing condition for reduced curve and Gorenstein singularities. In §6 we give it the following geometric interpretation analogous to [13, Thm. 1.2] in the hypersurface case.

Theorem 0.1. *Let X be a reduced equidimensional complex analytic singularity which is free in codimension one. Then X satisfies Saito's normal crossing condition if and only if X is a normal crossing divisor in codimension one.* □

The additional freeness hypothesis replaces the fact that any reduced hypersurface is a free divisor in codimension one. Our generalization of freeness is motivated by Aleksandrov–Terao theorem (see [1, §2 Thm.] and [38, Prop. 2.4]) stating that freeness of a reduced hypersurface is equivalent to Cohen–Macaulayness of the Jacobian ideal. We call a reduced Gorenstein singularity free if the ω-Jacobian ideal is a Cohen–Macaulay ideal (see Definition 6.1). In case of complete intersections of codimension k Pol [28, Thm. 4.5] showed that freeness is equivalent to the projective dimension of multilogarithmic differential k forms being equal to (or equivalently bounded by) $k-1$. Her approach is a direct generalization of the one taken in [13].

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1. Regular and logarithmic differential forms

Fix a complete valued field k of characteristic 0 and let A be a local analytic k-algebra of dimension $r \geq 1$. In particular A is Noetherian, Henselian and catenary (see [14, II.§0.1,§6.2]). Informally we refer to A as a *singularity*.

If A admits a positive grading in the sense of Scheja and Wiebe (see [34, §3]) then we call it a *quasihomogeneous singularity*. This means that $\mathfrak{m}_A$ is generated by eigenvectors of an *Euler derivation* $\chi \in \mathrm{Der}_k(A, \mathfrak{m}_A)$ with positive rational eigenvalues $w_1, \dots, w_n$. In this case one can write $\chi = \sum_{i=1}^n w_i x_i \partial_{x_i}$. If $w_1 = \cdots = w_n$ then we call the grading a *standard grading* and A a *homogeneous singularity*.

We denote by $Q(-)$ the total ring of fractions and abbreviate $L := Q(A)$. Let

$$R = k\langle\langle x_1, \dots, x_n \rangle\rangle$$

denote the regular ring of convergent power series over k in n variables $x_1, \dots, x_n$. It is a formal power series ring in case the valuation is trivial. For a suitable n, pick a finite k-algebra homomorphism

$$R \to A \tag{1.1}$$

of codimension $m = n - r$.

1.1. **Kersken's regular differential forms.** We begin by reviewing Kersken's description of regular differential forms (see [19, 18, 20]). Denote by $\Omega_A := \Omega_{A/k}$ the universally finite differential algebra of A over k (see [21, §11]). In particular, $\Omega_A = \bigoplus_{p\in\mathbb{N}} \Omega_A^p$ is graded with differential $d\colon \Omega_A \to \Omega_A[1]$ of degree 1. Let $C(A)$ be the (unaugmented) Cousin complex of A

$$C(A)\colon 0 \to C^0(A) \to C^1(A) \to \cdots$$

with respect to A-active sequences (see [19, §2]). It is a resolution of A if and only if A is Cohen–Macaulay and a (minimal) injective resolution if and only if A is Gorenstein (see [37]). Setting $C_\Omega(A) := C(A) \otimes_A \Omega_A$, the residue complex of A is the complex of graded (Ω_A, d)-modules

$$D_\Omega(A) := \underline{\mathrm{Hom}}_{\Omega_R}(\Omega_A, C_\Omega(R))[m;m]$$

where $\underline{\mathrm{Hom}}_{\Omega_R}$ denotes graded Hom_{Ω_R} and $[m;m]$ signifies a shift by m of both the Ω_R-module and Cousin complex grading. Notably this definition is independent of the choice of (1.1) (see [18, (3.3)]). We write δ both for the Cousin differential of $C(R)$ and induced differentials. The 0th cohomology of $D_\Omega(A)$ with respect to δ is a graded (Ω_A, d)-module

$$\omega_A := H^0(D_\Omega(A), \delta),$$

the complex of *regular differential forms* over A (see [18, p. 442]). For any graded Ω_R-module M one can identify (see [18, (3.6)])

$$\underline{\mathrm{Hom}}_{\Omega_R}(M, C_\Omega(R)) = \mathrm{Hom}_R(M[n], \Omega_R^n \otimes_R C(R)). \tag{1.2}$$

Since $C(R)$ is an injective resolution of R, this implies that $C_\Omega(R)$ is an injective resolution of Ω_R. It follows that (see [18, §6])

$$\omega_A = \underline{\mathrm{Ext}}^m_{\Omega_R}(\Omega_A, \Omega_R)[m]$$

which has graded components

$$\omega_A^p = \mathrm{Ext}^m_R(\Omega_A^{r-p}, \Omega_R^n) = \mathrm{Hom}_A(\Omega_A^{r-p}, \omega_A^r) \tag{1.3}$$

due to (1.2), adjunction of $-\otimes_A A$ and $\mathrm{Hom}_R(A, -)$, and since $\mathrm{Hom}_R(A, C(R)^q) = 0$ for $q < m$.

Kersken [18, §5] constructs a *trace form*[1] $c_A \in \omega_A^0$. In case (1.1) is a *Noether normalization* (see [14, II.§2.2]), $c_A \in \omega_A^0 = \underline{\mathrm{Hom}}_{\Omega_R}(\Omega_A, \Omega_R)$ restricts to (see [18, (5.1.4)])

$$c_A|_{A\otimes_R \Omega_R} = \mathrm{Tr}_{A/R} \otimes_R \Omega_R\colon A \otimes_R \Omega_R \to \Omega_R \tag{1.4}$$

where $\mathrm{Tr}_{A/R} \in \mathrm{Hom}_R(A, R)$ is the trace of A over R (see [30, (10.3)]). It induces a unique *trace map* of complexes of (Ω_A, d)-modules (see [18, (5.6)])

$$\gamma_A\colon C_\Omega(A) \to D_\Omega(A), \quad 1 \mapsto c_A$$

which is an isomorphism at regular primes of A (see [18, (5.7.2)]).

If A is reduced and equidimensional then

$$\Omega_A \otimes_A L = C^0_\Omega(A) \xrightarrow[\cong]{\gamma_A^0} D^0_\Omega(A) \tag{1.5}$$

[1] Its construction uses that k has characteristic 0.

is an isomorphism. It serves to identify ω_A with its preimage

$$\sigma_A := (\gamma_A^0)^{-1}(\omega_A), \tag{1.6}$$

the complex of *regular (meromorphic) differential forms* over A. Under the identification (1.3) becomes

$$\sigma_A^p = \operatorname{Hom}_A(\Omega_A^{r-p}, \sigma_A^r). \tag{1.7}$$

Composing $\Omega_A \to \Omega_A/T(\Omega_A)$ with $H^0(\gamma_A)$ yields a map

$$c_A\colon \Omega_A \to \omega_A \tag{1.8}$$

which is an isomorphism at regular primes of A (see [18, (5.7.3)]). We denote its cokernel by

$$N_A := \operatorname{coker} c_A. \tag{1.9}$$

The preceding objects then fit into a commutative diagram

$$\begin{tikzcd}
& \Omega_A \otimes_A L \arrow[r, "\gamma_A^0", "\cong"'] & D_\Omega^0(A) \\
& \sigma_A \arrow[u, hook] \arrow[r, "\cong"] & \omega_A \arrow[u, hook] \\
\Omega_A \arrow[uur] \arrow[ur] \arrow[urr, "c_A"'] & &
\end{tikzcd} \tag{1.10}$$

where the leftmost map is the canonical one. In particular, its degree-0 part $A \hookrightarrow L$ factors through an inclusion

$$c_A^0\colon A \hookrightarrow \sigma_A^0 \cong \omega_A^0. \tag{1.11}$$

If (1.1) is a presentation $R \twoheadrightarrow A$ with kernel $\mathfrak{a}$ then (see [21, Props. 3.8, 11.9])

$$\Omega_A^p = \Omega_R^p/(\mathfrak{a}\Omega_R^p + d\mathfrak{a} \wedge \Omega_R^{p-1}) = \bigwedge^p \Omega_A^1. \tag{1.12}$$

In other words, Ω_A is an exterior differential algebra. It follows that

$$D_\Omega(A) = \operatorname{Ann}_{C_\Omega(R)}(\mathfrak{a}\Omega_R + d\mathfrak{a} \wedge \Omega_R)[m; m]. \tag{1.13}$$

Elements of $C_\Omega(R)$ can be represented by residue symbols (see [19, §2]), which lie by definition in the image of some map

$$\Phi_{f_1,\dots,f_q}\colon (\Omega_R^p/\langle f_1,\dots,f_q\rangle\Omega_R^p)_g \hookrightarrow C_\Omega^q(R), \quad \overline{\xi}/g \mapsto \begin{bmatrix} \xi/g \\ f_1,\dots,f_q \end{bmatrix}, \tag{1.14}$$

where $f_1,\dots,f_q,g$ is an R-sequence. Injectivity of this map follows from [19, (2.6)] and Wiebe's Theorem (see [21, E.21]) using that the Ω_R^p are free R-modules. The (induced) Cousin differential δ operates as (see [19, (2.5)])

$$\delta\begin{bmatrix} \xi/g \\ f_1,\dots,f_q \end{bmatrix} = \begin{bmatrix} \xi \\ f_1,\dots,f_q,g \end{bmatrix}.$$

Thus, elements of $\ker\delta$ are of the form $\begin{bmatrix} \xi \\ f_1,\dots,f_q \end{bmatrix}$ where $\overline{\xi} \in \Omega_R^p/\langle f_1,\dots,f_q\rangle\Omega_R^p$. One may assume that $f_1,\dots,f_m \in \mathfrak{a}$ after multiplying ξ by a suitable transition determinant (see [19, (2.5.3)]). Combined with (1.13) this yields the explicit description (see [20, (1.2)])

$$\omega_A^p = \left\{ \begin{bmatrix} \xi \\ f_1, \dots, f_m \end{bmatrix} \;\middle|\; \xi \in \Omega_R^{p+m},\ f_1, \dots, f_m \in \mathfrak{a} \text{ } R\text{-sequence}, \right. \\ \left. \mathfrak{a}\xi \equiv 0 \equiv d\mathfrak{a} \wedge \xi \mod \langle f_1, \dots, f_m \rangle \Omega_R \right\}. \tag{1.15}$$

1.2. **Aleksandrov's multilogarithmic residue.** In the following we describe Aleksandrov's generalization (see [3]) to complete intersections of (0.2) in relation with Kersken's description of regular differential forms in §1.1. To this end, consider $A = R/\mathfrak{a}$ with $\mathfrak{a} = \langle h_1, \dots, h_m \rangle$ generated by an R-sequence $h_1, \dots, h_m$. Then (see [18, p. 445])

$$\gamma_A^q \colon \begin{bmatrix} \overline{\xi}/\overline{s} \\ \overline{f}_1, \dots, \overline{f}_q \end{bmatrix} \mapsto \begin{bmatrix} d\underline{h} \wedge \xi / s \\ \underline{h}, f_1, \dots, f_q \end{bmatrix} \tag{1.16}$$

where $d\underline{h} := dh_1 \wedge \dots \wedge dh_m$. In particular,

$$c_A = \gamma_A^0(1) = \begin{bmatrix} d\underline{h} \\ \underline{h} \end{bmatrix}. \tag{1.17}$$

The following types of differential forms with simple poles where introduced by Saito (see [29]) and implicitly by Aleksandrov (see [3]). Notably the multilogarithmic differential forms of Aleksandrov and Tsikh (see [4, 5]) not considered here have arbitrary poles (see [28, Appendix B] for details).

Definition 1.1. Let $\underline{h} = h_1, \dots, h_m$ be an R-sequence and set $h := h_1 \cdots h_m$. Then the *logarithmic differential forms* along $\langle h \rangle$ and the *multilogarithmic differential forms* along $\underline{h}$ are defined respectively by

$$\Omega_R(\log \langle h \rangle) := \left\{ \omega \in \frac{1}{h} \Omega_R \;\middle|\; dh \wedge \omega \in \Omega_R \right\},$$
$$\Omega_R(\log \underline{h}) := \left\{ \omega \in \frac{1}{h} \Omega_R \;\middle|\; \forall j = 1, \dots, m \colon dh_j \wedge \omega \in \sum_{i=1}^m \frac{h_i}{h} \Omega_R \right\}.$$

Lemma 1.2. *Let $\underline{h} = h_1, \dots, h_m$ be an R-sequence.*

(a) *An alternative definition of logarithmic differential forms reads*

$$\Omega_R(\log \langle h \rangle) = \left\{ \omega \in \frac{1}{h} \Omega_R \;\middle|\; \forall j = 1, \dots, m \colon dh_j \wedge \omega \in \frac{h_j}{h} \Omega_R \right\}. \tag{1.18}$$

In particular, $\Omega_R(\log \langle h \rangle) \subseteq \Omega_R(\log \underline{h})$ with equality for $m = 1$.

(b) *There is an inclusion*

$$dh_i \wedge \Omega_R(\log \langle h \rangle) \subseteq \Omega_R(\log \langle h/h_i \rangle).$$

(c) *If $m \leq 2$ then*

$$\Omega_R(\log \langle h \rangle) \cap \sum_{i=1}^m \frac{h_i}{h} \Omega_R = \sum_{i=1}^m \Omega_R(\log \langle h/h_i \rangle).$$

Proof.

(a) For $\omega \in \Omega_R(\log \langle h \rangle)$, we have

$$\sum_{i=1}^m \frac{h}{h_i} dh_i \wedge (h\omega) = h dh \wedge \omega \in h\Omega_R$$

with $dh_i \wedge (h\omega) \in \Omega_R$. Note that the factors $h_1, \dots, h_m$ of h are pairwise coprime because they form an R-sequence. It follows that $dh_i \wedge (h\omega) \in h_i\Omega_R$ for $i = 1, \dots, m$. Conversely, this latter condition implies that $dh \wedge \omega = \sum_{i=1}^m \frac{dh_i}{h_i} \wedge (h\omega) \in \Omega_R$.

(b) For $\omega \in \Omega_R(\log \langle h \rangle)$, (a) yields

$$dh_j \wedge dh_i \wedge \omega \in \frac{h_i}{h}\Omega_R \cap \frac{h_j}{h}\Omega_R = \frac{h_i h_j}{h}\Omega_R$$

for $i \neq j$ and hence $dh_i \wedge \omega \in \Omega_R(\log \langle h/h_i \rangle)$.

(c) Let $\sum_{i=1}^m \omega_i \in \Omega_R(\log \langle h \rangle)$ with $\omega_i \in \frac{h_i}{h}\Omega_R$ and set $\eta_i := \frac{h}{h_i}\omega_i \in \Omega_R$. By (a) and (b), we have $dh_j \wedge \sum_{i \neq j} \omega_i \in \frac{h_j}{h}\Omega_R$ and hence $\sum_{i \neq j} h_i dh_j \wedge \eta_i \in h_j\Omega_R$ for $j = 1, \dots, m$. Since $m \leq 2$ this implies that $dh_j \wedge \eta_i \in h_j\Omega_R$ and hence $dh_j \wedge \omega_i \in \frac{h_i h_j}{h}\Omega_R$ for $i \neq j$. Thus, $\omega_i \in \Omega(\log \langle h/h_i \rangle)$ for $i = 1, \dots, m$. ☐

The following sequences appear in [3, §4, Lem. 1, §6, Thm. 2].

Proposition 1.3. *Let $\underline{h} = h_1, \dots, h_m$ be an R-sequence. Then there is a commutative diagram with exact top row (and exact bottom row if $m \leq 2$)*

$$\begin{tikzcd} 0 \arrow[r] & \sum_{i=1}^m \frac{h_i}{h}\Omega_R \arrow[r] & \Omega_R(\log \underline{h}) \arrow[r, "\rho_{\underline{h}}"] & \omega_A \arrow[r] & 0 \\ 0 \arrow[r] & \sum_{i=1}^m \Omega_R(\log \langle h/h_i \rangle) \arrow[r] \arrow[u, hook] & \Omega_R(\log \langle h \rangle) \arrow[r, "\rho'_{\underline{h}}"] \arrow[u, hook] & \omega_A \arrow[u, equal] \end{tikzcd} \tag{1.19}$$

where $\rho_{\underline{h}}$ denotes the composition

$$\begin{tikzcd} \Omega_R(\log \underline{h}) \arrow[r, hook, "h\cdot"] & \Omega_R \arrow[r] & \Omega_R/\langle \underline{h} \rangle \Omega_R \arrow[r, hook, "\Phi_{\underline{h}}"] & \omega_A, \\ \omega = \frac{\eta}{h} \arrow[rrr, mapsto] & & & \begin{bmatrix} \eta \\ \underline{h} \end{bmatrix} = z, \end{tikzcd} \tag{1.20}$$

with $\Phi_{\underline{h}}$ from (1.14).

Proof. By (1.15) and Definition 1.1 the map $\rho_{\underline{h}}$ is well-defined. Using [19, (2.5.3)] and Wiebe's Theorem (see [21, E.21]), any element of ω_A can be rewritten as in (1.15) with $f_1, \dots, f_m = \underline{h}$. The vanishing conditions in (1.15) reduce to

$$dh_j \wedge \xi \equiv 0 \mod \langle \underline{h} \rangle \Omega_R.$$

Thus, the map $\rho_{\underline{h}}$ is surjective with kernel arising from the middle map in (1.20). If $m \leq 2$, then the left square in (1.19) is cartesian due to Lemma 1.2.(c). ☐

We deduce the following characterization of multilogarithmic differential forms appearing in [3, Thm. 1] (see also [4, Prop. 2.1] or [5, Prop. 1.1]).

Corollary 1.4. *Let $\underline{h} = h_1, \dots, h_m$ be an R-sequence such that $A = R/\langle \underline{h} \rangle$ is reduced. For any $\omega \in \Omega_R(\log \underline{h})$ there is a $g \in R$ with $\overline{g} \in A^{\mathrm{reg}}$, a $\xi \in \Omega_R$, and $\eta_i \in \frac{h_i}{h}\Omega_R$ for $i = 1, \dots, m$, such that*

$$g\omega = \frac{d\underline{h}}{h} \wedge \xi + \sum_{i=1}^m \eta_i. \tag{1.21}$$

Conversely, any $\omega \in \Omega_{R,h}$ admitting a representation (1.21) *lies in $\Omega_R(\log \underline{h})$.*

Proof. Let ω and z be as in (1.20). By the isomorphism (1.5) and by (1.16), there is a $g \in R$ and a $\xi \in \Omega_R$ as in the claim such that

$$\rho_{\underline{h}}(g\omega) = \left[\frac{g\eta}{\underline{h}}\right] = gz = \gamma_A^0(\overline{\xi}) = \left[\frac{d\underline{h} \wedge \xi}{\underline{h}}\right] = \rho_{\underline{h}}\left(\frac{d\underline{h}}{h} \wedge \xi\right). \tag{1.22}$$

Then (1.21) follows from the exact sequence (1.19). Conversely let $\omega = \frac{\eta}{h} \in \Omega_{R,h}$ satisfy (1.21). Then $\eta \in \Omega_R$ with

$$gdh_j \wedge \eta = \sum_{i=1}^m dh_j \wedge (h\eta_i) \in \sum_{i=1}^m h_i \Omega_R$$

and hence $dh_j \wedge \eta \in \sum_{i=1}^m h_i \Omega_R$ for $j = 1, \dots, m$ since $h_1, \dots, h_m, g$ is an R-sequence. It follows that $\omega \in \Omega_R(\log \underline{h})$. □

Remark 1.5.

(a) For $m = 1$ the upper and lower sequences in (1.19) coincide by Definition 1.1.

(b) It follows from (1.21) and (1.22) that $(\gamma_A^0)^{-1} \circ \rho_{\underline{h}}$ coincides with Aleksandrov's multiple residue defined as in (0.1) (see [3, §4]).

(c) Aleksandrov claims exactness of the bottom row for any m and surjectivity of $\rho'_{\underline{h}}$ in (1.19) (see [3, Thm. 2]). However Pol showed that in general $\rho'_{\underline{h}}$ is not surjective (see [28, Prop. 4.14]).

2. Saito's normal crossing condition

In addition to the hypotheses of §1 we shall assume from now on that k is algebraically closed and that A is r-equidimensional. The integral closure of A in $L = Q(A)$,

$$\nu_A \colon A \hookrightarrow \tilde{A}, \tag{2.1}$$

is a finite k-algebra homomorphism (see [14, II.§7.2]), the *normalization* of A. Denote by $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ the minimal primes of A and set

$$A_i := A/\mathfrak{p}_i, \quad L_i := Q(A_i).$$

Then $\dim A_i = r$ by r-equidimensionality of A. Since A is reduced,

$$\mathfrak{p}_i A_{\mathfrak{p}_i} = 0, \quad L_i = A_{\mathfrak{p}_i}. \tag{2.2}$$

For the same reason (see [14, II.§7.2]),

$$A \hookrightarrow \prod_{i=1}^s A_i \hookrightarrow \prod_{i=1}^s \tilde{A}_i = \tilde{A} \hookrightarrow \prod_{i=1}^s L_i = L \tag{2.3}$$

where each $\tilde{A}_i = \widetilde{A_i}$ is a local analytic k-algebra. Note that $L = Q(\tilde{A})$ and $L_i = Q(\tilde{A}_i)$. The objects of §1.1 can be defined verbatim for $\tilde{A}$ compatible with the product decomposition (2.3). In particular, $\gamma_{\tilde{A}} = \bigoplus_{i=1}^s \gamma_{\tilde{A}_i}$ and

$$\omega_{\tilde{A}} = \bigoplus_{i=1}^s \omega_{\tilde{A}_i}, \quad \sigma_{\tilde{A}} = \bigoplus_{i=1}^s \sigma_{\tilde{A}_i}.$$

For any $\mathfrak{q} \in \operatorname{Spec} \tilde{A}$ lying over $\mathfrak{p} = A \cap \mathfrak{q} \in \operatorname{Spec} A$,

$$\dim A_{\mathfrak{p}} = r - \dim A/\mathfrak{p} = r - \dim \tilde{A}/\mathfrak{q} = \dim \tilde{A}_{\mathfrak{q}} \tag{2.4}$$

using that A and $\tilde{A}$ are r-equidimensional and catenary (see [25, Prop. 2.5.10])

Proposition 2.1. *There is a commutative diagram*

$$
\begin{tikzcd}
\Omega_{\tilde A} \arrow[r, "c_{\tilde A}"] & \omega_{\tilde A} \arrow[d, hook] & \sigma_{\tilde A} \arrow[l, "\gamma^0_{\tilde A}|"', "\cong"] \arrow[r, hook] \arrow[d, hook] & \Omega_{\tilde A} \otimes_{\tilde A} L \\
\Omega_A \arrow[u, "\wedge\, d\nu_A"] \arrow[r, "c_A"] & \omega_A & \sigma_A \arrow[l, "\gamma^0_A|"', "\cong"] \arrow[r, hook] & \Omega_A \otimes_A L \arrow[u, "\cong"']
\end{tikzcd} \tag{2.5}
$$

where the horizontal compositions are the canonical maps.

Proof. Let (1.1) be a Noether normalization of A; composed with (2.1) it gives a Noether normalization of $\tilde A$. Setting $m = 0$ in (1.3) it serves to compute both ω_A and $\omega_{\tilde A}$. Note that $A \otimes_R Q(R) = L = \tilde A \otimes_R Q(R)$ and hence (see [30, §10])

$$\operatorname{Tr}_{\tilde A/R}|_A = \operatorname{Tr}_{A/R}. \tag{2.6}$$

There is a natural map of complexes of graded (Ω_A, d)-modules $D_\Omega(\tilde A) \to D_\Omega(A)$. By (1.4) and (2.6) it maps $c_{\tilde A}|_{\tilde A \otimes_R \Omega_R} \mapsto c_A|_{A\otimes_R \Omega_R}$. Together with the left claimed injectivity in diagram (2.5) this implies that $c_{\tilde A} \mapsto c_A$ (see [19, (5.1)]). The commutativity of diagram (2.5) follows using diagram (1.10).

The inclusion (2.1) has torsion cokernel, so applying $\operatorname{Hom}_R(-, \Omega^n_R)$ first gives

$$\omega^r_{\tilde A} \hookrightarrow \omega^r_A \tag{2.7}$$

due to (1.3). Consider the short exact sequence (see [21, Cor. 11.8, Prop. 11.17])

$$0 \longrightarrow T^1(\tilde A/A) \longrightarrow \tilde A \otimes_A \Omega^1_A \xrightarrow{\tilde A \otimes d\nu_A} \Omega^1_{\tilde A} \longrightarrow \Omega^1_{\tilde A/A} \longrightarrow 0. \tag{2.8}$$

Applying $\bigwedge^p$ to (2.8), which is right-exact and commutes with base change, (1.12) gives a short exact sequence

$$0 \longrightarrow T^p(\tilde A/A) \longrightarrow \tilde A \otimes_A \Omega^p_A \xrightarrow{\tilde A \otimes \bigwedge^p d\nu_A} \Omega^p_{\tilde A} \longrightarrow \Omega^p_{\tilde A/A} \longrightarrow 0 \tag{2.9}$$

where $T^p(\tilde A/A)$ is the image of $T^1(\tilde A/A) \otimes_A \Omega^{p-1}_A$ (see [11, Prop. A.2.2]). Both Ω^1_A and $\Omega^1_{\tilde A}$ have rank r (see [32, (4.4)]). By finiteness of $\tilde A$ over A, $\Omega^1_{\tilde A/A}$ is the universal differential module which is compatible with localization and hence $\Omega^1_{\tilde A/A} \otimes_{\tilde A} L = 0$. It follows that $T^p(\tilde A/A)$ and $\Omega^p_{\tilde A/A}$ are torsion. In particular, this gives the right vertical isomorphism in diagram (2.5) and, since $\omega^r_{\tilde A}$ is torsion-free, we have

$$\operatorname{Hom}_{\tilde A}(T^p(\tilde A/A), \omega^r_{\tilde A}) = 0 = \operatorname{Hom}_{\tilde A}(\Omega^p_{\tilde A/A}, \omega^r_{\tilde A}). \tag{2.10}$$

Now (2.7) yields the upper inclusion and (2.9) and (2.10) the lower inclusion in the following diagram

$$
\begin{tikzcd}
\omega^p_{\tilde A} \arrow[d, equal] & \operatorname{Hom}_A(\Omega^{r-p}_A, \omega^r_{\tilde A}) \arrow[r, hook] \arrow[d, equal] & \operatorname{Hom}_A(\Omega^{r-p}_A, \omega^r_A) \arrow[d, equal] \\
\operatorname{Hom}_{\tilde A}(\Omega^{r-p}_{\tilde A}, \omega^r_{\tilde A}) \arrow[r, hook] & \operatorname{Hom}_{\tilde A}(\tilde A \otimes_A \Omega^{r-p}_A, \omega^r_{\tilde A}) & \omega^p_A
\end{tikzcd} \tag{2.11}
$$

which proves injectivity of the vertical maps in diagram (2.5). ∎

The following fact stated by Kersken (see [20, p.6]) goes back to a result of Serre (see [24, p. 5]).

Proposition 2.2. *If A is normal then ω_A is a reflexive A-module.*

Proof. By Serre's criterion, normality of A is equivalent to conditions (R_1) and (S_2). Let (1.1) be a presentation $R \twoheadrightarrow A$ with kernel $\mathfrak{a}$ and let $\mathfrak{q} \in \operatorname{Spec} A$.

First assume that $\operatorname{depth} A_\mathfrak{q} \leq 1$. Then $\dim A_\mathfrak{q} \leq 1$ by (S_2) and $A_\mathfrak{q}$ is regular by (R_1). It follows that (1.8) induces an isomorphism $\omega_{A,\mathfrak{q}} \cong \Omega_{A,\mathfrak{q}}$ and that $\Omega_{A,\mathfrak{q}} = \bigwedge \Omega^1_{A,\mathfrak{q}}$ is free (see [18, (5.7.3)] and [32, (8.7)]). In particular, $\omega_{A,\mathfrak{q}}$ is reflexive in this case.

Then assume that $\operatorname{depth} A_\mathfrak{q} \geq 2$ and let $\mathfrak{p} \in \operatorname{Spec} R$ be the preimage of $\mathfrak{q}$. Since R is Cohen–Macaulay, $\operatorname{grade}(\mathfrak{a}, R) = m$ (see [8, Thm. 2.1.2.(b)]) and there is an R-sequence $\underline{f} = f_1, \ldots, f_m \in \mathfrak{a}$. Then $R_\mathfrak{p}/\langle \underline{f} \rangle \twoheadrightarrow A_\mathfrak{p} = A_\mathfrak{q}$ and since $R_\mathfrak{p}$ and hence $R_\mathfrak{p}/\langle \underline{f} \rangle$ is Cohen–Macaulay (see [8, Thm. 2.1.3.(a)])

$$\operatorname{grade}(\mathfrak{p}, R_\mathfrak{p}/\langle \underline{f} \rangle) = \dim(R_\mathfrak{p}/\langle \underline{f} \rangle) \geq \dim A_\mathfrak{q} \geq \operatorname{depth} A_\mathfrak{q} \geq 2.$$

Using $\Omega^0_A = A$ and $\Omega^n_R \cong R$ in (1.3), $\omega^r_A \cong \operatorname{Hom}_R(A, R/\langle \underline{f} \rangle)$ (see [8, Lem. 1.2.4]). It follows that (see [8, Ex. 1.4.19])

$$\operatorname{depth} \omega^r_{A,\mathfrak{q}} = \operatorname{grade}(\mathfrak{q}, \omega^r_{A,\mathfrak{q}}) = \operatorname{grade}(\mathfrak{p}, \operatorname{Hom}_{R_\mathfrak{p}}(A_\mathfrak{q}, R_\mathfrak{p}/\langle \underline{f} \rangle)) \geq 2.$$

Thus, reflexivity of ω^r_A and then of ω^p_A for all p follows (see [8, Prop. 1.4.1.(b)]). ☐

Corollary 2.3. *If A is normal then $\sigma^0_A = \Omega^0_A = A$.*

Proof. Using (1.9) and (1.11) it suffices to show that $N^0_A = 0$. By hypothesis, A satisfies Serre's conditions (R_1) and (S_2). By (R_1), N^0_A has support of codimension at least 2 (see [18, (5.7.3)]). Let $\mathfrak{q} \in \operatorname{Spec} A$ with $\dim A_\mathfrak{q} \geq 2$. By (S_2) and Proposition 2.2, both $A_\mathfrak{q}$ and $\omega^0_{A,\mathfrak{q}}$ have depth at least 2 (see [8, Prop. 1.4.1.(b).(ii)]). Then $\operatorname{depth} N^0_{A,\mathfrak{q}} \geq 1$ by the Depth Lemma (see [8, Prop. 1.2.9]) and hence $\mathfrak{q} \notin \operatorname{Ass} N^0_A$. Thus, $\operatorname{Ass} N^0_A = \emptyset$ and $N^0_A = 0$ as claimed. ☐

In the hypersurface case, the inclusion $\omega^0_{\tilde{A}} \hookrightarrow \omega^0_A$ in diagram (2.5) corresponds to the inclusion (0.3) using Corollary 2.3. This motivates the following

Definition 2.4. We say that A satisfies *Saito's normal crossing condition (SNCC)* if $\omega^0_{\tilde{A}} = \omega^0_A$. By *SNCC at* $\mathfrak{p} \in \operatorname{Spec} A$ we mean that $\omega^0_{\tilde{A},\mathfrak{p}} = \omega^0_{A,\mathfrak{p}}$.

We first note that SNCC is a codimension-one condition.

Proposition 2.5. *The equality $\omega^p_{\tilde{A}} = \omega^p_A$ holds true if and only if it holds true in codimension one. In particular, SNCC is a codimension-one condition.*

Proof. Assume that the inclusion $\omega^p_{\tilde{A}} \hookrightarrow \omega^p_A$ in diagram (2.5) is an equality at primes of codimension 1; denote by W^p_A its cokernel. Since W^p_A is torsion, W^p_A has support of codimension at least 2. Let $\mathfrak{p} \in \operatorname{Spec} A$ with $\dim A_\mathfrak{p} \geq 2$ and pick any $\mathfrak{q} \in V(\mathfrak{p}\tilde{A}) \subseteq \operatorname{Spec} \tilde{A}$. In particular, $\mathfrak{q} \cap A \supseteq \mathfrak{p}$ and hence $\dim \tilde{A}_\mathfrak{q} = \dim A_{\mathfrak{q}\cap A} \geq \dim A_\mathfrak{p} \geq 2$ using (2.4). By Serre's condition (S_2) for $\tilde{A}$ then also $\operatorname{depth} \tilde{A}_\mathfrak{q} \geq 2$. Thus, $\operatorname{depth} \omega^p_{\tilde{A},\mathfrak{q}} \geq 2$ by Proposition 2.2 (see [8, Prop. 1.4.1.(b).(ii)]). It follows that (see [36, IV.B.1.Prop. 12] and [8, Prop. 1.2.10.(a)])

$$\operatorname{depth} \omega^p_{\tilde{A},\mathfrak{p}} = \operatorname{grade}(\mathfrak{p}, \omega^p_{\tilde{A},\mathfrak{p}}) = \operatorname{grade}(\mathfrak{p}\tilde{A}, \omega^p_{\tilde{A},\mathfrak{p}}) = \min\{\operatorname{depth} \omega^p_{\tilde{A},\mathfrak{q}} \mid \mathfrak{q} \in V(\mathfrak{p}\tilde{A})\} \geq 2.$$

Since $\operatorname{depth} \omega^p_{A,\mathfrak{p}} \geq 1$ by diagram (1.10), the claim follows as in the proof of Corollary 2.3. ☐

Now we show that SNCC descends to any union of irreducible components. For any subset $I \subseteq \{1, \ldots, s\}$, set

$$A_I := A/\mathfrak{a}_I, \quad \mathfrak{a}_I := \bigcap_{i \in I} \mathfrak{p}_i. \tag{2.12}$$

Note that A_I is reduced with minimal primes $\mathfrak{p}_i/\mathfrak{a}_I$, $i \in I$.

Proposition 2.6. *If $\omega^p_{\tilde{A}} = \omega^p_A$ then $\omega^p_{\tilde{A}_I} = \omega^p_{A_I}$. In particular, SNCC descends from A to A_I for any subset $I \subseteq \{1, \dots, s\}$.*

Remark 2.7. Proposition 2.6 plays the role of the inclusion

$$\Omega^1(\log(D_1 + D_2)) \subseteq \Omega^1(\log D)$$

for irreducible components D_1 and D_2 of a hypersurface D used in [13, Ex. 3.3].

The proof of Proposition 2.6 relies on the following two lemmas.

Lemma 2.8. *For any subset $I \subseteq \{1, \dots, s\}$, we have $\omega^p_{A_I} = \operatorname{Hom}_A(\Omega^{r-p}_{A_I}, \omega^r_A)$.*

Proof. Let (1.1) be a Noether normalization of A; composed with $A \twoheadrightarrow A_I$ it gives a Noether normalization of A_I. Using (1.3) and Hom-tensor-adjunction, we compute that

$$\omega^r_{A_I} = \operatorname{Hom}_R(A_I, \omega^r_R) = \operatorname{Hom}_A(A_I, \operatorname{Hom}_R(A, \omega^r_R)) = \operatorname{Hom}_A(A_I, \omega^r_A)$$

and hence that

$$\omega^p_{A_I} = \operatorname{Hom}_{A_I}(\Omega^{r-p}_{A_I}, \omega^r_{A_I}) = \operatorname{Hom}_{A_I}(\Omega^{r-p}_{A_I}, \operatorname{Hom}_A(A_I, \omega^r_A)) = \operatorname{Hom}_A(\Omega^{r-p}_{A_I}, \omega^r_A). \qquad \square$$

Replacing A in (2.12) by $\tilde{A}$, $\tilde{\mathfrak{p}}_j = \prod_{i \neq j} \tilde{A}_i$, $j = 1, \dots, s$, are the minimal primes, $\tilde{\mathfrak{a}}_I = \prod_{i \notin I} \tilde{A}_i$ and

$$\tilde{A}_I = \tilde{A}/\tilde{\mathfrak{a}}_I = \prod_{i \in I} \tilde{A}_i = \widetilde{A_I}.$$

Lemma 2.9. *The natural surjections $A_I \otimes_A \Omega^p_A \twoheadrightarrow \Omega^p_{A_I}$ and $A_I \otimes_A \Omega^p_{\tilde{A}} \twoheadrightarrow \Omega^p_{\tilde{A}_I}$ have torsion kernels $T^p(A_I/A)$ and $\tilde{T}^p(A_I/A)$, respectively.*

Proof. By definition, $T^0(A_I/A) = 0$ and $\tilde{T}^0(A_I/A)$ is torsion by (2.2). In particular,

$$A_I \otimes_A \Omega^p_{\tilde{A}} \twoheadrightarrow \tilde{A}_I \otimes_{\tilde{A}} \Omega^p_{\tilde{A}} \tag{2.13}$$

has torsion a kernel. By (2.2), $\mathfrak{a}_I/\mathfrak{a}_I^2$ is torsion and surjects onto $T^1(A_I/A)$ (see [21, Cor. 11.10]). Therefore $T^p(A_I/A)$ is torsion for all $p \geq 1$ (see the proof of Proposition 2.1). Replacing A by $\tilde{A}$ also $T^p(\tilde{A}_I/\tilde{A})$ is torsion for all $p \geq 1$. By the Snake Lemma applied to

$$\begin{tikzcd}
0 \arrow[r] & \tilde{T}^p(A_I/A) \arrow[r] \arrow[d] & A_I \otimes_A \Omega^p_{\tilde{A}} \arrow[r] \arrow[d, two heads] & \Omega^p_{\tilde{A}_I} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T^p(\tilde{A}_I/\tilde{A}) \arrow[r] & \tilde{A}_I \otimes_{\tilde{A}} \Omega^p_{\tilde{A}} \arrow[r] & \Omega^p_{\tilde{A}_I} \arrow[r] & 0,
\end{tikzcd}$$

$\tilde{T}^p(A_I/A)$ is an extension of the torsion kernel of (2.13) and $T^p(\tilde{A}_I/\tilde{A})$. $\square$

Proof of Proposition 2.6. Using (1.3), Hom-tensor-adjunction, torsion-freeness of ω^r_A, Lemmas 2.9 and 2.8, we compute

$$\begin{aligned}
\operatorname{Hom}_A(A_I, \omega^p_A) &= \operatorname{Hom}_A(A_I, \operatorname{Hom}_A(\Omega^{r-p}_A, \omega^r_A)) \\
&= \operatorname{Hom}_A(A_I \otimes_A \Omega^{r-p}_A, \omega^r_A) \\
&= \operatorname{Hom}_A(\Omega^{r-p}_{A_I}, \omega^r_A) = \omega^p_{A_I}
\end{aligned} \tag{2.14}$$

and similarly $\operatorname{Hom}_A(A_I, \omega^p_{\tilde{A}}) = \omega^p_{\tilde{A}_I}$. Thus, $\operatorname{Hom}_A(A_I, -)$ applied to the inclusion $\omega^p_{\tilde{A}} \hookrightarrow \omega^p_A$ in diagram (2.5) yields the corresponding with A replaced by A_I. The claim follows. $\square$

Finally, we show that SNCC is compatible with analytic triviality.

Proposition 2.10. *Assume that*

$$A = A'\hat{\otimes}R'',$$

where A' satisfies the hypotheses on A, $\dim A' = r-1$ and $R'' = k\langle\langle x\rangle\rangle$ is regular. Then $\omega_A^0 = \omega_{A'}^0\hat{\otimes}R''$. In particular, A satisfies SNCC if and only if A' does.

Proof. Let (1.1) for A' be a Noether normalization

$$R' = k\langle\langle x_1,\dots,x_{r-1}\rangle\rangle \hookrightarrow A'. \tag{2.15}$$

A Noether normalization and a normalization of A can be obtained by applying $-\hat{\otimes}R''$ to (2.15) and to (2.1) for A' (see [14, III.§5]), that is,

$$R = R'\hat{\otimes}R'' \hookrightarrow A = A'\hat{\otimes}R'' \hookrightarrow \tilde{A} = \tilde{A}'\hat{\otimes}R''.$$

This leads to decompositions (see [14, III.§5.10])

$$\Omega_R^r = \Omega_{R'}^{r-1}\hat{\otimes}\Omega_{R''}^1,\quad \Omega_A^r = \Omega_{A'}^r\hat{\otimes}R'' \oplus \Omega_{A'}^{r-1}\hat{\otimes}\Omega_{R''}^1,$$

where $\Omega_{R''}^1$ and Ω_R^r are free of rank 1 and $\Omega_{A'}^r = \bigwedge^r\Omega_{A'}^1$ and hence $\Omega_{A'}^r\hat{\otimes}R''$ is torsion since $\operatorname{rk}\Omega_{A'}^1 = \dim A' = r-1$ (see [32, (8.8)]). Note that the analytic tensor products over R', R'' and over A', R'' coincide due to finiteness of A' over R' (see [14, III.§5.10]). Using (1.3) and flatness of $R'\to R$, we deduce

$$\begin{aligned}
\omega_A^0 &= \operatorname{Hom}_R(\Omega_A^r,\Omega_R^r)\\
&= \operatorname{Hom}_{R'\hat{\otimes}R''}(\Omega_{A'}^{r-1}\hat{\otimes}\Omega_{R''}^1,\Omega_{R'}^{r-1}\hat{\otimes}\Omega_{R''}^1)\\
&= \operatorname{Hom}_{R'\otimes_{R'}R\otimes_{R''}R''}(\Omega_{A'}^{r-1}\otimes_{R'}R\otimes_{R''}\Omega_{R''}^1,\Omega_{R'}^{r-1}\otimes_{R'}R\otimes_{R''}\Omega_{R''}^1)\\
&= \operatorname{Hom}_{R'\otimes_{R'}R}(\Omega_{A'}^{r-1}\otimes_{R'}R,\Omega_{R'}^{r-1}\otimes_{R'}R)\\
&= \operatorname{Hom}_{R'}(\Omega_{A'}^{r-1},\Omega_{R'}^{r-1}\otimes_{R'}R)\\
&= \operatorname{Hom}_{R'}(\Omega_{A'}^{r-1},\Omega_{R'}^{r-1})\otimes_{R'}R\otimes_{R''}R''\\
&= \omega_{A'}^0\hat{\otimes}R''
\end{aligned}$$

and similarly $\omega_{\tilde{A}}^0 = \omega_{\tilde{A}'}^0\hat{\otimes}R''$. It follows that the inclusions $\omega_{\tilde{A}'}^0\hookrightarrow\omega_{A'}^0$ and $\omega_{\tilde{A}}^0\hookrightarrow\omega_A^0$ correspond via $-\hat{\otimes}R''$ and $-\otimes_{R''}k$. ☐

3. Fractional ideals and ramification

Our approach to SNCC in case of curve and Gorenstein singularities uses that the inclusion $\omega_{\tilde{A}}^r\hookrightarrow\omega_A^r$ is given by the conductor ideal (see (4.2) and Lemma 5.1 below). With the latter we recall the basics on fractional ideals.

Definition 3.1. A (regular) *fractional ideal* of A is an A-submodule M of $L = Q(A)$ such that there exist $a, b\in A^{\mathrm{reg}}$ with $aM\subseteq A$ and $b\in M$.

Since A is Noetherian the first condition is equivalent to M being finitely generated. For any two fractional ideals $M, N\subset L$ of A one can identify

$$\operatorname{Hom}_A(M,N) = N :_L M \subseteq L,\quad \varphi\mapsto\frac{\varphi(m)}{m},\quad m\in M\cap A^{\mathrm{reg}},$$

with a fractional ideal of A. The functor $\operatorname{Hom}_A(-,-)$ is inclusion-reversing (inclusion-preserving) in the first (second) argument on fractional ideals of A. In particular, the dualizing operation

$$-^{-1} := \operatorname{Hom}(-,A)$$

is inclusion-reversing on fractional ideals of A. By (2.3), $Q(A)_\mathfrak{p} = Q(A_\mathfrak{p})$ and localization at $\mathfrak{p} \in \operatorname{Spec} A$ turns fractional ideals of A into fractional ideals of $A_\mathfrak{p}$. The localization of (2.1) at $\mathfrak{p} \in \operatorname{Spec} A$ is the normalization

$$\nu_{A,\mathfrak{p}} \colon A_\mathfrak{p} \hookrightarrow \tilde{A}_\mathfrak{p} = \widetilde{A_\mathfrak{p}}$$

of $A_\mathfrak{p}$ (see [16, Prop. 2.1.6]). If M is a fractional ideal of A then

$$\operatorname{End}_A(M) \subseteq \tilde{A}$$

by the determinantal trick (see [16, Lem. 2.1.8]). The *conductor (ideal)*

$$C_{\tilde{A}/A} := \operatorname{Ann}_A(\tilde{A}/A) = \tilde{A}^{-1} \tag{3.1}$$

is the largest ideal of A which is also an ideal of $\tilde{A}$. Multiplying the denominators of a (finite) set of A-module generators of $\tilde{A}$ yields an element $b \in A^{\mathrm{reg}} \cap C_{\tilde{A}/A}$ showing that $C_{\tilde{A}/A}$ is a fractional ideal of A.

Both in case of curve and Gorenstein singularities the normalization will be unramified as a consequence of SNCC (see Propositions 4.5 and 5.9 below). Denote by $F_A^i(M)$ the ith *Fitting ideal* of an A-module M. Then the *ramification ideal* of the normalization (2.1) is defined by

$$I_{\tilde{A}/A} := F_{\tilde{A}}^0(\Omega^1_{\tilde{A}/A}).$$

Lemma 3.2. *For any $\mathfrak{p} \in \operatorname{Spec} A$,*

$$(C_{\tilde{A}/A})_\mathfrak{p} = C_{\tilde{A}_\mathfrak{p}/A_\mathfrak{p}}, \quad (\Omega^1_{\tilde{A}/A})_\mathfrak{p} = \Omega^1_{\tilde{A}_\mathfrak{p}/A_\mathfrak{p}}, \quad (I_{\tilde{A}/A})_\mathfrak{p} = I_{\tilde{A}_\mathfrak{p}/A_\mathfrak{p}},$$

and following statements are equivalent:

(a) $\tilde{A}_\mathfrak{p}$ is unramified over $A_\mathfrak{p}$.
(b) $\Omega^1_{\tilde{A}_\mathfrak{p}/A_\mathfrak{p}} = 0$.
(c) $I_{\tilde{A}_\mathfrak{p}/A_\mathfrak{p}} = \tilde{A}_\mathfrak{p}$.

In particular, $\Omega^1_{\tilde{A}/A} = 0$ if and only if $A_i = \tilde{A}_i$ for $i = 1, \dots, s$.

Proof. By finiteness of $\tilde{A}$ over A, the conductor (3.1) commutes with flat base change and $\Omega^1_{\tilde{A}/A}$ is the universal differential module which commutes with base change. Fitting ideals commute with flat base change. The first claim and the equivalences follow (see [21, Prop. 6.8]). In particular, $\Omega^1_{\tilde{A}/A} = 0$ if and only if $\tilde{A}$ is unramified over A. Since $k = \overline{k}$, this is equivalent to

$$A_i/\mathfrak{m}_{A_i} = \tilde{A}_i/\mathfrak{m}_{\tilde{A}_i} = \tilde{A}_i/\mathfrak{m}_A\tilde{A}_i = \tilde{A}_i/\mathfrak{m}_{A_i}\tilde{A}_i$$

and hence to $A_i = \tilde{A}_i$ for $i = 1, \dots, s$ by Nakayama's Lemma. ☐

4. Curve singularities

Keeping all hypotheses of §2, we assume in addition that $r = \dim A = 1$. Informally we refer to A as a *curve (singularity)* with *branches* $A_1, \dots, A_s$ and we call it *plane* if

$$\operatorname{edim} A := \dim_k(\mathfrak{m}_A/\mathfrak{m}_A^2) \leq 2.$$

By Serre's normality criterion, the $\tilde{A}_i$ in (2.3) are regular and hence (see [14, II.§5.3])

$$\tilde{A}_i = k\langle\langle t_i \rangle\rangle.$$

We denote by $e_1, \dots, e_s \in \tilde{A}$ the primitive idempotents with $\tilde{A}e_i = \tilde{A}_i$.

For curve singularities we characterize SNCC numerically in terms of the De Rham cohomology of ω_A and the δ-*invariant* of A

$$\delta_A := \dim_k(\tilde{A}/A).$$

Proposition 4.1. *If A is a curve singularity then*

$$\dim_k H^1(\omega_A) \leq \delta_A$$

with equality equivalent to SNCC.

Proof. We set $\lambda_A := \dim_k N_A^0$ (see (1.9)). Then (see [20, (4.5) Satz]),

$$\dim_k H^1(\omega_A) = \mu_A - \lambda_A + s - 1.$$

Using Milnor's formula $\mu_A = 2\delta_A - s + 1$ (see [9, Prop. 1.2.1.1)]) this gives

$$\dim_k H^1(\omega_A) = 2\delta_A - \lambda_A.$$

By Corollary 2.3, the degree-0 part of the leftmost square in diagram (2.5) reads

$$\begin{tikzcd} \tilde{A} \arrow[r, equal, "c^0_{\tilde{A}}"] & \omega^0_{\tilde{A}} \arrow[d, hook] \\ A \arrow[u, hook] \arrow[r, hook, "c^0_A"] & \omega^0_A. \end{tikzcd}$$

Thus, $\lambda_A = \delta_A + \dim_k(\omega_A^0/\omega_{\tilde{A}}^0)$ and the claim follows. $\square$

Our goal is to show that the only curve singularities satisfying SNCC are plane normal crossing. For convenience we extend this notion as follows. Denote the fiber product of the $\tilde{A}_i$ over k by

$$A \hookrightarrow \tilde{A}' := \tilde{A}_1 \times_k \cdots \times_k \tilde{A}_s \hookrightarrow \tilde{A}.$$

Definition 4.2. We call a curve singularity A *normal crossing* if $A = \tilde{A}'$.

If A is normal crossing then $\mathfrak{m}_A = \mathfrak{m}_{\tilde{A}}$, $A_i = \tilde{A}_i$ for $i = 1, \dots, s$, $\operatorname{edim} A = s$ and

$$C_{\tilde{A}/A} = \begin{cases} A, & \text{if } s = 1, \\ \mathfrak{m}_A = \mathfrak{m}_{\tilde{A}}, & \text{if } s \geq 2. \end{cases} \tag{4.1}$$

We will first investigate the Gorenstein property of normal crossing curve singularities using the well-known results collected in the following lemma. The statement on regularity goes back to Jacobinski in far greater generality (see [17]).

Lemma 4.3.

(a) $A \subseteq \mathfrak{m}_A^{-1}$ and, unless A is regular, $\mathfrak{m}_A^{-1} \subseteq \tilde{A}$.
(b) A is Gorenstein if and only if $\dim_k(\mathfrak{m}_A^{-1}/A) = 1$.

Proof.
(a) If $\mathfrak{m}_A^{-1} \subsetneq \operatorname{End}_A(\mathfrak{m}_A)$ then there is a surjection $\mathfrak{m}_A \twoheadrightarrow A$. Since A is projective it splits and hence $\mathfrak{m}_A = xA \oplus I$ for some $x \in A^{\mathrm{reg}}$ Then $xI \subseteq xA \cap I = 0$ implies $I = 0$. It follows that $\mathfrak{m}_A = \langle x \rangle$ and A is regular.
(b) Any $x \in \mathfrak{m}_A \cap A^{\mathrm{reg}}$ induces an isomorphism

$$\operatorname{Ext}^1_A(k, A) \cong \operatorname{Hom}_A(k, A/xA) \cong (xA :_A \mathfrak{m}_A)/xA \xleftarrow[\cong]{\cdot x} \mathfrak{m}_A^{-1}/A. \qquad \square$$

Proposition 4.4. *A normal crossing curve singularity is Gorenstein if and only if it is plane.*

Proof. We may assume that A is singular, that is, $s \geq 2$. By (4.1) and Lemma 4.3.(a), $\mathfrak{m}_A^{-1} = \tilde{A}$ and hence

$$\mathfrak{m}_A^{-1}/A \cong (\tilde{A}/\mathfrak{m}_{\tilde{A}})/(A/\mathfrak{m}_A) \cong k^s/k \cong k^{s-1}.$$

By Lemma 4.3.(b), A is therefore Gorenstein if and only if $\operatorname{edim} A = s \leq 2$. $\square$

We now give a characterization of SNCC for curve singularities. The proof relies on the identity (see [23, Lem. 3.2])

$$\omega^r_{\tilde{A}} = C_{\tilde{A}/A}\omega^r_A. \tag{4.2}$$

We abbreviate $\mathrm{Der} := \mathrm{Der}_k$ to denote k-linear derivations.

Proposition 4.5. *A curve singularity A satisfies SNCC if and only if*

(a) A has regular branches, that is, $A_i = \tilde{A}_i$ for $i = 1, \dots, s$, and
(b) any k-derivation $A \to \omega^1_A$ factors through $\omega^1_{\tilde{A}}$, or equivalently,

$$\mathrm{Der}(A) = \mathrm{Der}(A, C_{\tilde{A}/A})$$

in case A is Gorenstein.

If A is Gorenstein and singular then (b) holds true if

$$C_{\tilde{A}/A} = \mathfrak{m}_A \tag{4.3}$$

and conversely (b) implies (4.3) if in addition A is quasihomogeneous.

Proof. Recall from the proof of Proposition 2.1 that $T^1(\tilde{A}/A)$ and $\Omega^1_{\tilde{A}/A}$ in (2.8) are torsion. So dualizing the short exact sequence

$$0 \to (\tilde{A} \otimes_A \Omega^1_A)/T^1(\tilde{A}/A) \to \Omega^1_{\tilde{A}} \to \Omega^1_{\tilde{A}/A} \to 0$$

obtained from (2.8) with the torsion-free module $\omega^1_{\tilde{A}}$ yields the following expansion of diagram (2.11) in case $r = 1$ and $p = 0$.

$$\begin{tikzcd}
& 0 \arrow[r] & \mathrm{Der}(A, \omega^1_{\tilde{A}}) \arrow[r] \arrow[d, equal] & \mathrm{Der}(A, \omega^1_A) \arrow[d, equal] & \\
& 0 \arrow[r] & \mathrm{Hom}_A(\Omega^1_A, \omega^1_{\tilde{A}}) \arrow[r] \arrow[dd, equal] & \mathrm{Hom}_A(\Omega^1_A, \omega^1_A) \arrow[d, equal] & \\
& \omega^0_{\tilde{A}} \arrow[d, equal] & & \omega^0_A & \\
0 \arrow[r] & \mathrm{Hom}_{\tilde{A}}(\Omega^1_{\tilde{A}}, \omega^1_{\tilde{A}}) \arrow[r] & \mathrm{Hom}_{\tilde{A}}(\tilde{A} \otimes_A \Omega^1_A, \omega^1_{\tilde{A}}) \arrow[r] & \mathrm{Ext}^1_{\tilde{A}}(\Omega^1_{\tilde{A}/A}, \omega^1_{\tilde{A}}) \arrow[r] & 0
\end{tikzcd} \tag{4.4}$$

The upper inclusion comes from the universal property of Ω^1_A. Its surjectivity is condition (b) and reads $\mathrm{Der}(A) = \mathrm{Der}(A, C_{\tilde{A}/A})$ for Gorenstein A due to (4.2). Since $\omega^1_{\tilde{A}}$ is a canonical module of $\tilde{A}$ by (1.3) and $\mathrm{Ext}^1_{\tilde{A}}(\Omega^1_{\tilde{A}/A}, \omega^1_{\tilde{A}})$ is the dual of $\Omega^1_{\tilde{A}/A}$ (see [8, Thm. 3.3.10]), surjectivity of the lower inclusion is equivalent to $\Omega^1_{\tilde{A}/A} = 0$ and hence to condition (a) by Lemma 3.2. Therefore the diagram proves the first claim.

The remaining claims are due to the following facts. If A is singular then $C_{\tilde{A}/A} \subseteq \mathfrak{m}_A$ and $\mathrm{Der}(A) \subseteq \mathrm{Der}(A, \mathfrak{m}_A)$ (see [35, (1.1)]). If A is quasihomogeneous then $\chi(A) = \mathfrak{m}_A$ for some Euler derivation $\chi \in \mathrm{Der}(A, \mathfrak{m}_A)$ (see [22] for a converse). $\square$

Remark 4.6. Let A be a Gorenstein curve singularity.

(a) Combining the degree-0 part of the leftmost square in diagram (2.5) with diagram (4.4) using (1.11) and (4.2) yields commutative diagram

$$\begin{tikzcd}
\tilde{A} \arrow[r, "c^0_{\tilde{A}}"] & \omega^0_{\tilde{A}} \arrow[r, hook] \arrow[d, hook] & \mathrm{Der}(A, C_{\tilde{A}/A}) \arrow[d, hook] \\
A \arrow[u, hook] \arrow[r, hook, "c^0_A"] & \omega^0_A \arrow[r, "\cong"'] & \mathrm{Der}(A).
\end{tikzcd}$$

The image of the bottom row is the module Δ of *trivial derivations* (see [23, §3] or [20, §5]). Condition (b) in Proposition 4.5 can therefore be rephrased as

$$\mathrm{Der}(A)/\Delta \to \mathrm{Der}(A, A/C_{\tilde{A}/A})$$

being the zero map.

(b) Proposition 2.6 can be deduced from Proposition 4.5 as follows. It suffices to show that condition (b) in Proposition 4.5 descends from A to A_I for any subset $I \subseteq \{1, \dots, s\}$. By (2.3), there is a commutative diagram

$$\begin{tikzcd}
\tilde{A} \arrow[r, two heads, "\tilde{\pi}_I"] & \tilde{A}_I \\
A \arrow[u, hook] \arrow[r, two heads, "\pi_I"] & A_I \arrow[u, hook]
\end{tikzcd}$$

and any $\delta_I \in \mathrm{Der}(A_I)$ lifts to a $\delta \in \mathrm{Der}(A)$ preserving $\mathfrak{a}_I$. For $x_I \in A_I$, pick $x \in A$ with $\pi_I(x) = x_I$. Assuming $\delta(x) \in C_{\tilde{A}/A}$, we compute using 4.5.(b) for A that

$$\delta_I(x_I)\tilde{A}_I = \pi_I(\delta(x))\tilde{\pi}_I(\tilde{A}) = \tilde{\pi}_I(\delta(x)\tilde{A}) \subseteq \tilde{\pi}_I(A) = A_I$$

and hence $\delta_I(x_I) \in C_{\tilde{A}_I/A_I}$ which is 4.5.(b) for A_I.

We now examine SNCC for normal crossing curve singularities.

Lemma 4.7. *A normal crossing curve singularity satisfies condition* (b) *of Proposition 4.5 if and only if it is plane.*

Proof. The canonical module ω^1_A of A is an ideal (see [8, Prop. 3.3.18]). With $A = \tilde{A}'$ also this ideal is standard graded and thus isomorphic to A or to $\mathfrak{m}_A$. Using Proposition 4.4, (4.2) and (4.1), this implies that

$$\omega^1_A \cong \begin{cases} A, & \text{if } s \le 2, \\ \mathfrak{m}_A, & \text{if } s \ge 3, \end{cases} \qquad \omega^1_{\tilde{A}} = \begin{cases} \omega^1_A, & \text{if } s = 1, \\ \mathfrak{m}_A \omega^1_A, & \text{if } s \ge 2. \end{cases}$$

If A is singular then $\mathrm{Der}(A) \subseteq \mathrm{Der}(A, \mathfrak{m}_A)$ (see [35, (1.1)]) and $\chi(A) = \mathfrak{m}_A$ for some Euler derivation $\chi \in \mathrm{Der}(A, \mathfrak{m}_A)$. Therefore condition (b) of Proposition 4.5 holds true if and only if $s \le 2$. $\square$

Our starting point for understanding SNCC for general curve singularities are two examples that occur in the proof of the main theorem in [13].

Example 4.8.

(a) In [13, Ex. 3.3.(2)], A is a plane quasihomogeneous curve defined by $\mathfrak{a} = \langle x_2(x_2 - x_1^p)\rangle$ where $p \ge 1$. Its normalization is given by $x_1 = (t_1, t_2)$, $x_2 = (0, t_2^p)$ and

$$C_{\tilde{A}/A} = \langle (t_1^p, t_2^p)\rangle = \langle x_1^p, x_2\rangle.$$

By Proposition 4.5, A satisfies SNCC if and only if $p = 1$.

(b) In [13, Ex. 3.3.(3)], A is the line arrangement defined by $\mathfrak{a} = \langle x_1x_2(x_1 - x_2)\rangle$. Its normalization is given by $x_1 = (t_1, 0, t_3)$, $x_2 = (0, t_2, t_3)$ and

$$C_{\tilde{A}/A} = \big\langle (t_1^2, t_2^2, t_3^2)\big\rangle = \big\langle x_1^2, x_2^2\big\rangle.$$

By Proposition 4.5, SNCC does not hold.

Both statements above are shown in loc. cit. by a different argument due to Saito.

Generalizations of Example 4.8 appear under the following conditions.

Lemma 4.9. *Let A be a non-normal crossing curve singularity different from that in Example 4.8.(a) with $s \geq 2$ branches. Assume that A_I is normal crossing for all $I \subset \{1, \dots, s\}$ with $|I| = s-1$. Then A is the union of $s-1 \geq 2$ coordinate axes and a diagonal as defined by (4.7). In particular, A is homogeneous and Gorenstein of embedding dimension $n = \operatorname{edim} A = s-1$ with conductor $C_{\tilde{A}/A} = \mathfrak{m}_A^2$.*

Proof. With $s \geq 2$ also $n \geq 2$ and $A_i = \tilde{A}_i$ for $i = 1, \dots, s$. Set $J := \{1, \dots, s-1\}$. Then A_J is normal crossing but A is not. Thus, there is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{m}_{A_s} \arrow[r] & \tilde{A}' \arrow[r] & \tilde{A}'_J \arrow[r] & 0 \\
0 \arrow[r] & \mathfrak{a}_J \arrow[r] \arrow[u, hook] & A \arrow[r] \arrow[u, hook] & A_J \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

in which the leftmost inclusion is strict. For any $j \in J$, both A_J and $A_{\{1,\dots,s\}\setminus\{j\}}$ are normal crossing. So there is an element $x_j \in \mathfrak{m}_A$ inducing uniformizers of A_j and A_s but zero in $\mathfrak{m}_{A_i}$ for any $i \neq j, s$. Additional generators of A can be chosen from $\mathfrak{a}_J \subseteq \mathfrak{m}_{A_s}^2$. The inclusion $A \subseteq \tilde{A}'$ is then given by

$$x_i = \begin{cases} u_it_ie_i + v_it_se_s, & i = 1, \dots, s-1, \\ w_it_s^{p_i}e_s, & i = s, \dots, n, \end{cases} \tag{4.5}$$

where the $u_i \in A_i^*$ and the $v_i, w_i \in A_s^*$ are units, $p_i \geq 2$, and $n \geq s-1$. If $n \geq s$, we may assume that $p := p_s$ is minimal and replace t_s to absorb w_s. For $i < s$, we replace x_i and t_i to absorb v_i and u_i. For $i > s$ and $j < s$, we have

$$x_i = w_it_s^{p_i}e_s = w_it_s^{p_i-p}t_s^pe_s = w_i(t_je_j + t_se_s)(t_je_j + t_se_s)^{p_i-p}t_s^pe_s = w_i(x_j)x_j^{p_i-p}x_s$$

which makes x_i redundant.

So we may finally assume that $u_i = v_i = w_i = 1$ and $n \leq s$ in (4.5). This leaves the following two cases extending Example 4.8.

$$n = s \geq 2, \quad p \geq 2, \quad x_i = \begin{cases} t_ie_i + t_se_s, & i = 1, \dots, s-1, \\ t_s^pe_s, & i = n, \end{cases} \tag{4.6}$$

$$n = s-1 \geq 2, \quad x_i = t_ie_i + t_se_s, \; i = 1, \dots, n. \tag{4.7}$$

For $n = 2$, (4.6) and (4.7) define the curve singularities from parts (a) and (b) of Example 4.8, respectively. For $n \geq 3$, (4.6) reduces to (4.7) since $x_n = x_1x_2^{p-1}$ is redundant. Then Lemma 4.10 below concludes the proof. □

Lemma 4.10. *The curve singularity A defined by (4.7) is homogeneous and Gorenstein.*

Proof. It follows from (4.7) that $A = R/\mathfrak{a}$ is defined by $\mathfrak{a} = \langle x_k(x_i - x_j) \mid k \neq i, j\rangle$, and hence homogeneous, and that the conductor equals $C_{\tilde{A}/A} = \mathfrak{m}_{\tilde{A}}^2 = \mathfrak{m}_A^2$. By Lemma 4.3.(a), $\mathfrak{m}_A^{-1}/A$ can be seen as a subquotient in

$$\mathfrak{m}_{\tilde{A}}^2 = C_{\tilde{A}/A} \subseteq A \subseteq \mathfrak{m}_A^{-1} \subseteq \tilde{A}.$$

Due to homogeneity of A this is a chain of standard graded ideals. Then with $\tilde{A}/\mathfrak{m}_{\tilde{A}}^2$ also $\mathfrak{m}_A^{-1}/A$ is non trivial at most in degrees 0 and 1. It follows from (4.7) that $\mathfrak{m}_A^{-1}$ and A have equal constant parts. Setting $t := \sum_{i=1}^s t_i e_i$, we have $t \cdot x_i = x_i^2 \in A$ for $i = 1, \dots, n$ and hence $t \in \mathfrak{m}_A^{-1} \setminus A$. On the other hand, $x_1, \dots, x_n, t$ is a k-basis of the linear part of $\tilde{A}$ with $x_1, \dots, x_n \in A$. Thus, t represents a k-basis of $\mathfrak{m}_A^{-1}/A$ and A is Gorenstein by Lemma 4.3.(b). $\square$

We can finally show that SNCC characterizes plane normal crossings among all curve singularities.

Proposition 4.11. *A curve singularity satisfies SNCC if and only if it is plane normal crossing.*

Proof. Plane normal crossing curve singularities are Gorenstein and therefore satisfy SNCC by (4.1) and Proposition 4.5. Conversely, let A be a curve singularity with s branches satisfying SNCC. If $s = 1$ then $A = A_1 = \tilde{A}_1 = \tilde{A}'$ by Proposition 4.5.(a). We now proceed by induction on s assuming $s \geq 2$. Due to Proposition 2.6 and the induction hypothesis, A_I is normal crossing for all $I \subset \{1, \dots, s\}$ with $|I| = s - 1$. The only curve singularity in Example 4.8.(a) satisfying SNCC is plane normal crossing. The conclusion of Lemma 4.9 contradicts to Proposition 4.5. Therefore A must be normal crossing and hence plane by Lemma 4.7. $\square$

5. Gorenstein singularities

Keeping all hypotheses of §2, we assume in addition that A is Cohen–Macaulay and Gorenstein at $\mathfrak{p} \in \operatorname{Spec} A$. By (1.3), ω_A^r is then a canonical module of A and hence (see [8, Thms. 3.3.5.(b), 3.3.7])

$$\omega_{A,\mathfrak{p}}^r = \omega_{A_\mathfrak{p}}^r \cong A_\mathfrak{p}. \tag{5.1}$$

In particular, $-^{-1} := \operatorname{Hom}_{A_\mathfrak{p}}(-, A_\mathfrak{p})$ corresponds to the duality $\operatorname{Hom}_{A_\mathfrak{p}}(-, \omega_{A,\mathfrak{p}}^r)$ on maximal Cohen–Macaulay modules.

Lemma 5.1. *Let A be Cohen–Macaulay and Gorenstein at $\mathfrak{p} \in \operatorname{Spec} A$. Then*

$$\omega_{\tilde{A},\mathfrak{p}}^r = C_{\tilde{A}_\mathfrak{p}/A_\mathfrak{p}} \omega_{A,\mathfrak{p}}^r \cong C_{\tilde{A}_\mathfrak{p}/A_\mathfrak{p}}.$$

Proof. Let (1.1) be a Noether normalization. By (1.3) and Hom-tensor-adjunction,

$$\omega_{\tilde{A}}^r = \operatorname{Hom}_R(\tilde{A}, \Omega_R^r) = \operatorname{Hom}_A(\tilde{A}, \operatorname{Hom}_R(A, \Omega_R^r)) = \operatorname{Hom}_A(\tilde{A}, \omega_A^r).$$

By finiteness of $\tilde{A}$ over A and (5.1), localization at $\mathfrak{p}$ turns this into

$$\omega_{\tilde{A},\mathfrak{p}}^r = \operatorname{Hom}_{A_\mathfrak{p}}(\tilde{A}_\mathfrak{p}, \omega_{A,\mathfrak{p}}^r) = \operatorname{Hom}_{A_\mathfrak{p}}(\tilde{A}_\mathfrak{p}, A_\mathfrak{p})\omega_{A,\mathfrak{p}}^r = C_{\tilde{A}_\mathfrak{p}/A_\mathfrak{p}} \omega_{A,\mathfrak{p}}^r. \qquad \square$$

Definition 5.2. The *Jacobian* and *ω-Jacobian (ideal)* of A are defined by

$$J_A := F_A^r(\Omega_A^1), \quad J'_A := \operatorname{Ann} \operatorname{coker} c_A^r = \operatorname{im}(c_A^r \otimes (\omega_A^r)^{-1}). \tag{5.2}$$

The ideals in (5.2) satisfy inclusion relations (see [26, Prop. 3.1])

$$J_A \subseteq J'_A \subseteq C_{\tilde{A}/A}. \tag{5.3}$$

The second inclusion is due to Lemma 5.1 and the degree-r part of the leftmost square in diagram (2.5).

Remark 5.3. Since Ω^1_A has rank r (see [32, (4.4)]), $J_{A,\mathfrak{p}_i} = F^r_{A_{\mathfrak{p}_i}}(\Omega^1_{A,\mathfrak{p}_i}) = A_{\mathfrak{p}_i}$ for $i = 1, \dots, s$ and J_A contains a regular element of A by prime avoidance. It follows that both J_A and J'_A are fractional ideals of A. In case of J'_A this follows also from c_A being an isomorphism at regular primes of A (see [18, (5.7.3)]) and Serre's reducedness criterion. If A is a complete intersection then $J_A = J'_A$ (see [33, Lem. 3.1] or [27, Prop. 1] and [26, Prop. 3.2] for a converse).

The statement of [13, Prop. 3.4] for hypersurface singularities generalizes by replacing the Jacobian by the ω-Jacobian.

Lemma 5.4. *Let A be Cohen–Macaulay and Gorenstein at $\mathfrak{p} \in \operatorname{Spec} A$. Then*

$$\sigma^0_{A,\mathfrak{p}} = (J'_{A,\mathfrak{p}})^{-1}$$

as fractional ideals of $A_\mathfrak{p}$.

Proof. We use (1.6) to identify ω_A with σ_A. By (5.2) and the Gorenstein hypothesis this turns $c^r_{A,\mathfrak{p}}$ into a map $\Omega^r_{A,\mathfrak{p}} \twoheadrightarrow J'_{A,\mathfrak{p}}\sigma^r_{A,\mathfrak{p}}$ with torsion cokernel. Then (1.7) localized at $\mathfrak{p}$ becomes $\sigma^0_{A,\mathfrak{p}} = \operatorname{Hom}_{A,\mathfrak{p}}(J'_{A,\mathfrak{p}}\sigma^r_{A,\mathfrak{p}}, \sigma^r_{A,\mathfrak{p}}) = (J'_{A,\mathfrak{p}})^{-1}$. □

Definition 5.5. We call A *free at* $\mathfrak{p} \in \operatorname{Spec} A$ if A is Cohen–Macaulay, $A_\mathfrak{p}$ is Gorenstein and $J'_{A,\mathfrak{p}}$ is a Cohen–Macaulay ideal. We say that A is *free* if it is free at $\mathfrak{m}_A$.

The Aleksandrov–Terao theorem (see [1, §2 Thm.] and [38, Prop. 2.4]) generalizes as follows.

Proposition 5.6. *Let A be Cohen–Macaulay and Gorenstein at $\mathfrak{p} \in \operatorname{Spec} A$. Then freeness of A at $\mathfrak{p}$ with $A_\mathfrak{p} \neq J'_{A,\mathfrak{p}}$ is equivalent to $A_\mathfrak{p}/J'_{A,\mathfrak{p}}$ being Cohen–Macaulay of dimension* $\dim A_\mathfrak{p} - 1$.

Proof. By Remark 5.3, $J'_{A,\mathfrak{p}} \subsetneq A_\mathfrak{p}$ is a fractional ideal of $A_\mathfrak{p}$ (see §3). In particular, it contains an element of $A^{\mathrm{reg}}_\mathfrak{p} \setminus A^*_\mathfrak{p}$ and hence $\operatorname{ht} J'_{A,\mathfrak{p}} \geq 1$. The claim follows (see [15, Satz 4.13] and [8, Thm. 2.1.2.(a)]). □

By (5.3), (3.1), Corollary 2.3, and Propositions 2.1, there is an ascending chain of fractional ideals

$$J'_A \subseteq C_{\tilde{A}/A} \subseteq A \subseteq \tilde{A} = \sigma^0_{\tilde{A}} \subseteq \sigma^0_A. \tag{5.4}$$

We deduce the following generalization of [13, Cor. 3.7].

Corollary 5.7. *Let A be Cohen-Macaulay and free at $\mathfrak{p} \in \operatorname{Spec} A$. Then A satisfies SNCC at $\mathfrak{p}$ if and only if $J'_{A,\mathfrak{p}} = C_{\tilde{A}_\mathfrak{p}/A_\mathfrak{p}}$.*

Proof. By reflexivity of $\tilde{A}$ (see [13, Lem. 2.8]), (3.1) and Lemma 5.4, the first and last inclusions in (5.4) localized at $\mathfrak{p} \in \operatorname{Spec} A$ are duals of each other. □

We recall an identity of ideals due to Piene (see [27, Cor. 1]) in case of a smooth normalization.

Lemma 5.8. *Let A be Cohen-Macaulay and let $\mathfrak{p} \in \operatorname{Spec} A$ such that $A_\mathfrak{p}$ is Gorenstein and $\tilde{A}_\mathfrak{p}$ is regular. Then $I_{\tilde{A}_\mathfrak{p}/A_\mathfrak{p}} C_{\tilde{A}_\mathfrak{p}/A_\mathfrak{p}} = \tilde{A} J'_{A,\mathfrak{p}}$.*

Proof. Since $\tilde{A}_\mathfrak{p}$ is regular, $\Omega^1_{\tilde{A},\mathfrak{p}}$ is locally free of rank r (see [32, (4.4),(8.7)]). The map $\tilde{A} \otimes d\nu_A$ from (2.8) is a presentation of $\Omega^1_{\tilde{A}/A}$. Using Lemma 3.2, it follows that $I_{\tilde{A}_\mathfrak{p}/A_\mathfrak{p}} \Omega^r_{\tilde{A},\mathfrak{p}}$ is the image of the map

$$\tilde{A} \otimes \bigwedge^r d\nu_{A,\mathfrak{p}} \colon \tilde{A} \otimes_A \Omega^r_{A,\mathfrak{p}} \to \Omega^r_{\tilde{A},\mathfrak{p}}$$

obtained by localizing the map $\tilde{A} \otimes \bigwedge^r d\nu_A$ from (2.9) at $\mathfrak{p}$. Together with the degree-r part of the leftmost square in diagram (2.5) localized at $\mathfrak{p}$ this map fits into a commutative diagram

$$\begin{tikzcd}
 & \Omega^r_{\tilde{A},\mathfrak{p}} \arrow[r, "c^r_{\tilde{A},\mathfrak{p}}", "\cong"'] & \omega^r_{\tilde{A},\mathfrak{p}} \arrow[d, hook] \\
 & \Omega^r_{A,\mathfrak{p}} \arrow[u, "\bigwedge^r d\nu_{A,\mathfrak{p}}"'] \arrow[r, "c^r_{A,\mathfrak{p}}"] \arrow[dl] & \omega^r_{A,\mathfrak{p}} \\
\tilde{A} \otimes_A \Omega^r_{A,\mathfrak{p}} \arrow[uur, "\tilde{A}\otimes\bigwedge^r d\nu_{A,\mathfrak{p}}"] \arrow[urr, "\tilde{A}\otimes c^r_{A,\mathfrak{p}}"'] & &
\end{tikzcd}$$

where $c^r_{\tilde{A},\mathfrak{p}}$ is an isomorphism since $\tilde{A}_\mathfrak{p}$ is regular (see [18, (5.7.3)]). Using Lemma 5.1 and (5.1) it follows that

$$\begin{aligned}
I_{\tilde{A}_\mathfrak{p}/A_\mathfrak{p}} C_{\tilde{A}_\mathfrak{p}/A_\mathfrak{p}} \omega^r_{A,\mathfrak{p}} = I_{\tilde{A}_\mathfrak{p}/A_\mathfrak{p}} \omega^r_{\tilde{A},\mathfrak{p}} &= \operatorname{im}\Big(c^r_{\tilde{A},\mathfrak{p}} \circ \tilde{A} \otimes \bigwedge^r d\nu_{A,\mathfrak{p}}\Big) \\
&= \operatorname{im}(\tilde{A} \otimes c^r_{A,\mathfrak{p}}) = \tilde{A} \operatorname{im} c^r_{A,\mathfrak{p}} = \tilde{A} J'_{A,\mathfrak{p}} \omega^r_{A,\mathfrak{p}}.
\end{aligned}$$

The claim follows by (5.1). □

The following result generalizes [13, Lem. 4.2].

Proposition 5.9. *Let A be Cohen–Macaulay and free at $\mathfrak{p} \in \operatorname{Spec} A$ such that $\tilde{A}_\mathfrak{p}$ is regular. Then A satisfies SNCC at $\mathfrak{p}$ if and only if $J'_{A,\mathfrak{p}}$ is an ideal of $\tilde{A}_\mathfrak{p}$ and $\tilde{A}_\mathfrak{p}$ is unramified over $A_\mathfrak{p}$.*

Proof. By Lemma 5.1, (1.12) and regularity of $\tilde{A}_\mathfrak{p}$ (see [18, (5.7.3)]),

$$C_{\tilde{A}_\mathfrak{p}/A_\mathfrak{p}} \cong \omega^r_{\tilde{A},\mathfrak{p}} \cong \Omega^r_{\tilde{A},\mathfrak{p}} = \bigwedge^r \Omega^1_{\tilde{A},\mathfrak{p}}$$

is locally free of rank 1 (see [32, (4.4),(8.7)]). By Corollary 5.7, SNCC for A at $\mathfrak{p}$ is equivalent to $J'_{A,\mathfrak{p}} = C_{\tilde{A}_\mathfrak{p}/A_\mathfrak{p}}$. By Lemma 5.8, this is equivalent to $\tilde{A}_\mathfrak{p} J'_{A,\mathfrak{p}} = J'_{A,\mathfrak{p}}$ and $I_{\tilde{A}_\mathfrak{p}/A_\mathfrak{p}} = \tilde{A}_\mathfrak{p}$. The claim follows using Lemma 3.2. □

6. Complex analytic spaces

In order to consider analytic spaces, we need in addition to the hypotheses of §2 that k is non-discretely valued. Therefore we assume that $k = \mathbb{C}$ and consider (germs of) complex analytic spaces.

Let X be a reduced r-equidimensional complex analytic space with normalization $\nu_X : \tilde{X} \to X$. Then there is an $\mathscr{O}_X$-coherent graded (Ω_X, d)-module ω_X and a trace map $c_X : \Omega_X \to \omega_X$ (see [7]). The *Jacobian* and *ω-Jacobian (ideals)* J_X and J'_X of X are defined as in (5.2). Taking stalks at $x \in X$ leads to the corresponding objects for $A = \mathscr{O}_{X,x}$. By a *complex analytic singularity* we mean the germ of a complex analytic space.

Definition 6.1. We say that a reduced equidimensional complex analytic space X satisfies *Saito's normal crossing condition (SNCC)* or that X is *free* if $A = \mathscr{O}_{X,x}$ satisfies the corresponding property for all $x \in X$ (see Definition 2.4 and Definition 5.5). We say that X satisfies a property in codimension (up to) c if it does outside of an analytic subset of codimension at least $c + 1$. We define the corresponding properties for complex analytic singularities by requiring them for some representative.

Remark 6.2. That X satisfies SNCC means that the inclusion of coherent $\mathscr{O}_X$-modules $(\nu_X)_*\omega^0_{\tilde{X}} \hookrightarrow \omega^0_X$ is an equality (see [7, p.195, Ex. i)]). In particular, SNCC is an open condition.

Freeness is an open condition as well. In fact, Cohen–Macaulay loci of coherent $\mathscr{O}_X$-modules are open (see [31, Satz 7]) and the Gorenstein locus of a Cohen–Macaulay X is the open set where the coherent $\mathscr{O}_X$-module ω^r_X is locally free of rank 1 (see [8, Thm. 3.3.7.(a)]).

Both SNCC and freeness are satisfied in codimension 0, that is, generically.

The following is the analytic version of Proposition 2.5.

Proposition 6.3. *A reduced equidimensional complex analytic singularity X satisfies SNCC if it does in codimension one.*

Proof. Assume that X satisfies SNCC in codimension one and replace X by a representative. Let $x \in X$ and set $A := \mathscr{O}_{X,x}$. Consider the coherent $\mathscr{O}_X$-module $\mathscr{F} = \omega^0_X/(\nu_X)_*\omega^0_{\tilde{X}}$ and the coherent $\mathscr{O}_X$-ideal $\mathscr{I} = \operatorname{Ann}\mathscr{F}$. By hypothesis and Remark 6.2, $V(\mathscr{I}) = \operatorname{Supp}\mathscr{F}$ and hence $V(\mathscr{I}_x)$ has codimension at least 2. In particular, for any $\mathfrak{p} \in \operatorname{Spec} A$ with $\operatorname{ht}\mathfrak{p} = 1$, $\operatorname{Ann}(\mathscr{F}_x) = \mathscr{I}_x \notin \mathfrak{p}$ and hence $\omega^0_{A,\mathfrak{p}}/\omega^0_{\tilde{A},\mathfrak{p}} = (\mathscr{F}_x)_\mathfrak{p} = 0$. In other words, A satisfies SNCC in codimension one. Then $\mathscr{O}_{X,x} = A$ satisfies SNCC due to Proposition 2.5. This means that X satisfies SNCC at x. Therefore X satisfies SNCC as claimed. ☐

In case of smooth irreducible components our results from §4 apply to a transversal curve singularity.

Proposition 6.4. *Let X be a reduced equidimensional complex analytic singularity with smooth local irreducible components in codimension one. If X satisfies SNCC then it must be a normal crossing divisor in codimension one.*

Proof. Set $r := \dim X$ and denote by $m := n - r$ the codimension of X in some smooth ambient space $(\mathbb{C}^n, 0)$. We may freely move the base point of the germ X to a general point in codimension one. Let Z be the reduced singular locus of X. We may assume that $Z \neq \emptyset$ is smooth of codimension one and that the irreducible components $X_1, \dots, X_s$ of X are smooth containing Z. By Proposition 2.6, SNCC descends to any union of irreducible components of X. We may therefore assume that $2 \leq s \leq 3$ and that $X_1 \cup \dots \cup X_{s-1}$ is a normal crossing divisor. Then there are local coordinates such that

$$\begin{aligned} Z &= \{x_1 = \dots = x_{m+1} = 0\}, \\ X_i &= \{x_1 = \dots = \widehat{x}_i = \dots = x_{m+1} = 0\},\ i = 1, \dots, s-1. \end{aligned} \tag{6.1}$$

By the implicit function theorem, there is a $j \in \{1, \dots, m+1\}$ such that

$$X_s = \{x_i = y_i(x_j, x_{m+2}, \dots, x_n) \mid j \neq i = 1, \dots, m+1\}.$$

If $y_i \neq 0$ then we may write $y_i = x_j^{p_i} u_i$ with $u_i(0, x_{m+2}, \dots, x_n) \neq 0$. We may then assume that the latter and hence also u_i is a unit. Dividing x_i by u_i results in $u_i = 1$ leaving (6.1) unchanged. This makes the defining equations of $X_1, \dots, X_s$, and hence of X, independent of $x_{m+2}, \dots, x_n$. Then X becomes a product $X = C \times Z$ where C is a curve in the transversal slice $\{x_{m+2} = \dots = x_n = 0\}$. By Proposition 2.10, with X also C satisfies SNCC. Then Proposition 4.11 forces C to be plane normal crossing. In particular, $s = 2$ and X is a normal crossing divisor. ☐

Example 6.5. The free divisor $D = \{xy(x + y)(x + xz) = 0\}$ has smooth reduced singular locus $Z = \{x = y = 0\}$ and 4 smooth local irreducible components at points of Z. However it is not analytically trivial along Z in codimension one.

We are finally ready to prove our main result.

Proof of Theorem 0.1. Suppose first that X satisfies SNCC. By Proposition 6.3, SNCC for X is a codimension-one condition. We may therefore assume that X is free and that $\tilde{X}$ is smooth. Proposition 5.9 then implies that ν_X is unramified. By Lemma 3.2 this means that X has smooth local irreducible components. Proposition 6.4 then forces X to be a normal crossing divisor in codimension one. The converse implication follows from Propositions 2.10, 4.11, and 6.3. ☐

We conclude with an application of our approach to splayed divisors. By a *divisor* we mean a reduced hypersurface singularity. Let $D_1, D_2 \subset (\mathbb{C}^{r+1}, 0)$ be divisors. Then D_1 and D_2 are called *splayed* (see [12]) if

$$D_1 \cong D_1' \times (\mathbb{C}^{r_2+1}, 0), \quad D_2 \cong (\mathbb{C}^{r_1+1}, 0) \times D_2'.$$

for divisors $D_i' \subset (\mathbb{C}^{r_i+1}, 0)$ for $i = 1, 2$ under some isomorphism $(\mathbb{C}^{r+1}, 0) \cong (\mathbb{C}^{r_1+1}, 0) \times (\mathbb{C}^{r_2+1}, 0)$. In this case we call the union $D_1 \cup D_2$ a *splayed divisor.* In other words, splayed divisors are *product unions*

$$D_1' \uplus D_2' := D_1' \times (\mathbb{C}^{r_2+1}, 0) \cup (\mathbb{C}^{r_1+1}, 0) \times D_2'$$

of divisors (see [10, §3]). Aluffi and Faber characterized splayedness in terms of logarithmic differential forms (see [6, Thm. 2.12]). Passing to the residual part of these forms yields a characterization in terms of regular differential forms.

Proposition 6.6. *Let $D_i = V(h_i) \subseteq (\mathbb{C}^{r+1}, 0)$ for $i = 1, 2$ be divisors. If D_1 and D_2 are splayed then the natural map*

$$\omega^0_{D_1 \sqcup D_2} = \omega^0_{D_1} \oplus \omega^0_{D_2} \to \omega^0_D \tag{6.2}$$

is an isomorphism. The converse holds true if $D = D_1 \cup D_2$ is free.

Proof. The map in (6.2) is obtained using (2.14) by applying $\operatorname{Hom}_{\mathscr{O}_D}(-, \omega^0_D)$ to the inclusion

$$\mathscr{O}_{D_1 \sqcup D_2} = \mathscr{O}_{D_1} \times \mathscr{O}_{D_2} \hookleftarrow \mathscr{O}_D. \tag{6.3}$$

If D_1 and D_2 have a common irreducible component D', which is not the case if they are are splayed, then applying $\operatorname{Hom}_{\mathscr{O}_D}(-, \omega^0_D)$ to the commutative diagram

$$\begin{tikzcd}
\mathscr{O}_{D_1} \times \mathscr{O}_{D_2} \arrow[d, two heads] & \mathscr{O}_D \arrow[l, hook'] \arrow[d, two heads] \\
\mathscr{O}_{D'} \times \mathscr{O}_{D'} & \mathscr{O}_{D'} \arrow[l, "{(\mathrm{id},\mathrm{id})}"']
\end{tikzcd}$$

and using (2.14) yields a commutative diagram

$$\begin{tikzcd}
\omega^0_{D_1} \oplus \omega^0_{D_2} \arrow[r] & \omega^0_D \\
\omega^0_{D'} \oplus \omega^0_{D'} \arrow[u, hook] \arrow[r, "+"] & \omega^0_{D'} \arrow[u, hook]
\end{tikzcd}$$

whose top row is (6.2). As $\omega^0_{D'} \neq 0$ this shows that (6.2) is not injective in this case. Therefore we may assume that D_1 and D_2 do not have a common irreducible component. Then (6.3) has a torsion cokernel and (6.2) is an inclusion since ω^0_D is torsion-free.

As in Proposition 2.1 there is a commutative diagram

$$\begin{tikzcd} \sigma^0_{D_1} \oplus \sigma^0_{D_2} \arrow[r, hook] \arrow[d, "\cong"] & \sigma^0_D \arrow[d, "\cong"] \\ \omega^0_{D_1} \oplus \omega^0_{D_2} \arrow[r, hook] & \omega^0_D. \end{tikzcd} \tag{6.4}$$

In fact, using (1.17) and (6.3) one computes that

$$c_{D_1} + c_{D_2} = \begin{bmatrix} dh_1 \\ h_1 \end{bmatrix} + \begin{bmatrix} dh_2 \\ h_2 \end{bmatrix} \mapsto \begin{bmatrix} h_2 dh_1 + h_1 dh_2 \\ h_1 h_2 \end{bmatrix} = \begin{bmatrix} d(h_1 h_2) \\ h_1 h_2 \end{bmatrix} = c_D$$

by the lower inclusion in (6.4). By [6, Thm. 2.2], D_1 and D_2 are splayed if and only if the natural inclusion of Jacobian ideals

$$J_D \hookrightarrow h_2 J_{D_1} \oplus h_1 J_{D_2} \tag{6.5}$$

is an equality. Lemma 5.4 identifies the upper inclusion in (6.4) as the dual of (6.5) and the first claim follows. Indeed, dualizing $\mathscr{O}_{D_1} = \mathscr{O}_D / h_1 \mathscr{O}_D$ over $\mathscr{O}_D$ yields

$$\operatorname{Hom}_{\mathscr{O}_D}(\mathscr{O}_{D_1}, \mathscr{O}_D) = \ker(h_1 \colon \mathscr{O}_D \to \mathscr{O}_D) = h_2 \mathscr{O}_D = h_2 \mathscr{O}_{D_1}$$

and hence by Hom-tensor-adjunction

$$\operatorname{Hom}_{\mathscr{O}_D}(-, \mathscr{O}_D) = \operatorname{Hom}_{\mathscr{O}_{D_1}}(-, \operatorname{Hom}_{\mathscr{O}_D}(\mathscr{O}_{D_1}, \mathscr{O}_D)) = h_2 \operatorname{Hom}_{\mathscr{O}_{D_1}}(-, \mathscr{O}_{D_1})$$

on $\mathscr{O}_{D_1}$-modules. Conversely, if D is free then J_D is reflexive and hence

$$(\sigma^0_D)^{-1} = J_D \hookrightarrow h_2 J_{D_1} \oplus h_1 J_{D_2} \hookrightarrow h_2 \cdot (\sigma^0_{D_1})^{-1} \oplus h_1 \cdot (\sigma^0_{D_2})^{-1} = (\sigma^0_{D_1} \oplus \sigma^0_{D_2})^{-1}.$$

Thus, dualizing an equality in (6.4) yields an equality in (6.5). ☐

Remark 6.7. If the divisors D_1 and D_2 have no common irreducible component then

$$\tilde{D} \twoheadrightarrow D_1 \sqcup D_2 \twoheadrightarrow D$$

and condition (6.2) can be seen as a weak form of SNCC.

References

[1] A. G. Aleksandrov. Nonisolated Saito singularities. *Mat. Sb. (N.S.)*, 137(179)(4):554–567, 576, 1988.

[2] A. G. Aleksandrov. Nonisolated hypersurface singularities. In *Theory of singularities and its applications*, volume 1 of *Adv. Soviet Math.*, pages 211–246. Amer. Math. Soc., Providence, RI, 1990.

[3] A. G. Aleksandrov. Multidimensional residue theory and the logarithmic de Rham complex. *J. Singul.*, 5:1–18, 2012. DOI: 10.5427/jsing.2012.5a

[4] Aleksandr G. Aleksandrov and Avgust K. Tsikh. Théorie des résidus de Leray et formes de Barlet sur une intersection complète singulière. *C. R. Acad. Sci. Paris Sér. I Math.*, 333(11):973–978, 2001.

[5] Aleksandr G. Aleksandrov and Avgust K. Tsikh. Multi-logarithmic differential forms on complete intersections. *J. Sib. Fed. Univ. Math. Phys.*, 1(2):105–124, 2008.

[6] Paolo Aluffi and Eleonore Faber. Splayed divisors and their Chern classes. *J. Lond. Math. Soc. (2)*, 88(2):563–579, 2013.

[7] Daniel Barlet. Le faisceau $\omega^{\cdot}_X$ sur un espace analytique X de dimension pure. In *Fonctions de plusieurs variables complexes, III (Sém. François Norguet, 1975–1977)*, volume 670 of *Lecture Notes in Math.*, pages 187–204. Springer, Berlin, 1978.

[8] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993.

[9] Ragnar-Olaf Buchweitz and Gert-Martin Greuel. The Milnor number and deformations of complex curve singularities. *Invent. Math.*, 58(3):241–281, 1980. DOI: 10.1007/BF01390254

[10] James Damon. Higher multiplicities and almost free divisors and complete intersections. *Mem. Amer. Math. Soc.*, 123(589):x+113, 1996.

[11] David Eisenbud. *Commutative algebra with a view toward algebraic geometry*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[12] Eleonore Faber. Towards Transversality of Singular Varieties: Splayed Divisors. *Publ. Res. Inst. Math. Sci.*, 49(3):393–412, 2013. DOI: 10.4171/PRIMS/109

[13] Michel Granger and Mathias Schulze. Normal crossing properties of complex hypersurfaces via logarithmic residues. *Compos. Math.*, 150(9):1607–1622, 2014. DOI: 10.1112/S0010437X13007860

[14] H. Grauert and R. Remmert. *Analytische Stellenalgebren*. Springer-Verlag, Berlin, 1971. Unter Mitarbeit von O. Riemenschneider, Die Grundlehren der mathematischen Wissenschaften, Band 176. DOI: 10.1007/978-3-642-65033-8

[15] Jürgen Herzog and Ernst Kunz, editors. *Der kanonische Modul eines Cohen-Macaulay-Rings*. Lecture Notes in Mathematics, Vol. 238. Springer-Verlag, Berlin, 1971. Seminar über die lokale Kohomologietheorie von Grothendieck, Universität Regensburg, Wintersemester 1970/1971.

[16] Craig Huneke and Irena Swanson. *Integral closure of ideals, rings, and modules*, volume 336 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2006.

[17] H. Jacobinski. Two remarks about hereditary orders. *Proc. Amer. Math. Soc.*, 28:1–8, 1971. DOI: 10.1090/S0002-9939-1971-0272807-7

[18] M. Kersken. Der Residuenkomplex in der lokalen algebraischen und analytischen Geometrie. *Math. Ann.*, 265(4):423–455, 1983. DOI: 10.1007/BF01455946

[19] Masumi Kersken. Cousinkomplex und Nennersysteme. *Math. Z.*, 182(3):389–402, 1983. DOI: 10.1007/BF01179758

[20] Masumi Kersken. Reguläre Differentialformen. *Manuscripta Math.*, 46(1-3):1–25, 1984. DOI: 10.1007/BF01185193

[21] Ernst Kunz. *Kähler differentials*. Advanced Lectures in Mathematics. Friedr. Vieweg & Sohn, Braunschweig, 1986.

[22] Ernst Kunz and Walter Ruppert. Quasihomogene Singularitäten algebraischer Kurven. *Manuscripta Math.*, 22(1):47–61, 1977. DOI: 10.1007/BF01182066

[23] Ernst Kunz and Rolf Waldi. Über den Derivationenmodul und das Jacobi-Ideal von Kurvensingularitäten. *Math. Z.*, 187(1):105–123, 1984. DOI: 10.1007/BF01163171

[24] Joseph Lipman. Dualizing sheaves, differentials and residues on algebraic varieties. *Astérisque*, (117):ii+138, 1984.

[25] Qing Liu. *Algebraic geometry and arithmetic curves*, volume 6 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné, Oxford Science Publications.

[26] Anna Oneto and Elsa Zatini. Jacobians and differents of projective varieties. *Manuscripta Math.*, 58(4):487–495, 1987. DOI: 10.1007/BF01277606

[27] Ragni Piene. Ideals associated to a desingularization. In *Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978)*, volume 732 of *Lecture Notes in Math.*, pages 503–517. Springer, Berlin, 1979. DOI: 10.1007/bfb0066659

[28] Delphine Pol. A characterization of freeness for complete intersections. *arXiv.org*, 1512.06778, 2015.

[29] Kyoji Saito. Theory of logarithmic differential forms and logarithmic vector fields. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 27(2):265–291, 1980.

[30] G. Scheja and U. Storch. *Lokale Verzweigungstheorie*. Institut des Mathématiques, Université de Fribourg, Fribourg, 1974. Vorlesungen über Kommutative Algebra (Wintersemester 1973/74), Schriftenreihe des Mathematischen Institutes der Universität Freiburg, No. 5.

[31] Günter Scheja. Fortsetzungssätze der komplex-analytischen Cohomologie und ihre algebraische Charakterisierung. *Math. Ann.*, 157:75–94, 1964. DOI: 10.1007/BF01362668

[32] Günter Scheja and Uwe Storch. Differentielle Eigenschaften der Lokalisierungen analytischer Algebren. *Math. Ann.*, 197:137–170, 1972. DOI: 10.1007/BF01419591

[33] Günter Scheja and Uwe Storch. Residuen bei vollständigen Durchschnitten. *Math. Nachr.*, 91:157–170, 1979. DOI: 10.1002/mana.19790910113

[34] Günter Scheja and Hartmut Wiebe. Über Derivationen von lokalen analytischen Algebren. In *Symposia Mathematica, Vol. XI (Convegno di Algebra Commutativa, INDAM, Rome, 1971)*, pages 161–192. Academic Press, London, 1973.

[35] Günter Scheja and Hartmut Wiebe. Über Derivationen in isolierten Singularitäten auf vollständigen Durchschnitten. *Math. Ann.*, 225(2):161–171, 1977.

[36] Jean-Pierre Serre. *Algèbre locale. Multiplicités*, volume 11 of *Cours au Collège de France, 1957–1958, rédigé par Pierre Gabriel. Seconde édition, 1965. Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1965.

[37] Rodney Y. Sharp. The Cousin complex for a module over a commutative Noetherian ring. *Math. Z.*, 112:340–356, 1969. DOI: 10.1007/BF01110229

[38] Hiroaki Terao. Arrangements of hyperplanes and their freeness. I. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 27(2):293–312, 1980.

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SIMPLE DYNAMICS AND INTEGRABILITY FOR SINGULARITIES OF HOLOMORPHIC FOLIATIONS IN DIMENSION TWO

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ABSTRACT. In this paper we study the dynamics of a holomorphic vector field near a singular point in dimension two. We consider those for which the set of separatrices is finite and the orbits are closed off this analytic set. We assume that none of the singularities arising in the reduction of the foliation has a zero eigenvalue. Under these hypotheses we prove that one of the following cases occurs: (i) there is a holomorphic first integral, (ii) the induced foliation is a pull-back of a hyperbolic linear singularity, (iii) there is a formal Liouvillian first integral. For a germ with closed leaves off the set of separatrices we prove that the existence of a holomorphic first integral is equivalent to the existence of some closed leaf arbitrarily close to the singularity. For this we do not need to assume any non-degeneracy hypothesis on the reduction of singularities. We also study some examples illustrating our results and we prove a characterization of pull-backs of hyperbolic singularities in terms of the dynamics of the leaves off the set of separatrices.

1. INTRODUCTION AND MAIN RESULTS

In this paper we resume the subject of *dynamics versus integrability* for a singularity of holomorphic vector field in dimension two (see [9, 27]). Some references in this subject are results of H. Poincaré, G. Darboux ([13]) (for polynomial vector fields in the complex plane) and more recently [16].

A modern starting point is the following theorem of Mattei-Moussu ([16]): *A germ of a holomorphic vector field at the origin of $\mathbb{C}^2$ admits a holomorphic first integral if, and only if, it has only finitely many leaves accumulating at the singularity and all other leaves are closed.* Also notable is the point of view adopted in [1] where the authors suppose the existence of an uniform bound for the volume of the orbits of the vector field. A holomorphic vector field X defined in a neighborhood $U \subset \mathbb{C}^2$ of the origin $0 \in \mathbb{C}^2$, with an isolated singularity at the origin, defines a germ of holomorphic foliation with a singularity at the origin, and conversely. In this paper we shall adopt the foliation terminology. We shall refer to a germ of a holomorphic foliation $\mathcal{F}$ as induced such a pair (X, U) where X is a holomorphic vector field defined a neighborhood U of the origin $0 \in \mathbb{C}^2$, singular at the origin $X(0) = 0$. Recall that a *separatrix* is an invariant irreducible analytic curve containing the singularity. Throughout this paper we will only consider germs of foliations with a finite number of separatrices, called *non-dicritical* singularities. In this case, we shall say that a leaf of $\mathcal{F}$ (i.e., an orbit of (X, U) for U small enough) *is closed off the set of separatrices* if either it is a separatrix, or it is not a separatrix but accumulates only at the union of separatrices. In few words, it accumulates at no leaf which is not contained in a separatrix. We then characterize those germs of foliations, under the additional hypothesis that they belong to the class of *generalized curves*, meaning that the reduction of singularities does not exhibit final singularities with a null eigenvalue. Before stating our first result we shall state a few notions. Recall that a germ of a singular holomorphic foliation $\mathcal{F}$ at the origin $0 \in \mathbb{C}^2$ is defined by a germ of a holomorphic one-form ω at the origin. We shall assume that $\operatorname{sing}(\omega) = \{0\}$. A holomorphic

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first integral for $\mathcal{F}$ is a germ of a (non-constant) holomorphic function $\mathcal{O}_2 \ni f\colon (\mathbb{C}^2,0) \to (\mathbb{C},0)$ such that $\omega \wedge df = 0$. In terms of Saito-De Rham division lemma, this is equivalent to say that $\omega = g df$ for some germ $g \in \mathcal{O}_2$, provided that we take $f \in \mathcal{O}_2$ as a reduced germ. The function g is necessarily a unit. Thus, if we write $\eta = \frac{dg}{g}$ then we have a germ of a *closed* holomorphic one-form such that $d\omega = \eta \wedge \omega$. In general, the germ $\mathcal{F}$ admits a *Liouvillian first integral* if there is a closed *meromorphic* one-form germ η such that $d\omega = \eta \wedge \omega$. Such a form η is called a *generalized integrating factor* for ω. In this case we say that the first integral is the Liouvillian function F defined by the differential algebraic equation $dF = \frac{\omega}{\exp \int \eta}$. This is all discussed in [31, 28]. We shall now introduce a slightly more general notion:

Definition 1.1. We shall say that $\mathcal{F}$ admits a *formal Liouvillian first integral* $\hat{F}$ if there is a *formal generalized integrating factor* $\hat{\eta}$ which is a formal closed meromorphic one-form such that $(*)\, d\omega = \hat{\eta} \wedge \omega$.

We may rewrite (*) as $d(\frac{\omega}{\exp \int \hat{\eta}}) = 0$, so that the formal Liouvillian first integral is defined by $d\hat{F} = \frac{\omega}{\exp \int \hat{\eta}}$. By a *formal meromorphic* one-form we mean a formal expression $\hat{\eta} = \hat{A} dx + \hat{B} dy$ where $\hat{A}, \hat{B}$ are quotient of formal functions $\hat{A} = \hat{a_1}/\hat{a_2}, \hat{B} = \hat{b_1}/\hat{b_2}$, $\hat{a_j}, \hat{b_j} \in \hat{\mathcal{O}_2}$ ([12]). With these notions we can state:

Theorem 1.2. *Let $\mathcal{F}$ be a germ of a non-dicritical generalized curve at $0 \in \mathbb{C}^2$. Assume that the leaves of $\mathcal{F}$ are closed off the set of separatrices. Then we have three possibilities:*

1. *$\mathcal{F}$ admits a holomorphic first integral.*
2. *$\mathcal{F}$ is a holomorphic pull-back of a hyperbolic (linearizable) singularity.*
3. *$\mathcal{F}$ admits a formal Liouvillian first integral.*

Possibility (3) really occurs, indeed, there is a number of examples which correspond to this last situation. We shall refer to these foliations as *of formal Liouvillian type*. Some information about these foliations is given in § 5. Indeed, the formal one-form $\hat{\eta}$ is actually convergent except in the so called *exceptional case*, which we will detail later on.

The foliation is already in case (2) if some singularity in the reduction of the singularities of the foliation is a non-resonant singularity. More generally, we are in case (2) if there is some non-resonant map in the virtual holonomy group of any separatrix of $\mathcal{F}$. Indeed, from the proof we give for Theorem 1.2 we obtain:

Theorem 1.3. *For a germ of a generalized curve holomorphic foliation $\mathcal{F}$ at the origin $0 \in \mathbb{C}^2$ assume that the following conditions are true:*

1. *There is only a finite number of separatrices and all leaves are closed off the set of separatrices.*
2. *Some separatrix has a holonomy map which is not a resonant map.*

Then $\mathcal{F}$ is the pull-back of a hyperbolic singularity.

We stress that the second hypothesis means that there is some separatrix of $\mathcal{F}$ whose local holonomy is of the form $f(z) = e^{2\pi\sqrt{-1}\lambda} z + a_{k+1} z^{k+1} + \ldots$, where $\lambda \in \mathbb{C} \setminus \mathbb{Q}$. We may assume, instead of (2), the weaker condition that the *virtual* holonomy of some separatrix contains some non-resonant map.

The hypotheses in Theorems 1.2 and 1.3 depend on the concept of reduction of singularities, detailed in Section 2. In short, $\mathcal{F}$ is a *generalized curve* if its reduction of singularities only produces singularities with non-zero eigenvalues. It is *non-dicritical* if there are only finitely many separatrices. The necessity of the generalized curve hypothesis in Theorems 1.2 and 1.3 is discussed in Examples 5.3 and 5.4.

We may conclude that we are in case (1) if arbitrarily close to the union of the separatrices we can find some closed leaf. Indeed, for the next result we do not need to assume that the singularity is a generalized curve:

Theorem 1.4. *Let $\mathcal{F}$ be a germ of a non-dicritical foliation at $0 \in \mathbb{C}^2$. Assume that the leaves of $\mathcal{F}$ are closed off the set of separatrices and there is a closed leaf arbitrarily close to the origin. Then $\mathcal{F}$ admits a holomorphic first integral.*

Outline of the proofs:

The proofs are based on a product of two points:

- A description of subgroups of germs of one-dimensional complex diffeomorphisms with closed orbits off the fixed point: these groups are finite, abelian linearizable generated by a hyperbolic map and a periodic (rational) rotation, or solvable discrete (cf. Proposition 4.2).
- A description of the singularities in the reduction of singularities of $\mathcal{F}$ by the blowing-up process.

We apply the above to the holonomy groups arising in the reduction of singularities of $\mathcal{F}$ and to some enriched groups called *virtual holonomy groups*. The possible combinations of these larger groups are also studied in order to prove that they are all solvable of a same type. For this we consider the connection between two such groups associated to adjacent components of the exceptional divisor of the reduction of singularities. This connection is given by the so called *Dulac correspondence* in suitable cases. When there is a closed leaf arbitrarily close to the singular point it is proven that all these groups have a closed orbit and then are finite. This is the case that correspond to the holomorphic first integral (cf. [16], [9]). It is also proven that if some of these virtual holonomy groups contains a map whose linear part is not periodic, then it must be hyperbolic and all these groups are abelian generated by a hyperbolic map and a rational rotation. This case corresponds to (2) in Theorem 1.2 via techniques from [5]. Finally, in the remaining case all the singularities in the reduction process are resonant as well as all the holonomies are solvable. In this case, by techniques from [26] or [21] we are able to construct a formal Liouvillian first integral. This construction is detailed in the Appendix § 9.

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2. Reduction of singularities in dimension two ([30])

Fix now a germ of holomorphic foliation with a singularity at the origin $0 \in \mathbb{C}^2$. Choose a representative $\mathcal{F}(U)$ for the germ $\mathcal{F}$, defined in an open neighborhood U of the origin, such that 0 is the only singularity of $\mathcal{F}(U)$ in U. The *Theorem of reduction of singularities* of Seidenberg ([30]) asserts the existence of a proper holomorphic map $\sigma\colon \tilde{U} \to U$ which is a finite composition of quadratic blowing-up's, starting with a blowing-up at the origin, such that the pull-back foliation $\tilde{\mathcal{F}} := \sigma^*\mathcal{F}$ of $\mathcal{F}$ by σ satisfies:

(1) The *exceptional divisor* $E(\mathcal{F}) = \sigma^{-1}(0) \subset \widetilde{U}$ can be written as $E(\mathcal{F}) = \bigcup_{j=1}^m D_j$, where each irreducible component D_j is diffeomorphic to an embedded projective line $\mathbb{C}P(1)$ introduced as a divisor of the successive blowing-up's ([7]).

(2) $\mathrm{sing}\tilde{\mathcal{F}} \subset \mathrm{E}$ is a finite set, and any singularity $\tilde{p} \in \mathrm{sing}\tilde{\mathcal{F}}$ is *irreducible* i.e., belongs to one of the following categories:
   (a) $xdy - \lambda ydx + \text{h.o.t.} = 0$ and λ is not a positive rational number, i.e. $\lambda \notin \mathbb{Q}_+$ (*non-degenerate singularity*),
   (b) $y^{p+1}dx - [x(1+\lambda y^p) + \text{h.o.t.}]\, dy = 0$, $p \geq 1$. This case is called a *saddle-node* ([18]).

A singularity is a *generalized curve* if its reduction of singularities produces only non-degenerate (i.e., no saddle-node) singularities ([6]). We call the lifted foliation $\tilde{\mathcal{F}}$ the *desingularization* or *reduction of singularities* of $\mathcal{F}$. The foliation is non-dicritical iff $E(\mathcal{F})$ is invariant by $\tilde{\mathcal{F}}$. Any two components D_i and D_j, $i \neq j$, of the exceptional divisor, intersect (transversely) at at most one point, which is called a *corner*. There are no triple intersection points.

3. Holonomy and virtual holonomy groups

Let now $\mathcal{F}$ be a holomorphic foliation with (isolated) singularities on a complex surface M (we have in mind here, the result of a reduction of singularities process). Denote by $\mathrm{sing}(\mathcal{F})$ the singular set of $\mathcal{F}$. Given a leaf L_0 of $\mathcal{F}$ we choose any base point $p \in L_0 \subset M \setminus \mathrm{sing}(\mathcal{F})$ and a transverse disc $\Sigma_p \Subset M$ to $\mathcal{F}$ centered at p. The holonomy group of the leaf L_0 with respect to the disc Σ_p and to the base point p is image of the representation $\mathrm{Hol}\colon \pi_1(\mathrm{L_0, p}) \to \mathrm{Diff}(\Sigma_\mathrm{p}, \mathrm{p})$ obtained by lifting closed paths in L_0 with base point p, to paths in the leaves of $\mathcal{F}$, starting at points $z \in \Sigma_p$, by means of a transverse fibration to $\mathcal{F}$ containing the disc Σ_p ([4]). Given a point $z \in \Sigma_p$ we denote the leaf through z by L_z. Given a closed path $\gamma \in \pi_1(L_0, p)$ we denote by $\tilde{\gamma}_z$ its lift to the leaf L_z and starting (the lifted path) at the point z. Then the image of the corresponding holonomy map is $h_{[\gamma]}(z) = \tilde{\gamma}_z(1)$, i.e., the final point of the lifted path $\tilde{\gamma}_z$. This defines a diffeomorphism germ map $h_{[\gamma]}\colon (\Sigma_p, p) \to (\Sigma_p, p)$ and also a group homomorphism $\mathrm{Hol}\colon \pi_1(\mathrm{L_0, p}) \to \mathrm{Diff}(\Sigma_\mathrm{p}, \mathrm{p})$. The image $\mathrm{Hol}(\mathcal{F}, \mathrm{L_0}, \Sigma_\mathrm{p}, \mathrm{p}) \subset \mathrm{Diff}(\Sigma_\mathrm{p}, \mathrm{p})$ of such homomorphism is called the *holonomy group* of the leaf L_0 with respect to Σ_p and p. By considering any parametrization $z\colon (\Sigma_p, p) \to (\mathbb{D}, 0)$ we may identify (in a non-canonical way) the holonomy group with a subgroup of $\mathrm{Diff}(\mathbb{C}, 0)$. It is clear from the construction that the maps in the holonomy group preserve the leaves of the foliation. Nevertheless, this property can be shared by a larger group that may therefore contain more information about the foliation in a neighborhood of the leaf. The *virtual holonomy group* of the leaf with respect to the transverse section Σ_p and base point p is defined as ([5], [8])

$$\mathrm{Hol^{virt}}(\mathcal{F}, \Sigma_\mathrm{p}, \mathrm{p}) = \{\mathrm{f} \in \mathrm{Diff}(\Sigma_\mathrm{p}, \mathrm{p}) \big| \mathrm{L_z} = \mathrm{L_{f(z)}}, \forall \mathrm{z} \in (\Sigma_\mathrm{p}, \mathrm{p})\}.$$

The virtual holonomy group contains the holonomy group and consists of the map germs that preserve the leaves of the foliation.

Fix now a germ of holomorphic foliation with a singularity at the origin $0 \in \mathbb{C}^2$, with a representative $\mathcal{F}(U)$ as above. Let Γ be a separatrix of $\mathcal{F}$. By Newton-Puiseaux parametrization theorem, $\Gamma \setminus \{0\}$ is biholomorphic to a punctured disc $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$. In particular, we may choose a loop $\gamma \in \Gamma \setminus \{0\}$ generating the (local) fundamental group $\pi_1(\Gamma \setminus \{0\})$. The corresponding holonomy map h_γ is defined in terms of a germ of complex diffeomorphism at the origin of a local disc Σ transverse to $\mathcal{F}$ and centered at a non-singular point $q \in \Gamma \setminus \{0\}$. This map is well-defined up to conjugacy by germs of holomorphic diffeomorphisms, and is generically referred to as *local holonomy* of the separatrix Γ. The connection between the dynamics of the leaves and the local holonomy is stated as follows:

Lemma 3.1. *Let $\mathcal{F}$ be a germ of a holomorphic foliation at the origin $0 \in \mathbb{C}^2$. Assume that $\mathcal{F}$ has only a finite number of separatrices and that there is a neighborhood V of the origin such that on V each leaf of the foliation is closed off the set of separatrices. Let $\Gamma \subset V$ be a separatrix of $\mathcal{F}\big|_V$, $p \in \Gamma \setminus \{0\}$ and Σ_p a small disc transverse to the foliation and centered at p.*

Then:

(1) *The orbits of the local holonomy of* Γ *and of the virtual holonomy group of* Γ *are closed off the origin.*
(2) *A leaf that accumulates at* Γ *properly and is a closed leaf in* V*, induces for the virtual holonomy group* $\mathrm{Hol}^{\mathrm{virt}}(\mathcal{F}, \Sigma_{\mathrm{p}}, \mathrm{p})$ *a pseudo-orbit which is closed.*

In what follows we consider the following situation: $\mathcal{F}$ is a foliation as in Theorem 1.2. We perform the reduction of singularities for $\mathcal{F}$ obtaining:

(1) A proper map $\sigma\colon \tilde{U} \to U$ which is a finite composition of quadratic blow-ups.
(2) A foliation $\tilde{\mathcal{F}} = \sigma^*(\mathcal{F})$ with only irreducible singularities of non-degenerate type.
(3) An invariant exceptional divisor $E(\mathcal{F}) = \sigma^{-1}(0) = \bigcup\limits_{j=1}^{r} D_j$.

Lemma 3.2. *Let* $q = D_i \cap D_j$ *be a (non-degenerate) corner singularity. Given small transverse discs* Σ_j *and* Σ_i *with* $\Sigma_j \cap D_j = \{q_j\}$ *and* $\Sigma_i \cap D_i = \{q_i\}$*, nonsingular points close enough to* q*, then we have: any local leaf of* $\tilde{\mathcal{F}}$ *that accumulates properly at the origin of* Σ_i *also accumulates properly at the origin of* Σ_j*.*

A combination of Lemmas 3.1 and 3.2 actually shows that:

Proposition 3.3. *Let* $\mathcal{F}$ *be as in Theorem 1.2. Then, all virtual holonomy groups* $\mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}, \mathrm{D_j})$ *of the components of* $D_j \subset E(\mathcal{F})$ *are groups with closed orbits off the origin. If moreover* $\mathcal{F}$ *has a closed leaf arbitrarily close to the origin, then each virtual holonomy group* $\mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}, \mathrm{D_j})$ *exhibits a closed pseudo-orbit arbitrarily close to the origin.*

4. Groups of complex diffeomorphisms

Let $\mathrm{Diff}(\mathbb{C}, 0)$ denote the group of germs at the origin $0 \in \mathbb{C}$ of holomorphic diffeomorphisms. It is a well-known result that a a finite group of germs of complex diffeomorphisms is analytically conjugate to a cyclic group generated by a rational rotation. We shall now study the connection between our dynamical hypothesis and the classification of the possible holonomy groups arising in the reduction of singularities. We start with the case of a sole irreducible singularity. This is done in what follows (cf. Lemma 6.1).

4.1. **Non-resonant maps and Pérez-Marco results.** A germ of a complex diffeomorphism f at the origin $0 \in \mathbb{C}$ writes $f(z) = e^{2\pi\sqrt{-1}\lambda} z + a_{k+1} z^{k+1} + \ldots$. The linear part $f'(0) = e^{2\pi\sqrt{-1}\lambda}$ does not depend on the coordinate system. We shall say that the germ $f \in \mathrm{Diff}(\mathbb{C}, 0)$ is *resonant* if $\lambda \in \mathbb{Q}^*$. If $\lambda \notin \mathbb{R}$ then $|f'(0)| \neq 1$ and the germ is *hyperbolic*. In the hyperbolic case the diffeomorphism is *analytically linearizable*, i.e., conjugated to its linear part by a germ of a map ([2]). In particular, its dynamics is one of an attractor or of a repeller. If $|f'(0)| = 1$, then we have $f'(0) = e^{2\pi\sqrt{-1}\theta}$ for some $\theta \in \mathbb{R}$. If $f'(0)$ is a root of the unity (i.e., if $\lambda \in \mathbb{Q}$) then f is called *resonant* and the dynamics of f is well-known ([2, 3]). In particular, if f is not linearizable, the orbits are closed off the origin, but no orbit is closed. If $f'(0)$ is not a root of the unity then we have $\lambda \in \mathbb{R} \setminus \mathbb{Q}$. In this case we shall say that the diffeomorphism if *non-resonant*. Assume that the map is not analytically linearizable. Given a representative defined in an open connected subset $0 \in U \subset \mathbb{C}$ the *stable set* of f in U is defined by $K(U, f) = \bigcap\limits_{j=0}^{\infty} f^{-j}(U)$ According to Pérez-Marco [22, 23])). It is compact, connected and not reduced to $\{0\}$. Any point of $K(U, f) \setminus \{0\}$ is recurrent (that is, a limit point of its orbit). Moreover, *there is an orbit in* $K(U, f)$ *which*

*It is common to refer to a map as a non-resonant map in case $\lambda \in \mathbb{R} \setminus \mathbb{Q}$. This may cause some confusion in our current framework. That is why we only define the resonant maps. All other maps are non-resonant for us.

accumulates at the origin and no non-trivial orbit of f converges to the origin. Such a map f will also be referred to in this paper as a *Pérez-Marco* map germ.

4.2. **Groups with closed orbits off the origin.** We shall now study the case of groups modeling the holonomy and virtual holonomy groups appearing in the reduction of singularities. The following definition will be useful.

Definition 4.1. A group $G \subset \mathrm{Diff}(\mathbb{C}, 0)$ of germs of holomorphic diffeomorphisms will be called *resonant* if each map $g \in G$ is a resonant germ. This is equivalent to the fact that G has a set of generators consisting only of resonant maps.

Denote by $\xi \subset \mathbb{C}$ the subset of roots of the unity. Our main result is:

Proposition 4.2. *Let $G \subset \mathrm{Diff}(\mathbb{C}, 0)$ be a finitely generated subgroup such that pseudo-orbits are closed off the origin in any small neighborhood of the origin $0 \in \mathbb{C}$.*

Then we have the following possibilities:

(1) *G is a finite cyclic group, generated by a rational rotation.*
(2) *G is abelian analytically linearizable generated by a periodic rotation and a hyperbolic map.*
(3) *G is resonant, either abelian or solvable non-abelian. In the non-abelian case G is formally conjugate to a subgroup of $\{(z \mapsto \frac{az}{(1+bz^k)^{\frac{1}{k}}})\,;\ a \in \xi, b \in \mathbb{C}\}$, for some $k \in \mathbb{N}$. In this case the subgroup $G_1 \subset G$ of maps tangent to the identity is discrete of the form $(z \mapsto \frac{z}{(1+\beta z^k)^{\frac{1}{k}}})\,;\ \beta \in \mathbb{C}$, where all the β belong to a set of type $\{n_1\beta_1 + n_2\beta_2; n_1, n_2 \in \mathbb{Z}\}$ for some $\beta_1, \beta_2 \in \mathbb{C}$.*

In particular, if G contains some non-resonant map, then it is as in (2).

Proof. By Nakai density theorem, the group G must be solvable. In particular, G is abelian or it is formally conjugate to a subgroup of the group $\mathbb{H}_k = \{(z \mapsto \frac{az}{(1+bz^k)^{\frac{1}{k}}})\,;\ a \neq 0, b \in \mathbb{C}\}$, for some $k \in \mathbb{N}$ ([10], [15]). Notice that $\mathbb{H}_k$ is a finite ramified covering of the group of homographies $\mathbb{H}_1$ by a map $z \mapsto z^k$ ([10]). If G is finite then G is as in (1) as it is well-known. Assume that G contains some hyperbolic diffeomorphism, say a map $f \in G$ whose multiplier is of the form $f'(0) = e^{2\pi\sqrt{-1}\alpha}$ where $\alpha \in \mathbb{C} \setminus \mathbb{R}$. In this case we claim that G is abelian. Indeed, assume that G is not abelian. Then G contains some nontrivial commutator and therefore some nontrivial flat element $g \in G$, $g = z + cz^\ell + \text{h.o.t.}$ for some $c \neq 0$. By what we have observed above there is a homography fixing the origin $T(z) = \frac{\lambda z}{1+\mu z}$ such that $(f(z))^k = T(z^k)$. From this we get $f'(0) = \lambda^{\frac{1}{k}}$. Since f is hyperbolic we have that $1 \neq \lambda = T'(0)$. Therefore T is conjugated to a linear map by another homography. Consequently, we may assume that $f(z) = f'(0).z$ and $g(z) = \frac{z}{(1+\beta z^k)^{\frac{1}{k}}}$. By a ramified covering map (ramified change of coordinates) $Z = \frac{1}{z^k}$ we consider the subgroup corresponding to $\langle f, g\rangle$ and which is generated by a homothety $(Z \mapsto \mu Z)$, with $|\mu| \neq 1$, and a translation $(Z \mapsto Z + \beta)$. It is well known that such a group has no orbit closed off the origin. The same then holds for the group G that contains the subgroup generated by f, g above, contradiction.

The above shows that in case G contains a hyperbolic map, it must be abelian, without flat elements. Since it contains a hyperbolic (analytically linearizable) map, the group G is analytically linearizable, so that it embeds as a subgroup of the multiplicative group $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$. Again, because G has orbits closed off the origin, G must then be generated by a hyperbolic map and a rational rotation (see the proof of Lemma 8 in [5] for a similar situation). The group G is then as in (2).

Now for the final part of the proposition, we may therefore proceed assuming that G contains no hyperbolic map. We claim:

Claim 4.3. *The group G contains no non-resonant map $f \in G$, i.e., there is no map $f \in G$ with multiplier $f'(0) = e^{2\pi\sqrt{-1}\theta}$ where $\theta \in \mathbb{R} \setminus \mathbb{Q}$.*

proof of the Claim. Assume by contradiction that there is $f \in G$ a nonresonant map. If f is analytically linearizable then no orbit is closed off the origin, indeed such orbits are dense on circles centered at the origin in some linearizing coordinates. Thus this case is excluded. Assume therefore that $f \in G$ is a Pérez-Marco map. In this case by Pérez-Marco result in Section 4.1 there is a pseudo-orbit which is not closed off the origin, contradiction. This case is also excluded then. $\square$

Assume now that G is not abelian. Let us now conclude that the group is as in (3). Every map in the group G is resonant. We embed $G \hookrightarrow \mathbb{H}_k = \{(z \mapsto \frac{\alpha z}{(1+\beta z^k)^{\frac{1}{k}}})\,;\ \alpha \neq 0, \beta \in \mathbb{C}\}$. This embedding is analytic unless the group is *exceptional*, in which case it already has the desired form (cf. [10] page 460 Theorem 1, see also Example 5.7). Assume then that the embedding is analytic. Given any map $g \in G$ we write $g(z) = \frac{az}{(1+bz^k)^{\frac{1}{k}}} \in G$. Since g is resonant we have $a \in \xi$. Since G is solvable, the subgroup $G_1 \subset G$ of flat elements, is abelian and analytically conjugated to a group of the form $(z \mapsto \frac{z}{(1+\beta z^k)^{1/k}})\,;\ \beta \in \mathbb{C}$. In particular, G_1 acts like a group of translations in the line $\mathbb{C}$. Since the orbits of G are closed orbits off the origin, we conclude that G_1 must be discrete so that all the β belong to a set of type $\{n_1\beta_1 + n_2\beta_2; n_1,\, n_2 \in \mathbb{Z}\}$ for some $\beta_1, \beta_2 \in \mathbb{C}$. This shows that G is as in (3). $\square$

From the proof of Proposition 4.2 we actually get:

Corollary 4.4. *Let $G \subset \mathrm{Diff}(\mathbb{C}, 0)$ be a (not necessarily finitely generated) subgroup such that pseudo-orbits are closed off the origin in any small neighborhood of the origin $0 \in \mathbb{C}$. Then:*

1. *Any finitely generated subgroup $H \subset G$ with a non-trivial closed pseudo-orbit is finite.*
2. *If the group G contains a map which is not a resonant map then G is abelian linearizable generated by a hyperbolic attractor and a periodic rotation.*

Proof. We apply Proposition 4.2. If a subgroup $H \subset G$ contains a non-trivial closed pseudo-orbit then it cannot contain any flat element (i.e., any element tangent to the identity). In particular, H is abelian and its resonant maps are periodic. Moreover, there are no non-resonant maps: a non-resonant map $f \in H$ is of the form $f(z) = e^{2\pi i\lambda}z + a_{k+1}z^{k+1} + \dots$ with $\lambda \notin \mathbb{Q}$. If $\lambda \in \mathbb{C} \setminus \mathbb{R}$ then f is hyperbolic and linearizable. This map cannot have a finite orbit off the origin. If $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ then by the proof of Proposition 4.2 we know that f cannot have all its orbits closed off the origin. We conclude that H is abelian consisting only of periodic maps. If H is finitely generated then it is finite. This proves the first part of the lemma. Let us now assume that G contains some map $f \in G$ which is non-resonant. This map is necessarily hyperbolic as we have seen above. But then G is abelian by Proposition 4.2 because in all other cases the group G is resonant. Applying the result of this same proposition we conclude that G is generated by g and some rational rotation. $\square$

5. Examples

In this section we perform a construction and give some examples related to our main results. We also discuss some possible extensions and a related question.

Example 5.1. We shall now construct an example of a fully-resonant foliation $\mathcal{F}$ with closed leaves off the origin, non-dicritical and a generalized curve, but without a holomorphic first integral[†]. Fixed $a \in \mathbb{C} \setminus \{0\}$ we consider the subgroup $G \subset \mathbb{H}_2$ of maps of the form $z \mapsto \frac{\xi z}{\sqrt{1+naz^2}}$ where $n \in \mathbb{Z}$ and $\xi^4 = 1$. The group G has discrete pseudo-orbits off the origin, indeed, it is generated by the periodic maps $f(z) = \frac{iz}{\sqrt{1+az^2}}$ and $g(z) = iz$, where $i^2 = -1$. The group G is finitely generated by periodic maps (f has order 4 and g has order 2), but it has infinite order because $g \circ f(z) = \frac{-z}{\sqrt{1+az^2}}$.

The map f is conjugate to the local holonomy map of the separatrix $(y = 0)$ of a linearizable saddle-type singularity q_f with a holomorphic first integral, say of the form $xdy + 4ydx = 0$. Similarly g is conjugate to the holonomy of a separatrix $(y = 0)$ of a linearizable saddle singularity q_g with holomorphic first integral, of the form $xdy + 2ydx = 0$. Finally, the map $h = (g \circ f)^{-1}$ is conjugate to the holonomy of a separatrix $(y = 0)$ of a non-linearizable resonant saddle-type singularity q_h of the form $\omega_{k,\ell} = kxdy + \ell y(1 + \frac{\sqrt{-1}}{2\pi} x^\ell y^k)dx = 0$ where $\ell = 1$ and $k = 2$.

According to [14] we can construct a germ of a holomorphic foliation $\mathcal{F}$ at the origin $0 \in \mathbb{C}^2$, having three separatrices contained in lines, and which can be reduced with a single blowing-up at the origin. The blow-up foliation $\tilde{\mathcal{F}}(1)$ then has exactly three singularities in the invariant projective line $E(\mathcal{F})(1)$, and the holonomy group of the leaf $L_0 = E(1) \setminus \operatorname{sing}(\tilde{\mathcal{F}}(1))$ is conjugated to the group generated by f, g and $h = (g \circ f)^{-1}$, which is the group G. The singularities of $\tilde{\mathcal{F}}(1)$ are locally conjugated to q_f, q_g and q_h with the above mentioned separatrices contained in the exceptional divisor. All the dynamics of the foliation $\mathcal{F}$ is then described by its projective holonomy, i.e., by the holonomy of the leaf L_0 of the blow-up foliation $\tilde{\mathcal{F}}(1)$. In particular, $\mathcal{F}$ has closed leaves off the set of separatrices. Nevertheless, because group G is not abelian, $\mathcal{F}$ is not given by a closed meromorphic one-form. The foliation admits a Liouvillian first integral. Indeed, the group G embeds into $\mathbb{H}_2$, $\mathcal{F}$ is non-dicritical reduced with a single blow-up and it is a generalized curve ([29] Chapter I, §5, pages 185-188 or [21]). This is also proved as follows: There is a system of coordinate charts $\{U_j, (x_j, y_j)\}_{j \in J}$ covering a neighborhood of L_0 in the blow-up $\tilde{\mathbb{C}}^2_0$, such that:

- $E(\mathcal{F})(1) \cap U_j = L_0 \cap U_j \subset \{y_j = 0\}$.
- On each open subset U_j the blow-up foliation $\tilde{\mathcal{F}}(1)$ is given by $dy_j = 0$.
- If $U_i \cap U_j \neq \emptyset$ then $U_i \cap U_j$ is connected and in this intersection we have $y_j = \phi_{ij}(y_i)$ for some map $\phi_{ij} \in \mathbb{H}_2$.

Then we can write on each U_j the lifted one-form $\tilde{\omega} = \pi^*(\omega)$ as $\tilde{\omega}\big|_{U_j} = g_j dy_j$ for some meromorphic function g_j on U_j. Then we define $\tilde{\eta}$ on each U_j by $\tilde{\eta}\big|_{U_j} = 2\frac{dy_j}{y_j} + \frac{dg_j}{g_j}$. The extension of $\tilde{\eta}$ to the singularities q_f, q_g and q_h is then proved as in [8] or else [26]. This shows the existence of a closed meromorphic one-form $\tilde{\eta}$ in a neighborhood of the projective line $E(\mathcal{F})(1)$ in the space $\tilde{\mathbb{C}}^2_0$. This form satisfies $d\tilde{\omega} = \tilde{\eta} \wedge \tilde{\omega}$. Projecting this one-form into a one-form η in a neighborhood of the origin $0 \in \mathbb{C}^2$ we get a generalized integrating factor for ω. Thus $\mathcal{F}$ admits a Liouvillian first integral. Another (much more general) way of constructing the form η is given in [21] and it is based on the notion of symmetry for the group G.

Notice that in Example 5.1 above, one of the singularities has a non-periodic holonomy. This seems to be an unavoidable situation if one looks for groups which are not finite, but with closed pseudo-orbits off the origin as projective holonomy groups. This fact together with Theorem 1.3 in [27] and Theorem 1.1 in [9], suggests the following question:

Question 5.2. *Given a germ of a foliation $\mathcal{F}$ at $0 \in \mathbb{C}^2$ such that:*

1. *$\mathcal{F}$ is a non-dicritical generalized curve.*
2. *The leaves of $\mathcal{F}$ are closed off the separatrices.*

[†]I am grateful to the anonymous referee for showing me Example 5.1.

(3) *Each separatrix has a periodic local holonomy map.*

Does $\mathcal{F}$ admit a holomorphic first integral?

Example 5.3. This example suggests the possibility of extending the conclusion of Theorem 1.2 for singularities which are not generalized curves. We consider a germ of a saddle-node singularity $\mathcal{F}$, given by $xdy - y^{k+1}dx + \ldots = 0$ at $0 \in \mathbb{C}^2$. According to [18] there is a formal diffeomorphism $\hat{\phi} \in \hat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ such that $\phi^*(\mathcal{F})$ is given by $\mathcal{S}_{k,a} : x(1+ay^k)dy - y^{k+1}dx = 0$, for some $a \in \mathbb{C}$. The *formal model* $\mathcal{S}_{k,a}$ admits the Liouvillian first integral given by the generalized integrating factor $\eta = d\log(xy^{k+1})$. In particular, the saddle-node $\mathcal{F}$ admits a formal Liouvillian first integral. An example with closed leaves off the set of separatrices is given by $\omega = xdy - y^2dx = 0$ at the origin $0 \in \mathbb{C}^2$. Integration of $\Omega = \frac{1}{xy^2}\omega$ gives the first integral $f = xe^{\frac{-1}{y}}$. The leaves are closed off the strong separatrix $(y = 0)$.

Example 5.4. This example is related to Question 5.2 above formulated. We construct a germ of a foliation $\mathcal{F}$ at $0 \in \mathbb{C}^2$ such that:

(1) $\mathcal{F}$ is non-dicritical.
(2) The leaves of $\mathcal{F}$ are closed off the separatrices.
(3) Each separatrix has a periodic local holonomy map.
(4) $\mathcal{F}$ does not admit a holomorphic first integral.

Nevertheless:

(4) $\mathcal{F}$ is not a generalized curve.

We consider the subgroup $G \subset \mathrm{Diff}(\mathbb{C}, 0)$ generated by $f(z) = \frac{z}{1-z}$ and $g(z) = -z$. This group is solvable, finite discrete pseudo-orbits off the origin. Indeed, it leaves invariant the function $\varphi(z) = \cos(\frac{2\pi}{z})$. We show that this group corresponds to the holonomy group of the projective line of the blowing-up of a non-dicritical singularity germ $\mathcal{F}$ at the origin $0 \in \mathbb{C}^2$. Indeed, we first consider the map $h = f \circ g$, i.e., $h(z) = \frac{-z}{1+z}$. This is a periodic map since $h \circ h = \mathrm{Id}$. Thus, we have $f \circ g \circ h = \mathrm{Id}$. Moreover, each diffeomorphism above corresponds to the holonomy of a germ of irreducible singularity as follows:

- f is conjugate to the map $z \mapsto \frac{z}{1+2\pi z}$, which is the holonomy map of the strong separatrix $(y = 0)$ of the saddle-node $q_f : xdy - y^2dx = 0$, evaluated at the transverse disc $\Sigma : (x = 1)$.
- $g(z) = -z$ is the holonomy of the separatrix $(y = 0)$ of the singularity with holomorphic first integral $q_g : xy^2$.
- h is also the holonomy of a separatrix of a singularity with first integral $q_h : xy^2$.

Then, according to [14] we can construct a germ of a holomorphic foliation at the origin $0 \in \mathbb{C}^2$, having three separatrices, and which can be reduced with a single blowing-up at the origin. The foliation $\tilde{\mathcal{F}}(1)$ then has exactly three singularities in the invariant projective line $E(\mathcal{F})(1)$, and the holonomy group of the leaf $L_0 = E(\mathcal{F})(1) \setminus \mathrm{sing}(\tilde{\mathcal{F}}(1))$ is conjugated to the group generated by f, g and h, which is the group G. The singularities of $\tilde{\mathcal{F}}(1)$ are locally conjugated to q_f, q_g and q_h. In particular, the saddle-node has its strong manifold contained in the projective line $E(\mathcal{F})(1)$ and the separatrix associated to this singularity at 0 is the central manifold, which has trivial holonomy map. The foliation $\mathcal{F}$ then has closed leaves off the set of separatrices, and periodic holonomy for each of its separatrices. Nevertheless, it does not admit a holomorphic first integral (it is not a generalized curve).

Example 5.5 (resonant singularities cf. [17])**.** According to Martinet-Ramis ([17]) a resonant non-linearizable singularity is formally isomorphic to an unique equation

$$\omega_{p/q,k,\lambda} := p(1 + (\lambda - 1)u^k)ydx + q(1 + \lambda u^k)xdy,$$

where $p, q, k \in \mathbb{N}, \lambda \in \mathbb{C}$ and $u := x^p y^q$. Moreover $p/q, k, \lambda$ are the formal invariants of the equation. By introducing integral numbers $n, m \in \mathbb{Z}$ such that $mp - nq = 1$ we can rewrite $\omega_{p/q,k,\lambda} = (1+(\lambda - mp)u^k)(pydx + qxdy) + pqu^k(nydx + mxdy)$. This last expression admits the integrating factor $h_{p/q,k,\lambda} = pqxyu^k$, this means that the one-form $\frac{1}{h_{p/q,k,\lambda}}\omega_{p/q,k,\lambda} := \Omega_{p/q,k,\lambda}$ is closed and meromorphic, with poles of order $kp+1$ in $(x=0)$ and $kq+1$ in $(y=0)$. In particular we can state:

Claim 5.6. *There is a single formal meromorphic closed one-form η with simple poles in $(y=0)$ such that $d\omega_{p/q,k,\lambda} = \eta \wedge \omega_{p/q,k,\lambda}$. This form is $\eta = dh_{p/q,k,\lambda}/h_{p/q,k,\lambda}$.*

Proof. Indeed, since $\Omega_{p/q,k,\lambda}$ is closed we conclude that $\eta_0 := dh_{p/q,k,\lambda}/h_{p/q,k,\lambda}$ satisfies the equation $d\omega_{p/q,k,\lambda} = \eta \wedge \omega_{p/q,k,\lambda}$. Now assume that ω is a closed meromorphic formal one-form as in the statement. We have $\eta - \eta_0 = g.\Omega_{p/q,k,\lambda}$ for some meromorphic function g such that $dg \wedge \Omega_{p/q,k,\lambda} = 0$. If g is not constant then $\Omega_{p/q,k,\lambda}$ admits a formal meromorphic first integral. This is not possible, because it does not admit a holomorphic first integral (see for instance [16]). Therefore g must be constant. Because both η and η_0 have simple poles, this implies that $g\Omega_{p/q,k,\lambda}$ has simple poles, therefore $g = 0$. □

Example 5.7 (exceptional case)**.** According to [10] a subgroup $G \subset \mathrm{Diff}(\mathbb{C}, 0)$ is called *exceptional* if it is formally conjugated to a group $G_{\xi,k}, 0 < k \in \mathbb{N}, \xi \in \mathbb{C}$, generated by the maps $f_\xi : z \mapsto \xi z$ and $g_k : z \mapsto \frac{z}{(1-kz^k)^{\frac{1}{k}}}$, with $\xi^k = -1$ and $(1)^{\frac{1}{k}} = 1$. In particular an exceptional group is a solvable non-abelian group, formally conjugated to a discrete subgroup of $\mathbb{H}_k = \{(z \mapsto \frac{az}{(1+bz^k)^{\frac{1}{k}}});\ a \neq 0, b \in \mathbb{C}\}$. A non-exceptional group is *formally rigid* (cf. [10] Theorem 1 page 460)[‡]. Moreover we have:

Any non-abelian solvable subgroup $G \subset \mathrm{Diff}(\mathbb{C}, 0)$ is formally conjugated to a subgroup of some $\mathbb{H}_k$, and this conjugation is analytic if G is not exceptional ([10],[15]).

Thus, in our Proposition 4.2 the only possibility for the group G to be not analytically conjugated to a subgroup of some $\mathbb{H}_k$ is that either G is abelian, or G is exceptional, i.e., formally equivalent to some $G_{\xi,k}$. In the exceptional case the group leaves invariant the formal function $\hat{\phi}(z) = \cos(\frac{2\pi}{z^k})$. We now extend the notion of exceptionality to germs of foliations:

Definition 5.8. A germ of a non-dicritical generalized curve $\mathcal{F}$ at $0 \in \mathbb{C}^2$ will be called *solvable exceptional* if every virtual holonomy group in the reduction of singularities of $\mathcal{F}$ is solvable (possibly abelian), and at least one virtual holonomy is solvable exceptional.

Concrete examples of non-formally rigid exceptional groups are found in [10] and [19], associated to certain cusp singularities. By a result due to Pérez-Marco and Yoccoz [24] any germ of a complex diffeomorphism $f \in \mathrm{Diff}(\mathbb{C}, 0)$ is conjugate to the local holonomy of a separatrix associated to a germ of a non-degenerate holomorphic foliation $\mathcal{F}(f) : xdy - \lambda ydx + \ldots = 0$, having two transverse separatrices. This completes previous results from Martinet-Ramis [17], by solving the "non-resonant" case. Adding to this the (local) synthesis result in [14] we conclude that:

Given an exceptional subgroup $G_{exc} \cong G_{\xi,k}$ there is a germ of a foliation $\mathcal{F}(G_{exc})$ at $0 \in \mathbb{C}^2$ such that:

- *$\mathcal{F}$ is a non-dicritical generalized curve, admitting a reduction with a single blow-up, and the exceptional divisor is an invariant projective line $E(\mathcal{F}) \cong \mathbb{P}^1$.*

[‡]The group G is *formally rigid* if given any formal conjugation with another group G' there is an analytic conjugation.

- *$\mathcal{F}$ exhibits three separatrices, all in general position.*
- *The (reduced) foliation $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}(1)$ exhibits three singularities, all non-degenerate, say* $\operatorname{sing}(\tilde{\mathcal{F}}) = \{\tilde{\mathrm{p}}_1, \tilde{\mathrm{p}}_2, \tilde{\mathrm{p}}_3\}$.
- *The holonomy of the leaf $L_0 = E(\mathcal{F}) \setminus \{\tilde{p}_1, \tilde{p}_2, \tilde{p}_3\}$ is conjugate to the group G_{exc}.*

To the list of properties above we can add:

- Assume that G_{exc} is not formally rigid, more precisely, assume that the formal embedding $G_{exc} \subset \mathbb{H}_k$ cannot be analytic. Then the virtual holonomy $H^{\mathrm{virt}} := \mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}, \mathrm{L}_0)$ of the leaf L_0 of $\tilde{\mathcal{F}}$ is conjugate to G_{exc}.

Indeed, H^{virt} contains G_{exc} and it is also solvable with closed orbits off the origin. If H^{virt} contains properly G_{exc} then H^{virt} is not exceptional, therefore it admits an analytic embedding into some $\mathbb{H}_k$. This embedding gives an analytic embedding of G_{exc} on $\mathbb{H}_k$.

Thus, under the above non-formal rigidity condition we can state:

- The virtual holonomy of the leaf $L_0 = E(\mathcal{F}) \setminus \{\tilde{p}_1, \tilde{p}_2, \tilde{p}_3\}$ is conjugate to the group G_{exc}.

Using the above and material in the Appendix § 9 we can state:

Proposition 5.9. *Let $\mathcal{F}$ be a germ of a solvable exceptional foliation at $0 \in \mathbb{C}^2$. Then $\mathcal{F}$ admits a formal first integral of Liouvillian type $\hat{\Phi}$. This first integral admits a transversely formal development along the separatrices of $\mathcal{F}$. Given a separatrix Γ and a transverse disc Σ to $\mathcal{F}$ and Γ, the restriction $\hat{\Phi}\big|_{\Gamma}$ can be written as $\cos(\frac{2\pi}{x^k})$ in suitable formal coordinates x, for some $k \in \mathbb{N}$.*

6. The irreducible case

Let us consider a germ of a holomorphic foliation $\mathcal{F}$ at the origin $0 \in \mathbb{C}^2$, a germ of an irreducible non-degenerate singularity. In suitable local coordinates we can write $\mathcal{F}$ as given by

$$x(1 + A(x, y))dy - \lambda y(1 + B(x, y))dx = 0,$$

for some holomorphic $A(x, y)$, $B(x, y)$ with $0 \neq \lambda \in \mathbb{C} \setminus \mathbb{Q}_+$, $A(0, 0) = B(0, 0) = 0$. In the normal form above, the separatrices are the coordinate axes. Let us denote by f the holonomy map (its class up to holomorphic conjugacy) of the separatrix $(y = 0)$. From the correspondence between the leaves of $\mathcal{F}$ and the orbits of f ([16, 17, 24]) and according to the well-known properties of f discussed in § 4.1 (see also [2, 3]) we conclude that the foliation $\mathcal{F}$ exhibits the following characteristics:

Lemma 6.1. *Let $\mathcal{F}$ be a germ of an irreducible non-degenerate singularity at the origin $0 \in \mathbb{C}^2$ as above. We have:*

(1) *In the hyperbolic case and in the resonant non-linearizable case, $\lambda \in \mathbb{Q}_-$, all leaves of $\mathcal{F}$ are closed off the set of separatrices, no leaf is closed.*
(2) *In the non-resonant (Siegel or Poincaré) case, $\lambda \in \mathbb{R} \setminus \mathbb{Q}$, $\mathcal{F}$ has always some leaves which are recurrent. Moreover, no leaf converges only to the set of separatrices, therefore if a leaf is closed off the set of separatrices then it is already a closed leaf.*

Proof. If the singularity is in the Poincaré domain then, since it is not a resonance (because $\lambda, 1/\lambda \notin \mathbb{N}$) it is analytically linearizable. We may therefore choose local coordinates (x, y) on $(\mathbb{C}^2, 0)$ such that the germ writes as $xdy - \lambda ydx = 0$. The holonomy of one of the coordinate axes with respect to a small disc $\Sigma : \{x = a\}$ is given by $h(y) = \exp(2\pi\sqrt{-1}\lambda)y$. Suppose that λ is irrational then the map h is an irrational rotation, and the leaves (not contained in the set of separatrices) are recurrent, therefore not closed off the set of separatrices.

□

7. Fully resonant singularities

The following notion is useful in our framework.

Definition 7.1 (fully resonant)**.** A germ of a generalized curve $\mathcal{F}$ at the origin $0 \in \mathbb{C}^2$ will be called *fully resonant* if every singularity arising in the reduction of singularities is a resonant singularity.

Lemma 7.2. *Let $\mathcal{F}$ be a germ of a non-dicritical generalized curve in a neighborhood of the origin $0 \in \mathbb{C}^2$. Suppose that for some representative $\mathcal{F}_U$ of $\mathcal{F}$ defined in a neighborhood U of the origin, all leaves are closed off the set of separatrices. Then we have two possibilities:*
(i) *$\mathcal{F}$ is a fully-resonant generalized curve.*
(ii) *The reduction of singularities of $\mathcal{F}$ exhibits some hyperbolic singularity, all the final singularities are linearizable. Moreover, given any separatrix Γ through the origin, and a transverse disc Σ meeting Γ at a point $q \neq 0$, the virtual holonomy group $\mathrm{Hol}^{\mathrm{virt}}(\mathcal{F}, \Sigma, \mathrm{q})$ contains a hyperbolic map. In particular, it is an abelian linearizable group generated by a hyperbolic map and a periodic map.*

Proof. We proceed by induction on the number $r \in \{0, 1, 2, ...\}$ of blowing-ups in the reduction of singularities for the germ $\mathcal{F}$.
Case 1. ($r = 0$). In this case the singularity is already irreducible. The result follows from Lemma 6.1.
Case 2 (Induction step). Assume that the result is proved for foliation germs that admit a reduction of singularities with a number of blowing-ups less greater than or equal to r. Suppose that the fixed germ $\mathcal{F}$ admits a reduction of singularities consisting of $r+1$ blowing-ups. Then we perform a first blow-up $\sigma_1 \colon \tilde{U}(1) \to U$ at the origin and obtain a lifted foliation $\tilde{\mathcal{F}}(1) = \sigma_1^*(\mathcal{F})$ with (first) exceptional divisor $E(\mathcal{F})(1) = \sigma_1^{-1}(0)$ consisting of a single embedded invariant projective line in $\tilde{U}(1)$ (by hypothesis the exceptional divisor is invariant by $\tilde{\mathcal{F}}(1)$). Given a leaf L of $\mathcal{F}$ in U we denote by $\tilde{L}(1)$ the lifting $\tilde{L}(1) = \sigma_1^{-1}(L)$ of L to $\tilde{U}(1)$ by the map $\sigma_1 \colon \tilde{U}(1) \to U$. Now, if a leaf L of $\mathcal{F}$ in U is closed in $U \setminus \mathrm{sep}(\mathcal{F}, \mathrm{U})$, then its lift $\tilde{L}(1)$ is closed in $\tilde{U}(1) \setminus \mathrm{sep}(\tilde{\mathcal{F}}(1), \tilde{\mathrm{U}}(1))$ (notice that for each singularity $\tilde{p} \in \mathrm{sing}(\tilde{\mathcal{F}}(1)) \subset \mathrm{E}(1)$ the set of local separatrices of $\tilde{\mathcal{F}}(1)$ through $\tilde{p}$ is formed by $E(\mathcal{F})(1)$ union the local branches through $\tilde{p}$, of the strict transform by $\sigma(1)$ of $\mathrm{sep}(\mathcal{F}, \mathrm{U})$). Given a singularity $\tilde{p} \in \mathrm{sing}(\tilde{\mathcal{F}}(1)) \subset \mathrm{E}(1)$ of $\tilde{\mathcal{F}}(1)$, since the blow-up map is proper, we can conclude that for any small enough neighborhood $\tilde{W}_{\tilde{p}}$ of $\tilde{p}$ in $\tilde{U}(1)$, a leaf $\tilde{L}_0$ of the restriction $\tilde{\mathcal{F}}(1)\big|_{\tilde{W}_{\tilde{p}}}$ is closed in $\tilde{W}_{\tilde{p}} \setminus \mathrm{sep}(\tilde{\mathcal{F}}(1), \tilde{\mathrm{p}})$ provided that it projects into a piece of leaf $\sigma_1(\tilde{L}_0)$ which is contained in a leaf L of $\mathcal{F}$ that is closed in $U \setminus \mathrm{sep}(\mathcal{F}, \mathrm{U})$. Furthermore, since the blow-up map defines a biholomorphism between $\mathbb{C}^2 \setminus \{0\}$ and the complement of the exceptional divisor $\tilde{\mathbb{C}}_0^2 \setminus E(1)$, we conclude that:

The leaves of $\tilde{\mathcal{F}}(1)\big|_{\tilde{W}_{\tilde{p}}}$ are closed off the set of local separatrices of $\tilde{\mathcal{F}}(1)$ through $\tilde{p}$. Thus, by the induction hypothesis, each singularity $\tilde{p} \in \mathrm{sing}(\tilde{\mathcal{F}}(1))$ in the first blow-up is fully-resonant or its reduction of singularities exhibits some hyperbolic singularity, all the final singularities are linearizable. Moreover, given any separatrix $\tilde{\Gamma}_{\tilde{p}}$ through this singularity, and a transverse disc $\tilde{\Sigma}_{\tilde{p}}$ meeting $\tilde{\Gamma}_{\tilde{p}}$ at a point $\tilde{p} \neq \tilde{q} = \tilde{\Sigma}_{\tilde{p}} \cap \Gamma_{\tilde{p}}$, the virtual holonomy group $\mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}(1), \Sigma_{\tilde{\mathrm{p}}}, \tilde{\mathrm{q}})$ is an abelian linearizable group generated by a hyperbolic map and a periodic map.

We have then two possibilities:
(a) All singularities in the first blow-up are fully-resonant. In this case, the original singularity is fully-resonant.

(b) Some singularity $\tilde{p} \in \mathrm{sing}(\tilde{\mathcal{F}}(1))$ in the first blow-up is not fully-resonant.
We shall consider this second possibility:

Claim 7.3. *Given a singularity $\tilde{p}_1 \in \operatorname{sing}(\tilde{\mathcal{F}}(1)) \subset \mathrm{E}(1)$ its reduction of singularities only produces linearizable singularities. Moreover, given any separatrix $\Gamma_{\tilde{p}_1}$ through $\tilde{p}_1$, and a transverse disc Σ meeting $\Gamma_{\tilde{p}_1}$ at a point $\tilde{p}_1 \neq \tilde{q}_1 = \Sigma \cap \Gamma_{\tilde{p}_1}$, the virtual holonomy group $\mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}(1), \Sigma, \tilde{\mathrm{q}}_1)$ is an abelian linearizable group generated by a hyperbolic map and a periodic map.*

Proof. At first sight it may seem that this is a straightforward consequence of the Induction hypothesis. Nevertheless, it is not clear that we are dealing with a singularity which is not fully-resonant. Let us see how to study the case $\tilde{p}_1$ is fully resonant. Since $E(\mathcal{F})(1)$ is invariant, the hyperbolic element in the virtual holonomy of the separatrix through $\tilde{p}$ contained in $E(\mathcal{F})(1)$ induces a hyperbolic element on the virtual holonomy of the separatrix through $\tilde{p}_1$ contained in the exceptional divisor $E(\mathcal{F})(1)$. This is done as follows. Given two points $\tilde{q}$ and $\tilde{q}_1$, close to $\tilde{p}$ and $\tilde{p}_1$ respectively, and transverse discs Σ and Σ_1 meeting $E(\mathcal{F})(1)$ at these points respectively, we can choose a simple path $\alpha\colon [0,1] \to E(1) \setminus \operatorname{sing}(\tilde{\mathcal{F}}(1))$ from $\tilde{q}$ to $\tilde{q}_1$. The holonomy map $h_\alpha\colon (\Sigma, \tilde{q}) \to (\Sigma_1, \tilde{q}_1)$ associated to the path α (recall that $E(\mathcal{F})(1)\setminus\operatorname{sing}(\tilde{\mathcal{F}}(1))$ is a leaf of $\tilde{\mathcal{F}}(1)$), induces a natural morphism for the virtual holonomy groups

$$\alpha^*\colon \mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}(1), \Sigma_1, \tilde{\mathrm{q}}_1) \to \mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}(1), \Sigma, \tilde{\mathrm{q}}),$$

by $\alpha^* : h \mapsto h_\alpha^{-1} \circ h \circ h_\alpha$. Since $h_{\alpha^{-1}} = (h_\alpha)^{-1}$ in terms of holonomy maps, we conclude that the above morphism is actually an isomorphism between the virtual holonomy groups. Thus the virtual holonomy group $\mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}(1)_{\tilde{\mathrm{p}}_1}, \mathrm{E}(\mathcal{F})(1), \Sigma_{\tilde{\mathrm{p}}_1}, \tilde{\mathrm{p}}_1)$ contains a hyperbolic map. Now we can use the Dulac correspondence in order to "pass" this hyperbolic map from the above virtual holonomy (of the separatrix contained in $E(\mathcal{F})(1)$) to the virtual holonomy of any separatrix of $\tilde{\mathcal{F}}_{\tilde{p}_1}$ (see the Appendix § 9). Indeed, because $\mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}(1)_{\tilde{\mathrm{p}}_1}, \mathrm{E}(\mathcal{F})(1), \Sigma_{\tilde{\mathrm{q}}_1}, \tilde{\mathrm{q}}_1)$ contains a hyperbolic element, according to Proposition 4.2 it must be linearizable, generated by this hyperbolic map and a periodic map. This already implies that all local holonomies arising in the reduction of singularities of $\tilde{q}$ are linearizable, therefore the corresponding singularities are linearizable. Because the singularities are linearizable, the Dulac map allows to pass the hyperbolic attractor from $E(\mathcal{F})(1)$ to any separatrix through $\tilde{q}$, proving in this way that any separatrix through $\tilde{q}$ contains a hyperbolic attractor in its virtual holonomy group[§]. Thus, also the virtual holonomy group associated to the separatrix $\tilde{\Gamma}$ of $\tilde{\mathcal{F}}(1)$ through $\tilde{p}_1$ contains some hyperbolic map. ☐

Now consider any separatrix Γ of $\mathcal{F}$ through the origin. Since the projective line $E(\mathcal{F})(1)$ in the first blow-up is invariant, the lift $\tilde{\Gamma}$ is the separatrix of some singularity $\tilde{p}$ of $\tilde{\mathcal{F}}(1)$. If $\mathcal{F}$ is not fully-resonant, then by the above, we conclude that the virtual holonomy group associated to this separatrix $\tilde{\Gamma}$ contains a hyperbolic map. Recall that the blow-up is a diffeomorphism off the origin and off the exceptional divisor, so that the maps in the virtual holonomy of $\tilde{\Gamma}$ induce maps in the disc Σ transverse to Γ in $\mathbb{C}^2$, but which are defined only in the punctured disc, i.e., off the origin. Nevertheless, since these projected maps are one-to-one, the classical Riemann extension theorem for bounded holomorphic maps shows that indeed such maps induce germs of diffeomorphisms defined in the disc Σ. These diffeomorphisms are the virtual holonomy maps of the separatrix Γ of $\tilde{\mathcal{F}}(1)$ evaluated at the transverse section Σ. Hence, by projecting the maps in $\mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}(1), \Sigma, \tilde{\mathrm{q}})$ we obtain hyperbolic maps in this virtual holonomy group as stated. Now the Induction Principle applies to finish the proof of the lemma. ☐

[§]The details of the construction of the Dulac map and the "passage" of (virtual) holonomy maps to virtual holonomy maps on adjacent components are are found in the Appendix § 9 and extensively explained in [8] and in [25] §2.3, pages 371 to 374.

8. Germs of foliations with closed leaves off the set of separatrices

In this section we prove Theorems 1.2, 1.3 and 1.4. We rely on Lemma 7.2 and on Lemmas 8.1 and 8.9 below.

Lemma 8.1. *Let $\mathcal{F}$ be a foliation germ as in Theorem 1.2. Then the following are equivalent:*

(1) *$\mathcal{F}$ admits a holomorphic first integral in some neighborhood of the origin.*
(2) *$\mathcal{F}$ is fully-resonant, has a closed leaf arbitrarily close to the origin and all singularities in the reduction of singularities are linearizable.*

Proof. Since (1) implies (2) is well-known (cf.[16],[27]), we prove the converse. Assume then that $\mathcal{F}$ (is as in Theorem 1.2 and moreover) has a closed leaf arbitrarily close to the origin and that all final singularities in the reduction process are resonant and linearizable. We must prove that $\mathcal{F}$ admits a holomorphic first integral.

We proceed by induction on the number $r \in \{0,1,2,...\}$ of blow-ups in the reduction of singularities for the germ $\mathcal{F}$.

Case 1. ($r=0$). In this case the singularity is already irreducible and resonant linearizable. Since it is resonant, it admits a holomorphic first integral.

Case 2. ($r-1 \implies r$). Assume that the result is proved for foliation germs that admit a reduction of singularities with a number of blow-ups smaller than r. Suppose that the fixed germ $\mathcal{F}$ admits a reduction of singularities consisting of r blow-ups. Let U be a small connected neighborhood of the origin where the leaves of $\mathcal{F}$ are closed off the set of separatrices. We also assume that for U arbitrarily small the foliation $\mathcal{F}$ exhibits a closed leaf in U. Then we proceed as in the proof of Lemma 7.2 from where we import the notation. Thus we perform a first blow-up $\sigma_1\colon \tilde{U}(1) \to U$ at the origin and obtain a lifted foliation $\tilde{\mathcal{F}}(1) = \sigma_1^*(\mathcal{F})$ with (first) exceptional divisor $E(\mathcal{F})(1) = \sigma_1^{-1}(0)$ consisting of a single embedded invariant projective line in $\tilde{U}(1)$ (by hypothesis the exceptional divisor is invariant by $\tilde{\mathcal{F}}(1)$). Given a leaf L of $\mathcal{F}$ in U we denote by $\tilde{L}(1)$ the lifting $\tilde{L}(1) = \sigma_1^{-1}(L)$ of L to $\tilde{U}(1)$ by the map $\sigma_1\colon \tilde{U}(1) \to U$. Now, if a leaf L of $\mathcal{F}$ in U is closed in $U \setminus \mathrm{sep}(\mathcal{F}, \mathrm{U})$, then its lift $\tilde{L}(1)$ is closed in $\tilde{U}(1) \setminus \mathrm{sep}(\tilde{\mathcal{F}}(1), \tilde{\mathrm{U}}(1))$ (notice that for each singularity $\tilde{p} \in \mathrm{sing}(\tilde{\mathcal{F}}(1)) \subset \mathrm{E}(1)$ the set of local separatrices of $\tilde{\mathcal{F}}(1)$ through $\tilde{p}$ is formed by $E(\mathcal{F})(1)$ union the local branches through $\tilde{p}$, of the strict transform by $\sigma(1)$ of $\mathrm{sep}(\mathcal{F}, \mathrm{U})$).

Given a singularity $\tilde{p} \in \mathrm{sing}(\tilde{\mathcal{F}}) \subset \mathrm{E}$ of $\tilde{\mathcal{F}}$, since the blow-up map is proper, we can conclude that for any small enough neighborhood $\tilde{W}_{\tilde{p}}$ of $\tilde{p}$ in $\tilde{U}$, a leaf $\tilde{L}_0$ of the restriction $\tilde{\mathcal{F}}\big|_{\tilde{W}_{\tilde{p}}}$ is closed in $\tilde{W}_{\tilde{p}}$ provided that it projects into a piece of leaf $\pi(\tilde{L}_0)$ which is contained in a leaf L of $\mathcal{F}$ that is closed in U. Similarly, a leaf $\tilde{L}_0$ is closed in $\tilde{W}_{\tilde{p}} \setminus E$ provided that it projects into a piece of leaf $\pi(\tilde{L}_0)$ which is contained in a leaf L of $\mathcal{F}$ that is closed in $U \setminus \{0\}$. By the Induction hypothesis, each singularity $\tilde{p} \in \mathrm{sing}(\tilde{\mathcal{F}})$ admits a holomorphic first integral say $\tilde{f}_{\tilde{p}}$ defined in $\tilde{W}_{\tilde{p}}$ if this last is small enough. Now we analyze the holonomy of the leaf $L_0 := E(\mathcal{F}) \setminus \mathrm{sing}(\tilde{\mathcal{F}})$. Choose a regular point $\tilde{q} \in E_0$ and a small transverse disc Σ to L_0 centered at $\tilde{q}$. The corresponding holonomy group representation will be denoted by $H := \mathrm{Hol}(\tilde{\mathcal{F}}, \Sigma, \tilde{\mathrm{q}}) \subset \mathrm{Diff}(\Sigma, \tilde{\mathrm{q}})$. We know that this group is finitely generated and by the invariance of $E(\mathcal{F})$ and the above argumentation and Lemma 3.1, we know that actually, *the orbits of the holonomy group H of the exceptional divisor are closed off the origin, one of which is closed*. Applying Corollary 4.4 we conclude that the holonomy group is finite. Since the virtual holonomy group preserves the leaves of the foliation, the arguments above already show that *the orbits of the virtual holonomy group H^{virt} are closed off the origin, one of which is closed*. The problem is we still do not know that the virtual holonomy group is finitely generated. Nevertheless, from Corollary 4.4 we obtain:

Claim 8.2. *Any finitely generated subgroup H of the virtual holonomy group H^{virt} is a finite group.*

Let us then proceed as follows: given the singularities $\{\tilde{p}_1,...,\tilde{p}_m\} = \mathrm{sing}(\tilde{\mathcal{F}}) \subset \mathrm{E}$, by induction hypothesis each singularity admits a local holomorphic first integral. Thus, there are small discs $D_j \subset E$, centered at the $\tilde{p}_j$ and such that in a neighborhood V_j of $\tilde{p}_j$ in the blow-up space $\tilde{\mathbb{C}_0^2}$, of product type $V_j = D_j \times \mathbb{D}_\epsilon$, we have a holomorphic first integral $g_j\colon V_j \to \mathbb{C}$, with $g_j(\tilde{p}_j) = 0$. Fix now a point $\tilde{p}_0 \in E \setminus \mathrm{sing}(\tilde{\mathcal{F}})$. Since $E(\mathcal{F})$ has the topology of the 2-sphere, we may choose a simply-connected domain $A_j \subset E$ such that $A_j \cap \{\tilde{p}_0, \tilde{p}_1,...,\tilde{p}_m\} = \{\tilde{p}_0, \tilde{p}_j\}$, for every $j = 1,...,m$. Since A_j is simply-connected, we may extend the local holomorphic first integral g_j to a holomorphic first integral $\tilde{g}_j$ for $\tilde{\mathcal{F}}$ in a neighborhood U_j of $D_j \cup A_j$, we may assume that U_j contains V_j. We observe that $\tilde{g}_j$ can be chosen to be *primitive*, i.e., it has connected fibers, therefore it cannot be written as $\tilde{g}_j = h^n$, for some holomorphic function h with $n \geq 2$. Now, given a local transverse section Σ_0 centered at $\tilde{p}_0$ and contained in U_j, we may introduce the *invariance group* of the restriction $g_j^0 := \tilde{g}_j\big|_{\Sigma_0}$ as the group

$$\mathrm{Inv}(\mathrm{g_j^0}) := \{\mathrm{f} \in \mathrm{Diff}(\Sigma_0, \tilde{\mathrm{p}}_0),\ \mathrm{g_j^0} \circ \mathrm{f} = \mathrm{g_j^0}\}.$$

In other words, the invariance group of g_j^0 is the group of germs of maps that preserve the fibers of g_j^0. Clearly $\mathrm{Inv}(\mathrm{g_j^0})$ is a finite (resonant) group ([16] Proposition 1.1. page 475). Let us now denote by $\mathrm{Inv}(\tilde{\mathcal{F}}, \Sigma_0) \subset \mathrm{Diff}(\Sigma_0, \tilde{\mathrm{p}}_0)$ the subgroup generated by the invariance groups $\mathrm{Inv}(\mathrm{g_j^0}), \mathrm{j} = 1,...,\mathrm{m}$. We call $\mathrm{Inv}(\tilde{\mathcal{F}}, \Sigma_0)$ the *global invariance group* of $\tilde{\mathcal{F}}$ with respect to $(\Sigma_0, \tilde{p}_0)$. Then, from the above we immediately obtain:

Claim 8.3. $\mathrm{Inv}(\tilde{\mathcal{F}}, \Sigma_0)$ *is a finite group.*

Proof. Indeed, first notice that $\mathrm{Inv}(\tilde{\mathcal{F}}, \Sigma_0)$ is finitely generated (by periodic maps). Since $\mathrm{Inv}(\tilde{\mathcal{F}}, \Sigma_0)$ preserves the leaves of $\tilde{\mathcal{F}}$ (recall that $\tilde{g}_j$ was chosen to be primitive) we have that $\mathrm{Inv}(\tilde{\mathcal{F}}, \Sigma_0) \subset \mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}, \Sigma_0, \tilde{\mathrm{p}}_0)$ and therefore by Corollary 4.4 $\mathrm{Inv}(\tilde{\mathcal{F}}, \Sigma_0)$ is a finite group. □

Notice that this global invariance group contains in a natural way the local invariance groups of the local first integrals g_j. Therefore, as observed in [16], once we have proved that the global invariance group $\mathrm{Inv}(\tilde{\mathcal{F}}, \Sigma_0)$ is finite, together with the fact that the singularities in $E(\mathcal{F})$ exhibit local holomorphic first integrals, we conclude as in [16] that the foliation $\tilde{\mathcal{F}}$ and therefore the foliation $\mathcal{F}$ has a holomorphic first integral.

□

As a consequence of the proof of the above lemma we have:

Lemma 8.4. *Let $\mathcal{F}$ be a foliation germ as in Theorem 1.2. Assume that $\mathcal{F}$ has a closed leaf arbitrarily close to the origin. Then $\mathcal{F}$ admits a holomorphic first integral in some neighborhood of the origin.*

Proof. The proof is based on Lemma 8.1 above and in the following claims:

Claim 8.5. *The foliation $\mathcal{F}$ is fully-resonant.*

Proof. Indeed, this is a consequence of the fact that any singularity in the reduction of singularities is such that the local induced foliation has closed leaves off the set of local separatrices and of Lemma 6.1. □

Claim 8.6. *Each virtual holonomy group in the reduction of singularities of $\mathcal{F}$ exhibits a closed pseudo-orbit arbitrarily close to the origin.*

Proof. This is a consequence of (what we have observed in the proof of) Proposition 3.3. □

Then, we conclude, as in the proof of Lemma 8.1, that each local holonomy map of a separatrix of a singularity in the reduction of singularities of $\mathcal{F}$, is a finite periodic map. This implies that all the singularities of the reduction of $\mathcal{F}$ are linearizable (and resonant). Applying now Lemma 8.1 we conclude. ☐

Lemma 8.7. *Let $\mathcal{F}$ be a foliation germ as in Theorem 1.2. Assume that some separatrix Γ of $\mathcal{F}$ contains some hyperbolic map in its virtual holonomy group. Then $\mathcal{F}$ is given by a closed meromorphic one-form with simple poles.*

Proof. We proceed by induction on the number $r \in \{0, 1, 2, ...\}$ of blowing-ups in the reduction of singularities for the germ $\mathcal{F}$.
Case 1. $(r = 0)$. In this case the singularity is already irreducible. Since it is not a saddle-node it can be written as $xdy - \lambda ydx + \ldots = 0$ for some $\lambda \in \mathbb{C} \setminus \mathbb{Q}_+$. We claim:

Claim 8.8. *The singularity is not a resonant singularity, i.e., $\lambda \notin \mathbb{Q}$.*

Proof of the Claim. Assume that we have $\lambda = -n/m \in \mathbb{Q}_-$ for some $n, m \in \mathbb{N}$ with $< n, m >= 1$. In this case we have two possibilities.

(1) The singularity is analytically linearizable. In this case we can write $nxdy + mydx = 0$. Then we have a holomorphic first integral $f = x^m y^n$ and any virtual holonomy map must preserve the fibers of f. This implies that any virtual holonomy map is actually a finite periodic map. This case is therefore excluded.

(2) The singularity is not analytically linearizable. As we have seen in Example 5.5, by [17] the foliation is formally isomorphic to an unique equation

$$\omega_{p/q,k,\lambda} := p(1 + (\lambda - 1)u^k)ydx + q(1 + \lambda u^k)xdy,$$

where $p, q, k \in \mathbb{N}, \lambda \in \mathbb{C}$ and $u := x^p y^q$ and $mp - nq = 1$. We can rewrite

$$\omega_{p/q,k,\lambda} = (1 + (\lambda - mp)u^k)(pydx + qxdy) + pqu^k(nydx + mxdy).$$

This last expression admits the integrating factor $h_{p/q,k,\lambda} = pqxyu^k$, this means that the one-form $\frac{1}{h_{p/q,k,\lambda}}\omega_{p/q,k,\lambda} := \Omega_{p/q,k,\lambda}$ is closed and meromorphic, with poles of order $kp+1$ in $(x = 0)$ and $kq + 1$ in $(y = 0)$. Now, if there is a hyperbolic map in the virtual holonomy of one of the separatrices (given by the axes) then this map clearly forces the closed meromorphic one-form to have simple poles along that separatrix, which is not the case, contradiction.

☐

Because the singularity is non-resonant we have $\lambda \in \mathbb{C} \setminus \mathbb{Q}$. We claim that this singularity is a hyperbolic singularity, i.e., $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Indeed, if $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ then we have two possibilities.

(i) The singularity is analytically linearizable. In this case we may assume that it is of the form $xdy - \lambda ydx = 0$. Then, the leaves are not closed off the origin, because a typical leaf has as closure the three-dimensional manifold given by $|y||x|^{-\lambda} = c$ for some $c > 0$.

(ii) The singularity is not analytically linearizable. In this case we must have $\lambda \in \mathbb{R}_-$ and the foliation is in the so called *Siegel domain*. In particular, there exactly are two separatrices and we may assume that it is of the form $xdy - \lambda y(1 + A(x, y))dx =$ for some $A(x, y)$ holomorphic with $A(0, 0) = 0$. Such a singularity has a local holonomy map for the separatrix $(y = 0)$ of the form $f(y) = \exp(2\pi\lambda)y + \ldots$. In particular, such a holonomy map is not a resonant map. By Lemma 6.1 or also by the considerations in the proof of Proposition 4.2 we know that the only possibility compatible with the fact that the leaves of $\mathcal{F}$ are closed off the origin, is that f is a hyperbolic map, i.e., $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

We conclude that the singularity is hyperbolic and that any separatrix has a hyperbolic holonomy map. The singularity is linearizable as $xdy - \lambda ydx = 0$ in suitable local coordinates. In these coordinates the foliation is given by the closed one-form $\Omega_\lambda = \frac{dy}{y} - \lambda \frac{dx}{x}$.

Case 2 (Induction step). Assume that the result is proved for foliation germs that admit a reduction of singularities with a number of blowing-ups less greater than or equal to r. Suppose that the fixed germ $\mathcal{F}$ admits a reduction of singularities consisting of $r+1$ blowing-ups. Then we perform a first blow-up $\sigma_1 \colon \tilde{U}(1) \to U$ at the origin and obtain a lifted foliation $\tilde{\mathcal{F}}(1) = \sigma_1^*(\mathcal{F})$ with (first) exceptional divisor $E(\mathcal{F})(1) = \sigma_1^{-1}(0)$ consisting of a single embedded invariant projective line in $\tilde{U}(1)$ (by hypothesis the exceptional divisor is invariant by $\tilde{\mathcal{F}}(1)$). Given a leaf L of $\mathcal{F}$ in U we denote by $\tilde{L}(1)$ the lifting $\tilde{L}(1) = \sigma_1^{-1}(L)$ of L to $\tilde{U}(1)$ by the map $\sigma_1 \colon \tilde{U}(1) \to U$. By hypothesis $\mathcal{F}$ has some separatrix Γ containing a hyperbolic map in its virtual holonomy. Let then $\tilde{p} \in \operatorname{sing}(\tilde{\mathcal{F}}(1))$ be a singularity exhibiting some separatrix $\tilde{\Gamma}_{\tilde{p}} = \overline{\sigma_1^{-1}(\Gamma \setminus \{0\})}$ not contained in the projective line $E(\mathcal{F})(1)$ and having a hyperbolic map in its virtual holonomy. By the Induction hypothesis the germ $\tilde{\mathcal{F}}(1)_{\tilde{p}}$ induced by $\tilde{\mathcal{F}}(1)$ at $\tilde{p}$, is given by a simple poles closed meromorphic one-form say $\tilde{\Omega}_{\tilde{p}}$. Since $\tilde{E}(1)$ is invariant, it contains a separatrix of the germ $\tilde{\mathcal{F}}(1)_{\tilde{p}}$. Because of the form $\tilde{\Omega}_{\tilde{p}}$, all the separatrices of $\tilde{\mathcal{F}}(1)_{\tilde{p}}$ contain hyperbolic maps in their virtual holonomy groups. Therefore, the separatrix of $\tilde{\mathcal{F}}(1)_{\tilde{p}}$ contained in $E(\mathcal{F})(1)$, contains a hyperbolic map for its virtual holonomy group. Thanks to the invariance of $E(\mathcal{F})(1)$ for $\tilde{\mathcal{F}}(1)$ this implies that each singularity $\tilde{q}$ of $\tilde{\mathcal{F}}(1)$ in $E(\mathcal{F})(1)$ contains a hyperbolic map in the virtual holonomy of the corresponding separatrix contained in $E(\mathcal{F})(1)$. Then, again by Induction hypothesis, each singularity $\tilde{q} \in E(1) \cap \operatorname{sing}(\tilde{\mathcal{F}}(1))$ is given by a closed meromorphic one-form $\tilde{\Omega}_{\tilde{q}}$ having simple poles. Now we focus on the leaf $L_0 = E(1) \setminus \operatorname{sing}(\tilde{\mathcal{F}}(1))$ and on its virtual holonomy group, which we shall denote simply by $\mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}(1), \mathrm{L}_0)$. This leaf contains therefore hyperbolic maps in its virtual holonomy group. In view of Proposition 4.2 the group $\mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}(1), \mathrm{L}_0)$ is abelian linearizable. Using this and the well-known techniques from [5] we can construct a simple poles closed meromorphic one-form $\tilde{\Omega}$ in a neighborhood of $E(\mathcal{F})(1)$, which defines $\tilde{\mathcal{F}}(1)$. Projecting this one-form onto a neighborhood of the origin $0 \in \mathbb{C}^2$, we obtain a closed meromorphic one-form Ω with simple poles, defining $\mathcal{F}$. The lemma is proved by Induction. $\square$

Lemma 8.9. *Let $\mathcal{F}$ be a foliation germ as in Theorem 1.2. Assume that $\mathcal{F}$ is not fully resonant. Then:*

(1) *Each separatrix contains some hyperbolic map in its virtual holonomy group.*
(2) *$\mathcal{F}$ is given by a closed meromorphic one form with simple poles.*

Proof. This is essentially a direct consequence of the lemma above. The idea is the following. Since $\mathcal{F}$ is not fully-resonant, it contains some singularity which is not resonant. As in the proof of Lemma 8.7, this singularity must be hyperbolic. The local holonomies of the separatrices of this singularity then are hyperbolic maps, which induce hyperbolic maps on the virtual holonomy of each separatrix of the foliation. By Lemma 8.7 we conclude. $\square$

Lemma 8.10. *Let $\mathcal{F}$ be a foliation germ as in Theorem 1.2. Assume that $\mathcal{F}$ is fully resonant. Then $\mathcal{F}$ admits a formal Liouvillian first integral.*

Proof. First we recall that all the virtual holonomy groups in the reduction of $\mathcal{F}$ are groups with closed orbits off the origin. Then, according to Proposition 4.2 these groups are solvable. Moreover, by hypothesis, there are no saddle-nodes in the reduction of singularities and all the projective lines are invariant. Then, as already mentioned in Example 5.1, using the techniques from [26], [8] or the more general techniques from [21] we can construct a formal generalized integrating factor for $\mathcal{F}$. We give the detailed proof in the Appendix § 9. □

Proof of Theorem 1.2. Let $\mathcal{F}$ be a germ of a non-dicritical generalized curve at $0 \in \mathbb{C}^2$. Assume that the leaves of $\mathcal{F}$ are closed off the set of separatrices. By hypothesis, there is a neighborhood U of the origin where the leaves are all closed off the set of separatrices. According to Lemma 8.9 we have only two possibilities:

(1) $\mathcal{F}$ is fully resonant.
(2) $\mathcal{F}$ is contains some hyperbolic singularity in its reduction of singularities and:
   (a) Each separatrix contains some hyperbolic map in its virtual holonomy group.
   (b) $\mathcal{F}$ is given by a closed meromorphic one form with simple poles.

We study the different possibilities:

Possibility 1. The singularity is fully-resonant. In this case, by Lemma 8.10 $\mathcal{F}$ admits a formal Liouvillian first integral.

Possibility 2. The singularity is a generalized curve which is not fully-resonant. Moreover, we have:

(1) $\mathcal{F}$ is given by a closed formal meromorphic one form with simple poles,
(2) Given any separatrix Γ through the origin, and a transverse disc Σ meeting Γ at a point $q \neq 0$, the virtual holonomy group $\mathrm{Hol}^{\mathrm{virt}}(\mathcal{F}, \Sigma, \mathrm{q})$ is an abelian linearizable group generated by a hyperbolic map and a periodic map.

As in [5] (page 440, paragraph after the proof Lemma 8) we can conclude that $\mathcal{F}$ is indeed a holomorphic pull-back of a linear hyperbolic singularity $xdy - \lambda ydx = 0$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. This ends the proof of Theorem 1.2. □

Proof of Theorem 1.3. According to Lemma 8.7 $\mathcal{F}$ is given by a closed meromorphic one-form with simple poles. The rest of the proof goes as in final part of the above proof of Theorem 1.2. □

Proof of Theorem 1.4. If we already know that $\mathcal{F}$ is a generalized curve then this is just the result of Lemma 8.4. Let us then prove that this is the case. Recall that, by hypothesis $\mathcal{F}$ is non-dicritical, its leaves are closed off the set of separatrices and $\mathcal{F}$ has a leaf which is closed on each small neighborhood of the origin. Assume that there is a saddle-node in the reduction of singularities of $\mathcal{F}$. Then the strong manifold of this saddle-node exhibits a non-trivial holonomy tangent to the identity, say of the form $z \mapsto z + a_{k+1}z^{k+1} + \dots$ for some $a_{k+1} \neq 0, k \in \mathbb{N}$. This map has no closed orbit. Because the exceptional divisor is invariant and connected, and thanks to Lemma 3.2, any given closed leaf must approach a saddle-node singularity by at least one of its separatrices. If it approaches by the strong separatrix then we have a contradiction with the above holonomy map dynamics. Therefore, the closed leaf must approach the saddle-node through the central separatrix. Nevertheless, thanks to the local description of the saddle-node, it is well-known that any leaf not contained in a separatrix and that accumulates properly at the central separatrix also accumulates properly at the strong separatrix. Therefore, again, we have a contradiction. This shows that the existence of a saddle-node is not possible under the additional hypothesis of existence of a closed leaf sufficiently close to the original singularity. Thus $\mathcal{F}$ is indeed a generalized curve. □

9. Appendix: Construction of generalized integrating factors

We shall now detail the construction of the formal generalized integrating factor indicated in the proof of Lemma 8.10. We shall adopt the notation of that section. We shall also denote by H_j (respectively, by H_j^{virt}) the holonomy group (respectively, the virtual holonomy group) of the component D_j of the divisor $E(\mathcal{F})$, $j = 1, ..., r$, which is by hypothesis invariant. We also denote by $D_j^* = D_j \setminus \operatorname{sing}(\mathcal{F})$. The virtual holonomy group H_j^{virt} has closed pseudo-orbits off the origin. This group is therefore solvable in the terms of Proposition 4.2. Fixed a regular point $q_j \in D_j - \operatorname{sing}(\tilde{\mathcal{F}}) \cap \mathrm{D_j}$, a small transverse disk $\Sigma_j \cong \mathbb{D}$, $\Sigma_j \cap D_j = \{q_j\}$ we have holonomy and virtual holonomy identifications $\mathrm{Hol}(\tilde{\mathcal{F}}, \mathrm{D_j}, \Sigma_\mathrm{j}) \cong \mathrm{H_j} \subset \mathrm{Diff}(\mathbb{C}, 0)$ and $\mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}, \mathrm{D_j}, \Sigma_\mathrm{j}) \cong \mathrm{H_j^{virt}} \subset \mathrm{Diff}(\mathbb{C}, 0)$. We recall the following result from groups of germs of complex diffeomorphisms in dimension one ([10], [21]):

Lemma 9.1. *Let $H \subset \mathrm{Diff}(\mathbb{C}, 0)$ be a subgroup. Then:*

(1) *H is abelian $\Leftrightarrow$ there exists a formal vector field ξ in one complex variable which is H-invariant, i.e., $g * \hat{\xi} = \hat{\xi}$, $\forall\, g \in H$.*
(2) *H is solvable $\Leftrightarrow$ there is a formal vector field $\hat{\xi}$ in one complex variable which is H-projectively invariant, i.e., for each $g \in H$ we have $g * \hat{\xi} = c_g \cdot \hat{\xi}$ for some $c_g \in \mathbb{C}^*$.*

As a consequence we have the following possibilities for H_j^{virt}:

(a) H_j is abelian $\Rightarrow$ there exists a formal vector field $\hat{\xi}_j$ in one complex variable $y_j \in \Sigma_j$, $y_j(\Sigma_j) = \mathbb{D}$, $y_j(q_j) = 0$, $\hat{\xi}_j$ writes in some formal coordinates $\hat{\xi}_j(\hat{z}) = \dfrac{\hat{z}^{k+1}}{1 + a\hat{z}^k} \dfrac{d}{d\hat{z}}$ such that: (a*) $g * \hat{\xi}_j = \hat{\xi}_j$, $\forall\, g \in H_j$,
(b) H_j is solvable non abelian $\Rightarrow$ there exists a formal vector field $\hat{\xi}_j$ such that: (b*) $g * \hat{\xi}_j = c_j \cdot \hat{\xi}_j$, $c_g \in \mathbb{C}^*$, $\forall\, g \in H_j$ and $c_g \neq 1$ for some $g \in H_j$. The vector field $\hat{\xi}_j$ writes in some formal coordinate $\hat{z}$ as $\hat{\xi}_j(\hat{z}) = \hat{z}^{k+1} \dfrac{d}{d\hat{z}}$.

Definition 9.2 (normalizing coordinates)**.** Let $H \subset \mathrm{Diff}(\mathbb{C}, 0)$ be solvable and $\hat{\xi}$ a projectively invariant as in Lemma 9.1 above. The vector field $\hat{\xi}_j$ writes in some formal coordinate $\hat{z}$ as $\hat{\xi}_j(\hat{z}) = \dfrac{\hat{z}^{k+1}}{1 + a\hat{z}^k} \dfrac{d}{d\hat{z}}$. Such coordinates are called *normalizing coordinates* for the group G.

Let ω be a holomorphic one-form defining $\mathcal{F}$ in a neighborhood $U \subset \mathbb{C}^2$ of the origin. Denote by $\tilde{\omega}$ the lift of ω by the reduction of singularities for $\mathcal{F}$, i.e., $\tilde{\omega} = \sigma^*(\omega)$ where $\sigma \colon \tilde{U} \to U$ is the morphism described in Section 2.

Lemma 9.3. *There exists a transversely formal 1-form $\hat{\eta}_j$ defined over D_j^* such that $d\tilde{\omega} = \hat{\eta}_j \wedge \tilde{\omega}$, $d\hat{\eta}_j = 0$, $\hat{\eta}_j$ has simple poles along D_j^* and along $(\tilde{\omega})_\infty \cup (\tilde{\omega})_0$.*

Moreover, if $C \subset (\tilde{\omega})_\infty \cup (\tilde{\omega})_0$ is an irreducible component with $C \cap D_j \neq \emptyset$, then either $\mathrm{Res_C}\hat{\eta}_\mathrm{j} = -\mathrm{ord}((\tilde{\omega})_\infty, \mathrm{C})$, or $\mathrm{Res_C}\hat{\eta}_\mathrm{j} = \mathrm{ord}(\tilde{\omega})_0$.

Proof. First we assume that H_j is abelian. We consider $\hat{\xi}_j$ as in (a) above. Condition (a*) allows as to extend $\hat{\xi}_j$ as a transversely formal global section $\hat{\tau}_j$ of the sheaf $\widehat{\mathrm{Sim}}(\mathcal{F}, D_j^*)$ of transversely formal symmetries associated to $\tilde{\mathcal{F}}$, over the open curve D_j^*. Indeed, this is just the usual *holonomy extension* of $\hat{\xi}$ as a constant vector field along the plaques of $\mathcal{F}$ near D_j^*. Then $\hat{h}_j = \tilde{\omega}(\hat{\tau}_j)$ is a transversely formal function defined over D_j^* and which satisfies $d\tilde{\omega} = \dfrac{d\hat{h}_j}{\hat{h}_j} \wedge \tilde{\omega}$ [21], so that we take $\hat{\eta}_j = \dfrac{d\hat{h}_j}{\hat{h}_j}$. This 1-form clearly satisfies the required properties.

Now we assume that H_j is solvable non abelian. We consider $\hat{\xi}_j$ as in (b). Condition (b*) allows the construction of a section $\hat{\tau}_j$ of the quotient sheaf $\widehat{\mathrm{Sim}}(\mathcal{F}, D_j^*)/\mathbb{C}^*$. Thus $\dfrac{d(\tilde{\omega}(\hat{\tau}_j))}{\tilde{\omega}(\hat{\tau}_j)} = \hat{\eta}_j$ is well-defined over D_j^* and has the required properties [21]. $\square$

Now we prove that $\hat{\eta}_j$ constructed in Lemma 9.3, extends to the singularities in $D_j \cap \mathrm{sing}(\tilde{\mathcal{F}})$. Let then $q_o \in \mathrm{sing}\tilde{\mathcal{F}} \cap \mathrm{D_j}$ be a singularity. If it is a corner, say $q_o = D_i \cap D_j$ is a corner then it has two separatrices, contained in D_i and D_j. Since q_o is not a saddle-node we have three distinct cases to consider:

(1) q_o admits a formal first integral. In this case by [16] q_o admits a holomorphic first integral so that $\tilde{\omega}$ admits a holomorphic integrating factor around q_o and q_o is analytically linearizable.

(2) q_o is non-resonant of the form $xdy - \lambda ydx + \text{h.o.t.} = 0$, $\lambda \notin \mathbb{Q}$: In this case the local holonomy around q_o is a non-periodic linear part so that H_j is analytically normalizable and we may assume that ($\widehat{\xi_j}$ and therefore) $\hat{\eta}_j$ is convergent.

(3) q_o is resonant not formally linearizable: In this case q_o admits the so called Martinet-Ramis formal normal forms [17]. In particular the 1-form $\tilde{\omega}$ admits a formal integrating factor $\hat{h}$ defined at q_o; that is, (*) $d\left(\dfrac{\tilde{\omega}}{\hat{h}}\right) = 0$ and $\hat{h}$ is a formal series at q_o. This equation (*) exhibits resommation properties for $\hat{h}$ so that by a Briot-Bouquet type argument [17],[18] $\hat{h}$ can be written $\hat{h}(x,y) = \sum\limits_{j=0}^{+\infty} a_j(x) y^j$, where $(x,y) \in U$ is a local coordinate centered at q_o, such that $D_j \cap U = \{y = 0\}$, $D_i \cap U = \{x = 0\}$, $a_j(x)$ is a holomorphic function converging in a small disk $\mathbb{D}_{q_o} \subset D_j$ centered at q_o, not depending on $j \in \mathbb{N}$.

Thus, in any of the three cases above, we conclude that there exists a transversely formal 1-form $\hat{\eta}_{q_o}$ defined over a small disk $q_o \in \mathbb{D}_{q_o} \subset D_j$ and with simple poles along the separatrices (so along D_i and D_j), such that $d\hat{\eta}_{q_o} = 0$ and $d\tilde{\omega} = \hat{\eta}_{q_o} \wedge \tilde{\omega}$. The difference $\hat{\eta}_j - \hat{\eta}_{q_o}$ writes therefore as $\hat{\eta}_j - \hat{\eta}_{q_o} = \hat{h} \cdot \tilde{\omega}$ for some transversely formal integrating factor $\hat{h}$ for $\tilde{\omega}$ (i.e., $d(\hat{h} \cdot \tilde{\omega}) = 0$) defined over the punctured disc $\mathbb{D}_{q_o}^* = \mathbb{D}_{q_o} \setminus \{q_o\}$.

Now we consider these three cases separately.

Case (1): There exists a local chart $(x,y) \in U$, $x(q_o) = y(q_o) = 0$ such that

$$\tilde{\omega}(x,y) = g(x,y)(nxdy + mydx)$$

for some $n, m \in \mathbb{N}^*$ and some holomorphic $g \in \mathcal{O}_2$. We consider the 1-form

$$\hat{\eta}_{q_o} = \frac{dg}{g} + \frac{d(xy)}{xy} = \frac{dg}{g} + \frac{dx}{x} + \frac{dy}{y},$$

which is meromorphic in U. Let also $\omega_o = n\dfrac{dy}{y} + \dfrac{dx}{x}$. Then we have $\hat{\eta}_j - \hat{\eta}_{q_o} = \hat{h} \cdot \tilde{\omega} = (\hat{h}xyg)\omega_o$ and since $d(\hat{h} \cdot \tilde{\omega}) = 0 = d\omega_o$ it follows that $d(\hat{h}xyg) \wedge \omega_o = 0$, that is, $\hat{f} = \hat{h}xyg$ is a transversely formal first integral for $\tilde{\mathcal{F}}$ over $\mathbb{D}_{q_o}^*$. Since $f_o = x^m y^n$ is already a *primitive* first integral for $\tilde{\mathcal{F}}$ around q_o (if we choose $\langle n, m \rangle = 1$) it follows that $\hat{f} = \hat{l}(f_o)$ for some one variable formal expression, that is, $\hat{f} = \hat{l}(x^m y^n)$ and since $\hat{f}$ is defined as a transversely formal expression over $\mathbb{D}_{q_o}^*$ which contains points of the form $(x, 0)$, $x \neq 0$ and since $x^m y^n = 0$ at these points, it follows that $\hat{l}$ is a formal series on the disk $\mathbb{D} \subset \mathbb{C}$ and therefore $\hat{f}$ extends as a transversely formal first integral along $\mathbb{D}_{q_o}$. It follows that ($\hat{h}$ and therefore) $\hat{\eta}_j$ extends as a transversely formal object to $\mathbb{D}_{q_o}$.

Case (2): There exists a formal linearization for $\tilde{\mathcal{F}}$ at q_o,

$$\tilde{\omega}(x,y) = g(x,y)(xdy - \lambda y(1+b(x,y))dx),$$

$b \in \mathcal{O}_2 \notin g$, $\lambda \in \mathbb{C}\backslash\mathbb{Q}$, $b(0,0)=0$, (x,y) is a holomorphic chart and we can find a formal chart $(x,\widehat{Y})$ at q_o, with $\widehat{Y}(x,y) = \sum\limits_{j=1}^{+\infty} a_j(x)y^j$, $a_j(x)$ holomorphic in $\mathbb{D}_{q_o}$, $\forall\, j$, such that, $\tilde{\omega}(x,\widehat{Y}) = \widehat{G}(x,\widehat{Y})\cdot(xd\widehat{Y} - \lambda\widehat{Y}\,dx)$ is linearized. We define $\hat{\eta}_{q_o} = \frac{d\widehat{G}}{\widehat{G}} + \frac{d(x\widehat{Y})}{x\widehat{Y}} = \frac{d\widehat{G}}{\widehat{G}} + \frac{dx}{x} + \frac{d\widehat{Y}}{\widehat{Y}}$ and $\widehat{\omega}_o = \dfrac{d\widehat{Y}}{\widehat{Y}} - \lambda\frac{dx}{x}$.

Therefore we may write $\hat{\eta}_j - \hat{\eta}_{q_o} = (\hat{h}\cdot x\cdot Y\cdot\widehat{G})\cdot\hat{\omega}_0$ and $d(\hat{h}\cdot x\cdot\widehat{Y}\cdot\widehat{G})\wedge\hat{\omega}_0 = 0$. We put $\hat{f} := \hat{h}x\cdot\widehat{Y}\cdot\widehat{G}$ and write $\hat{f} = \sum\limits_{j=0}^{+\infty} f_j(x)y^j$ where $f_j(x)$ is holomorphic in $\mathbb{D}^*_{q_o}$, $\forall\, j$. Then $d\hat{f}\wedge\hat{\omega}_0 = 0$ gives $(*)\ x\hat{f}_x + \lambda\widehat{Y}\cdot\hat{f}_{\widehat{Y}} = 0$ over $\mathbb{D}^*_{q_o}$; where by definition (notice that $\frac{\partial\widehat{Y}}{\partial x}$ and $\frac{\partial\widehat{Y}}{\partial y}$ are invertible elements of the ring of formal power series):

$$\hat{f}_x := \frac{\partial\hat{f}}{\partial x} = \sum_{j=0}^{+\infty} f_j'(x)y^j, \quad \hat{f}_y := \frac{\partial\hat{f}}{\partial y} = \sum_{j=1}^{+\infty} jf_j(x)y^{j-1} \ \ \hat{f}_{\widehat{Y}} := \hat{f}_x\left(\frac{\partial\widehat{Y}}{\partial x}\right)^{-1} + \hat{f}_y\left(\frac{\partial\widehat{Y}}{\partial y}\right)^{-1}$$

and

$$\frac{\partial\widehat{Y}}{\partial x} := \sum_{j=1}^{+\infty} a_j'(x)y^j, \quad \frac{\partial\widehat{Y}}{\partial y} := \sum_{j=1}^{+\infty} ja_j(x)y^{j-1}.$$

Thus by (*) we conclude that $\hat{f}_x = 0$, $\hat{f}_{\widehat{Y}} = 0 \Rightarrow \hat{f}_x = 0$, $\hat{f}_y = 0 \Rightarrow \hat{f} = f_o$ is a constant and therefore $\hat{f}$ extends naturally to $\mathbb{D}_{q_o}$. This shows that ($\hat{h}$ and therefore) $\hat{\eta}_j$ extends to $\mathbb{D}_{q_o}$.

Case (3): In this case we have local holomorphic coordinates $(x,y)\in U$ centered at q_o, such that $\tilde{\omega}(x,y) = g(x,y)[nxdy + my(1+b(x,y))dx]$ where $n,m\in\mathbb{N}^*$, $\langle n,m\rangle = 1$, $g,b\in\mathcal{O}_2$, $b(0,0)=0$ [17]. According to [17] and also from what we have observed above we may choose a formal coordinate system $(x,\widehat{Y})$ at q_0 , $\widehat{Y} = \sum\limits_{j=1}^{+\infty} a_j(x)y^j$, $a_j\in\mathcal{O}(\mathbb{D}_{q_o})\ \forall\, j$, such that if $\lambda = n/m$ then

$$\tilde{\omega}(x,\widehat{Y}) = \widehat{G}(x,\widehat{Y})[n(1+(\lambda-1)(x^m\widehat{Y}^n)^k)xd\widehat{Y} + m(1+\lambda(x^m\widehat{Y}^m)^k)\widehat{Y}\,dx].$$

In this case we define

$$\hat{\eta}_{q_o} = d\log[\widehat{G}(x,\widehat{Y})\cdot x^{m+1}\widehat{Y}^{m+1}] = \frac{d\widehat{G}}{\widehat{G}} + (m+1)\frac{dx}{x} + (n+1)\frac{d\widehat{Y}}{\widehat{Y}}$$

and

$$\widehat{\omega}_o = \frac{\tilde{\omega}}{\widehat{G}x^{n+1}\widehat{Y}^{n+1}} = -d\left(\frac{\widehat{Y}}{x^m\widehat{Y}^n}\right) + (x^m\widehat{Y}^n)^{k-1}\left[n(\lambda-1)\frac{d\widehat{Y}}{\widehat{Y}} + \lambda.m\frac{dx}{x}\right].$$

Since $\lambda = n/m$ it is a straightforward calculation to show that $d\hat{\omega}_o = 0$ and therefore if

$$\hat{f} = h\cdot\widehat{G}x^{m+1}\widehat{Y}^{n+1},$$

then $d\hat{v}\wedge\hat{\omega}_o = 0$. As in the case above, the fact that $\hat{\omega}_o$ admits no first integral outside one separatrix implies that $\hat{f}$ is constant and therefore $\hat{\eta}$ extends to $\mathbb{D}_{q_o}$. But we remark that $\hat{\eta}-\hat{\eta}_{q_o}$ has simple poles along $\mathbb{D}_{q_o}\subset D_j$ and $\hat{\omega}_o$ has poles of order $n+1\geq 2$ along $\mathbb{D}_{q_o}$, so that $\hat{\eta}-\hat{\eta}_{q_o} =$ const.. $\hat{\omega}_o \Rightarrow$ const.. $=0$ and therefore we have in fact concluded that if q_o is of type **(3)** then $\hat{\eta}_j$ extends as $\hat{\eta}_j = \hat{\eta}_{q_o}$ to q_o.

Summarizing the above discussion we obtain:

Proposition 9.4. *Given any component $D_j \subset E(\mathcal{F})$ there exists a transversely formal generalized integrating factor $\hat{\eta}_j$ for $\tilde{\omega}$ defined over D_j which also satisfies: the formal polar set $(\hat{\eta}_j)_\infty$ has order one.*

Now it remains to show how to construct the forms $\hat{\eta}_j$ in a compatible way, i.e., such that if $D_i \cap D_j = \{q\}$ then both forms bind up into a transversely form defined in $D_i \cup D_j$. For this we need the solvability of the virtual holonomy group H_j^{virt}, not only of the holonomy group H_j. The idea is basically the following: Take a component $D_i \subset E(\mathcal{F})$ that meets D_j at a corner singularity $q = D_i \cap D_j$. We may assume that q is resonant so that we are in Case 1 or 3 of the above argumentation. The difference $\hat{\alpha}_{ij} := \hat{\eta}_i - \hat{\eta}_j$ is a formal closed meromorphic one-form at q such that $\hat{\alpha}_{ij} \wedge \tilde{\omega} = 0$. Moreover, $\hat{\alpha}_{ij}$ is zero or it has only simple poles. Thus, we may assume that we are just in Case 1 of the above argumentation, i.e., that $\tilde{\mathcal{F}}$ has a holomorphic first integral at q. In this case the so called Dulac correspondence is defined as follows:

Choose a small neighborhood $\tilde{U}$ of q, where we take small transverse sections Σ_j to D_j and Σ_i to D_i. Denote by $\mathcal{F}(\Sigma_j)$ the collection of subsets $E \subset \Sigma_j$ such that E is contained in some leaf of $\tilde{\mathcal{F}}\big|_{\tilde{U}}$. Define $\mathcal{F}(\Sigma_i)$ in a similar way. Roughly speaking, the Dulac correspondence is a multivalued correspondence $\mathcal{D}_q : \Sigma_j \to \Sigma_i$, which is obtained by tracing the local leaves of $\tilde{\mathcal{F}}\big|_{\tilde{U}}$. Given any $x \in \Sigma_j$ the set of intersections of the local leaf of $\tilde{\mathcal{F}}\big|_{\tilde{U}}$ that contains x, with the transverse section Σ_j, is denoted by $L_x \cap \Sigma_j \in \mathcal{F}(\Sigma_j)$. The correspondence $\mathcal{D}_q$ associates to any point $z \in L_x \cap \Sigma_j$, the subset $\mathcal{D}_q(z) \subset L_x \cap \Sigma_i \in \mathcal{F}(\Sigma_i)$, usually defined by the some local normal form of $\tilde{\mathcal{F}}$ in $\tilde{U}$.

Given an element $h \in \mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}, \mathrm{D_j}, \Sigma_\mathrm{j})$, we associate h with a collection of elements

$$\{h^{\mathcal{D}}\} \subset \mathrm{Diff}(\Sigma_\mathrm{i}, \mathrm{q_i}) \subset \mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}, \mathrm{D_i}, \Sigma_\mathrm{i}),$$

each of which satisfies the following relation

$$h^{\mathcal{D}} \circ \mathcal{D}_q = \mathcal{D}_q \circ h\,,$$

called the *adjunction equation*. We remark that the adjunction equation is not exactly an equation, but rather an equality of sets or correspondences. More precisely, given any element $h \in \mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}, \mathrm{D_j}, \Sigma_\mathrm{j})$, each diffeomorphism $h^{\mathcal{D}} \in \mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}, \mathrm{D_i}, \Sigma_\mathrm{i})$ must satisfy, for every $x \in \Sigma_i$, the equality of sets $h^{\mathcal{D}}(\mathcal{D}_q(x)) = \mathcal{D}_q(h(x))$, where $\mathcal{D}_q(x) \subset L_x \cap \Sigma_i$ and $\mathcal{D}_q(h(x)) \subset L_x \cap \Sigma_j$ are subsets as above. This adjunction is adequately defined for the special case of singularities $\{q\} = D_i \cap D_j$ we are considering as we shall see in what follows. There are local holomorphic coordinates $(x,y) \in \tilde{U}$ such that $D_i \cap \tilde{U} = \{x = 0\}$, $\;D_j \cap \tilde{U} = \{y = 0\}$, and such that $\tilde{\mathcal{F}}\big|_{\tilde{U}}$ is given in the *normal form* as $nxdy + mydx = 0$ and $q : x = y = 0$, where $n/m \in \mathbb{Q}_+$ and $\langle n, m\rangle = 1$. We fix the local transverse sections as $\Sigma_j = \{x = 1\}$ and $\Sigma_i = \{y = 1\}$, such that $\Sigma_i \cap D_i = q_i \neq q$ and $\Sigma_j \cap D_j = q_j \neq q$. The local leaves of the foliation are given by $x^m y^n = $ const. The Dulac correspondence is the correspondence obtained by following these leaves

$$\mathcal{D}_q \colon \Sigma_j \to \Sigma_i,\ \mathcal{D}_q(x) = \{x^{m/n}\}.$$

from a local transverse section Σ_j to D_j to another transverse section Σ_i to D_i. Let be given a map f in the virtual holonomy H_i^{virt} of D_i. We search for a well-defined map $f^{\mathcal{D}_q} \in H_j^{\mathrm{virt}}$ in the virtual holonomy of D_j, such that it satisfies the " adjunction equation" $f^{\mathcal{D}_q} \circ \mathcal{D}_q = \mathcal{D}_q \circ f$.

The fact that we can construct both $\hat{\eta}_i$ and $\hat{\eta}_j$ in a compatible way, i.e., such that $\hat{\eta}_i$ and $\hat{\eta}_j$ agree as formal objects at q is a consequence of the following: (1) $\hat{\eta}_i$ and $\hat{\eta}_j$ are constructed in a compatible way (agreeing) with the virtual holonomy groups H_i^{virt} and H_j^{virt} respectively. (2) these virtual holonomy groups are related by the Dulac correspondence. Indeed, we can embed the virtual holonomy group of D_j into the virtual holonomy group of D_i. Thus, the solvability of the group H_i^{virt} means that, in a certain sense, both virtual holonomy groups are solvable and simultaneously written in formal normalizing coordinates. In particular, we can already choose the form $\hat{\eta}_i$ in such a way that it agrees with $\hat{\eta}_j$ as formal objects at q. The details of this construction and compatibility conditions are thoroughly discussed in [21]. There the author mentions the so called *zone holomorphe, zone logarithmique,...*. To such a zone, denoted by $\mathcal{Z}$, the author associates a holonomy pseudo-group $\text{Hol}(\mathcal{Z}, \text{f}_{\mathcal{Z}})$ which measures the obstruction to the integration of the foliation in a neighborhood of the zone $\mathcal{Z}$. The main point is that under our hypothesis, both components D_i and D_j are accumulated by analytic leaves and therefore both exhibit solvable virtual holonomy groups. On the other hand, any generalized holonomy $\text{Hol}(\mathcal{Z}, \text{f}_{\mathcal{Z}})$ constructed in [21] is contained in the virtual holonomy. This implies that the conditions of [21] are automatically satisfied by Proposition 3.3. Now we can finish the argumentation just by observing that from the above discussion we already conclude from Proposition 3.3 that the forms $\hat{\eta}_j$ can be constructed in a compatible way, resulting into a global transversely formal one-form $\tilde{\hat{\eta}}$ along the divisor $D = \bigcup\limits_j D_j$. Blowing down this one-form we obtain a transversely formal generalized integrating factor $\hat{\eta}$ for ω in a neighborhood of the origin $0 \in \mathbb{C}^2$.

Sketch of the proof of Proposition 5.9. We perform the reduction of singularities of the foliation $\mathcal{F}$. The first step is:

Claim 9.5. *All the virtual holonomy groups are exceptional, isomorphic.*

The next step is:

Claim 9.6. *There is a transversely formal function $\hat{\Phi}_j$ defined along $D_j^* = D_j \setminus \text{sing}(\tilde{\mathcal{F}})$, with the property below: Given any point $q \in D_j^*$ and a transverse disc Σ_q with $\Sigma_{\tilde{D}} \cap D_j = \{q\}$, we choose a formal normalizing coordinate $\hat{x}_q \in \Sigma_q$, centered at q, for the virtual holonomy group* $\text{Hol}^{\text{virt}}(\tilde{\mathcal{F}}, \text{D}_\text{j}, \Sigma_\text{q}), \text{q})$. *Then we have $\hat{\Phi}_j\big|_{\Sigma_q}(\hat{x}_q) = \cos(\frac{2\pi}{\hat{x}_q^{k_j}})$.*

In the case $\text{Hol}^{\text{virt}}(\tilde{\mathcal{F}}, \text{D}_\text{j}, \Sigma_\text{q}, \text{q})$ is exceptional we define $\hat{\Phi}_j\big|_{\Sigma_q}$ as $\hat{\Phi}_j(\hat{x}_q) = \cos(\frac{2\pi}{\hat{x}_q^{k_j}})$. Then:

Claim 9.7. *The function $\hat{\Phi}_j$ extends to each singularity $p \in D_j \cap \text{sing}(\tilde{\mathcal{F}})$, the result is a transversely formal Liouvillian function along D_j which is a first integral for $\tilde{\mathcal{F}}$.*

Proceeding as in the proof of Proposition 9.4 we obtain:

Claim 9.8. *Given any corner $p = D_i \cap D_j$ there is a constant $c_{ij} \in \mathbb{C}$ such that at p we have $\hat{\Phi}_i = c_{ij}\hat{\Phi}_j$ as formal objects.*

Since the exceptional divisor $E(\mathcal{F})$ contains no cycles we may choose a globally defined transversely formal function $\hat{\Phi}$ along $E(\mathcal{F})$ by suitable choices of constants $c_j \in \mathbb{C}$ and setting $\hat{\Phi} = c_j\hat{\Phi}_j$ whenever it makes sense. Blowing down $\hat{\Phi}$ we obtain the desired formal Liouvillian first integral. This completes the proof of Lemma 8.10. $\square$

References

[1] J.C. Alexander and A. Verjovsky: *First integrals for singular holomorphic foliations with leaves of bounded volume*, Holomorphic dynamics (Mexico, 1986), 1–10, Lecture Notes in Math. 1345, Springer-Verlag, Berlin - New York, 1988.

[2] F. Bracci: Local dynamics of holomorphic diffeomorphisms. Boll. UMI 7-B (2004), 609-636.

[3] C. Camacho: *On the local structure of conformal maps and holomorphic vector fields in* $\mathbb{C}^2$, Jounées Singuliéres de Dijon (Univ. Dijon, Dijon, 1978, 3, 83–94), Astérisque 59-60, Soc. Math. France, Paris 1978.

[4] C. Camacho and A. Lins Neto: Geometric theory of foliations, Birkhauser Inc., Boston, Mass., 1985. DOI: 10.1007/978-1-4612-5292-4

[5] C. Camacho, A. Lins Neto and P. Sad: *Foliations with algebraic limit sets*, Ann. of Math. (2) 136 (1992), 429–446. DOI: 10.2307/2946610

[6] C. Camacho, A. Lins Neto and P. Sad: Topological invariants and equidesingularization for holomorphic vector fields; J. of Diff. Geometry, vol. 20, no. 1 (1984), 143–174.

[7] C. Camacho and P. Sad: *Invariant varieties through singularities of holomorphic vector fields*, Ann. of Math. 115 (1982), 579–595.

[8] C. Camacho and B. Scárdua: *Foliations on complex projective spaces with algebraic limit sets*. Géométrie complexe et systèmes dynamiques (Orsay, 1995). Astérisque, 261, 57–88, Soc. Math. France, Paris 2000.

[9] C. Camacho and B. Scárdua: *A Darboux type theorem for germs of complex vector fields.* Ergodic Theory and Dynamical Systems, **35** (8), (2015), 2458-2473. DOI: 10.1017/etds.2014.46

[10] D. Cerveau et R. Moussu: *Groupes d'automorphismes de* $(\mathbb{C}, 0)$ *et équations différentielles* $ydy + \cdots = 0$; Bull. Soc. Math. France, **116** (1988), 459–488.

[11] R.C. Gunning: *Introduction to holomorphic functions of several variables*, vol. II, Local Theory. Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA, 1990

[12] H. Hironaka, H. Matsumara: *Formal functions and formal embeddings*; J. Math. Soc. Japan, Vol. 20, Nos.1-2, 1968. DOI: 10.2969/jmsj/02010052

[13] J.P. Jouanolou: *Équations de Pfaff algèbriques*, Lecture Notes in Math. 708, Springer-Verlag, Berlin, 1979.

[14] A. Lins Neto: *Construction of singular holomorphic vector fields and foliations in dimension two.* J. Differential Geom. 26 (1987), no. 1, 1 - 31.

[15] F. Loray: *Feuilletages holomorphes à holonomie résoluble*; These Université de Rennes I, 1994.

[16] J.F. Mattei and R. Moussu: *Holonomie et intégrales premières*, Ann. Sci. École Norm. Sup. (4) **13** (1980), 469–523.

[17] J. Martinet and J-P. Ramis: *Classification analytique des équations différentielles non linéaires resonnants du premier ordre*, Ann. Sci. École Norm. Sup. (4) 16 (1983), 571–621.

[18] J. Martinet and J-P. Ramis: *Problème de modules pour des équations différentielles non linéaires du premier ordre*, Publ. Math. Inst. Hautes Études Scientifiques, 55 (1982), 63–124. DOI: 10.1007/BF02698695

[19] Frank Loray, Rafik Meziani: *Classification de certains feuilletages associés à un cusp.* Boletim da Sociedade Brasileira de Matemática - Bulletin/Brazilian Mathematical Society March 1994, Volume 25, Issue 1, pp 93-106.

[20] I. Nakai: *Separatrices for nonsolvable dynamics on* $\mathbb{C}, 0$, Ann. Inst. Fourier (Grenoble) 44 (1994), 569–599. DOI: 10.5802/aif.1410

[21] E. Paul: *Formes singuliéres à holonomie résoluble*; J. Reine Angew Math. 514 (1999).

[22] R. Pérez-Marco: Sur une question de Dulac et Fatou. C.R. Acad. Sci. Paris 321 (1995), 1045-1048.

[23] R. Pérez-Marco: Fixed points and circle maps. Acta Math. 179 (1997), 243-294. DOI: 10.1007/BF02392745

[24] R. Pérez-Marco, J.-C. Yoccoz: *Germes de feuilletages holomorphes à holonomie prescrite.* Complex analytic methods in dynamical systems (Rio de Janeiro, 1992), Astérisque 222 (1994), 7, 345-371

[25] B. Scárdua: *Complex vector fields having orbits with bounded geometry*, Tohoku Math. J. (2) 54 (2002), no. 3, 367–392.

[26] B. Scárdua: *Integration of complex differential equations.* Journal of Dynamical and Control Systems, issue 1, vol. **5**, pp.1-50, 1999.

[27] B. Scárdua: *Dynamics and Integrability for Germs of Complex Vector Fields with Singularity in Dimension Two.* Qualitative Theory of Dynamical Systems, v.13, p.363 - 381, 2014. DOI: 10.1007/s12346-014-0122-z

[28] B. Scárdua: *Differential Algebra and Liouvillian first integrals of foliations*; Journal of Pure and Applied Algebra Volume 215, Issue 5, May 2011, Pages 764–788.

[29] B. Scárdua: *Transversely affine and transversely projective foliations.* Ann. Sc. École Norm. Sup., 4^e série, t.**30**, 1997, p.169-204

[30] A. Seidenberg: *Reduction of singularities of the differential equation* $Ady = Bdx$, Amer. J. Math. 90 (1968), 248–269. DOI: 10.2307/2373435

[31] M.F. Singer: *Liouvillian First Integrals of Differential Equations*; Transactions of the American Mathematical Society, vol. 333, number 2, october 1992, pp.673-688. DOI: 10.1090/S0002-9947-1992-1062869-X

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GEODESICS IN GENERALIZED FINSLER SPACES: SINGULARITIES IN DIMENSION TWO

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Abstract. We study singularities of geodesics flows in two-dimensional generalized Finsler spaces (pseudo-Finsler spaces). Geodesics are defined as extremals of a certain auxiliary functional whose non-isotropic extremals coincide with extremals of the action functional. This allows us to consider isotropic lines as (unparametrized) geodesics.

Introduction

This paper is a study of singularities of geodesics flows in generalized Finsler spaces (pseudo-Finsler spaces). This is a natural development of ongoing research on understanding the geometry of surfaces endowed with a signature-varying pseudo-Riemannian metric; see [8, 10, 11, 16, 19, 20, 21] and the references therein. One of the purposes of this paper is to compare singularities of geodesics flows in pseudo-Finsler and pseudo-Riemannian metrics. On the other hand, the interest in pseudo-Finsler spaces is motivated by physical applications; see, e.g., [1, 5].

Following [23], by a pseudo-Finsler space, we mean a manifold M, $\dim M = m$, with coordinates (x_i) endowed with a metric function $\overline{f}(x_i;\dot{x}_i) = \overline{F}(x_i;\dot{x}_i)^{\frac{1}{n}}$, where $\overline{F}\colon TM \to \mathbb{R}$ is positively homogeneous in $(\dot{x}_i)$ of degree n and smooth on the complement of the zero section of TM (a more detailed definition is given in Sections 1.1, 1.2). A well-known example is *Berwald–Moor* space $(M,\overline{f})$, where $\overline{f}(x_i;\dot{x}_i) = (\dot{x}_1\cdots\dot{x}_n)^{\frac{1}{n}}$, $n = m$; see, e.g., [6, 14, 23].

This paper starts with a discussion of the notion of geodesics in Finsler and pseudo-Finsler spaces with $n \geq 3$ (Section 1). Here, we use the variational definitions of geodesics [7, 23]. In contrast to pseudo-Riemannian spaces ($n = 2$), where naturally parametrized geodesics of all types (including isotropic) can be defined as extremals of the action functional, in pseudo-Finsler spaces a similar definition is not correct for isotropic lines. The solution of this problem is either to exclude isotropic lines from consideration or to find a natural extension of the definition of geodesics.

In the present paper, we choose the second way. Based on a simple variational property, we define geodesics as extremals of a certain auxiliary functional whose non-isotropic extremals coincide with extremals of the action functional. In this direction, we have the following result: in the case $m = 2$, all isotropic lines are (unparametrized) geodesics.

In Section 2, we consider singularities of the geodesic flows in pseudo-Finsler spaces $(M,\overline{f})$, where $m = 2$ and $\overline{F}$ is a polynomial in $(\dot{x}_i)$ of degree $n \geq 3$. The main results are presented in Section 2.2, where we consider the case $n = 3$ in detail. It is proved that singularities of the geodesic flow are connected with the degeneracy of the isotropic lines net. Namely, if the function $\overline{F}$ is generic, the manifold M contains two open domains M_+ and M_- separated by a curve M_0 so that, at every point $q \in M_+$ (resp., M_-), there exist 3 (resp., 1) different isotropic

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directions and isotropic lines which are tangent at $q \in M_0$. Singularities of the geodesic flow appear in the domain M_- and on the curve M_0.

Section 3 is devoted to a special case: pseudo-Finsler spaces $(M, \overline{f})$, where M is a surface in n-dimensional Berwald–Moor space. The corresponding function $\overline{F}$ is a non-generic polynomial in $(\dot{x}_i)$ of degree n. In this case, the domain $M_- = \emptyset$, and singularities of the geodesic flow appear on the curve M_0 only.

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1. Variational definition of geodesics

1.1. **Finsler spaces.** Consider a smooth (here and below, by smooth, we mean C^∞ unless otherwise stated) manifold M, $\dim M = m$, with coordinates (x_i) and a function $\overline{F}(x_i; \dot{x}_i)\colon TM \to \mathbb{R}$ that is positively homogeneous of degree $n \geq 2$ in $(\dot{x}_i)$ and smooth on the complement of the zero section of TM.

Define the function $\overline{f}(x_i; \dot{x}_i) = \overline{F}^{\frac{1}{n}}(x_i; \dot{x}_i)$, which is positively homogeneous of degree 1 in $(\dot{x}_i)$. The pair $(M, \overline{f})$ or, equivalently, $(M, \overline{F})$ is a *Finsler space* (in the classic sense), if the following conditions hold:

B. $\overline{F}(x_i; \dot{x}_i) > 0$ if $|\dot{x}_1| + \cdots + |\dot{x}_m| \neq 0$.

C. The Hessian of the function $\overline{f}^2$ with respect to $(\dot{x}_i)$ is positive definite, that is,

$$\sum_{i,j=1}^{m} \frac{\partial^2 \big(\overline{f}^2\big)}{\partial \dot{x}_i \partial \dot{x}_j} \, \xi_i \xi_j > 0 \quad \text{if} \quad \sum_{i=1}^{m} |\xi_i| \neq 0. \tag{1.1}$$

Here, we use the letters B and C to preserve the notations from the book [23], to which we shall refer. The quadratic form (1.1) is called *the fundamental tensor*, and $\overline{f}$ (positive and smooth on the complement of the zero section of TM) is called the *metric function* on M.

The metric function $\overline{f}(x_i; \cdot)$ defines a Minkowski norm on each tangent space T_xM. For a curve $\gamma\colon I \to M$, it allows us to define the length and the action functionals similarly to Riemannian metrics:

$$J^{(\nu)}(\gamma) = \int\limits_I \overline{f}^{\,\nu}(x_i; \dot{x}_i)\, dt = \int\limits_I \overline{F}^{\,\frac{\nu}{n}}(x_i; \dot{x}_i)\, dt, \quad \dot{x}_i = \frac{dx_i}{dt}, \tag{1.2}$$

with $\nu = 1$ (length) and $\nu = 2$ (action), see, e.g., [13, 23]. As in the Riemannian case, the length functional $J^{(1)}$ is invariant with respect to reparametrizations of γ, while $J^{(2)}$ is not.

Parametrized geodesics can be defined as extremals of the action functional $J^{(2)}$, the corresponding parametrization is called *natural* or *canonical* (it coincides with the arc-length parametrization, where $ds = \overline{f}$).

Non-parametrized geodesics can be defined as extremals of any one of the functionals $J^{(2)}$ and $J^{(1)}$. The difference between using $J^{(2)}$ and $J^{(1)}$ is the following. In the first case, we simply forget the natural parametrization of the extremals of $J^{(2)}$, while in the second case the Euler-Lagrange system with the Lagrangian $\overline{f}(x_i; \dot{x}_i)$ contains $m-1$ independent equations only [23]. This reflects the fact that the length functional $J^{(1)}(\gamma)$ is invariant with respect to reparametrizations of γ. Using this degree of freedom and assuming that we deal with continuously differentiable geodesics with definite tangent directions at all points, one can set (at least, locally) the parameter t equal to one of the coordinates x_i, and consequently, reduce the Euler-Lagrange system for $J^{(1)}(\gamma)$.

From now on, we use the following general notation. Let $\overline{\Phi}(x_i;\dot{x}_i)$ be a function on TM which is positively homogeneous of degree k in $(\dot{x}_i)$, then the formula

$$\Phi = \frac{\overline{\Phi}}{\dot{x}_1^k} \tag{1.3}$$

defines a function on the projectivized tangent bundle PTM. For instance, put $x = x_1$ and $y_i = x_i$, $p_i = dy_i/dx$ for $i = 2, \dots, m$. This yields

$$\frac{d}{dx}\left(\frac{\partial f}{\partial p_i}\right) = \frac{\partial f}{\partial y_i}, \quad f(x, y_i, p_i) = \frac{\overline{f}(x_i;\dot{x}_i)}{\dot{x}_1}, \quad i = 2, \dots, m. \tag{1.4}$$

The passage to equation (1.4) is the standard projectivization $\Pi\colon TM \to PTM$ of the tangent bundle. Moreover, non-parametrized geodesics can be defined as extremals of $J^{(\nu)}$ with arbitrary $\nu \geq 1$, on the basis of the following simple property (see, e.g., [15]).

Lemma 1. *Let $\psi\colon \mathbb{R} \to \mathbb{R}$ be a function such that $\psi \circ \overline{F}$ is C^2-smooth on the complement of the zero section of TM and $\psi'(s) \neq 0$ for all $s \neq 0$. Then non-parametrized extremals of the functional*

$$J_\psi(\gamma) = \int\limits_I \psi \circ \overline{F}(x_i;\dot{x}_i)\, dt, \quad \dot{x} = \frac{dx}{dt}, \quad \dot{y} = \frac{dy}{dt}, \tag{1.5}$$

coincide with non-parametrized extremals of $J_{\mathrm{id}}(\gamma)$*, where* id *is the identity map.*

Proof. The Euler-Lagrange equation of $J_\psi(\gamma)$ reads

$$\frac{d}{dt}\left(\psi' \circ \overline{F}(x_i;\dot{x}_i) \cdot \frac{\partial \overline{F}}{\partial \dot{x}_i}\right) = \psi' \circ \overline{F}(x_i;\dot{x}_i) \cdot \frac{\partial \overline{F}}{\partial x_i}, \quad i = 1, \dots, m. \tag{1.6}$$

In light of the condition $\overline{F}(x_i;\dot{x}_i) > 0$, every curve γ admits the arc-length parametrization, that is, $\overline{F}(x_i;\dot{x}_i) \equiv c \neq 0$ along γ. Using the arc-length parametrization in (1.5), after reducing the constant factor $\psi' \circ \overline{F}(x_i;\dot{x}_i) = \psi'(c)$ in both sides of (1.6), we get the Euler-Lagrange equation of the functional $J_{\mathrm{id}}(\gamma)$. □

Thus non-parametrized geodesics can be defined as extremals of $J_\psi(\gamma)$ with arbitrary function ψ from Lemma 1, in particular, of the functional $J_{\mathrm{id}}(\gamma)$, which is equal to $J^{(\nu)}(\gamma)$ with $\nu = n$ from (1.2). In classical Finsler spaces this extended definition of geodesics gives us nothing essentially new, but it can be useful for generalized Finsler spaces considered below.

1.2. **Generalized Finsler spaces.** A generalization of the notion of a Finsler space may be obtained if conditions B and C are dropped, such spaces are sometimes called *special Finsler* or *pseudo-Finsler*. Here we take the liberty to cite a passage from the classical book [23] (page 265):

> *Again, it should be remarked that very frequently the metric function which is given by a homogeneous Lagrangian function of a dynamical system does not always satisfy conditions B and C. The singularities which may occur as a result of the relaxation of condition C are usually ignored, but it is well possible that an investigation of these singularities in connection with physical applications cannot be avoided and might furthermore prove to be fruitful.*

From now on, we will consider pseudo-Finsler spaces $(M, \overline{F})$, where the function $\overline{F}$ is not assumed to satisfy conditions B and C.

The absence of condition B brings us to the existence of the *isotropic hypersurface* $\mathscr{F}$ given by the equation $\overline{F}(x_i;\dot{x}_i) = 0$ in TM or, equivalently, $F(x, y_i, p_i) = 0$ in PTM. The Euler-Lagrange equation for the functional $J^{(\nu)}(\gamma)$ with $\nu < n$ is not defined on $\mathscr{F}$, since the derivatives of

$\overline{F}^{\frac{\nu}{n}}(x_i;\dot{x}_i)$ are discontinuous on $\mathscr{F}$. This explains the advantage of the functional $J^{(n)}(\gamma)$ for the definition of geodesics compared to $J^{(\nu)}(\gamma)$ with $\nu < n$.

The Euler-Lagrange equation for the functional $J^{(n)}(\gamma)$ reads

$$\sum_{j=1}^{m}\frac{\partial^2\overline{F}}{\partial\dot{x}_i\partial\dot{x}_j}\,\ddot{x}_j+\sum_{j=1}^{m}\frac{\partial^2\overline{F}}{\partial\dot{x}_i\partial x_j}\,\dot{x}_j=\frac{\partial\overline{F}}{\partial x_i},\quad i=1,\dots,m, \tag{1.7}$$

or, equivalently,

$$\ddot{x}_i=\frac{\overline{H}_i(x_i;\dot{x}_i)}{\overline{H}(x_i;\dot{x}_i)},\quad \text{where}\quad \overline{H}=\det\left(\frac{\partial^2\overline{F}}{\partial\dot{x}_i\partial\dot{x}_j}\right),\quad i=1,\dots,m, \tag{1.8}$$

and $\overline{H}_i$ are the determinants defined from (1.7) by Cramer's rule. It is not hard to see that the functions $\overline{H}_i$ are positively homogeneous of degree n in $(\dot{x}_i)$ and $\overline{H}$ is positively homogeneous of degree $n-2$ in $(\dot{x}_i)$.

Similarly to (1.4), the projectivization $\Pi\colon TM\to PTM$ sends equation (1.8) to

$$p_i=\frac{dy_i}{dx},\quad \frac{dp_i}{dx}=\frac{dp_i}{dt}\left(\frac{dx}{dt}\right)^{-1}=\frac{1}{\dot{x}}\frac{d}{dt}\left(\frac{\dot{y}_i}{\dot{x}}\right)=\frac{\ddot{y}_i\dot{x}-\dot{y}_i\ddot{x}}{\dot{x}^3}= \frac{1}{\dot{x}^2}\frac{\overline{H}_i-p_i\overline{H}_1}{\overline{H}}=\frac{H_i-p_iH_1}{H},\quad i=2,\dots,m, \tag{1.9}$$

where the functions H, H_i are obtained from $\overline{H},\overline{H}_i$ by (1.3). Recall that, by formula (1.3), the independent variable x is the coordinate x_1. From Lemma 1, it follows that out of the isotropic hypersurface $\mathscr{F}$, integral curves of (1.9) coincide with integral curves of (1.4). However, equation (1.9) is defined in the whole space PTM.

Lemma 2. *$\mathscr{F}$ is an invariant hypersurface of both equations* (1.7) *and* (1.9)*. Moreover, in the case $m=2$ all isotropic lines are non-parametrized extremals of the functional $J^{(n)}$.*[1]

Proof. After straightforward transformations, equation (1.4) gives a direction field which is parallel to

$$F^{-\mu}X\left(\frac{\partial}{\partial x}+p\frac{\partial}{\partial y}\right)+F^{-\mu}\sum_{i=2}^{m}Y_i\frac{\partial}{\partial p_i}, \tag{1.10}$$

where $\mu=(m-1)(2-\frac{1}{n})$ and X, Y_i are smooth functions on PTM.

Since the vector field (1.10) is derived directly from the Euler-Lagrange equation (1.4), it is divergence-free in PTM except for the hypersurface $\mathscr{F}$ where the factor $F^{-\mu}$ is discontinuous, and the field is not defined. By Theorem 1 [8], $\mathscr{F}$ is an invariant hypersurface of the vector field

$$X\left(\frac{\partial}{\partial x}+p\frac{\partial}{\partial y}\right)+\sum_{i=2}^{m}Y_i\frac{\partial}{\partial p_i}, \tag{1.11}$$

which is obtained from (1.10) by eliminating the common factor $F^{-\mu}$. By Lemma 1, integral curves of (1.11) coincide with integral curves of (1.9); hence, $X=H$, $Y_i=H_i-p_iH_1$, and $\mathscr{F}$ is an invariant hypersurface of (1.9). We remark that every solution of (1.7) is obtained from a solution of (1.9) by choosing an appropriate parametrization $x_1(t)$. Thus $\mathscr{F}$ is an invariant hypersurface of equation (1.7) also.

Finally, consider the case $m=2$. Then $\dim PTM=3$ and $\dim\mathscr{F}=2$. The field of contact planes $y_2=p_2dx$ cuts a direction field on $\mathscr{F}$, which coincides with the restriction of the field

[1] This statement holds true only for $m=2$, while in the case $m>2$ some isotropic lines are geodesics and some of them are not. An example is given for $m=3$, $n=2$ (pseudo-Riemannian metrics) in [20].

(1.9) to $\mathscr{F}$. Hence the projection of integral curves from the surface $\mathscr{F} \subset PTM$ to M are isotropic lines and non-parametrized extremals of the functional $J^{(n)}$ simultaneously. $\square$

In accordance with the previous reasoning, one can give the following definition.

Definition 1. *The projections of integral curves of equation (1.9) from PTM to M distinguished from a point are non-parametrized geodesics in the pseudo-Finsler space $(M, \overline{F})$.*

By Lemma 2, in the case $m = 2$, all isotropic lines are geodesics in the sense of the given definition.

The natural parametrization of non-isotropic geodesics is defined by equation (1.6) with $\psi(s) = s^{\frac{2}{n}}$ and coincides with the arc-length parametrization. In the case $n > 2$, the natural parametrization of isotropic geodesics is not defined, while in the case $n = 2$ it is defined by equation (1.6) with $\psi = \mathrm{id}$.

2. Polynomial pseudo-Finsler metrics on 2-manifolds

From now on, we consider the case when $m = 2$ and the function $\overline{F}$ is a homogeneous polynomial of degree $n \geq 2$ in $(\dot{x}_i)$. Denote the coordinates on the manifold M by (x, y).

Consider a pseudo-Finsler space with the metric function $\overline{f} = \overline{F}^{\frac{1}{n}}$, where

$$\overline{F}(x, y; \dot{x}, \dot{y}) = \sum_{i=0}^{n} a_i(x, y)\dot{x}^{n-i}\dot{y}^i, \quad F(x, y; p) = \sum_{i=0}^{n} a_i(x, y)p^i, \tag{2.1}$$

the coefficients a_i smoothly depend on (x, y). Then equation (1.8) reads

$$\ddot{x} = \frac{\overline{H}_1}{\overline{H}}, \quad \ddot{y} = \frac{\overline{H}_2}{\overline{H}}, \quad \overline{H} = \begin{vmatrix} \overline{F}_{\dot{x}\dot{x}} & \overline{F}_{\dot{x}\dot{y}} \\ \overline{F}_{\dot{x}\dot{y}} & \overline{F}_{\dot{y}\dot{y}} \end{vmatrix}, \quad \overline{H}_1 = \begin{vmatrix} \overline{G}_1 & \overline{F}_{\dot{x}\dot{y}} \\ \overline{G}_2 & \overline{F}_{\dot{y}\dot{y}} \end{vmatrix}, \quad \overline{H}_2 = \begin{vmatrix} \overline{F}_{\dot{x}\dot{x}} & \overline{G}_1 \\ \overline{F}_{\dot{x}\dot{y}} & \overline{G}_2 \end{vmatrix}, \tag{2.2}$$

where $\overline{G}_1 = \overline{F}_x - \dot{x}\overline{F}_{\dot{x}x} - \dot{y}\overline{F}_{\dot{x}y}$ and $\overline{G}_2 = \overline{F}_y - \dot{x}\overline{F}_{x\dot{y}} - \dot{y}\overline{F}_{\dot{y}y}$.

Lemma 3. *The projectivization $\Pi\colon TM \to PTM$ sends equation (2.2) to*

$$p = \frac{dy}{dx}, \quad \frac{dp}{dx} = \frac{H_2 - pH_1}{H} = \frac{P}{\Delta}, \tag{2.3}$$

where

$$\begin{aligned} &\Delta(x, y; p) = nFF_{pp} - (n-1)F_p^2, \\ &P(x, y; p) = nF(F_y - F_{xp} - pF_{yp}) + (n-1)F_p(F_x + pF_y). \end{aligned} \tag{2.4}$$

Proof. Taking into account (1.9), it remains for us to establish the equality $\frac{H_2 - pH_1}{H} = \frac{P}{\Delta}$, where Δ, P are defined in (2.4). Let us prove that $H = (n-1)\Delta$ and $H_2 - pH_1 = (n-1)P$, i.e., $\overline{H} = \dot{x}^{2n-4}(n-1)\Delta$ and $\overline{H}_2 - p\overline{H}_1 = \dot{x}^{2n-2}(n-1)P$. Since both sides of the two last equalities can be treated as quadratic forms on $a_0, \dots, a_n$ with coefficients depending on $\dot{x}, \dot{y}$, it suffices to compare the coefficients of the monomials a_ia_j in the left- and right-hand sides.

Put $\varepsilon_{ij} = 1$ if $i \neq j$ and $\varepsilon_{ij} = \frac{1}{2}$ if $i = j$. Direct calculation shows that the coefficient of the monomial a_ia_j, $i + j = k$, in the expression $\overline{H} = \overline{F}_{\dot{x}\dot{x}}\overline{F}_{\dot{y}\dot{y}} - \overline{F}_{\dot{x}\dot{x}}^2$ is $\alpha_{ij}\varepsilon_{ij}\dot{x}^{2(n-1)-k}\dot{y}^{k-2}$, where

$$\begin{aligned} \alpha_{ij} = (n-i)(n-1-i)j(j-1) + (n-j)(n-1-j)i(i-1) - 2ij(n-i)(n-j) = \\ (n-1)\big(n(k^2 - k - 4ij) + 2ij\big). \end{aligned} \tag{2.5}$$

On the other hand, the coefficient of the monomial a_ia_j, $i + j = k$, in the expression

$$\Delta = nFF_{pp} - (n-1)F_p^2$$

is $\beta_{ij}\varepsilon_{ij}p^{k-2}$, where

$$\beta_{ij} = n(i(i-1)+j(j-1)) - 2ij(n-1) = n(k^2-k-4ij)+2ij. \tag{2.6}$$

From (2.5) and (2.6), we have $\alpha_{ij} = (n-1)\beta_{ij}$, that proves $\overline{H} = \dot{x}^{2n-4}(n-1)\Delta$. The proof of the equality $\overline{H}_2 - p\overline{H}_1 = \dot{x}^{2n-2}(n-1)P$ is similar. □

Remark 1. From formula (2.4), it follows that Δ and P are polynomials in p of degrees not greater than $2n-4$ and $2n-1$, respectively. For instance,

$$\Delta(x,y;p) = (2na_na_{n-2} - (n-1)a_{n-1}^2)p^{2n-4} + \cdots + 2na_0a_2 - (n-1)a_1^2.$$

For our further purposes, it is convenient to write equation (2.3) as the field

$$\Delta\left(\frac{\partial}{\partial x} + p\frac{\partial}{\partial y}\right) + P\frac{\partial}{\partial p}. \tag{2.7}$$

The field (2.7) is defined on the whole space PTM including the isotropic surface $\mathscr{F}$. The field of contact planes $dy = pdx$ defines on $\mathscr{F}$ a direction field whose integral curves correspond to isotropic lines, while all remaining integral curves of the field (2.7) (that do not belong entirely to the isotropic surface) correspond to non-isotropic geodesics.

In accordance with Definition 1, non-parametrized geodesics in the pseudo-Finsler space $(M,\overline{F})$ are the projections of integral curves of the field (2.7) from PTM to M distinguished from a point. Singularities of geodesics occur at the points of PTM where $\Delta(x,y;p)$ vanishes. To describe the locus of such points, we use the following lemma.

Lemma 4. *Given a polynomial*

$$\Phi(p) = \prod_{i=1}^{n}(p+\gamma_i), \quad \gamma_i \in \mathbb{R}, \quad n \geq 2, \tag{2.8}$$

consider the polynomial

$$\Delta(p) = n\Phi(p)\Phi''(p) - (n-1)\Phi'(p)^2. \tag{2.9}$$

Then the following statements hold:

(a) $\Delta \equiv 0$ *if and only if* $\gamma_1 = \cdots = \gamma_n$.
(b) *Suppose that* $\gamma_i \neq \gamma_j$ *for at least one pair* i,j. *Then* p *is a real root of the polynomial* Δ *if and only if* p *is a multiple root of the polynomial* Φ.
(c) *If* p *is a double root of* Φ *and* $n \geq 3$, *then* p *is a double root of* Δ.

Proof. The implications $\gamma_1 = \cdots = \gamma_n \Rightarrow \Delta \equiv 0$ in (a) and $\Phi(p) = \Phi'(p) = 0 \Rightarrow \Delta(p) = 0$ in (b) are trivial. The implication $\Delta \equiv 0 \Rightarrow \gamma_1 = \cdots = \gamma_n$ in (a) follows from (b). Indeed, assume that $\Delta \equiv 0$ holds and any two of the numbers $\gamma_1,\ldots,\gamma_n$ are not equal. By (b), $\Delta(p) = 0$ implies $\Phi(p) = 0$. Hence $\Phi \equiv 0$, which contradicts (2.8).

Statement (c) is also trivial: differentiating (2.9) twice, from $\Phi(p) = \Phi'(p) = 0$, we get

$$\Delta(p) = \Delta'(p) = 0 \text{ and } \Delta''(p) = (2-n)\Phi''(p)^2 \neq 0, \text{ if } \Phi''(p) \neq 0 \text{ and } n \neq 2.$$

It remains to prove the implication $\Delta(p) = 0$ and $p \in \mathbb{R} \Rightarrow \Phi(p) = \Phi'(p) = 0$ in statement (b). Assume that $\gamma_i \neq \gamma_j$ for at least one pair i,j and there exists $p_* \in \mathbb{R}$ such that $\Phi(p_*) \neq 0$. Making the change of variables $p \mapsto p - p_*$, without loss of generality we can assume that $p_* = 0$. Then $\Phi(0) = \gamma_1 \cdots \gamma_n \neq 0$, and

$$\Phi'(0) = \sum_{i=1}^{n}\alpha_i\Phi(0), \quad \Phi''(0) = 2\sum_{i<j}\alpha_i\alpha_j\Phi(0), \quad \alpha_i = \frac{1}{\gamma_i}.$$

Substituting the above formulae in (2.9), after straightforward transformations we get

$$\Delta(0) = \Phi^2(0)\Big(2n\sum_{i<j}\alpha_i\alpha_j - (n-1)\Big(\sum_{i=1}^{n}\alpha_i\Big)^2\Big) = -\Phi^2(0)\varphi_n(\alpha_1,\dots,\alpha_n), \tag{2.10}$$

where

$$\varphi_n(\alpha_1,\dots,\alpha_n) = n\sum_{i=1}^{n}\alpha_i^2 - \Big(\sum_{i=1}^{n}\alpha_i\Big)^2.$$

Let us prove that for any $n \geq 2$ the form $\varphi_n(\alpha_1,\dots,\alpha_n) \geq 0$ and $\varphi_n(\alpha_1,\dots,\alpha_n) = 0$ if and only if $\alpha_1 = \dots = \alpha_n$. Indeed, consider the vectors $\alpha = (\alpha_1,\dots,\alpha_n)$ and $\beta = (1,\dots,1)$ in n-dimensional Euclidean space with the standard inner product. Then the Cauchy–Schwarz inequality $(\alpha,\beta)^2 \leq (\alpha,\alpha)(\beta,\beta)$ gives the required assertion.

By our assumption, $\alpha_i \neq \alpha_j$ for at least one pair i, j. Then $\varphi_n(\alpha_1,\dots,\alpha_n) > 0$, and equality (2.10) implies that $\Delta(0) = 0 \Leftrightarrow \Phi(0) = 0$. Moreover, from (2.9) it follows that

$$\Delta(p_*) = \Phi(p_*) = 0 \;\Rightarrow\; \Phi'(p_*) = 0,$$

i.e., $p_* = 0$ is a multiple root of Φ. The lemma is proved. □

Remark 2. Obviously, the implication $\Phi = \Phi' = 0 \Rightarrow \Delta = 0$ holds true for all polynomial Φ, not necessarily (2.8). However, the inverse implication $\Delta = 0 \Rightarrow \Phi = 0$ is not valid if Φ has a complex root. The reason for this is easily ascertained: the inequality $\varphi_n(\alpha_1,\dots,\alpha_n) \geq 0$ is not valid if among the numbers α_i some are complex.

For example, consider the polynomial $\Phi = p^3 + p$ with the unique real root $p = 0$. Then the corresponding polynomial $\Delta = 2(3p^2 - 1)$ has two real roots, none of those coincides with 0. Moreover, the polynomial $\Phi = p^4 + 6p^2 + 1$ does not have real roots at all, while the corresponding polynomial $\Delta = 48(p^2 - 1)^2$ has two double roots $p = \pm 1$.

Lemma 4 gives a simple geometrical description of the singular locus of equation (2.3) for the domain $M' \subset M$, where the pseudo-Finsler space $(M, \overline{F})$ has m isotropic lines passing through every point of M', i.e., the polynomial $F(p)$ has m real roots (taking into account the multiplicity and possibly including $p = \infty$). For $(x, y) \in M'$, the function $\Delta(x, y; p)$ vanishes if and only if at least two of m isotropic lines are tangent at (x, y) and p is the corresponding tangential direction. We remark that this statement is not valid for the complement of M', where the polynomial $F(p)$ has complex roots.

This question will be considered in detail for $n = 3$.

2.1. **Pseudo-Riemannian metrics.** By Remark 1, in the case $n = 2$ (pseudo-Riemannian metrics) Δ is a zero degree polynomial in p, that is, Δ does not depend on p. Moreover, it is easy to check that $\Delta = -4D_{[F]}$, where $D_{[F]}$ means the discriminant of the quadratic polynomial F. Hence the locus of singularities of equation (2.3) coincides with the discriminant curve of the implicit differential equation $F(x, y; p) = 0$. It is not hard to see that the equation $\Delta(x, y) = 0$ defines an invariant surface of the field (2.7) filled with integral curves whose projections $PTM \to M$ are points forming the discriminant curve.

This property leads to a curious phenomenon: geodesics cannot pass through a point (x, y) of the discriminant curve in arbitrary tangential directions, but only in admissible directions p defined by the condition $P(x, y; p) = 0$. Generically, $P(x, y; p)$ is a cubic polynomial in p and the number of admissible directions is 1 or 3. Singularities of the geodesic flows in pseudo-Riemannian metrics are studied in detail (the interested reader is referred to the papers [8, 19, 20] devoted to 2-dimensional pseudo-Riemannian metrics; similar results for 3-dimensional pseudo-Riemannian metrics were announced in [16]).

It should be remarked that the case $n = 2$ is exceptional from the viewpoint of Finsler and pseudo-Finsler geometry ($n > 2$). In the case $n > 2$, Δ generically depends on p and the notion of admissible directions does not appear. Geodesics pass through every point of M in all possible directions, but some directions at some points are singular. In other words, only points of the space PTM may have the property of being singular.

In the rest of the paper, we consider the case $n = 3$ (cubic pseudo-Finsler metrics) in detail.

2.2. **Cubic pseudo-Finsler metrics.** Let $D_{[F]}$ and $D_{[\Delta]}$ be the discriminants of the cubic polynomial $F(x, y; p)$ and the quadratic polynomial $\Delta(x, y; p)$ in p, respectively. A direct calculation shows that $D_{[\Delta]} = -12D_{[F]}$.

Introduce the following stratification of the manifold M. The open domains M_+, M_- are defined by the conditions $D_{[F]} > 0$, $D_{[F]} < 0$, respectively. Generically, M_+, M_- are separated by the discriminant curve $M_0 \colon D_{[F]} = 0$, which consists of regular points (the cubic polynomial F has one prime root and one double root) and cusps (F has a triple root). By $M_{0,1}$ denote the set of all regular points of M_0, while $M_{0,0} = M_0 \setminus M_{0,1}$. The discriminant of the quadratic polynomial Δ is strictly negative in M_+; hence singularities of equation (2.3) occur only in M_- and M_0. Further we exclude from consideration the stratum $M_{0,0}$ of dimension zero, and consider only M_- and $M_{0,1}$.

In a neighborhood of every point of $M \setminus M_{0,0}$, the cubic polynomial F has at least one prime real root $p_*(x, y)$ smoothly depending on x, y. To simplify calculations, choose local coordinates such that the integral curves of the vector field $\frac{dy}{dx} = p_*(x, y)$ (one of three families of isotropic lines) become $x = \text{const}$. This yields $a_3(x, y) \equiv 0$ and

$$
\begin{aligned}
&F = ap^2 + 2bp + c, \quad \Delta = -2(ap + b)^2 + 6(ac - b^2), \quad D_{[F]} = 4a^2(b^2 - ac),\\
&M_\pm = \{\pm(b^2 - ac) > 0,\ a \neq 0\}, \quad M_{0,1} = \{b^2 - ac = 0,\ a \neq 0\},\\
&P = 3F(F_y - F_{xp} - pF_{yp}) + 2F_p(F_x + pF_y).
\end{aligned} \tag{2.11}
$$

2.2.1. *Singularities in the stratum M_-.* At every point in M_-, the quadratic equation $\Delta = 0$ has two prime real roots

$$
p_{1,2} = \frac{\pm\delta - b}{a}, \quad \delta = \sqrt{3(ac - b^2)}, \tag{2.12}
$$

and the domain M_- is filled with two transverse families of integral curves of the binary implicit differential equation $\Delta = 0$, which we shall call *singular lines* of the metric.

Consider the curves $S_i \subset M_-$ defined by the equations $P(x, y; p_i) = 0$, $i = 1, 2$, where P is defined in (2.11). They can be also considered as the branches of the locus $\operatorname{res}(\Delta, P) = 0$, where "res" means the resultant of two polynomials in p. In the space PTM, consider the corresponding curves

$$
\overline{S}_i = \{(x, y; p_i) : (x, y) \in S_i\}, \quad i = 1, 2,
$$

which consist of singular points of the field (2.7).

Let Γ_q denote the family of geodesics outgoing from a point $q = (x, y)$. The simplest type of singularities of the geodesic flow (codimension 0) is given in the following theorem.

Theorem 1. *Let $q \in M_-$ and $(q; p_i) \notin \overline{S}_i$, $i = 1, 2$. Then there exists a unique geodesic passing through the point q with tangential direction p_i: a semicubic parabola with the cusp at q. In particular, if $q \in M_- \setminus (S_1 \cup S_2)$, the family Γ_q contains two semicubic parabolas with tangential directions p_i, while geodesics with all remaining directions at q are smooth.*

Proof. If $P(q; p_i) \neq 0$, then, by the standard existence and uniqueness theorem, the field (2.7) has a unique integral curve γ_i passing through the point $(q; p_i)$. From the conditions $a \neq 0$, $ac - b^2 \neq 0$, it follows that Δ and Δ_p do not vanish simultaneously; see (2.11).

Hence the curve γ_i has first order tangency with the *vertical* direction (the vertical direction in the space PTM is called the p-direction, i.e., the kernel of the natural projection $PTM \to M$) at the point $(q; p_i)$, and the projection of the curve γ_i to M is a semicubic parabola with the cusp at q. $\square$

Example 1. Let F be given by formula (2.11) with $a = 1$ and $b = 0$. Then $\Delta = 2(3c - p^2)$ and $P = 7c_y p^2 + 4c_x p + 3cc_y$. The strata M_- and $M_{0,1}$ are defined by the conditions $c(x,y) > 0$ and $c(x,y) = 0$, respectively, and the curves S_i are $c_x = \pm 2c_y\sqrt{3c}$.

I. Put $c(x,y) = -x$ (Fig. 1, left). Then $S_1 = S_2 = \emptyset$ and the semiplane $x > 0$ (M_+) is filled with the net of isotropic lines $y = \pm\frac{2}{3}x^{\frac{3}{2}} + \text{const}$ (dashed curves), while the semiplane $x < 0$ (M_-) is filled with the net of singular lines $y = \pm\frac{2}{\sqrt{3}}(-x)^{\frac{3}{2}} + \text{const}$ (dotted curves). Cusps appear when geodesics (solid curves) are tangent to singular lines. We remark that geodesics pass from M_- to M_+ or vise versa through $M_{0,1}$ (the y-axis) without singularity if they intersect the y-axis with any non-isotropic tangential direction $p \neq 0$. Otherwise, equation (2.3) has a singularity. As we shall see in Section 2.2.2, at such points there exist a one-parameter family of geodesics outgoing in both domains M_+ and M_- with the common tangential direction $p = 0$, and the prolongation of geodesics through $M_{0,1}$ is not naturally defined.

II. Put $c(x,y) = \alpha y^2 - x$ with $\alpha \neq 0$. Then $S_i \neq \emptyset$, $i = 1, 2$, but both curves S_i do not pass through the origin. In a neighborhood of the origin that does not contain the curves S_i, geodesics are presented in Fig. 1 (right). Here, for definiteness, we assume $\alpha > 0$. All notations have the same meanings as before.

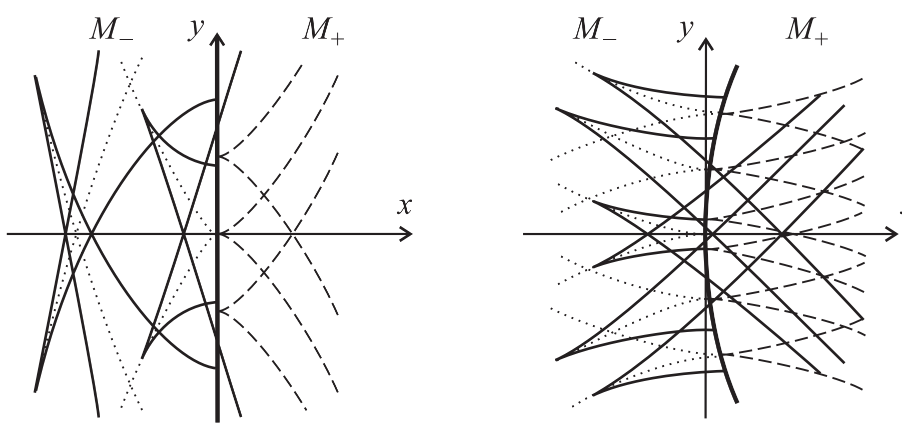

FIGURE 1. Example 1: I and II (left and right, resp.). The stratum $M_{0,1}$ (the y-axis on the left and the parabola on the right) is depicted with the bold solid line. Solid and dotted curves present geodesics and singular lines, respectively. Dashed curves present isotropic lines.

The next type of singularities of the geodesic flow in the domain M_- (codimension 1) is connected with vanishing of the field (2.7). This field belongs to a special class of vector fields whose singular points are not isolated, but form a manifold W of codimension 2 in the phase space. Yet such fields appear in many problems, see, e.g., [3, 8, 12, 17, 18, 22, 24]. It is convenient to expressed the above condition in the following algebraic form: the germs of all components of the field at every singular point belong to the ideal I (in the ring of smooth germs) generated by two of them.

The spectrum of the linear part of such a field (for brevity, we shall call it the spectrum of the field) contains only two non-zero eigenvalues $\lambda_{1,2}$, which play a prominent role in establishing the local normal form of the field. For instance, all components of the field (2.7) belong to the ideal $I = \langle \Delta, P \rangle$, and the set of singular points $W = \overline{S}_1 \cup \overline{S}_2$. The eigenvalues $\lambda_{1,2}$ and the corresponding eigenvectors are described by the following lemma.

Lemma 5.

1. *The resonance $\lambda_1 + \lambda_2 = 0$ holds at all points $(q; p) \in \overline{S}_i$, i.e., $\lambda_{1,2}$ are real or pure imaginary numbers with opposite signs.*
2. *The following conditions are equivalent:*

2.1. The eigenvalues λ_1, λ_2 at $(q; p) \in \overline{S}_i$ are not equal to zero.

2.2. $\overline{S}_i$ is a regular curve transversal to the contact plane $pdx - dy = 0$ at $(q; p)$.

2.3. S_i is a regular curve and the direction p_i is transversal to S_i at q.

3. *Generically, at almost all points $(q; p) \in \overline{S}_i$, conditions 2.1 – 2.3 hold.*

Proof. Without loss of generality, suppose that $q = 0$ (the origin) belongs to S_1 and choose local coordinates centered at 0 that preserve the lines $x = \text{const}$ and send integral curves of the vector field $\frac{dy}{dx} = p_1(x, y)$ to parallel lines $y = \text{const}$. The existence of such local coordinates follows from the general fact: if V_1 and V_2 are smooth vector fields on the plane transversal at the point 0, then in a neighborhood of 0 there exist local coordinates such that integral curves of V_1 and V_2 coincide with the coordinate lines.

Then the polynomials F, Δ, $D_{[F]}$ have the form (2.11) and the identities $p_1 \equiv 0$ and $b(x, y) \equiv \delta(x, y)$ hold. Note that the first of them implies $3ac \equiv 4b^2$. From $b^2 - ac < 0$, it follows that $ac > 0$. Taking into account $3ac \equiv 4b^2$, we conclude that none of the coefficients a, b, c vanishes at 0. Below, we present the proof for the stratum S_1 given by the equation $3c(c_y - 2b_x) + 4bc_x = 0$. The proof for the stratum S_2 is similar.

Let Λ be the matrix of the linear part of the field (2.7) and Λ_1 be the matrix of the Pfaffian system $d\Delta = 0$, $dP = 0$, $pdx - dy = 0$ considered at arbitrary point $(q; p) \in \overline{S}_1$, that is, for $q \in S_1$ and $p = 0$:

$$\Lambda = \begin{pmatrix} \Delta_x & \Delta_y & \Delta_p \\ 0 & 0 & 0 \\ P_x & P_y & P_p \end{pmatrix}\Bigg|_{p=0}, \qquad \Lambda_1 = \begin{pmatrix} \Delta_x & \Delta_y & \Delta_p \\ P_x & P_y & P_p \\ 0 & -1 & 0 \end{pmatrix}\Bigg|_{p=0},$$

where

$$\begin{aligned} \Delta_x\big|_{p=0} &= 6(ac_x + a_x c) - 16bb_x, \quad \Delta_p\big|_{p=0} = -4ab, \\ P_x\big|_{p=0} &= c_x(3c_y - 2b_x) + 4bc_{xx} + 3c(c_{xy} - 2b_{xx}), \\ P_p\big|_{p=0} &= 4ac_x - 6a_x c + 2b(5c_y - 2b_x). \end{aligned} \tag{2.13}$$

1. To prove the first statement, it suffices to show that $\operatorname{tr}\Lambda = 0$. Taking into account the equality $3c(c_y - 2b_x) + 4bc_x = 0$ on S_1 and the identity $b \equiv \delta$ (that implies $3ac \equiv 4b^2$ in a neighborhood of the origin), from (2.13), we have

$$\operatorname{tr}\Lambda = (\Delta_x + P_p)\big|_{p=0} = 10[ac_x + b(c_y - 2b_x)] = 10\Big(ac_x - b\frac{4bc_x}{3c}\Big) = \frac{10c_x}{3c}(3ac - 4b^2) = 0.$$

Hence the characteristic equation of the matrix Λ reads $\lambda(\lambda^2 + |\Lambda_1|) = 0$; this yields the equation $\lambda^2 + |\Lambda_1| = 0$ for $\lambda_{1,2}$.

2. Differentiating the identity $4b^2 \equiv 3ac$ by x, we get $8bb_x \equiv 3(a_xc + ac_x)$. Using these identities and (2.13), we have

$$\Delta_x\big|_{p=0} = 6(ac_x + a_xc) - 16bb_x = 6(ac_x + a_xc) - 6(ac_x + a_xc) = 0,$$

$$|\Lambda_1| = -\Delta_p P_x\big|_{p=0} = 16ab^2c_{xx} - 8abb_xc_x + 12ab[c_xc_y + cc_{xy} - 2b_{xx}c] =$$
$$12a^2cc_{xx} - 3ac_x(a_xc + ac_x) + 12ab[c_xc_y + cc_{xy} - 2b_{xx}c].$$

Thus the condition $\lambda_{1,2} \neq 0$ is equivalent to $|\Lambda_1| \neq 0$ that, in turn, is equivalent to condition 2.2.

On the other hand, the curve S_1 is tangent to the direction $p = 0$ at the point $q = 0$ if and only if $[3c(c_y - 2b_x) + 4bc_x]'_x = 0$. Taking into account the equalities $4b^2 \equiv 3ac$ and $8bb_x \equiv 3(a_xc + ac_x)$, we get

$$[3c(c_y - 2b_x) + 4bc_x]'_x = 3(c_xc_y + cc_{xy}) - 6b_{xx}c + \frac{4b^2c_{xx} - 2bb_xc_x}{b} =$$
$$3(c_xc_y + cc_{xy}) - 6b_{xx}c + \frac{12acc_{xx} - 3(a_xc + ac_x)c_x}{4b} = \frac{|\Lambda_1|}{4ab}.$$

This proves that $\lambda_{1,2} \neq 0$ is equivalent to condition 2.3.

3. Generically, at almost all points $(q;p) \in \overline{S}_1$, the determinants

$$\begin{vmatrix} \Delta_x & \Delta_y \\ P_x & P_y \end{vmatrix}, \qquad |\Lambda_1| = \begin{vmatrix} \Delta_x & \Delta_p \\ P_x & P_p \end{vmatrix}$$

are not equal to zero. Hence S_1 and $\overline{S}_1$ are regular curves and, moreover, conditions 2.1 − 2.3 hold. □

Theorem 2. *Let $(q;p_i) \in \overline{S}_i$ be a generic singular point of the field (2.7). Then the germ (2.7) at $(q;p_i)$ is smoothly orbitally equivalent to*

$$\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta} + \xi\eta\frac{\partial}{\partial\zeta}, \quad \text{if} \quad \lambda_{1,2} \in \mathbb{R} \setminus 0, \tag{2.14}$$

$$\eta\frac{\partial}{\partial\xi} - \xi\frac{\partial}{\partial\eta} + (\xi^2 + \eta^2)\frac{\partial}{\partial\zeta}, \quad \text{if} \quad \lambda_{1,2} \in \mathbb{I} \setminus 0, \tag{2.15}$$

where $\mathbb{R}, \mathbb{I}$ are real and imaginary axes, respectively.

In the first case, there exist two geodesics passing through the point $q \in S_i$ with the tangential direction p_i, both of them smooth. In the second case, there are no geodesics passing through the point $q \in S_i$ with the tangential direction p_i.

Proof. Since $(q;p_i) \in \overline{S}_i$ is a generic singular point, $|\Lambda_1| \neq 0$. By Lemma 5, the eigenvalues $\lambda_{1,2} \neq 0$ and $\overline{S}_i$ is a regular curve consisting of singular points of the field (2.7). The linear part of the germ (2.7) at $(q;p_i)$ (and every singular point sufficiently close to $(q;p_i)$) is orbitally equivalent to the linear part of the field (2.14) or (2.15) if $|\Lambda_1| < 0$ or $|\Lambda_1| > 0$, respectively. Here we use the following terminology: two vector fields are called orbitally smoothly (resp., topologically) equivalent, if there exists a diffeomorphism (resp., homeomorphism) that conjugates their integral curves, i.e., orbits of their phase flows.[2]

Recall that the field (2.7) belongs to the class of vector fields whose singular points are not isolated, but form a manifold W of codimension 2 in the phase space (in our case, $W = \overline{S}_i$). Local normal forms of such fields were studied by many authors. In [21] (Appendix A), we present a brief survey of such results, which covers all cases with $\operatorname{Re}\lambda_{1,2} \neq 0$. This condition

[2] It slightly differs from the generally accepted definition of the orbital equivalence, where coincidence of the orientation of integral curves is also required. Our definition is naturally related to directions fields, whose integral curves do not have an orientation a priori.

is equivalent to the assumption that $W = \overline{S}_i$ is the local center manifold, and consequently, the phase portrait of (2.7) has a simple topological structure (we shall discuss it later on, in the proof of Theorem 4). For instance, in the case $\lambda_{1,2} \in \mathbb{R} \setminus 0$, the germ (2.7) with generic quadratic part is smoothly orbitally equivalent to (2.14). This result belongs to Roussarie [22]. The genericity condition is determined explicitly in [21] (Theorem 5.7). The case $\lambda_{1,2} \in \mathbb{I} \setminus 0$ is more complicated. However, in [3] (Chapter 2, Section 1.2) and [12] it is claimed that in this case the germ (2.7) with generic quadratic part is smoothly orbitally equivalent to (2.15).

We remark that the diffeomorphism that brings the germ (2.7) to the normal form (2.14) or (2.15) does not give a normal form of equation (2.3), since it does not preserve the contact structure $dy = pdx$. However, we need not a normal form of (2.3).

To prove the last statement of the theorem, we only need to consider the possible mutual relationships between the phase portrait of the germ (2.7) and the (x, y)-plane in the space PTM. Geodesics are obtained from those integral curves of the field (2.7) whose projection on the (x, y)-plane are distinguished from points. Moreover, isotropic geodesics correspond to those integral curves that belong to the isotropic hypersurface $\mathscr{F}$ (by Lemma 2, $\mathscr{F}$ is an invariant hypersurface of the field (2.7)).

We consider the real and imaginary cases separately.

The real case. The field (2.14) has the first integral $\xi\eta$. The invariant foliation $\xi\eta = \text{const}$ contains only two leaves $\xi = 0$ and $\eta = 0$ that pass through singular points of the field, while all remaining invariant leaves are hyperbolic cylinders $\xi\eta = \text{const} \neq 0$, which do not intersect the set of singular points. It is easy to see that, for every singular point of the field (2.14), there are only two integral curves passing through this point: the straight lines parallel to the ξ-axis and the η-axis, respectively.

We prove now that the eigenvectors with the eigenvalues $\lambda_{1,2} \neq 0$ are not vertical. Let e be an eigenvector of the matrix Λ with λ_i. Then $\Lambda e = \lambda_i e$, and $e = \alpha\partial_x + \beta\partial_p$, where $(\Delta_x\big|_{p=0} - \lambda_i)\alpha + \Delta_p\big|_{p=0}\beta = 0$. If the eigenvector e is vertical, i.e., $\alpha = 0$, $\beta \neq 0$, this equality yields $\Delta_p\big|_{p=0} = 0$. From (2.13), we have $a(0) = 0$ or $b(0) = 0$. This contradicts the fact (established in the proof of Lemma 5) that none of the coefficients a, b, c vanishes at 0.

From the considerations above, it follows that the field (2.7) has only two integral curves passing through the given point $(q; p_i)$, both of them smooth and having non-vertical tangential directions. Projecting these integral curves from PTM to M, we get two smooth geodesics passing through the point $q \in S_i$ with the tangential direction p_i; see Fig. 2 (left).

The imaginary case. The field (2.15) has the first integral $\xi^2 + \eta^2$. The invariant foliation $\xi^2 + \eta^2 = \text{const}$ contains a one-dimensional degenerate leaf $\xi = \eta = 0$, which consists of singular points of the field (2.15) and a one-parameter family of two-dimensional leaves (elliptic cylinders $\xi^2 + \eta^2 = \text{const} \neq 0$), which do not intersect the set of singular points. The elliptic cylinders are filled with helix-like integral curves, whose projections to M have cusps; see Fig. 2 (right).

To complete the proof, observe that, in both the real and imaginary cases, the curve S_i itself is not a geodesic, since $\overline{S}_i$ is transversal to the contact plane $pdx - dy = 0$ (statement 2.2 in Lemma 5). Consequently, $\overline{S}_i$ is not a lift of a curve on M. □

Example 2. Let F be given by formula (2.11) with $a = 1$, $b = 0$, $c(x, y) = \alpha y^2 - x$. Then $\Delta = 2(3c - p^2)$, $P = 12\alpha y p^2 - 4p + 6\alpha y c$, and the curves S_i are the connected components of the graph $x = \alpha y^2 - \frac{1}{48}/\alpha^2 y^2$ lying in the upper and lower semiplanes. A straightforward calculation shows that $p_i\big|_{S_i} = 12\alpha y(\alpha y^2 - x)$; hence the direction p_i is tangent to the curve S_i at $x = \frac{47}{48}/\sqrt{\alpha}$ only. By statement 2.3 in Lemma 5, the eigenvalues $\lambda_{1,2} \neq 0$ at all points of $\overline{S}_i$ if $\alpha < 0$ and at all points of $\overline{S}_i$ with $x \neq \frac{47}{48}/\sqrt{\alpha}$ if $\alpha \geq 0$.

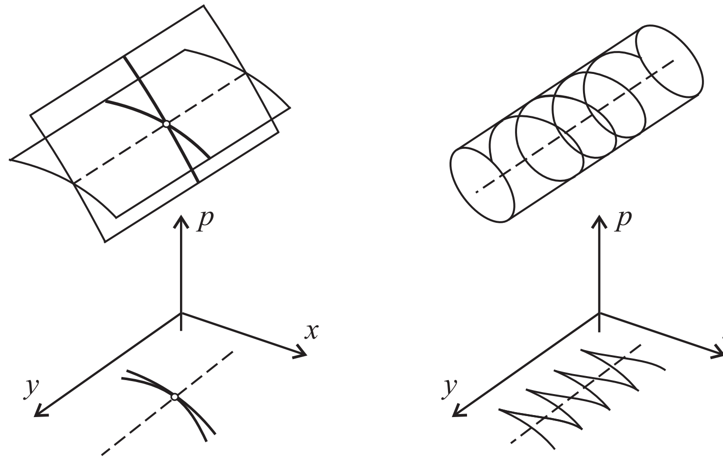

FIGURE 2. The phase portraits of the field (2.7) with the normal form (2.14) or (2.15) (left and right, resp.) and the projections of its integral curves to M. Dashed curves present $\overline{S}_i$ (up) and S_i (down).

In Fig. 3 (left and center), we present geodesics in the case $\alpha > 0$. Here both real and imaginary eigenvalues exist. The parts of S_i with real (imaginary) eigenvalues $\lambda_{1,2} \neq 0$ are presented as short-dashed (resp., long-dashed) lines. The dots represent the points of S_i with $x = \frac{47}{48}/\sqrt{\alpha}$, where $\lambda_{1,2} = 0$. In Fig. 3 (right), we present geodesics in the case $\alpha < 0$. Here only real eigenvalues exist, and the curves S_i are presented as short-dashed lines. The dots represent the points where geodesics intersect the curves S_i with the singular tangential direction p_i.

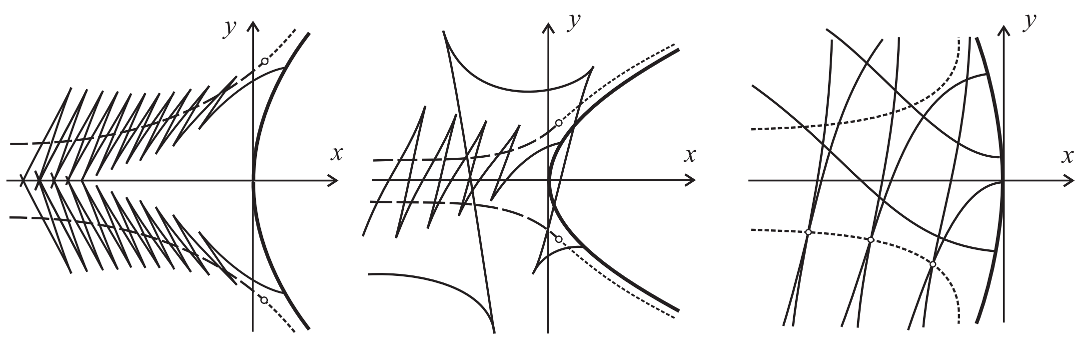

FIGURE 3. Example 2: $F = p^2 + c$, where $c = \alpha y^2 - x$ with $\alpha > 0$ (left, center) and $\alpha < 0$ (right). The solid lines present geodesics. $M_{0,1}$ is depicted with a bold solid line, S_i are depicted with short-dashed (long-dashed) lines if $\lambda_{1,2} \in \mathbb{R} \setminus 0$ ($\lambda_{1,2} \in \mathbb{I} \setminus 0$, respectively).

2.2.2. *Singularities in the stratum* $M_{0,1}$. In this section as before, we proceed in the local coordinates where F, Δ and $D_{[F]}$ have the form (2.11). At every point $q \in M_{0,1}$, the polynomial F has the double root $p_0 = -b/a$. It is easy to see that p_0 is also a double root of the polynomial Δ at q (this follows from Lemma 4 also). Thus $(q; p_0)$, $q \in M_{0,1}$, are singular points of both implicit differential equations $F = 0$ and $\Delta = 0$. From (2.4), it follows that $P(q; p_0) = 0$; hence $(q; p_0)$, $q \in M_{0,1}$, are singular points of the field (2.7).

Furthermore, we restrict ourselves to generic points $q \in M_{0,1}$ where $M_{0,1}$ is a regular curve and the isotropic direction p_0 is transverse to $M_{0,1}$. Then both implicit differential equations $F = 0$ and $\Delta = 0$ have Cibrario normal forms at such a point and their integral curves are semicubic parabolas lying on opposite sides of $M_{0,1}$ (in the domains M_+ and M_-, resp.), as is presented in Fig. 1.

Theorem 3. *Suppose that the isotropic direction p_0 is transverse to $M_{0,1}$ at $q \in M_{0,1}$. Then the germ (2.7) at its singular point $(q; p_0)$ is smoothly orbitally equivalent to*

$$3\xi\frac{\partial}{\partial\xi} + 2\eta\frac{\partial}{\partial\eta} + 0\frac{\partial}{\partial\zeta}, \tag{2.16}$$

and to p_0 corresponds a one-parameter family of geodesics outgoing from q into M_+ and M_-. There exist smooth local coordinates centered at q such that this family is

$$x = \alpha|\eta|^{\frac{3}{2}} + \eta^2 + \alpha\bar{X}_\alpha(\eta), \quad y = \alpha\eta|\eta|^{\frac{3}{2}} + \varepsilon\eta^3 + \alpha\bar{Y}_\alpha(\eta), \quad \varepsilon \neq 0, \tag{2.17}$$

where $\bar{X}_\alpha(\eta) = o\big(|\eta|^{\frac{3}{2}}\big)$ and $\bar{Y}_\alpha(\eta) = o\big(|\eta|^{\frac{5}{2}}\big)$ are C^2-smooth functions.

Here $\alpha > 0$ ($\alpha < 0$) corresponds to non-isotropic geodesics outgoing from q in M_+ (resp., M_-), while $\alpha = 0$ gives the isotropic geodesic, a semicubic parabola lying in M_+. The limit case $\alpha \to \infty$ corresponds to a unique smooth geodesic passing through q with the direction p_0. In a neighborhood of q, every non-isotropic geodesics outgoing from q in M_+ belongs to the curvilinear tongue-like sector bounded by the branches of the isotropic geodesic as it is presented in Figure 4 (left).

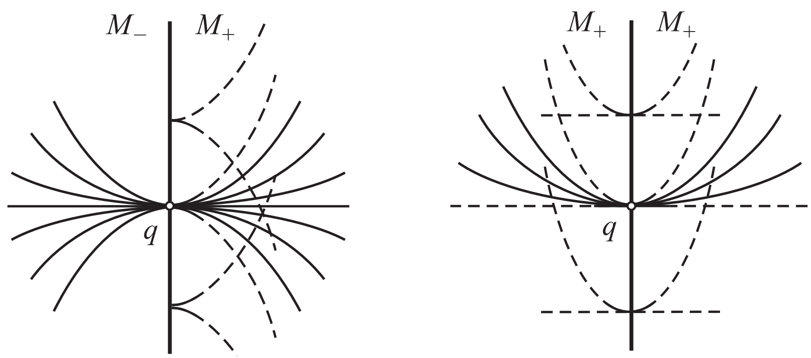

FIGURE 4. Illustrations of Theorem 3 (left) and Theorem 4 (right). The stratum $M_{0,1}$ is depicted with the bold solid line. Solid and dashed curves present non-isotropic and isotropic geodesics, respectively.

Proof. Without loss of generality, suppose that $q = 0$ (the origin of the (x, y)-plane) and choose local coordinates centered at 0 that preserve the lines $x = \text{const}$ and give $b(x, y) \equiv 0$. This can be done using an appropriate change of variables $y \mapsto yu(x, y)$, $u(0) \neq 0$. We remark that (unlike Lemma 5) we do not have the identity $p_1 \equiv 0$ nor $p_2 \equiv 0$ in a neighborhood of 0. Moreover, it is impossible to get any these identities using smooth changes of variables, since the integral curves of the implicit equation $\Delta = 0$ with roots p_1, p_2 have cusps on $M_{0,1}$.

In the local coordinates chosen above, we have

$$F = ap^2 + c, \quad \Delta = -2(ap)^2 + 6ac, \quad D_{[F]} = -4a^3c, \tag{2.18}$$
$$P = aa_yp^4 - 2aa_xp^3 + (7ac_y - 3a_yc)p^2 + (4ac_x - 6a_xc)p + 3cc_y.$$

The curve $M_{0,1}$ is given by $c(x, y) = 0$ and the direction $p_0 = 0$ at every $q \in M_{0,1}$. Hence the condition "the direction p_0 is transverse to $M_{0,1}$ at 0" is equivalent to $c_x(0) \neq 0$.

Substituting Δ and P from (2.18) into (2.7), one can find that the spectrum of the field (2.7) at every point $(q; 0)$, $q \in M_{0,1}$, is $(\lambda_1, \lambda_2, 0)$, where $\lambda_1 = 6ac_x$, $\lambda_2 = 4ac_x$. A straightforward computation shows that the corresponding eigenvectors are

$$e_1 = 2a\partial_x + 3c_y\partial_p, \quad e_2 = \partial_p, \quad e_0 = c_y\partial_x - c_x\partial_y. \tag{2.19}$$

Note that $\lambda_1 : \lambda_2 \equiv 3 : 2$ at all points $(q; 0)$, $q \in M_{0,1}$, the pair (λ_1, λ_2) is non-resonant and belongs to the Poincaré domain. Therefore, the germ (2.7) at 0 is smoothly orbitally equivalent to the linear field (2.16) (Theorem 5.5 in [21]). Moreover, comparing (2.7) and (2.16), one can see that the conjugating diffeomorphism $(x, y, p) \mapsto (\xi, \eta, \zeta)$ can be chosen in the form

$$x = 2a\xi + c_y\zeta + f_1(\xi, \eta, \zeta), \quad p = 3c_y\xi + \eta + f_2(\xi, \eta, \zeta), \quad y = -c_x\zeta + f_3(\xi, \eta, \zeta), \tag{2.20}$$

where a, c_x, c_y are evaluated at 0 and $f_i \in \mathfrak{M}^1$ ($\mathfrak{M}^k$, $k \geq 0$, is the ideal of k-flat functions in the ring of smooth functions).

The field (2.16) has the invariant foliation $\zeta = \text{const}$. The set of integral curves of (2.16) passing through the origin consists of the ξ-axis and the one-parameter family

$$\{\xi = \alpha|\eta|^{\frac{3}{2}}, \ \zeta = 0\}, \quad \alpha \in \mathbb{R}, \tag{2.21}$$

tending to the ξ-axis as $\alpha \to \infty$. Consider the possible mutual relationships between the phase portrait of the germ (2.7) at 0 and the (x, y)-plane in the space PTM using the eigenvectors (2.19). The integral curve of the field (2.7) corresponding to the ξ-axis in (2.16) has a non-vertical tangential direction at 0 (the eigenvector e_1); hence its projection to the (x, y)-plane is a smooth geodesic. On the contrary, the family (2.21) gives a family of integral curves of (2.7) with vertical tangential direction at 0 (the eigenvector e_2). The projections of these curves to the (x, y)-plane have a singularity at 0.

To establish the character of the singularity, substitute (2.21) in (2.20). This yields

$$x = 2a\alpha|\eta|^{\frac{3}{2}} + \bar{f}_{1,\alpha}(\eta), \quad p = \eta + \bar{f}_{2,\alpha}(\eta),$$

where $\bar{f}_{i,\alpha}(\eta) = f_i(\alpha|\eta|^{\frac{3}{2}}, \eta, 0)$. Observe that the functions $\bar{f}_{1,\alpha} = o\big(|\eta|^{\frac{3}{2}}\big)$ and $\bar{f}_{2,\alpha} = o(\eta)$ are C^2 and C^1, resp. Denote the sign of η by $s(\eta)$, then we have the equation

$$dy = pdx = \big(\eta + \bar{f}_{2,\alpha}(\eta)\big)\big(3a\alpha|\eta|^{\frac{1}{2}}s(\eta) + \bar{f}'_{1,\alpha}(\eta)\big)\, d\eta = \big(3a\alpha|\eta|^{\frac{3}{2}} + g_\alpha(\eta)\big)\, d\eta,$$

where $g_\alpha = o\big(|\eta|^{\frac{3}{2}}\big)$ is C^1-smooth. Integrating, we get

$$y = \tfrac{6}{5}a\alpha|\eta|^{\frac{5}{2}}s(\eta) + h_\alpha(\eta) = \tfrac{6}{5}a\alpha\eta|\eta|^{\frac{3}{2}} + h_\alpha(\eta),$$

where $h_\alpha = o\big(|\eta|^{\frac{5}{2}}\big)$ is C^2-smooth. The scaling $y \mapsto \frac{5}{3}y$, $\alpha \mapsto \pm 2a\alpha$ yields

$$x = \alpha|\eta|^{\frac{3}{2}} + X_\alpha(\eta), \quad y = \alpha\eta|\eta|^{\frac{3}{2}} + Y_\alpha(\eta), \tag{2.22}$$

where $X_\alpha = o\big(|\eta|^{\frac{3}{2}}\big)$ and $Y_\alpha = o\big(|\eta|^{\frac{5}{2}}\big)$ are C^2-smooth, and $\alpha > 0$ ($\alpha < 0$) corresponds to the domain M_+ (M_-, resp.). The asymptotic formula (2.22) makes sense for all real $\alpha \neq 0$.

In order to take care of the omitted case $\alpha = 0$, recall that the isotropic surface $\mathscr{F}$ is an invariant surface of the field (2.7) (Lemma 2) and contains its singular points (Lemma 4). Hence, in the normal coordinates (ξ, η, ζ), the surface $\mathscr{F}$ contains the ζ-axis and intersects every invariant leaf $\zeta = \text{const}$ in a certain integral curve of (2.16). For instance, $\mathscr{F}$ intersects the leaf $\zeta = 0$ in an integral curve of the family (2.21), which corresponds to an isotropic geodesic passing through 0.

On the other hand, we know that the implicit differential equation $F = ap^2 + c = 0$, which described the isotropic lines in $(M, \overline{F})$, has a Cibrario normal form at 0. Hence there exists a unique isotropic geodesic passing through 0, the semicubic parabola

$$x = \eta^2, \quad y = \eta^3 N(\eta), \quad N(0) \neq 0, \tag{2.23}$$

lying in the domain M_+.

From the uniqueness of the isotropic geodesic passing through 0, it follows that the lift of (2.23) is the curve of the family (2.22) with $\alpha = 0$ and $Y_0(\eta) = \eta^3 N(\eta)$. Using the representation $N(\eta) = N_1(\eta^2) + \eta N_2(\eta^2)$ and the change of variables $y \mapsto N_1(0)(y - x^2 N_2(x))/N_1(x)$, we get $N(\eta) \equiv N(0)$. It is not hard to check that the number $\varepsilon = N(0)$ is a mutual invariant of the curves (2.22) and (2.23). $\square$

In Example 1, we considered $F = p^2 + c$ with $c = -x$ and $c = \alpha y^2 - x$. In both cases the isotropic direction $p_0 = 0$ is transverse to the curve $M_{0,1}$ given by the equation $x = 0$ and $x = \alpha y^2$, respectively, and the conditions of Theorem 3 hold true.

Example 3. We consider here the case $F = p^2 - x$ in more detail. The field (2.7) is

$$-2(3x + p^2)\Big(\frac{\partial}{\partial x} + p\frac{\partial}{\partial y}\Big) - 4p\frac{\partial}{\partial p}. \tag{2.24}$$

It is easy to check that the isotropic surface $\mathscr{F}$ given by $p^2 = x$ is an invariant surface of the field (2.24) and the unique isotropic line passing through 0 is given by $x = p^2$, $y = \frac{2}{3}p^3$. Integrating the equation $dp/dx = 2p/(3x + p^2)$, we get the family $x = \alpha|p|^{\frac{3}{2}} + p^2$, where α is the constant of integration, and a single integral curve $p = 0$, which gives the smooth non-isotropic geodesic $y = 0$.

Integrating the relation $dy = pdx = p\big(\frac{3}{2}\alpha|p|^{\frac{1}{2}}s(p) + 2p\big)dp = \big(\frac{3}{2}\alpha|p|^{\frac{3}{2}} + 2p^2\big)dp$, we get $y = \frac{3}{5}\alpha p|p|^{\frac{3}{2}} + \frac{2}{3}p^3 + c_1$, where c_1 is the second constant of integration. The family of geodesics outgoing from $q = 0$ is characterized by $c_1 = 0$. The scaling $y \to \frac{5}{3}y$ brings this family to the form (2.17) with $\bar{X}_\alpha \equiv \bar{Y}_\alpha \equiv 0$ and $\eta = p$:

$$x = \alpha|\eta|^{\frac{3}{2}} + \eta^2, \quad y = \alpha\eta|\eta|^{\frac{3}{2}} + \tfrac{10}{9}\eta^3. \tag{2.25}$$

For $\alpha = 0$, formula (2.25) gives the isotropic geodesic. For $\alpha \to \infty$, the curves (2.25) tend to the smooth geodesic $y = 0$.

Remark 3. Theorem 3 shows that the extension of geodesics through the curve $M_{0,1}$ is not uniquely defined. Indeed, all geodesics of the family (2.17) have the same tangential direction at $q \in M_{0,1}$ and almost all of then have a singularity of the same type at q. So, a curve given by formula (2.17) with any $\alpha \neq 0$ does not have any advantages in comparison with the curve consisting of two bows (2.17) with α_1 if $(x, y) \in M_+$ and α_2 if $(x, y) \in M_-$.

3. Surfaces in Berwald–Moor spaces

Consider the space $\mathbb{R}^n$, $n \geq 3$, with the coordinates $(x_1, \dots, x_n)$ equipped with the *Berwald–Moor metric* $ds = (dx_1 \cdots dx_n)^{\frac{1}{n}}$, and a smooth two-dimensional surface $M \subset \mathbb{R}^n$ parametrized by $x_i = f_i(x, y)$, $i = 1, \dots, n$. The Berwald–Moor metric of the ambient space defines a two-dimensional pseudo-Finsler space $(M, \overline{F})$ with the metric function $\overline{f} = \overline{F}^{\frac{1}{n}}$, where

$$\overline{F}(x, y; \dot{x}, \dot{y}) = \prod_{i=1}^{n}\big(f_{ix}(x, y)\dot{x} + f_{iy}(x, y)\dot{y}\big), \quad f_{ix} = \frac{\partial f_i}{\partial x}, \quad f_{iy} = \frac{\partial f_i}{\partial y}, \tag{3.1}$$

and n families of isotropic lines

$$f_i(x,y) = \text{const}, \quad i = 1, \dots, n. \tag{3.2}$$

Given $q \in M$, the isotropic direction p is called *simple* (*double* or *multiple*) if there exist *only one* (*only two* or *more than one*, resp.) isotropic lines (3.2) passing through q with given direction p. By Lemma 4, singularities of the geodesic flow occur at the points $q \in M$ that have at least one multiple isotropic direction.

Remark 4. In the case $n = 3$, we have a cubic pseudo-Finsler space $(M, \overline{F})$. But unlike Section 2.2, the function $\overline{F}$ given by (3.1) is not generic. The corresponding cubic polynomial $F(x,y;p)$ at every point $q \in M$ has n real roots (taking into account the multiplicity and including the root $p = \infty$), and $M_- = \emptyset$. Hence singularities of geodesics appear only at the points where at least two of three isotropic lines (3.2) are tangent. Here the stratum $M_{0,1}$ consists of the points where two isotropic lines are tangent (the *double* isotropic direction) and the third one is transversal to them (the *simple* isotropic direction).

From now on, we assume that the functions $f_i(x,y)$ have non-degenerate differentials and every point $q \in M$ may have simple or double isotropic directions only (the number of double isotropic directions can vary from 0 to $[\frac{n}{2}]$). Moreover, assume that the tangency of isotropic lines with double isotropic directions has first order. Consider geodesics passing through a point q with a double isotropic direction p_0 satisfying the above conditions.

Without loss of generality assume that $q = 0$ (the origin in the (x,y)-plane) and p_0 corresponds to the isotropic lines $f_1(x,y) = 0$ and $f_2(x,y) = 0$, where $f_{2y}(0) \neq 0$. Making the change of variable $y \mapsto f_2(x,y)$, we transform the metric function (3.1) into a similar one with $f_2(x,y) = y$ and $f_{1x}(0) = 0$, $f_{1y}(0) \neq 0$, $f_{1xx}(0) \neq 0$. The double isotropic direction p_0 becomes $p = 0$ and, moreover, in a neighborhood of 0, $p = 0$ is the double isotropic direction at all points $q \in M_{0,1} = \{f_{1x}(x,y) = 0\}$.

By the condition $f_{1xx}(0) \neq 0$, $M_{0,1}$ is a smooth curve transversal to the x-axis. Making the change of variable $x \mapsto f_{1x}(x,y)$, we transform $M_{0,1}$ into $x = 0$ and the metric function (3.1) into a similar one with $f_{1x} = xa(x,y)$, $f_{1y} = b(x,y)$, $f_2(x,y) = y$, where a, b are smooth functions non-vanishing at 0. So, we get

$$F(x,y;p) = p(ax+bp)G, \quad G(x,y;p) = \prod_{i=3}^{n}(f_{ix} + f_{iy}p), \quad G(0,0;0) \neq 0. \tag{3.3}$$

Substituting (3.3) in (2.4), we get

$$\begin{aligned} \Delta &= \big((1-n)(ax)^2 + 2(2-n)abxp + 2(2-n)(bp)^2\big)\, G^2 + \Delta_0, \\ P &= ap\big((n-2)bp - ax + P_0\big)\, G^2, \end{aligned} \tag{3.4}$$

where $\Delta_0 \in \langle x^3, x^2p, xp^2, p^3 \rangle$ and $P_0 \in \langle x^2, xp, p^2 \rangle$ (both ideals are in the ring of smooth functions on x, y, p). Formula (3.4) shows that all components of the field (2.7) vanish on the line $\{x = p = 0\}$ and the spectrum of (2.7) at any point of this line has three zero eigenvalues. This does not allow us to establish a normal form similar to Theorem 3.

To overcome this problem, consider the blowing up

$$\mathcal{B}\colon (x,y,u) \mapsto (x,y,p), \quad p = xu, \quad u \in \mathbb{R}P^1 = \mathbb{R} \cup \infty. \tag{3.5}$$

The mapping $\mathcal{B}$ is one-to-one except on the plane $\Pi = \{(x,y,u)\colon x = 0\}$, whose image is the line $\mathcal{B}(\Pi) = \{(x,y,p)\colon x = p = 0\}$. The mapping $\mathcal{B}$ is a local diffeomorphism at all points of the (x,y,u)-space except Π. It has an inverse defined on $\mathbb{R}^3 \setminus \mathcal{B}(\Pi)$ given by

$$\mathcal{B}^{-1}(x,y,p) = \Big(x, y, \frac{p}{x}\Big).$$

Observe that there is no geodesic that coincides with the line $\mathcal{B}(\Pi)$. A straightforward calculation shows that the field (2.7) corresponds to a smooth field in the (x, y, u)-space (away of Π) that, after dividing by the common factor xG^2, is

$$x\big(A+\ldots\big)\left(\frac{\partial}{\partial x}+xu\frac{\partial}{\partial y}\right)+u(n-2)\big(2(bu)^2+3abu+a^2+\ldots\big)\frac{\partial}{\partial u}, \tag{3.6}$$

where

$$A(x,y,u)=(1-n)a^2+2(2-n)(abu+(bu)^2);$$

here and below the dots mean terms that belong to the ideal $\langle x\rangle$.

Remark 5. There exists $\varepsilon>0$ such that $A(x,y,u)\neq 0$ for all u if $|x|+|y|<\varepsilon$. Indeed, consider $A(x,y,u)$ as a quadratic polynomial on u with the discriminant

$$D=(2-n)^2(ab)^2-2(1-n)(2-n)(ab)^2=n(2-n)(ab)^2,$$

which is strictly negative if x, y are sufficiently close to zero.

Dividing the field (3.6) by $(A+\ldots)$, we get

$$x\left(\frac{\partial}{\partial x}+xu\frac{\partial}{\partial y}\right)+u(n-2)(U+\ldots)\frac{\partial}{\partial u},\quad U(x,y,u)=\frac{2(bu)^2+3abu+a^2}{A(x,y,u)}. \tag{3.7}$$

Remark 6. The plane $x=0$ is invariant for the fields (3.6) and (3.7). Moreover, it is filled with *vertical* (i.e., parallel to the u-direction) straight integral lines of these fields, whose projections to the (x, y)-plane along the u-axis are points on the y-axis.

Lemma 6. *Geodesics can pass through a point $q\in M$ lying on the y-axis with the direction $p=0$ only with the following admissible values u:*

$$u_0=0,\quad u_1=-a/2b,\quad u_2=-a/b. \tag{3.8}$$

Proof. By the standard existence and uniqueness theorem, for every point (x, y, u) such that $x=0$ and $U(x,y,u)\neq 0$, there exists a unique integral curve of the field (3.7) passing through this point. By Remark 6, it is a vertical straight line, whose projection to the (x, y)-plane is a point on the y-axis. Hence geodesics can pass through a point $q\in M$ lying on the y-axis with the direction $p=0$ only with $u=0$ or u such that $2(bu)^2+3abu+a^2=0$. This gives the three values in (3.8). □

Lemma 7. *The set of singular points of the field (3.7) consists of three mutually disjoint curves*

$$W_i^c=\{(x,y,u)\colon x=0,\ u=u_i(y)\},\quad i=0,1,2.$$

On every curve W_i^c, the linear part of the field (3.7) has the constant spectrum $(1,\lambda,0)$, where $\lambda=\frac{n-2}{n}$ if $i=1$ or $\lambda=\frac{n-2}{1-n}$ if $i\in\{0,2\}$. In both cases, ∂_u is the eigenvector with λ.

Proof. The first statement is trivial. All other statements are by direct calculations. □

Theorem 4. *Suppose that the functions $f_i(x,y)$ have non-degenerate differentials and p_0 is a double isotropic direction at $q\in M$ such that the corresponding isotropic lines have first order of tangency at q. Then the field (3.7) at its singular points $(q;u_i)$ has local orbital normal forms indicated in Table 1 and to p_0 corresponds a one-parameter family of geodesics outgoing from q. There exist smooth local coordinates centered at q such that this family consists of C^2-smooth non-isotropic geodesics*

$$y=x^2+Y(x,\alpha|x|^\lambda),\quad Y(x,\alpha|x|^\lambda)=o(x^2),\quad \lambda=\tfrac{n-2}{n},\quad \alpha\in\mathbb{R}, \tag{3.9}$$

where $Y(\cdot,\cdot)$ is a C^∞-smooth function, together with two C^∞-smooth isotropic geodesics

$$y=0\quad\text{and}\quad y=2x^2+o(x^2). \tag{3.10}$$

In a neighborhood of q, every geodesic of the family (3.9) belongs to the curvilinear tongue-like sector bounded by the curves (3.10) as it is presented in Figure 4 (right).

	Orbital normal form	
	topological	C^∞-smooth
W_1^c	$\xi\frac{\partial}{\partial\xi}+\eta\frac{\partial}{\partial\eta}+0\frac{\partial}{\partial\zeta}$	$\xi\frac{\partial}{\partial\xi}+\lambda\eta\frac{\partial}{\partial\eta}+0\frac{\partial}{\partial\zeta}$, where $\lambda=\frac{n-2}{n}$
W_0^c and W_2^c	$\xi\frac{\partial}{\partial\xi}-\eta\frac{\partial}{\partial\eta}+0\frac{\partial}{\partial\zeta}$	$\xi(n-1+\Phi_1(\rho,\zeta))\frac{\partial}{\partial\xi}-\eta(n-2+\Phi_2(\rho,\zeta))\frac{\partial}{\partial\eta}+\rho\Psi(\rho,\zeta)\frac{\partial}{\partial\zeta}$, where $\rho=\xi^{n-2}\eta^{n-1}$; $(n-1)\xi\frac{\partial}{\partial\xi}-(n-2)\eta\frac{\partial}{\partial\eta}+\rho\frac{\partial}{\partial\zeta}$, if $\Psi(0,0)\neq 0$.

TABLE 1. Local orbital normal forms of the field (3.7).

Proof. Choose local coordinates so that $q\in M$ is the origin and consider the field (3.7) in a neighborhood of its singular points $(0,0,u_i)$, $i=0,1,2$, where u_i are given by formula (3.8). By Lemma 7, at all singular points the condition $\operatorname{Re}\lambda_{1,2}\neq 0$ holds, and every curve W_i^c, $i=0,1,2$, is the center manifold of this field. Moreover, there exist also 2-dimensional unstable manifolds if $i=1$ and a pair of 1-dimensional stable and unstable manifolds if $i=0,2$. Hence all topological normal forms in Table 1 trivially follow from the reduction principle [2, 4, 9].

Indeed, the reduction principle asserts that the germ (3.7) is orbitally topologically equivalent to the direct product of the standard 2-dimensional node (if $i=1$) or saddle (if $i=0,2$) and the restriction of the field to the center manifold W_i^c. Since the restriction of the field (3.7) to every center manifold W_i^c, $i=0,1,2$, is identically zero, this gives us the topological normal forms in Table 1.

We establish now the smooth normal forms in the cases $i=1$ and $i=0,2$ separately.

The case $i=1$. By Lemma 7, the linear part of the field (3.7) at any point on W_1^c has spectrum $\big(1,\lambda,0\big)$ with $\lambda=\frac{n-2}{n}$. Then Theorem 5.5 in [21] asserts that the germ (3.7) at any point on W_1^c is orbitally C^∞-smoothly equivalent to

$$(\xi+\varphi(\zeta)\eta^{1/\lambda})\frac{\partial}{\partial\xi}+\lambda\eta\frac{\partial}{\partial\eta}+0\frac{\partial}{\partial\zeta}, \tag{3.11}$$

where $\varphi(\zeta)\equiv 0$ if the number $1/\lambda$ is not integer (non-resonant case).

Assume $1/\lambda$ is an integer and prove that $\varphi(\zeta)\equiv 0$ iff for every point $\omega_*\in W_1^c$ the field (3.7) has a C^∞-smooth integral curve passing through ω_* with the vertical tangential direction ∂_u. By Remark 6, such integral curves exist (the vertical straight lines); hence this establishes the equality $\varphi(\zeta)\equiv 0$ in the remaining cases $1/\lambda=3$ ($n=3$) and $1/\lambda=2$ ($n=4$).

For this, note that the field (3.11) has the invariant foliation $\zeta=\text{const}$. Every invariant leaf contains a single integral curve $\eta=0$ corresponding to eigendirection with the eigenvalue 1 and the one-parameter family of integral curves

$$\xi=\eta^{1/\lambda}(\alpha+\varphi(\zeta)\ln|\eta|),\quad \alpha\in\mathbb{R}, \tag{3.12}$$

corresponding to eigendirections with the eigenvalue λ. All curves (3.12) are $C^{1/\lambda-1}$-smooth (but not $C^{1/\lambda}$-smooth at zero) if $\varphi(\zeta)\neq 0$ and C^∞-smooth if $\varphi(\zeta)=0$. Without loss of generality, assume that the point $\omega_*\in W_1^c$ in the (x,y,u)-space corresponds to $(0,0,\zeta_*)$ in the (ξ,η,ζ)-space. The equality $\varphi(\zeta_*)=0$ is equivalent to the existence of at least one C^∞-smooth integral curve of the field (3.11) with tangential direction ∂_η lying on the invariant leaf $\{\zeta=\zeta_*\}$. To complete the proof, remark that the eigendirection ∂_η of (3.11) corresponds to the eigendirection ∂_u of (3.7).

The cases $i = 0, 2$. By Lemma 7, the linear part of the field (3.7) at all points on the curves W_0^c and W_2^c has spectrum $(\lambda_1, \lambda_2, 0)$, where $\lambda_1 = 1$ and $\lambda_2 = \frac{n-2}{1-n}$. This gives the resonance

$$\mu\lambda_1 + \nu\lambda_2 = 0 \tag{3.13}$$

with the resonant monomial $\rho = \xi^\mu \eta^\nu$, where we set $\mu = n-2$ and $\nu = n-1$. Everything that we say below is true as well for arbitrary relatively prime $\mu, \nu \in \mathbb{N}$.

The resonance (3.13) does not allow us to get a normal form with one identically zero component (as we have in the case $i = 1$) even in the finite-smooth category; see the discussion in [8] (Section 3.2). Moreover, (3.13) generates two infinite series of resonances

$$(1 + l\mu)\lambda_1 + l\nu\lambda_2 = \lambda_1, \quad l\mu\lambda_1 + (1 + l\nu)\lambda_2 = \lambda_2, \quad l = 1, 2, \dots,$$

and consequently, an infinite number of resonant monomials in the corresponding (orbital) Poincaré–Dulac normal form:

$$\xi(\nu + \Phi_1(\rho, \zeta))\frac{\partial}{\partial \xi} - \eta(\mu + \Phi_2(\rho, \zeta))\frac{\partial}{\partial \eta} + \rho\Psi(\rho, \zeta)\frac{\partial}{\partial \zeta}; \tag{3.14}$$

see, e.g., [8] (Section 3.2) or [17] (Section 5).

Moreover, if in addition, $\Psi \neq 0$ at a point ω_*, the germ (3.14) at ω_* is smoothly orbitally equivalent to

$$\nu\xi\frac{\partial}{\partial \xi} - \mu\eta\frac{\partial}{\partial \eta} + \rho\frac{\partial}{\partial \zeta}. \tag{3.15}$$

The normal form (3.15) was first established by Roussarie [22] in the partial case $\mu = \nu = 1$ in the C^∞-smooth category. For arbitrary integers μ, ν, the proof (in the finite-smooth category) can be found in [17] (Section 5). Combining the methods from [22] and [17], one can establish the normal form (3.15) with arbitrary μ, ν in the C^∞-smooth category also.

Completion of the proof. Integral curves of the field (3.7) passing through $(0, 0, u_1)$ correspond to integral curves of the field $\xi\frac{\partial}{\partial \xi} + \lambda\eta\frac{\partial}{\partial \eta}$ lying on the invariant leaf $\{\zeta = 0\}$: a single curve that coincides with the η-axis and one-parameter family $\{\eta = \alpha|\xi|^\lambda,\ \zeta = 0\}$, $\alpha \in \mathbb{R}$. Comparing the germ (3.7) at $(0, 0, u_1)$ with its normal form $\xi\frac{\partial}{\partial \xi} + \lambda\eta\frac{\partial}{\partial \eta}$, one can see that the conjugating diffeomorphism $(x, y, u) \mapsto (\xi, \eta, \zeta)$ can be chosen in the form

$$x = \xi, \quad u = u_1 + \eta + c_1\xi + c_2\zeta + \varphi(\xi, \eta, \zeta), \quad y = \zeta + \psi(\xi, \eta, \zeta), \quad \varphi, \psi \in \mathfrak{M}^2, \tag{3.16}$$

where $\mathfrak{M}^k$, $k \geq 0$, is the ideal of k-flat functions in the ring of smooth functions. Substituting $\xi = \zeta = 0$ in (3.16) and taking into account $p = ux$, we get $x = p = 0$. Hence the η-axis does not correspond to a geodesic.

Substituting $\eta = \alpha|\xi|^\lambda$ and $\zeta = 0$ in (3.16) and taking into account $x = \xi$ and $p = ux$, we get $p = x(u_1 + f(x, \alpha|x|^\lambda))$ with a certain smooth function $f \in \mathfrak{M}^0$. This gives the relation

$$dy = p\,dx = x(u_1 + f(x, \alpha|x|^\lambda))dx,$$

where $xf(x, \alpha|x|^\lambda) = o(x)$ is a C^1-smooth function. Integrating, we get $y = \frac{u_1}{2}x^2 + Y(x, \alpha|x|^\lambda)$. Here $Y(\cdot, \cdot)$ is a smooth function and $Y(x, \alpha|x|^\lambda) = o(x^2)$ is C^2-smooth. After the scaling $y \to 2y/u_1$, we get the family (3.9).

The topological and smooth orbital normal forms in Table 1 show that the field (3.7) has only two integral curves passing through its singular point $(0, 0, u_i)$, where $i = 0$ or 2. Moreover, one of these integral curves is a straight vertical line, whose projection to the (x, y)-plane is a point (see Remark 6 and Lemma 7). Another integral curve has a non-vertical tangential direction at $(0, 0, u_i)$; hence its projection to the (x, y)-plane is regular.

Thus all of the admissible values u_0 and u_2 give a smooth geodesic passing through the point q with tangential direction $p = 0$. It is not hard to see that these geodesics are isotropic lines, which are solutions of differential equations $p = 0$ and $ax + bp = 0$, respectively (see formula (3.3)). Taking into account (3.8), after the scaling $y \to 2y/u_1$ we get (3.10). $\square$

Remark 7. The normal forms $3\xi\frac{\partial}{\partial\xi} + 2\eta\frac{\partial}{\partial\eta}$ and $\xi\frac{\partial}{\partial\xi} + \frac{n-2}{n}\eta\frac{\partial}{\partial\eta}$ in Theorems 3, 4 are valid also in the analytic category; see, e.g., [25]. Therefore, in the analytic case, formulae (2.17) and (3.9) present Puiseux series for geodesics.

Example 4. Consider geodesics on the surface $z = y - 2x^2$ in the Berwald–Moor space (x, y, z) with the metric $ds = (dx\,dy\,dz)^{\frac{1}{3}}$. This yields

$$F(x, y; p) = p(p - 4x), \quad \Delta = -2(p^2 - 4xp + 16x^2), \quad P = -4p(p + 4x), \tag{3.17}$$

and the equation of geodesics (2.3) reads

$$\frac{dp}{dx} = \frac{2p(p + 4x)}{p^2 - 4xp + 16x^2}. \tag{3.18}$$

The isotropic lines are solutions of the differential equation $p(p - 4x) = 0$. This gives two families of isotropic lines $y = \text{const}$ and $y = 2x^2 + \text{const}$, which have first order tangency on the line $x = 0$. Substituting them into (3.18), one can see that they are geodesics.

Consider the geodesics outgoing from the point $q = 0$ with the double isotropic directions $p_0 = 0$. (Recall that, for every $p \neq 0$, there exists a unique geodesic passing through q with tangential direction p; we exclude such geodesics from further consideration.) The isotropic geodesics $y = 0$ and $y = 2x^2$ (formula (3.10)) separate the (x, y)-plane into four parts: the upper domain $y > 2x^2$, the semiplane $y < 0$ and two tongue-like sectors between them. See Fig. 4 (right); the isotropic geodesics $y = 0$ and $y = 2x^2$ are depicted with dashed lines.

Theorem 4 claims that there exists a one-parameter family of geodesics outgoing from q with the double isotropic directions $p_0 = 0$ into the tongue-like sectors (non-isotropic family (3.9)) and there are no geodesics outgoing from q with the double isotropic directions $p_0 = 0$ into two remaining parts of the plane. Geodesics of the family (3.9) correspond the admissible value $u_1 = 2$ (compare formulae (3.3), (3.8) and (3.17)) and they can be presented as the Puiseux series

$$y = t^6 + 3t^6 \sum_{i \geq 4} \frac{a_i}{i+3} t^{i-3}, \quad p = \frac{dy}{dx} = \frac{1}{3t^2}\frac{dy}{dt} = 2t^3 + \sum_{i \geq 4} a_i t^i, \quad \text{where} \quad x = t^3.$$

Substituting the above expression for p in (3.18), we obtain recurrence relations for the unknown coefficients a_i.

Namely, $a_i = 0$ for all odd i (this also follows from the fact that the surface $z = y - 2x^2$ is symmetric with respect to the plane $x = 0$). For even i, we have $24a_6 + 4a_4^3 = 0$, $48a_8 + 14a_4^2 a_6 = 0$, $72a_{10} + 16(a_4 a_6^2 + a_4^2 a_8) = 0$, etc. In general,

$$12(2i - 4)a_{2i} + b_{2i} = 0, \quad i = 3, 4, 5, \ldots, \tag{3.19}$$

where b_{2i} is a polynomial on a_{2j} with $j < i$ with zero free term. This shows that the coefficient a_4 is arbitrary, and all a_{2i} with $i \geq 3$ are uniquely defined by equations (3.19). This gives the one-parameter non-isotropic family (3.9). In particular, $a_4 = 0$ gives $a_{2i} = 0$ for all $i \geq 3$, and the corresponding solution $y = x^2$ presents the unique C^∞-smooth geodesic of non-isotropic family (3.9) (the corresponding value of the parameter is $\alpha = 0$).

References

[1] *Antonelli P. L., Miron R.*, Lagrange and Finsler Geometry. Applications to Physics and Biology. Kluwer Academic Publishers, 1996. DOI: 10.1007/978-94-015-8650-4

[2] *Arnol'd V. I.*, Geometrical methods in the theory of ordinary differential equations. Springer-Verlag, New York, 1988.

[3] *Arnol'd V. I., Givental' A. B.*, Symplectic geometry. Encyclopedia of Mathematical Sciences, Dynamical Systems IV (Springer, Berlin, 1985), pp. 5–131.

[4] *Arnol'd V. I., Il'yashenko Yu. S.*, Ordinary differential equations, Dynamical systems I. Encyclopaedia Math. Sci., vol. 1, Springer-Verlag 1988.

[5] *Asanov G. S.*, Finsler Geometry, Reltivity and Gauge Theories. Reidel, Dordrecht, 1985. DOI: 10.1007/978-94-009-5329-1

[6] *Balan V., Neagu M.*, Jet single-time Lagrange geometry and its applications. John Wiley & Sons, Inc., Hoboken, NJ, 2011. DOI: 10.1002/9781118143759

[7] *Bao D., Chern S.-S., Shen Z.*, An Introduction to Riemann-Finsler Geometry. Graduate Texts in Mathematics, 200. Springer-Verlag, New York, 2000. DOI: https://doi.org/10.1007/978-1-4612-1268-3

[8] *Ghezzi R., Remizov A. O.*, On a class of vector fields with discontinuities of divide-by-zero type and its applications to geodesics in singular metrics. Journal of Dynamical and Control Systems, 18:1 (2012), pp. 135–158. DOI: 10.1007/s10883-012-9137-4

[9] *Hirsch M. W., Pugh C. C., Shub M.*, Invariant manifolds. Lecture Notes in Mathematics, Vol. 583. Springer-Verlag, Berlin-New York, 1977. DOI: 10.1007/BFb0092042

[10] *Khesin B., Tabachnikov S.*, Pseudo-Riemannian geodesics and billiards. Advances in Math. 221 (2009), pp. 1364–1396. DOI: 10.1016/j.aim.2009.02.010

[11] *Kossowski M., Kriele M.*, Smooth and discontinuous signature type change in general relativity. Class. Quantum Grav. 10 (1993), pp. 2363–2371. DOI: 10.1088/0264-9381/10/11/019

[12] *Martinet J.*, Sur les singularités des formes différentielles, Ann. Inst. Fourier, 1970, 20, no 1, pp. 95–178. DOI: 10.5802/aif.340

[13] *Matsumoto M.*, Two-dimensional Finsler spaces whose geodesics constitute a family of special conic sections. J. Math. Kyoto Univ. 35 (1995), no. 3, pp. 357–376.

[14] *Matsumoto M., Shimada H.*, On Finsler spaces with 1-form metric. II. Berwald-Moor's metric $L = (y^1 y^2 \cdots y^n)^{1/n}$. Tensor (N.S.) 32 (1978), no. 3, pp. 275–278.

[15] *Mikeš J., Hinterleitner I., Vanžurová A.*, One remark on variational properties of geodesics in pseudoriemannian and generalized Finsler spaces. In: Geometry, integrability and quantization, Softex, Sofia, 2008, pp. 261–264. DOI: 10.7546/giq-9-2008-261-264

[16] *Pavlova N. G., Remizov A. O.*, Geodesics on hypersurfaces in the Minkowski space: singularities of signature change. Russian Math. Surveys, 2011, 66:6 (402), pp. 193–194. DOI: 10.4213/rm9449

[17] *Remizov A. O.*, Multidimensional Poincaré construction and singularities of lifted fields for implicit differential equations. J. Math. Sci. (N.Y.) 151:6 (2008), pp. 3561–3602. DOI: 10.1007/s10958-008-9043-1

[18] *Remizov A. O.*, Codimension-two singularities in 3D affine control systems with a scalar control. Mat. Sb., 199:4 (2008), 143–158. DOI: 10.1070/sm2008v199n04abeh003935

[19] *Remizov A. O.*, Geodesics on 2-surfaces with pseudo-Riemannian metric: singularities of changes of signature. Mat. Sb., 200:3 (2009), 75–94. DOI: 10.1070/sm2009v200n03abeh004001

[20] *Remizov A. O.*, On the local and global properties of geodesics in pseudo-Riemannian metrics. Differential Geometry and its Applications, 39 (2015), 36–58. DOI: 10.1016/j.difgeo.2015.01.003

[21] *Remizov A. O., Tari F.*, Singularities of the geodesic flow on surfaces with pseudo-Riemannian metrics. Geometriae Dedicata, 185 (2016), no. 1, 131–153. DOI: 10.1007/s10711-016-0172-2

[22] *Roussarie R.*, Modèles locaux de champs et de formes. Asterisque, vol. 30 (1975), pp. 1–181.

[23] *Rund H.*, The differential geometry of Finsler spaces. Die Grundlehren der Mathematischen Wissenschaften, Bd. 101. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1959, xiii+284 pp.

[24] *Sotomayor J., Zhitomirskii M.*, Impasse singularities of differential systems of the form $A(x)x' = F(x)$. J. Diff. Equations 169 (2001), no. 2, pp. 567–587. DOI: 10.1006/jdeq.2000.3908

[25] *Voronin S. M.*, The Darboux–Whitney theorem and related questions. In: Nonlinear Stokes phenomenon (Yu.S. Ilyashenko, ed.). Adv. Sov. Math. 14, Providence (1993), pp. 139–233.

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DECOMPOSITION THEOREM FOR SEMI-SIMPLES

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Abstract. We use standard constructions in algebraic geometry and homological algebra to extend the decomposition and hard Lefschetz theorems of T. Mochizuki and C. Sabbah so that they remains valid without the quasi-projectivity assumptions.

1. Introduction

M. Kashiwara [Ka] has put-forward a series of conjectures concerning the behavior of holonomic semi-simple D-modules on a complex algebraic variety under proper push-forward and under taking nearby/vanishing cycles.

Inspired by this conjecture, T. Mochizuki [Mo] has proved Kashiwara conjectures in the very important case where one assumes the holonomic D-modules to be regular. Mochizuki's work built on earlier work by C. Sabbah [Sa]. Because of the regularity assumptions (see [Sa, p.2-3, Remark 6]) for more context), part of their results can be expressed, via the Riemann-Hilbert correspondence, in the form of Theorem 2.1.1 below.

The methods employed in [Mo, Sa] are essentially analytic. Moreover, [Mo, Sa] are placed in the context of projective morphisms of quasi projective manifolds, so that Theorem 2.1.2 below, which generalizes Theorem 2.1.1, is not directly affordable by their methods: one would first need to extend aspects of their theory of polarizable pure twistor D-modules from projective manifolds to complex algebraic varieties. To my knowledge, this extension is not in the literature.

V. Drinfeld [Dr] has shown that an arithmetic conjecture by A. de Jong implies, rather surprisingly and again under the regularity assumption, Kashiwara's conjectures. Drinfeld's proof uses also algebraic geometry for varieties over finite fields. Note that [Dr] allows for arbitrary characteristic-zero coefficients. de Jong's conjecture has been proved by D. Gaitsgory [Ga] and by G. Böckle and C. Khare [Bo-Ka].

The combination of the work in [Dr, Ga, Bo-Ka] yields an arithmetic proof of Theorems 2.1.1 and of 2.1.2 below.

The purpose of this note is to provide a proof of Theorem 2.1.2 that stems directly from Theorem 2.1.1 and uses only simple reductions based on standard constructions in algebraic geometry.

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2. Decomposition and relative hard Lefschetz for semi-simples

2.1. **Statement.** A variety is a separated scheme of finite type over the field of complex numbers $\mathbb{C}$. For the necessary background concerning what follows, the reader may consult [dCM]. Given a variety Y, we work with the rational and complex constructible derived categories $D(Y,\mathbb{Q})$ and $D(Y,\mathbb{C})$ endowed with the middle-perversity t-structures, whose hearts, i.e. the respective categories of perverse sheaves on Y, are denoted by $P(Y,\mathbb{Q})$ and $P(Y,\mathbb{C})$, respectively. The simple objects in $P(Y,\mathbb{Q})$ and in $P(Y,\mathbb{C})$ have the form $IC_S(L)$, where S is an irreducible closed

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subvariety of Y, L is a simple (i.e. irreducible) complex/rational local system defined on some dense open subset of the regular part of S, and IC stands for intersection complex. We say that $K \in D(Y, \mathbb{Q})$ is semi-simple if it is isomorphic to the finite direct sum of shifted simple perverse sheaves as above: $K \cong \oplus_b {}^{\mathfrak{p}}\mathcal{H}^b(K)[-b] \cong \oplus_b \oplus_{(S,L)\in EV_b} IC_S(L)[-b]$, where ${}^{\mathfrak{p}}\mathcal{H}^b$ denotes the b-th perverse cohomology sheaf functor, and EV_b is a uniquely determined finite set of pairs (S, L) as above. Similarly, with $\mathbb{C}$-coefficients.

Our starting point is the following result of T. Mochizuki [Mo, §14.5 and §14.6], which generalizes one of C. Sabbah [Sa]. In fact, they both work in the more refined setting of polarized pure twistor D-modules and their results have immediate and evident counterparts in the setting of the constructible derived category, which is the one of this note.

Theorem 2.1.1. *Let $f : X \to Y$ be a projective map of irreducible quasi projective nonsingular varieties. If $K \in P(X, \mathbb{C})$ is semi-simple, then $f_*K \in D(Y, \mathbb{C})$ is semi-simple. The relative hard Lefschetz theorem holds.*

Even if the methods in [Mo] seem to require the smoothness and quasi projectivity assumptions, as well as $\mathbb{C}$-coefficients, one can deduce the following more general statement. We have nothing to say concerning the refined context of polarizable pure twistor D-modules.

Theorem 2.1.2. *Let $f : X \to Y$ be a proper map of varieties. If $K \in P(X, \mathbb{Q})$ is semi-simple, then $f_*K \in D(Y, \mathbb{Q})$ is semi-simple. If f is projective, then the relative hard Lefschetz theorem holds.*

We first show how to deduce the $D(Y, \mathbb{C})$-version of Theorem 2.1.2 from Theorem 2.1.1. Then we show how the $D(Y, \mathbb{C})$-version implies formally the $D(Y, \mathbb{Q})$-version.

The reader should have no difficulty in replacing $\mathbb{Q}$ with any field of characteristic zero and proving the same result.

2.2. **Proof of Theorem 2.1.2 for $D(Y, \mathbb{C})$.** Theorem 2.1.1 is stated for $\mathbb{C}$-coefficients. In this section, we use this statement to deduce Theorem 2.1.2 for $\mathbb{C}$-coefficients, i.e. to deduce Corollary 2.2.1 below.

The theorem will be reduced to several special cases, where we progressively relax the hypotheses on f, from projective, to quasi projective, to proper, and on X and Y, from smooth quasi projective, to quasi projective, to arbitrary. These conditions will be denoted symbolically by $(f_{proj}, X^{sm}_{qp}, \ldots)$. For example, we summarize the hypotheses of Theorem 2.1.1 graphically as follows:

$$(f_{proj}, X^{sm}_{qp}, Y^{sm}_{qp}) \qquad (f \text{ projective, } X \text{ and } Y \text{ smooth and quasi projective}).$$

Our goal is to establish Corollary 2.2.1 as an immediate consequence of the five following claims.

(1) *Theorem 2.1.1 holds for $(f_{proj}, X^{sm}_{qp}, Y_{qp})$.*
Choose any closed embedding $g : Y \to \mathbb{U}$ of Y into a Zariski-dense open subvariety $\mathbb{U} \subseteq \mathbb{P}$ of some projective space. Apply Theorem 2.1.1 to $h := g \circ f$ and observe that, modulo the natural identification of the objects in $D(Y, \mathbb{C})$ with the ones in $D(\mathbb{U}, \mathbb{C})$ supported on Y, we have $h_*K = f_*K$.

(2) *Theorem 2.1.1 holds for $(f_{proj}, X_{qp}, Y_{qp})$.*
Pick a resolution of the singularities $g : Z \to X$ of X with g projective. Let $X^o \subseteq X_{reg} \subseteq X$ be a dense Zariski open subset on which the simple local system M is defined and over which g is an isomorphism. Let $IC_Z(M) \in P(Z, \mathbb{C})$ be the intersection complex on Z with coefficients in the local system M transplanted to $g^{-1}(U^o)$. Apply 1. to g and h. Observe that $IC_X(M)$ is a direct summand of $g_*IC_Z(M)$. Deduce

that $f_*IC_X(M)$ is a direct summand of $h_*IC_Z(M)$ so that the first part of Theorem 2.1.1 holds for $(f_{proj}, X_{qp}, Y_{qp})$. In order to prove the second part of Theorem 2.1.1, i.e. the relative hard Lefschetz theorem for f, we argue as in [dCM], Lemma 5.1.1: we do not need self-duality to conclude: the argument gives injectivity; by dualizing we get surjectivity for the dual of the hard Lefschetz maps; this dualized map is the hard Lefschetz map for f, $IC_X(M)^\vee$ and the f-ample $\eta \in H^2(X,\mathbb{C})$; by switching the roles of M and $M^\vee$, we see that the relative hard Lefschetz theorem maps are isomorphisms. (N.B.: we may impose self-duality artificially, by replacing M with $M \oplus M^\vee$ and reach the same conclusion.)

(3) *Theorem 2.1.1 holds for* (f_{proj}, X_{qp}, Y).
Let $Y = \cup_i Y_i$ be an affine open covering. Let $f_i : X_i := f^{-1}(Y_i) \to Y_i$ be the obvious maps. By 2., the relative Hard Lefschetz holds for f_i. Since the relative hard Lefschetz maps are defined over Y and they are isomorphisms over the Y_i, the relative hard Lefschetz holds for f over Y. By the Deligne-Lefschetz criterion [De], we have $f_*K \cong \oplus_b {}^{\mathfrak{p}}\mathcal{H}^b(f_*K)[-b]$. It remains to show that the $P^b := {}^{\mathfrak{p}}\mathcal{H}^b(f_*K)$ are semi-simple. By 2., the $P^b_{|Y_i}$ are semi-simple after restriction to the open affine Y_i. By a repeated use of the the splitting criterion [dCM], Lemma 4.1.3[1] applied in the context of a Whitney stratification of Y w.r.t. which the P^b are cohomologically constructible, we deduce that the P^b split as direct sum of intersection complexes with coefficients in some local systems. (Note that [dCM], Assumption 4.1.1 is fulfilled in view of [dCM], Remark 4.1.2, because we already know that P_b splits as desired over the open Y_i.) We need to verify that these local systems are semi-simple. Since a local system on an integral normal variety is semisimple if and only if it is semisimple after restriction to a Zariski dense open subvariety, the desired semi-simplicity can be checked by restriction to the chosen affine covering of Y, where we can apply 2.

(4) *Theorem 2.1.1 holds for* (f_{proj}, X, Y).
As it was pointed out in 3., the relative hard Lefschetz can be verified on an affine covering $Y = \cup_i Y_i$. The resulting X_i are then quasi-projective and we can apply 3. For the semisimplicity of the direct image $f_*IC_X(M)$, we take a Chow envelope $g : Z \to X$ of X (Z quasi projective, g projective and birational); we produce $IC_Z(M)$ as above and we deduce the semisimplicity of $f_*IC_X(M)$ from the one –established in 3.– of $h_*IC_Z(M)$, as it was done in 2.

(5) *The semisimplicity statement in Theorem 2.1.1 holds for* (f_{proper}, X, Y).
Take a Chow envelope $g : Z \to X$ of f (g birational, g and $h := f \circ g$ projective). Produce $IC_Z(M)$ as above. Apply 4. and deduce that $f_*IC_X(M)$ is a direct summand of the semi-simple $h_*IC_Z(M)$.

The above, together with the obvious remark that it is enough to prove Theorem 2.1.2 in the case when X, Y are irreducible and $K = IC_X(M)$, yields the following

Corollary 2.2.1. *Theorem 2.1.2 holds for $\mathbb{C}$-coefficients.*

2.3. **Theorem 2.1.2 for $D(Y,\mathbb{C})$ implies the same for $D(Y,\mathbb{Q})$.** Let f be projective. Then we have the relative hard Lefschetz for $\mathbb{C}$-coefficients, hence for $\mathbb{Q}$-coefficients as well. By the Deligne-Lefschetz criterion, we have the isomorphism $f_*K \cong \oplus_b {}^{\mathfrak{p}}\mathcal{H}^b(f_*K)[-b]$ in $D(Y,\mathbb{Q})$. We

[1] Let P be a perverse sheaf on a variety Z; let $Z = U \coprod Z$ be Whitney-stratified in such a way that $U \subseteq Z$ is open and union of strata, $S \subseteq Z$ is a closed stratum, and P is cohomologically constructible with respect to the stratification; Lemma 4.1.3 in [dCM] is an iff criterion for the splitting of P into the intermediate extension $j_{!*}(P_{|U})$ to Z of the restriction $P_{|U}$ of P to U, direct sum a local system on S placed in cohomological degree minus the codimension of the stratum; the criterion is local in the classical and even in the Zariski topology

need to show that each $P^b := {}^{\mathfrak{p}}\mathcal{H}^b(f_*K)[-b]$ is semi-simple in $P(Y,\mathbb{Q})$. Note that extending the coefficients from $\mathbb{Q}$ to $\mathbb{C}$ is a t-exact functor $D(Y,\mathbb{Q}) \to D(Y,\mathbb{C})$. In particular, the formation of P^b is compatible with complexification. By arguing as in point 3. of the previous section, we see that each P^b is a direct sum of intersection complexes $IC_S(L)$, where the L are rational local systems (note that [dCM], Assumption 4.1.1 is now fulfilled in view of [dCM], Remark 4.1.2, because we already know that the complexification of P^b splits as desired over Y). We need to verify that each L is a semi-simple rational local system. We know its complexification is, hence so is L, in fact: let $0 \to L' \to L \to L'' \to 0$ be an extension of rational locally constant sheaves on S^o; it is classified by an element $e \in H^1(S^o, L''^* \otimes L')$; this element becomes trivial after complexification, hence it is trivial over $\mathbb{Q}$.

If f is proper, we take a Chow envelope $g : Z \to X$ of f, we set $h := f \circ g$ and we deduce semisimplicity of f_* from the semisimplicity of h_* (h is projective) as in point 5. of the previous section.

References

[Bo-Ka] G. Böckle, C. Khare. Mod ℓ representations of arithmetic fundamental groups. II. A conjecture of A. J. de Jong. Compos. Math. 142 (2006), no. 2, 271-294.

[dCM] M. A. de Cataldo, L. Migliorini. The Hodge theory of algebraic maps. Ann. Sci. cole Norm. Sup. (4) 38 (2005), no. 5, 693-750.

[De] P. Deligne. Théorème de Lefschetz et critères de dégénérescence de suites spectrales. (French) Inst. Hautes Études Sci. Publ. Math. No. 35 1968 259-278.

[Dr] V. Drinfeld. On a conjecture of Kashiwara. Mathematical Research Letters 8, 713-728 (2001). DOI: 10.4310/MRL.2001.v8.n6.a3

[Ga] D. Gaitsgory. On de Jong's conjecture. Israel J. Math. 157 (2007), 155-191. DOI: 10.1007/s11856-006-0006-2

[Ka] M. Kashiwara. Semi-simple holonomic D-modules Topological field theory, primitive forms and related topics (M. Kashiwara et al. editors), 267-271. Progr. Math., vol. 160. Boston: Birkhäuser, 1998.

[Mo] T. Mochizuki. Asymptotic behaviour of tame harmonic bundles and an application to pure twistor D-modules. I. Mem. Amer. Math. Soc. 185 (2007), no. 869, xii+324 pp.

[Sa] C. Sabbah. Polarizable twistor D-modules. Astrisque No. 300 (2005), vi+208 pp.

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