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THE BILIPSCHITZ GEOMETRY OF THE A_k SURFACE SINGULARITIES

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Abstract. Although it has been known for over half a century that analytic varieties are topologically conical in the neighborhood of a singular point, it has only become clear in the last decade that they need not be metrically conical. This paper explores that phenomenon in the case of the A_k singularities.

1. Introduction

It has been known for a long time that a complex analytic variety $V \subset \mathbb{C}^n$ is locally conical in two essentially different ways. First, near any point, V lies within arbitrarily small conical neighborhoods of its Zariski tangent cone, a complex analytic cone of the same dimension as V. Secondly, near any point, V is homeomorphic to the real cone over its link, even as an embedded variety. It is natural to ask whether either of these statements might be strengthened.

For example, the smooth part of V inherits a Riemannian metric from $\mathbb{C}^n$ which extends to a metric on V (often called the *inner metric*), and similarly the real cone over the link of V carries an induced metric. One sees quickly that it is too much to expect that locally V be isometric to the real cone over its link, but one might ask if it is bilipschitz to the cone over its link (or, for that matter, to any real cone.) Examples due to Brasselet (see [1]) show that this is not the case for real surface singularities. However, complex curve singularities are always bilipschitz to the cone over over their link [11], the tangent cone of a complex analytic variety has the same dimension as the variety at any point, and it seemed possible that the same might be true for complex surface singularities.

In the last decade, however, it has been become apparent that although topologically conical, a variety of dimension greater than one is often not metrically conical in any reasonable way. A lovely theory has emerged [9] that makes connections with some results in local complex analytic geometry from over thirty years ago. This paper, which is wholly expository, explores these phenomena through a simple (in fact, the simplest) example in which everything is explicitly computable.

2. Notation and Definitions

We collect some definitions and notational conventions that we will use.

Since we are interested in local properties of a variety $V \subset \mathbb{C}^n$ near a point $p \in V$, we will translate p to the origin 0, so assume that $0 \in V$ and that this is the point in which we are interested. We then typically suppress subscripts involving p (or 0).

If A is any subset of $\mathbb{C}^n$ and $s \in \mathbb{C}$ a number, we write sA for the set $\{sa\colon\ a \in A\}$. We define the *real* (respectively, *complex*) *cone* over A (based at 0) to be the sets

$$\mathrm{Cone}_{\mathbb{R}} A = \{sa\colon\ s \in \mathbb{R}, 0 \le s \le 1,\ a \in A\}$$

$$\mathrm{Cone}_{\mathbb{C}} A = \{sa\colon\ s \in \mathbb{C}, |s| \le 1,\ a \in A\}.$$

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A subset of C^n is said to a *real* (or *complex*) *cone* if it is either a cone over a subset of $\mathbb{C}^n$ or the extension $\mathbb{R}\mathrm{Cone}_{\mathbb{R}}A$ (resp., $\mathbb{C}\mathrm{Cone}_{\mathbb{C}}A$) of such to be closed under multiplication by all reals (or complexes). For any $\epsilon > 0$ real, we write S_ϵ and B_ϵ for the sphere and ball, respectively, of radius ϵ in $\mathbb{C}^n$ centered at the origin. In particular, $\epsilon S_1 = S_\epsilon$ and B_ϵ is the complex (and real) cone over S_ϵ.

We let CV denote the Zariski tangent cone to V at 0. It has a natural complex analytic structure. As a set,

$$CV = \{v \in C^n : \text{there exist } x_i \in V,\ s_i \in \mathbb{C}, x_i \to \mathit{0} \text{ with } s_i x_i \to v\}.$$

If $0 < \delta < \pi$ is a small positive real number, the δ-*conical* neighborhood of CV, denoted $\mathcal{N}_\delta(CV)$ is the set $\{x \in C^n : x \neq 0$ such that there exists $v \in CV$ such that the angle between real segments $\mathrm{Cone}_{\mathbb{R}}\{x\}$ and $\mathrm{Cone}_{\mathbb{R}}\{v\}$ is less than $\delta\}$. A variety is locally well-approximated by its Zarisiki tangent cone in the sense that, near any of its points, V lies within arbitrarily small conical neighborhoods of CV. That is, given $\delta > 0$, there exists $\epsilon > 0$ such that

$$B_\epsilon \cap V \subset B_\epsilon \cap \mathcal{N}_\delta(CV).$$

This is a very old result, and the proof follows from directly from fact, easily established from the definition of CV, that for any ϵ,

$$CV \cap S_\epsilon = \lim_{t \to 0}(\frac{1}{t} V \cap S_\epsilon).$$

It has been known for over half a century (see[15]) that for each sufficiently small $\epsilon > 0$, there is a homeomorphism $h : B_\epsilon \to B_\epsilon$ with $h(\mathit{0}) = \mathit{0}$ such that

$$h(V \cap B_\epsilon) = \mathrm{Cone}_{\mathbb{R}}(V \cap S_\epsilon).$$

In particular, for sufficiently small ϵ, the sets $V \cap S_\epsilon$ are homeomorphic and any one is called the *link* of V at $\mathit{0}$.

A variety $V \subset \mathbb{C}^n$ inherits two notions of distance from $\mathbb{C}^n$. The first, the so-called *outer metric*, assigns the distance between two points $x, y \in V \subset \mathbb{C}^n$ to be their distance in $\mathbb{C}^n$ (that is, $\|x - y\|$). The second, the *inner metric*, assigns the distance between x and y to the distance on V with respect to the metric on V induced by that on $\mathbb{C}^n$. (This is the infimum of the lengths of real-analytic paths in V connecting x and y, or equivalently the extension to V of the induced Riemannian metric on the smooth points of V.) A map between two metric spaces is said to be an *isometry* if it preserves distances between points. A map is said to be *bilipschitz* if the distortion between the images of any two points is bounded above and below by a non-zero constant. More precisely, a map $h : V \to W$ between two varieties V, W with metrics d_V and d_W is a *bilipschitz* homeomorphism if there exists a nonzero constant $K > 0$ such that $\frac{1}{K} d_V(x, y) \leq d_W(h(x), h(y)) \leq K d_V(x, y)$ for any $x, y \in V$. Unless explicitly stated otherwise, the metric on a variety is taken to be the inner metric. Two varieties V and W are said to be *bilipschitz equivalent* if there is a bilipschitz map taking one onto the other. A variety V is said to be *metrically conical* if it is bilipschitz equivalent to a cone over its link.

We have seen that locally a variety $V \subset \mathbb{C}^n$ is wedged between a complex cone and a real cone. The bilipschitz behavior of V depends on the behavior of $V \cap S_\epsilon$ as $\epsilon \to 0$. Rescaling $V \cap S_\epsilon$ gives

$$\frac{1}{\epsilon}(V \cap S_\epsilon) = \frac{1}{\epsilon} V \cap S_1.$$

So, we want to study the behavior of the degeneration

$$\frac{1}{\epsilon} V \cap S_1 \to CV \cap S_1$$

as $\epsilon \to 0$. On the other hand, work in the late 1970s and early 1980s by Henry, Lê, Teissier and others (see [12], [13]. [14]) established that the complex-analytic behavior of this degeneration is detected by the limits of tangent spaces to a variety V at the origin, the so-called *Nash cone*. As a result, the Nash cone is linked to the bilipschitz behavior of a cone and the failure of metric conicality.

3. The A_k-singularities

Consider the A_k family of surface singularities. Fix the local equations:

$$V_k = \{(x, y, z) \in \mathbb{C}^3 : xy - z^{k+1} = 0\}.$$

For all $k > 2$, the tangent cone

$$CV_k = \{(x, y, z) \in \mathbb{C}^3 : xy = 0\}$$

is the union of the planes $\{x = 0\}$ and $\{y = 0\}$. Thus, given any $\delta > 0$, we can choose $\epsilon > 0$ such that $V_k \cap B_\epsilon$ lies entirely within a δ-conical neighborhood of $\{xy = 0\}$.

Since V_k has an isolated singularity at the origin, $V_k \cap S_\epsilon$ will be a smooth, necessarily three-dimensional, manifold for $\epsilon > 0$ sufficiently small. An easy computation shows this holds without restriction on $\epsilon > 0$, so we can take $\epsilon = 1$. Fittingly, since the A_k are arguably the simplest and best understood surface singularities, the manifolds $V_k \cap S$ are among the simplest and best understood three-dimensional manifolds: the lens spaces.

Definition. The *lens space* $L(p, q)$ (where p and q are coprime integers) can be defined in one of three equivalent ways.

1. $L(p, q)$ is the quotient of the three-sphere $S^3 = \{(u, v) \in \mathbb{C}^2 : |u|^2 + |v|^2 = 1\}$ by the $\mathbb{Z}/p$ action $(u, v) \mapsto (\zeta u, \zeta^q v)$ where $\zeta = e^{2\pi i/p}$.
2. $L(p, q)$ is the space obtained from a solid three-dimensional ball in $\mathbb{R}^3$ by identifying each point on the upper hemisphere of the boundary 2-sphere to a point on the lower hemisphere as follows: rotate the point on the upper hemisphere clockwise through angle $2\pi q/p$ and identify with the point on the lower hemisphere immediately below.
3. $L(p, q)$ is the space obtained by attaching two disjoint solid tori along their boundaries so that so that the meridian (a $(0, 1)$ curve) of one goes to a $(p, -q)$ curve (that is a curve wrapping p times along the longitude and q times in the opposite direction of the meridian) of the other.

The equivalence of the three definitions is sometimes established in elementary topology classes (see [17] or [18]) and is a pleasant exercise (the biggest nuisance is keeping the orientations straight). Details can be found, for example, in Rolfsen [17] or Thurston [18]. The following result and proof are classical. We shall reprove it in a way that gives more metric information shortly.

Proposition 3.1. *The link $V_k \cap S_1$ is homeomorphic to the lens space $L(k + 1, k)$.*

Proof. (Due to du Val [10]). V_k is parameterized by

$$(s, t) \mapsto (s^{k+1}, t^{k+1}, st).$$

This is a $k+1$ to 1 map with (s, t) and $(\eta^r s, \eta^{kr} t)$ mapping to the same point where $\eta = e^{2\pi i/(k+1)}$ and $0 < r \leq k + 1$. Hence $V_k \cap S_1$ is the quotient of

$$\Sigma_k \equiv \{(s, t) \in \mathbb{C}^2 : |s|^{2(k+1)} + |t|^{2(k+1)} + |st|^2 = 1\}$$

by the $\mathbb{Z}/(k + 1)$ action $(u, v) \mapsto (\eta u, \eta^k v)$. One checks easily that for all $k > 1$, the manifold Σ_k is diffeomorphic to the three-sphere S^3 (the map being radial projection: any real ray in $\mathbb{C}^2$ from the origin to a point of Σ_k meets S_1 in precisely one point and conversely). Hence $V_k \cap S_1 \approx L(k + 1, k)$. □

Note that if $\eta = e^{2\pi i/(k+1)}$ as above, then $\eta^k = \eta^{-1}$. So, $L(k+1, k) = L(k+1, -1)$.

Thus $V_k \cap B_1$ is homeomorphic to a cone over the lens space $V_k \cap S_1 \approx L(k+1, k)$. As discovered by Birbrair, Fernandes, and Neumann [5], it is not, however, bilipschitz to a cone over $L(k+1, k)$.

To investigate this, we consider the rescaled deformation to the tangent cone. That is,

$$\frac{1}{t} V_k \cap S_1 \to CV_k \cap S_1$$

as $t \in \mathbb{C}$ tends to 0. Here, everything is compact, and convergence is pointwise. For every $t > 0$, the left hand side is homeomorphic to the lens space $L(k+1, k)$. This is because scaling by t is a homeomorphism and $t(\frac{1}{t} V_k \cap S_1) = V_k \cap S_t$. Since $CV_k = \{xy = 0\}$, the right-hand side $CV_k \cap S_1$ is the union of $S_1 \cap \{x = 0\}$ and $S_1 \cap \{y = 0\}$, which is the union of the unit three-dimensional sphere S^3_{yz} centered at origin of the yz-coordinate plane and the unit three-dimensional sphere S^3_{xz} centered at the origin of the xz-coordinate plane. These two three-spheres meet in the unit-circle S^1_z in the z-axis. So, topologically, we have

$$\text{Lens space } L(k+1, k) \to \text{Union of two 3–spheres } S^3_{xz} \cup S^3_{yz}.$$

We want to understand this degeneration metrically.

Let $f_k := xy - z^{k+1}$ denote the local equation for V_k. We have

$$(x, y, z) \in \frac{1}{t} V_k \iff f_k(t(x, y, z)) = 0 \iff t^2(xy - t^{k-1} z^{k+1}) = 0.$$

We package this as a hypersurface in the usual manner.

$$W = \{(t, x, y, z) \in \mathbb{C}^4 : F_k = xy - t^{k-1} z^{k+1} = 0\} \subset \mathbb{C}^4.$$

For fixed t, we let

$$W_t = \{(x, y, z) \in \mathbb{C}^3 : (t, x, y, z) \in W\}.$$

Clearly, $\frac{1}{t} V_k = W_t$ and $W_0 = CV_k$. The intersection of W with the tube

$$\{(t, x, y, z) \in C^4 : |x|^2 + |y|^2 + |z|^2 = 1\}$$

tracks the rescaled (to radius 1) intersection of V_k with spheres of radius t as $t \to 0$. We know that $W_t \cap S_1$ is homeomorphic to $L(k+1, k)$, and $W_0 \cap S_1$ is the union of the unit three-sphere in the xz-plane and the unit three-sphere in the yz-plane.

The simplicity of the equations allows direct computation to offer insight. Write

$$W_t \cap S_1 = X_t \cup Y_t$$

where

$$X_t = \{(x, y, z) \in W_t \cap S_1, |x| \le |y|\}$$

and

$$Y_t = \{(x, y, z) \in W_t \cap S_1, |y| \le |x|\}.$$

Since we cannot have both x and y be equal to 0 in $W_t \cap S_1$, note that $y \ne 0$ in X_t and $x \ne 0$ in Y_t. This allows us to display both sets as graphs. In particular, X_t is the graph $x = t^{k-1} z^{k+1}/y$ with $|x|^2 + |y|^2 + |z|^2 = 1, |x| \le |y|$ (and similarly Y_t is the graph $y = t^{k-1} z^{k+1}/x$, with $|x|^2 + |y|^2 + |z|^2 = 1, |y| \le |x|$).

Proposition 3.2. *The sets X_t and Y_t have common boundary a two-torus.*

Proof. The boundary of both X_t and Y_t is the set

$$\{(x,y,z) \in W_t \cap S_1, |x| = |y|\}.$$

Set $y = re^{i\theta}, z = se^{i\phi}$. Note that x is uniquely determined by the choice of y and z, and that neither y nor z can equal zero in ∂X_t. The positive number r is determined uniquely by the positive number s, since $r^2 = |t|^{k-1}s^{k+1}$. Finally, the positive number s is also uniquely determined, because the constraint $|x|^2 + |y|^2 + |z|^2 = 1$ gives $2|t|^{k-1}s^{k+1} + s^2 = 1$, and there is a unique positive solution the latter (since the left side is strictly increasing for $s > 0$). Call it s_0, and let r_0 be such that $r_0^2 = |t|^{k-1}s_0^{k+1}$. On the other hand, $0 \le \theta < 2\pi$ and $0 \le \phi < 2\pi$ are arbitrary, so the subset $\{(x,y,z) : y = r_0e^{i\theta}, z = s_0e^{i\phi}\}$ is manifestly a torus (that is, a set homeomorphic to $S^1 \times S^1$), and the latter is $\partial X_t = \partial Y_t$. □

Proposition 3.3. *The sets X_t and Y_t are solid tori, disjoint except for their common boundary which is a two-torus.*

Proof. It is clear that X_t and Y_t are disjoint except for their common boundary which is a two-torus by proposition 3.2 above. Since $y \neq 0$ in X_t, we can write

$$\begin{aligned} X_t = \{(x,y,z) \quad &= \quad (t^{k-1}z^{k+1}/y, y, z), \text{with} \\ &\quad\ (|t|^{k-1}|z|^{k+1}/|y|) \le |y|, \text{and} \\ &\quad\ \left(|t|^{k-1}|z|^{k+1}/|y|\right)^2 + |y|^2 + |z|^2 = 1\}. \end{aligned}$$

As in the proof of Proposition3.2, set $y = re^{i\theta}, z = se^{i\phi}$. Then

$$\begin{aligned} X_t = \{(x,y,z) \quad &= \quad ((t^{k-1}s^{k+1}/r)e^{(k+1)\theta - \phi}, re^{i\theta}, se^{i\phi}), \\ &\quad\ |t|^{k-1}s^{k+1} \le r^2, \\ &\quad\ |t|^{2(k-1)}s^{2(k+1)}/r^2 + r^2 + s^2 = 1\}. \end{aligned}$$

The last displayed equation can be rewritten as $r^4 - (1-s^2)r^2 + |t|^{2(k-1)}s^{2(k+1)} = 0$ whence

$$r^2 = \frac{1}{2}\left((1-s^2) \pm \sqrt{(1-s^2)^2 - 4|t|^{2(k-1)}s^{2(k+1)}}\right).$$

One checks that choosing a minus sign in the equation above rules out $|t|^{k-1}s^{k+1} \le r^2$ for small $|t|$, whence r is the positive square root:

$$r = \sqrt{\frac{1}{2}\left((1-s^2) + \sqrt{(1-s^2)^2 - 4|t|^{2(k-1)}s^{2(k+1)}}\right)}.$$

In particular, $r = r(s)$ is uniquely determined by s and as s increases from 0 to s_0, $r = r(s)$ decreases from 1 to $r_0 > 0$ where r_0 and s_0 are as in the proof of Proposition 3.2 (that is r_0 is the positive square root of $|t|^{k-1}s_0^{k+1}$ where s_0 is the unique solution of $2|t|^{k-1}s^{k+1} + s^2 = 1$). So, for fixed t, we have

$$X_t = \{\left(\frac{(t^{k-1}s^{k+1}}{r(s)}e^{(k+1)\theta - \phi}, r(s)e^{i\theta}, se^{i\phi}\right), 0 \le s \le s_0, 1 \ge r(s) \ge r_0\}$$

Since $r(s)$ is monotone decreasing and strictly positive on the interval $[0, s_0]$, this displays X_t as a solid torus. By symmetry, Y_t is also a solid torus. □

We are now ready to describe metrically the degeneration of the rescaled links $\frac{1}{t}V_k \cap S_1$ to the link $CV_k \cap S_1$ of the tangent cone.

Proposition 3.4. *For each $t \neq 0$, the link $\frac{1}{t}V_k \cap S_1$ is the union of two congruent solid tori X_t, Y_t in S_1 disjoint except for their common boundary $\partial X_t = \partial Y_t$. With suitable framings, a meridian on one corresponds to a $(k+1, -k)$ curve on the other (so that their union is the lens space $L(k+1, k)$). As t tends to zero, the torus $X_t \cap Y_t$ shrinks to the unit circle in z-axis, X_t to $\{x = 0\} \cap S_1$, Y_t to $\{y = 0\} \cap S_1$ and $X_t \cup Y_t$ tends to the union of two three-spheres in the unit 5-sphere S_1 intersecting in a circle of radius one in the z-axis.*

Proof. Note that the proof of Proposition 3.3 quickly yields framings of the solid tori X_t and Y_t. In particular, we see immediately that a meridian on the torus ∂X_t is a $(k+1, 1)$ curve on the same torus thought of as ∂Y_t. Since $L(k+1, -1) = L(k+1, k)$ (see the remark following Proposition 3.1), this gives an alternative proof of Proposition 3.1. (Alternatively, it yields an unusual proof of the equivalence of characterizations 1 and 3 in the definitions of the lens space $L(k+1, k)$.)

The remaining assertions of the proposition follow from the equations for X_t, Y_t and $\partial X_t = \partial Y_t$ in the proofs of Propositions 3.2 and 3.3. □

Note that as the torus $\{|x| = |y|\} \cap \frac{1}{t}V_k \cap S_1$ shrinks to the circle $\{x = y = 0, |z| = 1\}$ the topology encoded in how X_t and Y_t are identified along their common boundary is lost and the lens space $L(k+1, k)$ simplifies to two three-spheres meeting only along a geodesic circle.

The collapse of the torus to a circle in the deformation $\frac{1}{t}V_k \cap S_1 \to CV_k \cap S_1$ is an obstruction to metric conicality. For it corresponds to the separating set $\{|x| = |y|\} \cap V_k$ (that is, a set Z that separates V_k, but has $\dim CZ < \dim CV_k$). Alternatively, any choice of meridians in the tori $\{|x| = |y|\} \cap \frac{1}{t}V_k$ that vary smoothly with t gives a choking horn (see [4]).

4. Limits of Tangent Spaces

The phenomenon detailed in the last section with the varieties $V_k \subset \mathbb{C}^3$ whereby a torus collapses onto a circle as $\frac{1}{t}V_k \cap S_1 \to CV_k \cap S_1$, resulting in a loss of topology, is quite general, and turns out to be linked to a phenomenon elucidated in the late 1970s and early 1980s by Lê, Henry, Teissier and others, namely the structure of limiting tangent spaces to a variety $V \subset \mathbb{C}^n$ at a singular point.

Just as considering the limits of secants to a variety at a singular point gives a geometrically significant object (namely, the Zariski tangent cone), one can usefully consider limits of other geometric obects associated to points of a variety as one tends to a singular point. In particular, we can consider the set of limits of tangent spaces at smooth points of a variety V as one tends to a singular point, the so-called *Nash cone*, denoted $N(V)$. Whitney had originally shown that any limit of tangent spaces to the tangent cone of a variety is, in fact, a limiting tangent space to the variety (that is, $N(CV) \subset N(V)$), but not conversely. The limits of tangent spaces to V which are not limits of tangent spaces to the tangent cone reveal features of the local geometry of a variety which are not captured by the tangent cone and, hence, are the parts of the Nash cone of particular interest. In the case of surfaces, the following result, due to Lê and Henry [11] in the case of an isolated singularity and to Lê [12] in general, characterizes the "extra" limiting tangent spaces to a surface in $\mathbb{C}^n$. Lê, Teissier and others [13, 14] have generalized these results to algebraic varieties of arbitrary dimension and codimension. The nicest formulation is in terms of the conormal cone, which coincides with the Nash cone in the case of hypersurfaces.

Theorem 4.1. *Suppose that V is an algebraic surface, $0 \in V \subset \mathbb{C}^3$. There exists a finite (possibly empty) set of lines $\ell_1, \dots, \ell_r \subset CV$, $0 \in \ell_i$ for all $1 \leq i \leq r$, (called* **exceptional lines***) such that*

$$N(V) = N(CV) \bigcup \left(\bigcup_{i=1}^{r} N(\ell_i) \right)$$

where $N(\ell_i)$ denotes the pencil of all planes in $\mathbb{C}^3$ containing ℓ_i.

The theorem is proved in [13] by considering the deformation of V to CV as we did with V_k. The exceptional lines correspond to the lines along which the deformation is not equisingular in a well-defined sense. (More precisely, take a local equation $\{f = 0\}$ for V, and consider the hypersurface $W \subset \mathbb{C}^4$ with equation $F = 0$ where $F(t, x, y, z) = f(tx, ty, tz)/t^r$, with r being the multiplicity of f at the origin. In the case when the singularity is isolated, one then examines where Whitney condition a) does not hold along the stratum $W \cap \{t = 0\}$.) The exceptional lines are explicitly computable because there are a number of useful equivalent characterizations of them. In the case $V_k = \{(x, y, z) \in \mathbb{C}^3 : xy - z^{k+1} = 0\}$, no machinery is needed, and direct computation establishes the following.

Proposition 4.2. *The Nash cone $N(V_k)$ consists all complex planes containing the z-axis. (So $N(CV_k)$ is the union of the coordinate planes $\{x = 0\}$ and $\{y = 0\}$, and there is one exceptional line, the z-axis.)*

Proof. Direct computation establishes that the limit of tangent spaces to V_k along any path $\mathbf{u}(t) = (x(t), y(t), z(t))$ tending to the origin on V_k is well-defined (and either $\{x = 0\}$ or $\{y = 0\}$) as long as $\mathbf{u}(t)$ is not tangent to the z-axis. Conversely, one easily constructs paths $\mathbf{u}(t) \subset V_k$ tangent to the z-axis along which the tangent planes to V_k tend to any prescribed plane containing the z-axis. □

Now, let us return to the general situation where $V \subset \mathbb{C}^3$ is a surface, and consider the deformation

$$\frac{1}{t} V \cap S_1 \to CV \cap S_1.$$

If $\ell_1, \ldots, \ell_r \subset CV$ are exceptional lines, then

$$\ell_i \cap S_1$$

are real circles. It may be, as in the case of the V_k that one or more of these circles is the locus along which a two-dimensional torus (or higher genus surface) in the rescaled link collapses. In these instances, we have an obstruction to metric conicality. In other cases, (such as

$$V = \{x^2 + y^k - z^k = 0\}, k > 2,$$

where $\ell_1, \ldots \ell_k$ are the k exceptional lines in $\{x = 0\}$ corresponding the k factors of $y^k - z^k$), the circles $\ell_i \cap S_1$ do not represent loci along onto which a two-dimensional surface retracts and do not obstruct metric conicality.

Several conclusions emerge from these observations. First, if V has an isolated singularity, and there are no exceptional lines, then V is metrically conical. Second, some exceptional lines obstruct metric conicality, whereas others do not. Since exceptional lines are easily computable, it would be useful to have some effective criterion to tell the two cases apart. Third, it would be useful to have a catalog of possible topological and metric degenerations along exceptional lines. This would give another way to classify surface singularities.

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