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A NOTE ON THE MOND CONJECTURE AND CROSSCAP CONCATENATIONS

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Abstract. We prove the Mond conjecture relating the codimension of a map germ from $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$ with its image Milnor number for bigerms resulting from the operation of simultaneous augmentation and monic concatenation. We then define a new operation, the crosscap concatenation, in order to obtain new examples of multigerms where the Mond conjecture can be tested.

1. Introduction

In recent years a new impulse in the study of classification of singularities of map germs $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ with $S = \{x_1, \ldots, x_s\}$ under $\mathcal{A}$-equivalence (changes of coordinates in source and target) has taken place, specially regarding multigerms (when $s > 1$). (We consider complex analytic maps when $\mathbb{K} = \mathbb{C}$ and smooth maps when $\mathbb{K} = \mathbb{R}$.) Some classifications of multigerms have been carried out as in [7], where Hobbs and Kirk classify certain multigerms from surfaces to $\mathbb{R}^3$ using the complete transversal's method. Other classifications have been used in different contexts such as Vassiliev type invariants (see [6, 14, 2, 3], for example), where multigerms up to codimension 2 are needed. However, the classical singularity theory techniques used to classify monogerms are hard to deal with when working with multigerms.

A different approach to classify multigerms consists in defining operations in order to obtain germs and multigerms from other germs in lower dimensions and codimensions. In [4], Cooper, Wik Atique and Mond defined the operations of augmentation, monic concatenation and binary concatenation. They proved that any minimal corank codimension 1 multigerm with (n, p) in Mather's nice dimensions and $n \geq p - 1$ can be obtained using these operations starting from monogerms and one bigerm with $p = 1$. However, these operations fail to give complete lists of codimension 2 multigerms. To this purpose, in [15], Oset Sinha, Ruas and Wik Atique defined other operations, a simultaneous augmentation and monic concatenation and a generalised concatenation which includes the monic and binary concatenations as particular cases. They proved that any codimension 2 multigerm of minimal corank in Mather's nice dimensions and $n \geq p-1$ can be obtained using these new operations from monogerms and some special multigerms with $p \leq 2$.

Another active field of research regarding classification of germs is to prove the Mond conjecture relating the deformation-theoretic codimension (the $\mathcal{A}_e$-codimension) of a germ with the topology of a stable perturbation of it. Mond proved in [13] that given a finitely determined map germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ with $(n, n+1)$ in Mather's nice dimensions ($n < 15$), the image of a stable perturbation has the homotopy type of a wedge of n-spheres. The number of spheres in the wedge is called the image Milnor number and is denoted by μ_I. De Jong and Van

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Straten ([5]) and Mond ([13]) proved that

$$\mathcal{A}_e - codim(f) \leq \mu_I(f) \tag{1}$$

for the case $n = 2$. Since then only partial results have been obtained such as [4] where Cooper, Mond and Wik Atique proved this relation for corank 1 codimension 1 germs, [11] where Houston and Kirk proved it for some corank 1 monogerms from $\mathbb{C}^3$ to $\mathbb{C}^4$ or [1] where Altintas proves it for some families of corank 2 germs. In fact, Altintas defines a generalisation of augmentation and proves it for any germ obtained in this way. The conjecture that the relation 1 is satisfied whenever the pair $(n, n+1)$ is in Mather's nice dimensions is known as the Mond conjecture.

In this paper we prove the Mond conjecture for corank 1 bigerms obtained by the operation of simultaneous augmentation and monic concatenation defined in [15]. We then define a new type of generalised concatenation, *crosscap concatenation*, to provide a new source of examples to test the Mond conjecture.

Section 2 contains some basic definitions and preliminaries. In Section 3 we prove the Mond conjecture for the operation of simultaneous augmentation and concatenation. Finally, in Section 4 we define the crosscap concatenations and give a formula to obtain the codimension of the resulting multigerms. We give some new examples of multigerms from $\mathbb{C}^4$ to $\mathbb{C}^5$ which can be tested for the Mond conjecture.

2. Notation and definitions

Let $\mathcal{O}_n^p$ be the vector space of map germs with n variables and p components. When $p = 1$, $\mathcal{O}_n^1 = \mathcal{O}_n$ is the local ring of germs of functions in n-variables and $\mathcal{M}_n$ its maximal ideal. The set $\mathcal{O}_n^p$ is a free $\mathcal{O}_n$-module of rank p. A multigerm is a germ of an analytic (complex case) or smooth (real case) map $f = \{f_1, \ldots, f_r\} : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ where $S = \{x_1, \ldots, x_r\} \subset \mathbb{K}^n$, $f_i : (\mathbb{K}^n, x_i) \to (\mathbb{K}^p, 0)$ and $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. Let $\mathcal{M}_n\mathcal{O}_n^p$ be the vector space of such map germs. Let $\theta_{\mathbb{K}^n,S}$ and $\theta_{\mathbb{K}^p,0}$ be the $\mathcal{O}_n$-module of germs at S of vector fields on $\mathbb{K}^n$ and $\mathcal{O}_p$-module of germs at 0 of vector fields on $\mathbb{K}^p$ respectively. Let $\theta(f)$ be the $\mathcal{O}_n$-module of germs $\xi : (\mathbb{K}^n, S) \to T\mathbb{K}^p$ such that $\pi_p \circ \xi = f$ where $\pi_p : T\mathbb{K}^p \to \mathbb{K}^p$ denotes the tangent bundle over $\mathbb{K}^p$.

Define $tf : \theta_{\mathbb{K}^n,S} \to \theta(f)$ by $tf(\chi) = df \circ \chi$ and $wf : \theta_{\mathbb{K}^p,0} \to \theta(f)$ by $wf(\eta) = \eta \circ f$. The $\mathcal{A}_e$-tangent space of f is defined as $T\mathcal{A}_e f = tf(\theta_{\mathbb{K}^n,S}) + wf(\theta_{\mathbb{K}^p,0})$. Finally we define the $\mathcal{A}_e$-codimension of a germ f, denoted by $\mathcal{A}_e$-cod(f), as the $\mathbb{K}$-vector space dimension of

$$N\mathcal{A}_e(f) = \frac{\theta(f)}{T\mathcal{A}_e f}.$$

A vector field germ $\eta \in \theta_{\mathbb{K}^p,0}$ is called *liftable over* f, if there exists $\xi \in \theta_{\mathbb{K}^n,S}$ such that $df \circ \xi = \eta \circ f$ $(tf(\xi) = wf(\eta))$. The set of vector field germs liftable over f is denoted by Lift(f) and is an $\mathcal{O}_p$-module. When $\mathbb{K} = \mathbb{C}$ and f is complex analytic, $Lift(f) = Derlog(V)$ where V is the discriminant of f and $Derlog(V)$ is the $\mathcal{O}_p$-module of tangent vector fields to V.

Next we give the definitions of the operations mentioned throughout the paper:

Definition 2.1. [8] *Let $h : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ be a map-germ with a 1-parameter unfolding $H : (\mathbb{K}^n \times \mathbb{K}, S \times \{0\}) \to (\mathbb{K}^p \times \mathbb{K}, 0)$ which is stable as a map-germ, where $H(x, \lambda) = (h_\lambda(x), \lambda)$, such that $h_0 = h$. Let $g : (\mathbb{K}^q, 0) \to (\mathbb{K}, 0)$ be a function-germ. Then, the augmentation of h by H and g is the map $A_{H,g}(h)$ given by $(x, z) \mapsto (h_{g(z)}(x), z)$.*

Definition 2.2. *Suppose $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ is non-stable of finite $\mathcal{A}_e$-codimension and has a 1-parameter stable unfolding $F(x, \lambda) = (f_\lambda(x), \lambda)$. Let $k \geq 0$ and $g : (\mathbb{K}^p \times \mathbb{K}^k, 0) \to (\mathbb{K}^p \times \mathbb{K}, 0)$ be the fold map $(X, v) \mapsto (X, \Sigma_{j=1}^k v_j^2)$ (when $k = 0$ $g(X) = (X, 0)$). Then the multigerm $\{F, g\}$ is called the monic concatenation of f.*

Definition 2.3. *Given germs $f_0 : (\mathbb{C}^m, S) \to (\mathbb{C}^a, 0)$ and $g_0 : (\mathbb{C}^l, T) \to (\mathbb{C}^b, 0)$ with 1-parameter stable unfoldings $F(y, s) = (f_s(y), s)$ and $G(x, s) = (g_s(x), s)$, the multigerm h with $|S| + |T|$ branches defined by*

$$\begin{cases} (X, y, s) \mapsto (X, f_s(y), s) \\ (x, Y, s) \mapsto (g_s(x), Y, s) \end{cases} \tag{2}$$

is called the binary concatenation of f_0 and g_0.

3. Augmentation and concatenations and the Mond conjecture

In [8, Theorem 3.3], Houston states the following: Let F be a 1-parameter stable unfolding of a finitely determined f, then

$$\mathcal{A}_e - cod(A_{F,\phi}(f)) \geq \mathcal{A}_e - cod(f)\tau(\phi) \tag{3}$$

where τ is the Tjurina number and with equality if F or ϕ is quasihomogeneous.

He then uses this theorem to prove in [9, Theorem 6.7] that if $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ is a finitely determined map germ satisfying the Mond conjecture, F is a 1-parameter stable unfolding of it and ϕ defines an isolated hypersurface singularity, then if f or ϕ is quasihomogeneous

$$\mathcal{A}_e - cod(A_{F,\phi}(f)) \leq \mu_I(A_{F,\phi}(f)) \tag{4}$$

with equality if both f and ϕ are quasihomogeneous. In the proof he uses the fact that f being quasihomogeneous implies that F is quasihomogeneous in order to apply Theorem 3.3 from [8].

However, in [10, Theorem 4.4] he proves a slightly more general version of Theorem 3.3 from [8] and points out that if ϕ is not quasihomogeneous and F is, the unfolding parameter must have a non-zero weight for the result to hold. He defines the concept of substantial unfolding: Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be a map germ and $F(x, \lambda) = (f_\lambda(x), \lambda)$ a 1-parameter unfolding. We say that F is a substantial unfolding if λ is contained in $d\lambda(Lift(F))$.

Therefore the inequality (4) holds if ϕ is quasihomogeneous or F is a substantial unfolding and equality is reached when both hypotheses are satisfied at the same time.

In [15] a new operation was defined which merges two other ones, it is a simultaneous augmentation and monic concatenation. The authors proved the following

Theorem 3.1. [15] *Suppose $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ has a 1-parameter stable unfolding*

$$F(x, \lambda) = (f_\lambda(x), \lambda).$$

Let $g : (\mathbb{K}^p \times \mathbb{K}^{n-p+1}, 0) \to (\mathbb{K}^p \times \mathbb{K}, 0)$ be the fold map $(X, v) \mapsto (X, \Sigma_{j=p+1}^{n+1} v_j^2)$. Then,

i) the multigerm $\{A_{F,\phi}(f), g\}$, where $\phi : \mathbb{K} \to \mathbb{K}$, has

$$\mathcal{A}_e - cod(\{A_{F,\phi}(f), g\}) \geq \mathcal{A}_e - cod(f)(\tau(\phi) + 1),$$

where τ is the Tjurina number of ϕ. Equality is reached when ϕ is quasi-homogeneous and $\langle dZ(i^*(Lift(A_{F,\phi}(f))))\rangle = \langle dZ(i^*(Lift(F)))\rangle$ where $i : \mathbb{K}^p \to \mathbb{K}^{p+1}$ is the canonical immersion $i(X_1, \ldots, X_p) = (X_1, \ldots, X_p, 0)$ and dZ represents the last component of the target vector fields.

ii) $\{A_{F,\phi}(f), g\}$ has a 1-parameter stable unfolding.

The condition on ϕ to reach equality can be replaced by F being a substantial unfolding since the proof uses Houston's result (3).

Our main result in this section is

Theorem 3.2. *Suppose $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ satisfies the Mond conjecture and has a 1-parameter substantial stable unfolding $F(x, \lambda) = (f_\lambda(x), \lambda)$. Let $g : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1} \times \mathbb{C}, 0)$ be the immersion $X \mapsto (X, 0)$. Suppose that $\langle dZ(g^*(Lift(A_{F,\phi}(f))))\rangle = \langle dZ(g^*(Lift(F)))\rangle$ where dZ represents the last component of the target vector fields. Then, the multigerm $\{A_{F,\phi}(f), g\}$, where $\phi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, satisfies the Mond conjecture, i.e.*

$$\mathcal{A}_e - cod(\{A_{F,\phi}(f), g\}) \leq \mu_I(\{A_{F,\phi}(f), g\}).$$

Equality is reached if both f and ϕ are quasihomogeneous.

Proof. By the proof of Theorem 3.1 we know that $\{A_{F,\phi}(f), g\}$ has a 1-parameter stable unfolding

$$\begin{cases} (f_{\phi(z)+\delta}(x), z, \delta) \\ (X, 0, \delta) \end{cases} . \tag{5}$$

Define $Af_\delta(x, z) := (f_{\phi(z)+\delta}(x), z)$, which is a stable perturbation of $A_{F,\phi}(f)$ (see [8, Theorem 3.8]). By definition, $\mu_I(\{A_{F,\phi}(f), g\}) = rkH_{n+1}(D(Af_\delta) \cup D(g))$ where $D(f)$ stands for the image of f. Since $D(g) = g(\mathbb{C}^{n+1})$, $D(Af_\delta) \cap D(g) = D(f_\delta)$ where f_δ is a stable perturbation of f, therefore $rkH_n(D(Af_\delta) \cap D(g)) = \mu_I(f)$. Consider the Mayer-Vietoris exact sequence (considering an appropriate collar extension for $D(Af_\delta)$ and $D(g)$ along their intersection):

$$\longrightarrow H_{n+1}(D(Af_\delta) \cap D(g)) \longrightarrow H_{n+1}(D(Af_\delta)) \oplus H_{n+1}(D(g)) \longrightarrow$$
$$\longrightarrow H_{n+1}(D(Af_\delta) \cup D(g)) \longrightarrow H_n(D(Af_\delta) \cap D(g)) \longrightarrow \dots$$

Clearly $rkH_{n+1}(D(g)) = 0$. Since $D(Af_\delta) \cap D(g)$ is homotopy equivalent to a wedge of n-spheres it has non zero homology only in dimensions 0 and n so $rkH_{n+1}(D(Af_\delta) \cap D(g)) = 0$ and the sequence is in fact a short exact sequence. By the exactness of the sequence and the First Isomorphism Theorem we obtain $\mu_I(\{A_{F,\phi}(f), g\}) = \mu_I(A_{F,\phi}(f)) + \mu_I(f)$.

Finally we have that

$$\mathcal{A}_e - cod(\{A_{F,\phi}(f), g\}) = \mathcal{A}_e - cod(f)(\tau(\phi) + 1), \text{ by Theorem 3.1,} \tag{6}$$
$$= \mathcal{A}_e - cod(A_{F,\phi}(f)) + \mathcal{A}_e - cod(f), \text{ by (3),} \tag{7}$$
$$\leq \mu_I(A_{F,\phi}(f)) + \mathcal{A}_e - cod(f), \text{ by (4),} \tag{8}$$
$$\leq \mu_I(A_{F,\phi}(f)) + \mu_I(f), \text{ by Mond's conjecture for } f, \tag{9}$$
$$= \mu_I(\{A_{F,\phi}(f), g\}), \text{ by the Mayer-Vietoris argument.} \tag{10}$$

The first inequality turns into equality if ϕ is quasihomogeneous and the second inequality turns into equality when f is quasihomeogeneous. □

It seems probable that Theorem 6.7 in [9] is true for multigerms too, and in this case the above Theorem would be true when f is a multigerm. However, many of the proofs in [9] would have to be rewritten and we leave this for future work.

Example 3.3. i) *Consider $f_k(x, y) = (x^3 + y^{k+1}x, x^2, y)$ and the 1-parameter stable unfolding $F_k(x, y, \lambda) = (x^3 + y^{k+1}x + \lambda x, x^2, y, \lambda)$. We augment and concatenate them and obtain the family of bigerms*

$$\begin{cases} (x^3 + y^{k+1}x + z^{l+1}x, x^2, y, z) \\ (x, y, z, 0) \end{cases} \tag{11}$$

These bigerms have codimension $k(l + 1)$ and satisfy the Mond conjecture. These examples of bigerms from $\mathbb{C}^3$ to $\mathbb{C}^4$ were not known to satisfy the Mond conjecture up to now.

ii) *Consider* $f(u,v,x) = (u,v,x^3+ux,x^4+vx)$ *and the 1-parameter stable unfolding*

$$F(u,v,x,\lambda) = (u,v,x^3+ux,x^4+vx+\lambda x^2,\lambda).$$

We augment (with different augmenting functions) and concatenate it and obtain the bigerms

$$\begin{cases}(u,v,x^3+ux,x^4+vx+z^lx^2,z)\\(u,v,x,z,0)\end{cases} \tag{12}$$

which satisfy the Mond conjecture. These examples of bigerms from $\mathbb{C}^4$ *to* $\mathbb{C}^5$ *were not known to satisfy the Mond conjecture up to now.*

4. Crosscap concatenation

Definition 4.1. [15] *Let* $f:(\mathbb{K}^{n-s},S)\to(\mathbb{K}^{p-s},0)$, $s<p$, *be of finite* $\mathcal{A}_e$*-codimension and let* $F:(\mathbb{K}^n,S\times\{0\})\to(\mathbb{K}^p,0)$ *be a s-parameter stable unfolding of* f *with*

$$F(x_1,\dots,x_n) = (F_1(x_1,\dots,x_n),\dots,F_{p-s}(x_1,\dots,x_n),x_{n-s+1},\dots,x_n),$$

where $F_i(x_1,\dots,x_{n-s},0,\dots,0) = f_i(x_1,\dots,x_{n-s})$. *Suppose that* $\overline{g}:(\mathbb{K}^{n-p+s},T)\to(\mathbb{K}^s,0)$ *is stable. Then the multigerm* $\{F,g\}$ *is a generalised concatenation of* f *with* g, *where*

$$g = Id_{\mathbb{K}^{p-s}}\times\overline{g}.$$

A germ f is said to be a *suspension* of a germ f_0 if $f = id\times f_0$. An unfolding is said to be trivial if it is $\mathcal{A}$-equivalent to a suspension. In the previous definition g is a suspension of $\overline{g}$.

In [15], several examples of generalised concatenations in the equidimensional case were given, namely the cuspidal concatenation and the double fold concatenation. It was shown there that in order to obtain all codimension 2 multigerms this operation is necessary. The definition is very general and can only be controlled when studying a particular example. We give here a new type of generalised concatenation for the case $n = p-1$, a crosscap concatenation.

Definition 4.2. *Consider* $f:(\mathbb{K}^{n-3},S)\to(\mathbb{K}^{n-2})$ *with* $n\geq 3$, $F(x,\lambda) = (f_\lambda(x),\lambda)$ *a 3-parameter stable unfolding of* f *and*

$$g(x_1,...,x_{n-3},y,z,w) = (x_1,...,x_{n-3},y,z,w^2,zw),$$

a suspension of a crosscap. We call the multigerm $\{F,g\}$ *the crosscap concatenation of* f.

Definition 4.2 is independent up to $\mathcal{A}$-equivalence of the choice of parametrisation of g as long as it is an $(n-2)$-parameter suspension of a crosscap:

Proposition 4.3. *Given* $\tilde{g} = id_{\mathbb{K}^{n-2}}\times\tilde{g}_0$, *where* $\tilde{g}_0$ *is* $\mathcal{A}$*-equivalent to* (z,w^2,zw), *there exists a 3-parameter stable unfolding* F' *of* f *such that* $\{F',\tilde{g}\}$ *is* $\mathcal{A}$*-equivalent to* $\{F,g\}$.

Proof. Suppose we choose a different parametrisation

$$\tilde{g}(x_1,...,x_{n-3},y,z,w) = (x_1,...,x_{n-3},y,a(z,w),b(z,w),c(z,w))$$

such that $\tilde{g}$ is $\mathcal{A}$-equivalent to g. Since the suspensions $\tilde{g}$ and g are trivial $(n-2)$-parameter unfoldings of a crosscap, then $\tilde{g}$ and g are equivalent as unfoldings and there exist changes of coordinates ϕ and ψ such that $g = \phi\circ\tilde{g}\circ\psi$ and $\phi = id_{\mathbb{K}^{n-2}}\times\tilde{\phi}$ and $\psi = id_{\mathbb{K}^{n-2}}\times\tilde{\psi}$. We have that $g = (x_1,...,x_{n-3},y,\tilde{\phi}(a\circ\tilde{\psi},b\circ\tilde{\psi},c\circ\tilde{\psi}))$, so $\{F,\tilde{g}\}$ is $\mathcal{A}$-equivalent to $\{(f_\lambda(x),\tilde{\phi}(\lambda)),g\}$ which is $\mathcal{A}$-equivalent to $\{(f_{\tilde{\phi}^{-1}(\lambda)}(x),\lambda),g\}$ where $(f_{\tilde{\phi}^{-1}(\lambda)}(x),\lambda)$ is a 3-parameter stable unfolding of f. That is, given a different parametrisation $\tilde{g}$, there exists a 3-parameter stable unfolding F' of f such that $\{F',\tilde{g}\}$ is $\mathcal{A}$-equivalent to $\{F,g\}$. □

Theorem 4.4. *Let $f:(\mathbb{K}^{n-3},S)\to(\mathbb{K}^{n-2},0)$ with $n\geq 3$ and $\{F,g\}$ the crosscap concatenation of f, then*

$$\mathcal{A}_e-cod(\{F,g\})=dim_{\mathbb{K}}\frac{\mathcal{O}_n\oplus\mathcal{O}_n}{T_0},$$

where

$$T_0=\{(\xi_1,\xi_2);\xi_1=2wv_n(x,y,z,w)+\eta_n(x,y,z,w^2,zw)$$

and

$$\xi_2=-w\eta_{n-1}(x,y,z,w^2,zw)+zv_n(x,y,z,w)+\eta_{n+1}(x,y,z,w^2,zw)\},$$

η_{n-1}, η_n and η_{n+1} are the last three components of vector fields in $Lift(F)$ and $v_n\in\mathcal{O}_n$.

Proof. Similarly to the proofs of [4, Theorem 3.1] and [15, Theorems 4.3 and 4.12] the following sequence is exact

$$0\longrightarrow\frac{\theta(g)}{tg(\theta_n)+wg(Lift(F))}\longrightarrow N\mathcal{A}_e(\{F,g\})\longrightarrow N\mathcal{A}_e(F)\longrightarrow 0.$$

Since F is stable, $\dim_{\mathbb{K}}N\mathcal{A}_e(F)=0$, hence $\mathcal{A}_e$-cod$(\{F,g\})=\dim_{\mathbb{K}}\frac{\theta(g)}{tg(\theta_n)+wg(Lift(F))}$.

By projection to the last three components we have that $\frac{\theta(g)}{tg(\theta_n)+wg(Lift(F))}$ is isomorphic to $\frac{\mathcal{O}_n\oplus\mathcal{O}_n\oplus\mathcal{O}_n}{T}$, where

$$T=\left\{\begin{pmatrix}1&0\\0&2w\\w&z\end{pmatrix}\begin{pmatrix}v_{n-1}\\v_n\end{pmatrix};v_{n-1},v_n\in\mathcal{O}_n\right\}+d(Z,W_1,W_2)(wg(Lift(F)))$$

and $d(Z,W_1,W_2)$ represents the last three components of $wg(Lift(F))$.

Let

$$T_0=\{(\xi_1,\xi_2);(0,\xi_1,\xi_2)\in T\}=$$

$$=\{(\xi_1,\xi_2);\quad \begin{array}{l}\xi_1=2wv_n(x,y,z,w)+\eta_n(x,y,z,w^2,zw)\text{ and}\\ \xi_2=-w\eta_{n-1}(x,y,z,w^2,zw)+zv_n(x,y,z,w)+\eta_{n+1}(x,y,z,w^2,zw)\}\end{array}$$

where $\eta=(\eta_1,...,\eta_n,\eta_{n+1})\in Lift(F)$.

Let (g_{n-1},g_n,g_{n+1}) be the last three components of g and let

$$T_1=tg_{n-1}(\theta_n)+dZ(wg(Lift(F)))$$

The following sequence is exact (see Proposition 2.1 in [12] for a justification)

$$0\longrightarrow\frac{\mathcal{O}_n\oplus\mathcal{O}_n}{T_o}\xrightarrow{i^*}\frac{\mathcal{O}_n\oplus\mathcal{O}_n\oplus\mathcal{O}_n}{T}\xrightarrow{\pi^*}\frac{\theta(g_{n-1})}{T_1}\longrightarrow 0$$

where i is the inclusion and π is the projection. Since g_{n-1} is a submersion,

$$\mathcal{A}_e-cod\{F,g\}=\dim_{\mathbb{K}}\frac{\mathcal{O}_n\oplus\mathcal{O}_n}{T_o}.$$

□

Notice that the codimension (and so the resulting multigerm) depends on the choice of stable unfolding. This implies that there is little chance of proving the Mond conjecture for crosscap concatenations in general. However, each example may be studied separately. The following examples illustrate how the crosscap concatenation depends on the choice of stable unfolding.

Example 4.5. i) *Let* $f(x) = (x^2, x^3)$ *and the family of* 3*-parameter stable unfoldings* $F_l(x,y,z,w) = (x^2, x^3 + xy^l + xz, y, z, w)$, $l \geq 1$. *Concatenating with a crosscap we obtain the bigerms*

$$\{F_l, g\} : \begin{cases} (x^2, x^3 + xy^l + xz, y, z, w) \\ (x, y, z, w^2, zw) \end{cases} .$$

In this case

$$Lift(F_l) = \big\langle (0,0,0,0,1), (0,0,-1, lZ^{l-1}, 0), (0, Y, 0, X + Z^l + W_1, 0),$$
$$(-2X, 0, 0, 3X + Z^l + W_1, 0), (0, X^2 + XZ^l + XW_1, 0, Y, 0),$$
$$(2Y, 3X^2 + 4XW_1 + W_1^2 + 4XZ^l + 2Z^lW_1 + Z^{2l}, 0, 0, 0)\big\rangle .$$

The only standard generators of $\mathcal{O}_4 \oplus \mathcal{O}_4$ *missing from* T_0 *are* $(1,0)$,$(z,0)$,$\ldots$, $(z^{l-1},0)$ *and so* $\mathcal{A}_e$*-cod*$(\{F_l, g\}) = l$.

Now consider the 3*-parameter stable unfolding:*

$$F_\infty : (x^2, x^3 + xy, y, z, w).$$

$\{F_\infty, g\}$ *is not finitely determined. In fact,*

$$Lift(F_\infty) = \langle (0,0,0,0,1), (0,0,0,1,0), (0, Y, X + Z, 0, 0), (-2X, 0, 3X + Z, 0, 0),$$
$$(0, X^2 + XZ, Y, 0, 0), (2Y, 3X^2 + 4XZ + Z^2, 0, 0, 0)\big\rangle$$

The elements $(0, w^{2n+1})$, $n \in \mathbb{N}$, *do not belong to* T_0 *and so* $\mathcal{A}_e$*-cod*$(\{F_\infty, g\}) = \infty$.

ii) *Let* $f(x) = (x^2, x^{2k+1})$ *and consider the 3-parameter stable unfoldings*

$$F(x,z,w,y) = (x^2, x^{2k+1} + (y - z)x, y, z, w).$$

By doing the crosscap concatenation we obtain the codimension k *bigerms*

$$\begin{cases} (x^2, x^{2k+1} + (y-z)x, y, z, w) \\ (x, y, z, w^2, zw) \end{cases} \tag{13}$$

In fact, $Lift(F)$ *is generated by:*

$$\big\langle (0,0,0,0,1), (0,0,1,1,0), (0, -Y, 0, Z - W_1 + X^k, 0),$$
$$(2X, 0, 0, Z - W_1 + (2k+1)X^k, 0), (0, XZ - XW_1 + X^{k+1}, Y, 0, 0),$$
$$(2Y, Z^2 - 2ZW_1 + W_1^2 + (2k+1)X^{2k} + (2k+2)X^kZ - (2k+2)X^kW_1, 0, 0, 0)\big\rangle .$$

The only standard generators of $\mathcal{O}_4 \oplus \mathcal{O}_4$ *missing from* T_0 *are* $(0, w)$, $(0, wx), \ldots,$ $(0, wx^{k-1})$ *and so the codimension is* k.

These germs are $\mathcal{A}$*-equivalent to binary concatenations of the germs* (x^2, x^{2k+1}) *and* (w^2, w^3)*:*

$$\begin{cases} (x^2, x^{2k+1} + yx, y, z, w) \\ (x, y, z, w^2, w^3 + zw) \end{cases} \tag{14}$$

From [4] *we know that these examples satisfy the Mond conjecture.*

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