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FINE POLAR INVARIANTS OF MINIMAL SINGULARITIES OF SURFACES

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ABSTRACT. We consider the polar curves $P_{S,0}$ arising from generic projections of a germ $(S,0)$ of complex surface singularity onto $\mathbb{C}^2$. Taking $(S,0)$ to be a minimal singularity of normal surface (i.e., a rational singularity with reduced tangent cone), we give the δ-invariant of these polar curves, as well as the equisingularity-type of their generic plane projections, which are also the discriminants of generic projections of $(S,0)$.

These two pieces of equisingularity data for $P_{S,0}$ are described on the one hand by the geometry of the tangent cone of $(S,0)$, and on the other hand by the limit-trees introduced by T. de Jong and D. van Straten for the deformation theory of these minimal singularities. These trees give a combinatorial device for the description of the polar curve which makes it much clearer than in our previous note on the subject. This previous work mainly relied on a result of M. Spivakovsky. Here, we give a geometrical proof via deformations (on the tangent cone, and what we call Scott deformations) and blow-ups, although we need Spivakovsky's result at some point, extracting some other consequences of it along the way.

INTRODUCTION

The local polar varieties of any germ $(X,0)$ of a reduced complex analytic space were introduced by Lê D.T. and B. Teissier in [17]. In particular, the multiplicities of the *general* polar varieties are important analytic invariants of the germ $(X,0)$.

However, as also emphasised by these authors (see also [23] p. 430–431 and [24]), there is more information to be gained on the geometry of $(X,0)$ by considering not only the multiplicity but the (e.g., Whitney-) equisingularity class of these general polar varieties, which can also be shown to be an analytic invariant.

In this work, we will focus on the polar curves of a two-dimensional germ $(S,0)$.

Our reference on equisingularity theory for space curves will be the mémoire [8]. Of course, as opposed to the case of plane curves, there is no complete set of invariants attached to a germ of a space curve describing its equisingularity class. As a general rule, results on equisingularity beyond the case of plane curves only make sense by considering the constancy of invariants in given families. Here we look at the family of polar curves and will consider the following invariants:

Definition 0.1. Our equisingularity data for a germ of space curve consists of both:
(i) the value of the delta invariant of the curve, and
(ii) the equisingularity class of its generic plane projection.

We recall the definitions of these notions in the text (see Def. 6.1 and Def. 1.2). The constancy of these two invariants in a family of space curves ensures Whitney conditions and actually the stronger *equisaturation condition* (cf. [8]).

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In general, this is still partial information; for example, another interesting invariant for space curves, namely the semi-group of each branch, is completely independent of this equisaturation condition.

The purpose of this paper is to describe the equisingularity data in 0.1 for the *the general polar curve* of a class of normal surface singularities called *minimal*.

These *minimal singularities* were studied in any dimension by J. Kollár in [19]. In the case of normal surfaces, these are also the *rational singularities with reduced fundamental cycle* and were studied by M. Spivakovsky in [21] and T. de Jong and D. van Straten in [15].

For these surfaces, we prove the following:

Proposition $(*)$ **(cf. 5.5 for a more precise statement)**: *the general polar curve is a union of A_{n_i}-plane curves singularities*[1]*, where the n_i's and the contacts between these curves can be deduced from the resolution graph of the surface.*

This information gives in particular a complete description of part (ii) of the data in 0.1, i.e., of the general plane projection of the polar curve, which is also the discriminant of the general projection (the coincidence of these two concepts is a theorem, cf. section 1).

The information on the discriminant was already given in the note [5] as a consequence of a result of Spivakovsky, but the statement there was clumsy.

Here we give a much nicer device that allows us to read directly the information about of the discriminant (or the polar curve as well) from both the information contained in the tangent cone of these singularities and the information given by a graph deduced from the resolution graph, which is precisely the *limit tree* introduced by T. de Jong and D. van Straten in their study of the deformation theory for these minimal singularities (see [15]).

We also provide an inductive proof relying much more on the geometrical properties of these minimal singularities. This proof makes up the core of the paper. It still uses Spivakovsky's theorem, however, mainly through a characterisation of generic polar curves on the resolution which we deduce along the way.

The several plane branches of the polar curve lie in distinct planes in a bigger linear space, and the value of the delta invariant (part (i) in 0.1) gives some (partial) information on the configuration of these planes in the space. We explain how this delta invariant is easily computed from what we call the Scott deformation of the surface, which turns out to give a delta-constant deformation of the polar curve onto bunches of generic configurations of lines.

Organisation of the paper:

In Section 1, we recall the definition of the general polar curve $P_{S,0}$ of a germ of surface $(S,0)$, of the discriminant $\Delta_{S,0}$ of a generic projection of $(S,0)$ onto $\mathbb{C}^2$ and the important result that $\Delta_{S,0}$ is a *generic* projection of the curve $P_{S,0}$.

Section 2 gives the definition of minimal singularities in general, the particular case of normal surfaces, and their characterisation by their dual resolution graph. We then define, in Section 3, a notion of *height* on the vertices of this resolution graph, which was used in other places such as [21] and [15], and corresponds to the number of point blow-ups necessary to let the corresponding exceptional component appear. We also give there our convention in representing dual graphs with $\bullet$ and $*$ and define *reduced dual graphs* to be the ones in which the self-intersections for components of the tangent cone have the minimum absolute value.

In Section 4, we give the description of generic polar curves on a resolution of a minimal singularity as proved by M. Spivakovsky (Thm. 4.2). This result will play the following somehow different roles in the sections following it:

[1]Hence the information about the semi-group of the branches is obvious.

(i) Section 5 explains how, using the full strength of this result, one may derive quite quickly a description of the generic discriminant $\Delta_{S,0}$ (more precisely, of Proposition 5.5 for the polar curve). This sums-up the note [5] in an improved way, and a mistake in an example there is corrected.
(ii) In Section 6, we mention how, using a result of J. Giraud, Theorem 4.2 also permits one, at least theoretically, to deduce the δ-invariant for the general polar curve from the shape of its transform on the minimal resolution. This result is however not useful for concrete computations, for which we use another approach in Section 11.
(iii) In Section 7, we get, as a purely qualitative consequence of (i) and (ii), a characterisation of generic polar curves on the minimal resolution of the singularity $(S, 0)$. This will be the application of Spivakovsky's result we will use in the proof of our main result.

Sections 8 to 11 form the core of the text:

• in Section 8, the polar curve for the tangent cone of a minimal singularity is made geometrically explicit and through the process of deformation onto the tangent cone is also seen as "part" of the polar curve of the singularity.
• in Section 9, we recall what we need from the limit tree construction of de Jong and van Straten. With this,
• in Section 10, we give, and prove, our main theorem giving more details about the information in Proposition $(*)$ page 92 using the limit tree construction and the contribution of the tangent cone.
• in Section 11, we show how a special deformation of minimal singularities has a nice interpretation in our description of polar curves and also gives a nice method for computing the delta invariant of these, completing the information in Def. 0.1 (i).

This leads us to ask: can (part of) the deformation theory of these minimal singularities of surfaces be recovered from their discriminants?

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Several years have passed since I wrote the first version of this paper, and as it turned out, it happened to be useful to other people: I am very grateful to A. Pichon for her interest in this work, for inviting me to submit the paper to this Journal, and to the referee for his/her remarks.

1. Polar invariants of a surface singularity

1.1. **The general polar curve as an analytic invariant.** We recall here the definition of the local polar variety of a germ of surface following [17]:

Let $(S, 0)$ be a complex surface singularity $(S, 0)$, embedded in $(\mathbb{C}^N, 0)$: for any $(N-2)$-dimensional vector subspace D of $\mathbb{C}^N$, we consider a linear projection $\mathbb{C}^N \to \mathbb{C}^2$ with kernel D and denote by $p_D : (S, 0) \to (\mathbb{C}^2, 0)$ the restriction of this projection to $(S, 0)$.

Restricting ourselves to the D such that p_D is finite, and considering a small representative S of the germ $(S, 0)$, we define, as in [17] (2.2.2), the *polar curve* $C(D)$ of the germ $(S, 0)$ for the direction D, as the closure in S of the critical locus of the restriction of p_D to $S \setminus \mathrm{Sing}(S)$. It is a reduced analytic curve.

As explained in loc. cit., it makes sense to say that for an open dense subset of the Grassmann manifold $G(N-2, N)$ of $(N-2)$-planes in $\mathbb{C}^N$, the space curves $C(D)$ are *equisingular*, e.g., in terms of Whitney-equisingularity (or strong simultaneous resolution, but this is the same for families of space curves, cf. [8]). We call this equisingularity class the *general polar curve* for $(S,0)$ embedded in $\mathbb{C}^N$.

One may then compare the general polar curves obtained by two distinct embeddings of the surface into a $(\mathbb{C}^N,0)$ and it turns out that they are still Whitney-equisingular; this is essentially proved in [23] (see p. 430) in a much more general setting (arbitrary dimension and "relative" polar varieties). Summing up, we have:

Theorem 1.1. *The Whitney equisingularity-type of the general polar curve $C(D)$ depends only on the analytic type of the germ $(S,0)$.*

In this paper, following, in a sense, the program in [24], we want to study this invariant $C(D)$ for a special class of surface singularities.

1.2. **The generic discriminant as a derived invariant.** With the same notation as before, we define the *discriminant* Δ_{p_D} as (the germ at 0 of) the reduced analytic curve of $(\mathbb{C}^2,0)$ which is the image of the polar curve $C(D)$ by the finite morphism p_D.

Again, one may show that, for a generic choice of D, the discriminants obtained are *equisingular germs of plane curves*, and that this in turn defines an analytic invariant of $(S,0)$.

We will denote $\Delta_{S,0}$ the equisingularity class of the discriminant of a generic projection of $(S,0)$.

A first advantage of $\Delta_{S,0}$, as a germ of a plane curve, is that its equisingularity class is well-defined in terms of classical invariants such as the Puiseux pairs of the branches and the intersection numbers between branches (cf. e.g., the introduction of [8] for references on this subject).

As it turns out, there is a very nice relationship between the general polar curve and $\Delta_{S,0}$. For this we recall the following:

Definition 1.2. Let $(X,0) \subset (\mathbb{C}^N,0)$ be a germ of reduced curve. Then a linear projection $p : \mathbb{C}^N \to \mathbb{C}^2$ is said to be *generic* with respect to $(X,0)$ if the kernel of p does not contain any limit of bisecants to X (cf. [8] for an explicit description of the cone $C_5(X,0)$ formed by the limits of bisecants to $(X,0)$). For future reference, we will write $BS(X,0)$ for this cone denoted $C_5(X,0)$ in [8]).

Then the equisingularity type of the germ of plane curve $(p(X),0)$ image of $(X,0)$ by such a generic projection is uniquely defined by the saturation of the ring $\mathcal{O}_{X,0}$ (cf. [8]).

We now state the following transversality result (proved for curves on surfaces of $\mathbb{C}^3$ in [9] Theorem 3.12 and in general as the "lemme-clé" in [23] V (1.2.2)) relating polar curves and discriminants:

Theorem 1.3. *Let $p_D : (S,0) \to (\mathbb{C}^2,0)$ be as above, and $C(D) \subset (S,0) \subset (\mathbb{C}^N,0)$ be the corresponding polar curve. Then there is an open dense subset U of $G(N-2,N)$ such that for $D \in U$ the restriction of p to $C(D)$ is generic in the sense of Definition 1.2.*

Definition 1.4. Let us define $P_{S,0}$ to be not the *Whitney-equisingularity* class of the general polar curve as in Thm. 1.1, but the *equisaturation class* of the general polar curve (which may be a smaller class). As we recalled after Definition 0.1, this class is precisely given by the constancy of the invariants there. Then, the foregoing Theorem 1.3 states that $\Delta_{S,0}$ is *the generic plane projection* of $P_{S,0}$.

As stated in the introduction, the goal of this work is to determine $P_{S,0}$ completely.

2. Definition of minimal singularities

We begin with a definition valid in any dimension (following [19] § 3.4):

Definition 2.1. We call a singularity $(X, 0)$ *minimal* if it is reduced, Cohen-Macaulay, and the multiplicity and embedding dimension of $(X, 0)$ fulfill:
i) $\mathrm{mult}_0 X = \mathrm{emdim}_0 X - \dim_0 X + 1$, and
ii) the tangent cone $C_{X,0}$ of X at 0 is reduced.

Considering normal surfaces, one has the following characterisation:

Theorem 2.2. *Minimal singularities of normal surfaces are precisely the rational surface singularities with reduced fundamental cycles (with the terminology of* [2]*).*

Condition (i) follows for any rational surface singularity from Artin's formulas for multiplicities and embedding dimension (cf. [2]). Condition (ii) follows from the fact that the fundamental cycle of rational singularities is also the cycle defined by the maximal ideal. Conversely, the fact that minimal normal singularities are rational is proved in [19] 3.4.9. The proof that "reduced tangent cone" implies "reduced fundamental cycle" is easy (after our Thm. 3.2 or see, e.g., [26] p. 245).

Taking $(S, 0)$ to be a normal surface singularity and $\pi : (X, E) \to (S, 0)$ to be the *minimal* resolution of the singularity, one associates as usual to the exceptional curve configuration $E = \pi^{-1}(0)$ a dual graph Γ where each irreducible component L_i in E is represented by a vertex and two vertices are connected by an edge if, and only if, the corresponding components intersect.

Each vertex x of Γ (we will frequently abuse notation and write $x \in \Gamma$) is given a weight $w(x)$ defined as:

$$w(x) := -L_x^2,$$

where L_x^2 is the self-intersection of the corresponding component L_x on X.

For any rational surface singularity, it is well-known that all the L_i are smooth rational curves and that Γ is a tree. But in general, it takes some computation to check whether a given tree is the dual tree of a rational singularity (cf. [2]).

On the other hand, one reads at first sight from the dual graph that a surface singularity is minimal (cf. [21] II 2.3):

Remark 2.3. Let Γ be any weighted graph. Then, it is the dual graph of resolution of a minimal singularity if, and only if, Γ is a tree and, for each vertex $x \in \Gamma$, one has the following inequality:

$$w(x) \geq v(x),$$

where $v(x)$ denotes the valence of x, i.e., the number of edges attached to x.

3. More about the dual graphs

In the representation of the dual graph Γ of a minimal singularities, we will distinguish between the vertices with $w(x) = v(x)$ and the others.

Notation 3.1. In representing the dual graphs of minimal singularities, we chose to represent with a $\bullet$ the vertices with $w(x) = v(x)$, so that there is no need to mention the weight above them.

On the other hand, we enumerate as $x_1, \ldots, x_k$ the vertices with $w(x_i) > v(x_i)$, and let them be denoted by $*$ on the graph. One should then mention the weights of the (x_i) to define the graph.

In this work, we will pay much attention to the minimal singularities with the property that, for all vertices x_i with $w(x_i) > v(x_i)$, one has, in fact, the equality $w(x_i) = v(x_i) + 1$.

Let us call *reduced* the graphs with this property; it is then clear that in representing these dual graphs, there is no longer any need to mention the weights.

For example, saying that the graph in Figure 1 is *reduced* amounts to saying that

$$w(x_1) = w(x_n) = 2 \text{ and } w(x_i) = 3 \text{ for } 1 < i < n$$

(and the vertices with $\bullet$'s all have weight two here).

FIGURE 1.

The geometrical meaning of this distinction between vertices comes from:

Theorem 3.2 (Tyurina, cf. [25]). *Let $(S, 0)$ be a rational surface singularity and let $\pi : (X, E) \to (S, 0)$ be its minimal resolution. Let $b : S_1 \to S$ be the blow-up of 0 in S.*

Then there is a morphism $r : X \to S_1$ such that $\pi = b \circ r$ and a component L_i of $E = \pi^{-1}(0)$ is contracted to a point by r if, and only if, the intersection $(L_i \cdot Z) = 0$, where Z is the fundamental cycle.

Of course, the components of E which are not contracted by r are the strict transform by r of the components of the $\mathbb{P}(C_{S,0})$ appearing on S_1.

When $(S, 0)$ is a minimal singularity, the fundamental cycle is $Z = \sum_{x \in \Gamma} L_x$ and hence, for a given vertex $y \in \Gamma$, the intersection $(L_y \cdot Z)$ is just $v(y) - w(y)$.

This should justify:

Definition 3.3. Let $(S, 0)$ be a minimal normal surface singularity and Γ be the dual graph of its minimal resolution.

We will say that a vertex x in Γ has *height one* if $w(x) > v(x)$, which from the foregoing remarks means that the corresponding component L_x corresponds to a component of (the proj of) the tangent cone $C_{S,0}$. Hence we will denote by $\Gamma_{TC} = \{x_1, \ldots, , x_n\}$ the set of these vertices.

Then, we define the height of any vertex x in Γ as the number s_x defined by:

$$s_x := \text{dist}(x, \Gamma_{TC}) + 1,$$

where dist is the distance on the graph (number of edges on the geodesic between two vertices).

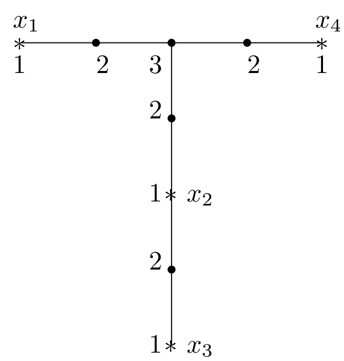
The reader should check that this height corresponds to the number of blow-ups necessary to make the corresponding component "appear".[2] The notation s_x here comes from [21] II 5.1 and was the one used in the previous work [5].

Example 3.4. As an example, we indicate the heights on the graph in Figure 2, where the (x_i) are, as before, the vertices of height one (with $*$'s):

We will also need the following:

[2]This latter notion is studied more systematically for any rational singularity as "desingularisation depth" in [18]; of course in this general case, it is not given directly from a distance!

FIGURE 2. Minimal graph with the heights for the vertices.



Definition 3.5. Let Γ be a minimal graph. The connected components Γ_i (for $i = 1, \dots, r$) of $\Gamma \setminus \Gamma_{TC}$ are called the *Tyurina components* of Γ.

Hence, Theorem 3.2 states that the blow-up S_1 of $(S, 0)$ has exactly r singularities (S_1, O_i) which are minimal singularities with dual resolution graph Γ_i.

4. A result of Spivakovsky

To state this result, we introduce further terminology:

Let $\pi : (X, E) \to (S, 0)$ be the minimal resolution of the singularity $(S, 0)$, where $E = \pi^{-1}(0)$ is the exceptional divisor, with components L_i. A *cycle* will be by definition a divisor with support on E, i.e., a linear combination $\sum a_i L_i$ with $a_i \in \mathbb{Z}$ (or $a_i \in \mathbb{Q}$ for a $\mathbb{Q}$-cycle).

Let Γ be the dual graph of the minimal resolution π and, for each vertex x, let s_x denote the *height* defined in Def. 3.3.

Definition 4.1. Let then x, y be two adjacent vertices on Γ; the edge (x, y) in Γ is called a *central arc* if $s_x = s_y$. A vertex x is called a *central vertex* if there are at least two vertices y adjacent to x such that $s_y = s_x - 1$ (cf. [21]).

We then define a $\mathbb{Q}$-cycle Z_Ω on the minimal resolution X of $(S, 0)$ by:

$$Z_\Omega = \sum_{x \in \Gamma} s_x L_x - Z_K, \tag{1}$$

where Γ is the dual graph, and Z_K is the numerically canonical $\mathbb{Q}$-cycle [3].

The theorem from [21] (Theorem 5.4) is now:

Theorem 4.2. *Let* $(S, 0)$ *be a* minimal *normal surface singularity. There is a open dense subset* U' *of the open set* U *of Theorem 1.3, such that, for all* $D \in U'$*, the strict transform* $C'(D)$ *of* $C(D)$ *on* X*:*
a) is a multi-germ of smooth curves intersecting each component L_x *of* E *transversally in exactly* $-Z_\Omega . L_x$ *points,*

[3]Uniquely defined by the condition that, for all $x \in \Gamma$, $Z_K . L_x = -2 - L_x^2$, since the intersection product on E is negative-definite.

b) goes through the point of intersection of L_x and L_y if and only if $s_x = s_y$ (point corresponding to a central arc *of the graph). Furthermore, the curves $C'(D)$, with $D \in U'$ do not share other common points (base points) and these base points are simple, i.e., the curves $C'(D)$ are separated when one blows up these points once.*

Referring to loc. cit. for unexplained terminology, let us make the following observation:

Remark 4.3. Blowing-up once the base points referred to in b) above, one gets a resolution X_N of the Nash blow-up of the germ $(S, 0)$. The map from X_N to the normalized Nash blow-up $N(S)$ is simply the contraction of the exceptional components which are not intersected by a branch of the generic polar curve.

5. First description of the polar curve and the discriminant

This section essentially describes the results obtained in [5] in an improved form. We refer to this note (Section 3) for the proofs of the following lemmas:

Lemma 5.1. *Let $(S, 0)$ be a* minimal *normal surface singularity and $\pi : X \to (S, 0)$ its minimal resolution. It is known that π is (the restriction to S of) a composition $\pi_1 \circ \cdots \circ \pi_r$ of point blow-ups. Then, this composition of blow-ups is also the minimal resolution of the generic polar curve $C(D)$ for $D \in U'$ as in Theorem 4.2.*

The following is a slightly more precise version of loc. cit. Lem. 3.2:

Lemma 5.2. *For $D \in U'$ as in Theorem 4.2, the polar curve $(C(D), 0)$ on $(S, 0)$ is a union of germs of curves of multiplicity two. In particular, it has only smooth branches and branches of multiplicity two, the latter being exactly those for which the strict transform goes through a central arc as in b) of Theorem 4.2.*

Let us now make a perhaps not so standard definition:

Definition 5.3. Let $(C_1, 0)$ and $(C_2, 0)$ be two analytically irreducible curve germs in $(\mathbb{C}^N, 0)$. We will *hereafter* call *contact* between the C_i simply the number of point blow-ups necessary to separate these two branches.

For the description of the polar curve, just recall that *an A_n-curve* is a curve analytically isomorphic to the plane curve defined by $x^2 + y^{n+1} = 0$:

Proposition 5.4. *Let $(S, 0)$ be a minimal surface singularity and $C = C(D)$ be a generic polar curve corresponding to D in the open set U' of thm. 4.2.*

Then, if $C = \cup_i C_i$ is the decomposition of C into analytic branches, denote by L_{C_i} the irreducible exceptional component on the minimal resolution of S which intersects the strict transform of C_i. It is unique except in the case of central arcs. In this case, just choose one between the two intersecting components. Then:
(i) *The contact between C_i and C_j in the sense of Def. 5.3 above is the minimum height in the chain between L_{C_i} and L_{C_j} (cf. Def. 3.3).*
(ii) *We may write rather C as a union of $C = \bigcup C_i$ of curves of multiplicity two by taking by pairs branches intersecting the same exceptional component on X that we will now denote L_{C_i}.*

Then, each C_i is a A_{n_i}-curve, where the number n_i equals $2.s(L_{C_i})$ if C_i goes through a central arc, and $2.s(L_{C_i}) - 1$ otherwise (which comprises the case of central vertices and components of height s equal to one).

We may obviously define the contact between these A_{n_i}-curves just by taking one branch in each, so that it is still given by (i).

Proof. The statement about the contact in (i) follows from Lemma 5.1. The first statement in (ii) is Lemma 5.2.

Any curve of multiplicity two is a A_n-curve, see, e.g., [4] p. 62. The statement about the n_i follows from (i) just like the statement about the contacts. □

The result in Prop. 5.4 gives a complete description of the equisingularity class of the discriminant plane curve in $(\mathbb{C}^2, 0)$ using Theorem 1.3 [4]:

Proposition 5.5. *The discriminant $\Delta_{p_D} = p_D(C(D))$ has exactly the same properties as the polar curve $C(D)$ in Prop. 5.4. This describes the generic discriminant $\Delta_{S,0}$ as a union of A_{n_i}-curves with the n_i and the contacts described in 5.4.*

Proof. The curves C_i in Prop. 5.4, being plane curves, are their own generic plane projections. Hence by Thm. 1.3, the image Δ_{p_D} of $C(D)$ by the *generic projection* p_D decomposes as the same union of A_{n_i}-curves.

We give here a direct argument to prove that the contact (in sense of Def. 5.3) between the branches in Δ_{p_D} is the same as the one in $C(D)$ (in [5], we invoked a bilipschitz invariance which is perhaps not obvious with our definition of contact).

Let us write down equations for a special case: considering a pair C_1, C_2 of branches in $C(D)$, we embed $C_1 \cup C_2$ into a $(\mathbb{C}^3, 0)$ and suppose there are coordinates so that C_1 is parameterised by $(x = t^{\varepsilon_1}, y = t^n, z = 0)$ and C_2 is parameterised by $(x = t^{\varepsilon_2}, y = 0, z = t^m)$, with $\varepsilon_i = 1$ or 2.

We then leave it to the reader to verify that the projection defined by $(x, y + z)$ is transverse to the cone of bisecants $BS(C, 0)$ of Def. 1.2 and that the contact in our sense is preserved.

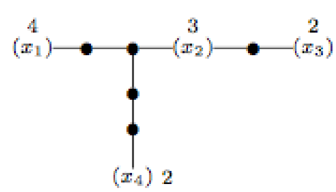
In the general case, the contact between C_1 and C_2 may be smaller, but the results remain valid with another parameterisation of C_2.

□

The foregoing description of the discriminant still involves the computation of the number of branches on each central vertex by Spivakovsky's formula. We will describe a much better and condensed one in Section 10, which does not involve any computation and is geometrically more significant. Before, the author would like to make amends to the readers of [5] for a mistake in the following:

Example 5.6 (Correct version of Example 1 in [5])**.** Consider $(S, 0)$ with dual graph Γ as in Figure 3, where, following the convention of Section 3, the • denote vertices with $w(x) = v(x)$, and the others form $\Gamma_{TC} = \{x_1, \ldots, x_4\}$ with the weights indicated on the graph.

FIGURE 3.



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[4] This is an equivalent, but more simply expressed, version of the statement in [5], Cor. 4.3.

The branches of the polar curve going through the components of Γ_{TC} are just four branches going through L_{x_1}, which gives in the equisingularity class $\Delta_{S,0}$ four distinct lines through the origin with contact one with any other branch of $\Delta_{S,0}$.

Then, we have two central vertices (of heights 3 and 2) and a central arc (with boundaries of height 2), which give, respectively, an A_5, an A_3, and an A_4-curve from Prop. 5.4 and 5.5 above.

The contact between the A_5 and the A_4 is *two* (and not 3 as claimed in loc. cit.) and their contact with the other A_3 is one.

Hence, using coordinates, a representative of the equisingularity class of $\Delta_{S,0}$ can be chosen to be:

$$\underbrace{(x^4+y^4)}_{\text{The two } A_1\text{'s}}(x^2+y^5)\underbrace{(x+y^2+iy^3)(x+y^2-iy^3)}_{\text{The } A_5}(y^2+x^4)=0.$$

6. The delta invariant of the polar curve

Definition 6.1. Let $(C,0)$ be a germ of a reduced complex curve. Let $n:\overline{C}\to(C,0)$ be its normalisation map, which provides a finite inclusion of the local ring $\mathcal{O}_{C,0}$ into the semi-local ring $\mathcal{O}_{\overline{C}}$.

The δ-invariant of $(C,0)$ is by definition $\delta(C,0):=\dim_{\mathbb{C}}\mathcal{O}_{\overline{C}}/\mathcal{O}_{C,0}$.

In the paper [14], J. Giraud gives a way to compute $\delta(C,0)$ for any curve on a *rational* surface singularity $(S,0)$ if one knows a resolution of the surface singularity where the strict transform C' of C is a multi-germ of smooth curves.

To quote this result, we need the following lemma, proved in loc. cit. 3.6.2:

Lemma 6.2. *Let $p:(X,E)\to(S,0)$ be a resolution of a normal surface singularity $(S,0)$, with $E=\pi^{-1}(0)=\cup_i E_i$. Let $D=\sum_i a_iE_i$ be a $\mathbb{Q}$-cycle on X.*

There exists a unique $\mathbb{Z}$-cycle $V=\sum_i \alpha_iE_i$ with the property that the intersection $(V\cdot E_i)$ is less than or equal to $(D\cdot E_i)$ for all i, and is a minimum among cycles with this property.

This $\mathbb{Z}$-cycle will be denoted as $\lfloor D\rfloor$.

(In the previous lemma, "minimum" means that any other $\mathbb{Z}$-cycle with this property has the form $\lfloor D\rfloor+W$ with W a cycle with non-negative coefficients.)

In the situation of Lemma 6.2, let us associate to any curve $(C,0)\subset(S,0)$ a $\mathbb{Q}$-cycle Z_C uniquely defined by the condition that, for all irreducible component E_i of E, the intersection number $(E_i\cdot Z_C)$ equals $(E_i\cdot C')$, where C' denotes the strict transform of C on X. We may then quote loc. cit. Cor. (3.7.2):

Theorem 6.3. *Let $p:(X,E)\to(S,0)$ be a resolution of a* rational *surface singularity. Let $(C,0)$ be a germ of a reduced curve on $(S,0)$, such that, denoting by C' the strict transform of C on X, C' is a multi-germ of smooth curves on X.*

Then, using the $\mathbb{Q}$-cycle Z_C associated C in the way defined above, and letting

$$D_C:=Z_C+\lfloor -Z_C\rfloor,$$

one has the following formula[5]*:*

$$\delta(C,0)=-\frac{1}{2}(Z_C\cdot(Z_C+Z_K))+\frac{1}{2}(D_C\cdot(D_C+Z_K)).$$

[5]Beware that, in loc. cit., the + before the second term in the right hand-side of the corresponding formula (5) is not properly printed, yet it *is* a plus. One should also read formula (3) there as $\underline{D}:=e(D_s)-\lceil e(D_s)\rceil=e(D_s)+\lfloor -e(D_s)\rfloor$, which agrees with my definition for D_C.

Thanks to Spivakovsky's theorem 4.2, we may apply the foregoing to a general polar curve of a minimal singularity $(C, 0) \subset (S, 0)$, X the minimal resolution of $(S, 0)$, and $Z_C = -Z_\Omega$. As a corollary to these two theorems, we have:

Corollary 6.4. *Let $(S, 0)$ be a minimal singularity of a normal surface (hence, rational by Thm. 2.2). The δ-invariant of the generic polar curve is a topological invariant of $(S, 0)$, i.e., depends only on the data of the weighted resolution graph.*

Applying the formula in 6.3 to get δ for the polar curve in concrete cases leads to huge computations, except in some simple examples:

Example 6.5. Let $(S, 0)$ be the singularity at the vertex of the cone over a rational normal curve of degree n. It is the minimal singularity whose (dual) resolution graph has only one vertex of weight n. Assume that $n \geq 3$. Check that, using E to denote the irreducible exceptional divisor, one has $Z_\Omega = (2n-2)/nE$, $Z_K = -(n-2)/nE$, $\lfloor Z_\Omega \rfloor = 2E$ and hence $\delta(C, 0) = 3n - 6$.

In Section 8, we will obtain the result of the foregoing example (and more) from a geometric argument, with no use of the theorems above. The problem of computing δ for the general polar curve of any minimal singularity is solved in 11.4.

7. A characterisation of the generic polar curve in a resolution

As a consequence of the results of Sections 5 and 6, we get the following characterisation for generic polar curves on the minimal resolution of the surface:

Theorem 7.1. *Let $(S, 0)$ be a minimal normal surface singularity, and X the minimal resolution of $(S, 0)$. Let $C(D)$ be* any polar curve *of $(S, 0)$ with the property that its strict transform $C'(D)$ on X is exactly as depicted in Thm. 4.2.*

Then $C(D)$ is a generic polar curve $P_{S,0}$ *as defined in Def. 1.4, i.e., has the generic invariants defined in 0.1 of the introduction.*

Proof. The description of Prop. 5.4 rests only on the the shape of the polar curve in the resolution X, and gives in particular the datum (ii) in 0.1 (cf. Def 1.4 and Prop. 5.5). Giraud's theorem 6.3 gives the value of the delta invariant also from the data of the resolution. Considering the linear system of polar curves, our special polar curve is now equisingular in the sense of Def. 0.1 to the generic polar curve. □

Remark 7.2. We explained in [6] how such characterisations of "general" curves on a resolution may be useful; here, it will be used in Rem. 10.6.

We also need the following inductive property for which we will use[6] the explicit form of the cycle Z_Ω in (1) before Spivakovsky's Thm. 4.2:

Proposition 7.3. *Let $(S, 0)$ be a minimal singularity of a normal surface, with dual resolution graph Γ. Let S_1 be the blow-up of $(S, 0)$ at 0 and O_i a singular point of S_1. Let $\Gamma_i \subset \Gamma$ be the Tyurina component corresponding to O_i as in Def. 3.5. Let Z_{Ω_i} be the cycle associated to Γ_i as Z_Ω is associated to Γ in Thm. 4.2.*

Then, for every vertex $x \in \Gamma_i$, the corresponding component L_x on X satisfies the following intersection property:

$$(Z_\Omega \cdot L_x) = (Z_{\Omega_i} \cdot L_x). \tag{2}$$

This means that the corresponding component L_x is intersected by exactly the same number of branches of the generic polar curve for $(S, 0)$ or for (S_1, O_i), and the central arcs in Γ_i are obviously also central arcs in Γ.

[6] Ideally, we would have liked not to do so; see precisely (a) of the proof of this proposition.

Proof. Although the assertion in (2) follows easily from the explicit form of the cycles Z_Ω and Z_{Ω_i} (cf. (1) p. 97), we distinguish between:

(a) the components L_x with $w(x) - v_{\Gamma_i}(x) \geq 2$. Since $x \in \Gamma_i$, $w(x) = v_\Gamma(x)$; hence the property in Γ_i implies that x is a central vertex in Γ. Hence L_x bears components of the strict transform of the general polar curve of $(S, 0)$, and here we know no method other than computing to prove (2).

(b) the central components L_x in Γ_i (central vertex or boundary of a central arc). Then, it is also central in Γ, and we believe (2) should be understood without any reference to the cited formula, using the following remark in [21], p. 459 (first lines): "in the neighbourhood of L_x, $\tilde{\Omega}$ is generated by sections whose zero set is contained in the exceptional divisor near L_x". □

8. The contribution of the tangent cone in the polar curve

In Section 5, we said that $P_{S,0}$ was formed by A_n-curves. Here we explain how the bunches of A_1-curves arise, and will be more precise about their geometry.

8.1. Discriminant and polar curve for cones over Veronese curves.

Remark 8.1. Let $(S, 0)$ be the singularity of the cone over the rational normal curve of degree m in $\mathbb{P}^m_\mathbb{C}$, whose dual graph has just one vertex, with weight m.

Denoting by P_m the polar curve for a generic projection of $(S, 0)$ onto $(\mathbb{C}^2, 0)$, it is just the cone over the critical set of the projection of the rational normal curve onto $\mathbb{P}^1_\mathbb{C}$, which is a set of $2m - 2$ distinct points by the Hurwitz formula.

Hence we know that here P_m is given by $(2m - 2)$ lines in $(\mathbb{C}^{m+1}, 0)$ with:
i) δ-invariant $3m - 6$ as computed in Example 6.5, from Giraud's formula.
ii) obviously a set of $2m - 2$ distinct lines in $(\mathbb{C}^2, 0)$ as generic plane projection, denoted δ_m.

We can say more on the geometry of P_m in this case, and re-find the value of δ:

Lemma 8.2. *The general polar curve P_m of the singularity of a cone over a Veronese curve of degree $m \geq 3$ is a set of $(2m-2)$ lines in $(\mathbb{C}^{m+1}, 0)$, which has the generic (i.e, minimum) value of the δ-invariant for* any set *of $2m - 2$ lines in $(\mathbb{C}^{m+1}, 0)$, and this value is $3m - 6$.*

Proof. (a) We will denote by $V = v_m(\mathbb{P}^1)$ the rational normal curve of degree m in $\mathbb{P}^m_\mathbb{C}$ and by $G_p(m-2, m)$ the Grassmann manifold of subspaces of codimension two in this $\mathbb{P}^m_\mathbb{C}$, and consider the map

$$G_p(m - 2, m) \to \mathrm{Hilb}^{2m-2}_V$$

onto the Hilbert scheme parameterising the set of $2m - 2$-points in V, which assign to each Λ the critical subscheme of the projection along Λ.

Using a result of H. Flenner and M. Manaresi (in [11] 3.3-3.5), this map is generically finite, and since both spaces have dimension $2m - 2$ and the target space is irreducible, the image of this map is dense.

(b) Now, from a result of G.M. Greuel in [13] (3.3), a set of r-lines through the origin in $\mathbb{C}^{m+1}$, corresponding to a set $p_1, \ldots, p_r$ of points in $\mathbb{P}^m_\mathbb{C}$, has the generic δ invariant, if for all d in some bounded set of integers, their images $v_d(p_1), \ldots, v_d(p_r)$ by the corresponding Veronese embedding $v_d : \mathbb{P}^m_\mathbb{C} \to \mathbb{P}^{N_d}_\mathbb{C}$ span a projective space of maximal dimension.

If we take $V \subset \mathbb{P}^m_\mathbb{C}$ to be a Veronese curve, one may always find such generic sets of points on V since by composing the Veronese embeddings in Greuel's condition with the Veronese embedding defining V, this amounts to a genericity condition for points in $\mathbb{P}^1_\mathbb{C}$.

Hence, there is a non-trivial open subset $U \subset \mathrm{Hilb}_V^{2m-2}$ with the property that the cone over this set of points has the minimum delta invariant. Applying (a) gives that these points actually occur as critical locus.

(c) A formula for the delta invariant for such a generic configuration of r lines in $\mathbb{C}^n$ is given by Greuel in loc. cit. We leave it to the reader to check that it gives $3m-6$ in our situation. □

8.2. **Geometry of the tangent cone of a minimal singularity.**

Remark 8.3. Let $(S,0)$ be a minimal normal surface singularity with embedding dimension N, and $C_{S,0}$ be its tangent cone in $(\mathbb{C}^N,0)$.

Then, if $\mathbb{P}:\mathbb{C}^N\setminus\{0\}\to\mathbb{P}_{\mathbb{C}}^{N-1}$ denotes the standard projection, the projective curve $\mathbb{P}(C_{S,0})$ is a connected, non-degenerate[7] curve of minimal degree in $\mathbb{P}_{\mathbb{C}}^{N-1}$. Indeed, condition (i) in Def. 2.1 immediately passes to $\mathbb{P}(C_{S,0})$.

It then follows by a standard argument (cf., e.g., [3], p. 67–68) that each of its irreducible components is a rational normal curve of a linear subspace of $\mathbb{P}_{\mathbb{C}}^{N-1}$.

Let Γ be the dual graph of the minimal resolution of $(S,0)$. From Tyurina's Thm. 3.2 and the remarks following it, an irreducible component L_{x_i} of $\mathbb{P}(C_{S,0})$ corresponds to a vertex x_i with $w(x_i)>v(x_i)$ in Γ and it is easy to compute that the degree $m(x_i)$ of the rational normal curve L_{x_i} is precisely $w(x_i)-v(x_i)$.

Conclusion 8.4. Hence the tangent cone $C_{S,0}$ is embedded in $(\mathbb{C}^N,0)$ as a union of cones over rational normal curves of degree m_i intersecting along singular lines.

8.3. **Scheme-theoretic critical spaces and discriminants.** To study deformations of polar curves and discriminants, we need a scheme-theoretic definition for these objects, introduced by B. Teissier in [22] through the use of Fitting ideals.

Further, when non-isolated singularities occur, the right objects for deformations are not polar curves but critical spaces, which also contain the singular locus of the surface.

Definition 8.5. We call $C^F(S,0)$ the *critical space* of a generic projection p of a surface $(S,0)$ onto $(\mathbb{C}^2,0)$ as defined by the Fitting ideal $F_0(\Omega_p)$ in $\mathcal{O}_{S,O}$ and $\Delta^F_{S,0}$ its image as defined by $F_0(p_*(\mathcal{O}_{C^F(S,0)}))$ in $\mathcal{O}_{\mathbb{C}^2,0}$.

Beware, $C^F(S,0)$ always contains $\mathrm{Sing}(S)$, which, if $\mathrm{Sing}(S)$ is not reduced to $\{0\}$, makes $C^F(S,0)$ even set-theoretically bigger than the polar curve $P_{S,0}$ defined in Section 1. But by a Bertini type theorem, one gets that:

Remark 8.6. (i) For a generic projection p of any reduced surface $(S,0)$, the intersection of the $C^F(S,0)$ with $S\setminus\mathrm{Sing}(S)$ is reduced, and hence the divisorial part $\operatorname{div} C^F(S,0)$ is formed of the generic polar curve $P_{S,0}$ and of (possibly non reduced) components of $\mathrm{Sing}(S)$. The same is true for $\operatorname{div}\Delta^F_{S,0}$.

(ii) In particular, if $(S,0)$ is an isolated singularity $\operatorname{div} C^F(S,0)$ and $\operatorname{div}\Delta^F_{S,0}$ coincide with the $P_{S,0}$ and $\Delta_{S,0}$ defined in Section 1.[8]

Lemma 8.7. *Let $(S,0)$ be a minimal normal surface singularity, with tangent cone $C_{S,0}$, and Γ the dual graph of the minimal resolution of S. Recall that we then denote Γ_{TC} the set of vertices x_i in Γ with $w(x_i)>v(x_i)$.*

[7] This means not contained in a hyperplane of $\mathbb{P}_{\mathbb{C}}^{N-1}$.

[8] But as explicitly proved in [7] 3.5.2, the Fitting critical curves and discriminants for minimal singularities do have embedded components as soon as the multiplicity is bigger than 3.

Here, we denote by $m(x_i)$ the difference $w(x_i)-v(x_i)$, and we have just seen that $C_{S,0}$ is made of cones over rational normal curves of degree $m(x_i)$ intersecting along singular lines. Hence, considering a projection of $\mathbb{C}^N$ onto $\mathbb{C}^2$ restricted to $C_{S,0}$ we get that:

$$\operatorname{div} C^F_{C_{S,0}} = \bigcup_{x_i \in \Gamma_{TC}} P_{m(x_i)} \cup \textit{the singular lines in } C_{S,0} \textit{ with some multiplicity,}$$

and

$$\operatorname{div} \Delta^F_{C_{S,0}} = \bigcup_{x_i \in \Gamma_{TC}} \delta_{m(x_i)} \cup \textit{non reduced lines,}$$

where the P_m and δ_m were defined in Rem. 8.1 and Lem. 8.2.

8.4. **Deformations of polar curves and discriminants.** We first recall what we need from the construction of the deformation of $(S,0)$ onto its tangent cone $C_{S,0}$, as described in [12], Chap. 5: let M be the blow-up of $(0,0)$ in $S\times\mathbb{C}$, and $\rho : M \to \mathbb{C}$ the flat map induced by composing the blow-up map with the second projection. One then shows that: for all $t \neq 0$, the fiber $M_t := \rho^{-1}(t)$ is isomorphic to S and M_0 is the sum of the two divisors on M, namely $\mathbb{P}(C_{S,0}\oplus 1)+S_1$, where S_1 stands for the blow-up of S in 0. To this deformation, we will apply the following:

Proposition 8.8. *Let $\rho : X \to \mathbb{D}$ be a flat map, with a section σ so that the germs $(X_t,\sigma(t))$ are isolated singularities for $t \neq 0$, X_0 is a reduced possibly non-isolated singularity, and $\dim X_t$ is two for all t.*

Then, reducing the disk $\mathbb{D}$, one may find a projection $p : X \to \mathbb{C}^2 \times \mathbb{D}$ compatible with ρ so that, for all $t\in\mathbb{D}$, the polar curve of $p_t : X_t \to \mathbb{C}^2\times\{t\}$ is generic, and its image is also the generic discriminant $\Delta_{X_t,0}$ as defined in Section 1.

The proposition above is well-known to specialists and may be deduced from more general results (see also [1], Th. 3.1).

Applying the proposition to the foregoing deformation $\rho : M \to \mathbb{D}$ gives that $P^F_{S,0}$ deforms onto $P^F_{C_{S,0}}$ and the same statement for the Fitting discriminants. The description of the generically reduced branches of $P^F_{C_{S,0}}$ in Lem. 8.7 now implies:

Corollary 8.9. *Let $(S,0)$ be a minimal singularity, with notation as in Lem. 8.7, and let us denote by L_{x_i} the component of $\mathbb{P}(C_{S,O})$ corresponding to $x_i \in \Gamma_{TC}$. Then:*
(i) *The generic polar curve $P_{S,0}$ of $(S,0)$ contains a union:*

$$P_{TC} = \bigcup_{x_i\in\Gamma_{TC}} P_{m(x_i)}$$

of generic configuration of lines $P_{m(i)}$ as described in Lem. 8.2. The bunch $P_{m(x_i)}$ in $P_{TC}\subset P_{S,0}$ is by definition the set of branches of $P_{S,0}$ which are deformed onto the (scheme-theoretically) smooth branches $P_{m(x_i)} \subset P^F_{C_{S,O}}$ of Lem. 8.7.
(ii) *The same statement is true for the generic discriminant $\Delta_{S,0}$ of $(S,0)$:*

writing $\Delta_{TC} = \cup_{x_i\in\Gamma_{TC}}\delta_{m(x_i)}$, with $\delta_{m(x_i)}$ standing for $2m(x_i)-2$ lines in $(\mathbb{C}^2,0)$, we may just as well say that these smooth branches with pairwise distinct tangents just form a Δ_{TC} part in $\Delta_{S,0}$.
(iii) *Denote by S_1 the blow-up of $(S,0)$. The strict transforms on S_1 of the smooth curves in $P_{m(x_i)} \subset P_{S,0}$ intersect the exceptional divisor only in L_{x_i} and this intersection is transverse.*

Proof. (i) A curve deforming onto a smooth curve is certainly smooth, hence locally a line. In Lem. 8.2, we said the P_m-curves are characterised by the minimality of δ. By semi-continuity of this δ applied to the family deforming onto $P_{m(x_i)} \subset P_{C_{S,O}}$, we get the full conclusion for the curves in $P_{S,0}$. (ii) is follows directly from (i).

(iii) Let us denote by ρ the deformation onto the tangent cone as recalled at the beginning of Section 8.4. The fiber $\rho^{-1}(0)$ contains the blow-up S_1 of $(S,0)$ intersecting $\mathbb{P}(C_{S,O} \oplus 1)$ in $\mathbb{P}(C_{S,O})$. Since the lines $P_{m(x_i)}$ in $C_{S,O}$ are transverse to the Veronese curve L_{x_i} in the $\mathbb{P}(C_{S,O})$ at infinity, it also follows that the strict transforms of the curves in $P_{m(x_i)} \subset P_{S,0}$ are transverse to the corresponding exceptional component $L_{x_i} \subset \mathbb{P}(C_{S,O})$ on the blow-up S_1. □

9. Limit trees

We proceed to identify the remaining part in $P_{S,0}$ besides the P_{TC}-part just exhibited. The following *limit tree* construction introduced by T. de Jong and D. van Straten in [15] will turn out to be very relevant to this description. Precisely, using the height function we defined in 3.3, one finds in loc. cit. (1.13):

Definition 9.1. Let Γ be the dual graph of a minimal resolution of a minimal singularity of a normal surface. A *limit equivalence relation* $\sim$ is an equivalence relation on the vertices of Γ satisfying the following two conditions:

(a) Vertices x with height $s_x = 1$, i.e., with $w(x) > v(x)$, belong to different equivalence classes,

(b) for every vertex x in Γ with height $s_x = k+1$, $k \geq 1$, there is exactly *one vertex* y connected to x with height $s_y = k$ and $y \sim x$.

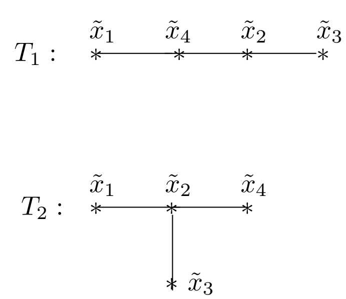
Then, the tree $T = \Gamma/\sim$ is a called a *limit tree* associated to Γ.

It is clear that any equivalence class contains exactly one vertex x_i of height one, so that we denote these equivalences classes as vertices $\widetilde{x}_i$ in T.

In fact, we only make this construction in the particular case of minimal singularities with *reduced graphs* in the sense of notation 3.1, so that the definition above really correspond to the definition in loc. cit.[9]

Starting with Γ as in Example 3.4 , one may associate non-isomorphic limit trees to the same reduced graph Γ, depending on the equivalence classes chosen, namely:

FIGURE 4. Two distinct limit trees for the dual graph in Figure 2.



Notation 9.2. For any pair x, y of vertices on the dual graph Γ, we denote by $C(x,y)$ the (minimal) chain on Γ joigning them (including the end points). This is unique since Γ is a tree.

[9]For the non-reduced case, one has to use the extended resolution graph of loc. cit. to build the limit tree, to really get a bijection between vertices of T and element of the set $\mathcal{H}$ considered in loc. cit. But, again, we will not use this.

We define the length $l(x,y)$ to be the number of vertices on $C(x,y)$ and the overlap $\rho(x,y;z)$ as the number of vertices on $C(x,z) \cap C(y,z)$.

As in [15], we attach to a limit tree T the following data:

- for any edge $(\tilde{x},\tilde{y})$ of T, the length $l(x,y)$, where x,y are the corresponding vertices of height one in the resolution graph Γ,
- for any pair of adjacent edges $(\tilde{x},\tilde{z})$ and $(\tilde{z},\tilde{y})$ in T, the overlap $\rho(x,y;z)$.

We use the notation (T,l,ρ) for the data above. In loc. cit. Lemma (1.16), it is shown that these data determine uniquely the resolution graph Γ.

10. Description of the polar curve using the limit tree

The following is our main result; we formulate it for the polar curve $P_{S,0}$, reminding the reader that this implies the analogous statements for the discriminant $\Delta_{S,0}$:

Theorem 10.1. *Let $(S,0)$ be a minimal singularity of a normal surface. Let Γ be the dual graph of the minimal resolution of S.*

Let Γ^r be the reduced graph associated to Γ in the sense of notation 3.1, i.e., the same graph with the weights of the x_i of height one reduced to $v(x_i)+1$, and let $(S^r,0)$ be a minimal singularity with dual resolution graph Γ^r.

Then the generic polar curve $P_{S,0}$ decomposes into:

$$P_{S,0} = P_{TC} \cup P_{S^r}$$

where the contact between any line in P_{TC} and any branch in $P_{S^r,0}$ is one and P_{TC} was described in Cor. 8.9 as the "contribution of the tangent cone".

Let T be the limit tree for Γ^r, as defined in Section 9 and (T,l,ρ) the set of data (length and overlap) associated to it at the end of that section.

These data give the following easy description of P_{S^r} (as a union of A_n-curves):

- *each edge $(\tilde{x_i},\tilde{x_j})$ in the limit tree T defines exactly one $A_{l_{i,j}}$-curve in P_{S^r}, where $l_{i,j}$ stands for $l(\tilde{x_i},\tilde{x_j})$.*
- *For each pair of adjacent edges $(\tilde{x_i},\tilde{x_j})$ and $(\tilde{x_j},\tilde{x_k})$, the contact (Def. 5.3) between the corresponding $A_{l_{i,j}}$ and $A_{l_{j,k}}$-curves in P_{S^r} is exactly the overlap $\rho(i,k;j)$.*
- *For non adjacent edges $(\tilde{x_i},\tilde{x_j})$ and $(\tilde{x_k},\tilde{x_l})$, the contact between the corresponding $A_{l_{i,j}}$ and $A_{l_{k,l}}$-curves in P_{S^r} is the* minimum *of the contacts between adjacent edges on the chain joining them.*

Let us first illustrate this on the following:

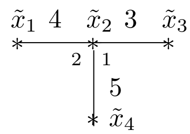
Example 10.2. (i) For a minimal singularity $(S,0)$ with dual graph as in Figure 2, p. 97, using any of the limit trees in Figure 4, we get: $P_{S,0} = A_5 \cup A_5' \cup A_3$, with contact three between the two A_5 and contact one between the A_5's and the A_3.
(ii) For Example 5.6, the description of the discriminant was already given there. It is now more directly seen from the limit tree in Figure 5 given below together with the data (l,ρ), where the lengths l are put above the edges and the ρ as smaller numbers in-between a pair of edges (following the same convention as in [15] (1.19)).

The rest of this section is devoted to the proof of Thm. 10.1 above.

First we recall the following well-known:

Lemma 10.3. *Let $(S,0)$ be a minimal singularity of a normal surface and m be the multiplicity of $(S,0)$. Then the multiplicity of the generic polar curve (resp. discriminant) is $2m-2$.*

FIGURE 5. Limit tree for the reduced graph associated to the graph in Example 5.6



Proof. This is easily deduced from the following two facts (we refer, e.g., to [7] (3.9) and § 5):
(a) for any normal surface $(S, 0)$ and any projection $p : S \to \mathbb{C}^2$ whose degree equals the multiplicity m of the surface, the multiplicity of the discriminant Δ_p is $m + \mu - 1$, where μ is the Milnor number of a generic hyperplane section of $(S, 0)$.
(b) When $(S, 0)$ is minimal, $\mu = m - 1$. □

The proof of Thm. 10.1 is by induction on the maximal height of the vertices in Γ:

A) Initial step – The maximal height in Γ is one. We prove the result by a direct argument (independent from Spivakovsky's theorem). Now, all the vertices x_i in Γ are in Γ_{TC}, and the minimal resolution X of $(S, 0)$ is the first blow-up.
(a)We know from the deformation onto the tangent cone that each exceptional component E_{x_i} bears the strict transform of $(2n_i - 2)$ smooth branches of the polar curve, cutting E_{x_i} transversely at general points, with $n_i = w_i - v_i$ (cf. 8.9 (iii)).
(b) A general theorem of J. Snoussi ([20], Thm. 6.6), valid for any normal surface singularity, describes the base points of the linear system of polar curves on the first normalized blow-up of $(S, 0)$. In our situation, the blow-up is already normal and even smooth, and hence Snoussi's theorem implies that here the bases points are exactly the singular points of the exceptional divisor, i.e., the intersection points of two components E_{x_i} and E_{x_j}.

Let N be the number of vertices in Γ; then Γ has $N - 1$ edges (it is a tree), which represent the intersections points of exceptional components.

By Bertini's Theorem, the part of the generic polar curve $P_{S,0}$ whose strict transform goes through a base point is singular, i.e., has multiplicity at least two.

Hence, adding the contributions of the smooth branches in (a) and the singular curves in (b), the multiplicity $m(P_{S,0}, 0)$ of the polar curve satisfies the inequality:

$$m(P_{S,0}, 0) \geq \sum_{i=1}^{N} (2n_i - 2) + 2(N - 1). \tag{3}$$

Comparing this to the equality $m(P_{S,0}, 0) = 2m - 2$ of Lemma 10.3 above, where the multiplicity m of $(S, 0)$ equals the $\sum_{i=1}^{N} n_i$, proves that (3) is in fact an equality.

Hence, each point of intersection of two exceptional components bears the strict transform of a curve of multiplicity exactly two on $(S, 0)$. We now claim that the curve in question is a A_2-curve singularity on $(S, 0)$. Let C be such a curve.

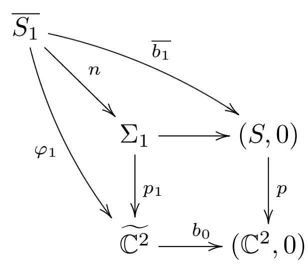
Then, the multiplicity $m(C, 0) = 2$ is the intersection number of C with a generic hyperplane section of $(S, 0)$. This intersection number may be computed on X as the intersection number of the strict transform C' with the reduced exceptional divisor (which is the cycle defined by the maximal ideal of $(S, 0)$). Since we know C' intersects two exceptional components, the intersection of C' with each one should be transverse.

Hence C is a branch of multiplicity two resolved in one blow-up, i.e., an A_2-curve. This completes the proof of the initial step A.

B) The induction step – We use first the following general lemma in [7], 6.1:

Lemma 10.4. *Let $(S,0)$ be any normal surface singularity and $p:(S,0)\to(\mathbb{C}^2,0)$ any projection with degree equal to the multiplicity $\nu = m(S,0)$.*

Then, denoting by $b_0:\widetilde{\mathbb{C}^2}\to(\mathbb{C}^2,0)$ the blow-up of the origin, and by Σ_1 the analytic fiber product of b_0 and p above $(\mathbb{C}^2,0)$, one proves that the normalisation of Σ_1 coincides with the normalised blow-up $\overline{S_1}$ of $(S,0)$, which yields the following commutative diagram:



where $\varphi_1:\overline{S_1}\to\widetilde{\mathbb{C}^2}$ is the composition of the pulled-back projection p_1 with the normalisation n.

The following formula can then be obtained for the discriminant Δ_{φ_1}:

$$\Delta_{\varphi_1} = (\Delta_p)' + (\nu - r)E, \tag{4}$$

where $(\Delta_p)'$ is the strict transform of the discriminant of p, E denotes the reduced exceptional divisor, ν is the multiplicity of the germ $(S,0)$ and r the number of branches of a general hyperplane section of $(S,0)$.

We refer to loc. cit. for the proof, we just make precise that the discriminants in the equality of the lemma are the divisorial parts of Fitting discriminants as defined in Section 8.3, which are hence allowed to have non-reduced components.

Here, $(S,0)$ being a minimal singularity, the blow-up S_1 is already normal (cf. e.g., [7], Thm. 5.9), so that $\overline{S_1}$ is just S_1. The generic projection that we consider certainly fulfills the property $\deg p = m(S,0)$ as a necessary condition. Since the general hyperplane section of a minimal singularity of multiplicity ν is just ν lines (cf., e.g., loc. cit., lem. 5.4), formula (4) in the above lemma simply reads:

$$\Delta_{\varphi_1} = (\Delta_p)',$$

and similarly, denoting by $C(D)$ the polar curve of the projection p, $C'(D)$ its strict transform on S_1 and C_{φ_1} the polar curve for the projection φ_1, we get:

$$C'(D) = C_{\varphi_1}. \tag{5}$$

From Thm. 3.2 (see also Def. 3.5), we know that the singularities O_i of S_1 are minimal singularities whose resolution graphs are the Tyurina components Γ_i.

Localising the result of (5) in O_i yields the following:

Conclusion 10.5. Let $C(D)$ be a generic polar curve for $(S,0)$ and $C'(D)$ its strict transform on the blow-up S_1 of S at 0. Let O_i be a singular point of S_1. We proved that the part of $C(D)'$ going through O_i is the polar curve for the projection φ_1 obtained of the germ (S_1,O_i) onto a plane, as in the lemma above.

Remark 10.6. To apply induction, we need to know that the projection

$$\varphi_1:(S_1,O_i)\to(\mathbb{C}^2,0),$$

in question is generic, i.e., has the generic polar curve.

Counting multiplicity as in A) gives that this projection has degree equal to the multiplicity of (S_1, O_i), but this is not enough to prove that the polar curve is generic. This will be proved thanks to the results of Section 7.

Indeed, once we know from Conclusion 10.5 that $C'(D)$ is a polar curve for (S_1, O_i), we may use Prop. 7.3 to see that the strict transform of $C'(D)$ on X, which is also part of the strict transform of $C(D)$, actually fulfills the conditions of the characterisation in Thm. 7.1. Then:

Conclusion 10.7. With the same notation as in Conclusion 10.5, the part of $C'(D)$ going through O_i is the *generic polar curve* P_{S_1,O_i} for the germ (S_1, O_i).

Now, the induction hypothesis applied to each (S_1, O_i) yields that P_{S_1,O_i} is a union of A_n-curves described by a limit tree T_i for Γ_i as stated in Theorem 10.1.

C) Reconstructing $P_{S,0}$ from its strict transform

Let $(S, 0)$ be a minimal surface singularity and let S_1 be its blow-up, and E the exceptional divisor with components $E_1, \dots, E_r$. We will denote by $O_1, \dots, O_s$ the singular points of S_1 and by $Q_1, \dots, Q_t$ the points of intersection of components of E which are not singular points of S_1.

We already know that the generic polar curve $P_{S,0}$ of $(S, 0)$ is precisely made of:

(1) A_1-curves in number $\sum_{i=1}^{r}(m_i - 1)$, whose strict transforms intersect each E_i as $(2m_i - 2)$ lines going through generic points of E_i, for $i = 1, \dots, r$,
(2) A_2-curves singularities in number t, each one having its strict transform on S_1 intersecting a different point Q_i defined above,
(3) curves whose strict transforms go through the singular points O_i of S_1.

The first two points are proved by the same reasoning as in step A). Step B) applied to curves in (3) for each O_i gives the description of the strict transforms of these curves as A_n-curves described by the limit tree T_i associated to (S_1, O_i).

The corresponding description, for all the curves in (3) whose strict transforms go through the same O_i, on $(S, 0)$ itself, is then obtained by adding 2 to all the n's and one to the ρ by elementary properties of these A_n-curves and our Def. 5.3 of the contact.

But now from [15] (1.18), we know that the data associated to limit trees T_i of Γ_i are related to T exactly the same way (length:= length-2, overlap := overlap -1).

This completes the proof by induction for the first two points of Theorem 10.1, the last point follows by definition of the contact.

11. Scott deformations and polar invariants

The following was first proved by de Jong and van Straten in [15] Thm. 2.13:

Theorem 11.1. *Let $(S, 0)$ be a minimal singularity of a normal surface with multiplicity m. Let S_1 be the blow-up of 0 in S, with singular points $O_1, \dots, O_r$. Then there exists a one-parameter deformation $\rho : X \to \mathbb{D}$ of $(S, 0)$ on the Artin component such that X_s for $s \neq 0$ has $r + 1$ singular points isomorphic respectively to the (S_1, O_i) for $i = 1, \dots, r$ and to the cone over the rational normal curve of degree m.*

This has to be compared to a standard result for plane curves, attributed to C. A. Scott in [16], where a proof is also given (see p. 460):

Lemma 11.2. *Let $(C, 0) \in (\mathbb{C}^2, 0)$ be a plane curve singularity of multiplicity m. Let O_i for $i = 1, \dots, r$ be the singularities of the first blow-up C_1 of $(C, 0)$. Then there exists a one-parameter δ-constant deformation Γ of $(C, 0)$ such that Γ_s for $s \neq 0$ is a plane curve which has $r + 1$ singular points isomorphic respectively to the (C_1, O_i) for $i = 1, \dots, r$ and to an ordinary m-tuple point.*

Beyond the formal analogy between Thm. 11.1 and Lem. 11.2, de Jong and van Straten prove the result in Thm. 11.1 for the more general class of *sandwiched singularities* as a consequence of their theory of *decorated curves*: all the deformations of these surface singularities can be obtained from deformations of *decorated curves* associated to the singularity. In particular, the Scott deformation of a decorated curve (conveniently adjusted) gives rise to the deformation in Thm. 11.1.

As an application of our description for generic discriminants in Thm. 10.1, however, we get first a new relation between these two deformations:

Corollary 11.3. *Let the notation be the same as in Thm. 11.1. We will also call the deformation X the* Scott deformation *of the surface* $(S, 0)$.

Considering a projection p of X in $\mathbb{D} \times C^2$ as in Prop. 8.8, i.e., compatible with ρ and such that the discriminant $\Delta(p_t : X_t \to \mathbb{C}^2)$ is the generic discriminant Δ_t for all the singularities in X_t, for all t, one gets a deformation $\rho' : \Delta \to \mathbb{D}$ of the generic discriminant $\Delta_{S,0}$ of $(S, 0)$, which is exactly the Scott deformation of this curve as defined in Lem. 11.2.

Proof. In our proof in Section 10, it is proved that the discriminant of (S_1, O_i) is the part of the strict transform of the discriminant of $(S, 0)$ going through the image of O_i in the plane (it is of course also obvious from the result there).

The discriminant of the cone over the m-th Veronese curve is a $2m-2$-tuple ordinary point in the plane (cf. Rem. 8.1). This is indeed the last singularity occurring in the Scott deformation of $\Delta_{S,0}$, since, by Lem. 10.3, the multiplicity of the $\Delta_{S,0}$ is $2m - 2$. □

Considering polar curves in this Scott deformation, we get the more interesting:

Theorem 11.4. *Let the notation be as in Cor. 11.3. Then, the polar curve for $p_t : X_t \to (\mathbb{C}^2, 0)$ is also the generic polar curve P_{X_t} (which is a multi-germ of space curves for $t \neq 0$). Further, P_{X_t} is a δ-constant deformation of the generic polar curve $P_{S,0}$. Hence iterating Scott deformations, one may compute the δ-invariant of $P_{S,0}$ as sum of δ-invariants for sets of generic lines P_m as in Lem. 8.2.*

Proof. In Theorem 11.1, the deformation X_t is said to belong to the Artin-component of $(S, 0)$. This means that it has a simultaneous resolution, in which (cf. Lem 5.1) the P_{X_t} are also resolved. One then has a *normalisation in family* for the family P_{X_t}, which is equivalent to "δ-constant" (cf. [22], p. 609). □

We illustrate the second statement in Thm 11.4 by giving:

Example 11.5. Taking a singularity with graph as in Figure 6, and applying twice the Scott deformation of the surface, one gets two cones over a cubic and two cones over a conic. Hence the polar curve deforms onto two P_3's and two P_2's (in the notation of Lem. 8.2), which gives 8 for the δ-invariant.[10]

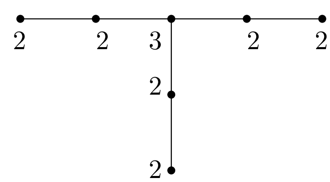
Let us end this with the following:

Remark 11.6. The information on (the resolution graph of) $(S, 0)$ given by the generic discriminant $\Delta_{S,0}$ is of course partial: e.g., one may permute the Tyurina components in the resolution graph of $(S, 0)$ or the weights on the tangent cone without changing $\Delta_{S,0}$. However, when one looks at deformations on $(S, 0)$, we believe the information on the discriminant is most valuable:

(a) As a very basic occurence of this: a family of normal surfaces S_t with constant generic discriminant $\Delta_{S_t,0}$ is Whitney-equisingular and in particular has constant topological type (encoded by the minimal resolution graph). As a consequence of our result, these three equisingularity

[10] Beware that $\delta(P_2) = 1$ is not given by the formula $\delta(P_n) = 3n - 6$, valid for $n \geq 3$.

Figure 6. Graph with weights on the vertices for example 11.5



conditions are in fact equivalent for minimal singularities of surfaces (see also [1], Th. 3.6. and Cor. 4.3).

(b) Much more generally, one can deform the discriminant $\Delta_{S,0}$ and ask which deformation of $(S,0)$ "lies above" the curve-deformation: for example, can one deduce the existence of the Scott deformation of the surface $(S,0)$ in the sense of Thm. 11.1 as deformation "lying above" the Scott deformation of $\Delta_{S,0}$?

This would give a description of some deformation theory of the surface through an invariant which, as opposed to the birational join construction of Spivakovsky or the decorated tree construction of De Jong and Van Straten, is uniquely defined from $(S,0)$.

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