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THE NASH PROBLEM AND ITS SOLUTION: A SURVEY

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Abstract. The goal of this survey is to give a historical overview of the Nash Problem of arcs in arbitrary dimension, as well as its solution. This problem was stated by J. Nash around 1963 and has been an important subject of research in singularity theory. In dimension two the problem has been solved affirmatively by J. Fernández de Bobadilla and M. Pe Pereira in 2011. In 2002 S. Ishii and J. Kollár gave a counterexample in dimension four and higher, and in May 2012 T. de Fernex settled (negatively) the last remaining case — that of dimension three. After some history, we give an outline of the solution of the Nash problem for surfaces by Fernández de Bobadilla and Pe Pereira. We end this survey with the latest series of counterexamples, as well as the Revised Nash problem, both due to J. Johnson and J. Kollár.

1. Introduction

In this paper, $\Bbbk$ is an algebraically closed field of characteristic 0 (see Remark 1.7 below for the case of positive characteristic).

1.1. **The statement of the problem.** Let X be a singular algebraic variety over $\Bbbk$ and $\pi : \tilde{X} \longrightarrow X$ a *divisorial* resolution of singularities of X (this means that $\tilde{X}$ is a smooth variety and the **exceptional set** $E =: \pi^{-1}(Sing\ X)$ is a **divisor**, that is, is of pure codimension one). Let

$$E = \bigcup_{i \in \Delta} E_i \tag{1}$$

be the decomposition of E into its irreducible components. The set E has two kinds of irreducible components: essential and inessential. For each i let μ_i denote the divisorial valuation determined by E_i.

Definition 1.1. *We say that E_i is an* **essential divisor** *if for any other resolution $\pi' : (X', E') \to (X, Sing\ X)$ the center of μ_i on X' is an irreducible component of E'. The divisor E_i is* **inessential** *if it is not essential.*

Remark 1.2. *Intuitively, an irreducible divisor is essential if it appears, as an irreducible component, on every resolution of X.*

In general (that is, when $\dim\ X \geqslant 3$*) it is quite difficult to show that a given component is essential (see [32] for a discussion of this question as well as some sufficient conditions for essentiality and [3] and [17] for new criteria of essentiality). In dimension two there exists a unique minimal resolution $\tilde{X}$ of X and each irreducible exceptional divisor of $\tilde{X}$ is essential.*

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In order to study the resolution $\tilde{X}$ of X, J. Nash (around 1963, published in 1995 [26]) introduced the space X_∞^{sing} of *arcs* meeting the singular locus $Sing\ X$.

Definition 1.3. *An* ***arc*** *is a* $\Bbbk$*-morphism from* $Spec\ \Bbbk[[t]]$ *to* X*.*

Let X_∞^{sing} *be the set of arcs whose origin (that is, the image of the closed point) belongs to the singular locus of* X*.*

Intuitively, such an arc should be thought of as a parametrized formal curve, contained in X and meeting the singular locus of X. The analogue of an arc in complex analysis is a test map from a small disk around the origin on the complex plane to X. We will also need to consider more general arcs, which are morphisms from $Spec\ K[[t]]$ to X, where K is a field extension of $\Bbbk$; they are called K**-arcs**.

Let us denote the closed point (the origin) of $Spec\ \Bbbk[[t]]$ by 0 and the generic point by η.

An arc can be lifted to any resolution:

Lemma 1.4. *Let* $f : \tilde{X} \to X$ *be a resolution of the singularities. Every arc* $\alpha : Spec\ K[[t]] \to X$ *such that* $\alpha(\eta) \notin Sing(X)$ *can be lifted to an arc* $\tilde{\alpha} : Spec\ \Bbbk[[t]] \to \tilde{X}$*.*

The proof comes from the fact the resolution map π is proper (it is a special case of the valuative criterion of properness).

Nash showed that X_∞^{sing} has finitely many irreducible components, F_i, called *families of arcs*, and defined the following map:

Definition 1.5 (Nash [26])**.** *Let*

$$\mathcal{N} : \{\textit{irreducible components of } X_\infty^{sing}\} \to \{\textit{essential divisors of } \tilde{X}\}$$

be the map sending a family F_i *to the exceptional divisor* E_i *such that the generic arc of* F_i *has lifting to the resolution, passing through a general point of the component* E_i*.*

(see §2.2 for more details).

He showed that this map, now called the **Nash map**, is injective. The celebrated **Nash problem**, posed in [26], is the question whether the Nash map is surjective.

Let us fix a divisorial resolution of singularities $\tilde{X} \to X$ and let $E = \pi^{-1}(Sing\ X)$. Consider the decomposition (1) of E into irreducible components, as above. Let $\Delta' \subset \Delta$ denote the set which indexes the essential divisors.

M. Lejeune-Jalabert [20], inspired by Nash's original paper [26], proposed the following decomposition of the space X_∞^{sing}: for $i \in \Delta'$, let C_i be the set of arcs whose strict transform in $\tilde{X}$ intersects the essential divisor E_i transversally but does not intersect any other exceptional divisor E_j. M. Lejeune-Jalabert shows that $X_\infty^{sing} = \bigcup\limits_{i \in \Delta'} \overline{C_i}$ and the set $\overline{C_i}$ is an irreducible algebraic subvariety of the space of arcs; therefore the families of arcs are among the $\overline{C_i}$'s. Moreover there are as many $\overline{C_i}$ as essential divisors E_i. Then the Nash problem reduces to showing that the $\overline{C_i}$, $i \in \Delta'$, are precisely the irreducible components of X_∞^{sing}, that is, to proving $card(\Delta')(card(\Delta') - 1)$ non-inclusions:

Problem 1.6. *Is it true that* $\overline{C_i} \not\subset \overline{C_j}$ *for all* $i \neq j$*?*

Remark 1.7. *All of the above definitions make sense also when char* $\Bbbk > 0$*, with the following modification. An arc family is said to be* **good** *if its general element is not entirely contained in* $Sing\ X$*. When char* $\Bbbk = 0$ *it can be shown that all the arc families are good (*[16]*, Lemma 2.12). If the singularities of* X *are isolated (say,* $Sing(X) = \{\xi_1, \dots, \xi_l\}$*) then the only arcs* *contained in $Sing(X)$ are the trivial ones which map Spec $K[[t]]$ identically to one of the ξ_i. Every such arc is in the closure of every other arc passing through ξ_i. Hence the arcs contained in ξ_i belong to the closure of every irreducible component of X_∞^{sing} lying over ξ_i and cannot form an irreducible component by themselves. This proves that for X with isolated singularities X_∞^{sing} has no bad components, regardless of char $\Bbbk$. If char $\Bbbk > 0$ and* $\dim$ *$Sing(X) \geqslant 1$, there may exist some bad families, and the Nash map is only defined on the set of good families. With this in mind, the Nash problem remains the same: is the Nash map, defined on the set of good families, surjective? See* [37] *for some recent work on the Nash problem in positive characteristic.*

1.2. **Some partial answers in dimension 2.** Before the work of Fernández de Bobadilla — Pe Pereira, the Nash problem for surfaces has been answered affirmatively in the following special cases: for A_n singularities by Nash, for minimal surface singularities by A. Reguera [34] (with other proofs by J. Fernandez-Sanchez [7] and C. Plénat [29]), for sandwiched singularities by M. Lejeune-Jalabert and A. Reguera (cf. [21] and [35]), for toric vareties in all dimensions by S. Ishii and J. Kollar [16] (using earlier work of C. Bouvier and G. Gonzalez-Sprinberg [1] and [2]), for a family of non-rational surface singularities by P. Popescu-Pampu and C. Plénat ([31]), for quotients of $\mathbb{C}^2$ by an action of finite group [27] by M. Pe Pereira in 2010 based on the work [5] of J. Fernández de Bobadilla (other proofs for $\mathbf{D_n}$ in 2004 by Plénat [30], for E_6 in 2010 by C. Plénat and M. Spivakovsky [33], (with a method that works for some normal hypersurface singularities), and by M. Leyton-Alvarez (2011) for E_6 and E_7, by applying the method for the following classes of normal hypersurfaces in $\mathbb{C}^3$: hypersurfaces $S(p, h_q)$ given by the equation $z^p + h_q(x, y) = 0$, where h_q is a homogeneous polynomial of degree q without multiple factors, and $p \geqslant 2$, $q \geqslant 2$ are two relatively prime integers [23]). A. Reguera [37] gave an affirmative answer to the Nash problem for rational surface singularities simultaneously and independently from the work [6].

See the bibliography for a (hopefully) complete list of references on the subject.

In 2011, J. Fernández de Bobadilla and M. Pe Pereira [6] showed that the answer is positive for any surface singularity. The main aim of this paper is to give an outline of their proof. Before going further into the details, we need to recall some earlier results that lead to the final proof.

The rest of the paper is organized as follows: §2 is dedicated to the work preceding the paper [6]; in §3 an outline of the proof is given. §4 contains a brief discussion of the Nash problem in dimension three and higher.

2. Previous results on the Nash problem

2.1. **The Wedge problem** [18]**.** In 1980, M.Lejeune-Jalabert proposed to look at the Nash problem from a new point of view. She formulated in [18] what is now called "the wedge problem", which is related to a "Curve Selection Lemma" in the space of arcs.

Let X be a singular algebraic variety over $\Bbbk$.

Let us first define **wedge**:

Definition 2.1. *Let K be a field extension of $\Bbbk$. A K-wedge on X is a $\Bbbk$-morphism*

$$\omega : Spec(K[[t, s]]) \to X$$

which maps the set $\{t = 0\}$ to Sing X.

The wedge ω induces two arcs on X as follows: a K-arc obtained by restricting ω to the set $\{s=0\}$ (this arc is called the **special arc** of ω), and a $K((s))$-arc, obtained by restricting ω to the set $Spec(K[[t,s]])\setminus\{s=0\}$ (this arc is called the **general arc** of ω). We regard ω as a deformation of its special arc to its general arc or, alternatively, as an arc in the space of arcs X_∞^{sing}.

The wedge is said to be **centered** at an arc γ_0 if its special arc is γ_0.

Let $(X,0)$ be a germ of a normal surface singularity, and let $\pi:(\tilde{X},E)\to(X,0)$ be its minimal (and so divisorial) resolution, with $E=\bigcup E_j=\pi^{-1}(0)$. Let E_i,E_j be irreducible components of E (they are essential as X is a surface). Let C_i and C_j be as above. Then if $C_j\subset\overline{C_i}$, E_j is not in the image of Nash map. If one had Curve Selection lemma in the space of arcs X_∞^{sing}, the inclusion above would just mean that one has a $\Bbbk$-wedge with special arc in C_j and generic arc in C_i. Then the morphism ω would not lift to the resolution $\tilde{X}$ as it has an indeterminacy at 0.

M. Lejeune-Jalabert proposed the following problem:

Problem 2.2. *For all irreducible essential divisors of the minimal resolution, any $\Bbbk$-wedge centered at $\gamma_i\in C_i$ can be lifted to $\tilde{X}$.*

It is not trivial to generalize the classical Curve Selection Lemma to the case of infinite-dimensional varieties such as X_∞^{sing}. A. Reguera proved a Curve Selection Lemma for X_∞^{sing} thus establishing the equivalence between the the Nash and the wedge problems. The wedges appearing in A. Reguera's theorem are K-wedges rather than $\Bbbk$-wedges, where K is an extension of $\Bbbk$ of infinite transcendence degree. This work of A. Reguera and its corollaries are discussed in §2.3. §2.2 is dedicated to an interpretation of the space of arcs in terms of representable functors. This interpretation is due to S. Ishii and J. Kollár [16]. It has been a great step in the resolution of the problem.

2.2. **Arc spaces as representable functors** [16].

Let X be a reduced scheme of finite type over $\Bbbk$.

Definition 2.3. *Let $\Bbbk\subset K$ be a field extension. A morphism $Spec(K[[t]]/(t^{n+1}))\to X$ is called an n-jet of X over K and a morphism $Spec\ (K[[t]])\to X$ is called a K-arc of X. Let us denote the closed point (the origin) of $Spec\ K[[t]]$ by 0 and the generic point by η.*

Let $\mathcal{S}ch/\Bbbk$ be the category of $\Bbbk$-schemes and Set the category of sets. Define a contravariant functor

$$F_m:\mathcal{S}ch/\Bbbk\to Set$$

by

$$F_m(Y)=Hom_{\Bbbk}(Y\times_{Spec\ k}Spec(\Bbbk[[t]]/(t^m)),X)$$

Then, F_m is *representable* by a scheme X_m of finite type over $\Bbbk$. This means, by definition, that

$$Hom_{\Bbbk}(Y,X_m)=Hom_{\Bbbk}(Y\times_{Spec\ k}Spec(\Bbbk[[t]]/(t^m)),X)$$

for a $\Bbbk$-scheme Y.

This X_m is called the scheme of n-jets of X. The canonical surjection

$$\Bbbk[[t]]/(t^m)\to\Bbbk[[t]]/(t^{m-1})$$

induces a morphism $\phi_m:X_m\to X_{m-1}$. Define $\rho_m=\phi_1\circ\cdots\circ\phi_m:X_m\to X$. A point $x\in X_m$ gives an m-jet $\alpha_x:Spec\ K[[t]]/(t^m)\to X$ and $\rho_m(x)=\alpha_x(0)$, where K is the residue field at x.

Let $X_\infty = \varprojlim X_m$ and call it the space of arcs of X. X_∞ is not of finite type over $\Bbbk$ but it is a $\Bbbk$-scheme. Denote the canonical projection $X_\infty \to X_m$ by η_m and the natural map $X_\infty \to X$ by ρ; it is the composition $\rho_m \circ \eta_m$ $\forall m$. A point $x \in X_\infty$ with residue field K gives an arc $\alpha_x : Spec\ K[[t]] \to X$ with $\rho(x) = \alpha_x(0)$.

The scheme X_∞^{sing} defined earlier is nothing but the subscheme of X_∞ consisting of those arcs α for which $\alpha(0) \in Sing(X)$.

Lemma 1.4 applies equally well to K-arcs: any K-arc not contained in $Sing(X)$ has a unique lifting to any resolution of singularities $\tilde{X}$.

Let $\overline{C_i}$ be the closure of the set of arcs α that lift to a general point of a component E_i and such that $\alpha(\eta) \notin Sing(X)$ and $\alpha(0) \in Sing(X)$. Let γ_i denote the generic point of the closed irreducible set $\overline{C_i}$ and $\Bbbk_i$ the residue field of the local ring $\mathcal{O}_{X_\infty^{sing},\gamma_i}$.

Theorem 2.4 (Nash [26])**.** *The* **Nash map**

$$\mathcal{N} : \{C_i\} \to \{\textit{essential components of } \tilde{X}\}$$

given by $C_i \to E_i$ is injective.

In [16], after the reformulation of Nash problem (in any dimension), two beautiful results are shown: a positive answer to Nash problem for toric varieties in any dimension and a counter-exemple in dimension 4 and higher.

2.3. **A Curve Selection Lemma in X_∞^{sing} [36].** In the paper [36], A. Reguera has shown that a positive answer to the wedge problem is equivalent to the surjectivity of the Nash map. She has also extended the wedge problem to all dimensions. Note that she does not assume the singular varieties to be normal. More precisely, she proves the following:

Theorem 2.5. *Let X be a singular variety.*
Let E_i be an essential divisor over X. Let γ_i be the generic point of $\overline{C_i}$ (the closure of the set of arcs lifting transversally to E_i), $\Bbbk_i$ its residue field. The following are equivalent:

(1) *E_i belongs to the image of the Nash map.*
(2) *For any resolution of singularities $p : \tilde{X} \to X$ and for any field extension K of $\Bbbk_i$, any K-wedge whose special arc maps to γ_i, and whose generic arc maps to X_∞^{sing}, lifts to $\tilde{X}$.*
(3) *There exists a resolution of singularities $p : \tilde{X} \to X$ satisfying the conclusion of* (2)*.*

To prove this she needed a Curve Selection lemma for X_∞^{sing} for curves defined over K. This field is of infinite transcendence degree over $\Bbbk$, so it is quite difficult to work with. J. Fernández de Bobadilla [5] and M. Lejeune-Jalabert with A. Reguera [22] have shown, independently, that one may replace K by $\Bbbk$ in A. Reguera's theorem, provided that $\Bbbk$ is uncountable.

2.4. **The Nash Problem is a topological problem** [5]**.** In this paper, J. Fernández de Bobadilla looks at normal surface singularities, and the hypotheses of normality and dimension 2 are essential. He first gives the definition of wedges that realize an adjacency between two essential divisors.

Definition 2.6. *Let E_u and E_v be two essential divisors, and C_u and C_v the families of arcs associated to these divisors.*
A K-wedge realizes an adjacency from E_u to E_v if its generic arc belongs to C_u and its special arc belongs to C_v^o (i.e. it is transverse to E_v in a general point of E_v).

Note that if such a wedge exists, then C_v is not in the image of Nash map. This statement can be interpreted as the easy part of the Theorem of the previous section ($2 \implies 1$): a wedge realizing the adjacency cannot be lifted to any resolution.

J. Fernández de Bobadilla proves the following theorem:

Theorem 2.7. *Let $(X, 0)$ be a normal surface singularity defined over an uncountable algebraically closed field $\Bbbk$ of characteristic 0. Let E_v be an essential irreducible component of the exceptional divisor of a resolution. Then the following are equivalent:*

(1) *The set C_v is in the Zariski closure of C_u, where E_u is another component of the exceptional divisor.*
(2) *Given any proper closed subset $\mathcal{Z} \subset \overline{C_u}$, there exists an algebraic $\Bbbk$-wedge realizing an adjacency from E_u to E_v and avoiding $\mathcal{Z}$.*
(3) *There exists a formal $\Bbbk$-wedge realizing an adjacency from E_u to E_v.*
(4) *Given any proper closed subset $\mathcal{Z} \subset \overline{C_u}$, there exists a finite morphism realizing an adjacency from E_u to E_v and avoiding $\mathcal{Z}$.*

If the base field is $\mathbb{C}$ the following further conditions are equivalent to those above:

(5) *Given any convergent arc $\gamma \in C_u^o$ there exists a convergent $\mathbb{C}$-wedge realizing an adjacency from E_u to γ and avoiding the set Δ_u of arcs lifting to singular points of E_u or not transversal to E_u.*
(6) *Given any convergent arc $\gamma \in C_u^o$ there exists a convergent $\mathbb{C}$-wedge realizing an adjacency from E_u to γ.*
(7) *Given any convergent arc $\gamma \in C_u^o$ there exists a finite morphism realizing an adjacency from E_u to γ and avoiding Δ_u.*

See [5] for the definition of finite morphism realizing an adjacency from E_u to γ.

Sketch of the proof:
For 1) $\Rightarrow$ 2) J. Fernández de Bobadilla uses A. Reguera's results to obtain a K-wedge realizing an adjacency from E_u to E_v, with $\Bbbk \subset K$ an extension of $\Bbbk$. Then he uses a specialization process to obtain a $\Bbbk$-wedge realizing an adjacency from E_u to E_v and avoiding $\mathcal{Z}$. One can find a similar specialization process in [22] in which the authors characterize essential components that belong to the image of the Nash map and deduce that an irreducible exceptional divisor which is not uniruled is in the image of the Nash map (for uncountable fields).
For 4) $\Rightarrow$ 1), he needs to introduce some technical tools. First, he gets an algebraic $\Bbbk$-wedge using Popescu's theorem and Artin type approximation to replace the first formal $\Bbbk$-wedge. Then by Stein Factorization he obtains a finite morphism realizing an adjacency from E_u to E_v and avoiding $\mathcal{Z}$. He finally reduces to the case of $\Bbbk = \mathbb{C}$, and shows 1) in that case. For this, he proves a property that he calls "moving wedges":

Lemma 2.8. *Given two convergent arcs $\gamma, \gamma' \in C_v^o$, there exists a finite morphism realizing an adjacency from E_u to γ if and only if there exists a finite morphism realizing an adjacency from E_u to γ'.*

He uses the Lemma to prove the following theorem:

Theorem 2.9. *The set of adjacencies between exceptional divisors of a normal surface singularity is a combinatorial property of the singularity: it only depends on the dual weighted graph of* *the minimal good resolution. In the complex analytic case this means that the set of adjacencies only depends on the topological type of the singularity and not on the complex structure.*

The last important paper needed to understand the proof in dimension two is due to M. Pe Pereira [27], which gives an affirmative solution to the Nash problem for quotient singularities of surfaces. In that paper she has, in particular, introduced some useful tools needed in [6]. We will discuss them in the following section.

3. Solution of the Nash problem for surfaces

Theorem 3.1. *Let $\Bbbk$ be an algebraically closed field of characteristic 0 and $(X, 0)$ a normal singular surface over $\Bbbk$.*
The Nash map associated to $(X, 0)$ is bijective.

In [5] (7.2 p. 163), J. Fernández de Bobadilla shows that the families of arcs are stable under base change and so is the bijectivity of Nash map. Thus it remains to prove the theorem for complex normal surface singularities.

Let $(X, 0)$ be a normal surface singularity over $\mathbb{C}$.

The proof proceeds by contradiction.

Let $E = \bigcup_{i=0}^{n} E_i$ be the decomposition of E into irreducible components. Suppose there are two families $\overline{C_0}$ and $\overline{C_i}$ associated with two essential divisors E_0 and E_i of the minimal resolution such that $\overline{C_0} \subset \overline{C_i}$.

3.1. Definition of representatives of arcs and wedges.

The first ingredient is the definition of Milnor representative of arcs and wedges.

From now on, replace X by its underlying complex-analytic space. By abuse of notation, we will continue to denote this space by X. Let $\pi : \tilde{X} \to X$ be the minimal resolution of X.

Let us recall Milnor's work on isolated singularities.
Let B_ε denote the closed ball in $\mathbb{C}^N$ centered at the origin of radius ε and let S_ε be its boundary sphere. There exists for X a Milnor radius ε_0 such that all the spheres S_ε are transverse to X and $X \cap S_\varepsilon$ is a closed subset of S_ε for all $0 < \varepsilon \leqslant \varepsilon_0$. Let us call $X_{\varepsilon_0} = X \cap B_{\varepsilon_0}$ the Milnor representative of X. Let $\tilde{X}_{\varepsilon_0}$ be the minimal resolution of singularities of X_{ε_0}; $\tilde{X}_{\varepsilon_0}$ is nothing but the preimage of X_{ε_0} under π. Under these conditions X_{ε_0} has a conical structure and $\tilde{X}_{\varepsilon_0}$ admits E as a deformation retract.

Consider an arc $\gamma : (\mathbb{C}, 0) \to X_{\varepsilon_0}$. It is proved in [27] and [6] that there exists $\varepsilon \leqslant \varepsilon_0$ such that, restricted to X_ε, γ becomes a **Milnor arc**:

Definition 3.2. Milnor arc

A Milnor representative of γ is a map of the form

$$\gamma|_U : U \to X_\varepsilon$$

such that $\gamma|_U$ is a proper morphism, U is diffeomorphic to a closed disk, $\gamma^{-1}(\partial X_\varepsilon) = \partial U$ and the mapping $\gamma_{|U}$ is transverse to any sphere $S_{\varepsilon'}$ for $\varepsilon' \leqslant \varepsilon$. The radius ε is called a Milnor radius for γ.

Let $\alpha : (\mathbb{C}^2, 0) \to X_\varepsilon$ be an analytic wedge such that $\alpha(t, s) = \alpha_s(t)$ is the generic arc and $\alpha_0(t) = \gamma(t)$ is the special arc.
Let $\gamma\mid_U : U \to X_\varepsilon$ be a Milnor Representative of γ.

For the disk D_δ of radius δ around the origin in the complex plane we will use the notation ${D^o}_\delta = D_\delta \setminus \{0\}$.

Proposition 3.3. Milnor wedge

There exist $\delta > 0$ small enough, an open set $\mathcal{U} \subset U \times D_\delta$ and a map

$$\beta : U \times D_\delta \to X_\varepsilon \times D_\delta$$

$$(t, s) \quad \to (\alpha_s(t), s)$$

such that

- *$\alpha_0(t) = \gamma\mid_U$ is a Milnor representative of α_0.*
- *the restriction $\beta\mid_{\mathcal{U}^o} : \mathcal{U}^o \to X^o_\varepsilon \times D^o_\delta$ is a proper and finite morphism of analytic spaces and its image is a closed 2-dimensional closed analytic subset of $X^o_\varepsilon \times D^o_\delta$.*
- *the set $U_s = \mathcal{U} \cap \mathbb{C} \times \{s\}$ is diffeomorphic to a disk for all s.*
- *for any $s \in D_\delta$, $\beta_{U\times\{s\}}$ is transverse to $S_\varepsilon \times D^o_\delta$ (this means that every $x \in \partial U_s$ is a regular point of $\beta_{U\times\{s\}}$ and the vector space $d\beta_{U\times\{s\}}$ is transverse to the tangent space of $S_\varepsilon \times D^o_\delta$ at $\beta_{U\times\{s\}}(x)$.*
- *$\mathcal{U}$ is a smooth manifold with boundary $\beta^{-1}(\partial X_\varepsilon \times D^o_\delta)$*

Definition 3.4. *The map β restricted to $\mathcal{U}$ is a* **Milnor representative** *of the wedge α, whose special arc is $\gamma\mid_U$.*

Remark 3.5. *One has to prove that such a representative does exist, in particular that the set U can be taken to be differomorphic to a disk. See* [27] *or* [6].

Aiming for contradiction, we now consider a Milnor representative $\alpha : \mathcal{U} \to X_\varepsilon$ of an analytic wedge, realizing the adjacency from E_i to E_0, that is, a wedge such that the generic arc $\alpha_s(t)$ belongs to C_i and the special arc $\gamma(t)$ belongs to C_0.

Remark 3.6. *These definitions of representatives are a key point in the proof of the theorem. Let $\alpha_s : U_s \to X_\varepsilon$ be a generic arc of the wedge. By construction, U_s is a disk and thus has Euler characteristic equal to one. The aim of the rest of the proof is to show that the Euler Characteristic of U_s is bounded above by an expression less or equal to 0, and thus get the contradiction.*

We have the following lemma:

Lemma 3.7. *The mapping $\alpha_s : U_s \to X_\varepsilon$ is injective.*

Proof. The map α_s is a smooth deformation of $\alpha_0 : U_0 \to X_\varepsilon$. But the map $\alpha_0 : U_0 \to X_\varepsilon$ is injective since by construction it is transversal to every S_μ for $\mu \leqslant \varepsilon$, so α_0 is an injective and smooth mapping.
Moreover, for all $s \in D^o_\delta$ we have $\beta^{-1}(\partial B_\varepsilon \times \{s\}) \cong S^1$. The degree of a map of S^1 to itself is upper semi-continuous under smooth deformation, thus the map

$$\alpha_s\mid_{\partial U_s} : S^1 \to S^1$$

is of degree one. By Definition 3.4 and Proposition 3.3, α_s has no critical points on ∂U_s; this implies that $\alpha_s\mid_{\partial U_s}$ is one–to–one.
Hence α_s is a local homeomorphism and so is generically one–to–one. □

3.2. **Eliminating the indeterminacy of $\tilde{\alpha}$.** Let $\tilde{\beta}$ be the meromorphic map defined as the composition of $\sigma^{-1} \circ \beta$ with $\sigma = (\pi, id\,|_{D_\delta})$:

$$\begin{tikzcd} & \tilde{X}_\varepsilon \times D_\delta \arrow[d, "\sigma"] \\ \mathcal{U} \arrow[ur, dashed, "\tilde{\beta}"] \arrow[r, "\beta"] & X_\varepsilon \times D_\delta \end{tikzcd}$$

The indeterminacy locus of $\sigma^{-1} \circ \beta$ is of codimension 2. Thus we may assume that, shrinking the radius δ, if necessary, $(0,0)$ is the only indeterminacy point of $\tilde{\beta}$.

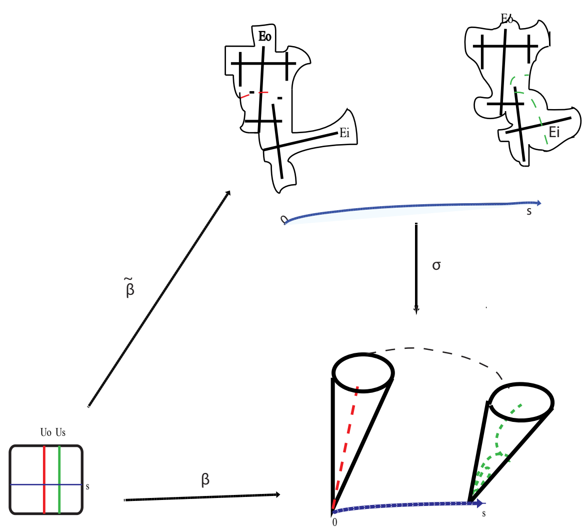

FIGURE 1 . Wedge representative

Moreover there exists a unique meromorphic lifting $\tilde{\alpha}$ of α such that:

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d] & \tilde{X}_\varepsilon \arrow[d, "\pi"] \\ \mathcal{U} \arrow[ur, dashed, "\tilde{\alpha}"] \arrow[r, "\alpha"] & X_\varepsilon \end{tikzcd}$$

Let $H = \beta(\mathcal{U})$ the image of $\mathcal{U}$ by β; it is an analytic subvariety of dimension two (as β is finite and proper). Let Y be the analytic Zariski closure of $\sigma^{-1}(H \backslash (\{0\} \times D_\delta))$ and let $Y_s = Y \cap (\tilde{X}_\varepsilon \times \{s\})$. The surface Y is reduced and is a Cartier divisor in the smooth threefold $\tilde{X} \times D_\delta$. One can prove the following ([5], p. 7):

(1) for all $s \in D_\delta^o$ one has $\tilde{\beta}(t,s) = (\tilde{\alpha}_s(t), s)$
(2) $Y \cap (\tilde{X}_\varepsilon \times D_\delta^o) = \tilde{\beta}(\mathcal{U} \backslash U_0)$.

Thus

$$Y_s = \tilde{\alpha}_s(U_s).$$

Lemma 3.8. *The mapping $\tilde{\alpha}_s : U_s \to Y_s$ is the morphism of normalization of Y_s.*

Proof. First, by the previous Lemma, α_s is generically one–to–one and proper. Hence so is $\tilde{\alpha}_s$. As U_s is a smooth disk, the mapping $\tilde{\alpha}_s : U_s \to Y_s$ is thus the morphism of normalization of Y_s. □

Definition 3.9. Returns
Elements of the set $\alpha_s^{-1}(0) \setminus \{(0,s)\}$ are called **returns**. *Their images by α_s are 0 and by $\tilde{\alpha}_s$ points of the exceptional set E.*

The curve $Y_0 = Y \cap (\tilde{X}_\varepsilon \times \{0\})$ does not need to be reduced. It contains $Z_0 := \tilde{\alpha_0}(U_0)$ and a sum of the exceptional components E_i with suitable multiplicities, which can be explicitly described as follows. For any point $\xi \in X$, let f_ξ denote the local defining equation of Y_0 near ξ.

We have a unique factorization

$$f_\xi = g_{n+1} \prod_{i=0}^{n} g_i^{a_i}$$

where $g_{n+1} = 0$ is the local defining equation of Z_0 near ξ and $g_i = 0$ the local defining equation of E_i near ξ (if $\xi \notin E_i$, we take $g_i = 1$, and similarly for g_{n+1}). It is very easy to see that, given two points $\xi, \xi' \in E_i$, one obtains the same exponent a_i from the local equations at ξ and ξ'; in other words, a_i is determined by E_i and not by the choice of the point ξ. We express this situation by the equation $Y_0 = Z_0 + \sum a_i E_i$; the analytic space Y_0 is reduced along $Z_0 \setminus E$.

Since Y_s is a deformation of Y_0, we have $b_i := Y_s.E_i = Y_0.E_i$; that is

$$M.(a_0, ..., a_n)^t = (1 + b_0, b_1, ..., b_n),$$

where M is the self-intersection matrix of E (the curve E_0 plays a special role in this equality because $Z_0.E_0 = 1$ and $Z_0.E_i = 0$ for $i \neq 0$). Note that the b_i's correspond to the number of returns that lift to E_i. By linear algebra, one obtains that $a_0 \neq 0$ (i.e. E_0 belongs to Y_0) and $b_0 = 0$, that is, Y_s must not intersect E_0.

3.3. **End of the proof.** As explained before, to obtain a contradiction we want to show that U_s has non-positive Euler characteristic. To do this, Fernández de Bobadilla and Pe Pereira give an upper bound on $\chi(U_s)$ in terms of $\chi(Y_s)$, $\chi(Y_0)$ and the possible returns.

Recall that $Y_0 = Z_0 + \sum_i a_i E_i$. We construct a tubular neighborhood of E in the following way.

Define $E_i^o = E_i \setminus Sing(Y_0^{red})$. Let $Sing(Y_0^{red}) = \{p_0, p_1, ..., p_m\}$, where $p_0 = Z_0 \cap E$. Let B_k be a small ball in $\tilde{X}$ centered at p_k. For $j \in \{0, \dots, n\}$, let T_j be a tubular neighborhood of E_j, small enough so that its intersection with each B_k is transverse. Let T_{n+1} be a tubular neighborhood of Z_0, small enough so that its intersection with B_0 is transverse. Let

$$W_j = T_j \setminus \left(\bigcup_{k=0}^{m} B_k \right).$$

All the neighborhoods are chosen so that

$$\chi(U_s) = \sum_{j=0}^{n+1} \chi(\tilde{\alpha_s}^{-1}(Y_s \cap W_j)) + \sum_{k=0}^{m} \chi(\tilde{\alpha_s}^{-1}(Y_s \cap B_k)). \tag{2}$$

We do not need to count $\chi(Y_s \cap T_j \cap B_k)$ since by the assumed transversality each of these intersections is a finite union of circles and thus

$$\chi(Y_s \cap T_j \cap B_k) = 0. \tag{3}$$

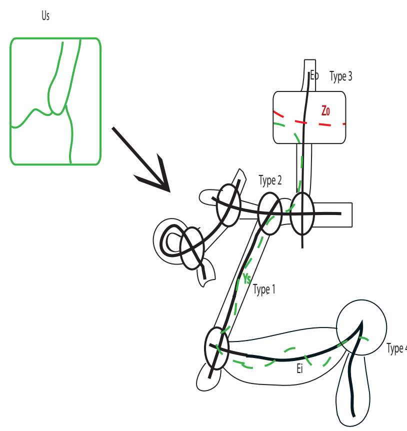

FIGURE 2 . Normalization map

It remains to bound above each summand on the right hand side of (2). To do this, we first consider the special case when $\tilde{X}_\varepsilon$ is a *good resolution* of X_ε, that is, when the exceptional set E has *normal crossings*. We divide the summands appearing in (2) into three types and deal separately with each type.

- Type 1: Terms of the form $\chi(\tilde{\alpha_s}^{-1}(Y_s \cap W_j))$. If $j \leqslant n$, by Hurwitz formula, we have

$$\chi(\tilde{\alpha_s}^{-1}(Y_s \cap W_j)) \leqslant a_i \chi(E_j^o) \tag{4}$$

  as the maps $\tilde{\alpha_s}^{-1}(Y_s \cap W_j) \to E_j \setminus \left(\bigcup_{k=0}^{m} B_k\right)$ are branched covers of degree a_i. For $j = n+1$, $Z_0 \setminus p_0$ is homeomorphic to a punctured disk, so Hurwitz formula gives

$$\chi(\tilde{\alpha_s}^{-1}(Y_s \cap W_{n+1})) \leqslant \chi(Z_0 \setminus p_0) = 0. \tag{5}$$

- Type 2: Terms of the form $\chi(\tilde{\alpha_s}^{-1}(Y_s \cap B_k))$ such that $k > 0$, $p_k \notin Y_s$ and $B_k \cap E$ has only normal crossings. Let (x, y) be local coordinates at p_k such that $f(x, y) = xy$ is a local defining equation of the set $E \cap B_k$. Let $Y_s^1, \ldots, Y_s^q$ be the connected components of the set $Y_s \setminus B_k$. Since the only connected orientable surfaces with boundary having positive Euler characteristic are disks, and in view of (3), we only have to be careful about those Y_s^l which are homeomorphic to disks.

  As Y_s is a deformation of Y_0, the boundary of such a component Y_s^l either deforms to $V(x) \cap S_\varepsilon$ or to $V(y) \cap S_\varepsilon$. This implies that Y_s^l must intersect either $V(x)$ or $V(y)$. In this case one has,

$$\chi(\tilde{\alpha_s}^{-1}(Y_s \cap B_k)) \leqslant \sum_{p \in Y_s \cap Y_0 \cap B_k} I_p(Y_s, Y_0^{red}). \tag{6}$$

- Type 3: Finally, we will show that $\chi(\tilde{\alpha_s}^{-1}(Y_s \cap B_0)) \leqslant a_0 - 1$. Indeed, as Y_0 is reduced locally at Z_0 let us suppose that the local equation at $Z_0 \cup E_0$ is of the form $f_0 = xy^{a_0} = 0$. Let Y_s^l be an irreducible component of $Y_s \cap B_0$ whose normalization is a disk. Then as Y_s is a deformation of Y_0, the boundary of that component Y_s^l either deforms to $V(x) \cap \partial B$ or to $V(y) \cap \partial B_0$. This implies that Y_s^l must intersect either $V(x)$ or $V(y)$. Therefore, as in the case of Type 2, we have

$$\chi(\tilde{\alpha_s}^{-1}(Y_s \cap B_0)) \leqslant \sum_{p \in Y_s \cap Y_0 \cap B_0} I_p(Y_s, Y_0^{red}) \leqslant a_0 + 1. \tag{7}$$

As Y_s is a deformation of Y_0, there exists a connected component F of $Y_s \cap B_0$ whose boundary contains a circle K_s deforming to $V(x) \cap \partial B_0$. If $\partial F = K_s$ then, by the connectedness of Y_s, $Y_s \cap \partial B_0$ does not contain a circle deforming to $V(y) \cap \partial B_0$, which is impossible. Thus $K_s \subsetneq \partial F$, so ∂F must be a union of at least two circles. In particular, the normalization of F cannot be a disk. Since there are at least two circles in $Y_s \cap \partial B_0$ which bound a connected component of $Y_s \cap B_0$ having non-positive Euler characteristic, the inequality (7) remains true after we subtract 2 from the right hand side:

$$\chi(\tilde{\alpha_s}^{-1}(Y_s \cap B_0)) \leqslant a_0 - 1. \tag{8}$$

Combining (2), (4), (5), (6) and (8), we obtain

$$\chi(U_s) \leqslant a_0 - 1 + \sum_{i=0}^{n} a_i(\chi(E_i) - E_i.(Y_0^{red} - E_i)) + \sum_{p \in Y_s \cap (Y_0 \setminus B_0)} I_p(Y_s, Y_0^{red})$$

Rearranging the sum one has

$$\chi(U_s) \leqslant \sum_i a_i(2 - 2g_i + E_i.E_i).$$

This last sum is less or equal to 0 as each member is less than or equal to 0. We have proved that the disk U_s has non-positive Euler characteristic, which gives the desired contradiction. This completes the proof in the case when the minimal resolution $\tilde{X}$ is a good resolution.

We now briefly sketch the proof in the general case, that is, when E is not necessarily normal crossings.

The main difference with the normal crossings case is that now we must take more care to bound the terms in (2) of the form $\chi(\tilde{\alpha_s}^{-1}(Y_s \cap B_k))$ such that $B_k \cap E$ does not have normal crossings (in particular, $k > 0$). Assume that $Y_s \cap B_k \neq \emptyset$. Suppose, too, that Y_s does not pass through p_k (if not, one can reduce the problem to this case by suitably deforming Y_s).

To study the inequality (2), we use the following numerical characters of the singularities of the reduced exceptional set E. For each E_i consider the set of irreducible components of the germ of E_i at each point of $Sing(Y_0^{red})$. We denote by ν_i the total number of local branches of E_i at all the singular points of $Sing(Y_0^{red})$, by μ_i the sum of Milnor numbers of all these local branches, and by η_i the sum of all the pairwise intersection number between the branches.

Fix a sequence of point blowings up of $\tilde{X}_\varepsilon$ under which the total preimage of $E \cap B_k$ is normal crossings, and replace $\tilde{X}_\varepsilon$ by the resulting manifold. The Euler characteristic of the preimage of Y_s is equal to that of Y_s.

Analyzing the blown up surface by techniques similar to the ones used in the case of good resolution, we obtain the inequality

$$\chi(U_s) \leqslant a_0 - 1 + \sum_i a_i(\chi(E_i) + \nu_i - \mu_i - \eta_i - E_i.(Y_0^{red} - E_i)) + \sum_{p \in Y_s \cap (Y_0 \setminus B_0)} I_p(Y_s, Y_0^{red}).$$

Rearranging the sum one has

$$\chi(U_s) \leqslant \sum_i a_i(2 - 2g_i - \mu_i - \eta_i + E_i.E_i)$$

This last sum is less or equal to 0 as each member is less than 0. Contradiction.

3.4. **The non-normal case.** As before, thanks to Lefschetz Principle, Fernández de Bobadilla and Pe Pereira reduce the problem to the complex case.

Let X be a complex algebraic surface, not necessarily normal, and let $\pi : \tilde{X} \to X$ be its minimal resolution of the singularities of X. Let $E := \pi^{-1}(Sing(X))$. Let $E = \bigcup_{i=0}^{n} E_i$ be the decomposition of E into irreducible components.

Definition 3.10. *We say that E_i is of the* **first kind** *if* $\dim \pi(E_i) = 0$ *and of the* **second kind** *if* $\dim \pi(E_i) = 1$.

A priori, we have four types of possible adjacencies: an arc family of the first kind could be adjacent to one of the first or second kind and an arc family of the second kind could be adjacent to one of the first or second kind. The fact that a family of the second kind cannot be adjacent to another one of either first or second kind follows easily from the continuity of the wedge which realizes this hypothetical adjacency. The fact that an arc family C_i of the first kind cannot be adjacent to another family C_j of the first kind follows from the normal case: such an adjacency would induce an adjacency of the preimage of C_i to the preimage of C_j in the normalization of X, which is impossible by the normal case. To settle the last remaining case, that of an arc family C_i of the first kind adjacent to an arc family C_j of the second kind, J. Fernández de Bobadilla and M. Pe Pereira use plumbing to construct an auxiliary normal surface singularity (X', ξ') and two distinct Nash families C'_i and C'_j on X' such that C'_i is adjacent to C'_j, again contradicting the normal case.

4. Higher dimensions

For singularities of higher dimensions, the Nash Problem enunciated as above is false, though a few positive results have been proved: in [16], S. Ishii and J. Kollar give an affirmative answer for toric varieties in all dimensions. Affirmative answers for a family of singularities in dimension higher than 2 by P. Popescu-Pampu and C. Plénat ([32]) and another family by M. Leyton-Alvarez [23] (2011).

In [16], S. Ishii and J. Kollár give a counterexample to the Nash problem in dimension greater than or equal to 4: the hypersurface

$$x^3 + y^3 + z^3 + u^3 + w^6 = 0$$

which has a resolution with two irreducible exceptional components. These are essential, as one is the projectivization of the tangent cone at the singular point (hence it clearly corresponds to a Nash family), and the other one is not uniruled. Then the authors construct geometrically a wedge whose generic arc is in the Nash family, and whose special arc is in the second family.

In May 2012, T. de Fernex gave a counterexample in dimension 3 ([3], 2012). The equation is

$$(x^2 + y^2 + z^2)w + x^3 + y^3 + z^3 + w^5 + w^6 = 0 \tag{9}$$

In the algebraic setting, he can prove that the two exceptional components obtained after two blowing-ups are essential. But as an analytic variety, the hypersurface obtained from (9) by blowing up the origin is locally isomorphic to the non-degenerate quadratic cone, hence it admits a small resolution; this implies that the second exceptional component is not essential, so the counterexample does not apply in the analytic category. Deforming the equation (9), de Fernex obtains a counterexample to the Nash problem in dimension 3, valid in both the algebraic and the analytic setting:

$$(x^2+y^2)w+x^3+y^3+z^3+w^5+w^6=0.$$

An even more recent paper on the Nash problem is due J. Johnson and J. Kollár [17]. In that paper, J. Johnson and J. Kollár gives a new family of counterexamples to the Nash problem in dimension 3, called $c\mathcal{A}_1$-type singularities:

$$x^2+y^2+z^2+t^m=0$$

with m odd, $m>3$. These singularities are isolated and have only one Nash family, but two of the exceptional components in the resolution are essential.
Moreover, J. Johnson and J.Kollár formulates the Revised Nash problem, which we now explain.

Definition 4.1. *Let X be a variety over a field k, $k\subset K$ a field extension of k and ϕ : Spec $K[[t]]\to X$ an arc such that Supp $\phi^{-1}(Sing(X))=\{0\}$. A* **sideways deformation** *of ϕ is an extension of ϕ to a morphism Φ : Spec $K[[t,s]]\to X$ such that*

$$Supp\ \Phi^{-1}(Sing(X))=\{(0,0)\}.$$

Definition 4.2. *We say that X is* **arcwise Nash-trivial** *if every general arc in X_∞^{sing} has a sideways deformation.*

Definition 4.3. *Let X be a variety over k. A divisor over X is called* **very essential** *if the following holds. Let $p: Y\to X$ be a proper birational morphism such that Y is $\mathbb{Q}$-factorial and has only arcwise Nash-trivial singularities. Then $center_Y E$ is an irreducible component of $p^{-1}(Sing(X))$.*

In fact in the three counterexamples above, the components corresponding to Nash families are given precisely by the very essential divisor. Imitating and conceptualizing the proofs of non-essentiality appearing in the above counterexamples, one can show that divisors appearing in the image of the Nash map are very essential. We are lead to the following problem:

Problem 4.4. *Is the Nash map surjective onto the set of very essential divisors?*

In April 2014, when the present paper was well into the refereeing process, Tommaso de Fernex and Roi Docampo [4] made further significant progress on the Nash problem. They defined the notion of **terminal valuations** over X (where X is a variety of any dimension) and showed that any divisor associated to a terminal valuation is in the image of the Nash map. Restricting this result to the case dim $X=2$ provides a new and completely algebraic proof of the Theorem of Fernández de Bobadilla – Pe Pereira. We acknowledge this very important paper even though we did not have a chance to discuss it in detail.

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