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ON REAL ANTI-BICANONICAL CURVES WITH ONE DOUBLE POINT ON THE 4-TH REAL HIRZEBRUCH SURFACE

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Abstract. We list up all the candidates for the real isotopy types of real anti-bicanonical curves with one real nondegenerate double point on the 4-th real Hirzebruch surface $\mathbb{RF}_4$ by enumerating the connected components of the moduli space of real 2-elementary K3 surfaces of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$. We also list up all the candidates for the non-increasing simplest degenerations of real nonsingular anti-bicanonical curves on $\mathbb{RF}_4$. We find an interesting correspondence between the real isotopy types of real anti-bicanonical curves with one real nondegenerate double point on $\mathbb{RF}_4$ and the non-increasing simplest degenerations of real nonsingular anti-bicanonical curves on $\mathbb{RF}_4$. This correspondence is very similar to the one provided by the rigid isotopic classification of real sextic curves on $\mathbb{RP}^2$ with one real nondegenerate double point by I. Itenberg.

1. Introduction

This paper is a continuation of [10] and [11]. In [10] the moduli spaces of "$(\mathcal{D}\mathbb{R})$-nondegenerate" real K3 surfaces with non-symplectic holomorphic involutions (namely, real 2-elementary K3 surfaces) are formulated and it is shown that the connected components of such a moduli space are in **one to one correspondence** with the isometry classes of integral involutions of the K3 lattice of certain type (see Theorem 2.11 below and [10] for more precise statements). As its applications, we obtain the real isotopic classifications (more precisely, deformation classifications) of real nonsingular sextic curves on the real projective plane $\mathbb{RP}^2$, real nonsingular curves of bidegree $(4, 4)$ on the real $\mathbb{P}^1 \times \mathbb{P}^1$ (hyperboloid, ellipsoid), and real nonsingular anti-bicanonical curves on the real Hirzebruch surfaces $\mathbb{RF}_m$ $(m = 1, 4)$. Here the m-th Hirzebruch surface $\mathbb{F}_m$ $(m \geq 0)$ means the ruled surface over $\mathbb{P}^1$ having an exceptional section s with $s^2 = -m$. $\mathbb{F}_4$ has 2 real structures (anti-holomorphic involutions). The real part of $\mathbb{F}_4$ is homeomorphic to the 2-dimensional torus or the empty set. On the other hand, $\mathbb{F}_3$ has a unique real structure and its real part is homeomorphic to the Klein's bottle (see [2]).

We say a complex curve A on a real Hirzebruch surface is **real** if its anti-holomorphic involution can be restricted to the curve A. We say two real curves $\mathbb{R}A$, $\mathbb{R}A'$ on a real nonsingular surface $\mathbb{R}B$ are **real isotopic** if there exists a continuous map $\Phi : \mathbb{R}B \times [0, 1] \to \mathbb{R}B$ (a "real isotopy from $\mathbb{R}A$ to $\mathbb{R}A'$") such that $\Phi_t := \Phi(\ ,t) : \mathbb{R}B \to \mathbb{R}B$ is a homeomorphism for any $t \in [0, 1]$, $\Phi_0 = \mathrm{id}_{\mathbb{R}B}$, and $\Phi_1(\mathbb{R}A) = \mathbb{R}A'$. Moreover, two real curves $\mathbb{R}A$, $\mathbb{R}A'$ in a fixed class on a real nonsingular surface $\mathbb{R}B$ are **rigidly isotopic** if there exists a real isotopy $\Phi : \mathbb{R}B \times [0, 1] \to \mathbb{R}B$ from $\mathbb{R}A$ to $\mathbb{R}A'$ such that $\Phi_t(\mathbb{R}A)$ is contained in the same class for any $t \in [0, 1]$.

Using the same method as above, we obtain the real isotopic classifications of real nonsingular anti-bicanonical curves on the real Hirzebruch surfaces $\mathbb{RF}_2$ and $\mathbb{RF}_3$ (see also Theorem 2.13 and Remark 2.26 of this paper) in [11]. Especially, all the connected components of the moduli space of real 2-elementary K3 surfaces of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$, which are defined below,

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are enumerated in [11]. Here we should remark that any real 2-elementary K3 surfaces of type $(S,\theta) \cong ((3,1,1), -\mathrm{id})$ are $(\mathcal{D}\mathbb{R})$-**nondegenerate** in the sense of [10].

Any 2-elementary K3 surface (X,τ) of type $S \cong (3,1,1)$ has an elliptic fibration and a unique reducible fiber $E+F$, where the curves E and F are defined in [11] (see also Subsection 2.2 of this paper). There are two types of F, which are reducible and irreducible. Let A be the fixed point set (curve) of τ on X, and $\pi : X \to Y := X/\tau$ be the quotient map. A intersects $\pi(E)$ at a single point. If we contract the curve $\pi(E)$ on Y to a point, then we get the 3-th Hirzebruch surface $\mathbb{F}_3$, and the image of the curve A is a real **nonsingular** anti-bicanonical curve on $\mathbb{F}_3$. This enables us to enumerate up (see [11]) all the real isotopy types of real nonsingular anti-bicanonical curves on $\mathbb{R}\mathbb{F}_3$.

However, on the other hand, if we contract the curve $\pi(F)$ on Y to a point, then we get the 4-th Hirzebruch surface $\mathbb{F}_4$, and the image of the curve A is real anti-bicanonical and has **one real double point**. Even though a real 2-elementary K3 surface of type $(S,\theta) \cong ((3,1,1), -\mathrm{id})$ is always $(\mathcal{D}\mathbb{R})$-**nondegenerate**, the double point is possibly **degenerate**, namely, a real cusp point. Thus, the main difficulty of the real isotopic classification of real anti-bicanonical curves on $\mathbb{F}_4$ with one real double point is that the connected components (equivalently, the isometry classes of integral involutions of the K3 lattice) of the moduli space ([10]) of real 2-elementary K3 surfaces of type $(S,\theta) \cong ((3,1,1), -\mathrm{id})$ cannot distinguish degenerate and nondegenerate double points, equivalently, reducible F and irreducible F. Hence, the connected components cannot distinguish the topological types (node, cusp, or isolated point) of the real double points (Remark 2.25). This problem was left to the readers in [11] (see Remark 8).

Thus, the aim of this paper is: *Classify the real isotopy types of real anti-bicanonical curves with one real nondegenerate double point on* $\mathbb{R}\mathbb{F}_4$.

First we list up all the candidates for the real isotopy types of real anti-bicanonical curves with one real nondegenerate double point on $\mathbb{R}\mathbb{F}_4$ (Theorem 2.24) using some well-known topological interpretations ([10], [11]) of some arithmetic invariants of integral involutions of the K3 lattice.

Unfortunately, the realizability of each real isotopy type listed in Theorem 2.24 has not been resolved in this paper. We only know that at least one of real isotopy types with nondegenerate double points can be realized for **each** isometry class (see Remark 2.25).

In order to distinguish the real isotopy types, we should remove real 2-elementary K3 surfaces which yield anti-bicanonical curves with degenerate double points on $\mathbb{R}\mathbb{F}_4$ from the moduli space (period domain) in the sense of [10]. We follow Itenberg's argument ([4],[5],[6]) for the rigid isotopic classification of real sextic curves on $\mathbb{R}\mathbb{P}^2$ with one nondegenerate double point. Lemma 4.6 provides a sufficient condition for the double point to be non-degenerate.

Moreover, according to his papers [4] and [5], real curves of degree 6 on $\mathbb{R}\mathbb{P}^2$ with one nondegenerate double point are obtained by "non-increasing simplest degenerations" of real nonsingular curves of degree 6 on $\mathbb{R}\mathbb{P}^2$. Hence, we next list up the candidates for the non-increasing simplest degenerations of real nonsingular anti-bicanonical curves on $\mathbb{R}\mathbb{F}_4$ (Theorem 3.4).

Then, we find an obvious interesting correspondence between the real isotopy types of curves with one real nondegenerate double point on $\mathbb{R}\mathbb{F}_4$ (Theorem 2.24) and the non-increasing simplest degenerations of nonsingular curves on $\mathbb{R}\mathbb{F}_4$ (Theorem 3.4). See Remark 3.5. We also get some similar properties (Lemma 3.7) to the degenerations of real nonsingular curves of degree 6 on $\mathbb{R}\mathbb{P}^2$.

In the final section 4, we review and confirm the periods of marked real 2-elementary K3 surfaces $((X,\tau,\varphi),\ \alpha)$ of type (S,θ) satisfying $\alpha \circ \varphi_* \circ \alpha^{-1} = \psi$ for a fixed integral involution ψ, and give some further problems which are inspired by the argument in [4] and [5].

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2. Real 2-elementary K3 surfaces of type $(S, \theta) \cong ((3,1,1), -\mathrm{id})$

2.1. Real 2-elementary K3 surfaces.

Definition 2.1 (Real 2-elementary K3 surface)**.** A triple (X, τ, φ) is called a **real K3 surface with non-symplectic (holomorphic) involution** (or **real 2-elementary K3 surface**) if
(1) (X, τ) is a K3 surface X with a non-symplectic holomorphic involution τ, i.e., "2-elementary K3 surface" ([8]).
(2) φ is an anti-holomorphic involution on X.
(3) $\varphi \circ \tau = \tau \circ \varphi$

Note that any K3 surface with a non-symplectic holomorphic involution is algebraic.

For a real K3 surface with non-symplectic involution (X, τ, φ), we call $\widetilde{\varphi} := \tau \circ \varphi = \varphi \circ \tau$ the **related (anti-holomorphic) involution** of φ ([10]). The triple $(X, \tau, \tau \circ \varphi)$ is also a real K3 surface with non-symplectic involution.

For a 2-elementary K3 surface (X, τ), we denote by $H_{2+}(X, \mathbb{Z})$ the fixed part of

$$\tau_* : H_2(X, \mathbb{Z}) \to H_2(X, \mathbb{Z}).$$

Note that $H_{2+}(X, \mathbb{Z}) \subset N(X)$, where $N(X)$ is the Picard lattice of X.

We fix an even unimodular lattice $\mathbb{L}_{K3}$ of signature $(3, 19)$. The isometry class of such lattices is unique (the K3 lattice).

Let $\alpha : H_2(X, \mathbb{Z}) \to \mathbb{L}_{K3}$ be an isometry (marking). If we temporary set $S := \alpha(H_{2+}(X, \mathbb{Z}))$, then S is a primitive (,i.e., $\mathbb{L}_{K3}/S$ is free,) hyperbolic (i.e., S is of signature $(1, \operatorname{rank} S - 1)$) 2-elementary (i.e., $S^*/S \cong (\mathbb{Z}/2\mathbb{Z})^a$ for some nonnegative integer a) sublattice of $\mathbb{L}_{K3}$.

Now let $S \quad (\subset \mathbb{L}_{K3})$ be a primitive hyperbolic 2-elementary sublattice of the K3 lattice $\mathbb{L}_{K3}$.

Definition 2.2. We set $r(S) := \operatorname{rank} S$. The non-negative integer $a(S)$ is defined by the equality

$$S^*/S \cong (\mathbb{Z}/2\mathbb{Z})^{a(S)}.$$

We define

$$\delta(S) := \begin{cases} 0 & \text{if } z \cdot \sigma(z) \equiv 0 \bmod 2 \quad (\forall z \in \mathbb{L}_{K3}) \\ 1 & \text{otherwise,} \end{cases}$$

where we define $\sigma : \mathbb{L}_{K3} \to \mathbb{L}_{K3}$ to be the unique integral involution whose fixed part is S.

It is known that the triplet

$$(r(S), a(S), \delta(S))$$

determines the isometry class of the lattice S ([8]).

If S and S' are primitive hyperbolic 2-elementary sublattices of the K3 lattice $\mathbb{L}_{K3}$, and S is isometric to S'; then there exists an automorphism f of $\mathbb{L}_{K3}$ such that $f(S') = S$ ([1], [7]). Hence, if (X, τ) and (X', τ') are two 2-elementary K3 surfaces, $\alpha : H_2(X, \mathbb{Z}) \to \mathbb{L}_{K3}$ is an isometry, and $H_{2+}(X, \mathbb{Z})$ is isometric to $H_{2+}(X', \mathbb{Z})$; then there exists an isometry (marking) $\alpha' : H_2(X', \mathbb{Z}) \to \mathbb{L}_{K3}$ such that $\alpha'(H_{2+}(X', \mathbb{Z})) = \alpha(H_{2+}(X, \mathbb{Z}))$. Thus, only the isometry class of $H_{2+}(X, \mathbb{Z})$ is essential for 2-elementary K3 surfaces (X, τ). Henceforth, we often fix an isometry class, equivalently, the invariants $(r(S), a(S), \delta(S))$ instead of fixing a particular sublattice.

We quote the following formulation from [10]. We additionally fix a half-cone $V^+(S)$ of the cone $V(S) := \{x \in S \otimes \mathbb{R} \mid x^2 > 0\}$. We also fix a fundamental chamber $\mathcal{M} \subset V^+(S)$ for the group $W^{(-2)}(S)$ generated by reflections in all elements with square (-2) in S. This is equivalent to fixing a fundamental subdivision

$$\Delta(S) = \Delta(S)_+ \cup -\Delta(S)_+$$

of all elements with square -2 in S. $\mathcal{M}$ and $\Delta(S)_+$ define each other by the condition

$$\mathcal{M} \cdot \Delta(S)_+ \geq 0.$$

Let (X, τ) be a 2-elementary K3 surface, and $\alpha : H_2(X, \mathbb{Z}) \to \mathbb{L}_{K3}$ be a marking such that $\alpha(H_{2+}(X, \mathbb{Z})) = S$. We always assume that $\alpha_{\mathbb{R}}^{-1}(V^+(S))$ contains a hyperplane section of X and the set $\alpha^{-1}(\Delta(S)_+)$ contains only classes of effective curves of X.

Now let θ be an integral involution of S.

Definition 2.3 (the action of φ on $H_{2+}(X, \mathbb{Z})$)**.** A real 2-elementary K3 surface (X, τ, φ) is called that of **type** (S, θ) if there exists an isometry (marking)

$$\alpha : H_2(X, \mathbb{Z}) \cong \mathbb{L}_{K3}$$

such that $\alpha(H_{2+}(X, \mathbb{Z})) = S$ and the following diagram commutes:

$$\begin{array}{ccc} H_{2+}(X, \mathbb{Z}) & \xrightarrow{\alpha} & S \\ {\scriptstyle \varphi_*}\downarrow & & \downarrow{\scriptstyle \theta} \\ H_{2+}(X, \mathbb{Z}) & \xrightarrow{\alpha} & S. \end{array}$$

Definition 2.4 (marked real 2-elementary K3 surfaces)**.** A pair $((X, \tau, \varphi),\ \alpha)$ of a real 2-elementary K3 surface (X, τ, φ) of type (S, θ) and an isometry (marking) $\alpha : H_2(X, \mathbb{Z}) \cong \mathbb{L}_{K3}$ such that

- $\alpha(H_{2+}(X, \mathbb{Z})) = S$,
- $\alpha \circ \varphi_* = \theta \circ \alpha$ on $H_{2+}(X, \mathbb{Z})$,
- $\alpha_{\mathbb{R}}^{-1}(V^+(S))$ contains a hyperplane section of X and
- the set $\alpha^{-1}(\Delta(S)_+)$ contains only classes of effective curves of X

is called a **marked real 2-elementary K3 surface of type** (S, θ).

Note that we have $\theta(V^+(S)) = -V^+(S)$ and $\theta(\Delta(S)_+) = -\Delta(S)_+$ for a marked real 2-elementary K3 surface of type (S, θ).

Definition 2.5 (Integral involution ψ of $\mathbb{L}_{K3}$ of type (S, θ))**.** Let S be a hyperbolic 2-elementary sublattice of $\mathbb{L}_{K3}$, $\theta : S \to S$ be an integral involution of the lattice S, and $\psi : \mathbb{L}_{K3} \to \mathbb{L}_{K3}$ be an integral involution of the lattice $\mathbb{L}_{K3}$ such that the following diagram commutes:

$$\begin{array}{ccc} S & \subset & \mathbb{L}_{K3} \\ \theta \downarrow & & \downarrow \psi \\ S & \subset & \mathbb{L}_{K3}. \end{array}$$

We say such a pair $(\mathbb{L}_{K3}, \psi)$ (or ψ) an **integral involution of** $\mathbb{L}_{K3}$ **of type** (S, θ).

Let $((X,\tau,\varphi),\ \alpha)$ be a marked real 2-elementary K3 surface of type (S,θ). If we set

$$\psi := \alpha\circ\varphi_*\circ\alpha^{-1} : \mathbb{L}_{K3}\to\mathbb{L}_{K3},$$

then we have $\psi(S)=S$, and $\psi(x)=\theta(x)$ for every $x\in S$ because $\alpha^{-1}(x)\in H_{2+}(X,\mathbb{Z})$. Hence, $(\mathbb{L}_{K3},\psi)$ is an integral involution of $\mathbb{L}_{K3}$ of type (S,θ).

Definition 2.6 (the associated integral involution)**.** We call the integral involution ψ of $\mathbb{L}_{K3}$ of type (S,θ) **the associated integral involution** of $\mathbb{L}_{K3}$ with a marked real 2-elementary K3 surface $((X,\tau,\varphi),\ \alpha)$ of type (S,θ) if the following diagram commutes:

$$\begin{array}{ccc} H_2(X,\mathbb{Z}) & \xrightarrow{\alpha} & \mathbb{L}_{K3} \\ {\scriptstyle\varphi_*}\downarrow & & \downarrow{\scriptstyle\psi} \\ H_2(X,\mathbb{Z}) & \xrightarrow{\alpha} & \mathbb{L}_{K3}. \end{array}$$

Note that the fixed part $\mathbb{L}_{K3}^{\psi}$ of ψ is hyperbolic for any associated integral involution of $\mathbb{L}_{K3}$.

Definition 2.7 (($\mathcal{D}\mathbb{R}$)-nondegenerate, [10])**.** $x\in N(X)\otimes\mathbb{R}$ is **nef** if $x\neq 0$ and $x\cdot C\geq 0$ for any effective curve on X. We say that a 2-elementary K3 surface (X,τ) of type S is **($\mathcal{D}$)-degenerate** if there exists $h\in\mathcal{M}$ such that h is not nef. This is equivalent to the existence of an exceptional curve (i.e., irreducible and having negative self-intersection) with square -2 on the quotient surface $Y:=X/\tau$. This is also equivalent to have an element $\delta\in N(X)$ with $\delta^2=-2$ such that $\delta=(\delta_1+\delta_2)/2$ where $\delta_1\in S$, $\delta_2\in S^{\perp}_{N(X)}$ and $\delta_1^2=\delta_2^2=-4$.

A real 2-elementary K3 surface $(X,\tau,\ \varphi)$ of type (S,θ) is **($\mathcal{D}\mathbb{R}$)-degenerate** if there exists a real element $h\in S_-\cap\mathcal{M}$ which is not nef, where we set $S_\pm := \{x\in S\,|\,\theta(x)=\pm x\}$. This is equivalent to have an element $\delta\in N(X)$ with $\delta^2=-2$ such that $\delta=(\delta_1+\delta_2)/2$ where $\delta_1\in S$, $\delta_2\in S^{\perp}_{N(X)}$ and $\delta_1^2=\delta_2^2=-4$, and δ_1 must be orthogonal to an element $h\in S_-\cap\mathrm{int}(\mathcal{M})$ with $h^2>0$. Here $\mathrm{int}(\mathcal{M})$ denote the interior part of $\mathcal{M}$, i. e., the polyhedron $\mathcal{M}$ without its faces.

Let $\Delta(S,L)^{(-4)}$ be the set of all elements δ_1 in S such that $\delta_1^2=-4$ and there exists $\delta_2\in S^{\perp}_L$ such that $(\delta_2)^2=-4$ and $\delta=(\delta_1+\delta_2)/2\in L$. Let $W^{(-4)}(S,L)\subset O(S)$ be the group generated by reflections in all roots from $\Delta(S,L)^{(-4)}$, and $W^{(-4)}(S,L)_{\mathcal{M}}$ be the stabilizer subgroup of $\mathcal{M}$ in $W^{(-4)}(S,L)$. Let G be the subgroup generated by reflections s_{δ_1} in all elements $\delta_1\in\Delta(S,L)^{(-4)}$ which are contained either in S_+ or in S_- and satisfy $s_{\delta_1}(\mathcal{M})=\mathcal{M}$. (Then G is a subgroup of $W^{(-4)}(S,L)_{\mathcal{M}}$.)

Definition 2.8 (Isometries with respect to the group G)**.** Let $(\mathbb{L}_{K3},\psi_1)$ and $(\mathbb{L}_{K3},\psi_2)$ be two integral involutions of $\mathbb{L}_{K3}$ of type (S,θ). An **isometry with respect to the group** G from $(\mathbb{L}_{K3},\psi_1)$ to $(\mathbb{L}_{K3},\psi_2)$ means an isometry $f:\mathbb{L}_{K3}\to\mathbb{L}_{K3}$ such that $f(S)=S$, $f|_S\in G$, and the following diagram commutes:

$$\begin{array}{ccc} \mathbb{L}_{K3} & \xrightarrow{f} & \mathbb{L}_{K3} \\ \downarrow{\scriptstyle\psi_1} & & {\scriptstyle\psi_2}\downarrow \\ \mathbb{L}_{K3} & \xrightarrow{f} & \mathbb{L}_{K3}. \end{array}$$

In the above definition, remark that $\theta\circ f|_S = f|_S\circ\theta$ on S, hence, $f|_S$ is at least an automorphism of (S,θ). However, we require the condition "$f|_S\in G$".

Definition 2.9. We say two integral involutions $(\mathbb{L}_{K3},\psi_1)$ and $(\mathbb{L}_{K3},\psi_2)$ of type (S,θ) are **isometric with respect to the group** G if there exists an isometry with respect to the group G from $(\mathbb{L}_{K3},\psi_1)$ to $(\mathbb{L}_{K3},\psi_2)$.

By an **automorphism** of an integral involution $(\mathbb{L}_{K3}, \psi)$ of type (S, θ) **with respect to the group** G we mean an isometry with respect to the group G from $(\mathbb{L}_{K3}, \psi)$ to itself. Namely, an isometry $f : \mathbb{L}_{K3} \to \mathbb{L}_{K3}$ which satisfies that

$$\psi \circ f = f \circ \psi, \ f(S) = S \text{ and } f|_S \in G.$$

Definition 2.10 (analytically isomorphic with respect to G)**.** Two marked real 2-elementary K3 surfaces $((X, \tau, \varphi), \ \alpha)$ and $((X', \tau', \varphi'), \alpha')$ of type (S, θ) are **analytically isomorphic with respect to the group** G if there exists an analytic isomorphism $f : X \to X'$ such that $f \circ \tau = \tau' \circ f$, $f \circ \varphi = \varphi' \circ f$ and $\alpha' \circ f_* \circ \alpha^{-1}|S \in G$.

By considering their associated integral involutions, we get a natural map from the moduli space of marked real 2-elementary K3 surfaces of type (S, θ) to the set of isometry classes with respect to G of integral involutions of $\mathbb{L}_{K3}$ of type (S, θ) such that the fixed part $\mathbb{L}_{K3}^{\psi}$ of ψ is hyperbolic.

Theorem 2.11 ([10])**.** *The natural map above gives a bijective correspondence between the connected components of the moduli space of ($\mathcal{D}\mathbb{R}$)-nondegenerate marked real 2-elementary K3 surfaces of type (S, θ) and the set of isometry classes with respect to G of integral involutions of $\mathbb{L}_{K3}$ of type (S, θ) such that the fixed part $\mathbb{L}_{K3}^{\psi}$ of ψ is hyperbolic.*

2.2. **Elliptic fibrations of 2-elementary K3 surfaces of type $S \cong (3, 1, 1)$.** We quote some basic facts from [1]. Let (X, τ) be a 2-elementary K3 surface of type S with invariants

$$(r(S), a(S), \delta(S)) = (3, 1, 1),$$

and $A := X^{\tau}$ be the fixed point set (curve) of τ. Then we have

$$A = A_0 \cup A_1 \ \ \text{(disjoint union)},$$

where A_0 is a nonsingular rational curve ($\cong \mathbb{P}^1$) with $A_0^2 = -2$ and A_1 is a nonsingular curve of genus 9. (X, τ) has a structure of an elliptic pencil $|E + F|$ with its section A_0 and the unique reducible fiber $E + F$ having the following properties below:

- **(i):** E is an irreducible nonsingular rational curve with $E^2 = -2$ and $E \cdot A_0 = 1$.
- **(ii):** $E \cdot F = 2, \ F^2 = -2, \ F \cdot A_0 = 0,$ and F is either an irreducible nonsingular rational curve (type **IIa**), or the union of two irreducible nonsingular rational curves F' and F'' which are conjugate by τ and $F' \cdot F'' = 1$ (type **IIb**) (that is, the reducible fiber $E + F$ corresponds to the extended root system $\widetilde{\mathbb{A}_1}$ or $\widetilde{\mathbb{A}_2}$). We have $(F')^2 = (F'')^2 = -2$. See also §2.4 of [1].
- **(iii):** The classes $[A_0]$, $[E]$ and $[F]$ generate the lattice $H_{2+}(X, \mathbb{Z})$ ($\cong S$). Moreover, $A_1 \cdot E = 1, \ A_1 \cdot F = 2$. The Gram matrix of the lattice $H_{2+}(X, \mathbb{Z})$ with respect to the basis $[E]$, $[F]$ and $[A_0]$ is as follows:

	$[E]$	$[F]$	$[A_0]$
$[E]$	-2		
$[F]$	2	-2	
$[A_0]$	1	0	-2

We next consider the quotient complex surface $Y := X/\tau$ and let $\pi : X \to Y$ be the quotient map. Then, A, considered in Y, is contained in $|-2K_Y|$ (an anti-bicanonical curve on Y).

We define the curves:

$$e := \pi(E) \quad \text{and} \quad f := \pi(F).$$

If F is the union of two nonsingular rational curves F' and F'' which are conjugate by τ and $F' \cdot F'' = 1$, then

$$f = \pi(F) = \pi(F' \cup F'') = \pi(F') = \pi(F'').$$

We use the same symbols A_0 and A_1 for their images in Y by π. Then, the Picard group $\mathrm{Pic}(Y)$ of Y is generated by the classes $[e]$, $[f]$ and $[A_0]$. The Gram matrix of $\mathrm{Pic}(Y)$ with respect to the basis $[e]$, $[f]$ and $[A_0]$ is as follows:

$$\begin{array}{cccc} & [e] & [f] & [A_0] \\ [e] & -1 & & \\ [f] & 1 & -1 & \\ [A_0] & 1 & 0 & -4 \end{array}$$

For A_1, we have

$$A_1 \cdot e = 1 \quad \text{and} \quad A_1 \cdot f = 2.$$

We have:
(1) F is a nonsingular rational curve if and only if A_1 intersects f in **two distinct points**.
(2) F is a union of two nonsingular rational curves if and only if A_1 **touches** f.

Remark 2.12 ([11])**.** For any **real** 2-elementary K3 surface (X, τ, φ) of type (S, θ) with $S \cong (3, 1, 1)$, we have $\theta = -\mathrm{id}$ and $G = \{\mathrm{id}\}$, where id stands for the identity map on S.

2.3. **Enumeration of the connected components of the moduli space.** For any 2-elementary K3 surface (X, τ) of type $S \cong (3, 1, 1)$, all exceptional curves on Y are exactly the curves e, f and A_0. Hence, (X, τ) is $(\mathcal{D})$-nondegenerate and any real 2-elementary K3 surface (X, τ, φ) of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$ is $(\mathcal{D}\mathbb{R})$-nondegenerate (see Definition 2.7). By Theorem 2.11, we have:

Theorem 2.13 ([11]CTheorem 1)**.** *The connected components of the moduli space (in the sense of* [10]*) of marked real* 2*-elementary K3 surfaces* $((X, \tau, \varphi),\ \alpha)$ *of type* $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$ *are in bijective correspondence with the isometry classes of integral involutions of* $\mathbb{L}_{K3}$ *of type* $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$ *with respect to* $G = \{\mathrm{id}\}$*.*

The complete isometry invariants of integral involutions ψ of $\mathbb{L}_{K3}$ of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$ are the data

$$(r(\psi),\ a(\psi),\ \delta_{\psi S},\ H(\psi)). \tag{2.1}$$

See [10], Subsection 2.3 (after the equation (2.20)) for the definition of the invariant $\delta_{\psi S}$ and $H(\psi)$. Remark that $H(\psi)$ is a subgroup of the discriminant group $S_-/2S_- = S/2S = \mathbb{Z}/2\mathbb{Z}(\alpha([F]))$. Hence, $H(\psi) = 0$ or isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

All realizable data (2.1) are enumerated in [11]. We have 12 data of "Type 0" ($\Rightarrow H(\psi) = 0$), 12 data of "Type Ia" , 39 data of "Type Ib" with $H(\psi) = 0$, and 39 data of "Type Ib" with $H(\psi) \cong \mathbb{Z}/2\mathbb{Z}$. See also Table 1 for $H(\psi) = 0$ case and Table 2 for $H(\psi) \cong \mathbb{Z}/2\mathbb{Z}$ case, where 0 or 1 in each cell stands for the value of $\delta_{\psi S}$.

Thus we have exactly **102** connected components of the moduli of real 2-elementary K3 surfaces of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$. By [10], we find how invariants of an involution ψ and those of its "related integral involution" $\sigma \circ \psi$ are calculated from each other. Here $\sigma : \mathbb{L}_{K3} \to \mathbb{L}_{K3}$ is defined to be the integral involution of $\mathbb{L}_{K3}$ whose fixed part is S. Identifying related pairs of anti-holomorphic involutions on each K3 surface, we have exactly **51** connected components.

$a(\psi)$ / $r(\psi)$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
9									1									
8								1		$0,1$								
7							1		1		1							
6						1		1		$0,1$		1						
5					1		1		1		1		1					
4				1		$0,1$		1		$0,1$		1		$0,1$				
3			1		1		1		1		1		1		1			
2		$0,1$		1		0		1		$0,1$		1		0		1		
1	1		1						1		1						1	
0		0								0								0

TABLE 1. All the data $(r(\psi),\ a(\psi),\ \delta_{\psi S})$ for $H(\psi) = 0$.

$a(\psi)$ / $r(\psi)$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
10										1								
9									$0,1$		1							
8								1		1		1						
7							1		$0,1$		1		1					
6						1		1		1		1		1				
5					$0,1$		1		$0,1$		1		$0,1$		1			
4				1		1		1		1		1		1		1		
3			1		0		1		$0,1$		1		0		1		$0,1$	
2		1						1		1						1		1
1	0								0								0	

TABLE 2. All the data $(r(\psi),\ a(\psi),\ \delta_{\psi S})$ for $H(\psi) \cong \mathbb{Z}/2\mathbb{Z}$.

2.4. **The regions A_+ and A_- in the real part $Y(\mathbb{R})$ of Y.** Let (X, τ, φ) be a real 2-elementary K3 surface of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$. We use the notation in Subsection 2.2. Let $X_\varphi(\mathbb{R})$ denote the real part of (X, φ), i.e., the fixed point set of φ, and let $Y(\mathbb{R})$ be the real part of the quotient surface Y with the real structure $\varphi_{\mathrm{mod}\ \tau}$.

The real part

$$\mathbb{R}A = \mathbb{R}A_0 \cup \mathbb{R}A_1$$

of the branch curve A divides $Y(\mathbb{R})$ into two regions [1] A_+ and A_-.

Either A_+ or A_- is doubly covered by the real part $X_\varphi(\mathbb{R})$, and the other by the real part $X_{\widetilde{\varphi}}(\mathbb{R})$, where we set $\widetilde{\varphi} := \tau \circ \varphi$. Since $X_\varphi(\mathbb{R})$ is always non-empty ([11], p.27), $Y(\mathbb{R})$, A_+, and A_- are also non-empty. Regions $A_\pm$ could have several connected components.

We distinguish two regions A_+ and A_- as follows.

Definition 2.14 (The regions A_+ and A_-)**.** For a real 2-elementary K3 surface (X, τ, φ) of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$, we define the regions A_+ and A_- $(\subset Y(\mathbb{R}))$ as follows:

$$\begin{cases} A_+ & := \pi(X_\varphi(\mathbb{R})) \quad \text{if} \quad H(\psi) \cong \mathbb{Z}/2\mathbb{Z}, \\ A_- & := \pi(X_\varphi(\mathbb{R})) \quad \text{if} \quad H(\psi) = 0. \end{cases}$$

[1] These two regions are called "positive curves" in [10].

Note that if $H(\psi) \cong \mathbb{Z}/2\mathbb{Z}$, then $H(\sigma \circ \psi) = 0$.

An important topological characterization of the invariant $H(\psi)$ is as follows.

Lemma 2.15 ([10])**.** *Suppose that the curve F is irreducible. Then we have $H(\psi) \cong \mathbb{Z}/2\mathbb{Z}$ if and only if $[F_\varphi(\mathbb{R})] = 0$ in $H_1(X_\varphi(\mathbb{R}); \mathbb{Z}/2\mathbb{Z})$, where we set $F_\varphi(\mathbb{R}) := X_\varphi(\mathbb{R}) \cap F$.*

2.5. **Real anti-bicanonical curves with one real double point on $\mathbb{R}\mathbb{F}_4$.** We now contract the exceptional curve $f = \pi(F)$ to a point. We get a blow up $\mathrm{bl} : Y \to \mathbb{F}_4$, where $\mathbb{F}_4$ is the 4-th Hirzebruch surface.

We set

$$s := \mathrm{bl}(A_0), \quad A_1' := \mathrm{bl}(A_1), \quad c := \mathrm{bl}(e),$$

and we have

$$\mathrm{bl}(A) = \mathrm{bl}(A_0) + \mathrm{bl}(A_1) = s + A_1' \quad \in |-2K_{\mathbb{F}_4}|,$$

i.e., $\mathrm{bl}(A)$ is an anti-bicanonical curve, where s is the exceptional section of $\mathbb{F}_4$ with $s^2 = -4$ and c is a fiber of the fibration $\mathbb{F}_4 \to s$ with $c^2 = 0$. Since $s \cdot A_1' = 0$, A_1' **does not intersect the section** s.

We have $-2K_{\mathbb{F}_4} \sim 12c + 4s$. It follows that $A_1' \in |12c + 3s|$.

Let

$$\mathbb{R}\mathrm{bl}(A) = \mathbb{R}s \cup \mathbb{R}A_1'$$

be the real part of the curve $\mathrm{bl}(A)$, where $\mathbb{R}s$ and $\mathbb{R}A_1'$ are the real parts of s and A_1' respectively.

We set

$$P_0 := \mathrm{bl}(f) \ \ (\in \mathbb{R}c).$$

Since $A_1 \cdot f = 2$ in Y, A_1' has **one real double point** P_0. If A_1 intersects with f at two distinct points in Y, then they are real points or non-real conjugate points. In the former case P_0 is a **real node**, and in the latter case it is a **real isolated point**. Anyway it is a **nondegenerate** double point. If A_1 touches to f in Y, then P_0 is a **real cusp**. (a **degenerate** double point)

Thus, there are three topological types of the curve $\mathbb{R}A_1'$ near the double point P_0.

Node case: P_0 is a real node of $\mathbb{R}A_1'$.

Cusp case: P_0 is a real cusp of $\mathbb{R}A_1'$. (degenerate double point)

Isolated point case: P_0 is a real isolated point of $\mathbb{R}A_1'$.

See Figure 1 below.

We have $A_1' \cdot s = 0$ and $A_1' \cdot c = 3$ (A_1' is a trigonal curve). Since $f \cdot A_0 = 0$ in Y, the section s does not meet the double point P_0. We may assume that e does not pass through any intersection point of A_1 and f in Y. (See Figure 1.) Since $A_1 \cdot e = 1$ in Y, the intersection point of A_1 with e is **real** and does not meet f. Via the map bl, this point goes to a real intersection point of A_1' with c with multiplicity 1. Thus we set

$$P_1 := \mathrm{bl}(A_1 \cap e). \quad \text{(the intersection point)}$$

Since $A_1' \cdot c = 3$, A_1' intersects with c at P_0 with multiplicity 2 and at P_1 with multiplicity 1.

Since $A_1' \cap s = \emptyset$, any **non-contractible (possibly real singular)** components of $\mathbb{R}A_1'$ are "parallel" to $\mathbb{R}s$. See Figure2, 3 and 4. But two types of non-contractible singular components in Figure2 are real isotopic, and three types of non-contractible components in Figure4 are real isotopic.

Definition 2.16. We call a connected component of $\mathbb{R}A_1'$ an **oval** if it has no real singular points and contractible in $\mathbb{R}\mathbb{F}_4$ (a torus), namely, realizes 0 in $H_1(\mathbb{R}\mathbb{F}_4; \mathbb{Z})$.

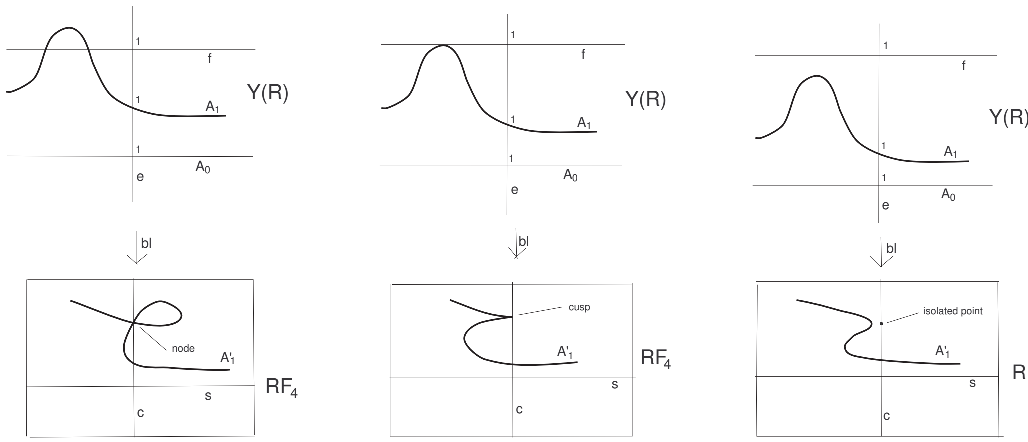


FIGURE 1. The real double point P_0 of the curve A_1'

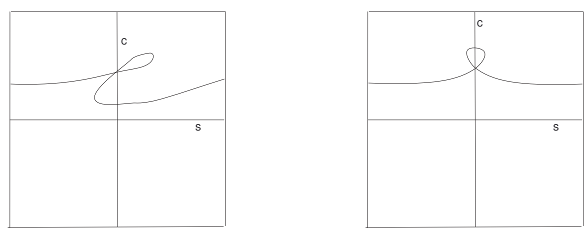


FIGURE 2. A non-contractible component with a node of type **Node (1)**

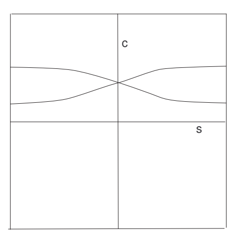


FIGURE 3. A non-contractible component with a node of type **Node (*)**

Remark 2.17. Since $A_1' \cdot c = 3$, A_1' is a **trigonal** curve on $\mathbb{F}_4$. See [3], [12] for related results. We see that the interior of each oval of $\mathbb{R}A_1'$ does not contain any other ovals. Moreover, the ovals are canonically ordered according to their projections to the base $\mathbb{R}s$.

We now get the following possibilities.

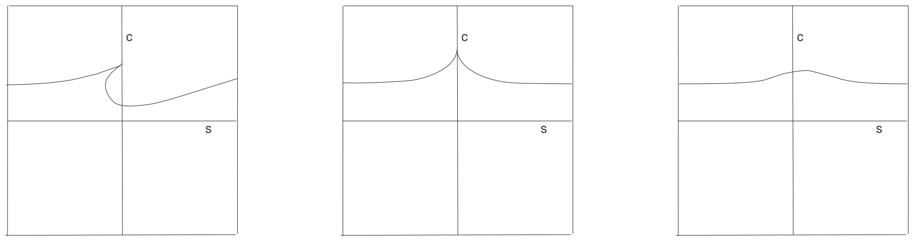


FIGURE 4. A non-contractible component with a **real cusp** or without singular points

Node case:

Node (1) case: We consider the case when **both** the node P_0 and the intersection point P_1 are contained in **the same** connected singular component of $\mathbb{R}A_1'$. Then $\mathbb{R}A_1'$ meets $\mathbb{R}c$ at only P_0 and P_1, and the singular component is **not contractible**. See Figure 2.

Note that $\mathbb{R}A_1'$ might have some ovals. Since $A_1' \cdot c = 3$, the interior of any oval of $\mathbb{R}A_1$ does not contain other ovals. The interior of the node also does not contain ovals.

The non-contractible singular component containing P_0 **and** P_1 and **the section** $\mathbb{R}s$ divide $\mathbb{RF}_4$ (a torus) into three parts, which are **the interior of the node** and **two non-contractible regions**.

Definition 2.18 (The regions R_1 and R_2 in Node (1) case)**.** Let R_1 denote **the non-contractible region which is connected with the interior of the node in the blow up of** $\mathbb{RF}_4$, and let R_2 denote the other non-contractible region. We define the integers α and β as follows:

$$\begin{aligned} \alpha &:= \#\{\text{ovals contained in } R_1\}, \\ \beta &:= \#\{\text{ovals contained in } R_2\}. \end{aligned} \tag{2.2}$$

See **the left** figure of Figure 5.

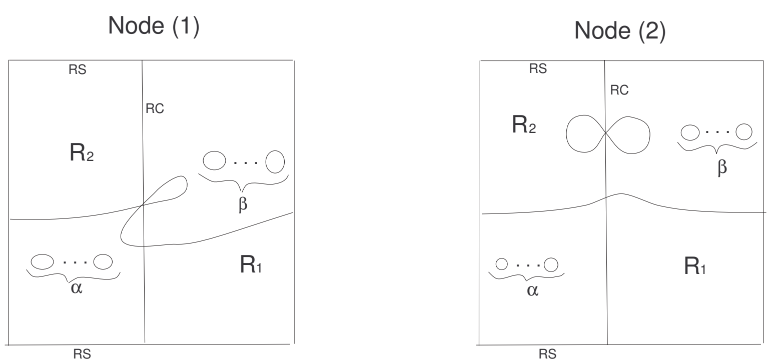


FIGURE 5. The regions R_1 and R_2 in **Node (1)** or **Node (2)** case

Node (2) case and Node (*) case: When the node P_0 and the intersection point P_1 respectively are contained in **different** connected components of $\mathbb{R}A_1'$, the component containing P_1 is nonsingular and not contractible like the rightmost figure of Figure 4 above. The component containing P_0 can be either **contractible** (the left figure of Figure 6) or **non-contractible** (Figure 3).

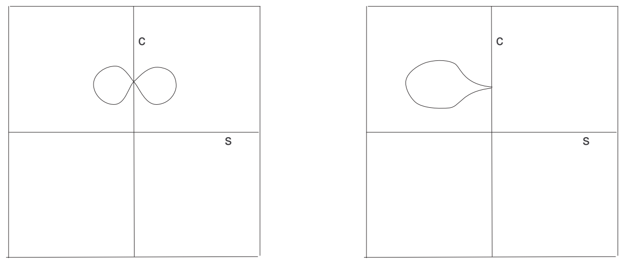


FIGURE 6. Contractible components with a real node or a real cusp

Node (2) case: When the component containing the node P_0 is **contractible** (the left figure of Figure 6), $\mathbb{R}A_1'$ might have some ovals. **The component containing** P_1 and **the section** $\mathbb{R}s$ divide $\mathbb{R}\mathbb{F}_4$ into two regions.

Definition 2.19 (The regions R_1 and R_2 in Node (2) case)**.** Let R_1 denote **the region which does not contain the contractible component containing the node** P_0, and let R_2 denote the other region. (Since $A_1' \cdot c = 3$, the interior of the contractible component containing P_0 and the interior of any oval of $\mathbb{R}A_1$ cannot contain any other ovals.)

We define the integers α and β by (2.2). See **the right** figure of Figure 5.

Node (*) case: If the component containing the node P_0 is **non-contractible** (see Figure 3), then $\mathbb{R}A_1'$ has **no ovals** (see Figure 7).

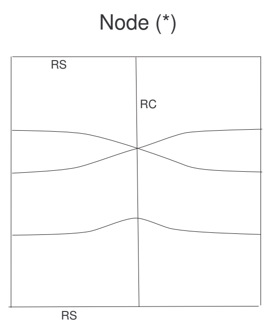


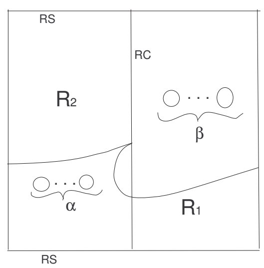
FIGURE 7. **Node (*)**

Cusp case:

Cusp (1): When both the cusp P_0 and the point P_1 are contained in **the same** connected component of $\mathbb{R}A_1'$, $\mathbb{R}A_1'$ meets $\mathbb{R}c$ at only P_0 and P_1, and the component containing P_0 and P_1 is not contractible (the leftmost figure of Figure 4). $\mathbb{R}A_1'$ might have some ovals. **The non-contractible component containing** P_0 **and** P_1 and **the section** $\mathbb{R}s$ divide $\mathbb{R}\mathbb{F}_4$ into two regions (the left figure of Figure 8). **One of these regions goes to a non-orientable region via the blow up of** $\mathbb{R}\mathbb{F}_4$**.**

Definition 2.20 (The regions R_1 and R_2 in Cusp (1) case)**.** Let R_2 denote the region which goes to a **non-orientable region** via the blow up of $\mathbb{R}\mathbb{F}_4$. and let R_1 denote the other region. (Since $A_1' \cdot c = 3$, the interior of any oval of $\mathbb{R}A_1$ does not contain any other ovals.)

We define the integers α and β by (2.2). See the left figure of Figure 8.



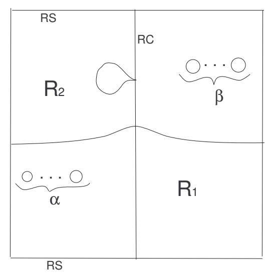


FIGURE 8. The regions R_1 and R_2 in **Cusp (1)** or **Cusp (2)** case

Cusp (2) When the cusp P_0 and P_1 respectively are contained in **different** connected components of $\mathbb{R}A_1'$, the component containing P_1 is **not contractible** (the rightmost figure of Figure 4). The component containing the cusp P_0 should be **contractible** (the right figure of Figure 6). (If not, then this component would be like the middle figure of Figure 4 and the number of the intersection points of $\mathbb{R}A_1'$ with $\mathbb{R}c$ would be even. This contradicts with $A_1' \cdot c = 3$.) $\mathbb{R}A_1'$ might have some ovals. **The component containing** P_1 and **the section** $\mathbb{R}s$ divide $\mathbb{R}\mathbb{F}_4$ into two regions.

Definition 2.21 (The regions R_1 and R_2 in Cusp (2) case)**.** Let R_1 denote the region which does not contain the contractible component which contains the cusp P_0, and let R_2 denote the other region. (the right figure of Figure 8) (Since $A_1' \cdot c = 3$, the interior of any oval of $\mathbb{R}A_1$ does not contain any ovals.)

We define the integers α and β by (2.2). See the right figure of Figure 8 above.

Isolated point case:

In this case the connected component containing P_1 is nonsingular and non-contractible like the right figure of Figure 4. $\mathbb{R}A_1'$ might have some ovals. **The component containing** P_1 and **the section** $\mathbb{R}s$ divide $\mathbb{R}\mathbb{F}_4$ into two regions.

Definition 2.22 (The regions R_1 and R_2 in Isolated point case)**.** Let R_1 denote the region which **does not contain the isolated point**, and let R_2 denote the other region. (Since $A_1' \cdot c = 3$, the interior of any oval of $\mathbb{R}A_1$ does not contain any other ovals.)

We define the integers α and β by (2.2). See Figure 9.

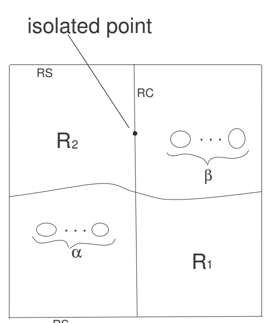


FIGURE 9. The regions R_1 and R_2 in **Isolated point** case

From the above argument, we have:

Proposition 2.23. *The real isotopy type of the singular connected component, which has one double point, of the curve* $\mathbb{R}A_1'$ *on* $\mathbb{R}\mathbb{F}_4$ *is one of the following* 6 *types:*

- Node (1), Node (2), Node (*),
- Cusp (1), Cusp (2), *or*
- Isolated point.

See TABLE 3. □

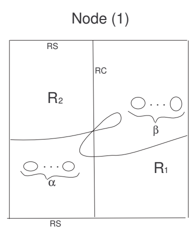
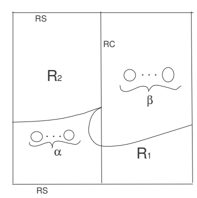
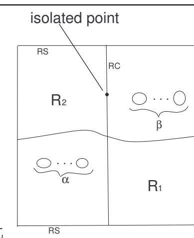
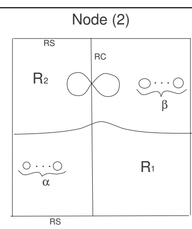
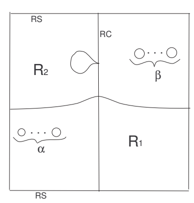
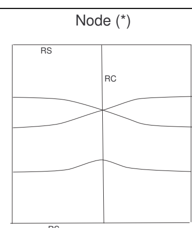
<table>
<tr><td>I</td><td>Node (1)

</td><td>Cusp (1)

</td><td>Isolated point

</td></tr>
<tr><td>II</td><td>Node (2)

</td><td>Cusp (2)

</td><td></td></tr>
<tr><td>III</td><td>Node (*)

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TABLE 3. Real isotopy types of the singular component of the curve $\mathbb{R}A_1'$.

2.6. **Topology of the real parts of K3 surfaces X viewed via the blow up $Y \to \mathbb{F}_4$.** We determine the topology of the real parts $X_\varphi(\mathbb{R})$ and $X_{\widetilde{\varphi}}(\mathbb{R})$ of a real 2-elementary K3 surface (X, τ, φ) of type $(S, \theta) \cong ((3,1,1), -\mathrm{id})$, and the real part $Y(\mathbb{R})$ of the quotient surface Y with the real structure $\varphi_{\mathrm{mod}\ \tau}$. Recall Proposition 2.23.

I. Node (1), **Cusp (1)** and **Isolated point** cases.

In these cases, by the definitions of the regions $A_\pm$ and R_1, R_2, we see that:

- A_+ is homeomorphic to the disjoint union of (an annulus with α holes) and (β disks), and
- A_- is homeomorphic to the disjoint union of (((an annulus $\setminus D^2$)$\cup$ **Möbius band**) with β holes) and (α disks).

For example, see Figure 10 for **Node (1)** case.

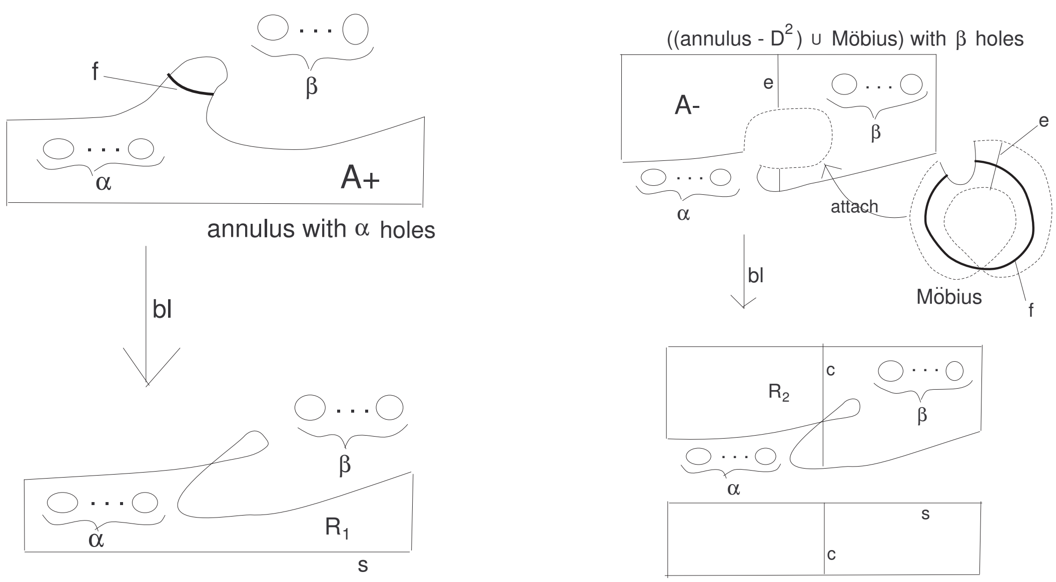

FIGURE 10. The regions A_+ and A_- in **Node (1)** case

Suppose that the invariant $H(\psi) = 0$ for the involution φ.

If F is **irreducible**, then we have $[X_\varphi(\mathbb{R}) \cap F] \neq 0$ and $\pi(X_\varphi(\mathbb{R})) = A_-$. On the other hand, for $\widetilde{\varphi}$, we have $[X_{\widetilde{\varphi}}(\mathbb{R}) \cap F] = 0$ and $\pi(X_{\widetilde{\varphi}}(\mathbb{R})) = A_+$.
We can say that

$$\begin{aligned} \alpha &= \#\{\text{ovals whose } \textbf{interiors} \text{ are contained in } \mathrm{bl}(A_-)\}, \text{ and} \\ \beta &= \#\{\text{ovals whose } \textbf{interiors} \text{ are contained in } \mathrm{bl}(A_+)\}. \end{aligned}$$

Thus we have

$$X_\varphi(\mathbb{R}) \sim \Sigma_{2+\beta} \cup \alpha S^2.$$

Moreover, we have $(r(\psi), a(\psi), \delta_{\psi S}) \neq (10, 10, 0),\ (10, 8, 0)$,

$$r(\psi) = 9 + \alpha - \beta, \quad a(\psi) = 9 - \alpha - \beta.$$

On the other hand, we have

$$X_{\widetilde{\varphi}}(\mathbb{R}) \sim \Sigma_{1+\alpha} \cup \beta S^2.$$

Hence, we have $H(\sigma \circ \psi) \cong \mathbb{Z}/2\mathbb{Z}$ and

$$r(\sigma \circ \psi) = 10 - \alpha + \beta, \quad a(\sigma \circ \psi) = 10 - \alpha - \beta.$$

We omit the cusp and isolated point cases.

II. Node (2) and **Cusp (2)** cases.

In these cases, by the definitions of the regions $A_\pm$ and $R_1,\ R_2$, we see that

- A_+ is homeomorphic to the disjoint union of (an annulus with α holes) and (($\beta+1$) disks), and
- A_- is homeomorphic to the disjoint union of (((an annulus $\setminus D^2$)$\cup$ **Möbius band**) with ($\beta+1$) holes) and (α disks).

For example, See Figure 11 for **Node (2)** case.

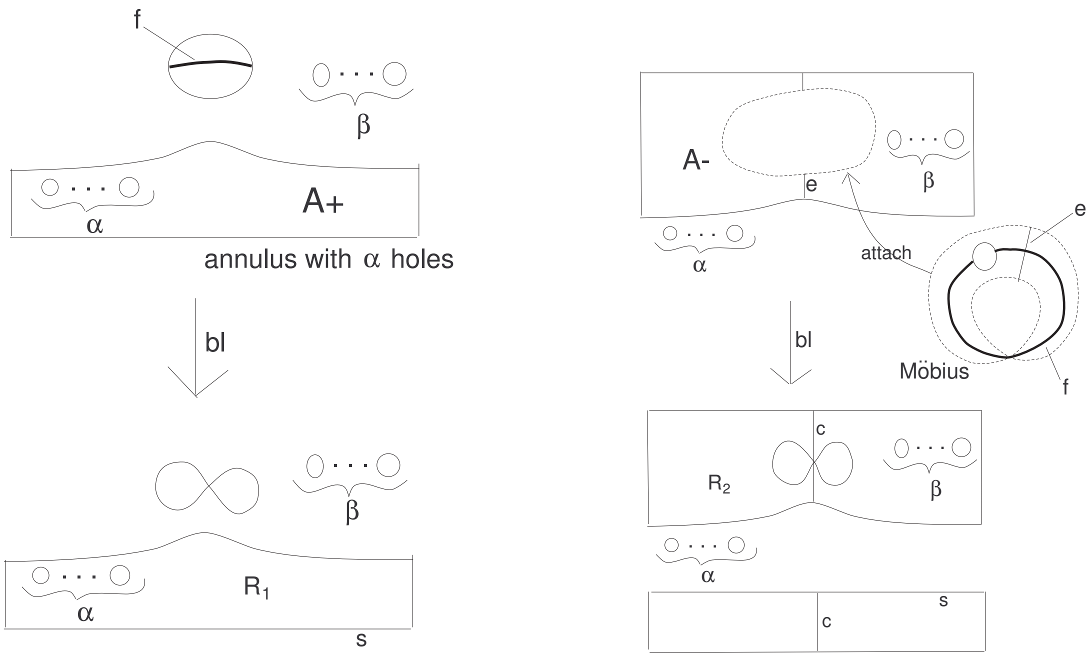

Figure 11. The regions A_+ and A_- in **Node (2)** case

Suppose that $A_- = \pi(X_\varphi(\mathbb{R}))$, namely, the invariant $H(\psi) = 0$ for the involution φ. Then we have

$$X_\varphi(\mathbb{R}) \sim \Sigma_{2+(\beta+1)} \cup \alpha S^2.$$

Moreover, we have $(r(\psi), a(\psi), \delta_{\psi S}) \neq (10, 10, 0),\ (10, 8, 0)$,

$$r(\psi) = 8 + \alpha - \beta, \quad a(\psi) = 8 - \alpha - \beta.$$

On the other hand, we have $A_+ = \pi(X_{\widetilde{\varphi}}(\mathbb{R}))$ and

$$X_{\widetilde{\varphi}}(\mathbb{R}) \sim \Sigma_{1+\alpha} \cup (\beta+1)S^2,$$

Moreover, we have $H(\sigma \circ \psi) \cong \mathbb{Z}/2\mathbb{Z}$ and

$$r(\sigma \circ \psi) = 11 - \alpha + \beta, \quad a(\sigma \circ \psi) = 9 - \alpha - \beta.$$

We omit the cusp cases.

III. Node (*) case.

In this case, we see that

- A_+ is homeomorphic to $D^2 \setminus 2D^2$ and
- A_- is the disjoint union of an **Möbius band** and an annulus.

See Figure 12.

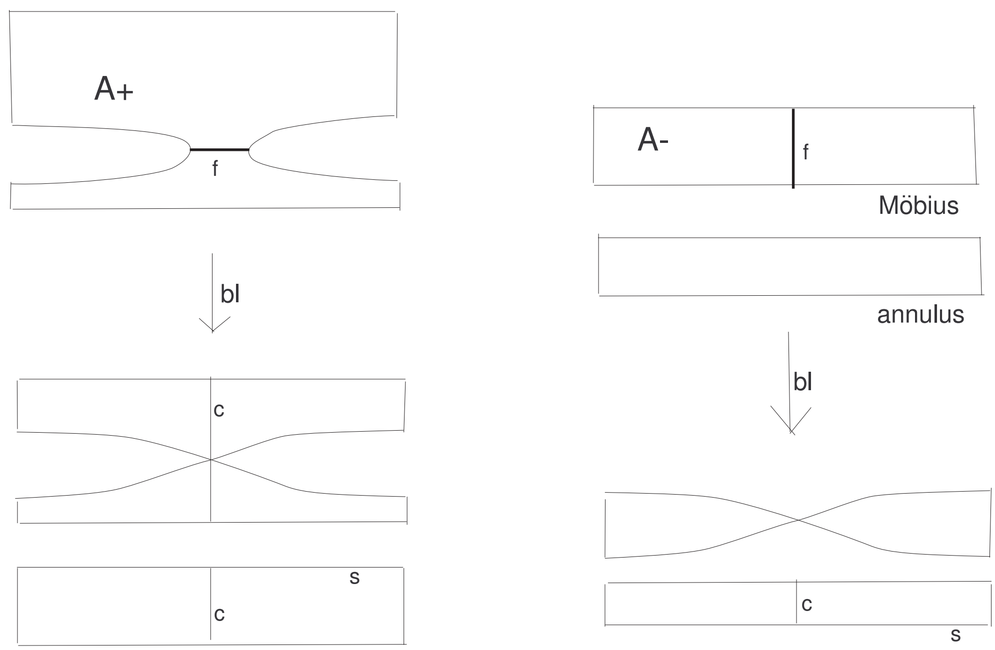

FIGURE 12. The regions A_+ and A_- in **Node (*)** case

Suppose that, for φ, $A_- = \pi(X_\varphi(\mathbb{R}))$. Then we see that

$$X_\varphi(\mathbb{R}) \sim T^2 \cup T^2,$$

and

$$A_+ = \pi(X_{\widetilde{\varphi}}(\mathbb{R})), \quad X_{\widetilde{\varphi}}(\mathbb{R}) \sim \Sigma_2.$$

Moreover, we have

$$H(\psi) = 0 \quad \text{and} \quad (r(\psi), a(\psi), \delta_{\psi S}) = (10, 8, 0).$$

On the other hand, we have

$$H(\sigma \circ \psi) \cong \mathbb{Z}/2\mathbb{Z}$$

and

$$(r(\sigma \circ \psi), a(\sigma \circ \psi), \delta_{\sigma \circ \psi S}) = (9, 9, 0).$$

2.7. **Real isotopy types of real anti-bicanonical curves $\mathbb{R}\mathrm{bl}(A)$ with one real double point on $\mathbb{R}A_1'$ on $\mathbb{RF}_4$.** In Node and Cusp cases, the number of connected components of $\mathbb{R}A_1'$ equals to that of connected components of $\mathbb{R}A_1$. Hence, we have

$$1 \leq \#\{\text{Connected components of } \mathbb{R}A_1'\} \leq 10.$$

In **Node (1)** and **Cusp (1)** cases, we have

$$0 \leq \alpha + \beta \leq 9.$$

If for φ, $A_- = \pi(X_\varphi(\mathbb{R}))$, namely, the invariant $H(\psi) = 0$, then $\alpha + \beta = 9 - a(\psi)$.

In **Node (2)** and **Cusp (2)** cases, we have

$$0 \leq \alpha + \beta \leq 8.$$

If for φ, $A_- = \pi(X_\varphi(\mathbb{R}))$, namely, the invariant $H(\psi) = 0$, then $\alpha + \beta = 8 - a(\psi)$.

In Isolated point case, the number of connected components of $\mathbb{R}A_1'$ equals to that of connected components of $\mathbb{R}A_1$ plus 1.

Hence, we have

$$2 \leq \#\{\text{Connected components of } \mathbb{R}A_1'\} \leq 11.$$

Hence, we have

$$0 \leq \alpha + \beta \leq 9.$$

If for φ, $A_- = \pi(X_\varphi(\mathbb{R}))$, namely, the invariant $H(\psi) = 0$, then $\alpha + \beta = 8 - a(\psi)$.

We already have all the isometry classes. Recall TABLE 1 AND TABLE 2 in Subsection 2.3.

Theorem 2.24. *We have the following.*

- *For* **each** *isometry class with $H(\psi) = 0$, the real isotopy type of a real anti-bicanonical curve $\mathbb{R}\mathrm{bl}(A) = \mathbb{R}s \cup \mathbb{R}A_1'$ on $\mathbb{RF}_4$ with* **one real nondegenerate double point** *on $\mathbb{R}A_1'$ on $\mathbb{RF}_4$ is one of the data listed up in Table 4.*
- *For* **each** *isometry class with $H(\psi) \cong \mathbb{Z}/2\mathbb{Z}$, the real isotopy type of a real anti-bicanonical curve $\mathbb{R}\mathrm{bl}(A) = \mathbb{R}s \cup \mathbb{R}A_1'$ on $\mathbb{RF}_4$ with* **one real nondegenerate double point** *on $\mathbb{R}A_1'$ on $\mathbb{RF}_4$ is one of the data listed up in Table 5.*

Note that the isometry class No.k and the isometry class No.k′ are related integral involutions for each $k = 1, \ldots, 50$. The isometry class $(10, 8, 0, H(\psi) = 0)$ and $(9, 9, 0, H(\psi) \cong \mathbb{Z}/2\mathbb{Z})$ are also related integral involutions. □

Isometry class of type $((3,1,1),-\mathrm{id})$						Node (1)		Isolated point		Node (2)		Node (*)
No.	$r(\psi)$	$a(\psi)$	$\delta_{\varphi S}$	g	k	α	β	α	β	α	β	
1	1	1	1	10	0	0	8	0	8	0	7	
2	2	0	0	10	1	1	8	1	8	1	7	
3	2	2	0	9	0	0	7	0	7	0	6	
4	2	2	1	9	0	0	7	0	7	0	6	
5	3	1	1	9	1	1	7	1	7	1	6	
6	3	3	1	8	0	0	6	0	6	0	5	
7	4	2	1	8	1	1	6	1	6	1	5	
8	4	4	1	7	0	0	5	0	5	0	4	
9	5	3	1	7	1	1	5	1	5	1	4	
10	5	5	1	6	0	0	4	0	4	0	3	
11	6	2	0	7	2	2	5	2	5	2	4	
12	6	4	0	6	1	1	4	1	4	1	3	
13	6	4	1	6	1	1	4	1	4	1	3	
14	6	6	1	5	0	0	3	0	3	0	2	
15	7	3	1	6	2	2	4	2	4	2	3	
16	7	5	1	5	1	1	3	1	3	1	2	
17	7	7	1	4	0	0	2	0	2	0	1	
18	8	2	1	6	3	3	4	3	4	3	3	
19	8	4	1	5	2	2	3	2	3	2	2	
20	8	6	1	4	1	1	2	1	2	1	1	
21	8	8	1	3	0	0	1	0	1	0	0	
22	9	1	1	6	4	4	4	4	4	4	3	
23	9	3	1	5	3	3	3	3	3	3	2	
24	9	5	1	4	2	2	2	2	2	2	1	
25	9	7	1	3	1	1	1	1	1	1	0	
26	9	9	1	2	0	0	0	0	0			
27	10	0	0	6	5	5	4	5	4	5	3	
28	10	2	0	5	4	4	3	4	3	4	2	
29	10	2	1	5	4	4	3	4	3	4	2	
30	10	4	0	4	3	3	2	3	2	3	1	
31	10	4	1	4	3	3	2	3	2	3	1	
32	10	6	0	3	2	2	1	2	1	2	0	
33	10	6	1	3	2	2	1	2	1	2	0	
	10	8	0	2	1							$T^2 \cup T^2$
34	10	8	1	2	1	1	0	1	0			
35	11	1	1	5	5	5	3	5	3	5	2	
36	11	3	1	4	4	4	2	4	2	4	1	
37	11	5	1	3	3	3	1	3	1	3	0	
38	11	7	1	2	2	2	0	2	0			
39	12	2	1	4	5	5	2	5	2	5	1	
40	12	4	1	3	4	4	1	4	1	4	0	
41	12	6	1	2	3	3	0	3	0			
42	13	3	1	3	5	5	1	5	1	5	0	
43	13	5	1	2	4	4	0	4	0			
44	14	2	0	3	6	6	1	6	1	6	0	
45	14	4	0	2	5	5	0	5	0			
46	14	4	1	2	5	5	0	5	0			
47	15	3	1	2	6	6	0	6	0			
48	16	2	1	2	7	7	0	7	0			
49	17	1	1	2	8	8	0	8	0			
50	18	0	0	2	9	9	0	9	0			

TABLE 4. Candidates for real isotopy types of real anti-bicanonical curves $\mathbb{R}\mathrm{bl}(A)$ with one real **nondegenerate** double point on $\mathbb{R}A_1'$ on $\mathbb{RF}_4$ for each isometry class of type $(S,\theta) \cong ((3,1,1),-\mathrm{id})$ with $H(\psi)=0$

Isometry class of type $((3,1,1),-\mathrm{id})$						Node (1)		Isolated point		Node (2)		Node (*)
No.	$r(\psi)$	$a(\psi)$	$\delta_{\varphi S}$	g	k	α	β	α	β	α	β	
1’	18	2	1	1	8	0	8	0	8	0	7	
2’	17	1	0	2	8	1	8	1	8	1	7	
3’	17	3	0	1	7	0	7	0	7	0	6	
4’	17	3	1	1	7	0	7	0	7	0	6	
5’	16	2	1	2	7	1	7	1	7	1	6	
6’	16	4	1	1	6	0	6	0	6	0	5	
7’	15	3	1	2	6	1	6	1	6	1	5	
8’	15	5	1	1	5	0	5	0	5	0	4	
9’	14	4	1	2	5	1	5	1	5	1	4	
10’	14	6	1	1	4	0	4	0	4	0	3	
11’	13	3	0	3	5	2	5	2	5	2	4	
12’	13	5	0	2	4	1	4	1	4	1	3	
13’	13	5	1	2	4	1	4	1	4	1	3	
14’	13	7	1	1	3	0	3	0	3	0	2	
15’	12	4	1	3	4	2	4	2	4	2	3	
16’	12	6	1	2	3	1	3	1	3	1	2	
17’	12	8	1	1	2	0	2	0	2	0	1	
18’	11	3	1	4	4	3	4	3	4	3	3	
19’	11	5	1	3	3	2	3	2	3	2	2	
20’	11	7	1	2	2	1	2	1	2	1	1	
21’	11	9	1	1	1	0	1	0	1	0	0	
22’	10	2	1	5	4	4	4	4	4	4	3	
23’	10	4	1	4	3	3	3	3	3	3	2	
24’	10	6	1	3	2	2	2	2	2	2	1	
25’	10	8	1	2	1	1	1	1	1	1	0	
26’	10	10	1	1	0	0	0	0	0			
27’	9	1	0	6	4	5	4	5	4	5	3	
28’	9	3	0	5	3	4	3	4	3	4	2	
29’	9	3	1	5	3	4	3	4	3	4	2	
30’	9	5	0	4	2	3	2	3	2	3	1	
31’	9	5	1	4	2	3	2	3	2	3	1	
32’	9	7	0	3	1	2	1	2	1	2	0	
33’	9	7	1	3	1	2	1	2	1	2	0	
	9	9	0	2	0	1	0	1	0			Σ_2
34’	9	9	1	2	0	1	0	1	0			
35’	8	2	1	6	3	5	3	5	3	5	2	
36’	8	4	1	5	2	4	2	4	2	4	1	
37’	8	6	1	4	1	3	1	3	1	3	0	
38’	8	8	1	3	0	2	0	2	0			
39’	7	3	1	6	2	5	2	5	2	5	1	
40’	7	5	1	5	1	4	1	4	1	4	0	
41’	7	7	1	4	0	3	0	3	0			
42’	6	4	1	6	1	5	1	5	1	5	0	
43’	6	6	1	5	0	4	0	4	0			
44’	5	3	0	7	1	6	1	6	1	6	0	
45’	5	5	0	6	0	5	0	5	0			
46’	5	5	1	6	0	5	0	5	0			
47’	4	4	1	7	0	6	0	6	0			
48’	3	3	1	8	0	7	0	7	0			
49’	2	2	1	9	0	8	0	8	0			
50’	1	1	0	10	0	9	0	9	0			

TABLE 5. Candidates for real isotopy types of real anti-bicanonical curves $\mathbb{R}\mathrm{bl}(A)$ with one real **nondegenerate** double point on $\mathbb{R}A_1'$ on $\mathbb{R}\mathbb{F}_4$ for each isometry class of type $(S,\theta) \cong ((3,1,1),-\mathrm{id})$ with $H(\psi) \cong \mathbb{Z}/2\mathbb{Z}$

Remark 2.25. By Theorem 2.24, we find that each isometry class of integral involutions of the K3 lattice $\mathbb{L}_{K3}$ of type $(S,\theta) \cong ((3,1,1),-\mathrm{id})$ may contain several real isotopy types (for example, Node (1), Isolated point, and Node (2)) of real anti-bicanonical curves $\mathbb{R}\mathrm{bl}(A)$ with one real nondegenerate double point on $\mathbb{R}A_1'$ on $\mathbb{RF}_4$. Hence, the realizability of all the real isotopy types listed in Table 4,5 has not been solved yet. However, we can take a real 2-elementary K3 surface for which F is irreducible in the same connected component of the moduli (see [11], notes after Theorem 2). Hence, for each isometry class, at least one of real isotopy types is realizable. Especially, Node (*) with the isometry class $(10,8,0,H(\psi)=0)$ is realizable. It is conjectured that Node (1) with $(\alpha,\beta)=(1,0)$ and Isolated point with $(\alpha,\beta)=(1,0)$ do not exist for the isometry class $(9,9,0,H(\psi)\cong\mathbb{Z}/2\mathbb{Z})$. See also Remark 3.5 below.

Remark 2.26 ([11])**.** On the other hand, if we contract the exceptional curve $e=\pi(E)$ in Y, then we get a map onto the 3-th Hirzebruch surface $\mathbb{F}_3$: $\mathrm{bl}_1: Y \to \mathbb{F}_3$. Then $s := \mathrm{bl}_1(A_0)$ is the exceptional section of $\mathbb{F}_3$ with $s^2=-3$ and $c := \mathrm{bl}_1(f)$ is a fiber. We have

$$\mathrm{bl}_1(A) = s + \mathrm{bl}_1(A_1) \in |-2K_{\mathbb{F}_3}|.$$

$A_1 := \mathrm{bl}_1(A_1)$ is a real **nonsingular** curve of genus 9. The real isotopic classification of $\mathbb{R}A_1$ on $\mathbb{RF}_3$ was already done in [11], Theorem 1.

In order to **distinguish** the real isotopy types of real anti-bicanonical curves with one real double point on $\mathbb{RF}_4$, we expect that Itenberg's argument of the rigid isotopic classification of real curves of degree 6 on $\mathbb{RP}^2$ with one nondegenerate double point are helpful. See [4],[5] and [6], and also the last section 4 of this paper. This classification corresponds to that of real **nonsingular** curves A in $|-2K_{\mathbb{F}_1}|$ on the first real Hirzebruch surface $\mathbb{RF}_1$ when we blow up $\mathbb{P}^2$ at the nondegenerate double point to $\mathbb{F}_1$. The double coverings X of $\mathbb{F}_1$ ramified along the nonsingular curves A are real 2-elementary K3 surfaces of type $(S,\theta)\cong(\langle 2\rangle\oplus\langle -2\rangle,-\mathrm{id})$. Moreover, the classification of real curves of degree 6 on $\mathbb{RP}^2$ with one nondegenerate double point is related to that of "non-increasing simplest degenerations" (conjunctions and contractions) of real **nonsingular** curves of degree 6 on $\mathbb{RP}^2$. See [4] and [5].

Thus we next pay attention to the degenerations of real nonsingular anti-bicanonical curves on $\mathbb{RF}_4$.

3. Degenerations of nonsingular real anti-bicanonical curves on $\mathbb{RF}_4$

3.1. **Review of nonsingular real anti-bicanonical curves on $\mathbb{RF}_4$.**

The contents of this subsection are quoted from the last section of [10].

Let $\mathbb{U}$ be the even unimodular lattice of signature $(1,1)$ (the hyperbolic plane). Consider real 2-elementary K3 surfaces (X,τ,φ) of type $(S,\theta)\cong(\mathbb{U},-\mathrm{id})$. All these real 2-elementary K3 surfaces are $(\mathcal{D})$-nondegenerate.

Let A be the fixed point set of τ. Then A is a real nonsingular curve. We have $Y := X/\tau = \mathbb{F}_4$. Let $\pi: X \to \mathbb{F}_4$ be the quotient map. We use the same symbol A for its image in $\mathbb{F}_4$ by π. Then $A \in |-2K_{\mathbb{F}_4}|$. Let s be the exceptional section with $s^2=-4$ of $\mathbb{F}_4$, and c the fiber of the fibration $f: \mathbb{F}_4 \to s$, where $c^2=0$. We have $-2K_{\mathbb{F}_4} \sim 12c+4s$. The nonsingular curve A has **two irreducible components** s and A_1;

$$A = s \cup A_1 \quad \text{(disjoint union)}.$$

Conversely, any nonsingular curve A_1 in $|12c+3s|$ gives a nonsingular curve $A = s + A_1$ in $|-2K_{\mathbb{F}_4}|$.

We set $C := \pi^*(c)$ and $E := \pi^*(s)/2$ in $H_2(X,\mathbb{Z})$. Then $C^2=0$, $E^2=-2$ and $C\cdot E=1$. C and E generate the fixed part of $\tau_*: H_2(X,\mathbb{Z}) \to H_2(X,\mathbb{Z})$. Hence, $S \cong \mathbb{Z}C+\mathbb{Z}E \cong \mathbb{U}$.

An isometry class of an integral involution $(\mathbb{L}_{K3}, \psi)$ of type $(S, \theta) \cong (\mathbb{U}, -\mathrm{id})$ is determined (see [10]) by the data

$$(r(\psi),\ a(\psi),\ \delta_\psi = \delta_{\psi S}). \tag{3.1}$$

The complete list of the data (3.1) is given in Section 7 (Figure 30) of [10].

There are 14 isometry classes with $\delta_\psi = 0$ and 49 isometry classes with $\delta_\psi = 1$. Thus we have 63 classes.

If we identify related integral involutions, there are 10 isometry classes with $\delta_\psi = 0$ and 27 isometry classes with $\delta_\psi = 1$. Thus we have 37 classes.

If $(r(\psi), a(\psi), \delta_\psi) = (10, 10, 0)$, then $\mathbb{R}\mathbb{F}_4 = \emptyset$. Hence, $\mathbb{R}A = \emptyset$.

If $(r(\psi), a(\psi), \delta_\psi) \neq (10, 10, 0)$, then $\mathbb{R}\mathbb{F}_4$ is not empty and homeomorphic to a 2-torus. We have $\mathbb{R}A \supset \mathbb{R}s \neq \emptyset$. The real curve $\mathbb{R}A_1$ is contained in the open cylinder $\mathbb{R}\mathbb{F}_4 \setminus \mathbb{R}s$. The region $A_- = \pi(X_\varphi(\mathbb{R}))$ with the invariants (3.1) has the real isotopy type given in Figure 13.

When $(r(\psi), a(\psi), \delta_\psi) \neq (10, 10, 0)$ and $\neq (10, 8, 0)$, we set

$$g := (22 - r(\psi) - a(\psi))/2 \quad \text{and} \quad k := (r(\psi) - a(\psi))/2.$$

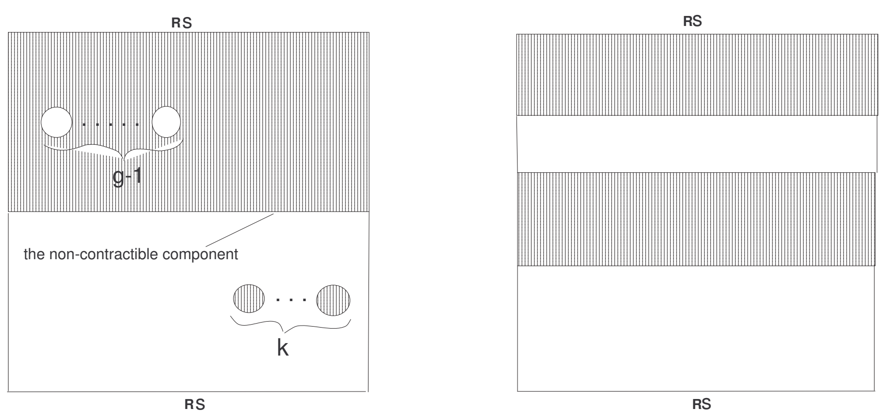

FIGURE 13. A region A_- with $(r(\psi), a(\psi), \delta_\psi) \neq (10, 8, 0), (10, 10, 0)$ and a region A_- with $(r(\psi), a(\psi), \delta_\psi) = (10, 8, 0)$

Since all real 2-elementary K3 surfaces of type $(S, \theta) \cong (\mathbb{U}, -\mathrm{id})$ are $(\mathcal{D})$-nondegenerate, by Theorem 2.11, we have:

Theorem 3.1 ([10], Theorem 27)**.** *A connected component of the moduli of the regions*

$$A_- := \pi(X_\varphi(\mathbb{R})),$$

curves $A \in |-2K_{\mathbb{F}_4}|$, up to the action of the automorphisms group of $\mathbb{F}_4$ over $\mathbb{R}$ is determined by the data (3.1), equivalently, the real isotopy types of A_- and the invariants $\delta_\psi = \delta_{\psi S}$. All the data are given in Section 7 (Figure 30) of [10]. See also Figure 13.

3.2. **Degenerations of nonsingular real anti-bicanonical curves on $\mathbb{RF}_4$.** We next introduce the notions of "non-increasing simplest degenerations" (conjunctions, contractions) of **nonsingular** real anti-bicanonical curves on $\mathbb{RF}_4$ as analogies of Section 3 (pp.284–285) of [4].

As stated in the previous subsection, a nonsingular curve in $|-2K_{\mathbb{F}_4}|$ has two irreducible components s and A_1, where A_1 is a nonsingular curve in $|12c+3s|$.

Let C_0 be a real curve in $|12c+3s|$ on $\mathbb{F}_4$ with one nondegenerate double point, and

$$C_t \quad (-\varepsilon < t < \varepsilon)$$

be a smoothing (, where every C_t $(t \neq 0)$ is a real nonsingular curve in $|12c+3s|$ on $\mathbb{F}_4$,) such that

$$\sharp\{\text{ovals of a nonsingular curve } C_{t_{-1}}\ \} \ \geq\ \sharp\{\text{ovals of a nonsingular curve } C_{t_1}\ \}$$

for any $t_{-1} < 0$ and any $t_1 > 0$.

We call such a family C_t $(t_{-1} \leq t \leq 0)$ a **non-increasing simplest degeneration** of $C_{t_{-1}}$ to C_0.

We do not know whether non-increasing simplest degenerations of $C_{t_{-1}}$ to C_0 are realizable for **any pair** $C_{t_{-1}}$ and C_0.

We define 8 kinds of non-increasing simplest degenerations.

Definition 3.2 (Conjunctions 1), 2), 1'), 2') and Contractions 3), 3')). First we fix **an isometry class** of integral involutions of type $(S,\theta) \cong (\mathbb{U}, -\mathrm{id})$ with $(r(\psi), a(\psi), \delta_\psi) \neq (10,8,0)$ and $\neq (10,10,0)$.

Take a corresponding real 2-elementary K3 surfaces (X, τ, φ) of type $(S,\theta) \cong (\mathbb{U}, -\mathrm{id})$. Let $\pi : X \to X/\tau = \mathbb{F}_4$ be the quotient map. Then we get a real nonsingular curve $A = s + A_1$ in $|-2K_{\mathbb{F}_4}|$ where A_1 is a real nonsingular curve in $|12c+3s|$ on $\mathbb{RF}_4$. Suppose that *the fixed point set $X_\varphi(\mathbb{R})$ is homeomorphic to $\Sigma_g \cup kS^2$.* Then the region $\pi(X_\varphi(\mathbb{R}))$ $(\subset \mathbb{RF}_4)$ is the disjoint union of an annulus with $(g-1)$ holes and k disks. The boundary of the annulus with $(g-1)$ holes consists of (see Figure 13) *the non-contractible component of $\mathbb{R}A_1$, $\mathbb{R}s$, and the $(g-1)$ empty ovals.*

- **Conjunction 1)** The conjunction of the non-contractible component and one of the $(g-1)$ empty ovals. (Recall that the annulus with $(g-1)$ holes is covered by the fixed point set of the anti-holomorphic involution φ.)
- **Conjunction 1')** The conjunction of the non-contractible component and one of the k empty ovals. (Remark that the region (annulus) surrounded by the non-contractible component, $\mathbb{R}s$ and the k empty ovals is covered by the fixed point set of the related involution $\widetilde{\varphi}$ of the anti-holomorphic involution φ.)
- **Conjunction 2)** The conjunction of two of the $(g-1)$ empty ovals.
- **Conjunction 2')** The conjunction of two of the k empty ovals.
- **Contraction 3)** The contraction of one of the $(g-1)$ empty ovals.
- **Contraction 3')** The contraction of one of the k empty ovals.

Definition 3.3 (Conjunctions 4), 4')). Consider **the isometry class** of integral involutions of type $(S,\theta) \cong (\mathbb{U}, -\mathrm{id})$ with $(r(\psi), a(\psi), \delta_\psi) = (9,9,1)$ or $(11,9,1)$. Remark that these two involutions are **related**.

If $(r(\psi), a(\psi), \delta_\psi) = (9,9,1)$, then we have $g = 2,\ k = 0$. Then the region $\pi(X_\varphi(\mathbb{R}))$ $(\subset \mathbb{RF}_4)$ is an annulus with one hole. The boundary of the annulus consists of *the non-contractible component of $\mathbb{R}A_1$, $\mathbb{R}s$, and the empty oval.*

- **Conjunction 4)** The empty oval conjuncts with itself and becomes the union of two real lines (Node (*)) on $\mathbb{RF}_4$.

If $(r(\psi), a(\psi), \delta_\psi) = (11, 9, 1)$, then we have $g = 1,\ k = 1$. Then the region $\pi(X_\varphi(\mathbb{R}))$ $(\subset \mathbb{RF}_4)$ is the disjoint union of an annulus and one disk. The boundary of the annulus consists of *the non-contractible component of* $\mathbb{R}A_1$ *and* $\mathbb{R}s$.

- **Conjunction 4')** The empty oval conjuncts with itself and becomes the union of two real lines (Node (*)) on $\mathbb{RF}_4$.

See Figure 14. Compare with Figure 13.

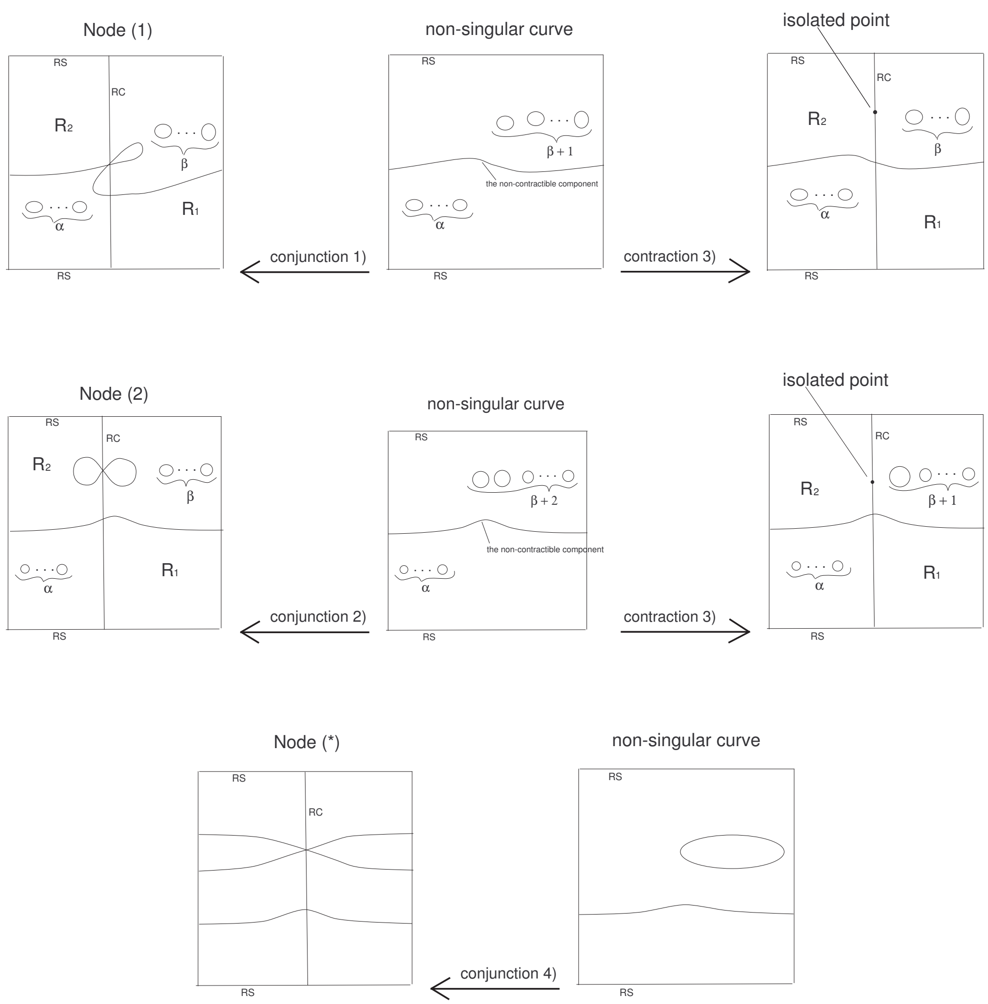

FIGURE 14. Non-increasing simplest degenerations of nonsingular real anti-bicanonical curves on $\mathbb{RF}_4$.

isometry class of type $(\mathbb{U}, -\mathrm{id})$ (nonsingular curve)						conjunction 1) → Node (1)	conjunction 2) → Node (2)	contraction 3) → Isolated point	
$r(\psi)$	$a(\psi)$	δ_ψ	g	k	$g-1$	α, β	α, β	α, β	No.
1	1	1	10	0	9	0, 8	0, 7	0, 8	1
2	0	0	10	1	9	1, 8	1, 7	1, 8	2
2	2	0	9	0	8	0, 7	0, 6	0, 7	3
2	2	1	9	0	8	0, 7	0, 6	0, 7	4
3	1	1	9	1	8	1, 7	1, 6	1, 7	5
3	3	1	8	0	7	0, 6	0, 5	0, 6	6
4	2	1	8	1	7	1, 6	1, 5	1, 6	7
4	4	1	7	0	6	0, 5	0, 4	0, 5	8
5	3	1	7	1	6	1, 5	1, 4	1, 5	9
5	5	1	6	0	5	0, 4	0, 3	0, 4	10
6	2	0	7	2	6	2, 5	2, 4	2, 5	11
6	4	0	6	1	5	1, 4	1, 3	1, 4	12
6	4	1	6	1	5	1, 4	1, 3	1, 4	13
6	6	1	5	0	4	0, 3	0, 2	0, 3	14
7	3	1	6	2	5	2, 4	2, 3	2, 4	15
7	5	1	5	1	4	1, 3	1, 2	1, 3	16
7	7	1	4	0	3	0, 2	0, 1	0, 2	17
8	2	1	6	3	5	3, 4	3, 3	3, 4	18
8	4	1	5	2	4	2, 3	2, 2	2, 3	19
8	6	1	4	1	3	1, 2	1, 1	1, 2	20
8	8	1	3	0	2	0, 1	0, 0	0, 1	21
9	1	1	6	4	5	4, 4	4, 3	4, 4	22
9	3	1	5	3	4	3, 3	3, 2	3, 3	23
9	5	1	4	2	3	2, 2	2, 1	2, 2	24
9	7	1	3	1	2	1, 1	1, 0	1, 1	25
9	9	1	2	0	1	0, 0	impossible	0, 0	26
10	0	0	6	5	5	5, 4	5, 3	5, 4	27
10	2	0	5	4	4	4, 3	4, 2	4, 3	28
10	2	1	5	4	4	4, 3	4, 2	4, 3	29
10	4	0	4	3	3	3, 2	3, 1	3, 2	30
10	4	1	4	3	3	3, 2	3, 1	3, 2	31
10	6	0	3	2	2	2, 1	2, 0	2, 1	32
10	6	1	3	2	2	2, 1	2, 0	2, 1	33
10	8	1	2	1	1	1, 0	impossible	1, 0	34
11	1	1	5	5	4	5, 3	5, 2	5, 3	35
11	3	1	4	4	3	4, 2	4, 1	4, 2	36
11	5	1	3	3	2	3, 1	3, 0	3, 1	37
11	7	1	2	2	1	2, 0	impossible	2, 0	38
12	2	1	4	5	3	5, 2	5, 1	5, 2	39
12	4	1	3	4	2	4, 1	4, 0	4, 1	40
12	6	1	2	3	1	3, 0	impossible	3, 0	41
13	3	1	3	5	2	5, 1	5, 0	5, 1	42
13	5	1	2	4	1	4, 0	impossible	4, 0	43
14	2	0	3	6	2	6, 1	6, 0	6, 1	44
14	4	0	2	5	1	5, 0	impossible	5, 0	45
14	4	1	2	5	1	5, 0	impossible	5, 0	46
15	3	1	2	6	1	6, 0	impossible	6, 0	47
16	2	1	2	7	1	7, 0	impossible	7, 0	48
17	1	1	2	8	1	8, 0	impossible	8, 0	49
18	0	0	2	9	1	9, 0	impossible	9, 0	50

TABLE 6. Conjunction 1), Conjunction 2) and Contraction 3)

We list up candidates for possible non-increasing simplest degenerations for each of 63 isometry classes of integral involutions of type $(S, \theta) \cong (\mathbb{U}, -\mathrm{id})$ with the invariant $(r(\psi), a(\psi), \delta_\psi)$.

Theorem 3.4. *We can enumerate up the following candidates:*

- *For Conjunction 1) Conjunction 2) Contraction 3), see* TABLE 6.
- *For Conjunction 1') Conjunction 2') Contraction 3'), see* TABLE 7.
- *For Conjunction 4) and 4'), see* TABLE 8. □

isometry class of type $(\mathbb{U}, -\mathrm{id})$ (nonsingular curve)						conjunction 1') → Node (1)	conjunction 2') → Node (2)	contraction 3') → Isolated point	
$r(\psi)$	$a(\psi)$	δ_ψ	g	k	$g-1$	α, β	α, β	α, β	No.
19	1	1	1	9	0	0, 8	0, 7	0, 8	1'
18	0	0	2	9	1	1, 8	1, 7	1, 8	2'
18	2	0	1	8	0	0, 7	0, 6	0, 7	3'
18	2	1	1	8	0	0, 7	0, 6	0, 7	4'
17	1	1	2	8	1	1, 7	1, 6	1, 7	5'
17	3	1	1	7	0	0, 6	0, 5	0, 6	6'
16	2	1	2	7	1	1, 6	1, 5	1, 6	7'
16	4	1	1	6	0	0, 5	0, 4	0, 5	8'
15	3	1	2	6	1	1, 5	1, 4	1, 5	9'
15	5	1	1	5	0	0, 4	0, 3	0, 4	10'
14	2	0	3	6	2	2, 5	2, 4	2, 5	11'
14	4	0	2	5	1	1, 4	1, 3	1, 4	12'
14	4	1	2	5	1	1, 4	1, 3	1, 4	13'
14	6	1	1	4	0	0, 3	0, 2	0, 3	14'
13	3	1	3	5	2	2, 4	2, 3	2, 4	15'
13	5	1	2	4	1	1, 3	1, 2	1, 2	16'
13	7	1	1	3	0	0, 2	0, 1	0, 2	17'
12	2	1	4	5	3	3, 4	3, 3	3, 4	18'
12	4	1	3	4	2	2, 3	2, 2	2, 3	19'
12	6	1	2	3	1	1, 2	1, 1	1, 2	20'
12	8	1	1	2	0	0, 1	0, 0	0, 1	21'
11	1	1	5	5	4	4, 4	4, 3	4, 4	22'
11	3	1	4	4	3	3, 3	3, 2	3, 3	23'
11	5	1	3	3	2	2, 2	2, 1	2, 2	24'
11	7	1	2	2	1	1, 1	1, 0	1, 1	25'
11	9	1	1	1	0	0, 0	impossible	0, 0	26'
10	0	0	6	5	5	5, 4	5, 3	5, 4	27'
10	2	0	5	4	4	4, 3	4, 2	4, 3	28'
10	2	1	5	4	4	4, 3	4, 2	4, 3	29'
10	4	0	4	3	3	3, 2	3, 1	3, 2	30'
10	4	1	4	3	3	3, 2	3, 1	3, 2	31'
10	6	0	3	2	2	2, 1	2, 0	2, 1	32'
10	6	1	3	2	2	2, 1	2, 0	2, 1	33'
10	8	1	2	1	1	1, 0	impossible	1, 0	34'
9	1	1	6	4	5	5, 3	5, 2	5, 3	35'
9	3	1	5	3	4	4, 2	4, 1	4, 2	36'
9	5	1	4	2	3	3, 1	3, 0	3, 1	37'
9	7	1	3	1	2	2, 0	impossible	2, 0	38'
8	2	1	6	3	5	5, 2	5, 1	5, 2	39'
8	4	1	5	2	4	4, 1	4, 0	4, 1	40'
8	6	1	4	1	3	3, 0	impossible	3, 0	41'
7	3	1	6	2	5	5, 1	5, 0	5, 1	42'
7	5	1	5	1	4	4, 0	impossible	4, 0	43'
6	2	0	7	2	6	6, 1	6, 0	6, 1	44'
6	4	0	6	1	5	5, 0	impossible	5, 0	45'
6	4	1	6	1	5	5, 0	impossible	5, 0	46'
5	3	1	7	1	6	6, 0	impossible	6, 0	47'
4	2	1	8	1	7	7, 0	impossible	7, 0	48'
3	1	1	9	1	8	8, 0	impossible	8, 0	49'
2	0	0	10	1	9	9, 0	impossible	9, 0	50'

TABLE 7. Conjunction 1'), Conjunction 2') and Contraction 3')

isometry class of type $(\mathbb{U}, -\mathrm{id})$ (nonsingular curve)						conjunction 4)
$r(\psi)$	$a(\psi)$	δ_ψ	g	k	$g-1$	
9	9	1	2	0	1	Node (*)

isometry class of type $(\mathbb{U}, -\mathrm{id})$ (nonsingular curve)						conjunction 4')
$r(\psi)$	$a(\psi)$	δ_ψ	g	k	$g-1$	
11	9	1	1	1	0	Node (*)

TABLE 8. Conjunctions 4) and 4')

Remark 3.5. We observe an obvious interesting correspondence between the data of Theorem 2.24 (TABLES 4 — 5) and those of Theorem 3.4 (TABLES 6 — 8).

If we prove the existence of all the non-increasing simplest degenerations listed in TABLES 6, 7, and 8, then we can say that every real isotopy type with an anti-holomorphic involution in Theorem 2.24 except Node (1) with $(\alpha, \beta) = (1, 0)$ and Isolated point with $(\alpha, \beta) = (1, 0)$ in the isometry class $(9, 9, 0, H(\psi) \cong \mathbb{Z}/2\mathbb{Z})$ can be obtain by certain non-increasing simplest degeneration of a real nonsingular curve in $|12c + 3s|$ on $\mathbb{F}_4$. Recall Remark 2.25.

To state the following lemma 3.7, we distinguish the two anti-holomorphic involutions $\varphi_\pm$ on (X, τ) of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$ as follows.

Definition 3.6 (Definition of $\varphi_\pm$)**.** We define the anti-holomorphic involutions $\varphi_\pm$ on a 2-elementary K3 surface (X, τ) of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$ such that

$$\pi(X_{\varphi_\pm}(\mathbb{R})) = A_\pm$$

respectively. Here recall Definition 2.14 of the two regions $A_\pm$.

Thus, if $H(\psi) = 0$ for a real 2-elementary K3 surface (X, τ, φ) of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$, then we set $A_- = \pi(X_\varphi(\mathbb{R}))$, and

$$\varphi_- := \varphi \quad \text{and} \quad \varphi_+ := \widetilde{\varphi} = \tau \circ \varphi.$$

Comparing the data in Theorem 2.24 and Theorem 3.4, we have:

Lemma 3.7. *Fix an* **isometry class** *of integral involutions of* $\mathbb{L}_{K3}$ *of type* $(S, \theta) \cong (\mathbb{U}, -\mathrm{id})$ *with* $(r(\psi), a(\psi), \delta_\psi) \neq (10, 8, 0)$, $(10, 10, 0)$, *and take a corresponding real 2-elementary K3 surface of type* $(S, \theta) \cong (\mathbb{U}, -\mathrm{id})$ *and a real* **nonsingular** *curve* $A = s + A_1$ *in* $|-2K_{\mathbb{F}_4}|$ *where* A_1 *is a real nonsingular curve in* $|12c + 3s|$ *on* $\mathbb{RF}_4$.

Then we have the following:

- *Take real curves* A_1' *with* **one nondegenerate double point** *on* $\mathbb{F}_4$ *from the degenerations of types 1)—3) of the real* **nonsingular** *curve* A_1. **Choose** *real 2-elementary K3 surfaces* (X, τ, φ_-) *of type* $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$. *(See Definition 3.6 and also Remark 4.7 below.)*

  Then, for all such marked real 2-elementary K3 surfaces $((X, \tau, \varphi_-),\ \alpha)$ *obtained from* A_1', *their associated integral involutions* $\alpha \circ (\varphi_-)_* \circ \alpha^{-1}$ *of* $\mathbb{L}_{K3}$ *are* **isometric** *with respect to* $G = \{\mathrm{id}\}$.
- *Take real curves* A_1' *with* **one nondegenerate double point** *on* $\mathbb{F}_4$ *from the degenerations of types 1')—3') of the real* **nonsingular** *curve* A_1. **Choose** *real 2-elementary K3 surfaces* (X, τ, φ_-) *of type* $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$. *(See Definition 3.6 and also Remark 4.7 below.)*

  Then, for all such marked real 2-elementary K3 surfaces $((X, \tau, \varphi_-),\ \alpha)$ *obtained from* A_1', *their associated integral involutions* $\alpha \circ (\varphi_-)_* \circ \alpha^{-1}$ *of* $\mathbb{L}_{K3}$ *are* **isometric** *with respect to* $G = \{\mathrm{id}\}$. □

Lemma 3.7 is an analogy of Proposition 3.3 of [4].

4. APPENDIX: PERIOD DOMAINS AND FURTHER PROBLEMS

We use the terminology defined in Subsection 2.1, and review the formulations of period domains of marked real 2-elementary K3 surfaces in [10], [4].

Lemma 4.1. *Let $((X,\tau,\varphi),\ \alpha)$ and $((X',\tau',\varphi'),\ \alpha')$ be two marked real 2-elementary K3 surfaces of type (S,θ). Let $(\mathbb{L}_{K3},\psi)$ and $(\mathbb{L}_{K3},\psi')$ be their associated integral involutions respectively. Suppose that f is an* **isometry** *with respect to the group G from $(\mathbb{L}_{K3},\psi)$ to $(\mathbb{L}_{K3},\psi')$. Then $((X',\tau',\varphi'),\ f^{-1}\circ\alpha')$ is also a marked real 2-elementary K3 surface of type (S,θ), and its associated integral involution is also $(\mathbb{L}_{K3},\psi)$.*

Especially, if f is an **automorphism** *of $(\mathbb{L}_{K3},\psi)$ with respect to the group G, then $((X,\tau,\varphi),\ f\circ\alpha)$ is also a marked real 2-elementary K3 surface of type (S,θ), and its associated integral involution is also $(\mathbb{L}_{K3},\psi)$. (Recall Definition 2.9.)*

Proof. We have $\psi'\circ f = f\circ\psi$, $f(S)=S$, and $f|_S\in G$. We have $\alpha'\circ{\varphi'}_*\circ{\alpha'}^{-1}\circ f = f\circ\alpha\circ\varphi_*\circ\alpha^{-1}$, and hence, $f^{-1}\circ\alpha'\circ{\varphi'}_*\circ{\alpha'}^{-1}\circ f=\alpha\circ\varphi_*\circ\alpha^{-1}$. We have

$$(f^{-1}\circ\alpha')\circ{\varphi'}_*\circ(f^{-1}\circ\alpha')^{-1}=\alpha\circ\varphi_*\circ\alpha^{-1}=\psi.$$

Here $(f^{-1}\circ\alpha'):H_2(X',\mathbb{Z})\to\mathbb{L}_{K3}$ is an isometry with

$$(f^{-1}\circ\alpha')(H_{2+}(X',\mathbb{Z}))=f^{-1}(S)=S$$

and $(f^{-1}\circ\alpha')\circ{\varphi'}_*=f^{-1}\circ(\alpha'\circ{\varphi'}_*)=f^{-1}\circ(\psi'\circ\alpha')=\psi\circ f^{-1}\circ\alpha'$. Since $f^{-1}\circ\alpha'(H_{2+}(X',\mathbb{Z}))=S$ and $\psi|_S=\theta$, we have

$$(f^{-1}\circ\alpha')\circ{\varphi'}_*=\theta\circ f^{-1}\circ\alpha'\text{on } H_{2+}(X',\mathbb{Z}).$$

Hence,

$$(f^{-1}\circ\alpha'):H_2(X',\mathbb{Z})\to\mathbb{L}_{K3}$$

is another marking of (X',τ',φ'). If we take this new marking of (X',τ',φ'), then its associated integral involution is ψ, which is the same as $((X,\tau,\varphi),\ \alpha)$. Moreover, since $f|_S\in G$, we have $f(\mathcal{M})=\mathcal{M}$. Hence, if we set $\beta:=(f^{-1}\circ\alpha')$, then $\beta_{\mathbb{R}}^{-1}(V^+(S))$ contains a hyperplane section of X' and the set $\beta^{-1}(\Delta(S)_+)$ contains only classes of effective curves of X'. □

Let us **fix** an integral involution $(\mathbb{L}_{K3},\psi)$ of type (S,θ) through this subsection.

We set

$$\Omega_\psi:=\{v\ (\neq 0)\in\mathbb{L}_{K3}\otimes\mathbb{C}\mid v\cdot v=0,\ v\cdot\overline{v}>0,\ v\cdot S=0,\ \psi_{\mathbb{C}}(v)=\overline{v}\}/\mathbb{R}^\times.$$

Let $((X,\tau,\varphi),\ \alpha)$ be a marked real 2-elementary K3 surface of type (S,θ) satisfying

$$\alpha\circ\varphi_*\circ\alpha^{-1}=\psi,$$

namely, ψ is **the associated integral involution with** $((X,\tau,\varphi,)\ \alpha)$. We denote by

$$H\ \ (\subset H_2(X,\mathbb{C}))$$

the Poincare dual of the complex 1-dimensional space $H^{2,0}(X)$. Then we have the complex 1-dimensional subspace $\alpha_{\mathbb{C}}(H)$ in $\mathbb{L}_{K3}\otimes\mathbb{C}$. Then we have

$$\alpha_{\mathbb{C}}(H)\ \in\Omega_\psi.$$

Definition 4.2 (Periods)**.** We say $\alpha_{\mathbb{C}}(H)$ the **period** of a marked real 2-elementary K3 surface $((X,\tau,\varphi),\ \alpha)$ of type (S,θ) satisfying $\alpha\circ\varphi_*\circ\alpha^{-1}=\psi$.

By Lemma 4.1, all marked real 2-elementary K3 surfaces whose associated integral involutions are isometric to $(\mathbb{L}_{K3},\psi)$ with respect to G can be found in Ω_ψ if we change their markings appropriately.

A point in Ω_ψ is not necessarily the period of some marked real 2-elementary K3 surface of type (S,θ) satisfying $\alpha\circ\varphi_*\circ\alpha^{-1}=\psi$.

Definition 4.3 (Equivalence)**.** We say a point $[v]$ $(\in \Omega_\psi)$ is **equivalent** to a point $[v']$ $(\in \Omega_\psi)$ if $[v'] = f_{\mathbb{C}}([v])$ for an automorphism f of $(\mathbb{L}_{K3}, \psi)$ of type (S, θ) with respect to the group G.

Lemma 4.4. *If a point $[v]$ $(\in \Omega_\psi)$ is equivalent to $[v']$ $(\in \Omega_\psi)$ and $[v]$ is the period of some marked real 2-elementary K3 surface $((X, \tau, \varphi),\ \alpha)$ of type (S, θ) satisfying $\alpha \circ \varphi_* \circ \alpha^{-1} = \psi$, then $[v']$ is also the period of a marked real 2-elementary K3 surface $((X, \tau, \varphi),\ \alpha')$ of type (S, θ) satisfying $(\alpha') \circ \varphi_* \circ (\alpha')^{-1} = \psi$ where α' is* **some another marking** *of (X, τ, φ).*

Proof. Since $[v]$ is equivalent to $[v']$, we have $[v'] = f_{\mathbb{C}}([v])$ for an automorphism f of $(\mathbb{L}_{K3}, \psi)$ of type (S, θ) with respect to the group G. By Lemma 4.1, $((X, \tau, \varphi),\ f \circ \alpha)$ is also a marked real 2-elementary K3 surface of type (S, θ) satisfying $(f \circ \alpha) \circ \varphi_* \circ (f \circ \alpha)^{-1} = \psi$. Moreover, the period of $((X, \tau, \varphi),\ f \circ \alpha)$ is

$$(f \circ \alpha)_{\mathbb{C}}(H) = f_{\mathbb{C}}(\alpha_{\mathbb{C}}(H)) = f_{\mathbb{C}}([v]) = [v']$$

where H $(\subset H_2(X, \mathbb{C}))$ is the Poincare dual of $H^{2,0}(X)$. It is sufficient that we set $\alpha' = f \circ \alpha$. $\square$

Remark 4.5 ([10])**.** By the global Torelli theorem, if two periods are equivalent, then corresponding marked real 2-elementary K3 surfaces are analytic isomorphic (see Definition 2.10). The converse is also true.

The domain Ω_ψ has two connected components which are interchanged by $-\psi$. We see $-\psi$ is an **automorphism** of $(\mathbb{L}_{K3}, \psi)$ with respect to the group G. Hence, by Lemma 4.4 and Remark 4.5, it is sufficient that we investigate the quotient space $\Omega_\psi / -\psi$. We set

$$\mathbb{L}_\pm := \{x \in \mathbb{L}_{K3} \mid \psi(x) = \pm x\}.$$

Note that the lattices $\mathbb{L}_\pm$ depend on the integral involution ψ.

For $[v] \in \Omega_\psi$ $(v \in \mathbb{L}_{K3} \otimes \mathbb{C})$, we have the decomposition $v = v_+ + v_-$, where $v_\pm \in \mathbb{L}_\pm \otimes \mathbb{R}$.

We restrict ourselves the case when $S \subset \mathbb{L}_-$, namely, $\theta = -\mathrm{id}$, and set

$$\mathbb{L}_{-,S} := \mathbb{L}_- \cap S^\perp.$$

$\mathbb{L}_{-,S}$ also depends on the integral involution ψ.

Since $v_- \in \mathbb{L}_{-,S} \otimes \mathbb{R}$ and $v_+^2 = v_-^2 > 0$, we see that $\mathbb{L}_+,\ \mathbb{L}_{-,S}$ is a hyperbolic lattice.

Let $\mathcal{L}_+,\ \ \mathcal{L}_{-,S}$ be the hyperbolic spaces obtained from $\mathbb{L}_+ \otimes \mathbb{R}$, $\mathbb{L}_{-,S} \otimes \mathbb{R}$ respectively. Then we have

$$\Omega_\psi / -\psi \ = \ \mathcal{L}_+ \times \mathcal{L}_{-,S} \quad \text{(a direct product)}.$$

We now fix a primitive hyperbolic 2-elementary sublattice S with $S \cong (3, 1, 1)$ of the K3 lattice $\mathbb{L}_{K3}$ and set $\theta := -\mathrm{id}$. Remark that $G = \{\mathrm{id}\}$ for this case (Remark 2.12).

We use the terminology and facts stated in Subsection 2.2. For a real 2-elementary K3 surface (X, τ, φ) of type $(S, \theta) \cong ((3, 1, 1), -\mathrm{id})$, the fixed point curve A is a disjoint union of A_0 and A_1. X has an elliptic fibration with the exceptional section A_0. Its unique reducible fiber is $E + F$. There are two kinds of the curve F, which is irreducible or not. The classes $[A_0]$, $[E]$ and $[F]$ generate $H_{2+}(X, \mathbb{Z})$. We have an orthogonal decomposition

$$\mathbb{Z}([A_0],\ [E] + [F]) \oplus \mathbb{Z}([F])$$

of $H_{2+}(X, \mathbb{Z})$. The subgroups $\mathbb{Z}([A_0],\ [E] + [F])$, $\mathbb{Z}([F])$ are isometric to the hyperbolic plane and the lattice $\langle -2 \rangle$ respectively.

We fix an integral involution $(\mathbb{L}_{K3}, \psi)$ of type $(S, \theta) \cong ((3,1,1), -\mathrm{id})$. Moreover, we fix a decomposition $S = \mathbb{U} \oplus \langle -2 \rangle$, where $\mathbb{U}$ is a sublattice of S isometric to the hyperbolic plane. Let $\mathcal{F}$ be the generator of $\langle -2 \rangle$ with $\mathcal{M} \cdot \mathcal{F} \geq 0$.

We consider marked real 2-elementary K3 surfaces $((X, \tau, \varphi), \alpha)$ of type $(S, \theta) \cong ((3,1,1), -\mathrm{id})$ (see Definition 2.4) such that

$$\alpha(\mathbb{Z}([A_0],\ [E] + [F])) = \mathbb{U}, \quad \alpha([F]) = \mathcal{F},$$

and

$$\alpha \circ \varphi_* \circ \alpha^{-1} = \psi.$$

Any real 2-elementary K3 surface (X, τ, φ) of type $(S, \theta) \cong ((3,1,1), -\mathrm{id})$ has such a marking α.

We want to know a criterion for the double point of a real anti-bicanonical curve $\mathbb{R}\mathrm{bl}(A)$ with one real double point on $\mathbb{R}A_1'$ on $\mathbb{R}\mathbb{F}_4$ to be **nondegenerate**. At present we can prove the following lemma.

Lemma 4.6. *For $[\omega] \in \Omega_\psi/-\psi$, $[\omega]$ is the period of a marked real 2-elementary K3 surface of type $(S, \theta) \cong ((3,1,1), -\mathrm{id})$ (see Definition 2.4) such that $\alpha(\mathbb{Z}([A_0],\ [E]+[F])) = \mathbb{U}, \quad \alpha([F]) = \mathcal{F}$, and $\alpha \circ \varphi_* \circ \alpha^{-1} = \psi$ obtained from a real anti-bicanonical curve $\mathbb{R}\mathrm{bl}(A)$ with one real* **nondegenerate** *double point on $\mathbb{R}A_1'$ on $\mathbb{R}\mathbb{F}_4$* **if** *there are no $\mathbf{v}$ $(\neq \pm\mathcal{F})$ in $\mathbb{L}_{K3}$ satisfying that $\mathbf{v} \cdot \omega = 0,\ \mathbf{v} \cdot \mathbb{U} = 0$, and $\mathbf{v}^2 = -2$.*

Proof. For a real anti-bicanonical curve with one real degenerate double point on $\mathbb{R}\mathbb{F}_4$, the corresponding marked real 2-elementary K3 surface $(X, \tau, \varphi, \alpha)$ has a unique reducible fiber $E + F$ where $F^2 = -2$, F is the union of two nonsingular rational curves F' and F''. Here F' and F'' are conjugate by τ and $F' \cdot F'' = 1$. Hence, $(F')^2 = (F'')^2 = -2$. (see Subsection 2.2) Both $[F']$ and $[F'']$ are orthogonal to $[A_0]$ and $[E] + [F]$. We set $\mathbf{v} := \alpha[F']$ or $\alpha[F'']$. Since $\mathbb{U} = \alpha(\mathbb{Z}([A_0],\ [E] + [F]))$, we have $\mathbf{v}$ $(\neq \pm\mathcal{F})$, $\mathbf{v} \cdot \omega = 0$, $\mathbf{v} \cdot \mathbb{U} = 0$, and $\mathbf{v}^2 = -2$ for the period $[\omega]$ of $(X, \tau, \varphi, \alpha)$. □

Problem 1 (cf. [4], the top of p.281)**.** Is the converse of Lemma 4.6 also true?

If the converse of Lemma 4.6 is true, then we can get the precise image $(\subset \mathcal{L}_+ \times \mathcal{L}_{-,S})$ of the period map on the set of all marked real 2-elementary K3 surfaces of type $(S, \theta) \cong ((3,1,1), -\mathrm{id})$ such that $\alpha(\mathbb{Z}([A_0],\ [E] + [F])) = \mathbb{U}$, $\alpha([F]) = \mathcal{F}$, and $\alpha \circ \varphi_* \circ \alpha^{-1} = \psi$ obtained from real anti-bicanonical curves $\mathbb{R}\mathrm{bl}(A)$ with one real **nondegenerate** double point on $\mathbb{R}A_1'$ on $\mathbb{R}\mathbb{F}_4$.

Let $\mathbf{v}$ be an element of $\mathbb{L}_+$ with square -2. The reflection on $\mathcal{L}_+$ with respect to the real hyperplane $\mathbf{v}^\perp$ is well-defined and it sends a point in $\Omega_\psi/-\psi$ to its equivalent point. Also, the reflection on $\mathcal{L}_{-,S}$ with respect to the real hyperplane $\mathbf{v}^\perp$ where $\mathbf{v}$ is an element of $\mathbb{L}_{-,S}$ with square -2 is well-defined and it sends a point in $\Omega_\psi/-\psi$ to its equivalent point. Let

$$\Omega_+ \quad (\text{respectively, } \Omega_{-,S})$$

be the open fundamental domains with respect to the groups generated by the reflections with respect to the real hyperplanes $\mathbf{v}^\perp$ satisfying $\mathbf{v}^2 = -2$ and $\mathbf{v} \in \mathbb{L}_+$ (respectively, $\mathbb{L}_{-,S}$) and we consider the direct product $\Omega_+ \times \Omega_{-,S}$.

If the converse of Lemma 4.6 is true, then the periods of marked real 2-elementary K3 surfaces $(X, \tau, \varphi, \alpha)$ of type $((3,1,1), -\mathrm{id})$ obtained from some real anti-bicanonical curves with one real **nondegenerate** double point on $\mathbb{R}\mathbb{F}_4$ and satisfying $\alpha \circ \varphi_* \circ \alpha^{-1} = \psi$ are contained in $\Omega_+ \times \Omega_{-,S}$ up to equivalence. Moreover, we should remove more $\omega = \omega_+ + \omega_-$ orthogonal to some $\mathbf{v} \in \mathbb{L}_{K3}$ such that $\mathbf{v}$ $(\neq \pm\mathcal{F})$, $\mathbf{v} \cdot \mathbb{U} = 0$, $\mathbf{v}^2 = -2$, $\mathbf{v} \notin \mathbb{L}_+$ and $\mathbf{v} \notin \mathbb{L}_{-,S}$. Hence, as the argument before Theorem 2.1 of Itenberg's paper [4], we should remove some (-6)-orthogonal real hyperplanes (extra "walls") from $\Omega_{-,S}$ or Ω_+. This seems to be the reason why the connected components of the moduli space in the sense of [10] (equivalently, the isometry classes of integral involutions of the K3 lattice) cannot distinguish the topological types (node or isolated point) of the real double points (Remark 2.25).

Problem 2 (cf. [4], Theorem 2.1)**.** Formulate some period domain, say it were $\Omega_*^{\mathbb{RF}_4}$, whose connected components (up to equivalence) are in **bijective correspondence** with the connected components of the moduli space (up to the action of the automorphism group of $\mathbb{F}_4$ over $\mathbb{R}$) of real anti-bicanonical curves with one real nondegenerate double point on $\mathbb{RF}_4$ which yield marked real 2-elementary K3 surfaces $((X, \tau, \varphi_-), \alpha)$ satisfying $\alpha \circ (\varphi_-)_* \circ \alpha^{-1} = \psi$.

Remark 4.7. For $(\mathbb{L}_{K3}, \psi)$, either $\mathbb{L}_+$ or $\mathbb{L}_{-,S}$ does not contain any element $\mathbf{v}$ such that

$$\mathbf{v} \equiv \mathcal{F} \pmod{2\mathbb{L}_{K3}}.$$

For the anti-holomorphic involution "φ_-" (recall Definition 3.6), $\mathbb{L}_{-,S}$ contains an element $\mathbf{v}$ such that $\mathbf{v} \equiv \mathcal{F} \pmod{2\mathbb{L}_{K3}}$. This phenomenon is similar to the argument in [4].

We are also interested in the correspondences of the Coxeter graphs (see [4], [5]) obtained from isometry classes of integral involutions of the K3 lattice $\mathbb{L}_{K3}$ of type $(S, \theta) \cong ((3,1,1), -\mathrm{id})$ and the non-increasing simplest degenerations of nonsingular curves. Problems concerning this topic are as follows.

We fix an integral involution $(\mathbb{L}_{K3}, \psi)$ of type $(S, \theta) \cong ((3,1,1), -\mathrm{id})$. Recall

$$\mathbb{L}_{\pm} := \{x \in \mathbb{L}_{K3} \mid \psi(x) = \pm x\},$$

$\mathbb{U}$, and $\mathbb{L}_{-,\mathbb{U}} := \mathbb{L}_- \cap \mathbb{U}^{\perp}$. $\mathbb{L}_{-,\mathbb{U}}$ is also hyperbolic. Let $\mathcal{L}_{-,\mathbb{U}}$ be the hyperbolic space obtained from $\mathbb{L}_{-,\mathbb{U}} \otimes \mathbb{R}$. The group generated by the reflections with respect to real hyperplanes $\mathbf{v}^{\perp}$ such that $\mathbf{v} \in \mathbb{L}_{-,\mathbb{U}}$ with $\mathbf{v}^2 = -2$ acts on the hyperbolic space $\mathcal{L}_{-,\mathbb{U}}$. Let $\widetilde{\Omega_-}$ be one of its fundamental domains which has a face orthogonal to F. Let C be the Coxeter graph of $\widetilde{\Omega_-}$.

Definition 4.8.
- Let C' be the graph which is obtained by removing all thick or dotted edges from C.
- Consider the group of symmetries of C' obtained from some automorphism of $(\mathbb{L}_{K3}, \psi, \mathbb{U})$. Let C'' be the quotient graph of C' by the action of the group.
- Let e be the vertex of C corresponding to F and let e' be the class (in C'') containing e.
- Let K be the connected component of C'' containing e'.

Problem 3 (cf. [4], Proposition 3.1)**.** Does the number (up to equivalence) of connected components of $\Omega_*^{\mathbb{RF}_4}$ coincide with that of vertices of the graph K ?

Degenerations and the graph P. We fix an isometry class of integral involutions of type $(S, \theta) \cong (\mathbb{U}, -\mathrm{id})$ with $(r(\psi), a(\psi), \delta_\psi) \neq (10, 8, 0)$, $(10, 10, 0)$. Then we get a real 2-elementary K3 surface (X, τ, φ) of type $(S, \theta) \cong (\mathbb{U}, -\mathrm{id})$ and a real **nonsingular** curve $A = s + A_1$ in $|-2K_{\mathbb{F}_4}|$ where A_1 is a real nonsingular curve in $|12c + 3s|$ on $\mathbb{RF}_4$.

We define the graphs P as follows (See Definition 3.2).

Definition 4.9.
- The **vertices** of P are all the **rigid isotopy classes** of real curves A_1' in $|12c+3s|$ with one nondegenerate double point on $\mathbb{F}_4$ obtained from the degenerations of types 1)—3) of the real nonsingular curve A_1 with φ.
- Two vertices of P are connected by an **edge** if one rigid isotopy class is obtained from the conjunction of the ovals $\mathcal{E}_1$ and the non-contractible component of A_1 (Conjunction 1)), and the other rigid isotopy class is obtained from the contraction of the oval $\mathcal{E}_1$ (Contraction 3)).

- Two vertices of P are connected by an **edge** if one rigid isotopy class is obtained from the conjunction of the ovals $\mathcal{E}_1$ and $\mathcal{E}_2$ of A_1 (Conjunction 2)), and the other rigid isotopy class is obtained from the contraction of the oval $\mathcal{E}_1$ (Contraction 3)).

Problem 4 (cf. [4], Proposition 3.4)**.** Fix a real **nonsingular** curve $A = s + A_1$ in $|-2K_{\mathbb{F}_4}|$ where A_1 is a real nonsingular curve in $|12c + 3s|$ on $\mathbb{RF}_4$ and φ as above.

Let us get an **arbitrary** real curve A_1' in $|12c + 3s|$ with one nondegenerate double point on $\mathbb{F}_4$ obtained from one of the degenerations of types 1)—3) of A_1 with φ, and construct the graph K (see Lemma 3.7 and Definition 4.8) from A_1' and φ_-. Then, is the graph K isomorphic to the graph P ?

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RESONANT BANDS, AOMOTO COMPLEX, AND REAL 4-NETS

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ABSTRACT. The resonant band is a useful notion for the computation of the nontrivial monodromy eigenspaces of the Milnor fiber of a real line arrangement. In this article, we develop the resonant band description for the cohomology of the Aomoto complex. As an application, we prove that real 4-nets do not exist.

1. Introduction

Combinatorial decisions of topological invariants are the central problems in the theory of hyperplane arrangements. Milnor fibers and their eigenspace decompositions have received a lot of attention and have been studied by diverse techniques ([23]) (e.g., Alexander polynomials, Hodge theory, nets and multinets, covering spaces, Salvetti complexes, characteristic and resonance varieties etc.) Among others, the authors follow the previous studies using real structures, ([24, 29, 30]) and Aomoto complexes over finite fields, [4, 7, 17, 20].

Concerning the relation between Milnor fibers and Aomoto complexes, two key results were obtained by Papadima and Suciu [20, 21].

$$\text{Monodromy eigenspaces} \overset{(1)}{\longleftrightarrow} \text{Aomoto complex} \overset{(2)}{\longleftrightarrow} \text{Multinets}$$

The first one is an upper bound for the rank of the eigenspaces in terms of the Betti numbers of the Aomoto complexes over finite fields [20]. It was subsequently used by many authors to prove vanishing theorems [1, 2, 8, 17]. The second one is the bijective correspondence between 3-nets and nonzero elements in the cohomology group of the Aomoto complex over $\mathbb{F}_3$. A degree one element of the Orlik-Solomon algebra over the finite field $\mathbb{F}_q$ is bijectively corresponding to the coloring (with q-colors) of the arrangement. Papadima and Suciu succeeded to translate the cocycle condition into combinatorics of coloring [21]. A deep relation between nontrivial eigenspaces and multinet structures had been conjectured. Papadima-Suciu's results provide a beautiful framework to understand the nontrivial eigenspaces via multinets.

If we restrict our attention to real arrangements, the real structure contains a lot of information about the topology of the complexification. The resonant band, introduced in [29, 30], is a useful tool for computing nontrivial eigenspaces of the Milnor fibers and local system cohomology groups. The purpose of this paper is to introduce the notion of resonant bands for the Aomoto complex (over any coefficient ring) of a real arrangement. Then combining resonant bands techniques with the above Papadima-Suciu's picture (over $\mathbb{F}_2$), we prove that real 4-nets do not exist, which is a partial answer to a conjecture that the Hessian arrangement is the only 4-net.

The paper is organized as follows. §2 is a summary of well known facts on multinets and Orlik-Solomon algebras. Especially, we describe in detail the transformation of the Orlik-Solomon algebra when we exchange the hyperplane at infinity, which will be used later. §3 is a summary of the recent work by Papadima-Suciu. The crucial result that we use later is Proposition 3.4. Proposition 3.4 translates the cocycle conditions of the Aomoto complex (over $\mathbb{F}_2$) into combinatorial structures of subarrangements. §4 is the main part of this paper. After recalling a description of the Aomoto complex in terms of chambers in §4.1 (following [27]), we introduce the notion of η-resonant band in §4.2. In the main theorem (Theorem 4.8), we prove that the cohomology of the Aomoto complex is isomorphic to a submodule of the free module generated by resonant bands under a certain non-resonant condition at infinity. When the coefficient ring of the Aomoto complex is $\mathbb{F}_2$, everything can be described in terms of combinatorics of subarrangements. This translation is done in §4.3. In §5, we prove the non-existence of real 4-nets. The key result is the Non-Separation Theorem 5.1 in §5.1 which concerns subarrangements corresponding to cocycles of the Aomoto complex over $\mathbb{F}_2$ with differential given by the diagonal element. The Non-Separation Theorem asserts that at an intersection of multiplicity 4, the subarrangement corresponding to a nontrivial cohomology class has a special ordering. This assertion heavily relies on the real structure. Therefore, at this moment, it seems hopeless to generalize our argument to the complex case. Assuming a real 4-net exists, it is easy to construct a subarrangement which contradicts the Non-Separation Theorem. Hence real 4-nets do not exist (§5.2). (This fact was first proved by Cordovil-Forge [6, Lem. 2.4]. Our arguments prove a little bit stronger version. See Remark 5.3.)

2. Preliminaries

2.1. **Conventions.** In this paper, three types of hyperplane arrangements appear: affine arrangements in $\mathbb{K}^\ell$, arrangements in the projective space $\mathbb{KP}^\ell$ and central arrangements in $\mathbb{K}^{\ell+1}$. It is better to distinguish them by notations ([18, 19]).

- $\mathcal{A} = \{H_1, \dots, H_n\}$ denotes an arrangement of affine hyperplanes in the affine ℓ-space $\mathbb{K}^\ell$.
- $\widetilde{\mathcal{A}} = c\mathcal{A} = \{\widetilde{H}_0, \widetilde{H}_1, \dots, \widetilde{H}_n\}$ denotes the coning of $\mathcal{A}$, which is a central hyperplane arrangement in $\mathbb{K}^{\ell+1}$. The hyperplane $\widetilde{H}_0$ is corresponding to the hyperplane at infinity of $\mathcal{A}$.
- $\overline{\mathcal{A}} = \{\overline{H}_0, \overline{H}_1, \dots, \overline{H}_n\}$ denotes the projectivization of $\widetilde{\mathcal{A}}$, which is a hyperplane arrangement induced by $\widetilde{\mathcal{A}}$ in the projective ℓ-space $\mathbb{KP}^\ell$.
- $\mathsf{d}_{\widetilde{H}_i}\widetilde{\mathcal{A}} = \{\mathsf{d}_{\widetilde{H}_i}\widetilde{H}_0, \dots, \widehat{\mathsf{d}_{\widetilde{H}_i}\widetilde{H}_i}, \dots, \mathsf{d}_{\widetilde{H}_i}\widetilde{H}_n\}$ denotes the deconing of $\widetilde{\mathcal{A}}$ with respect to the hyperplane $\widetilde{H}_i$. Note that $\mathsf{d}_{\widetilde{H}_0}\widetilde{\mathcal{A}} = \mathcal{A}$.

Other frequently used notations are:

- R: a commutative ring (unless stated otherwise),
- $\mathbb{K}$: a field,
- $M(\mathcal{A})$: the complexified complement of $\mathcal{A}$.

2.2. **Multinets.** In this subsection, we recall several facts on multinets.

Definition 2.1. *Let $\overline{\mathcal{A}} = \{\overline{H}_0, \dots, \overline{H}_n\}$ be a projective line arrangement in $\mathbb{CP}^2$. Let $k \geq 3$ and $d \geq 2$ be integers. A* (reduced) (k,d)-multinet *(or k-*multinet *for simplicity) on $\mathcal{A}$ is a pair $(\mathcal{N}, \mathcal{X})$, where $\mathcal{N}$ is a partition of $\mathcal{A}$ into k classes $\overline{\mathcal{A}} = \overline{\mathcal{A}}_1 \sqcup \cdots \sqcup \overline{\mathcal{A}}_k$ and $\mathcal{X} \subset \mathbb{CP}^2$ is a set of multiple points (called the base locus) such that*

- (i) $|\overline{\mathcal{A}}_1| = \cdots = |\overline{\mathcal{A}}_k| =: d$;
- (ii) $\overline{H} \in \overline{\mathcal{A}}_i$ *and* $\overline{H}' \in \overline{\mathcal{A}}_j$ $(i \neq j)$ *imply that* $\overline{H} \cap \overline{H}' \in \mathcal{X}$;
- (iii) *for all* $p \in \mathcal{X}$, $n_p := |\{\overline{H} \in \overline{\mathcal{A}}_i \mid \overline{H} \ni p\}|$ *is constant and independent of* i;
- (iv) *for any* $\overline{H}, \overline{H}' \in \overline{\mathcal{A}}_i$ $(i = 1, \dots, k)$, *there is a sequence* $\overline{H} = \overline{H}'_0, \overline{H}'_1, \dots, \overline{H}'_r = \overline{H}'$ *in* $\overline{\mathcal{A}}_i$ *such that* $\overline{H}'_{j-1} \cap \overline{H}'_j \notin \mathcal{X}$ *for* $1 \leq j \leq r$.

If $n_p = 1$ for every $p \in \mathcal{X}$, then $(\mathcal{N}, \mathcal{X})$ is called a net.

Note that in the previous Definition (ii) and (iii) implies (i), and, by [13], $\mathcal{N}$ and $\mathcal{X}$ determine each other. Moreover, note that if $(\mathcal{N}, \mathcal{X})$ is a (k,d)-net, then each $p \in \mathcal{X}$ has multiplicity k.

The next theorem, which combines results of Pereira and Yuzvinsky [22, 33], summarizes what is known about the existence of non-trivial multinets on arrangements (see also [23, 3, 32] for more results).

Theorem 2.2. *Let $\overline{\mathcal{A}}$ be a k-multinet, with base locus $\mathcal{X}$. Then*

(1) *If $|\mathcal{X}| > 1$, then $k = 3$ or 4.*
(2) *If there is a hyperplane $\overline{H} \in \overline{\mathcal{A}}$ such that $m_H > 1$, then $k = 3$.*
(3) *If $k = 4$, then $|\mathcal{X}| = d^2$ and $\overline{\mathcal{A}}$ is a $(4,d)$-net.*

Although several infinite families of multinets with $k = 3$ are known, only one multinet with $k = 4$ is known to exist: the $(4,3)$-net on the Hessian arrangement (which is defined over $\mathbb{Q}(\sqrt{-3})$). It is conjectured that the only $(4,d)$-net is the Hessian arrangement. In [10], it is proved that the Hessian is the unique $(4,d)$-net for $d \leq 6$ (hence for $|\mathcal{A}| \leq 24$). We will later prove that real $(4,d)$-nets do not exist, for any d.

2.3. **Orlik-Solomon algebra and Aomoto complex.** Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of affine hyperplanes in $\mathbb{K}^\ell$ and R be a commutative ring. Let $E_1 = \bigoplus_{j=1}^n Re_j$ be the free module generated by $e_1, e_2, \dots, e_n$, where e_i is a symbol corresponding to the hyperplane H_i. Let $E = \wedge E_1$ be the exterior algebra over R. The algebra E is graded via $E = \bigoplus_{p=0}^n E_p$, where $E_p = \wedge^p E_1$. The R-module E_p is free and has the distinguished basis consisting of monomials $e_S := e_{i_1} \wedge \cdots \wedge e_{i_p}$, where $S = \{i_1, \dots, i_p\}$ is running through all the subsets of $\{1, \dots, n\}$ of cardinality p and $i_1 < i_2 < \cdots < i_p$. The graded algebra E is a commutative DGA with respect to the differential ∂ of degree -1 uniquely defined by the conditions $\partial e_i = 1$ for all $i = 1, \dots, n$ and the graded Leibniz formula. Then for every $S \subset \{1, \dots, n\}$ of cardinality p

$$\partial e_S = \sum_{j=1}^{p} (-1)^{j-1} e_{S_j},$$

where S_j is the complement in S to its j-th element.

For every $S \subset \{1, \dots, n\}$, put $\cap S = \bigcap_{i \in S} H_i$ (possibly $\cap S = \emptyset$). The set of all intersections $L(\mathcal{A}) := \{\cap S \mid S \subset \{1, \dots, n\}\}$ is called the intersection poset. The subset $S \subset \{1, \dots, n\}$ is called *dependent* if $\cap S \neq \emptyset$ and the set of linear polynomials $\{\alpha_i \mid i \in S\}$ with $H_i = \alpha_i^{-1}(0)$, is linearly dependent.

Definition 2.3. *The* Orlik-Solomon ideal *of $\mathcal{A}$ is the ideal $I = I(\mathcal{A})$ of E generated by*

(1) *all e_S with $\cap S = \emptyset$ and*
(2) *all ∂e_S with S dependent.*

The algebra $A := A_R^\bullet(\mathcal{A}) = E/I(\mathcal{A})$ is called the Orlik-Solomon algebra *of $\mathcal{A}$.*

Clearly I is a homogeneous ideal of E whence A is a graded algebra and we can write $A = \bigoplus_{p \geq 0} A_R^p$, where $A_R^p = E_p/(I \cap E_p)$. If $\mathcal{A}$ is central, then for any $S \subset \mathcal{A}$, we have $\cap S \neq \emptyset$. Therefore, the Orlik-Solomon ideal is generated by the elements of type (2) from Definition 2.3. In this case, the map ∂ induces a well-defined differential $\partial\colon A_R^\bullet(\mathcal{A}) \longrightarrow A_R^{\bullet-1}(\mathcal{A})$.

Recall that [18, Cor. 3.73], for each p, we can write (Brieskorn decomposition)

$$A_R^p(\mathcal{A}) = \bigoplus_{X \in L_p(\mathcal{A})} A_R^p(\mathcal{A}_X), \tag{1}$$

where $L_p(\mathcal{A}) := \{X \in L(\mathcal{A}) \mid \operatorname{codim} X = p\}$ and $\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subset H\}$. See for example [18, Corollary 3.73].

Recall that the coning $\widetilde{\mathcal{A}} = c\mathcal{A} = \{\widetilde{H}_0, \widetilde{H}_1, \dots, \widetilde{H}_n\}$ of $\mathcal{A}$ is a central arrangement in $\mathbb{K}^{\ell+1}$. We denote the corresponding generators of the Orlik-Solomon algebra $A^\bullet_R(\widetilde{\mathcal{A}})$ by $\widetilde{e}_0, \widetilde{e}_1, \dots, \widetilde{e}_n$. The map

$$\iota\colon A^1_R(\mathcal{A}) \longrightarrow A^1_R(\widetilde{\mathcal{A}}) : e_i \longmapsto \widetilde{e}_i - \widetilde{e}_0,$$

induces an injective R-algebra homomorphism $\iota\colon A^\bullet_R(\mathcal{A}) \longrightarrow A^\bullet_R(\widetilde{\mathcal{A}})$ ([31]). The image of the embedding ι is equal to the subalgebra

$$A^\bullet_R(\widetilde{\mathcal{A}})_0 := \{\omega \in A^\bullet_R(\widetilde{\mathcal{A}}) \mid \partial(\omega) = 0\}$$

of $A^\bullet_R(\widetilde{\mathcal{A}})$.

Consider the deconing $\mathcal{A}' := \mathsf{d}_{\widetilde{H}_i}\widetilde{\mathcal{A}} = \{H'_0, \dots, \widehat{H'_i}, \dots, H'_n\}$ with respect to the hyperplane $\widetilde{H}_i \in \widetilde{\mathcal{A}}$. We denote the generators of the Orlik-Solomon algebra $A^\bullet_R(\mathcal{A}')$ by $e'_0, \dots, \widehat{e'_i}, \dots, e'_n$. Then the Orlik-Solomon algebras of $\mathcal{A}$ and $\mathcal{A}'$ are isomorphic. The explicit isomorphism is given by

$$e_j \longmapsto \begin{cases} e'_j - e'_0, & \text{if } 1 \le j \le n, j \ne i, \\ -e'_0, & \text{if } j = i. \end{cases}$$

Let us fix an element $\eta = \sum_{i=1}^n a_i e_i \in A^1_R(\mathcal{A})$. Since $\eta \wedge \eta = 0$,

$$0 \longrightarrow A^1_R(\mathcal{A}) \xrightarrow{\eta} A^2_R(\mathcal{A}) \xrightarrow{\eta} \cdots \xrightarrow{\eta} A^\ell_R(\mathcal{A}) \xrightarrow{\eta} 0$$

forms a cochain complex, $(A^\bullet_R(\mathcal{A}), \eta)$, which is called the *Aomoto complex*. By the above embedding ι, we can identify the Aomoto complex $(A^\bullet_R(\mathcal{A}), \eta)$ with $(A^\bullet_R(\widetilde{\mathcal{A}})_0, \widetilde{\eta})$, where

$$\widetilde{\eta} = \iota(\eta) = \sum_{i=1}^n a_i \widetilde{e}_i - (a_1 + \cdots + a_n)\widetilde{e}_0 \quad ([11]).$$

3. Mod p Aomoto complex and the Papadima-Suciu correspondence

In this section, we recall recent results by Papadima and Suciu [21]. They found a way of constructing a 3-net from a non-trivial element of the cohomology of Aomoto complex over $\mathbb{F}_3$. Let $\overline{\mathcal{A}} = \{\overline{H}_0, \dots, \overline{H}_n\}$ be a line arrangement on the projective plane $\mathbb{KP}^2$ with $3 \mid |\overline{\mathcal{A}}|$. Assume that there do not exist multiple points of multiplicity $\{3r \mid r \in \mathbb{Z}, r > 1\}$. Let

$$\widetilde{\eta}_0 := \sum_{i=0}^n \widetilde{e}_i \in A^1_{\mathbb{F}_3}(\widetilde{\mathcal{A}})_0$$

be the diagonal element. Then there is a natural bijective correspondence:

$$(H^1(A^\bullet_{\mathbb{F}_3}(\widetilde{\mathcal{A}})_0, \widetilde{\eta}_0) \setminus \{0\})/\mathbb{F}_3^\times \xrightarrow{\simeq} \left\{ \begin{array}{c} \text{Isomorphism classes of} \\ \text{3-net structures on } \overline{\mathcal{A}} \end{array} \right\}. \tag{2}$$

The correspondence is explicitly given by $H^1(A^\bullet_{\mathbb{F}_3}(\widetilde{\mathcal{A}})_0, \widetilde{\eta}_0) \ni \omega = \sum_{i=0}^n a_i \widetilde{e}_i \longmapsto (\overline{\mathcal{A}}_0, \overline{\mathcal{A}}_1, \overline{\mathcal{A}}_2)$, where $\overline{\mathcal{A}}_m = \{\overline{H}_i \mid a_i = m\}$ $(m = 0, 1, 2)$. The point of the above correspondence is that by using the local structures of the Orlik-Solomon algebra, we can translate the cocycle condition into the combinatorial conditions of $(\overline{\mathcal{A}}_0, \overline{\mathcal{A}}_1, \overline{\mathcal{A}}_2)$, which turn out to be exactly the defining conditions of 3-nets. Later we will employ a similar consideration for the Aomoto complex over $\mathbb{F}_2$ which we summarize in this section.

3.1. **A local lemma.** The next lemma (cf. [16, §3]) is useful for analyzing the map $\eta\colon A^1_R(\mathcal{A}) \longrightarrow A^2_R(\mathcal{A})$ by the Brieskorn decomposition (1).

Lemma 3.1. *Let $\mathcal{C}_s = \{H_1, \dots, H_s\}$ be a central arrangement in $\mathbb{K}^2$ (Figure 1). Let R be a commutative ring and $\eta = a_1e_1 + \cdots + a_se_s \in A^1_R(\mathcal{C}_s)$ be a degree one element of Orlik-Solomon algebra.*

(1) $\eta \wedge (e_i - e_j) = -(\sum_{i=1}^s a_i) \cdot e_i \wedge e_j$.
(2) *Let $\omega = b_1e_1 + \cdots + b_se_s \in A^1_R(\mathcal{A})$ be another element. Assume that ω and η are linearly independent (i.e., $c_1\eta + c_2\omega = 0, (c_1, c_2 \in R) \Longrightarrow c_1 = c_2 = 0$). Then $\eta \wedge \omega = 0$ if and only if $\sum_{i=1}^s a_i = \sum_{i=1}^s b_i = 0$.*

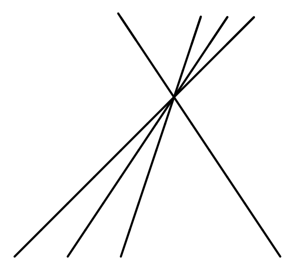

FIGURE 1. Central arrangement $\mathcal{C}_s$

Proof. (1) It is straightforward from the relation $e_{ij} = e_{ik} - e_{jk}$, where $e_{ij} := e_i \wedge e_j$.

(2) (Cf. [34, Proposition 2.1]) If $\sum_{i=1}^s b_i = 0$, then $\omega = b_1(e_1 - e_s) + \cdots + b_{s-1}(e_1 - e_{s-1})$. Then applying (1), we have $\eta \wedge \omega = -(\sum_{i=1}^s a_i) \cdot (b_1e_{1s} + \cdots + b_{s-1}e_{1,s-1})$. This is zero if $\sum_{i=1}^s a_i = 0$. Conversely, suppose $\eta \wedge \omega = 0$. Since $\mathcal{C}_s$ is central, we can apply the differential ∂. We have

$$0 = \partial(\eta \wedge \omega) = (\partial\eta)\omega - (\partial\omega)\eta.$$

By the assumption that η and ω are linearly independent, $\partial\eta = \partial\omega = 0$. $\square$

3.2. **Aomoto complex over $\mathbb{F}_p$.** Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of affine lines in $\mathbb{K}^2$. Choose a prime p such that $p|(n+1)$. Consider the Aomoto complex over $R = \mathbb{F}_p$ and the embedding $\iota\colon A^\bullet_{\mathbb{F}_p}(\mathcal{A}) \xrightarrow{\simeq} A^\bullet_{\mathbb{F}_p}(\widetilde{\mathcal{A}})_0 \subset A^\bullet_{\mathbb{F}_p}(\widetilde{\mathcal{A}})$. Since n is equal to -1 in $\mathbb{F}_p$, the image of the diagonal element $\eta_0 := e_1 + \cdots + e_n \in A^1_{\mathbb{F}_p}(\mathcal{A})$ is

$$\widetilde{\eta}_0 := \iota(\eta_0) = \widetilde{e}_0 + \widetilde{e}_1 + \cdots + \widetilde{e}_n \in A^1_{\mathbb{F}_p}(\widetilde{\mathcal{A}})_0.$$

We consider the first cohomology group of the Aomoto complex $(A^\bullet_{\mathbb{F}_p}(\mathcal{A}), \eta_0) \simeq (A^\bullet_{\mathbb{F}_p}(\widetilde{\mathcal{A}})_0, \widetilde{\eta}_0)$. Let $\widetilde{\omega} = \sum_{i=0}^n a_i\widetilde{e}_i \in A^1_{\mathbb{F}_p}(\widetilde{\mathcal{A}})_0$. Let us translate the relation $\widetilde{\eta} \wedge \widetilde{\omega} = 0$ in terms of the coefficients a_i of $\widetilde{\omega}$ by using the Brieskorn decomposition (1). For an intersection $X \in L_2(\widetilde{\mathcal{A}})$ of codimension two, let us define the localization at X by

$$\widetilde{\omega}|_X := \sum_{\widetilde{H}_i \in \widetilde{\mathcal{A}}_X} a_i\widetilde{e}_i. \tag{3}$$

Proposition 3.2. *With notation as above, $\widetilde{\eta}_0 \wedge \widetilde{\omega} = 0$ if and only if the following (i) and (ii) hold, for any $X \in L_2(\widetilde{\mathcal{A}})$.*

(i) *If $|\widetilde{\mathcal{A}}_X|$ is divisible by p, then $\sum_{\widetilde{H}_i \in \widetilde{\mathcal{A}}_X} a_i = 0$ in $\mathbb{F}_p$.*

(*ii*) *If* $|\widetilde{\mathcal{A}}_X|$ *is not divisible by* p, *then*

$$a_{i_1} = a_{i_2} = \cdots = a_{i_t},$$

where $\widetilde{\mathcal{A}}_X = \{\widetilde{H}_{i_1}, \widetilde{H}_{i_2}, \ldots, \widetilde{H}_{i_t}\}$. *(This is equivalent to that* $\widetilde{\omega}|_X$ *and* $\widetilde{\eta}_0|_X$ *are linearly dependent.)*

Proof. By the Brieskorn decomposition (1), $\widetilde{\eta}_0 \wedge \widetilde{\omega} = 0$ if and only if $\widetilde{\eta}_0|_X \wedge \omega|_X = 0$ for all $X \in L_2(\widetilde{\mathcal{A}})$. Using the Lemma 3.1 (2), it is equivalent to (*i*) and (*ii*) above. □

3.3. **Aomoto complex over $\mathbb{F}_2$ and subarrangements.** Now we consider the Aomoto complex over $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. Since the coefficient is either 0 or $1 \in \mathbb{F}_2$, elements of $A^1_{\mathbb{F}_2}(\widetilde{\mathcal{A}})$ can be identified with subarrangements of $\widetilde{\mathcal{A}}$.

Definition 3.3. *Let* $\widetilde{\mathcal{S}} \subset \widetilde{\mathcal{A}}$ *be a subset. Let us define an element corresponding to the subset by*

$$\widetilde{e}(\widetilde{\mathcal{S}}) := \sum_{\widetilde{H}_i \in \widetilde{\mathcal{S}}} \widetilde{e}_i \in A^1_{\mathbb{F}_2}(\widetilde{\mathcal{A}}).$$

For an affine arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$ *and a subset* $\mathcal{S} \subset \mathcal{A}$, *similarly we define*

$$e(\mathcal{S}) := \sum_{H_i \in \mathcal{S}} e_i \in A^1_{\mathbb{F}_2}(\mathcal{A}).$$

Obviously the diagonal element is $\widetilde{\eta}_0 = \widetilde{e}(\widetilde{\mathcal{A}})$ and $\widetilde{e}(\widetilde{\mathcal{S}}) + \widetilde{\eta}_0 = \widetilde{e}(\widetilde{\mathcal{A}} \setminus \widetilde{\mathcal{S}})$.
Applying Proposition 3.2 for $p = 2$, we have the following.

Proposition 3.4. *Let* $\widetilde{\mathcal{A}} = \{\widetilde{H}_0, \widetilde{H}_1, \ldots, \widetilde{H}_n\}$ *be central arrangement in* $\mathbb{K}^3$. *Let* $\widetilde{\mathcal{S}} \subset \widetilde{\mathcal{A}}$ *be a subset. Then* $\widetilde{\eta}_0 \wedge \widetilde{e}(\widetilde{\mathcal{S}}) = 0$ *if and only if the following* (*i*) *and* (*ii*) *hold, for any* $X \in L_2(\widetilde{\mathcal{A}})$.

(*i*) *If* $|\widetilde{\mathcal{A}}_X|$ *is even, then* $|\widetilde{\mathcal{S}}_X|$ *is also even.*
(*ii*) *If* $|\widetilde{\mathcal{A}}_X|$ *is odd, then either* $\widetilde{\mathcal{A}}_X = \widetilde{\mathcal{S}}_X$ *or* $\widetilde{\mathcal{S}}_X = \emptyset$.

Remark 3.5. *The existence of* $\widetilde{\omega} \in A^1_{\mathbb{F}_2}(\widetilde{\mathcal{A}})$ *such that* $\widetilde{\omega} \neq 0, \widetilde{\omega} \neq \widetilde{\eta}_0$ *and* $\widetilde{\eta}_0 \wedge \widetilde{\omega} = 0$ *is equivalent to the existence of a partition* $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}_1 \sqcup \widetilde{\mathcal{A}}_2$ *such that at each intersection* $X \in L_2(\widetilde{\mathcal{A}})$ *of codimension* 2, *(at least) one of the following is satisfied:*

(1) $\widetilde{\mathcal{A}}_X$ *is included in* $\widetilde{\mathcal{A}}_1$ *or in* $\widetilde{\mathcal{A}}_2$,
(2) $|(\widetilde{\mathcal{A}}_1)_X|$ *and* $|(\widetilde{\mathcal{A}}_2)_X|$ *are both even.*

The authors do not know any real essential arrangement which possesses the above partition. Hence, we do not know any real essential arrangement which satisfies $H^1(A^{\bullet}_{\mathbb{F}_2}(\widetilde{\mathcal{A}})_0, \widetilde{\eta}_0) \neq 0$.

Example 3.6. *Suppose that* $\overline{\mathcal{A}} = \overline{\mathcal{A}}_1 \sqcup \overline{\mathcal{A}}_2 \sqcup \overline{\mathcal{A}}_3 \sqcup \overline{\mathcal{A}}_4$ *is a 4-net. Then*

$$\begin{aligned}\widetilde{\eta}_0 \wedge \widetilde{e}(\widetilde{\mathcal{A}}_1 \cup \widetilde{\mathcal{A}}_2) &= \widetilde{\eta}_0 \wedge \widetilde{e}(\widetilde{\mathcal{A}}_1 \cup \widetilde{\mathcal{A}}_3)\\ &= \widetilde{\eta}_0 \wedge \widetilde{e}(\widetilde{\mathcal{A}}_1 \cup \widetilde{\mathcal{A}}_4)\\ &= 0\end{aligned}$$

These three elements satisfy a linear relation,

$$\widetilde{e}(\widetilde{\mathcal{A}}_1 \cup \widetilde{\mathcal{A}}_2) + \widetilde{e}(\widetilde{\mathcal{A}}_1 \cup \widetilde{\mathcal{A}}_3) + \widetilde{e}(\widetilde{\mathcal{A}}_1 \cup \widetilde{\mathcal{A}}_4) = \widetilde{\eta}_0,$$

and span a two dimensional subspace in $H^1(A^{\bullet}_{\mathbb{F}_2}(\widetilde{\mathcal{A}})_0, \widetilde{\eta}_0)$. *We obtain a well-known inequality:*

$$\dim H^1(A^{\bullet}_{\mathbb{F}_2}(\widetilde{\mathcal{A}})_0, \widetilde{\eta}_0) \geq 2 \quad ([9, 20]).$$

4. Resonant bands description of Aomoto complex

Resonant bands provide effective tools to compute local system cohomology groups and eigenspaces of Milnor monodromies. In this section, we give a description of the cohomology of the Aomoto complex in terms of resonant bands.

4.1. **Aomoto complex via chambers.** We first introduce several notions related to the real structure of line arrangements. (The notions are summarized in Example 4.4 and Figure 2.) Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of affine lines in $\mathbb{R}^2$. A connected component of $\mathbb{R}^2 \setminus \bigcup_{H \in \mathcal{A}} H$ is called a chamber. The set of all chambers is denoted by $\mathsf{ch}(\mathcal{A})$. Let $C, C' \in \mathsf{ch}(\mathcal{A})$. The line $H \in \mathcal{A}$ is said to separate C and C' when they are contained in opposite sides of H. The set of all lines separating C and C' is denoted by $\mathrm{Sep}(C, C')$. The set of chambers $\mathsf{ch}(\mathcal{A})$ is provided with a natural metric, the so-called adjacency distance, $d(C, C') = |\mathrm{Sep}(C, C')|$.

Let us fix a flag

$$\emptyset = \mathcal{F}^{-1} \subset \mathcal{F}^0 \subset \mathcal{F}^1 \subset \mathcal{F}^2 = \mathbb{R}^2,$$

(of affine subspaces with $\dim \mathcal{F}^i = i$, we also fix orientations of subspaces) satisfying the following conditions:

(i) (genericity) $\mathcal{F}^0$ is not contained in $\bigcup_{i=1}^n H_i$, and $\mathcal{F}^1$ intersects with $\bigcup_{i=1}^n H_i$ at distinct n points.
(ii) (near to ∞)
  - $\mathcal{F}^0$ does not separate n points $\mathcal{F}^1 \cap H_i$ $(i = 1, \dots, n)$ in $\mathcal{F}^1$.
  - $\mathcal{F}^1$ does not separate intersections of $\mathcal{A}$ in $\mathbb{R}^2$.

(See Figure 2 for example.) Each line H_i determines two half spaces $H_i^{\pm}$. We choose $H_i^{\pm}$ so that $\mathcal{F}^0 \in H_i^-$ for all $i = 1, \dots, n$. We also fix an orientation of $\mathcal{F}^1$ and after re-numbering the lines, if necessary, we may assume the following

$$\mathcal{F}^0 < H_1 \cap \mathcal{F}^1 < H_2 \cap \mathcal{F}^1 < \cdots < H_n \cap \mathcal{F}^1,$$

with respect to the ordering of $\mathcal{F}^1$.

Associated to such a flag $\mathcal{F} = \{\mathcal{F}^\bullet\}$, we define a subset of $\mathsf{ch}(\mathcal{A})$ as follows.

$$\mathsf{ch}^i_{\mathcal{F}}(\mathcal{A}) = \{C \in \mathsf{ch}(\mathcal{A}) \mid C \cap \mathcal{F}^{i-1} = \emptyset, C \cap \mathcal{F}^i \neq \emptyset\}.$$

We denote by $R[\mathsf{ch}^i_{\mathcal{F}}(\mathcal{A})] = \bigoplus_{C \in \mathsf{ch}^i_{\mathcal{F}}(\mathcal{A})} R \cdot [C]$, the free R-module generated by $C \in \mathsf{ch}^i_{\mathcal{F}}(\mathcal{A})$, where R is a commutative ring. It is known that $\mathrm{rank}_R A^i_R(\mathcal{A}) = |\mathsf{ch}^i_{\mathcal{F}}(\mathcal{A})|$ ([25]). We fix notations as follows.

Assumption 4.1. *Let us set*

$$\mathsf{ch}^0_{\mathcal{F}}(\mathcal{A}) = \{C_0\}, \ \mathsf{ch}^1_{\mathcal{F}}(\mathcal{A}) = \{C_1, \dots, C_n\}, \ \text{ and } \ \mathsf{ch}^2_{\mathcal{F}}(\mathcal{A}) = \{D_1, D_2, \dots, D_b\},$$

where $b = |\mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})|$. We can choose $C_1, \dots, C_n$ such that $\mathrm{Sep}(C_0, C_i) = \{H_1, H_2, \dots, H_i\}$, or equivalently, $C_i = H_1^+ \cap \cdots \cap H_i^+ \cap H_{i+1}^- \cap \cdots \cap H_n^-$, for all $i = 0, 1, \dots, n$ (see Figure 2).

When $1 \leq i < n$, the boundary of $C_i \cap \mathcal{F}^1$ consists of two points, $H_i \cap \mathcal{F}^1$ and $H_{i+1} \cap \mathcal{F}^1$, while $C_n \cap \mathcal{F}^1$ is a half-line and its boundary consists of a point $H_n \cap \mathcal{F}^1$.

Definition 4.2. *We use the notations above. Consider $\eta = \sum_{i=1}^n a_i e_i \in A^1_R(\mathcal{A})$.*

(1) *Define the R-homomorphisms* $\nabla_\eta\colon R[\mathsf{ch}^0_{\mathcal{F}}(\mathcal{A})] \longrightarrow R[\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})]$ *as follows.*

$$\begin{aligned}\nabla_\eta([C_0]) &= \sum_{C\in\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})}\left(\sum_{H_i\in\mathrm{Sep}(C_0,C)} a_i\right)\cdot[C]\\ &= \sum_{i=1}^{n}(a_1+\cdots+a_i)\cdot[C_i].\end{aligned}$$

(2) *Define the map*

$$\deg\colon \mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})\times\mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})\longrightarrow\{\pm1,0\},$$

as follows.

(*i*) *If $i<n$, then the segment $C_i\cap\mathcal{F}^1$ has two boundaries, say, $H_i\cap\mathcal{F}^1$ and $H_{i+1}\cap\mathcal{F}^1$.*

$$\deg(C_i,D)=\begin{cases}1 & \text{if } D\subset H_i^-\cap H_{i+1}^+,\\ -1 & \text{if } D\subset H_i^+\cap H_{i+1}^-,\\ 0 & \text{otherwise.}\end{cases}$$

(*ii*) *If $i=n$,*

$$\deg(C_n,D)=\begin{cases}-1 & \text{if } D\subset H_n^+,\\ 0 & \text{if } D\subset H_n^-.\end{cases}$$

(3) *Define the R-homomorphisms* $\nabla_\eta\colon R[\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})] \longrightarrow R[\mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})]$ *as follows.*

$$\nabla_\eta([C])=\sum_{D\in\mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})}\deg(C,D)\left(\sum_{H_i\in\mathrm{Sep}(C,D)}a_i\right)\cdot[D].$$

Proposition 4.3. *([27]) $(R[\mathsf{ch}^\bullet_{\mathcal{F}}(\mathcal{A})],\nabla_\eta)$ is a cochain complex. Furthermore, there is a natural isomorphism of cochain complexes,*

$$\varphi\colon (R[\mathsf{ch}^\bullet_{\mathcal{F}}(\mathcal{A})],\nabla_\eta)\xrightarrow{\simeq}(A^\bullet_R(\mathcal{A}),\eta). \tag{4}$$

In degree 1, the isomorphism is explicitly given by

$$R[\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})]\xrightarrow{\simeq}A^1_R(\mathcal{A}),\ [C_i]\longmapsto\varphi([C_i])=\begin{cases}e_i-e_{i+1} & \text{if } i<n,\\ e_n & \text{if } i=n.\end{cases} \tag{5}$$

In particular, we have

$$H^1(R[\mathsf{ch}^\bullet_{\mathcal{F}}(\mathcal{A})],\nabla_\eta)\simeq H^1(A^\bullet_R(\mathcal{A}),\eta).$$

The isomorphism (4) is natural in the sense that it respects Borel-Moore homology [27, 14, 28]. Recall that each chamber $C\in\mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})$ (with suitable orientation) determines a Borel-Moore 2-homology cycle $[C]\in H_2^{BM}(M(\mathcal{A}),R)$ of the complexified complement $M(\mathcal{A})$. The isomorphism (4), for $i=2$, is obtained by the composition

$$R[\mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})]\longrightarrow H_2^{BM}(M(\mathcal{A}),R)\xrightarrow{\simeq}H^2(M(\mathcal{A}),R)\simeq A^i_R(\mathcal{A}). \tag{6}$$

Example 4.4. *Let $\mathcal{A}=\{H_1,\dots,H_6\}$ be six affine lines as in Figure 2. We also fix a flag (with orientation) $\mathcal{F}=\{\mathcal{F}^0\subset\mathcal{F}^1\}$ (as in Figure 2). There are 16 chambers. We have*

$$\begin{aligned}\mathsf{ch}^0_{\mathcal{F}}(\mathcal{A})&=\{C_0\},\\ \mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})&=\{C_1,C_2,\dots,C_6\},\\ \mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})&=\{D_1,D_2,\dots,D_9\}.\end{aligned}$$

The degree maps are computed, as follows.

$\deg(C_i, D_j)$	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9
C_1	0	-1	0	0	1	0	1	1	0
C_2	-1	0	0	-1	-1	0	0	-1	0
C_3	1	1	1	1	1	1	0	1	1
C_4	-1	-1	-1	0	0	0	0	0	0
C_5	0	0	0	-1	-1	-1	0	0	0
C_6	0	0	0	0	0	0	-1	-1	-1

Consider $\eta = \sum_{i=1}^{6} a_i e_i \in A^1_R(\mathcal{A})$. *We will compute the map* ∇_η.

The first one $\nabla_\eta \colon R[\mathsf{ch}^0_{\mathcal{F}}(\mathcal{A})] \longrightarrow R[\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})]$ *is, by definition,*

$$\nabla_\eta([C_0]) = a_1 \cdot [C_1] + a_{12} \cdot [C_2] + a_{123} \cdot [C_3] + a_{1234} \cdot [C_4] + a_{12345} \cdot [C_5] + a_{123456} \cdot [C_6],$$

where $a_{ijk} = a_i + a_j + a_k$, *etc. The second one* $\nabla_\eta \colon R[\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})] \longrightarrow R[\mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})]$ *is given as follows.*

$$\nabla_\eta \begin{pmatrix} [C_1] \\ [C_2] \\ \vdots \\ [C_6] \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -a_4 & 0 & 0 & a_{1245} & 0 & a_{123456} & a_{12456} & 0 \\ -a_4 & 0 & 0 & -a_{45} & -a_{145} & 0 & 0 & -a_{1456} & 0 \\ a_{34} & a_{234} & a_{1234} & a_{345} & a_{1345} & a_{12345} & 0 & a_{13456} & a_{123456} \\ -a_3 & -a_{23} & -a_{123} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_3 & -a_{13} & -a_{123} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a_1 & -a_{13} & -a_{123} \end{pmatrix} \begin{pmatrix} [D_1] \\ [D_2] \\ \vdots \\ [D_9] \end{pmatrix}.$$

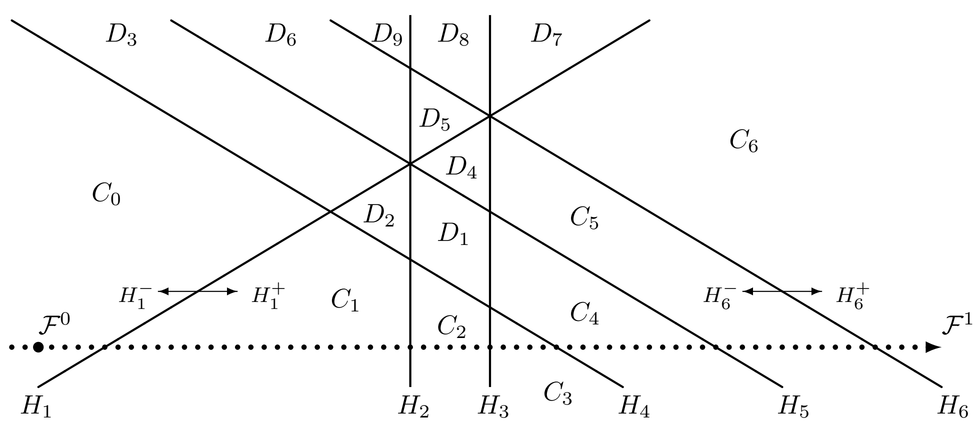

FIGURE 2. Example 4.4

4.2. **Aomoto complex via resonant bands.** Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of affine lines in $\mathbb{R}^2$. We fix the flag $\mathcal{F}$ as in §4.1. The cohomology of the Aomoto complex can be computed using chambers. In this subsection, we introduce the notion "η-resonant bands" which enables us to simplify the computation of cohomology. This can be regarded as "the Aomoto complex version" of the results in [29, 30].

Definition 4.5. *A* band *B is a region bounded by a pair of consecutive parallel lines H_i and H_{i+1}.*

Each band B contains two unbounded chambers $U_1(B), U_2(B) \in \mathsf{ch}(\mathcal{A})$. Since B intersects $\mathcal{F}^1$, we may assume that $B \cap \mathcal{F}^1 = U_1(B) \cap \mathcal{F}^1$ and $U_2(B) \cap \mathcal{F}^1 = \emptyset$. In other words, $U_1(B) \in \mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})$ and $U_2(B) \in \mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})$. The distance $d(U_1(B), U_2(B))$ is called the *length* of the band B.

Definition 4.6. *Let $\eta = \sum_{i=1}^n a_i e_i \in A^1_R(\mathcal{A})$. A band B is called η-*resonant *if*

$$\sum_{H_i \in \mathrm{Sep}(U_1(B), U_2(B))} a_i = 0.$$

We denote by $\mathrm{RB}_\eta(\mathcal{A})$ the set of all η-resonant bands.

We can extend U_1 to an injective R-module homomorphism $U_1 \colon R[\mathrm{RB}_\eta(\mathcal{A})] \hookrightarrow R[\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})]$. We denote by

$$\widetilde{\nabla}_\eta := -\nabla_\eta \circ U_1 \colon R[\mathrm{RB}_\eta(\mathcal{A})] \longrightarrow R[\mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})]$$

the composition of U_1 and ∇_η (multiplied by -1).

More precisely, to each η-resonant band $B \in \mathrm{RB}_\eta(\mathcal{A})$, we associate an element

$$\widetilde{\nabla}_\eta(B) \in R[\mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})]$$

as follows:

$$\widetilde{\nabla}_\eta(B) := -\nabla_\eta(U_1(B)) = \sum_{D \in \mathsf{ch}(\mathcal{A}), D \subset B} \left(\sum_{H_i \in \mathrm{Sep}(U_1(B), D)} a_i \right) \cdot [D].$$

Example 4.7. *Let $\mathcal{A} = \{H_1, \dots, H_6\}$ be an arrangement of lines as in Figure 2. There are three bands B_1, B_2, B_3, i.e., those defined by (H_2, H_3), (H_4, H_5) and (H_5, H_6), respectively. We have*

$$U_1(B_1) = C_2, U_2(B_1) = D_8, \; U_1(B_2) = C_4, U_2(B_2) = D_3, \quad \text{and} \quad U_1(B_3) = C_5, U_2(B_3) = D_6.$$

The band B_1 has length 4, while B_2 and B_3 have length 3. Let $\eta = a_1 e_1 + \cdots + a_6 e_6 \in A^1_R(\mathcal{A})$. The band B_1 is η-resonant if and only if $a_1 + a_4 + a_5 + a_6 = 0$. Then we have

$$\widetilde{\nabla}_\eta([B_1]) = a_4[D_1] + (a_4 + a_5)[D_4] + (a_1 + a_4 + a_5)[D_5].$$

Obviously the map U_1 induces $U_1 \colon \mathrm{Ker}(\widetilde{\nabla}_\eta) \longrightarrow \mathrm{Ker}(\nabla_\eta : R[\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})] \longrightarrow R[\mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})])$. Thus we have a natural map

$$\widetilde{U}_1 \colon \mathrm{Ker}(\widetilde{\nabla}_\eta) \longrightarrow H^1(R[\mathsf{ch}^\bullet_{\mathcal{F}}(\mathcal{A})], \nabla_\eta). \tag{7}$$

The above map $\widetilde{U}_1$ is neither injective nor surjective in general. The following is the main result concerning resonant bands which asserts that the map $\widetilde{U}_1$ above (7) is isomorphic under certain non-resonant assumption at infinity. This provides an effective way to compute $H^1(R[\mathsf{ch}^\bullet_{\mathcal{F}}(\mathcal{A})], \nabla_\eta)$. Indeed, normally, $|\mathrm{RB}_\eta(\mathcal{A})|$ is much smaller than $|\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})|$.

Theorem 4.8. *Let R be a commutative ring and $\eta = \sum_{i=1}^n a_i e_i \in A^1_R(\mathcal{A})$.*

(i) *Suppose that $\alpha := \sum_{i=1}^n a_i \in R^\times$ is invertible. Then the natural map $\widetilde{U}_1$ is injective.*
(ii) *We assume that R is an integral domain and $\alpha := \sum_{i=1}^n a_i \in R^\times$. Then $\widetilde{U}_1$ is an isomorphism.*
(iii) *Let R be an arbitrary commutative ring. If $\alpha := \sum_{i=1}^n a_i \in R^\times$ and all bands are η-resonant, then the natural map $\widetilde{U}_1$ is an isomorphism.*

Proof. (i) Let

$$\sum r_B \cdot [B] := \sum_{B \in \mathrm{RB}_\eta(\mathcal{A})} r_B \cdot [B] \in R[\mathrm{RB}_\eta(\mathcal{A})], \quad r_B \in R.$$

Suppose $\sum r_B \cdot [B] \in \operatorname{Ker} \widetilde{U}_1$, that is, $U_1(\sum r_B \cdot [B]) \in \operatorname{Im}\left(\nabla_\eta \colon R[\mathsf{ch}^0_{\mathcal{F}}(\mathcal{A})] \longrightarrow R[\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})]\right)$. Since $R[\mathsf{ch}^0_{\mathcal{F}}(\mathcal{A})] = R \cdot [C_0]$, there exists an element $s \in R$ such that

$$\sum r_B \cdot [U_1(B)] = s \cdot \nabla_\eta([C_0]). \tag{8}$$

Note that in the left hand side of (8), the chamber C_n does not appear, because C_n is not bounded by two parallel lines. By Definition 4.2, $\nabla_\eta([C_0]) = \sum_{i=1}^n (a_1 + \cdots + a_i) \cdot [C_i]$. The coefficient of $[C_n]$ is equal to $s \cdot (a_1 + \cdots + a_n) = s \cdot \alpha$. By the assumption that α is invertible, we have $s = 0$. Hence $\sum r_B \cdot U_1(B) = s \cdot \nabla_\eta([C_0]) = 0$, and, since U_1 is injective, we have $\sum r_B \cdot [B] = 0$.

Next we show the surjectivity of (7). Suppose that

$$\beta = \sum_{i=1}^n b_i \cdot [C_i] \in \operatorname{Ker}(\nabla_\eta \colon R[\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})] \longrightarrow R[\mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})]).$$

Consider the following element,

$$\begin{aligned} \beta' &= \beta - \frac{b_n}{\alpha} \cdot \nabla_\eta([C_0]) \\ &= \sum_{i=1}^{n-1} b'_i \cdot [C_i]. \end{aligned} \tag{9}$$

Obviously, β and β' represent the same element in $H^1(R[\mathsf{ch}^\bullet_{\mathcal{F}}(\mathcal{A})], \nabla_\eta)$. It is sufficient to show that $\beta' \in \operatorname{Im} U_1$.

Next we consider a chamber C_i $(i < n)$ such that H_i and H_{i+1} are not parallel. Then there is a unique chamber $D_p \in \mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})$ such that $\mathrm{Sep}(C_i, D_p) = \mathcal{A}$, which is called the "opposite chamber of C_i" in [26, Def. 2.1] and denoted by $D_p = C_i^\vee$. Then we consider the coefficient $c_{C_i^\vee}$ of $[C_i^\vee]$ in $\nabla_\eta(\beta') = \sum c_D \cdot [D]$. Since $C_i^\vee$ appears only in $\nabla_\eta([C_i])$ and $\nabla_\eta([C_n])$, and the coefficient of $[C_n]$ is already zero, we have $c_{C_i^\vee} = \alpha \cdot b'_i$. By the assumption that $\alpha \in R^\times$, $\nabla_\eta(\beta') = 0$, in particular $c_{C_i^\vee} = 0$, implies that $b'_i = 0$. So $\beta' = \sum_{i=1}^{n-1} b'_i \cdot [C_i]$ is a linear combination of C_i's $(i < n)$ such that H_i and H_{i+1} are parallel. So far, we only use the fact $\alpha \in R^\times$. If all bands are η-resonant, then we have already proved that β' is generated by $U_1(B)$ with $B \in \mathrm{RB}_\eta(\mathcal{A})$. Thus (iii) is proved.

Now we assume that R is an integral domain. We will prove (ii). Let C_i be a chamber such that the walls H_i and H_{i+1} are parallel. Let B be the corresponding band defined by H_i and H_{i+1}. Note that $C_i = U_1(B)$ and its opposite chamber is $U_2(B)$. Suppose that B is not an η-resonant band, that is, $\alpha' := \sum_{H_j \in \mathrm{Sep}(U_1(B), U_2(B))} a_j \neq 0$. Again consider the coefficient of $[U_2(B)]$ in $\nabla_\eta(\beta')$. $[U_2(B)]$ appears in $\nabla_\eta([C_i])$ and some other terms $\nabla_\eta([C_k])$ for k such that H_k and H_{k+1} are not parallel. However the coefficients of chambers of the second type in β' are already zero. Therefore the coefficient of $[U_2(B)]$ in $\nabla_\eta(\beta')$ is $-\alpha' \cdot b'_i$, which is zero. Since R is an integral domain, we have $b'_i = 0$. Hence β' is a linear combination of $U_1(B)$'s where $B \in \mathrm{RB}_\eta(\mathcal{A})$. This completes the proof of the surjectivity. $\square$

Remark 4.9. *Equation (7) and Theorem 4.8 are concerning the following homomorphism of cochains.*

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & R[\mathrm{RB}_\eta(\mathcal{A})] \arrow[r, "\widetilde{\nabla}_\eta"] \arrow[d, "\varphi_1"] & R[\mathsf{ch}(\mathcal{A})] \arrow[r] \arrow[d, "\varphi_2"] & 0 \\
0 \arrow[r] & A^0_R(\mathcal{A}) \arrow[r, "\eta"] & A^1_R(\mathcal{A}) \arrow[r, "\eta"] & A^2_R(\mathcal{A}) \arrow[r] & 0
\end{tikzcd}$$

The map $\widetilde{U}_1$ is nothing but the homomorphism $\mathrm{Ker}(\widetilde{\nabla}_\eta) \longrightarrow H^1(A^\bullet_R(\mathcal{A}), \eta)$ induced by φ_1. By Proposition 4.3 (especially, the explicit map (5)), the map φ_1 above is given by

$$[B] \longmapsto e_i - e_{i+1},$$

where B is a η-resonant band bounded by the lines H_i and H_{i+1}.

Example 4.10. *Let $R = \mathbb{F}_2$. Let $\mathcal{A} = \{H_1, \dots, H_6\}$ be an arrangement of affine lines as in Figure 3 (which is $\mathcal{A}(7,1)$ in [12]). Let $\eta = e_2 + e_3 + e_6 \in A^1_R(\mathcal{A})$ (the supporting lines of η are colored blue). There are three bands B_1 (bounded by H_1 and H_2), B_2 (bounded by H_3 and H_4), and B_3 (bounded by H_5 and H_6). $\mathrm{Sep}(U_1(B_1), U_2(B_1)) = \{H_3, H_4, H_5, H_6\}$ and two of the lines, H_3 and H_6, have non-zero coefficients in η. Hence B_1 is an η-resonant band. Similarly, B_2 and B_3 are η-resonant, and we have $\mathrm{RB}_\eta(\mathcal{A}) = \{B_1, B_2, B_3\}$. By definition, $\widetilde{\nabla}_\eta(B_1) = \widetilde{\nabla}_\eta(B_2) = \widetilde{\nabla}_\eta(B_3) = [D_1]$. Hence the kernel*

$$\mathrm{Ker}(\widetilde{\nabla}_\eta \colon \mathbb{F}_2[\mathrm{RB}_\eta(\mathcal{A})] \longrightarrow \mathbb{F}_2[\mathsf{ch}(\mathcal{A})])$$

is 2-dimensional (generated by $[B_1]-[B_2]$ and $[B_2]-[B_3]$). By Theorem 4.8, $H^1(A^\bullet_{\mathbb{F}_2}(\mathcal{A}), \eta) \simeq \mathbb{F}_2^2$.

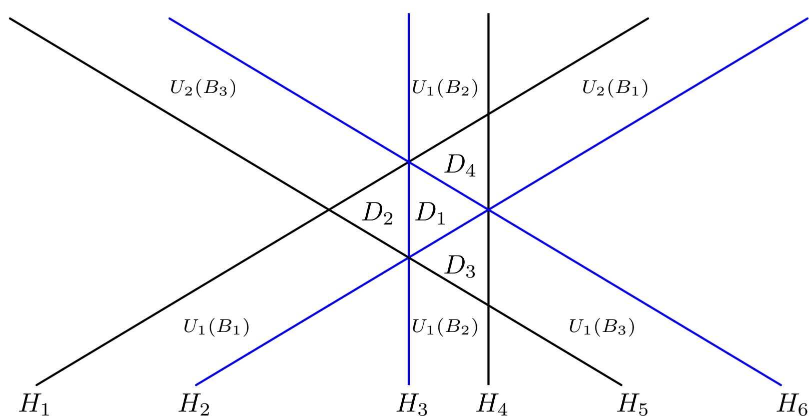

Figure 3. Example 4.10

Example 4.11. *We consider $\overline{\mathcal{A}} = \mathcal{A}(16,1) = \{\overline{H}_1, \dots, \overline{H}_{16}\}$ from the Grünbaum's catalogue [12], see Figure 4. Let us denote by $\mathcal{A} = \{H_2, H_3, \dots, H_{16}\}$ the deconing $\mathsf{d}_{\widetilde{H}_1}\widetilde{\mathcal{A}}$, the lower-left one in Figure 4. The affine arrangement $\mathcal{A}$ has 7 bands $B_1, \dots, B_7$. To indicate the choice of $U_1(B)$ and $U_2(B)$, we always put the name B of the band in the unbounded chamber $U_1(B)$.*

Let $R = \mathbb{Z}/8\mathbb{Z}$. Define $\widetilde{\eta}_1, \widetilde{\eta}_2 \in A^1_R(\widetilde{\mathcal{A}})_0$ by

$$\widetilde{\eta}_1 = \widetilde{e}_1 + \widetilde{e}_3 + \widetilde{e}_5 + \widetilde{e}_7 + \widetilde{e}_9 + \widetilde{e}_{11} + \widetilde{e}_{13} + \widetilde{e}_{15},$$
$$\widetilde{\eta}_2 = \widetilde{e}_2 + \widetilde{e}_4 + \widetilde{e}_6 + \widetilde{e}_8 + \widetilde{e}_{10} + \widetilde{e}_{12} + \widetilde{e}_{14} + \widetilde{e}_{16},$$

and set $\widetilde{\eta} := \widetilde{\eta}_1 + 6\widetilde{\eta}_2$.

Let $\eta = (e_3+e_5+e_7+\cdots+e_{15})+6(e_2+e_4+\cdots+e_{16}) \in A^1_R(\mathcal{A})$. *Then all* 7 *bands are* η*-resonant. Thus we can apply theorem Theorem 4.8 (iii). The kernel* $\operatorname{Ker}(\widetilde{\nabla}_\eta\colon R[\mathrm{RB}_\eta(\mathcal{A})] \longrightarrow R[\mathsf{ch}(\mathcal{A})])$ *is a free R-module generated by*

$$[B_1]+2[B_2]+3[B_3]+4[B_4]+5[B_5]+6[B_6]+7[B_7].$$

The corresponding element (via the correspondence Remark 4.9) in $A^1_R(\widetilde{\mathcal{A}})_0$ *is*

$$4(\widetilde{e}_2+\widetilde{e}_3)+3(\widetilde{e}_4-\widetilde{e}_7+\widetilde{e}_{13}-\widetilde{e}_{16})+2(\widetilde{e}_6+\widetilde{e}_9-\widetilde{e}_{11}-\widetilde{e}_{14})+(\widetilde{e}_5+\widetilde{e}_8-\widetilde{e}_{12}-\widetilde{e}_{15}).$$

By Theorem 4.8 (iii), the cohomology of the Aomoto complex

$$H^1(A^1_R(\widetilde{\mathcal{A}})_0,\widetilde{\eta}) \simeq H^1(A^1_R(\mathcal{A}),\eta) \simeq \operatorname{Ker}(\widetilde{\nabla}_\eta) \simeq R \simeq \mathbb{Z}/8\mathbb{Z}$$

is non-vanishing.

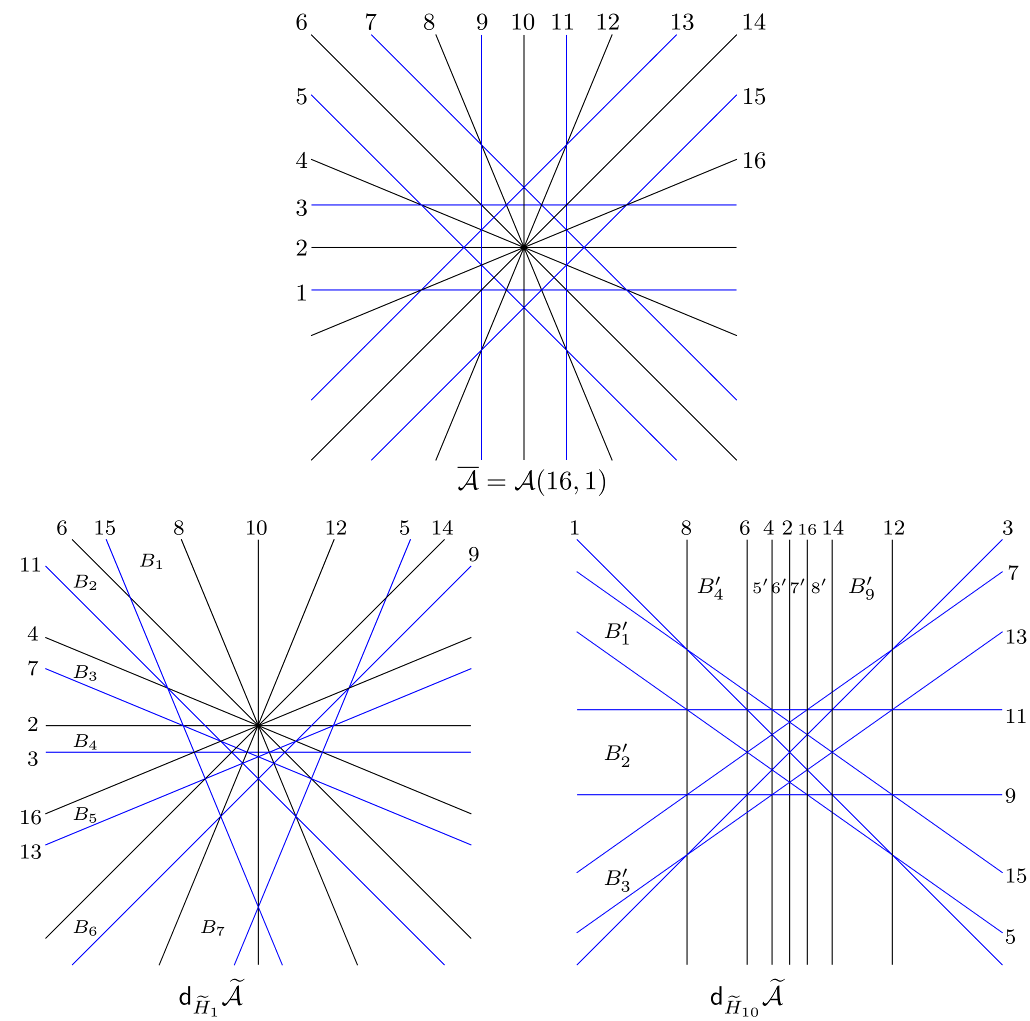

FIGURE 4. $\mathcal{A}(16,1)$ and deconings with respect to H_1 and H_{10}.

Remark 4.12. *Let us point out a possible relation between* $\mathbb{Z}/8\mathbb{Z}$*-resonance in Example 4.11 and isolated torsion points of order* 8 *in the characteristic variety of* $\mathcal{A}(16,1)$.

Let us denote $M = M(\mathcal{A}(16,1)) = \mathbb{CP}^2 \setminus \bigcup_{H \in \mathcal{A}(16,1)} H_{\mathbb{C}}$ the complexified complement. Recall that the character torus of M is

$$\mathbb{T} := \mathrm{Hom}(\pi_1(M), \mathbb{C}^\times) \simeq \{\boldsymbol{t} = (t_1, t_2, \dots, t_{16}) \in (\mathbb{C}^\times)^{16} \mid \prod_{i=1}^{16} t_i = 1\}.$$

We also define the essential open subset of $\mathbb{T}$ by

$$\mathbb{T}^\circ := \{\boldsymbol{t} = (t_1, \dots, t_{16}) \in \mathbb{T} \mid t_i \neq 1, \forall i = 1, \dots, 16\}.$$

The characteristic variety $\mathcal{V}^1(\mathcal{A}(16,1))$ of $\mathcal{A}(16,1)$ is the set of points in the character torus $\mathbb{T}$ such that the associated local system has non-vanishing first cohomology, i.e.,

$$\mathcal{V}^1(\mathcal{A}(16,1)) = \{\boldsymbol{t} \in \mathbb{T} \mid \dim H^1(M, \mathcal{L}_{\boldsymbol{t}}) \geq 1\}.$$

Let $\zeta = e^{2\pi i/8}$ and consider the following point,

$$\rho = (\zeta, \zeta^6, \zeta, \zeta^6, \zeta, \zeta^6, \zeta, \zeta^6, \zeta, \zeta^6, \zeta, \zeta^6, \zeta, \zeta^6, \zeta, \zeta^6) \in \mathbb{T}^\circ.$$

Let us recall quickly the resonant band algorithm for computing local system cohomology groups (see [30] for details). For a given local system $\mathcal{L}_{\boldsymbol{t}}$, we define the set $\mathrm{RB}_{\mathcal{L}_{\boldsymbol{t}}}(\mathcal{A})$ of $\mathcal{L}_{\boldsymbol{t}}$-resonant bands and the map $\nabla_{\mathcal{L}_{\boldsymbol{t}}} \colon \mathbb{C}[\mathrm{RB}_{\mathcal{L}_{\boldsymbol{t}}}(\mathcal{A})] \longrightarrow \mathbb{C}[\mathsf{ch}(\mathcal{A})]$. If $\mathcal{L}_{\boldsymbol{t}}$ has non-trivial monodromy around the line at infinity, then we have the isomorphism $H^1(M, \mathcal{L}_{\boldsymbol{t}}) \simeq \mathrm{Ker}(\nabla_{\mathcal{L}_{\boldsymbol{t}}})$.

Since $\mathcal{L}_\rho$ defined above has non trivial monodromy around any line, we can apply resonant band algorithm to any deconings. Here we exhibit two cases (although the results coincide logically), $\mathsf{d}_{\widetilde{H}_1}\widetilde{\mathcal{A}}$ and $\mathsf{d}_{\widetilde{H}_{10}}\widetilde{\mathcal{A}}$. (See Figure 4.)

- *The affine arrangement $\mathsf{d}_{\widetilde{H}_1}\widetilde{\mathcal{A}}$ has seven bands $B_1, \dots, B_7$, which are all $\mathcal{L}_\rho$-resonant. Then $\mathrm{Ker}(\nabla_{\mathcal{L}_\rho})$ is one dimensional and generated by the following element,*

$$\sin\left(\frac{\pi}{8}\right)[B_1] - \sin\left(\frac{\pi}{4}\right)[B_2] + \sin\left(\frac{3\pi}{8}\right)[B_3] - \sin\left(\frac{\pi}{2}\right)[B_4] \\ + \sin\left(\frac{3\pi}{8}\right)[B_5] - \sin\left(\frac{\pi}{4}\right)[B_6] + \sin\left(\frac{\pi}{8}\right)[B_7].$$

- *The affine arrangement $\mathsf{d}_{\widetilde{H}_{10}}\widetilde{\mathcal{A}}$ has nine bands $B_1', \dots, B_9'$, which are all $\mathcal{L}_\rho$-resonant. Then $\mathrm{Ker}(\nabla_{\mathcal{L}_\rho})$ is one dimensional and generated by the following element,*

$$[B_1] + \sqrt{2}[B_2] + [B_3] \\ - [B_4] + (1+\sqrt{2})[B_5] - (2+\sqrt{2})[B_6] \\ + (2+\sqrt{2})[B_7] + (1+\sqrt{2})[B_8] + [B_9].$$

Hence we have $\dim H^1(M, \mathcal{L}_\rho) = 1$. Furthermore, we can prove that ρ generates the essential part of the characteristic variety. More precisely, we have the following,

$$\mathcal{V}^1(\mathcal{A}(16,1)) \cap \mathbb{T}^\circ = \{\rho, \rho^2, \rho^3, \rho^5, \rho^6, \rho^7\}. \tag{10}$$

4.3. **Resonant bands over $\mathbb{F}_2$ and subarrangements.** Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of affine lines in $\mathbb{R}^2$. Let $\mathcal{S} \subset \mathcal{A}$ be a subset. Denote $e(\mathcal{S}) := \sum_{H_i \in \mathcal{S}} e_i \in A^1_{\mathbb{F}_2}(\mathcal{A})$.

Clearly, $e(\mathcal{S}) + e(\mathcal{A}) = e(\mathcal{A} \setminus \mathcal{S})$. Below is the summary of "subarrangement description of resonant band algorithm":

(a) Let B be a band of $\mathcal{A}$. Then $B \in \mathrm{RB}_{e(\mathcal{S})}(\mathcal{A})$ if and only if the number of lines in $\mathcal{S}$ separating $U_1(B)$ and $U_2(B)$ is even, i.e., $2 \mid |\mathcal{S} \cap \mathrm{Sep}(U_1(B), U_2(B))|$.

(b) $\widetilde{\nabla}_{e(\mathcal{S})}\colon \mathbb{F}_2[\mathrm{RB}_{e(\mathcal{S})}(\mathcal{A})] \longrightarrow \mathbb{F}_2[\mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})]$ is given by the following formula.

$$\widetilde{\nabla}_{e(\mathcal{S})}(B) = \sum_{C\in \mathsf{ch}(\mathcal{A}), C\subset B} |\mathcal{S}\cap \mathrm{Sep}(U_1(B), C)|\cdot [C].$$

(See Example 4.10). In particular, if we consider $\eta_0 = e(\mathcal{A}) = e_1 + e_2 + \cdots + e_n$, then we have

$$\widetilde{\nabla}_{\eta_0}(B) = \sum_{C\in \mathsf{ch}(\mathcal{A}), C\subset B} d(U_1(B), C)\cdot [C].$$

(c) Suppose that $|\mathcal{S}|$ is odd. Then we can apply Theorem 4.8, and we have an isomorphism

$$\Psi_{e(S)}\colon\ \mathrm{Ker}(\widetilde{\nabla}_{e(\mathcal{S})}) \xrightarrow{\simeq} H^1(A^{\bullet}_{\mathbb{F}_2}(\mathcal{A}), e(\mathcal{S})).$$

(d) Using Remark 4.9 (and Proposition 4.3 (especially, the explicit map (5))), the above isomorphism is given by

$$\Psi_{e(S)}([B]) = e_i + e_{i+1} \in A^1_{\mathbb{F}_2}(\mathcal{A}),$$

where B is a $e(\mathcal{S})$-resonant band determined by the lines H_i and H_{i+1}.

5. Non-existence of real 4-nets

5.1. **Aomoto complex for the diagonal element.** Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of affine lines in $\mathbb{R}^2$ with odd n. Let $\widetilde{\mathcal{A}} = \{\widetilde{H}_0, \widetilde{H}_1, \dots, \widetilde{H}_n\}$ be the coning of $\mathcal{A}$ and $\overline{\mathcal{A}} = \{\overline{H}_0, \overline{H}_1, \dots, \overline{H}_n\}$ be the projectivization. Recall that

$$\widetilde{\eta}_0 := \widetilde{e}(\widetilde{\mathcal{A}}) = \widetilde{e}_0 + \widetilde{e}_1 + \cdots + \widetilde{e}_n \in A^1_{\mathbb{F}_2}(\widetilde{\mathcal{A}})_0$$

is the diagonal element and $\eta_0 = e(\mathcal{A}) = e_1 + \cdots + e_n \in A^1_{\mathbb{F}_2}(\mathcal{A})$. Notice that n odd implies that the map Ψ_{η_0} is an isomorphism by §4.3 (c).

Choose a subset $\widetilde{\mathcal{S}} \subset \widetilde{\mathcal{A}}$. In the figures below, the lines in $\widetilde{\mathcal{S}}$ are colored in red. The other lines are black.

As we saw in Proposition 3.4, the relation $\widetilde{\eta}_0 \wedge \widetilde{e}(\widetilde{\mathcal{S}}) = 0$ is equivalent to "$|\overline{\mathcal{A}}_X|$ is even $\Longrightarrow |\overline{\mathcal{S}}_X|$ is even" and "$|\overline{\mathcal{A}}_X|$ is odd $\Longrightarrow$ either $\overline{\mathcal{S}}_X = \emptyset$ or $\overline{\mathcal{S}}_X = \overline{\mathcal{A}}_X$" for $\forall X \in L_2(\overline{\mathcal{A}})$. From this, it is easily seen that if the multiplicity is $|\overline{\mathcal{A}}_X| \le 3$, then $\overline{\mathcal{A}}_X$ is monocolor (either all red $\overline{\mathcal{S}}_X = \overline{\mathcal{A}}_X$ or all black $\overline{\mathcal{S}}_X = \emptyset$). However, when $|\overline{\mathcal{A}}_X| = 4$, there are four cases (Figure 5):

(*i*) $\overline{\mathcal{S}}_X = \emptyset$.
(*ii*) $\overline{\mathcal{S}}_X = \overline{\mathcal{A}}_X$.
(*iii*) $|\overline{\mathcal{S}}_X| = 2$ and the lines in $\overline{\mathcal{S}}_X$ are adjacent.
(*iv*) $|\overline{\mathcal{S}}_X| = 2$ and the lines in $\overline{\mathcal{S}}_X$ are separated by lines in $\overline{\mathcal{A}}_X \setminus \widetilde{\mathcal{S}}_X$.

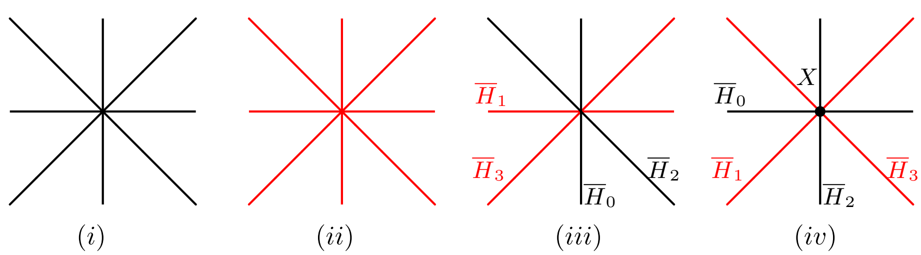

Figure 5. Local structures of $\overline{\mathcal{S}}_X$. (Members of $\overline{\mathcal{S}}_X$ are red, and $\overline{\mathcal{S}}_X$ equals $\{\overline{H}_1, \overline{H}_3\}$ in (*iii*) and (*iv*)).

The cases (iii) and (iv) are combinatorially identical. However, the real structures are different. This difference is crucial, actually, by using resonant bands, we can prove that (iv) can not happen ("Non Separation Theorem").

Theorem 5.1. *Let $\overline{\mathcal{S}} \subset \overline{\mathcal{A}}$. Suppose that $\widetilde{\eta}_0 \wedge \widetilde{e}(\widetilde{\mathcal{S}}) = 0$. Let $X \in \mathbb{RP}^2$ be an intersection of $\overline{\mathcal{A}}$ such that $|\overline{\mathcal{A}}_X| = 4$ and $|\overline{\mathcal{S}}_X| = 2$. Then the two lines of $\overline{\mathcal{S}}_X$ are adjacent as Figure 5 (iii). In particular, (iv) does not happen.*

Proof. Suppose that there exists $X \in \mathbb{RP}^2$ such that $\overline{\mathcal{A}}_X = \{\overline{H}_0, \overline{H}_1, \overline{H}_2, \overline{H}_3\}$ with $\overline{\mathcal{S}}_X = \{\overline{H}_1, \overline{H}_3\}$ arranging as (iv) in Figure 5.

First consider the deconing with respect to $\overline{H}_0$, we have $\mathcal{A} = \mathsf{d}_{\widetilde{H}_0}\widetilde{\mathcal{A}} = \{H_1, \dots, H_n\}$. Then

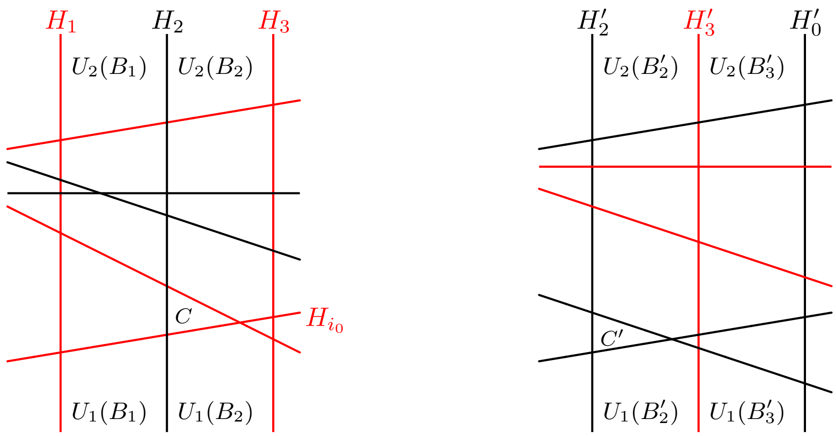

FIGURE 6. Deconings $\mathsf{d}_{\widetilde{H}_0}\widetilde{\mathcal{A}}$ and $\mathsf{d}_{\widetilde{H}_1}\widetilde{\mathcal{A}}$

$$\mathcal{S} = \{H_1, H_3, \dots\} \subset \mathcal{A}.$$

The lines H_1, H_2, H_3 are parallel (the left of Figure 6) and determine two bands B_1 (bounded by H_1 and H_2) and B_2 (bounded by H_2 and H_3). Note that $e(\mathcal{S}) = e_1 + e_3 + \dots \in A^1_{\mathbb{F}_2}(\mathcal{A})$. By the correspondence in §4.3 (d), we have

$$\Psi_{\eta_0}^{-1}(e(\mathcal{S})) = [B_1] + [B_2] + \dots,$$

in particular, both $[B_1]$ and $[B_2]$ appear. (Otherwise, e_1, e_3 can not appear.) On the other hand, we have the following relation

$$\widetilde{\nabla}_{\eta_0}(\Psi_{\eta_0}^{-1}(e(\mathcal{S})) = \widetilde{\nabla}_{\eta_0}([B_1]) + \widetilde{\nabla}_{\eta_0}([B_2]) + \dots = 0. \tag{11}$$

Choose a chamber C such that $C \subset B_2$ and $d(U_1(B_2), C) = 1$. Let $\mathrm{Sep}(U_1(B_2), C) = \{H_{i_0}\}$. The chamber C is adjacent to an unbounded chamber $U_1(B_2)$, hence, C is contained in at most two bands B_2 and B_{j_0}. Since $\widetilde{\nabla}_{\widetilde{\eta}}([B_2]) = [C] + \dots \in \mathbb{F}_2[\mathrm{RB}_{\widetilde{\eta}}(\mathcal{A})]$, by (11), $[C]$ must be cancelled by another resonant band B_{j_0} which appears in $\Psi_{\eta_0}^{-1}(e(\mathcal{S}))$. Thus we have

$$\Psi_{\eta_0}^{-1}(e(\mathcal{S})) = [B_1] + [B_2] + \dots + [B_{j_0}] + \dots.$$

Let H_{i_0} and H_{i_0+1} be walls of B_{j_0}. Then applying Ψ, we have

$$\begin{aligned} e(\mathcal{S}) &= (e_1 + e_2) + (e_2 + e_3) + \dots + (e_{i_0} + e_{i_0+1}) + \dots \\ &= e_1 + e_3 + \dots + e_{i_0} + \dots. \end{aligned}$$

Here note that e_{i_0} survives because B_{j_0} is the only band which has H_{i_0} as a wall. This implies $H_{i_0} \in \mathcal{S}$. Therefore, if $C \subset B_2$ and $d(U_1(B_2), C) = 1$, then $\mathrm{Sep}(U_1(B_2), C) \subset \mathcal{S}$. (Left hand side of Figure 6.) The same assertion holds for the opposite unbounded chamber $U_2(B_2)$.

Next we consider $\overline{\mathcal{S}}' := \overline{\mathcal{A}} \setminus \overline{\mathcal{S}}$. Since $\widetilde{e}(\widetilde{\mathcal{S}}') = \widetilde{\eta}_0 + \widetilde{e}(\widetilde{\mathcal{S}})$, $\widetilde{\eta}_0 \wedge \widetilde{e}(\widetilde{\mathcal{S}}') = 0$. In Figure 5 (iv), the roles of black and red lines exchange. Black lines are the members of $\overline{\mathcal{S}}'$ and red lines are not. We take the deconing with respect to $\widetilde{H}_1$, we have $\mathsf{d}_{\widetilde{H}_1}\widetilde{\mathcal{A}} = \{H_0', H_2', H_3', \dots, H_n'\}$ (Right hand side of Figure 6). Then $\mathcal{S}' = \{H_0', H_2', \dots\} \subset \mathsf{d}_{\widetilde{H}_1}\widetilde{\mathcal{A}}$. The lines H_0', H_2', H_3' are parallel and determine two bands B_2' (bounded by H_2' and H_3') and B_3' (bounded by H_3' and H_0'). By a similar argument to the previous case (deconing with respect to $\widetilde{H}_0$), we can conclude that if $C' \subset B_2'$ and $d(U_1(B_2'), C') = 1$, then $\mathrm{Sep}(U_1(B_2'), C') \subset \mathcal{S}'$. (Right hand side of Figure 6.) The same assertion holds for the opposite unbounded chamber $U_2(B_2')$.

The bands B_2 and B_2' are identical in the projective plane $\mathbb{RP}^2$. However, the colors of boundaries of unbounded chambers are different. This is a contradiction. Thus the case (iv) can not happen. □

5.2. **Real 4-nets do not exist.**

Theorem 5.2. *There does not exist a real arrangement $\overline{\mathcal{A}}$ that supports a 4-net structure.*

Proof. Suppose $\overline{\mathcal{A}}$ supports a 4-net structure with partition $\overline{\mathcal{A}} = \overline{\mathcal{A}}_1 \sqcup \overline{\mathcal{A}}_2 \sqcup \overline{\mathcal{A}}_3 \sqcup \overline{\mathcal{A}}_4$. There exists a multiple point $X \in \mathbb{RP}^2$ of $\overline{\mathcal{A}}$ with multiplicity 4 such that X is the intersection point of 4 lines $H_i \in \mathcal{A}_i$. Suppose that the lines are ordered as in Figure 7.

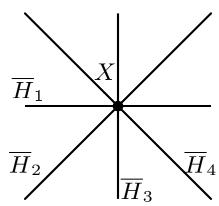

FIGURE 7. Local structure of a 4-net.

We can now define $\widetilde{\mathcal{S}} = \widetilde{\mathcal{A}}_1 \sqcup \widetilde{\mathcal{A}}_3$. Then as in Example 3.6, we have $\widetilde{\eta}_0 \wedge \widetilde{e}(\widetilde{\mathcal{S}}) = 0$. By definition, $\overline{\mathcal{S}}_X = \{\overline{H}_1, \overline{H}_3\}$ consists of two lines separated by the other two lines $\overline{H}_2$ and $\overline{H}_4$. Therefore (iv) in Figure 5 happens. This contradicts the Non-separation Theorem 5.1. □

Remark 5.3. *The non-existence of real* 4*-nets was proved in* [6, Lem. 2.4]. *Their proof relies on the metric structure of* $\mathbb{R}^2$. *So it does not apply to oriented matroids. Our arguments actually prove that there do not exist rank* 3 *oriented matroids (equivalently, pseudo-line arrangements in* $\mathbb{RP}^2$*) which have* 4*-net structures. The details are omitted.*

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ON THE ŁOJASIEWICZ EXPONENTS OF QUASI-HOMOGENEOUS FUNCTIONS

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Abstract. Quasi-homogeneous functions, and especially polynomials, enjoy some specific properties around the origin which allow to estimate the so-called Łojasiewicz exponents in a way quite similar to homogeneous functions. In particular we generalize a previous result for polynomials of two variables concerning the optimal Łojasiewicz gradient inequality at the origin.

1. Introduction

In his pioneering papers [13], [14], Łojasiewicz established that any analytic function f of n real variables satisfies an inequality of the form

$$\|\nabla f(x)\| \geq c|f(x) - f(a)|^{\beta} \tag{1.1}$$

for $\|x - a\|$ small enough with $c > 0$, $\beta \in (0, 1)$. This inequality, known as the Łojasiewicz gradient inequality, is useful to establish trend to equilibrium of the general solutions of gradient systems

$$u' + \nabla f(u) = 0 \tag{1.2}$$

and can also be used to evaluate the rate of convergence. It is therefore of interest to know as precisely as possible the connection between f and its gradient and in particular to determine the best(smallest) possible value of β in (1.1) when a is a critical point of f. This value is called the Łojasiewicz exponent at a. In [7] for instance, it was shown that if f is a homogeneous polynomial with degree $d \geq 2$, the Łojasiewicz exponent at the origin is exactly $1 - \frac{1}{d}$ when $n = 2$. This property is no longer true if $n > 2$.

On the other hand, Gwoździewicz [6] (see also [16]) considered the case of a real analytic function at an isolated zero and also found, in this case, an interesting relationship between various Łojasiewicz exponents, relative to different Łojasiewicz inequalities. In addition the case of general polynomials has been thoroughly investigated by D'Acunto and Kurdyka in [2] (see also [11, 12, 17]).

Our paper is concerned to the extension of the result from [7] and several estimates of Łojasiewicz exponents at the origin when f is a quasi-homogeneous map (see, for example, [1]). It is divided in 5 sections. In Section 2, we state and prove some preliminary results, mainly concerning the local behavior of quasi-homogeneous maps near the origin. Section 3 contains more information in the specific case where the origin is an isolated zero of f. Section 4 deals with the Łojasiewicz gradient exponent of quasi-homogeneous polynomials, in particular we generalize the main result of [7] (see also [4, 5, 9] for related results). Section 5 contains

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more precise estimates for quasi-homogeneous polynomials of two variables. These results are illustrated by typical examples and completed by a few remarks.

2. Definitions and preliminary results

We first recall the concept of a quasi-homogeneous map. Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. We say that $f\colon \mathbb{K}^n \to \mathbb{K}^k$ is a *(positively) quasi-homogeneous map with weight*

$$w := (w_1, w_2, \ldots, w_n) \in (\mathbb{R}_+ - \{0\})^n$$

and quasi-degree $d := (d_1, d_2, \ldots, d_k) \in (\mathbb{R}_+ - \{0\})^k$ if

$$f_i(t^{w_1}x_1, t^{w_2}x_2, \ldots, t^{w_n}x_n) = t^{d_i} f_i(x_1, x_2, \ldots, x_n) \tag{2.1}$$

for each $i = 1, 2, \ldots, k$, and all $t > 0$. Note that if $w_j = 1$ for $j = 1, 2, \ldots, n$ then the above definition means that the components f_i are homogeneous functions of degree d_i. When $k = 1$, a scalar function $f = f_1$ satisfying (2.1) is called a *(positively) quasi-homogeneous function with weight* $w = (w_1, w_2, \ldots, w_n)$ *and quasi-degree* $d = d_1$. In the sequel we shall drop for simplicity the word "positively". Note that any monomial $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ is a quasi-homogeneous function with arbitrary weight $w = (w_1, w_2, \ldots, w_n) \in (\mathbb{N} - \{0\})^n$ and quasi-degree $\langle w, \alpha\rangle := w_1\alpha_1 + w_2\alpha_2 + \cdots + w_n\alpha_n$. Moreover, we have

Proposition 2.1. *Let $f\colon \mathbb{K}^n \to \mathbb{K}$ be a polynomial function. Then f is quasi-homogeneous with weight $w \in (\mathbb{N} - \{0\})^n$ and quasi-degree $m \in \mathbb{N} - \{0\}$ if and only if all its constitutive monomials are quasi-homogeneous functions with weight w and quasi-degree m.*

Proof. Suppose that f is a quasi-homogeneous polynomial with weight w and quasi-degree m. We have the following finite expansion

$$f(x) := \sum_\alpha a_\alpha x^\alpha.$$

Then

$$\sum_\alpha a_\alpha t^{\langle w,\alpha\rangle} x^\alpha = t^m \sum_\alpha a_\alpha x^\alpha.$$

This gives

$$\sum_\alpha a_\alpha [t^{\langle w,\alpha\rangle - m} - 1] x^\alpha = 0$$

for all $x \in \mathbb{K}^n$ and for all $t > 0$. Since the field $\mathbb{K}$ is of characteristic 0, we get

$$\langle w, \alpha\rangle - m = 0$$

for all α provided $a_\alpha \neq 0$. In other words, all constitutive monomials of f are quasi-homogeneous functions with weight w and quasi-degree m. The converse is clear. $\square$

For a fixed weight $w := (w_1, w_2, \ldots, w_n) \in (\mathbb{R}_+ - \{0\})^n$ we set

$$\|x\|_w := \max_{j=1,2,\ldots,n} |x_j|^{\frac{1}{w_j}}$$

for $x := (x_1, x_2, \ldots, x_n) \in \mathbb{K}^n$. In the special case $w = w_0 = (1, \ldots 1)$, we recover the usual l^∞ norm on $\mathbb{K}^n$ and we set

$$\|x\| := \|x\|_{w_0} = \max_{j=1,2,\ldots,n} |x_j|.$$

Remark 2.2. (i) It is worth noting that, except in the special case $w = w_0 = (1, ...1)$, $\|\cdot\|_w$ is not a norm since it is not homogeneous of degree 1.

(ii) It is easy to see that $\|\cdot\|_w$ is a quasi-homogeneous function with weight w and with quasi-degree 1.

The following basic properties will be used throughout the text.

Proposition 2.3. *Let*

$$\begin{aligned} w_* &:= \min_{j=1,2,\ldots,n} w_j, \\ w^* &:= \max_{j=1,2,\ldots,n} w_j. \end{aligned}$$

Then the following hold

(i) *For all $\|x\| \geq 1$ we have*

$$\|x\|^{\frac{1}{w_*}} \geq \|x\|_w \geq \|x\|^{\frac{1}{w^*}}.$$

In particular, $\|x\| \to \infty$ if and only if $\|x\|_w \to \infty$.

(ii) *For all $\|x\| \leq 1$ we have*

$$\|x\|^{\frac{1}{w_*}} \leq \|x\|_w \leq \|x\|^{\frac{1}{w^*}},$$

In particular, $\|x\| \to 0$ if and only if $\|x\|_w \to 0$.

Proof. The proof of the proposition is straightforward from the definitions. □

In the sequel for $t > 0$, for any $w := (w_1, w_2, \ldots, w_n) \in (\mathbb{R}_+ - \{0\})^n$ and

$$x := (x_1, x_2, \ldots, x_n) \in \mathbb{K}^n$$

we denote

$$t \bullet x := (t^{w_1} x_1, t^{w_2} x_2, \ldots, t^{w_n} x_n),$$

and for $d := (d_1, d_2, \ldots, d_k) \in (\mathbb{R}_+ - \{0\})^k$ we set

$$\begin{aligned} d_* &:= \min_{i=1,2,\ldots,k} d_i, \\ d^* &:= \max_{i=1,2,\ldots,k} d_i. \end{aligned}$$

Let $f \colon \mathbb{K}^n \to \mathbb{K}^k$ be quasi-homogeneous with weight $w := (w_1, w_2, \ldots, w_n) \in (\mathbb{R}_+ - \{0\})^n$ and quasi-degree $d := (d_1, d_2, \ldots, d_k) \in (\mathbb{R}_+ - \{0\})^k$. If $f_i \equiv 0$ for some $i \in \{1, 2, \ldots, k\}$, then d_i can be replaced by any positive number. In the sequel we shall assume

$$\forall i \in \{1, 2, \ldots, k\}, \quad f_i \not\equiv 0. \tag{2.2}$$

It is easy to check that in this case d_i is uniquely defined by (2.1) for all $i \in \{1, 2, \ldots, k\}$. Then d_* and d^* are well defined.

The next two results summarize some important consequences of the quasi homogeneity property.

Proposition 2.4. *Let $f := (f_1, f_2, \ldots, f_k) \colon \mathbb{K}^n \to \mathbb{K}^k$ be a quasi-homogeneous map with weight $w := (w_1, w_2, \ldots, w_n)$ and quasi-degree $d := (d_1, d_2, \ldots, d_k)$ satisfying (2.2). Then the following properties are equivalent.*

(i) *The origin is an isolated zero of f.*

(ii) $f^{-1}(0) = \{0\}$.

Proof. It is clear that (ii) implies (i). Conversely if (ii) is not satisfied, let $a \neq 0$ be such that $f(a) = 0$. This implies that $f_1(a) = f_2(a) = \cdots = f_k(a) = 0$. Hence

$$f_i(t \bullet a) = t^{d_i} f_i(a) = 0$$

for $i = 1, 2, \ldots, k$ and all $t > 0$. Note that $\|t \bullet a\| \to 0$ as $t \to 0$. Thus, the origin is not an isolated zero of f, which is a contradiction. □

Proposition 2.5. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous quasi-homogeneous function with weight w and quasi-degree m. Suppose that $n \geq 2$. Then the following conditions are equivalent*

(i) $f^{-1}(0) = \{0\}$.
(ii) *f has a strict global extremum at the origin.*
(iii) *For each $\epsilon \geq 0$, $\{x \in \mathbb{R}^n \mid |f(x)| = \epsilon\}$ is a non-empty compact set.*
(iv) $\min_{\|x\|=1} |f(x)| > 0$.

Proof. (i) $\Rightarrow$ (ii) If $f^{-1}(0) = \{0\}$, then by connectedness f has a constant sign (for instance $f > 0$) on the unit Euclidian sphere $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ of dimension $(n-1)$ if $n > 1$. But for any $x \neq 0$, there is clearly $t > 0$ such that $y := t \bullet x$ is in $\mathbb{S}^{n-1}$. Indeed the euclidian norm of $t \bullet x$ is 0 for $t = 0$, tends to infinity with t and is a continuous function of t, hence it must take the value 1 for some finite positive t. Then $f(x) = f(t^{-1} \bullet y) = t^{-m} f(y) > 0$, which proves (ii).

(ii) $\Rightarrow$ (iv) Suppose, by contradiction, that $\min_{\|x\|=1} |f(x)| = 0$. Then there exists a point $a \in \mathbb{R}^n$ such that $\|a\| = 1$ and $f(a) = 0$. This implies that $f(t \bullet a) = 0$ for all $t > 0$, which contradicts (ii).

(iv) $\Rightarrow$ (iii) By contradiction, assume that the set $\{x \in \mathbb{R}^n \mid |f(x)| = \epsilon\}$ is not compact for some $\epsilon \geq 0$. This means that there exists a sequence $x^p \in \mathbb{R}^n, p \in \mathbb{N}$, such that $\|x^p\| \to \infty$ as $p \to \infty$ and $|f(x^p)| = \epsilon$. Let $t_p := \frac{1}{\|x^p\|_w} \to 0$. Then the sequence $|f(t_p \bullet x^p)| = t_p^m \epsilon$ tends to zero as $p \to \infty$. From the sequence of points $t_p \bullet x^p$ lying on the compact set $\{\|x\|_w = 1\}$ one can choose a subsequence convergent to some $a, \|a\|_w = 1$. Clearly, $f(a) = 0$ and $a \neq 0$, which contradicts (iv).

(iii) $\Rightarrow$ (i) If $f(a) = 0$ for some $a \neq 0$, then $f(t \bullet a) = 0$ for all $t > 0$. Consequently, by letting t tend to infinity, we obtain that the set $\{x \in \mathbb{R}^n \mid |f(x)| = 0\}$ is not compact, which contradicts (iii). □

3. The Łojasiewicz inequality for a quasi-homogeneous map which vanishes only at the origin

In this section we are interested in the first Łojasiewicz inequality which relates in general the size of $f(u)$ and the distance of u to the set $f^{-1}(0)$. However we essentially restrict our study to the case where this set is reduced to 0.

Proposition 3.1. *Let $f := (f_1, f_2, \ldots, f_k)\colon \mathbb{K}^n \to \mathbb{K}^k$ be a continuous quasi homogeneous map with weight $w := (w_1, w_2, \ldots, w_n)$ and quasi-degree $d := (d_1, d_2, \ldots, d_k)$. Then the following statements hold.*

(i) *There exists a positive constant c_1 such that*

$$\|f(x)\| \leq c_1 \|x\|_w^{d_*}, \quad \textit{as} \quad \|x\| \leq 1.$$

(ii) *If $f^{-1}(0) = \{0\}$, then there exists a positive constant c_2 such that*

$$c_2 \|x\|_w^{d^*} \leq \|f(x)\|, \quad \textit{as} \quad \|x\| \leq 1.$$

Proof. Consider the family of topological closed spheres

$$S_t := \{(x_1, x_2, \ldots, x_n) \in \mathbb{K}^n \mid \|x\|_w = t\}.$$

Then, by Proposition 2.3(i), for each $t > 0$ we have that S_t is a compact set. Let $x \in \mathbb{K}^n$ be such that $x \neq 0$ and $\|x\| \leq 1$. By Proposition 2.3(ii), we have $t := \frac{1}{\|x\|_w} \geq 1$. Note that $t \bullet x \in S_1$.

(i) Let

$$c_1 := \max_{i=1,2,\ldots,k} \max_{y \in S_1} |f_i(y)|.$$

We have

$$c_1 \geq |f_i(t \bullet x)| = |t^{d_i} f_i(x)| \quad \text{for} \quad i = 1, 2, \ldots, k.$$

It follows that

$$c_1 \|x\|_w^{d_i} \geq |f_i(x)| \quad \text{for} \quad i = 1, 2, \ldots, k.$$

Consequently,

$$c_1 \|x\|_w^{d_*} = c_1 \max_{i=1,2,\ldots,k} \|x\|_w^{d_i} \geq \|f(x)\|.$$

which proves (i).

(ii) Let $c_2 := \min_{y \in S_1} \|f(y)\| > 0$. By definition, we have

$$\begin{aligned} c_2 \quad &\leq \quad \|f(t \bullet x)\| = \max_{i=1,2,\ldots,k} |t^{d_i} f_i(x)| \ \leq \max_{i=1,2,\ldots,k} |t^{d_i}| \max_{i=1,2,\ldots,k} |f_i(x)| \\ &\leq \quad |t^{d^*}| \|f(x)\| \ = \ \frac{1}{\|x\|_w^{d^*}} \|f(x)\|, \end{aligned}$$

which proves (ii). The proposition is proved. □

Theorem 3.2. *Let $f := f^0 + f^1 + \cdots + f^l \colon \mathbb{K}^n \to \mathbb{K}^k$, where $f^0, f^1, \ldots, f^l$ are continuous quasi-homogeneous maps with weight $w = (w_1, w_2, \ldots, w_n)$ and quasi-degrees $d^0, d^1, \ldots, d^l$ respectively such that*

$$(d^0)^* < (d^1)_* \leq (d^2)_* \leq \cdots \leq (d^l)_*.$$

If the origin is an isolated zero of f^0 then there exists a positive constant c such that

$$c\|x\|_w^{(d^0)^*} \leq \|f(x)\|, \quad \textit{as} \quad \|x\| \ll 1.$$

Proof. By Proposition 3.1(ii), there exists a positive constant c_0 such that

$$c_0 \|x\|_w^{(d^0)^*} \leq \|f^0(x)\|, \quad \text{as} \quad \|x\| \ll 1.$$

On the other hand, from Proposition 3.1(i), there exist positive constants $c_1, c_2, \ldots, c_l$ such that for $i = 1, 2, \ldots, l$,

$$\|f^i(x)\| \leq c_i \|x\|_w^{(d^i)_*}, \quad \text{as} \quad \|x\| \ll 1.$$

We have for $\|x\| \ll 1$ the next estimate

$$\begin{aligned} \|f^1(x) + f^2(x) + \cdots + f^l(x)\| \quad &\leq \quad \|f^1(x)\| + \|f^2(x)\| + \cdots + \|f^l(x)\| \\ &\leq \quad c_1 \|x\|_w^{(d^1)_*} + c_2 \|x\|_w^{(d^2)_*} + \cdots + c_l \|x\|_w^{(d^l)_*}. \end{aligned}$$

Thus it follows from $(d^0)^* < (d^1)_* \leq (d^2)_* \leq \cdots \leq (d^l)_*$ that

$$\|f^1(x) + f^2(x) + \cdots + f^l(x)\| \quad \ll \quad \|x\|_w^{(d^0)^*}, \quad \text{as} \quad \|x\| \ll 1.$$

Therefore, we have for $\|x\| \ll 1$ the next inequality

$$\begin{aligned} \|f(x)\| \quad &\geq \quad \|f^0(x)\| - \|f^1(x) + f^2(x) + \cdots + f^l(x)\| \\ &\geq \quad c_0 \|x\|_w^{(d^0)^*} - c' \|x\|_w^{(d^0)^*} \qquad (0 < c' \ll c_0). \end{aligned}$$

This gives
$$\|f(x)\| \geq (c_0 - c')\|x\|_w^{(d^0)^*}, \quad \text{as} \quad \|x\| \ll 1,$$
which proves the theorem. □

The following is a direct consequence from Proposition 3.1 and Theorem 3.2:

Corollary 3.3. *Under the hypothesis of Theorem 3.2, there exists a positive constant c such that*
$$c\|x\|^{\frac{(d^0)^*}{w_*}} \leq \|f(x)\|, \quad \textit{as} \quad \|x\| \ll 1.$$

We define the *Łojasiewicz exponent* $\alpha_0(f)$ of the map f at the origin $0 \in \mathbb{K}^n$ as the infimum of the set of all real numbers $l > 0$ which satisfy the condition: there exists a positive constant c such that
$$c\|x\|^l \leq \|f(x)\|, \quad \text{as} \quad \|x\| \ll 1.$$
If the set of all the exponents is empty we put $\alpha_0(f) := +\infty$.

Corollary 3.4. *Let $f := (f_1, f_2, \ldots, f_k) \colon \mathbb{K}^n \to \mathbb{K}^k$ be a continuous quasi-homogeneous map with weight $w := (w_1, w_2, \ldots, w_n)$ and quasi-degree $d := (d_1, d_2, \ldots, d_k)$. Suppose that $f^{-1}(0) = \{0\}$. Then*
$$\frac{d_*}{w_*} \ \leq \ \alpha_0(f) \ \leq \ \frac{d^*}{w_*}.$$

Proof. It follows from Corollary 3.3 that
$$\alpha_0(f) \ \leq \ \frac{d^*}{w_*}.$$

In order to prove the left inequality, let i, j be such that $d_i = d_*$ and $w_j = w_*$. Take $a \in \mathbb{K}^n$ with the property that $a_j f_i(a) \neq 0$. Then, asymptotically as $t \to 0$, we have[1]
$$\begin{aligned} \|f(t \bullet a)\| &\simeq t^{d_*}, \\ \|t \bullet a\| &\simeq t^{w_*}. \end{aligned}$$
Consequently,
$$\|f(t \bullet a)\| \simeq \|t \bullet a\|^{\frac{d_*}{w_*}}.$$
By the definition of the Łojasiewicz exponent $\alpha_0(f)$, we find that
$$\frac{d_*}{w_*} \ \leq \ \alpha_0(f).$$
□

Example 3.5. (i) Let $f := (f_1 := x^2 + y^4, f_2 := (x^2 - y^4)^2) \colon \mathbb{R}^2 \to \mathbb{R}^2$. It is easy to check that f is a positive quasi-homogeneous map with weight $w := (2, 1)$ and quasi-degree $d := (4, 8)$. Moreover, $\alpha_0(f) = 4$ $(= \frac{d_*}{w_*})$.

(ii) Let $f := (f_1 := x^2 - y^4, f_2 := (x^2 + y^4)^2) \colon \mathbb{R}^2 \to \mathbb{R}^2$. Then f is a positive quasi-homogeneous map with weight $w := (2, 1)$ and quasi-degree $d := (4, 8)$. Moreover,
$$\alpha_0(f) = 8 \ (= \frac{d^*}{w_*}).$$

Corollary 3.6. *Let $f \colon \mathbb{K}^n \to \mathbb{K}$ be a continuous quasi-homogeneous function with weight w and quasi-degree m. If $f^{-1}(0) = \{0\}$ then*
$$\alpha_0(f) = \frac{m}{w_*}.$$

[1] Where $A \simeq B$ means that A/B lies between two positive constants.

Proof. The claim comes from $d^* = d_* = m$. □

4. The Łojasiewicz gradient inequality for quasi-homogeneous polynomials

We now consider the case $k = 1$ and let $f\colon \mathbb{K}^n \to \mathbb{K}$ be a C^1 quasi-homogeneous function with weight $w := (w_1, w_2, \ldots, w_n)$ and quasi-degree m :

$$f(t^{w_1}x_1, t^{w_2}x_2, \ldots, t^{w_n}x_n) = t^m f(x_1, x_2, \ldots, x_n).$$

We define the *Łojasiewicz gradient exponent* $\beta_0(f)$ of the map f at the origin $0 \in \mathbb{K}^n$ as the infimum of the set of all real numbers $l > 0$ which satisfy the condition: there exists a positive constant c such that

$$c|f(x)|^l \le \|\nabla f(x)\| \quad \text{for} \quad \|x\| \ll 1.$$

If the set of all the exponents is empty we put $\beta_0(f) := +\infty$. It is well-known (see [14]) that if f is analytic, then $\beta_0(f) < 1$.

We start with a general result valid for C^1 functions.

Theorem 4.1. *Let $f\colon \mathbb{K}^n \to \mathbb{K}$ be a C^1 quasi-homogeneous function with weight w and quasi-degree m. Then*

$$\max\{0, 1 - \frac{w^*}{m}\} \ \le \ \beta_0(f) \le 1.$$

Proof. Since f is a C^1-positive quasi-homogeneous function with weight w and quasi-degree $m \ge w^*$,

$$mt^{m-1}f(x) \quad = \quad \sum_{j=1}^n w_j t^{w_j-1} x_j \frac{\partial f}{\partial x_j}(t \bullet x).$$

In particular, we have the generalized Euler identity

$$mf(x) \quad = \quad \sum_{j=1}^n w_j x_j \frac{\partial f}{\partial x_j}(x). \tag{4.1}$$

As a consequence, there exists a positive constant c_1 such that

$$c_1|f(x)| \quad \le \quad \|x\|\|\nabla f(x)\| \quad \text{for all} \quad \|x\| \ll 1.$$

This implies that $\beta_0(f) \le 1$.

Next, let $l > 0$ and $c_2 > 0$ be such that

$$c_2|f(x)|^l \quad \le \quad \|\nabla f(x)\| \quad \text{for all} \quad \|x\| \ll 1. \tag{4.2}$$

We have for all $t > 0$ the following relation

$$t^{w_j}\frac{\partial f}{\partial x_j}(t \bullet x) \quad = \quad t^m \frac{\partial f}{\partial x_j}(x) \quad \text{for} \quad j = 1, 2, \ldots, n.$$

This shows that

$$\frac{\partial f}{\partial x_j}(t \bullet x) \quad = \quad t^{m-w_j}\frac{\partial f}{\partial x_j}(x) \quad \text{for} \quad j = 1, 2, \ldots, n. \tag{4.3}$$

Then we have for all $t > 0$ the following equations

$$\begin{aligned} f(t \bullet x) \quad &= \quad t^m f(x), \\ \frac{\partial f}{\partial x_j}(t \bullet x) \quad &= \quad t^{m-w_j}\frac{\partial f}{\partial x_j}(x) \quad \text{for} \quad j = 1, 2, \ldots, n. \end{aligned}$$

Since $f \not\equiv 0$ there is $a \in \mathbb{K}^n$ such that $0 \neq \nabla f^2(a) = 2f(a)\nabla f(a)$. This implies $f(a) \neq 0$ and $\nabla f(a) \neq 0$. Then, asymptotically as $t \to +0$, we have

$$\begin{array}{rcl} |f(t \bullet a)| & \simeq & t^m, \\ \|\nabla f(t \bullet a)\| & \simeq & t^{m-w_j} \quad \text{for some} \quad j \in \{1, 2, \ldots, n\}, \end{array}$$

Therefore (4.2) implies the existence of $c_3 > 0$ such that

$$c_3 t^{ml} \ \leq \ t^{m-w_j} \ \leq \ t^{m-w^*}$$

as $t \to +0$. This in turn implies that $ml \geq m - w^*$, which is equivalent to

$$l \geq 1 - \frac{w^*}{m}.$$

This, together with the definition of $\beta_0(f)$, implies the desired result. □

In the special case where 0 is the only critical point of f we have a more precise estimation as follows.

Corollary 4.2. *Let $f\colon \mathbb{K}^n \to \mathbb{K}$ be a C^1 quasi-homogeneous function with weight w and quasi-degree $m \geq w^*$. Suppose that $\nabla f^{-1}(0) = \{0\}$. Then*

$$1 - \frac{w^*}{m} \ \leq \ \beta_0(f) \ \leq \ 1 - \frac{w_*}{m}.$$

Proof. One has only to show that

$$\beta_0(f) \ \leq \ 1 - \frac{w_*}{m}.$$

Indeed, by the generalized Euler identity (4.1), there exists a positive constant c_1 such that

$$|f(x)| \leq c_1 \|\nabla f(x)\| \|x\|.$$

On the other hand, it follows from (4.3) that the following

$$\nabla f(x)\colon \mathbb{K}^n \to \mathbb{K}^n, \quad x \mapsto \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right),$$

is a continuous quasi-homogeneous map with weight w and quasi-degree

$$(m - w_1, m - w_2, \ldots, m - w_n).$$

Therefore, by Proposition 2.3 and then Proposition 3.1(ii), for all $\|x\| \ll 1$ we have

$$\begin{array}{rcl} |f(x)| & \leq & c_1 \|\nabla f(x)\| \|x\|_w^{w_*} \\ & \leq & c_2 \|\nabla f(x)\| \|\nabla f(x)\|^{\frac{w_*}{m-w_*}} \ = \ c_2 \|\nabla f(x)\|^{1+\frac{w_*}{m-w_*}}, \end{array}$$

for some $c_2 > 0$. Hence there exists a positive constant c such that

$$c|f(x)|^{1-\frac{w_*}{m}} \leq \|\nabla f(x)\| \quad \text{for} \quad \|x\| \ll 1.$$

Consequently, by the definition of the Łojasiewicz gradient exponent $\beta_0(f)$, we obtain

$$\beta_0(f) \ \leq \ 1 - \frac{w_*}{m},$$

which completes the proof. □

Example 4.3. (i) (see [10]). Let $f\colon \mathbb{C}^2 \to \mathbb{C}, (x, y) \mapsto x^3+3xy^k, k \geq 3$, be a complex polynomial. It is clear that f is a quasi-homogeneous polynomial with weight $w = (k, 2)$ and quasi-degree $m = 3k$. A direct computation shows that the origin in $\mathbb{C}^2$ is an isolated critical point of f. Moreover, it follows from the results in [10] and [18] that

$$\begin{aligned}\alpha_0(\nabla f) &= \frac{3k}{2} - 1,\\ \beta_0(f) &= \frac{\alpha_0(\nabla f)}{1+\alpha_0(\nabla f)} = 1 - \frac{2}{3k}.\end{aligned}$$

(ii) Let $f\colon \mathbb{C}^2 \to \mathbb{C}, (x, y) \mapsto x^4 - 4xy$, be a complex polynomial. It is clear that f is a quasi-homogeneous polynomial with weight $w = (1, 3)$ and quasi-degree $m = 4$. A direct computation shows that the origin in $\mathbb{C}^2$ is an isolated critical point of f. Moreover, it follows from the results in [10] and [18] that

$$\begin{aligned}\alpha_0(\nabla f) &= 1,\\ \beta_0(f) &= \frac{\alpha_0(\nabla f)}{1+\alpha_0(\nabla f)} = \frac{1}{2}.\end{aligned}$$

Remark 4.4. Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a complex polynomial function with an isolated singularity at 0. Then from the works of Teissier [18, Corollary 2] we have the following equation

$$\beta_0(f) = \frac{\alpha_0(\nabla f)}{1+\alpha_0(\nabla f)}.$$

Moreover, Gwoździewicz has remarked, [6], that the above relation fails to hold for some real polynomial functions with an isolated singularity at 0. However, we have the following.

Corollary 4.5. (see also [6, Theorem 1.3]) *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a quasi-homogeneous polynomial function with weight w and quasi-degree $m \geq w^*$. If $f^{-1}(0) = \{0\}$, then*

$$\begin{aligned}\alpha_0(f) &= \frac{m}{w_*},\\ \alpha_0(\nabla f) &= \alpha_0(f) - 1 = \frac{m}{w_*} - 1,\\ \beta_0(f) &= \frac{\alpha_0(\nabla f)}{1+\alpha_0(\nabla f)} = 1 - \frac{w_*}{m}.\end{aligned}$$

Proof. In fact, by Corollary 3.6, we have

$$\alpha_0(f) = \frac{m}{w_*}.$$

Then the remained relations follow from [6, Theorem 1.3]. We will give below a direct proof in order to keep our paper self-contained.

We first note that the origin is an isolated critical point of f. Indeed, if $\nabla f(a) = 0$ for some $a \neq 0$, then it follows easily from the generalized Euler identity (4.1) that $f(a) = 0$, which is a contradiction.

Without loss of generality, we may suppose that $f(x) \geq 0$ for all $x \in \mathbb{R}^n$ with equality if and only if $x = 0$. For each $\delta > 0$, the restriction of f to the sphere $\{x \in \mathbb{R}^n \mid \|x\|_E = \delta\}$ attains its minimum at least one point, where $\|x\|_E := \sqrt{\sum_{i=1}^n x_i^2}$. Let

$$\Gamma := \{u \in \mathbb{R}^n \mid f(u) = \min_{\|x\|_E = \|u\|_E} f(x)\}.$$

It follows from the Tarski-Seidenberg theorem (see, for example, [3, Theorem 2.3.4]), that Γ is semi-algebraic. Hence the Curve Selection Lemma [15] is applicable. Together with Lagrange's

Multipliers Theorem, this implies that there exists an analytic map

$$(\lambda, \varphi)\colon (-\epsilon, \epsilon) \to \mathbb{R} \times \mathbb{R}^n, \tau \mapsto (\lambda(\tau), \varphi(\tau)),$$

such that

(i) $\varphi(\tau) = 0$ if and only if $\tau = 0$;
(ii) $\varphi(\tau) \in \Gamma$ for all $\tau \in [0, \epsilon)$; and
(iii) $\nabla f(\varphi(\tau)) = \lambda(\tau)\varphi(\tau)$ for all $\tau \in [0, \epsilon)$.

Let $at^p, a > 0$, be the leading term of the Taylor expansion of $\|\varphi(\tau)\|^2$, and $bt^q, b \neq 0$, be that of $|f(\varphi(\tau))|^2$. Then, asymptotically as $t \to 0$, we have

$$|f(\varphi(\tau))| \simeq \|\varphi(\tau)\|^{\frac{q}{p}}.$$

Consequently, by the definition of $\alpha_0(f)$, we get

$$\alpha_0(f) \geq \frac{q}{p}.$$

On the other hand, we may assume (taking $\epsilon > 0$ small enough if necessary) that the function $\tau \mapsto \|\varphi(\tau)\|$ is strictly increasing. Together with the condition (i), we find that for each $x \in \mathbb{R}^n, \|x\| \ll 1$, there exists a positive number $\tau \in [0, \epsilon)$ satisfying the relation $\|\varphi(\tau)\|_E = \|x\|_E$. Hence,

$$|f(x)| = f(x) \geq \min_{\|u\|_E = \|x\|_E} f(u) = f(\varphi(\tau)) \simeq \|\varphi(\tau)\|^{\frac{q}{p}} = \|x\|^{\frac{q}{p}}.$$

By the definition, thus

$$\alpha_0(f) \leq \frac{q}{p}.$$

Therefore,

$$\alpha_0(f) = \frac{q}{p}.$$

Moreover, it follows from the generalized Euler identity (4.1) that

$$mf(\varphi(\tau)) = \sum_{j=1}^{n} w_j \varphi_j(\tau) \frac{\partial f}{\partial x_j}(\varphi(\tau)).$$

By the condition (iv), hence

$$\begin{aligned}
|mf(\varphi(\tau))| &= |\lambda(\tau)| \sum_{j=1}^{n} w_j [\varphi_j(\tau)]^2 \\
&= \frac{\|\nabla f(\varphi(\tau))\|}{\|\varphi(\tau)\|} \sum_{j=1}^{n} w_j [\varphi_j(\tau)]^2 \\
&\simeq \|\varphi(\tau)\| \|\nabla f(\varphi(\tau))\|.
\end{aligned}$$

In particular, we get

$$\|\nabla f(\varphi(\tau))\| \simeq \|\varphi(\tau)\|^{\frac{q}{p}-1} \simeq |f(\varphi(\tau))|^{1-\frac{p}{q}}.$$

By definitions, hence

$$\begin{aligned}
\alpha_0(\nabla f) &\geq \frac{q}{p} - 1 = \alpha_0(f) - 1 = \frac{m}{w_*} - 1, \\
\beta_0(f) &\geq 1 - \frac{p}{q} = 1 - \frac{1}{\alpha_0(f)} = 1 - \frac{w_*}{m}.
\end{aligned}$$

Then the corollary follows immediately from Corollaries 3.4 and 4.2. □

The following result is of general interest but we shall only use it to prove Theorem 4.7 below.

Lemma 4.6. *Let $f\colon \mathbb{K}^n \to \mathbb{K}$ be a C^1-function. For each k positive integer, consider the function $\widetilde{f}\colon \mathbb{K}^n \to \mathbb{K}, x \mapsto [f(x)]^k$. Suppose that there exist $c > 0$ and $\theta \in (0, 1]$ such that*

$$c|\widetilde{f}(x)|^{1-\theta} \le \|\nabla \widetilde{f}(x)\| \quad \textit{for} \quad \|x\| \ll 1.$$

Then

$$\frac{c}{k}|f(x)|^{1-k\theta} \le \|\nabla f(x)\| \quad \textit{for} \quad \|x\| \ll 1.$$

Proof. We have

$$\nabla \widetilde{f}(x) = k[f(x)]^{k-1}\nabla f(x).$$

Hence

$$c|[f(x)]^k|^{1-\theta} \le k|[f(x)]^{k-1}|\|\nabla f(x)\|.$$

This implies

$$\frac{c}{k}|f(x)|^{1-k\theta} \le \|\nabla f(x)\|,$$

concluding the proof of the lemma. □

The following is a generalization of [7, Theorem 2.1].

Theorem 4.7. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a quasi-homogeneous polynomial function with weight $w := (w_1, w_2)$ and quasi-degree m. Then there exists a positive constant c such that*

$$c|f(x,y)|^{1-\frac{w_*}{m}} \le \|\nabla f(x,y)\|, \quad \textit{as} \quad \|(x,y)\| \ll 1. \tag{4.4}$$

Proof. Without loss of generality, we may suppose that

$$1 \le w_* = w_1 \le w_2.$$

There are two cases to be considered.

Case 1. *m is divisible by w_1; i.e., $q := \frac{m}{w_1}$ is a positive integer number.*

Consider the following function

$$g(x,y) := f(x, y^{\frac{w_2}{w_1}}).$$

Then, by Proposition 2.1, we can see that g is a homogeneous polynomial on $\mathbb{R} \times \mathbb{R}_+$ of degree $q = \frac{m}{w_1}$. Indeed we can write for some finite set $S \subset \mathbb{N} \times \mathbb{N}$:

$$f(x,y) = \sum_{\alpha:=(\alpha_1,\alpha_2)\in S} a_\alpha x^{\alpha_1} y^{\alpha_2}$$

with

$$w_1\alpha_1 + w_2\alpha_2 = m = qw_1.$$

Hence

$$\alpha_2 = (q - \alpha_1)\frac{w_1}{w_2};$$

and therefore

$$f(x,y) = \sum_{\alpha\in S} a_\alpha x^{\alpha_1} y^{(q-\alpha_1)\frac{w_1}{w_2}}$$

which provides

$$g(x,y) = \sum_{\alpha\in S} a_\alpha x^{\alpha_1} y^{q-\alpha_1}.$$

It now follows from [7, Theorem 2.1] that there exists a positive constant c such that

$$c|g(x,y)|^{1-\frac{w_1}{m}} \le \|\nabla g(x,y)\|, \quad \text{as} \quad \|(x,y)\| \ll 1 \;\text{ and }\; y \ge 0.$$

On the other hand, by the definition

$$\begin{aligned}\frac{\partial g}{\partial x}(x,y) &= \frac{\partial f}{\partial x}(x, y^{\frac{w_2}{w_1}}),\\ \frac{\partial g}{\partial y}(x,y) &= \frac{w_2}{w_1} y^{\frac{w_2}{w_1}-1}\frac{\partial f}{\partial y}(x, y^{\frac{w_2}{w_1}}).\end{aligned}$$

Therefore, asymptotically as $(x, y) \to (0, 0)$ and $y \geq 0$,

$$\begin{aligned}c|f(x, y^{\frac{w_2}{w_1}})|^{1-\frac{w_1}{m}} &\leq \left\|\left(\frac{\partial f}{\partial x}(x, y^{\frac{w_2}{w_1}}), \frac{w_2}{w_1} y^{\frac{w_2}{w_1}-1}\frac{\partial f}{\partial y}(x, y^{\frac{w_2}{w_1}})\right)\right\|.\\ &\leq \left\|\left(\frac{\partial f}{\partial x}(x, y^{\frac{w_2}{w_1}}), \frac{\partial f}{\partial y}(x, y^{\frac{w_2}{w_1}})\right)\right\|,\end{aligned}$$

because $w_2 \geq w_1$.

Let $u := y^{\frac{w_2}{w_1}} \geq 0$. Then

$$c|f(x,u)|^{1-\frac{w_1}{m}} \leq \left\|\left(\frac{\partial f}{\partial x}(x,u), \frac{\partial f}{\partial y}(x,u)\right)\right\|.$$

By an entirely analogous argument but replacing $g(x, y) = f(x, y^{\frac{w_2}{w_1}})$ by $f(x, -y^{\frac{w_2}{w_1}})$ we can show that the above inequality also holds for all $u \leq 0$. These prove the theorem in Case 1.

Case 2. *m is not divisible by w_1.*

Let $\widetilde{f}(x, y) := [f(x, y)]^{w_1}$. Then it is clear that $\widetilde{f}(x, y)$ is a positive quasi-homogeneous polynomial with weight (w_1, w_2) and quasi-degree $\widetilde{m} := mw_1$. Since $\frac{\widetilde{m}}{w_1} = m$ is an integer number, by applying Case 1 for the polynomial $\widetilde{f}$ we get

$$\widetilde{c}|\widetilde{f}(x,y)|^{1-\frac{w_1}{\widetilde{m}}} \leq \|\nabla \widetilde{f}(x,y)\|, \quad \text{as} \quad \|(x,y)\| \ll 1,$$

for some $\widetilde{c} > 0$.

By Lemma 4.6, we get

$$\frac{\widetilde{c}}{w_1}|f(x,y)|^{1-\frac{w_1}{m}} \leq \|\nabla f(x,y)\|,$$

which completes the proof of the theorem. □

Remark 4.8. As we see in the next proposition, the result of Theorem 4.7 is no longer valid in dimensions $n > 2$.

Proposition 4.9. (Compare with [7, Remark 2.4]) *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by*

$$f(x,y,z) := x^4 + x^2z^2 - 2xy^2z + y^4 = x^4 + (xz - y^2)^2.$$

Then there exists a curve $\varphi : [0, \epsilon) \to \mathbb{R}^n, t \mapsto \varphi(t)$, such that

$$\|\nabla f[\varphi(t)]\| \ll |f[\varphi(t)]|^{1-\frac{1}{4}} \quad \textit{for} \quad 0 < t \ll 1.$$

In particular, $\beta_0(f) > 1 - \frac{w_*}{m} = 1 - \frac{1}{4}$.

Proof. It is clear that f is a weighted quasi-homogeneous polynomial with weight $w = (1, 1, 1)$ and quasi-degree $m = 4$. Moreover, f has non-isolated zero at the origin; namely,

$$f^{-1}(0) = \{(0, 0, t) \mid t \in \mathbb{R}\}.$$

Define the polynomial curve $\varphi : [0, \epsilon) \to \mathbb{R}^3, t \mapsto (x(t), y(t), z(t))$, by

$$\begin{aligned}x(t) &:= t^2,\\ y(t) &:= t + t^5,\\ z(t) &:= 1.\end{aligned}$$

One easily verifies that

$$\begin{aligned} f[\varphi(t)] &= t^8+4t^{12}+4t^{16}+t^{20},\\ \frac{\partial f}{\partial x}[\varphi(t)] &= -2t^{10},\\ \frac{\partial f}{\partial y}[\varphi(t)] &= 8t^7+12t^{11}+4t^{15},\\ \frac{\partial f}{\partial z}[\varphi(t)] &= -4t^8-2t^{12}. \end{aligned}$$

Hence, asymptotically as $t\to 0$,

$$\|\nabla f[\varphi(t)]\| \simeq t^7 \ll t^6 \simeq |f[\varphi(t)]|^{1-\frac{1}{4}}.$$

This completes the proof. □

Remark 4.10. The polynomial $x^4+x^2z^2-2xy^2z+y^4$ in the above proposition is a homogenization of $x^4+(x-y^2)^2$ by the new variable z. The last one is a polynomial in the class of polynomials which was considered by János Kollár ([8]).

The following is a direct consequence of Theorem 4.1 and Theorem 4.7.

Corollary 4.11. *Let $f\colon \mathbb{R}^2\to\mathbb{R}$ be a quasi-homogeneous polynomial function with weight $w:=(w_1,w_2)$ and quasi-degree m. Then*

$$\max\{0, 1-\frac{w^*}{m}\} \le \beta_0(f) \le 1-\frac{w_*}{m}.$$

5. Additional results, remarks and examples in dimension 2

In this section we will denote by $f\colon\mathbb{R}^2\to\mathbb{R}$ a quasi-homogeneous polynomial with weight $w=(w_1,w_2)$ and quasi-degree m such that

$$w_*=w_1\le w_2=w^*\le m.$$

We now apply Corollary 4.11 in special cases.

Corollary 5.1. *If the origin is an isolated zero of f_y then*

$$\beta_0(f)=1-\frac{w_2}{m}.$$

Proof. It is well known that f_y is quasi-homogeneous polynomial with weight $w=(w_1,w_2)$ and quasi-degree $m-w_2$. Since $f_y^{-1}(0)=\{0\}$, it follows from Proposition 2.5 that the polynomial f_y has a strict global extremum at the origin. Thus we can assume that $f_y>0$ on $\mathbb{R}^n-\{0\}$. By again Proposition 2.5, the set $\{f_y=1\}$ is nonempty compact. Hence

$$\infty > c := \max_{f_y(u,v)=1}|f(u,v)| > 0.$$

Take any $(x,y)\in\mathbb{R}^2$. Let $\epsilon := [f_y(x,y)]^{\frac{1}{m-w_2}}$. Then $\{(u,v)\in\mathbb{R}^2 \mid f_y(u,v)=\epsilon^{m-w_2}\}$ is a non-empty compact set. Moreover,

$$\begin{aligned} |f(x,y)| \le \max_{f_y(u,v)=\epsilon^{m-w_2}}|f(u,v)| &= \max_{f_y(\epsilon^{-w_1}u,\epsilon^{-w_2}v)=1}|f(u,v)|\\ &= \max_{f_y(\tilde u,\tilde v)=1}|f(\epsilon^{w_1}\tilde u,\epsilon^{w_2}\tilde v)|\\ &= \max_{f_y(\tilde u,\tilde v)=1}|f(\tilde u,\tilde v)|\epsilon^m\\ &= c[f_y(x,y)]^{\frac{m}{m-w_2}}. \end{aligned}$$

This gives

$$f_y(x,y) \geq c'|f(x,y)|^{1-\frac{w_2}{m}},$$

here $c' := c^{1-\frac{w_2}{m}} > 0$. Therefore

$$\|\nabla f(x,y)\| \geq |f_y(x,y)| \geq c'|f(x,y)|^{1-\frac{w_2}{m}}.$$

By the definition of $\beta_0(f)$, we get

$$\beta_0(f) \leq 1 - \frac{w_2}{m}.$$

Then, by Corollary 4.11, $\beta_0(f) = 1 - \frac{w_2}{m}$. □

Example 5.2. Let $f\colon \mathbb{R}^2 \to \mathbb{R}, (x,y) \mapsto y^3 + 3x^4y + 2x^6$ be a real polynomial. It is clear that f is a quasi-homogeneous polynomial with weight $w = (1,2)$ and quasi-degree $m = 6$. A direct computation shows that $f_y^{-1}(0) = \{(0,0)\}$. Hence $\beta_0(f) = 1 - \frac{2}{6} = \frac{2}{3}$.

Corollary 5.3. *Suppose that the origin is not an isolated zero of f_y. If there exists $(a,b) \in \mathbb{R}^2$ such that $f(a,b) \neq 0, f_x(a,b) \neq 0$ and $f_y(a,b) = 0$, then*

$$\beta_0(f) = 1 - \frac{w_1}{m}.$$

Proof. By the hypothesis, we have for all $t > 0$

$$\begin{array}{rcl} f(t^{w_1}a, t^{w_2}b) & \simeq & t^m, \\ f_x(t^{w_1}a, t^{w_2}b) & \simeq & t^{m-w_1}, \\ f_y(t^{w_1}a, t^{w_2}b) & \equiv & 0. \end{array}$$

Asymptotically as $t \to 0$, hence

$$\begin{array}{rcl} |f(t^{w_1}a, t^{w_2}b)| & \simeq & t^m, \\ \|\nabla f(t^{w_1}a, t^{w_2}b)\| & \simeq & t^{m-w_1}. \end{array}$$

This implies that

$$\|\nabla f(t^{w_1}a, t^{w_2}b)\| \quad \simeq \quad |f(t^{w_1}a, t^{w_2}b)|^{1-\frac{w_1}{m}}.$$

Then, by the definition of $\beta_0(f)$,

$$\beta_0(f) \geq 1 - \frac{w_1}{m}.$$

On the other hand, by Corollary 4.11, $\beta_0(f) \leq 1 - \frac{w_1}{m}$. Therefore $\beta_0(f) = 1 - \frac{w_1}{m}$. □

Example 5.4. Let $f\colon \mathbb{R}^2 \to \mathbb{R}, (x,y) \mapsto x^2y - y^2$ be a real polynomial. It is clear that f is a quasi-homogeneous polynomial with weight $w = (1,2)$ and quasi-degree $m = 4$. A direct computation shows that $f_y^{-1}(0) = \{x^2 - 2y = 0\}$ and the origin in $\mathbb{R}^2$ is an isolated critical point of f. Moreover, it is easy to see that the conditions of Corollary 5.3 are satisfied. Hence, $\beta_0(f) = 1 - \frac{1}{4} = \frac{3}{4}$.

Remark 5.5. (i) All results in this paper allow to compute the Łojasiewicz exponents for some functions which are not quasi-homogeneous, for instance, the function $f(x) := P(Ax)$, where P is a quasi-homogeneous polynomial of two variables and A a nonsingular 2×2 square matrix. As an example the polynomial $P(x,y) := ax^4 + by^2 + cx^2y$ is quasi-homogeneous with weight $(1,2)$ and quasi-degree 4. The polynomial $Q(x,y) = P(x, x+y)$ is not quasi-homogeneous if $bc \neq 0$.

(ii) On the other hand, there are of course polynomials of two variables which cannot be put in the form $= P(Ax)$ with, P, A as above. For instance the polynomial $Q(x,y) := x^2(1+y)$ is such that no polynomial $P = Q \circ A$ with A a nonsingular 2×2 square matrix is quasi-homogeneous.

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L^2-RIEMANN-ROCH FOR SINGULAR COMPLEX CURVES

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Abstract. We present a comprehensive L^2-theory for the $\overline{\partial}$-operator on singular complex curves, including L^2-versions of the Riemann-Roch theorem and some applications.

1. Introduction

The L^2-theory for the $\overline{\partial}$-operator is one of the central tools in complex analysis on complex manifolds, but still not very well developed on singular complex spaces. Just recently, considerable progress has been made in understanding the L^2-cohomology of singular complex spaces with isolated singularities. Let X be a Hermitian complex space of pure dimension n and with isolated singularities only. For simplicity, we assume that X is compact. Let $H^{p,q}_w(X^*)$ be the L^2-Dolbeault cohomology on the level of (p,q)-forms with respect to the $\overline{\partial}$-operator in the sense of distributions, denoted by $\overline{\partial}_w$ in the following, computed on $X^* = \operatorname{Reg} X$. Let $\pi : M \to X$ be a resolution of singularities with snc exceptional divisor, $Z := \pi^{-1}(\operatorname{Sing} X)$ the unreduced exceptional divisor. Then it has been shown by Øvrelid, Vassiliadou [ØV13] and the first author [Rup11, Rup14] by different approaches that there exists an effective divisor $D \geq Z - |Z|$ on M such that there are natural isomorphisms

$$\begin{aligned} H^{n,q}_w(X^*) &\cong H^{n,q}(M)\ , \\ H^{0,q}_w(X^*) &\cong \frac{H^q\big(M, \mathcal{O}(D)\big)}{H^q_{|Z|}\big(M, \mathcal{O}(D)\big)} \end{aligned} \tag{1.1}$$

for all $0 \leq q \leq n$. Here, $H^q_{|Z|}$ denotes the cohomology with support on $|Z|$. If $\dim X \leq 2$, then (1.1) holds with the divisor $D = Z - |Z|$ and $H^q_{|Z|}\big(M, \mathcal{O}(Z - |Z|)\big) = \{0\}$, so that (1.1) gives a nice smooth representation of the L^2-cohomology groups $H^{0,q}_w(X^*)$. In case $\dim X > 2$, it is conjectured that (1.1) holds with $D = Z - |Z|$ (see [Rup11]).

However, the L^2-theory for the $\overline{\partial}$-operator developed in [ØV13] and [Rup11, Rup14] applies only to $\dim X \geq 2$ (for $\dim X = 1$, (1.1) has been known before, see [Par89, PS91]). The purpose of the present paper is to give a complete L^2-theory for the $\overline{\partial}$-operator on a singular complex curve, including L^2-versions of the Riemann-Roch theorem, and to understand the appearance of the divisor $Z - |Z|$ in the case $\dim X = 1$.

Let us explain some of our results in detail. Let X be a Hermitian singular complex space[1] of dimension 1, i.e., a Hermitian complex curve, and $L \to X$ a Hermitian holomorphic line bundle. Let $\overline{\partial}_w : L^{p,q}(X^*, L) \to L^{p,q+1}(X^*, L)$ denote the weak extension of the Cauchy-Riemann operator $\overline{\partial} : \mathscr{D}^{p,q}(X^*, L) \to \mathscr{D}^{p,q+1}(X^*, L)$, i.e., the $\overline{\partial}$-operator in the sense of distributions. Here, $\mathscr{D}^{p,q}(X^*, L) := \mathscr{C}^\infty_{\text{cpt}}(X^*, \Lambda^{p,q} T^* X^* \otimes L)$ denotes the set of smooth differential forms with compact support in X^* and values in L, and $L^{p,q}(X^*, L)$ is the set of square-integrable forms with values in L and respect to the Hermitian metrics on X^* and L.

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[1] A Hermitian complex space (X, g) is a reduced complex space X with a metric g on the regular part such that the following holds: If $x \in X$ is an arbitrary point there exist a neighborhood $U = U(x)$, a holomorphic embedding of U into a domain G in $\mathbb{C}^N$ and an ordinary smooth Hermitian metric in G whose restriction to U is $g|_U$.

Let $H^{p,q}_w(X^*, L)$ denote the L^2-Dolbeault cohomology on X^* with respect to $\overline{\partial}_w$ and

$$h^{p,q}_w(X^*, L) := \dim H^{p,q}_w(X^*, L).$$

Note that the genus $g = g(X)$ of X and the degree $\deg(L)$ of L are well-defined, even in the presence of singularities (see Section 2.2). For a singular point $x \in \operatorname{Sing} X$, we define its modified multiplicity $\operatorname{mult}'_x X$ as follows: Let X_j, $j = 1, ..., m$, be the irreducible components of X in the singular point x. Then

$$\operatorname{mult}'_x X := \sum_{j=1}^{m} (\operatorname{mult}_x X_j - 1).$$

Note that regular irreducible components do not contribute to $\operatorname{mult}'_x X$. In Section 2.2, we recall the definition of the multiplicity $\operatorname{mult}_x X_j$ and present different ways to compute it.

Theorem 1.2 ($\overline{\partial}_w$-Riemann-Roch)**.** *Let X be a compact Hermitian complex curve with m irreducible components and $L \to X$ a holomorphic line bundle. Then*

$$h^{0,0}_w(X^*, L) - h^{0,1}_w(X^*, L) = m - g + \deg(L) + \sum_{x \in \operatorname{Sing} X} \operatorname{mult}'_x X, \tag{1.3}$$

and

$$h^{1,1}_w(X^*, L) - h^{1,0}_w(X^*, L) = m - g - \deg(L).$$

Theorem 1.2 is a corollary of Theorem 4.4 which we prove in Section 4. We also consider an L^2-dual version there, i. e., an L^2-Riemann-Roch theorem for the minimal closed L^2-extension of the $\overline{\partial}$-operator which we denote by $\overline{\partial}_s$ (see Section 2.1).

On singular complex curves, the $\overline{\partial}_s$-operator is of particular importance because of its relation to weakly holomorphic functions. Namely, the weakly holomorphic functions are precisely the $\overline{\partial}_s$-holomorphic L^2_{loc}-functions (for a localized version of the $\overline{\partial}_s$-operator, see Section 5). Let $H^{p,q}_{s,\text{loc}}(X^*)$ denote the L^2_{loc}-Dolbeault cohomology on X^* with respect to $\overline{\partial}_s$, and $\widehat{\mathcal{O}}_X$ the sheaf of germs of weakly holomorphic functions on X. Then:

Theorem 1.4. *Let X be a Hermitian complex curve. Then*

$$\begin{aligned} H^0(X, \widehat{\mathcal{O}}_X) &= H^{0,0}_{s,\text{loc}}(X^*), \\ H^1(X, \widehat{\mathcal{O}}_X) &\cong H^{0,1}_{s,\text{loc}}(X^*). \end{aligned}$$

If X is irreducible and compact, then $\dim H^0(X, \widehat{\mathcal{O}}_X) = 1$, $\dim H^1(X, \widehat{\mathcal{O}}_X) = g(X)$. We prove Theorem 1.4 in Section 5.

To exemplify the use of L^2-theory for the $\overline{\partial}$-operator on a singular complex space, in particular the L^2-Riemann-Roch theorem, we give in Section 6 two applications. There, we use our L^2-theory to give alternative proofs of two well-known facts. First, we show that each compact complex curve can be realized as a ramified covering of $\mathbb{CP}^1$. Second, we show that a positive holomorphic line bundle over a compact complex curve is ample, yielding that any compact complex curve is projective.

Let us clarify the relation to previous work of others. In the case of complex curves, (1.1) was in essence discovered by Pardon [Par89], and one can deduce parts of Theorem 4.4 and the second statement of Corollary 4.8 from Pardon's work by some additional arguments on the regularity of the $\overline{\partial}$-operator. The first part of Corollary 4.8 was discovered by Haskell [Has89], and from that one can deduce the second statement of Theorem 1.2 by use of L^2-Serre duality. Moreover, Theorem 1.2 was proved in essence by Brüning, Peyerimhoff and Schröder in [BPS90] and [Sch89] by computing the indices of the $\overline{\partial}_w$- and the $\overline{\partial}_s$-operator.

The new point in the present work is that we can put all the partial results mentioned above in the general framework of a comprehensive L^2-theory. From that, we draw also a new understanding of weakly holomorphic functions (Theorem 1.4) and of the divisor $Z - |Z|$. Moreover, all the previous work has been done only for forms with values in the trivial bundle (except of [Sch89]), whereas we incorporate line bundles. This is essential for applications as we illustrate by the examples mentioned above.

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2. Preliminaries

2.1. **Closed extensions of the Cauchy-Riemann operator.** Let X be a complex curve and $X^* := \operatorname{Reg} X$ the set of regular points. We assume that X is a Hermitian complex space in the sense that X^* carries a Hermitian metric γ which is locally given as the restriction of the metric of the ambient space when X is embedded holomorphically into some complex number space.

We denote by $\mathscr{D}^{p,q}(X^*)$ the smooth differential forms of degree (p,q) with compact support in X^* (test forms) and by $L^{p,q}(X^*)$ the set of square-integrable forms with respect to the metric γ on X^*.

Let $\overline{\partial}_s : L^{p,q}(X^*) \to L^{p,q+1}(X^*)$ be the minimal (strong) closed L^2-extension of the Cauchy-Riemann operator $\overline{\partial} : \mathscr{D}^{p,q}(X^*) \to \mathscr{D}^{p,q+1}(X^*)$, i. e., $\overline{\partial}_s$ is defined by the closure of the graph of $\overline{\partial}$ in $L^{p,q}(X^*) \times L^{p,q+1}(X^*)$. $\overline{\partial}_w : L^{p,q}(X^*) \to L^{p,q+1}(X^*)$ is the maximal (weak) closed L^2-extension of $\overline{\partial}$, i. e., $\overline{\partial}_w$ is defined in sense of distributions. We denote by $H^{p,q}_{w/s}(X^*)$ the Dolbeault cohomology with respect to $\overline{\partial}_w$ or $\overline{\partial}_s$, respectively, and by $h^{p,q}_{w/s}(X^*)$ the dimension of $H^{p,q}_{w/s}(X^*)$.

Let $\vartheta : \mathscr{D}^{p,q+1}(X^*) \to \mathscr{D}^{p,q}(X^*)$ be the formal adjoint of $\overline{\partial}$ and $\vartheta_{s/w} := \overline{\partial}^*_{w/s}$ the Hilbert-space adjoint of $\overline{\partial}_{w/s}$. This notation makes sense as $\vartheta_{w/s}$ is in fact the maximal (weak) or minimal (strong), respectively, L^2-extension of ϑ. Let $\overline{*} : L^{p,q}(X^*) \to L^{1-p,1-q}(X^*)$ be the conjugated Hodge-$*$-operator with respect to the metric γ. Then we have $\vartheta_{w/s} = -\,\overline{*}\,\overline{\partial}_{w/s}\,\overline{*}$.

Let $L \to X$ be a Hermitian holomorphic line bundle on X with an (arbitrary) metric on L which is smooth on the whole of X. We define $\mathscr{D}^{p,q}(X^*, L) := \mathscr{C}^\infty_{\mathrm{cpt}}(X^*, \Lambda^{p,q}T^*X^* \otimes L)$ as the smooth (p,q)-forms with compact support and values in L, and $L^{p,q}(X^*, L)$ as the Hilbert space of square-integrable forms with values in L. We consider the Cauchy-Riemann operator $\overline{\partial} : \mathscr{D}^{p,q}(X^*, L) \to \mathscr{D}^{p,q+1}(X^*, L)$ locally given by $\overline{\partial} : \mathscr{D}^{p,q}(X^*) \to \mathscr{D}^{p,q+1}(X^*)$. Since $\overline{\partial}$ commutes with the trivializations of the holomorphic line bundle, $\overline{\partial}$ is well defined. We get the weak and strong extensions $\overline{\partial}_w$, $\overline{\partial}_s : L^{p,q}(X^*, L) \to L^{p,q+1}(X^*, L)$ and the cohomology $H^{p,q}_{w/s}(X^*, L)$ as above.

In Section 3, we will study also the following other closed extensions of $\overline{\partial}$ besides the minimal $\overline{\partial}_s$ and the maximal $\overline{\partial}_w$. Let $D \Subset \mathbb{C}^n$ be a domain, and $X \subset D$ an analytic set of dimension one with $\operatorname{Sing} X = \{0\}$. We can interpret $\overline{\partial}_s$ as $\overline{\partial}_w$ with certain boundary conditions. The boundary of X^* consists of two parts, the singular point $\{0\}$ and the boundary at ∂D: $\partial X = \partial X^* \setminus \{0\}$. Therefore, there are two boundary conditions. Let $\overline{\partial}_{s,w}$ denote the closed L^2-extension which satisfies the boundary condition at $\{0\}$, i. e., $f \in \operatorname{dom} \overline{\partial}_{s,w}$ iff $f \in \operatorname{dom} \overline{\partial}_w$ and there is a sequence $\{f_j\}$ in $\operatorname{dom} \overline{\partial}_w$ such that $\operatorname{supp} f_j \cap \{0\} = \varnothing$, $f_j \to f$, and $\overline{\partial}_w f_j \to \overline{\partial}_w f$ in L^2. $\overline{\partial}_{w,s}$ denotes the extension which satisfies the boundary condition at ∂X, i. e., $f \in \operatorname{dom} \overline{\partial}_{w,s}$ iff $f \in \operatorname{dom} \overline{\partial}_w$ and there is a sequence $\{f_j\}$ in $\operatorname{dom} \overline{\partial}_w$ such that $\operatorname{supp} f_j \cap \partial X = \varnothing$, $f_j \to f$, and $\overline{\partial}_w f_j \to \overline{\partial}_w f$ in L^2. We define the adjoint operators

$$\vartheta_{s,w} := -\,\overline{*}\,\overline{\partial}_{s,w}\,\overline{*} \quad \text{and} \quad \vartheta_{w,s} := -\,\overline{*}\,\overline{\partial}_{w,s}\,\overline{*},$$

which we can realize as Hilbert-space adjoint operators (see [Rup14, Lem. 5.1]):

Lemma 2.1. *The Hilbert-space adjoints $\overline{\partial}^*_{s,w}$ and $\overline{\partial}^*_{w,s}$ satisfy the representations*

$$\overline{\partial}^*_{s,w} = \vartheta_{w,s} = -\,\overline{*}\,\overline{\partial}_{w,s}\,\overline{*} \qquad \text{and} \qquad \overline{\partial}^*_{w,s} = \vartheta_{s,w} = -\,\overline{*}\,\overline{\partial}_{s,w}\,\overline{*},$$

respectively.

2.2. **Resolution of complex curves, divisors, line bundles.** Every (reduced) complex space X (which is countable at infinity) has a resolution of singularities $\pi : M \to X$, i. e., there are a complex manifold M, a proper complex subspace S of X which contains the singular locus of X and a proper holomorphic map $\pi : M \to X$ such that the restriction $M \setminus \pi^{-1}(S) \to X \setminus S$ of π is biholomorphic, and $\pi^{-1}(S)$ is the locally finite union of smooth hypersurfaces (see [Hir64] and [Hir77, Thm. 7.1]).

If X is a compact complex curve, then such a resolution is given just by the normalization of the curve, and it is unique up to biholomorphism: Let $\pi_1 : M_1 \to X$ and $\pi_2 : M_2 \to X$ be two resolutions of X. Then $\psi := \pi_2^{-1} \circ \pi_1 : M_1 \setminus \pi_1^{-1}(\operatorname{Sing} X) \to M_2 \setminus \pi_2^{-1}(\operatorname{Sing} X)$ is biholomorphic and bounded in the singular locus. Yet, $\pi_i^{-1}(\operatorname{Sing} X)$ consist of isolated points. Therefore, ψ has a (bi-) holomorphic extension.

Let $\pi : M \to X$ be a resolution of a compact complex curve X. We define the *genus* of X by the genus of the resolution

$$g(X) := h^1(M) = \dim H^1(M, \mathcal{O}).$$

If X has more than one irreducible component, then M is not connected and $h^1(M)$ is the sum of the genera of the connected components. Since the resolution is unique up to biholomorphism, this is well-defined.

Throughout the article (except of Section 6.1), we will work with divisors on compact Riemann surfaces only. Therefore, there is no difference between Cartier and Weil divisors, and we can associate to each line bundle a divisor.

Let $L \to X$ be a holomorphic line bundle. Then the pull-back $\pi^* L \to M$ is well-defined by the pull-back of the transition functions of the line bundle. There is a divisor D on M associated to $\pi^* L$ such that $\mathcal{O}(\pi^* L) \cong \mathcal{O}(D)$[2] and $\deg \pi^* L = \deg D$. The uniqueness of the resolution (up to biholomorphism) implies the independence of $\deg \pi^* L$ from π, so that

$$\deg L := \deg \pi^* L$$

is also well-defined.

For any divisor D on M, there exists a holomorphic line bundle $L_D \to M$ associated to D such that $\mathcal{O}(L_D) \cong \mathcal{O}(D)$. The constant function $f = 1$ induces a meromorphic section s_D of L_D such that $\operatorname{div}(s_D) = D$. One can then identify sections in $\mathcal{O}(D)$ with sections in $\mathcal{O}(L_D)$ by $g \mapsto g \otimes s_D$, and we denote the inverse mapping by $s \mapsto s \cdot s_D^{-1}$. If Y is an effective divisor, then s_Y is a holomorphic section of L_Y and $\mathcal{O} \subset \mathcal{O}(Y)$. Hence, there is the natural inclusion $\mathcal{O}(D) \subset \mathcal{O}(D+Y)$ which induces the inclusion $\mathcal{O}(L_D) \subset \mathcal{O}(L_{D+Y})$ given by $s \mapsto (s \cdot s_D^{-1}) \otimes s_{D+Y}$. For $U \subset M$, we obtain the inclusion

$$L^{p,q}_{\mathrm{loc}}(U, L_D) \hookrightarrow L^{p,q}_{\mathrm{loc}}(U, L_{D+Y}),\, s \mapsto (s \cdot s_D^{-1}) \otimes s_{D+Y}. \tag{2.2}$$

Here, $L^{p,q}_{\mathrm{loc}}(U, L_D)$ denotes the locally square-integrable forms with values in L_D. This definition is independent of the chosen Hermitian metric on L_D. If M is compact, all metrics are equivalent and we get the inclusion

$$L^{p,q}(M, L_D) \subset L^{p,q}(M, L_{D+Y}). \tag{2.3}$$

Let $Z := \pi^{-1}(\operatorname{Sing} X)$ be the unreduced exceptional divisor and $|Z|$ the underlying reduced divisor. Then $\deg(Z - |Z|)$ is independent of the resolution as well. We will discuss some alternative ways to compute $\deg Z$.

[2] We denote by $\mathcal{O}(D)$ the sheaf of germs of holomorphic functions f such that $\operatorname{div}(f) + D \geq 0$.

Locally, the resolution is given by the Puiseux parametrization: Let A be an analytic set of dimension 1 in $\Omega \Subset \mathbb{C}^n$ with $\operatorname{Sing} A = \{0\}$ which is irreducible at 0. Shrinking Ω, there are coordinates $z, w_1, ..., w_{n-1}$ around 0 such that A is contained in the cone $\|w\| \le C|z|$, $w = (w_1, .., w_{n-1})$. The projection $\mathrm{pr}_z : A \to \mathbb{C}_z$ on the z-coordinate is a finite ramified covering. Let s be the number of the sheets of pr_z. Generic choice of the coordinates gives the same number of sheets s, called the multiplicity $\mathrm{mult}_0 A$ of A in $\{0\}$. There exists a parametrization $\pi : \Delta \to A, t \mapsto (t^s, w_1(t), ..., w_{n-1}(t))$, where $\Delta := \{t \in \mathbb{C} : |t| < 1\}$; cf. e. g. [Chi89, Sect. 6.1]. π is called the Puiseux parametrization. The unreduced exceptional divisor is just $Z = (\pi^{-1}(z)) = (t^s)$, and so $\deg Z = s$.

The number of sheets of the covering pr_z is also equal to the Lelong number $\nu([A], 0)$ of the positive closed current $[A]$ given by the integration over A (see [Chi89, Prop. 2 in § 3.15], [Dem12, Thm. 7.7] or [GH78, § 3.2]).

The tangent cone gives another way to compute $\mathrm{mult}_0 A$. For a holomorphic function f on Ω, let $f = \sum_{k=k_0}^{\infty} f_k$ be the decomposition in homogeneous polynomials f_k of degree k with $f_{k_0} \neq 0$ (choosing a smaller Ω) and $f^* := f_{k_0} \neq 0$ be the *initial homogeneous polynomial of f*. If A is given by the ideal sheaf $\mathscr{J}_A$, then

$$C_0(A) = \{\alpha \in \mathbb{C}^n : f^*(\alpha) = 0 \ \forall f \in \mathscr{J}_{A,0}\} \subset T_0\mathbb{C}^n$$

is called the *tangent cone* of A in 0 (cf. [Chi89, Sect. 8.4]). The natural projection $\mathbb{C}^n \setminus 0 \to \mathbb{CP}^{n-1}$ maps $C_0(A)$ on a projective variety $\widetilde{C_0}(A)$. The degree $\deg Y$ of a projective variety Y in $\mathbb{CP}^{n-1}$ of dimension p is defined as the class of Y in $H_{2p}(\mathbb{CP}^{n-1}, \mathbb{Z}) \cong \mathbb{Z}$, and $\mathrm{mult}_0 A = \deg \widetilde{C_0}(A)$ (see Sect. 2 of [GH78, § 1.3]). In the case of an irreducible complex curve A, note that $\widetilde{C_0}(A)$ is just a point of multiplicity $\mathrm{mult}_0 A$.

All in all, we have

$$\deg Z = \mathrm{mult}_0 A = \nu([A], 0) = \deg \widetilde{C_0}(A).$$

2.3. **Extension theorems.** We need the following extension theorem. Let Δ be the unit disc in $\mathbb{C}$ and $\Delta^* := \Delta \setminus \{0\}$.

Theorem 2.4 (L^2-extension)**.** *If $u \in L^{p,0}_{\mathrm{loc}}(\Delta)$ and $v \in L^{p,1}_{\mathrm{loc}}(\Delta)$ satisfy $\overline{\partial} u = v$ on Δ^* in the sense of distributions, then $\overline{\partial} u = v$ on Δ.*

A more general statement is true for domains in $\mathbb{C}^n$ and proper analytic subsets of arbitrary codimension, cf. e. g. [Rup09, Thm. 3.2].

If $A \subset \Omega \Subset \mathbb{C}^n$ is a pure dimensional analytic set, let $\widehat{\mathcal{O}} = \widehat{\mathcal{O}}_A$ be the normalization sheaf of $\mathcal{O}_A$ which is defined stalk-wise by the integral closure of $\mathcal{O}_{A,x}$ in the sheaf $\mathcal{M}_{A,x}$ of meromorphic functions for all $x \in A$. A function in $\widehat{\mathcal{O}}(U)$, $U \subset A$ open, is called *weakly holomorphic*. Weakly holomorphic functions are holomorphic in regular points of A and bounded in singular points. If A is locally irreducible, then weakly holomorphic functions are continuous in $\operatorname{Sing} A$ (cf. e. g. [GR84, § VI.4].)

The classical Riemann extension theorem generalizes to the following result (see e. g. [GR84, Sect. VII.4.1]).

Theorem 2.5 (Riemann extension)**.** *Let $A \subset \Omega \Subset \mathbb{C}^n$ be a pure dimensional analytic set. Every holomorphic function on $A^* :=$ $\operatorname{Reg} A$ which is bounded at $\operatorname{Sing} A$ is weakly holomorphic on A.*

3. Local L^2-theory of complex curves

In this section, we study the local L^2-theory of (locally) irreducible analytic curves in $\mathbb{C}^n$. By the remarks on the local structure of singular complex curves in Section 2.2 and Section 4, it follows that the studied situation is general enough. We will compute the L^2-Dolbeault cohomology by use of the Puiseux parametrization and will see why the term $\sum_{x\in \operatorname{Sing} X} \operatorname{mult}'_x X$ occurs in (1.3).

Let A be an irreducible analytic curve in $\Delta^n \subset \mathbb{C}^n_{zw_1\dots w_{n-1}}$ given by the Puiseux parametrization

$$\pi\colon \Delta \to \mathbb{C}^n\ ,\ \pi(t) := (t^s, w(t)),$$

where $w = (w_1, ..., w_{n-1})\colon \Delta \to \Delta^{n-1}$ is a holomorphic map such that each component w_i vanishes at least of the order $s+1$ in the origin. Here, Δ is the unit disk $\{t \in \mathbb{C} : |t| < 1\}$. We can assume that π is bijective, in particular, a resolution/normalization of A such that $\operatorname{mult}_0 A = s$. Further, we can assume that 0 is the only singular point of A.

For a regular point $(z_0, w_0) \in A^* := \operatorname{Reg} A$, let $t_0 \in \Delta^*$ be the preimage under π. Since π is biholomorphic on $\Delta^* := \Delta \setminus \{0\}$, $d\pi_{t_0}(\frac{\partial}{\partial t}) = st_0^{s-1}\frac{\partial}{\partial z} + \sum_{k=1}^{n-1} w'_k(t_0)\frac{\partial}{\partial w_k}$ is a non-vanishing tangent vector of A^* in (z_0, w_0), i. e.,

$$(1 + \|\tfrac{1}{s}t_0^{1-s}w'(t_0)\|^2)^{-1/2}\left(\frac{\partial}{\partial z} + \sum_{k=1}^{n-1}\frac{1}{s}t_0^{1-s}w'_k(t_0)\frac{\partial}{\partial w_k}\right)$$

is a normalized generator of $T_{(z_0,w_0)}A^*$ and $(1 + \|\frac{1}{s}t_0^{1-s}w'(t_0)\|^2)^{1/2}dz$ is a normalized generator of $T^*_{(z_0,w_0)}A^*$. Since w'_k vanishes at least of order s in the origin, we obtain $1 + \|\frac{1}{s}t^{1-s}w'(t)\|^2 \sim 1$ on Δ and $dV_{A^*} \sim i dz\wedge d\bar{z}$, where dV_{A^*} denotes the volume form on A^* induced by the standard Euclidean metric of $\mathbb{C}^n$. Using $\pi^*dz = d(\pi^*z) = dt^s = st^{s-1}dt$ and $\pi^*(dz\wedge d\bar{z}) = s^2|t|^{2(s-1)}dt\wedge d\bar{t}$, we get

$$\pi^*dV_{A^*} \sim |t|^{2(s-1)}dV_\Delta.$$

Let $\iota : A^* \to \Delta^*$ be the inverse of π. Then, $\iota(z, w)$ is the root $t = \sqrt[s]{z}$ with $w = w(t)$. We get $\iota^*(dt) = \frac{1}{s}z^{1/s-1}dz$ and $\iota^*(dt\wedge d\bar{t}) = \frac{1}{s^2}|z|^{2(1/s-1)}dz\wedge d\bar{z}$, i. e.,

$$\iota^*dV_{\Delta^*} \sim |z|^{2(1/s-1)}dV_{A^*}.$$

If g is a measurable function on A^*, we obtain

$$\int_{A^*}|g|^2dV_{A^*} = \int_\Delta |\pi^*g|^2\pi^*dV_{A^*} \sim \int_\Delta |\pi^*g|^2\cdot|t|^{2(s-1)}dV_\Delta.$$

Hence,

$$g \in L^{0,0}(A^*) \Leftrightarrow t^{s-1}\pi^*g \in L^{0,0}(\Delta).$$

For $(0, 1)$-forms and $(1, 1)$-forms, we have

$$\pi^*(gd\bar{z}) = \pi^*g\cdot\pi^*(d\bar{z}) = \bar{t}^{s-1}\pi^*(g)d\bar{t},$$

$$\pi^*(gdz\wedge d\bar{z}) = |t|^{2(s-1)}\pi^*(g)dt\wedge d\bar{t}.$$

Thus

$$\begin{aligned} &f \in L^{0,0}(A^*) \Leftrightarrow t^{s-1}\cdot\pi^*f \in L^{0,0}(\Delta),\\ &f \in L^{1,0}(A^*) \Leftrightarrow \pi^*f \in L^{1,0}(\Delta),\\ &f \in L^{0,1}(A^*) \Leftrightarrow \pi^*f \in L^{0,1}(\Delta),\ \text{and}\\ &f \in L^{1,1}(A^*) \Leftrightarrow t^{1-s}\cdot\pi^*f \in L^{1,1}(\Delta). \end{aligned} \tag{3.1}$$

On the other hand, if $v \in L^{0,0}(\Delta)$, we get

$$\infty > \int_\Delta |v|^2dV_\Delta = \int_{A^*}|\iota^*v|^2\iota^*dV_\Delta \sim \int_{A^*}|\iota^*v|^2\cdot|z|^{2(1/s-1)}dV_{A^*}.$$

Thus, $|z|^{1/s-1}\iota^* v$ is square-integrable on A^*. For each $(0,1)$-form $v d\bar{t} \in L^{0,1}(\Delta)$, we get

$$s\iota^*(v d\bar{t}) = \bar{z}^{1/s-1}\iota^*(v) d\bar{z} \in L^{0,1}(A^*),$$

and for each $(1,1)$-form $v dt \wedge d\bar{t} \in L^{1,1}(\Delta)$, we get $|z|^{1-\frac{1}{s}}\iota^*(v dt \wedge d\bar{t}) \in L^{1,1}(A^*)$.

So, if $f \in L^{0,1}(A^*)$, then $u := \pi^* f$ is in L^2, too. Since $\dim \Delta = 1$, there exists $v \in L^{0,0}(\Delta)$ with $\overline{\partial}_w v = u$. We set $g := \iota^* v$. Since $|z|^{1/s-1} g$ is in L^2 and $|z|^{2(1-1/s)}$ is bounded,

$$\|g\|^2_{L^2} = \int_{A^*} |z^{1/s-1} g|^2 \cdot |z|^{2(1-1/s)} dV_{A^*} \leq \|z^{1/s-1} g\|_{L^2} \cdot \|z^{2(1-1/s)}\|_{L^\infty} < \infty.$$

Hence, we get an L^2-solution for $\overline{\partial}_w g = f$ and

$$H^{0,1}_w(A^*) = L^{0,1}(A^*)/\mathcal{R}(\overline{\partial}_w) = 0.$$

In the same way, it is easy to compute

$$H^{1,1}_w(A^*) = 0.$$

We will now determine $H^{p,0}_w(A^*) = \ker(\overline{\partial}_w : L^{p,0} \to L^{p,1})$ by use of the L^2-extension theorem (Theorem 2.4). For this, let $\mathcal{O}_{L^2}(\Delta)$ be the square-integrable holomorphic functions on Δ, and let $\Omega^1_{L^2}(\Delta)$ be the holomorphic 1-forms with square-integrable coefficient. If $g \in L^{0,0}(A^*)$ and $\overline{\partial}_w g = 0$, then $v := \pi^* g \in |t|^{1-s} L^{0,0}(\Delta)$ and $\overline{\partial}_w v = 0$ on Δ^*. Therefore, $\overline{\partial}(t^{s-1} v) = 0$ on Δ^* and $t^{s-1} v \in L^{0,0}(\Delta)$. The extension theorem implies $\overline{\partial}(t^{s-1} v) = 0$ on Δ, i. e., v is a meromorphic function with a pole of order $s-1$ or less at the origin. We say $v \in t^{1-s}\mathcal{O}_{L^2}(\Delta)$. Since, on the other hand, $\iota^*(t^{1-s}\mathcal{O}_{L^2}(\Delta)) \subset \ker \overline{\partial}_w$, we conclude

$$H^{0,0}_w(A^*) \cong t^{1-s}\mathcal{O}_{L^2}(\Delta). \tag{3.2}$$

If $f \in L^{1,0}(A^*)$ and $\overline{\partial}_w f = 0$, then $u := \pi^* f \in L^{1,0}(\Delta)$ and $\overline{\partial}_w u = 0$ on Δ (using the extension theorem again). Hence, u is holomorphic on Δ and

$$H^{1,0}_w(A^*) \cong \Omega^1_{L^2}(\Delta).$$

To compute the cohomology groups $H^{*,*}_s(A^*)$, we use L^2-duality:

Lemma 3.3. *Let $\overline{\partial}_e$ denote either the weak or the strong closed extension of $\overline{\partial}$, and $\overline{\partial}_{e^c}$ the other one. For $p \in \{0,1\}$, let the range $\mathcal{R}(\overline{\partial}_e)$ of $\overline{\partial}_e : L^{p,0} \to L^{p,1}$ be closed. Then*

$$H^{p,1}_e(A^*) \cong H^{1-p,0}_{e^c}(A^*).$$

For the proof see e. g. [Rup14, Thm. 2.3].

Lemma 3.4. *For $p \in \{0,1\}$,*

$$H^{p,0}_s(A^*) \cong H^{1-p,1}_w(A^*) = 0 \text{ and}$$
$$H^{p,1}_s(A^*) \cong H^{1-p,0}_w(A^*).$$

Proof. Recall that $H^{1-p,1}_w(A^*) = 0$. This implies $L^{1-p,1}(A^*) = \mathcal{R}(\overline{\partial}_w)$ and, particularly, that the range of $\overline{\partial}_w : L^{1-p,0} \to L^{1-p,1}$ is closed. As $\vartheta_w = -\bar{*}\,\overline{\partial}_w\,\bar{*}$ and $\bar{*}$ is an isometric isomorphism, we conclude that the range of $\vartheta_w : L^{p,1} \to L^{p,0}$ is closed as well. This is equivalent to the range of $\overline{\partial}_s = \vartheta^*_w : L^{p,0} \to L^{p,1}$ being closed (standard functional analysis). Lemma 3.3 implies both isomorphisms. $\square$

To get the complete picture, we also need to understand the Dolbeault cohomology groups of the closed extensions $\overline{\partial}_{s,w}$ and $\overline{\partial}_{w,s}$, respectively.

Lemma 3.5. *For $p \in \{0,1\}$,*

$$H^{p,0}_{w,s}(A^*) = 0.$$

Proof. Let $f \in \ker \overline{\partial}_{w,s} = H^{p,0}_{w,s}(A^*)$. We have showed $\omega \cdot u := \omega \cdot \pi^* f \in L^{p,0}(\Delta)$ with $\omega(t) = t^{s-1}$ if $p = 0$ and $\omega(t) \equiv 1$ if $p = 1$. By the extension theorem, we conclude $\overline{\partial}_s(\omega \cdot u) = 0$ on Δ, where $\overline{\partial}_s$ denotes the (strong) closure of $\overline{\partial}_{\mathrm{cpt}} : \mathscr{C}^\infty_{\mathrm{cpt};p,0}(\Delta) \to \mathscr{C}^\infty_{\mathrm{cpt};p,1}(\Delta)$. The generalized Cauchy condition implies that the trivial extension of ωu to the complex plane is a holomorphic p-form with compact support (cf. [LM02, § V.3]). We deduce that $\omega u = 0$ and, hence, $f = 0$. □

Lemma 3.6.

$$H^{0,0}_{s,w}(A^*) \cong \mathcal{O}_{L^2}(\Delta) \text{ and}$$
$$H^{1,0}_{s,w}(A^*) \cong t^{s-1}\Omega^1_{L^2}(\Delta).$$

As $\mathcal{O}_{L^2}(\Delta) \cong \widehat{\mathcal{O}}_{L^2}(A)$, the first isomorphism implies that the $\overline{\partial}_{s,w}$-holomorphic functions on a singular complex curve are precisely the square-integrable weakly holomorphic functions.

Proof. First, we prove that $\mathcal{O}_{L^2}(\Delta) = \pi^*(\ker \overline{\partial}_{s,w} : L^{0,0}(A^*) \to L^{0,1}(A^*))$.

i) For $v \in \mathcal{O}_{L^2}(\Delta)$, we claim that $g := \iota^* v \in \ker \overline{\partial}_{s,w}$. To see that, choose smooth functions $\tilde{\chi}_k : \mathbb{R} \to [0,1]$ with $\tilde{\chi}_k|_{(-\infty,k]} = 0$, $\tilde{\chi}_k|_{[k+1,\infty)} = 1$ and $|\tilde{\chi}_k'| \le 2$. We get

$$(\tilde{\chi}_k \circ \log \circ |\log|)'(\rho) = \frac{\tilde{\chi}_k'(\log|\log \rho|)}{\rho \log \rho}.$$

We define $\chi_k : A^* \to [0,1], (z,w) \mapsto \tilde{\chi}_k(\log|\log|z||)$ (which is inspired by [PS91, p. 617]) and get $\operatorname{supp} \overline{\partial}\chi_k \subset A^* \cap \Delta^n_{\varepsilon_k}$, where $\varepsilon_k := \exp(-\exp(k)) \to 0$ if $k \to \infty$. As $v \in L^{0,0}(\Delta)$, we have $g \in z^{1-\frac{1}{s}} L^{0,0}(A^*) \subset L^{0,0}(A^*)$. Then $g \cdot \chi_k \to g$ in L^2. As a holomorphic function, v is bounded in a neighborhood of 0. Therefore,

$$\begin{aligned}
\|g\overline{\partial}\chi_k\|^2_{A^*} &= \left\| g \cdot \frac{\tilde{\chi}_k'(\log|\log|z||)}{|z|\log|z|} \overline{\partial}|z| \right\|^2_{A^* \cap \Delta^n_{\varepsilon_k}} \lesssim \left\| g \cdot \frac{1}{|z|\log|z|} \right\|^2_{A^* \cap \Delta^n_{\varepsilon_k}} \\
&\sim \left\| v \cdot \frac{|t|^{s-1}}{|t|^s \log|t|^s} \right\|^2_{\Delta_{\varepsilon_k}} \lesssim \left\| \frac{1}{|t|\log|t|} \right\|^2_{\Delta_{\varepsilon_k}} = \int_{\Delta_{\varepsilon_k}} \frac{1}{|t|^2 \log^2|t|} dV \\
&= 2\pi \int_0^{\varepsilon_k} \frac{\rho}{\rho^2 \log^2 \rho} d\rho \sim \left[-\frac{1}{\log \rho} \right]_0^{\varepsilon_k} \to 0, \text{ if } k \to \infty.
\end{aligned}$$

Hence, $\overline{\partial}(g\chi_k) = g\overline{\partial}\chi_k \to 0 = \overline{\partial}_w g$ in L^2. So, $g \in \operatorname{dom} \overline{\partial}_{s,w}$.

ii) $\pi^*(\ker \overline{\partial}_{s,w}) \subset \mathcal{O}_{L^2}(\Delta)$ (cf. the proof of Lem. 6.2 in [Rup14]): Let g be in $\ker \overline{\partial}_{s,w}$, i. e., there are g_j in $L^2(A^*)$ with $g_j \to g$, $\overline{\partial} g_j \to 0$ in $L^2(A^*)$ and $0 \notin \operatorname{supp} g_j$. Let $\chi \in \mathscr{C}^\infty_{\mathrm{cpt}}(\Delta, [0,1])$ be identically 1 on $\Delta_{1/2}$. We define $u := \chi\pi^* g$ and $u_j := \chi\pi^* g_j$. It follows that $t^{s-1}u_j \to t^{s-1}u$ and $\overline{\partial} u_j \to \overline{\partial} u$ in $L^2(\Delta)$. Let $P : L^2(\Delta) \to L^2(\Delta)$ be the Cauchy-operator on the punctured disc, i. e.,

$$[P(h)](t) := \frac{1}{2\pi \mathrm{i}} \int_{\Delta^*} \frac{h(\zeta)}{\zeta - t} d\zeta \wedge d\overline{\zeta}.$$

Since the support of u_j is away from 0 and $\partial\Delta$, we get $u_j = P\left(\frac{\partial u_j}{\partial \overline{\zeta}}\right)$. The L^2-continuity of P and $\overline{\partial} u_j \to \overline{\partial} u$ in L^2 imply that

$$u_j = P\left(\frac{\partial u_j}{\partial \overline{\zeta}}\right) \to P\left(\frac{\partial u}{\partial \overline{\zeta}}\right)$$

in L^2. Since t^{s-1} is bounded, we obtain $t^{s-1}u_j \to t^{s-1}P\left(\frac{\partial u}{\partial \overline{\zeta}}\right)$ and, hence, $u = P\left(\frac{\partial u}{\partial \overline{\zeta}}\right)$ in L^2. That yields $\pi^* g \in L^2(\Delta)$. With $\pi^* g \in t^{1-s}\mathcal{O}_{L^2}(\Delta)$ and the extension theorem, we conclude $\pi^* g \in \mathcal{O}_{L^2}(\Delta)$.

Second, we claim that $\ker(\overline{\partial}_{s,w} : L^{1,0}(A^*) \to L^{1,1}(A^*)) \cong t^{s-1}\Omega^1_{L^2}(\Delta)$. $f = gdz$ is in $\ker \overline{\partial}_{s,w}$ iff $g \in \ker \overline{\partial}_{s,w}$. This is equivalent to $\pi^* g \in \mathcal{O}_{L^2}(\Delta)$. Since $\pi^*(dz) = t^{s-1}dt$, we infer that $\pi^* : H^{1,0}_{s,w}(A^*) \to t^{s-1}\Omega^1_{L^2}(\Delta), \pi^* f = t^{s-1}\pi^* g dt$ is an isomorphism.

The Riemann extension theorem (Theorem 2.5) implies $\mathcal{O}_{L^2}(\Delta) \cong \widehat{\mathcal{O}}_{L^2}(A)$ and the last statement. □

Lemma 3.7. *For $p \in \{0,1\}$, $\mathcal{R}(\overline{\partial}_{w,s} : L^{p,0}(A^*) \to L^{p,1}(A^*))$ and $\mathcal{R}(\overline{\partial}_{s,w} : L^{p,0}(A^*) \to L^{p,1}(A^*))$ both are closed, and*

$$H^{p,1-q}_{w,s}(A^*) \cong H^{1-p,q}_{s,w}(A^*) \quad \text{for } q \in \{0,1\}.$$

Proof. Since $\overline{\partial}^*_{w,s} = \vartheta_{s,w}$, $\overline{\partial}^*_{s,w} = \vartheta_{w,s}$, $\vartheta_{s,w} = -\overline{*}\,\overline{\partial}_{s,w}\,\overline{*}$ and $\vartheta_{w,s} = -\overline{*}\,\overline{\partial}_{w,s}\,\overline{*}$ (see Lemma 2.1), it is easy to see that Lemma 3.3 holds for $\overline{\partial}_{w,s}$ and $\overline{\partial}_{s,w}$. We remark (cf. the proof of Lemma 3.4) that

$$\begin{aligned} \mathcal{R}(\overline{\partial}_{s,w}) \text{ is closed } &\Leftrightarrow \mathcal{R}(\vartheta_{w,s}) \text{ is closed} \\ &\Leftrightarrow \mathcal{R}(\overline{\partial}_{w,s}) \text{ is closed.} \end{aligned}$$

Therefore, it is enough to show that $\mathcal{R}(\overline{\partial}_{w,s})$ is closed.

Let $\varphi : \Delta^* \to \mathbb{R}$ be the smooth function defined by $\varphi(t) := (1-s)\log|t|^2$. Then

$$L^{p,q}(\Delta, \varphi) = t^{1-s} L^{p,q}(\Delta)$$

for the L^2-space $L^{p,q}(\Delta,\varphi)$ with weight $e^{-\varphi}$.

We set $T_1 := \pi^* \overline{\partial}_{w,s} \iota^* : L^{0,0}(\Delta,\varphi) \to L^{0,1}(\Delta)$. The extension theorem implies that T_1 is the (strong) closure of $\overline{\partial}_{\text{cpt}} : L^{0,0}(\Delta,\varphi) \to L^{0,1}(\Delta)$. Therefore, T_1^* is the weak closed extension of $\vartheta^{\varphi}_{\text{cpt}} : L^{0,1}(\Delta) \to L^{0,0}(\Delta,\varphi)$ which is defined by

$$(\overline{\partial}_{\text{cpt}}\alpha, \beta) = (\alpha, \vartheta^{\varphi}_{\text{cpt}}\beta)_{\varphi} = \int \langle \alpha, \vartheta^{\varphi}_{\text{cpt}}\beta\rangle e^{-\varphi} dV.$$

We set $\overline{*}_{\varphi} := e^{-\varphi}\,\overline{*}$. Then $T_2 := -\overline{*}_{\varphi} T_1^* \overline{*}$ is the weak closed extension of

$$\overline{\partial}_{\text{cpt}} : L^{1,0}(\Delta) \to L^{1,1}(\Delta, -\varphi)$$

because integration by parts implies $\vartheta^{\varphi}_{\text{cpt}} = -\overline{*}_{\varphi}\,\overline{\partial}_{\text{cpt}}\,\overline{*}$:

$$\begin{aligned} (\alpha, \overline{*}_{-\varphi}\,\overline{\partial}_{\text{cpt}}\,\overline{*}\,\beta)_{\varphi} &= \int \alpha \wedge \overline{*}_{\varphi}\,\overline{*}_{-\varphi}\,\overline{\partial}_{\text{cpt}}\,\overline{*}\,\beta = (-1)^{1-p}\int \alpha \wedge \overline{\partial}_{\text{cpt}}\,\overline{*}\,\beta \\ &= -\int \overline{\partial}_{\text{cpt}}\alpha \wedge \overline{*}\,\beta = -(\overline{\partial}_{\text{cpt}}\alpha, \beta). \end{aligned}$$

Hence, T_2 is $\overline{\partial}_w : L^{1,0}(\Delta) \to L^{1,1}(\Delta, -\varphi)$ in sense of distributions. Since for all

$$u \in L^{1,1}(\Delta, -\varphi) = t^{s-1} L^{1,1}(\Delta) \subset L^{1,1}(\Delta)$$

there is a $v \in L^{1,0}(\Delta)$ with $T_2 v = \overline{\partial}_w v = u$, the range of T_2 is closed. Thus, the range of T_1^* and the range of $\overline{\partial}_{w,s} = \iota^* T_1 \pi^* : L^{0,0}(A^*) \to L^{0,1}(A^*)$ are closed as well.

We set $S_1 := \pi^* \overline{\partial}_{w,s} \iota^* : L^{1,0}(\Delta) \to L^{1,1}(\Delta, -\varphi)$ and $S_2 := -\overline{*}\, S_1^*\, \overline{*}_{\varphi}$. Then S_2 is the weak closure of $\overline{\partial}_{\text{cpt}} : L^{0,0}(\Delta,\varphi) \to L^{0,1}(\Delta)$.

$$\begin{aligned} \mathcal{R}(S_2) &= \{u \in L^{0,1}(\Delta) : \exists v \in L^{0,0}(\Delta,\varphi) = t^{1-s}L^{0,0}(\Delta) \text{ with } S_2 v = u\} \\ &\supset \{u \in L^{0,1}(\Delta) : \exists v \in L^{0,0}(\Delta) \text{ with } \overline{\partial}_w v = u\} = L^{0,1}(\Delta). \end{aligned}$$

Therefore, $\mathcal{R}(S_2) = L^{0,1}(\Delta)$ is closed. This implies the claim. □

Summarizing, we computed (with $s = \mathrm{mult}_0 A$):

$$\begin{aligned} H^{0,0}_w(A^*) &\cong H^{1,1}_s(A^*) &&\cong t^{1-s}\mathcal{O}_{L^2}(\Delta),\\ H^{1,0}_w(A^*) &\cong H^{0,1}_s(A^*) &&\cong \Omega^1_{L^2}(\Delta),\\ H^{p,1}_w(A^*) &= H^{1-p,0}_s(A^*) &&= 0,\\ H^{0,0}_{s,w}(A^*) &\cong H^{1,1}_{w,s}(A^*) &&\cong \mathcal{O}_{L^2}(\Delta),\\ H^{1,0}_{s,w}(A^*) &\cong H^{0,1}_{w,s}(A^*) &&\cong t^{s-1}\Omega^1_{L^2}(\Delta), \text{ and}\\ H^{p,1}_{s,w}(A^*) &= H^{1-p,0}_{w,s}(A^*) &&= 0. \end{aligned} \tag{3.8}$$

4. L^2-cohomology of complex curves

We will prove Theorem 1.2 in this section. As a preparation, we consider the following local situation: Let A be a locally irreducible analytic set of dimension one in a domain $\Omega \Subset \mathbb{C}^n_{zw_1\ldots w_{n-1}}$ with $\mathrm{Sing}\, A = \{0\}$, let dV denote the volume form on $A^* := \mathrm{Reg}\, A$ which is induced by the Euclidean metric and let $z : A \to \mathbb{C}_z$ be the projection on the first coordinate. Let us mention (cf. e. g. Prop. in [Chi89, Sect. 8.1]):

Theorem 4.1. *The set of all tangent vectors at a point of a one-dimensional irreducible analytic set in $\mathbb{C}^n$ is a complex line.*

Thus, we can assume that $C_0(A) = \mathbb{C}_z \times \{0\} \subset \mathbb{C}_z \times \mathbb{C}^{n-1}_{w_1\ldots w_{n-1}}$, and, therefore, $dV \sim dz \wedge d\overline{z}$ (by shrinking Ω if necessary).

Let $\pi : M \to A$ be a resolution of A, $x_0 := \pi^{-1}(0)$. Then $Z = (\pi^*(z))$ is the unreduced exceptional divisor of the resolution. After shrinking A and M again, we can assume that M is covered by a single chart $\psi : M \to \mathbb{C}$ with $x_0 \in M$ and $\psi(x_0) = 0$. We set $\zeta := \pi^*(z)$ and get $Z = (\zeta)$. $|Z| = (\psi)$ implies $Z - |Z| = (\frac{\zeta}{\psi})$. We obtain

$$\pi^*(dz) = d(\pi^* z) = \frac{\partial \zeta}{\partial \psi} d\psi \sim \frac{\zeta}{\psi} d\psi.$$

Therefore, $\pi^*(dV) \sim \left|\frac{\zeta}{\psi}\right|^2 d\psi \wedge d\overline{\psi}$, and we conclude (recall the definition of line bundles L_D from Section 2.2):

$$\begin{aligned} f \in L^{p,q}(A^*) \Leftrightarrow \left|\frac{\zeta}{\psi}\right|^{1-p-q} \cdot \pi^* f \in L^{p,q}(M)\\ \Leftrightarrow \pi^* f \in L^{p,q}(M, L_{(1-p-q)(Z-|Z|)}), \end{aligned} \tag{4.2}$$

Nagase stated this equivalence already in Lem. 5.1 of [Nag90]. By use of the extension Theorem 2.4, we get:

$$\begin{aligned} f \in \mathrm{dom}\left(\overline{\partial}_w : L^{p,0}(A^*) \to L^{p,1}(A^*)\right)\\ \Leftrightarrow \pi^* f \in \mathrm{dom}\left(\overline{\partial}_w : L^{p,0}(M, L_{(1-p)(Z-|Z|)}) \to L^{p,1}(M, L_{p(|Z|-Z)})\right). \end{aligned} \tag{4.3}$$

The essential observation for the proof of Theorem 1.2 is the following:

Theorem 4.4. *Let X be a compact complex curve and $L \to X$ a holomorphic line bundle. Let $\pi : M \to X$ be a resolution of X with exceptional divisor Z, and D a divisor on M such that $\pi^* L \cong L_D$, i. e., $\mathcal{O}(\pi^* L) \cong \mathcal{O}(D)$. Then*

$$\begin{aligned} H^{0,0}_w(X^*, L) &\cong H^0(M, \mathcal{O}(Z - |Z| + D)),\\ H^{0,1}_w(X^*, L) &\cong H^1(M, \mathcal{O}(Z - |Z| + D)),\\ H^{1,0}_w(X^*, L) &\cong H^0(M, \Omega^1(D)) \cong H^1(M, \mathcal{O}(-D)), \text{ and}\\ H^{1,1}_w(X^*, L) &\cong H^1(M, \Omega^1(D)) \cong H^0(M, \mathcal{O}(-D)). \end{aligned}$$

In [Par89, §5], Pardon proved that $H^{0,q}_{(2),\mathrm{sm}}(X^*) \cong H^q(M, \mathcal{O}(Z-|Z|)$, where $H^{p,q}_{(2),\mathrm{sm}}(X^*)$ denotes the $\overline{\partial}$-cohomology with respect to smooth L^2-forms. We will use similar arguments here.

Proof. Let x_0 be in $\operatorname{Sing} X$, and let A be an open neighborhood of $x_0 = 0$ in X embedded locally in $\mathbb{C}^n$. We assume that $A = A_1 \cup \ldots \cup A_m$ with at x_0 irreducible analytic sets A_i. We obtain resolutions $\pi_i := \pi|_{\pi^{-1}(A_i)} : M_i \to A_i$ of A_i. The sets M_i are pairwise disjoint in M and, also, the support of the exceptional divisors Z_i of the resolution π_i. We get $Z|_{\pi^{-1}(A)} = \sum_{i=1}^m Z_i$ and $|Z||_{\pi^{-1}(A)} = \sum_{i=1}^m |Z_i|$. Therefore, the consideration in the local case (see (4.3)) implies that $\overline{\partial}_w : L^{p,0}(X^*, L) \to L^{p,1}(X^*, L)$ can be identified with

$$\overline{\partial}_w : L^{p,0}(M, L_{(1-p)(Z-|Z|)+D}) \to L^{p,1}(M, L_{p(|Z|-Z)+D}).$$

Hence,

$$H^{0,0}_w(X^*, L) \cong \ker(\overline{\partial}_w : L^{0,0}(M, L_{Z-|Z|+D}) \to L^{0,1}(M, L_D)) \cong H^0(M, \mathcal{O}(Z-|Z|+D))$$

and

$$H^{1,0}_w(X^*, L) \cong \ker(\overline{\partial}_w : L^{1,0}(M, L_D) \to L^{1,1}(M, L_{|Z|-Z+D})) \cong H^0(M, \Omega^1(D)).$$

Serre duality (see Theorem 2 in [Ser55, § 3.10]) implies

$$H^{1,0}_w(X^*, L) \cong H^0(M, \Omega^1(D)) \cong H^1(M, \mathcal{O}(-D)).$$

To prove the other two isomorphisms, consider the following general situation: Let E be a divisor on M, L_E the associated bundle, and let $\mathcal{L}^{p,q}_E$ denote the sheaf on M which is defined by $\mathcal{L}^{p,q}_E(U) := L^{p,q}_{\mathrm{loc}}(U, L_E)$ for each open set $U \subset M$. Let $E' \leq E$ be another divisor. Consider the $\overline{\partial}$-operator in the sense of distributions $\overline{\partial}_w : \mathcal{L}^{p,0}_E \to \mathcal{L}^{p,1}_{E'}$. Let $\mathcal{C}^{p,0}_{E,E'}$ denote the sheaf defined by

$$\mathcal{C}^{p,0}_{E,E'}(U) := \operatorname{dom}\left(\overline{\partial}_w : L^{p,0}_{\mathrm{loc}}(U, L_E) \to L^{p,1}_{\mathrm{loc}}(U, L_{E'})\right).$$

Then $\mathcal{C}^{p,0}_{E,E'}$ is fine and, in particular, $H^1(M, \mathcal{C}^{p,0}_{E,E'}) = 0$. We get the sequence

$$0 \to \Omega^p(E) \to \mathcal{C}^{p,0}_{E,E'} \xrightarrow{\overline{\partial}_w} \mathcal{L}^{p,1}_{E'} \to 0 \tag{4.5}$$

which is exact by the usual Grothendieck-Dolbeault lemma because there is an embedding $\mathcal{L}^{p,q}_{E'} \subset \mathcal{L}^{p,q}_E$ (induced by the natural inclusion $\mathcal{O}(E') \subset \mathcal{O}(E)$, see (2.2)).

This induces the long exact sequence of cohomology groups

$$0 \to \Gamma(M, \Omega^p(E)) \to \mathcal{C}^{p,0}_{E,E'}(M) \xrightarrow{\overline{\partial}_w} \mathcal{L}^{p,1}_{E'}(M) \to H^1(M, \Omega^p(E)) \to H^1(M, \mathcal{C}^{p,0}_{E,E'}) = 0.$$

Hence, $\mathcal{L}^{p,1}_{E'}(M)/\overline{\partial}_w \mathcal{C}^{p,0}_{E,E'}(M) \cong H^1(M, \Omega^p(E))$. We conclude

$$H^{0,1}_w(X^*, L) \cong \mathcal{L}^{0,1}_D(M)/\overline{\partial}_w \mathcal{C}^{0,0}_{Z-|Z|+D,D}(M) \cong H^1(M, \mathcal{O}(Z-|Z|+D))$$

and, using the Serre duality again,

$$H^{1,1}_w(X^*, L) \cong \mathcal{L}^{1,1}_{|Z|-Z+D}(M)/\overline{\partial}_w \mathcal{C}^{1,0}_{D,|Z|-Z+D}(M) \cong H^1(M, \Omega^1(D)) \cong H^0(M, \mathcal{O}(-D)).$$

□

Theorem 1.2 follows now as a simple corollary by use of the classical Riemann-Roch theorem for each connected component of the Riemann surface M, keeping in mind that by definition $g(M) = g(X)$, $\deg L = \deg \pi^* L = \deg D$ and $\operatorname{mult}'_x X = \sum_{p \in \pi^{-1}(x)} \deg_p(Z - |Z|)$.

To deduce also a Riemann-Roch theorem for the $\overline{\partial}_s$-cohomology, we can use the following L^2-version of Serre duality:

Theorem 4.6. *For each $p \in \{0,1\}$, the range of $\overline{\partial}_w : L^{p,0}(X^*, L) \to L^{p,1}(X^*, L)$ is closed. In particular, we get*

$$H^{p,q}_w(X^*, L) \cong H^{1-p,1-q}_s(X^*, L^{-1}).$$

Proof. Recall the following well-known fact. If $P : H_1 \to H_2$ is a densely defined closed operator between Hilbert spaces with range $\mathcal{R}(P)$ of finite codimension, then the range $\mathcal{R}(P)$ is closed in H_2 (see e. g. [HL84], Appendix 2.4).

As M is compact, Theorem 4.4 implies particularly that the range of $\overline{\partial}_w$ is finite codimensional and, therefore, closed. Since $\overline{\partial}_s$ is the adjoint of $-\overline{*}\,\overline{\partial}_w\,\overline{*}$, the range of

$$\overline{\partial}_s : L^{1-p,0}(X^*, L^{-1}) \to L^{1-p,1}(X^*, L^{-1})$$

is closed as well. That both ranges are closed implies the L^2-duality (cf. Lemma 3.3)

$$H_w^{p,q}(X^*, L) \cong \mathscr{H}_w^{p,q}(X^*, L) \cong \mathscr{H}_s^{1-p,1-q}(X^*, L^{-1}) \cong H_s^{1-p,1-q}(X^*, L^{-1}),$$

where $\mathscr{H}_{w/s}^{p,q}(X^*, L) := \ker \overline{\partial}_{w/s} \cap \ker \overline{\partial}_{w/s}^*$ denotes the space of $\overline{\partial}$-harmonic forms with values in L. $\square$

Therefore, Theorem 4.4 yields:

$$\begin{aligned}
H_s^{0,0}(X^*, L) &\cong H_w^{1,1}(X^*, L^{-1}) \cong H^0(M, \mathcal{O}(D)),\\
H_s^{0,1}(X^*, L) &\cong H_w^{1,0}(X^*, L^{-1}) \cong H^1(M, \mathcal{O}(D)),\\
H_s^{1,0}(X^*, L) &\cong H_w^{0,1}(X^*, L^{-1}) \cong H^1(M, \mathcal{O}(Z - |Z| - D)), \text{and}\\
H_s^{1,1}(X^*, L) &\cong H_w^{0,0}(X^*, L^{-1}) \cong H^0(M, \mathcal{O}(Z - |Z| - D)).
\end{aligned} \tag{4.7}$$

Haskell computed $H_{\text{cpt}}^{0,q}(X^*) \cong H^q(M, \mathcal{O}_M)$, where $H_{\text{cpt}}^{p,q}(X^*)$ denotes the $\overline{\partial}$-cohomology with respect to smooth forms with compact support (see Thm. 3.1 in [Has89]). From (4.7), we obtain the dual version of Theorem 1.2, i. e., the Riemann-Roch theorem for the $\overline{\partial}_s$-cohomology:

Corollary 4.8 ($\overline{\partial}_s$-Riemann-Roch)**.** *Let X be a compact complex curve with m irreducible components, $L \to X$ be a holomorphic line bundle, and $\pi : M \to X$ be a resolution of X. Then,*

$$\begin{aligned}
h_s^{0,0}(X^*, L) - h_s^{0,1}(X^*, L) &= m - g + \deg L, \textit{ and}\\
h_s^{1,1}(X^*, L) - h_s^{1,0}(X^*, L) &= m - g + \deg(Z - |Z|) - \deg L,
\end{aligned}$$

where Z is the exceptional divisor of the resolution.

In [BPS90], Brüning, Peyerimhoff and Schröder proved that $h_s^{0,0}(X^*) - h_s^{0,1}(X^*) = m - g$ and $h_w^{0,0}(X^*) - h_w^{0,1}(X^*) = m - g + \deg Z - |Z|$ by computing the indices of the differential operators $\overline{\partial}_s$ and $\overline{\partial}_w$. Schröder generalized this result for vector bundles in [Sch89].

5. **Weakly holomorphic functions**

In this section, we will prove Theorem 1.4 by studying weakly holomorphic functions and a localized version of the $\overline{\partial}_s$-operator.

Recalling the arguments at the beginning of Section 4, it is easy to see that the results of Section 3 generalize to arbitrary complex curves. In particular, the $\overline{\partial}_{s,w}$-holomorphic functions on a singular complex curve are precisely the square-integrable weakly holomorphic functions (cf. Lemma 3.6), and the $\overline{\partial}_{s,w}$-equation is locally solvable in the L^2-sense (combine Lemma 3.5 and Lemma 3.7).

Let X be a singular complex curve, $\mathcal{L}_X^{p,q}$ the sheaf of locally square-integrable forms, and let

$$\overline{\partial}_w : \mathcal{L}_X^{p,0} \to \mathcal{L}_X^{p,1}$$

be the $\overline{\partial}$-operator in the sense of distributions. For each open set $U \subset X$, we define $\overline{\partial}_{s,\text{loc}}$ on $L_{\text{loc}}^{p,0}(U)$ by $f \in \text{dom}\, \overline{\partial}_{s,\text{loc}}$ iff $f \in \text{dom}\, \overline{\partial}_w$ and $f \in \text{dom}\, \left(\overline{\partial}_{s,w} : L^{p,0}(V) \to L^{p,1}(V)\right)$ for all $V \Subset U$ (for more details, see [Rup14, Sect. 6]). Let $\mathcal{F}_X^{p,0}$ be the sheaf of germs defined by

$\mathcal{F}_X^{p,0}(U) := \operatorname{dom}\left(\overline{\partial}_{s,\mathrm{loc}} : L^{p,0}_{\mathrm{loc}}(U) \to L^{p,1}_{\mathrm{loc}}(U)\right)$, and let $\widehat{\mathcal{O}}_X$ denote the sheaf of germs of weakly holomorphic functions on X. Then our considerations above yield an exact sequence

$$0 \to \widehat{\mathcal{O}}_X = \ker \overline{\partial}_{s,\mathrm{loc}} \hookrightarrow \mathcal{F}_X^{0,0} \overset{\overline{\partial}_{s,\mathrm{loc}}}{\longrightarrow} \mathcal{L}_X^{0,1} \to 0. \tag{5.1}$$

The sheaves $\mathcal{F}_X^{0,0}$ and $\mathcal{L}_X^{0,1}$ are fine and so (5.1) is a fine resolution of $\widehat{\mathcal{O}}_X$. Let $H^{p,q}_{s,\mathrm{loc}}(X^*)$ denote the L^2_{loc}-Dolbeault cohomology on X^* with respect to the $\overline{\partial}_{s,\mathrm{loc}}$-operator. Using $\widehat{\mathcal{O}}_X = \pi_* \mathcal{O}_M$, we deduce from (5.1):

$$\begin{aligned} H^0(M, \mathcal{O}_M) &\cong H^0(X, \widehat{\mathcal{O}}_X) = H^{0,0}_{s,\mathrm{loc}}(X^*), \\ H^1(M, \mathcal{O}_M) &\cong H^1(X, \widehat{\mathcal{O}}_X) \cong H^{0,1}_{s,\mathrm{loc}}(X^*), \end{aligned}$$

where $\pi : M \to X$ is a resolution of X. That proves Theorem 1.4.

6. **Applications**

There are many applications of the classical Riemann-Roch theorem; we will transfer two of them to our situation to exemplify how the L^2-Riemann-Roch theorem can substitute the classical one on singular spaces.

6.1. **Compact complex curves as covering spaces of $\mathbb{CP}^1$.** Let X be a compact irreducible complex curve with $\operatorname{Sing} X = \{x_1, ..., x_k\}$, let $(h_i)_{x_i} \in \mathcal{O}_{x_i}$ be chosen such that $(h_i)_{x_i} \widehat{\mathcal{O}}_{x_i} \subset \mathcal{O}_{x_i}$ and let $U_i \subset X$ be a (Stein) neighborhood of x_i with $h_i \cdot \widehat{\mathcal{O}}(U_i) \subset \mathcal{O}(U_i)$ (for the existence of the h_i, see e. g. Thm. 6 and its Cor. in [Nar66, § III.2]). Choose an $x_0 \in X^*$ and a (Stein) neighborhood U_0 of x_0. We can assume that U_0, ..., U_k are pairwise disjoint.

We define a line bundle $L \to X$ as follows. Let $U_{k+1} = X^* \setminus \{x_0\}$ and choose $f_0 \in \mathcal{O}(U_0)$ such that f_0 is vanishing to the order $r := \operatorname{ord}_{x_0} f_0 \geq 1$ in x_0, which we will determine later, but has no other zeros. We also set $f_i := 1/h_i$ for $i = 1, ..., k$ and $f_{k+1} = 1$, and consider the Cartier divisor $\{(U_i, f_i)\}_{i=0,...,k+1}$ on X. Let $L \to X$ be the line bundle associated to this divisor. As the f_i have no zeros for $i > 0$, there exists a non-negative integer δ such that $\deg L = r - \delta$. Now choose $r := g(X) + \delta + 1$. It follows that $\deg L = g(X) + 1$. Give L an arbitrary smooth Hermitian metric.

There is a canonical way to identify holomorphic sections of L with meromorphic functions on X. A holomorphic section $s \in \mathcal{O}(L)$ is represented by a tuple $\{s_i\}_i$ where $s_j/f_j = s_l/f_l$ on $U_j \cap U_l$. This gives a meromorphic function $\Psi(s)$ by setting $\Psi(s) := s_j/f_j$ on U_j. Note that $\Psi(s)$ has zeros in the singular points $x_1, ..., x_k$ and may have a pole of order r at $x_0 \notin \operatorname{Sing} X$.

We can now apply our L^2-Riemann-Roch theorem. The $\overline{\partial}_s$-Riemann-Roch theorem, Corollary 4.8, implies $\dim H^{0,0}_s(X^*, L) \geq 1 - g(X) + \deg L = 2$. Therefore, there is a section $\tau \in L^{0,0}(X^*, L)$ with $\overline{\partial}_s \tau = 0$ and τ_{k+1} is non-constant, where $\tau = \{\tau_i\}_{i=0,..,k+1}$ is written in the trivialization as above. This means that $\tau_i \in L^{0,0}(X^* \cap U_i)$, $\overline{\partial}_s \tau_i = 0$, and $\tau_j/f_j = \tau_{k+1}$ is non-constant on $U_j \cap U_{k+1}$. Theorem 1.4 implies that $\tau_i \in \widehat{\mathcal{O}}(U_i)$, $i = 1, ..., k+1$. Now consider $\Psi(\tau)$ as defined above, i. e., $\Psi(\tau) = \tau_i/f_i$ on U_i. We conclude that $\Psi(\tau)/h_i \in \widehat{\mathcal{O}}(U_i)$, thus $\Psi(\tau) \in \mathcal{O}(U_i)$ for $i = 1, ..., k$. Moreover, $\Psi(\tau)$ is non-constant, so it cannot be holomorphic on the whole compact space X, thus must have a pole of some order $\leq r$ in x_0. Thus:

$$\Psi(\tau) : X \setminus \{x_0\} \to \mathbb{C}, \text{and}$$

$$\widetilde{\Psi}(\tau) : X \to \mathbb{CP}^1, x \mapsto \begin{cases} [\Psi(\tau)(x) : 1], & x \neq x_0 \\ [1 : \frac{1}{\Psi(\tau)(x)}], & x \in U_0 \end{cases}$$

are finite, open and, hence, analytic ramified coverings (Covering Lemma, see [GR84, Sect. VII.2.2]). In particular, $X \setminus \{x_0\}$ is Stein (use e. g. Thm. 1 in [GR79, § V.1]).

6.2. **Projectivity of compact complex curves.** A line bundle $L \to X$ on a compact complex space is called very ample if its global holomorphic sections induce a holomorphic embedding into the projective space $\mathbb{CP}^N$, i. e., if $s_0, ..., s_N$ is a basis of the space of holomorphic sections $\Gamma(X, \mathcal{O}(L))$, then the map

$$\Phi : X \to \mathbb{CP}^N \ , \ x \mapsto [s_0(x) : ... : s_N(x)], \tag{6.1}$$

given in local trivializations of the s_i, defines a holomorphic embedding of X in $\mathbb{CP}^N$. If some positive power of the line bundle has this property, then we say that it is ample. A compact complex space is called projective if there is an ample (and, hence, a very ample) line bundle on it.

A classical application of the Riemann-Roch theorem is that any compact Riemann surface is projective, and a line bundle on a Riemann surface is ample if its degree is positive (cf. e. g. [Nar92, Sect. 10]). This generalizes to singular complex curves:

Theorem 6.2. *Let X be a compact locally irreducible complex curve. If $L \to X$ is a holomorphic line bundle with $\deg L \gg 0$, then L is very ample. In particular, X is projective and each holomorphic line bundle on X is ample if its degree is positive.*

Clearly, this result is well-known and follows from more general sheaf-theoretical methods (vanishing theorems) once one knows that L is positive iff $\deg L > 0$ (cf. e. g. Thm. 4.4 in [Pet94, Sect. V.4.3] or Satz 2 in [Gra62, § 3]). Nevertheless, it seems interesting to us to present another proof of Theorem 6.2 which is based on the L^2-Riemann-Roch of singular complex curves. The assumption that X must be locally irreducible in Theorem 6.2 is not necessary. One can prove the result without this assumption easily by the same technique. Yet, to keep the notation simple, we present here only the locally irreducible case.

Let us make some preparations for the proof of Theorem 6.2. Let X be a connected complex curve and $\pi : M \to X$ a resolution of X. We choose a point $x_0 \in \operatorname{Sing} X$ and a small neighborhood $U \subset X$ of x_0 with $U^* := U \setminus \{x_0\} \subset \operatorname{Reg} X$. Assume X is irreducible at x_0. We define $p_0 := \pi^{-1}(x_0)$, $V := \pi^{-1}(U)$, and $V^* := V \setminus \{p_0\}$. We can assume that there is a chart $t : V \to \mathbb{C}$ such that the image of t is bounded.

The Riemann extension theorem implies that $\pi^{-1} : U \to V$ is weakly holomorphic or, briefly, $\tau := t \circ \pi^{-1} \in \widehat{\mathcal{O}}(U)$ (see Theorem 2.5). We show that τ generates the weakly holomorphic functions at x_0 in the following sense: Let $f \in \widehat{\mathcal{O}}(U)$. Then $f \circ \pi$ is holomorphic on V^* and bounded in p_0. This implies that $f \circ \pi$ is holomorphic on V, $f \circ \pi(t) = \sum_{\iota=0}^{\infty} a_\iota t^\iota$, and $f(x) = \sum a_\iota \tau(x)^\iota$ (by shrinking U and V if necessary). This allows to define the order $\operatorname{ord}_{x_0} f$ of vanishing of f in x_0 by $r \in \mathbb{N}_0$ if $a_r \neq 0$ and $a_\iota = 0$ for $\iota < r$. In particular,

$$\operatorname{ord}_{x_0} f = \operatorname{ord}_{p_0}(f \circ \pi).$$

Note that this definition does not depend on the resolution as different resolutions are biholomorphically equivalent.

The L^2-extension theorem (see Theorem 2.4) and (4.3) imply

$$f \in H^{0,0}_w(U) \Leftrightarrow t^{r_0} \cdot \pi^* f \in \mathcal{O}_{L^2}(V) \Leftrightarrow (\tau^{r_0} \cdot f \in \widehat{\mathcal{O}}(U) \text{ and } f \in L^2(U)), \tag{6.3}$$

where Z denotes the exceptional divisor of the resolution and $r_0 := \deg_{p_0}(Z - |Z|)$. In particular, we get the representation $f(x) = \sum_{\iota \geq -r_0} a_\iota \tau(x)^\iota$ and $\operatorname{ord}_{x_0} f := \operatorname{ord}_{p_0} \pi^* f \geq -r_0$ is again well-defined. f is weakly holomorphic iff $\operatorname{ord}_{x_0} f \geq 0$.

We denote by L_{x_0} the holomorphic line bundle on X which is trivial on $X \setminus \{x_0\}$ and is given by τ on U, i. e., the line bundle on X given by the open covering $U_1 := X \setminus \{x_0\}, U_0 := U$ and the transition function $g_{01} := \tau : U_0 \cap U_1 \to \mathbb{C}$. Then $\pi^* L_{x_0} \cong L_{p_0}$, where L_{p_0} is the holomorphic line bundle $L_{p_0} \to M$ associated to the divisor $\{p_0\}$.

Let $L \to X$ be any holomorphic line bundle, $L' := L \otimes L_{x_0}^{-1}$, and let s' be a section in $H^{0,0}_w(X^*, L')$. We can assume that L and L' are given by divisors $\{(U_j, f_j)\}$ and $\{(U_j, f'_j)\}$, respectively, where $\{U_j\}$ is an open covering of X with $U_0 = U$ and $x_0 \notin U_j$ for $j \neq 0$ and where

$f_j, f'_j \in \mathcal{M}(U_j)$ with $g_{jk} := f_j/f_k$ and $g'_{jk} := f'_j/f'_k$ in $\mathcal{O}(U_j \cap U_k)$ ($g_{j,k}$ and g'_{jk} are the transition functions of L and L', respectively).

We get $f_0 = f'_0 \cdot \tau$ and $f_j = f'_j$ for $j \neq 0$. There is a meromorphic function $\tilde{s} := \Psi(s') \in \mathcal{M}(X)$ representing s'. This meromorphic function is defined by $\tilde{s} = s'_j/f'_j$ on U_j, where s'_j is the trivialization of s' on U_j. We can define a section $s = \{s_j\} \in H^{0,0}_w(X^*, L)$ by $s_j = \tilde{s} \cdot f_j$. Thus $s_0 = s'_0 \cdot \tau$ and $s_j = s'_j$ for $j \neq 0$. Hence, $\operatorname{ord}_{x_0} s_0 = \operatorname{ord}_{x_0} s'_0 + 1$. Summarizing, we get an injective linear map

$$T : H^{0,0}_w(X^*, L \otimes L^{-1}_{x_0}) \to H^{0,0}_w(X^*, L),\ s' \mapsto s,$$

which we call the *natural inclusion*. It follows from the construction above and by use of (6.3) that each section $s \in H^{0,0}_w(X^*, L)$ with $\operatorname{ord}_{x_0} s_0 > -r_0$ is in the image of T.

As $H^1(M, \mathcal{O}(D')) = 0$ for a divisor D' with $\deg D' > 2g-2$ by the classical Riemann-Roch theorem (cf. e. g. [Nar92, Sect. 10]), Theorem 4.4 – more precisely,

$$H^{0,1}_w(X^*, L) \cong H^1(M, \mathcal{O}(Z - |Z| + D))$$

– implies the following vanishing theorem.

Theorem 6.4. *If $L \to X$ is a holomorphic line bundle on an irreducible compact complex curve X with $\deg L > 2g - 2 - \sum_{x \in \operatorname{Sing} X} \operatorname{mult}'_x X$, then $H^{0,1}_w(X^*, L) = 0$.*

As a preparation for the proof of Theorem 6.2, we get our main ingredient:

Lemma 6.5. *Let $L \to X$ be a holomorphic line bundle on a connected compact locally irreducible complex curve X with $\deg L > 2g - 1 - \sum_{x \in \operatorname{Sing} X} \operatorname{mult}'_x X$. Then the natural inclusion*

$$T : H^{0,0}_w(X^*, L \otimes L^{-1}_{x_0}) \to H^{0,0}_w(X^*, L)$$

is not surjective. If $\deg L > 2g + r_0 - 1 - \sum_{x \in \operatorname{Sing} X} \operatorname{mult}'_x X$, then there is a section $s \in H^{0,0}_w(X^*, L)$ which is weakly holomorphic on $U(x_0)$ and does not vanish in x_0.

Recall that $r_0 = \operatorname{mult}_{x_0} X - 1 = \deg_{p_0}(Z - |Z|)$.

Proof. i) As $\pi^*(L \otimes L^{-1}_{x_0}) \cong \pi^* L \otimes L^{-1}_{p_0}$, we get $\deg L \otimes L^{-1}_{x_0} = \deg L - 1 > 2g - 2 - \deg(Z - |Z|)$. The $\overline{\partial}_w$-Riemann-Roch theorem and $h^{0,1}_w(X^*, L) = 0 = h^{0,1}_w(X^*, L \otimes L^{-1}_{x_0})$ (using Theorem 6.4) yield

$$\begin{aligned} h^{0,0}_w(X^*, L \otimes L^{-1}_{x_0}) &= 1 - g + \deg(Z - |Z|) + \deg L \otimes L^{-1}_{x_0} \\ &< 1 - g + \deg(Z - |Z|) + \deg L = h^{0,0}_w(X^*, L). \end{aligned}$$

Therefore, the natural inclusion T cannot be surjective.

ii) The image of $T^{r_0} : H^{0,0}_w(X^*, L \otimes L^{-r_0}_{x_0}) \to H^{0,0}_w(X^*, L)$ are the sections s with $\operatorname{ord}_{x_0} s_0 \geq 0$, where s_0 is the trivialization of s over $U(x_0)$, i. e., the sections where s_0 is weakly holomorphic on $U(x_0)$. As $H^{0,0}_w(X^*, L \otimes L^{-r_0-1}_{x_0}) \to H^{0,0}_w(X^*, L \otimes L^{-r_0}_{x_0})$ is not surjective (use

$$\deg L \otimes L^{-r_0}_{x_0} = \deg L - r_0 > 2g - 1 - \deg(Z - |Z|)$$

and part (i)), there is a section

$$s' \in H^{0,0}_w(X^*, L \otimes L^{-r_0}_{x_0})$$

with $\operatorname{ord}_{x_0} s'_0 = -r_0$ and $\operatorname{ord}_{x_0}(T^{r_0}(s'))_0 = 0$. So, $s := T^{r_0}(s')$ is the section of $H^{0,0}_w(X^*, L)$ we were looking for. □

Proof of Theorem 6.2. Let X be a connected compact locally irreducible complex curve with $\operatorname{Sing} X = \{x_1, ..., x_k\}$, and $L \to X$ a line bundle with $\deg L \gg 0$. Following the classical arguments to show that the map Φ in (6.1) is a well-defined holomorphic embedding (see e. g. [Pet94, V.4, Thm. 4.4]), we have to prove:

(i) Φ is well-defined: For $x \in X$, there exists $s \in \Gamma(X, \mathcal{O}(L))$ such that $s(x) \neq 0$.

(ii) Φ is injective: For $x, y \in X$, $x \neq y$, there exists $s \in \Gamma(X, \mathcal{O}(L))$ such that $s(x) \neq 0$ and $s(y) = 0$.

(iii) Φ is an immersion: For $x \in X$, the differential $T_x\Phi$ is injective.

Since (obviously) Φ is closed, (ii) and (iii) imply that Φ is an embedding (see e. g. Sect. 1.2.7 in [GR84]).

We will prove the statements (i) and (iii) for singular points $x \in \operatorname{Sing} X$. The case of regular points is simpler and follows easily with the natural inclusion and Lemma 6.5. The statement (ii) can be seen just as (i) by imposing the additional condition that $s(y) = 0$ in what we do to prove the statement (i).

Let $\pi : M \to X$ be a resolution of singularities. Set $X^* = \operatorname{Reg} X$, $M^* = \pi^{-1}(X^*)$, $p_j := \pi^{-1}(x_j)$, and $r_j := \deg_{p_j}(Z - |Z|)$, where Z is the unreduced exceptional divisor of the resolution. Fix a $\mu \in \{1, ..., k\}$ and choose a neighborhood U_μ of x_μ such that there exist a resolution of the singularities $\pi : V_\mu \to U_\mu$ and a chart $t : V_\mu \to \mathbb{C}$ with $t \circ \pi^{-1}(x_\mu) = 0$, and set $\tau := t \circ \pi^{-1}$.

For each singularity x_j, we can choose a function $h_j \in \mathcal{O}(U_j)$ such that $h_j \cdot \widehat{\mathcal{O}}(U_j) \subset \mathcal{O}(U_j)$ for a neighborhood U_j of x_j small enough (see [Nar66, § III.2]). The number

$$\eta_j := \operatorname{ord}_{x_j} h_j$$

is important for our considerations because of the following fact. If f is a function on U_j with $\operatorname{ord}_{x_j} f \geq \eta_j$, then f/h_j is bounded at x_j ($\operatorname{ord}_{x_j} f/h_j \geq 0$); this implies $f/h_j \in \widehat{\mathcal{O}}(U_j)$ and, hence, $f \in \mathcal{O}(U_j)$. For the maximal ideal in $\mathcal{O}_{X,x_j}$, we get $\mathfrak{m}_{x_j} = \{f \in \mathcal{O}_{X,x_j} : \operatorname{ord}_{x_j} f > 0\}$ and $\{f \in \mathcal{O}_{X,x_j} : \operatorname{ord}_{x_j} f \geq 2\eta_j\} \subset \mathfrak{m}_{x_j}^2$.

We can choose a weakly holomorphic section $\sigma \in H^{0,0}_w(X^*, L)$ such that σ does not vanish in x_μ and $\operatorname{ord}_{x_j} \sigma \geq \eta_j$ for $j \neq \mu$. This section σ exists as we have the natural inclusion (see the construction above)

$$H^{0,0}_w\left(X^*, L \otimes L_{x_\mu}^{-r_\mu} \otimes \bigotimes_{j \neq \mu} L_{x_j}^{-\eta_j - r_j}\right) \to H^{0,0}_w(X^*, L),$$

and $\deg L \gg 0$ implies by Lemma 6.5 that the natural inclusion

$$H^{0,0}_w\left(X^*, L \otimes L_{x_\mu}^{-r_\mu - 1} \otimes \bigotimes_{j \neq \mu} L_{x_j}^{-\eta_j - r_j}\right) \to H^{0,0}_w\left(X^*, L \otimes L_{x_\mu}^{-r_\mu} \otimes \bigotimes_{j \neq \mu} L_{x_j}^{-\eta_j - r_j}\right)$$

is not surjective.

Note that σ is holomorphic on $X - \{x_\mu\}$ but just weakly holomorphic in x_μ. We will now modify σ so that it becomes holomorphic and non-vanishing in x_μ. Shrink U_μ such that $\sigma = \sum_{\iota \geq 0} a_\iota \tau^\iota$ on U_μ with $a_0 \neq 0$. Let $\sigma' := \sigma / a_0$ so that $\operatorname{ord}_{x_\mu}(\sigma' - 1) \geq 1$, i. e., $\sigma' - 1 = \sum_{\iota \geq 1} a'_\iota \tau^\iota$ on U_μ. Choose as above a $\tilde{\sigma} \in H^{0,0}_w(X^*, L)$ with $\operatorname{ord}_{x_\mu} \tilde{\sigma} = 1$ and $\operatorname{ord}_{x_j} \tilde{\sigma} \geq \eta_j$ for $j \neq \mu$. Let $\tilde{\sigma} = \sum_{\iota \geq 1} \tilde{a}_\iota \tau^\iota$ close to x_μ with $\tilde{a}_1 \neq 0$. We define $\sigma'' := \sigma' - \frac{a'_1}{\tilde{a}_1} \tilde{\sigma}$. Then, $\operatorname{ord}_{x_\mu}(\sigma'' - 1) \geq 2$ and $\operatorname{ord}_{x_j} \sigma'' \geq \eta_j$ for $j \neq \mu$. We repeat this procedure recursively to get a section $\xi = \{\xi_j\} \in H^{0,0}_w(X^*, L)$ with $\operatorname{ord}_{x_\mu}(\xi_\mu - 1) \geq \eta_\mu$ and $\operatorname{ord}_{x_j} \xi_j \geq \eta_j$ for $j \neq \mu$. Thus, ξ is a holomorphic section on X, non-vanishing in x_μ. That shows (i) for $x = x_\mu$.

We will prove (iii) for x_μ. Let $v \in T_{x_\mu} X = (\mathfrak{m}_{x_\mu} / \mathfrak{m}_{x_\mu}^2)^*$ satisfy $v \neq 0$, i. e., there exists an $f \in \mathfrak{m}_{x_\mu}$ with $v(f + \mathfrak{m}_{x_\mu}^2) \neq 0$. We claim there exists a $g \in \mathfrak{m}_{\Phi(x_\mu)}$ with $g \circ \Phi - f \in \mathfrak{m}_{x_\mu}^2$. Then $v(g \circ \Phi + \mathfrak{m}_{x_\mu}^2) = v(f + \mathfrak{m}_{x_\mu}^2) \neq 0$, i. e., $T_x\Phi(v) \neq 0$.

Proof of the claim: Replacing 1 with $f = \sum_{\iota \geq 1} f_\iota \tau^\iota$, we can repeat the procedure in (i) to construct a section $\xi = \{\xi_j\} \in H^{0,0}_w(X^*, L)$ with $\operatorname{ord}_{x_\mu}(\xi_\mu - f) \geq 2\eta_\mu$ and $\operatorname{ord}_{x_j} \xi_j \geq \eta_j$ for $j \neq \mu$. We get ξ is holomorphic, $\xi_\mu \in \mathfrak{m}_{x_\mu}$ and $\xi_\mu - f \in \mathfrak{m}_{x_\mu}^2$. Let Φ be defined by $\Phi(x) = [s_0(x):...:s_N(x)]$ with holomorphic sections $s_i = \{s_{ij}\}$ (see (6.1)). Hence, we can choose a vector $(g_0, ..., g_N) \in \mathbb{C}^{N+1}$

such that $\xi = \sum_i g_i s_i$. Because of (i), there exits an i_0 such that $c := s_{i_0}(x_\mu) \neq 0$ – we can assume $i_0 = 0$. We set $U := \{x \in U_\mu \colon s_{0\mu}(x) \neq 0\}$ and identify $\{[t_0{:}...{:}t_N] \colon t_0 = 1\} \subset \mathbb{CP}^N$ with $\mathbb{C}^N$ such that $\Phi|_U \colon U \to \mathbb{C}^N$ is defined by $\Phi(x) = \left(\frac{s_{1\mu}(x)}{s_{0\mu}(x)}, ..., \frac{s_{N\mu}(x)}{s_{0\mu}(x)}\right)$. Let $g \colon \mathbb{C}^N \to \mathbb{C}$ denote the holomorphic function $g(t_1, ..., t_N) := c \cdot (g_0 + \sum_{i=1}^N g_i t_i)$, i. e.,

$$s_{0\mu} \cdot (g \circ \Phi|_U) = c \sum_{i=0}^N g_i s_{i\mu} = c \cdot \xi_\mu$$

on U. Since $c = s_{0\mu}(x_\mu) \neq 0$ and since f and $\frac{c}{s_{0\mu}} - 1$ are in $\mathfrak{m}_{x_\mu}$, we get $g \in \mathfrak{m}_{\Phi(x_\mu)}$ and

$$g \circ \Phi - f = \frac{c}{s_{0\mu}} (\xi_\mu - f) + f \cdot \left(\frac{c}{s_{0\mu}} - 1\right) \in \mathfrak{m}^2_{x_\mu}. \qquad \square$$

For this proof, L has to satisfy

$$\deg L > 2g + \max\{\eta_j\} + \sum_{j=1}^k (\eta_j + r_j) - \deg(Z - |Z|) = 2g + k + \max\{\eta_j\} + \sum_{j=1}^k \eta_j.$$

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SYMMETRIES AND STABILIZATION FOR SHEAVES OF VANISHING CYCLES

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WITH AN APPENDIX BY JÖRG SCHÜRMANN

Abstract. We study symmetries and stabilization properties of perverse sheaves of vanishing cycles $\mathcal{PV}^\bullet_{U,f}$ of a regular function $f : U \to \mathbb{C}$ on a smooth $\mathbb{C}$-scheme U, with critical locus $X = \mathrm{Crit}(f)$. We prove four main results:

(a) If $\Phi : U \to U$ is an isomorphism fixing X and compatible with f, then the action of Φ_* on $\mathcal{PV}^\bullet_{U,f}$ is multiplication by $\det\bigl(\mathrm{d}\Phi|_{X^{\mathrm{red}}}\bigr) = \pm 1$.

(b) $\mathcal{PV}^\bullet_{U,f}$ depends up to canonical isomorphism only on $(X^{(3)}, f^{(3)})$, for $X^{(3)}$ the third-order thickening of X in U, and $f^{(3)} = f|_{X^{(3)}} : X^{(3)} \to \mathbb{C}$.

(c) If U, V are smooth $\mathbb{C}$-schemes, $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ are regular, $X = \mathrm{Crit}(f)$, $Y = \mathrm{Crit}(g)$, and $\Phi : U \to V$ is an embedding with $f = g \circ \Phi$ and $\Phi|_X : X \to Y$ an isomorphism, there is a natural isomorphism $\Theta_\Phi : \mathcal{PV}^\bullet_{U,f} \to \Phi|_X^*(\mathcal{PV}^\bullet_{V,g}) \otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Phi$, for P_Φ a principal $\mathbb{Z}/2\mathbb{Z}$-bundle on X.

(d) If (X, s) is an *oriented d-critical locus* in the sense of Joyce [23], there is a natural perverse sheaf $P^\bullet_{X,s}$ on X, such that if (X, s) is locally modelled on $\mathrm{Crit}(f : U \to \mathbb{C})$ then $P^\bullet_{X,s}$ is locally modelled on $\mathcal{PV}^\bullet_{U,f}$.

We also generalize our results to replace U, X by complex analytic spaces, and $\mathcal{PV}^\bullet_{U,f}$ by $\mathscr{D}$-modules or mixed Hodge modules.

We discuss applications of (d) to categorifying Donaldson–Thomas invariants of Calabi–Yau 3-folds, and to defining a 'Fukaya category' of Lagrangians in a complex symplectic manifold using perverse sheaves.

Contents

1. Introduction

Let U be a smooth $\mathbb{C}$-scheme and $f : U \to \mathbb{C}$ a regular function, and write $X = \mathrm{Crit}(f)$, as a $\mathbb{C}$-subscheme of U. Then one can define the *perverse sheaf of vanishing cycles* $\mathcal{PV}^\bullet_{U,f}$ on X. Formally, $X = \coprod_{c\in f(X)} X_c$, where $X_c \subseteq X$ is the open and closed $\mathbb{C}$-subscheme of points $x \in X$ with $f(x) = c$, and

$$\mathcal{PV}^\bullet_{U,f}|_{X_c} = \phi^p_{f-c}(A_U[\dim U])|_{X_c}$$

for each $c \in f(X)$, where $A_U[\dim U]$ is the constant perverse sheaf on U over a base ring A, and

$$\phi^p_{f-c} : \mathrm{Perv}(U) \longrightarrow \mathrm{Perv}(f^{-1}(c))$$

is the vanishing cycle functor for $f - c : U \to \mathbb{C}$. See §2 for an introduction to perverse sheaves, and an explanation of this notation.

This paper will prove four main results, Theorems 3.1, 4.2, 5.4 and 6.9. The first three give properties of the $\mathcal{PV}^\bullet_{U,f}$, which we may summarize as follows:

(a) Let U, f, X be as above, and write X^{red} for the reduced $\mathbb{C}$-subscheme of X. Suppose $\Phi : U \to U$ is an isomorphism with $f \circ \Phi = f$ and $\Phi|_X = \mathrm{id}_X$. Then Φ induces a natural isomorphism $\Phi_* : \mathcal{PV}^\bullet_{U,f} \to \mathcal{PV}^\bullet_{U,f}$.

Theorem 3.1 implies that $\mathrm{d}\Phi|_{TU|_{X^{\mathrm{red}}}} : TU|_{X^{\mathrm{red}}} \to TU|_{X^{\mathrm{red}}}$ has determinant

$$\det\bigl(\mathrm{d}\Phi|_{X^{\mathrm{red}}}\bigr) : X^{\mathrm{red}} \longrightarrow \mathbb{C} \setminus \{0\},$$

which is a locally constant map $X^{\mathrm{red}} \to \{\pm 1\}$, and $\Phi_* : \mathcal{PV}^\bullet_{U,f} \to \mathcal{PV}^\bullet_{U,f}$ is multiplication by $\det\bigl(\mathrm{d}\Phi|_{X^{\mathrm{red}}}\bigr)$.

In fact Theorem 3.1 proves a more complicated statement, which only requires Φ to be defined étale locally on U.

(b) Let U, f, X be as above, and write $I_X \subseteq \mathcal{O}_U$ for the sheaf of ideals of regular functions $U \to \mathbb{C}$ vanishing on X. For each $k = 1, 2, \ldots$, write $X^{(k)}$ for the k^{th} *order thickening of X in U*, that is, $X^{(k)}$ is the closed $\mathbb{C}$-subscheme of U defined by the vanishing of the sheaf of ideals I^k_X in $\mathcal{O}_U$. Write $f^{(k)} := f|_{X^{(k)}} : X^{(k)} \to \mathbb{C}$.

Theorem 4.2 says that the perverse sheaf $\mathcal{PV}^\bullet_{U,f}$ depends only on the third-order thickenings $(X^{(3)}, f^{(3)})$ up to canonical isomorphism.

As in Remark 4.5, étale locally, $\mathcal{PV}^\bullet_{U,f}$ depends only on $(X^{(2)}, f^{(2)})$ up to non-canonical isomorphism, with isomorphisms natural up to sign.

(c) Let U, V be smooth $\mathbb{C}$-schemes, $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ be regular, and $X = \mathrm{Crit}(f)$, $Y = \mathrm{Crit}(g)$ as $\mathbb{C}$-subschemes of U, V. Let $\Phi : U \hookrightarrow V$ be a closed embedding of $\mathbb{C}$-schemes with $f = g \circ \Phi : U \to \mathbb{C}$, and suppose $\Phi|_X : X \to Y$ is an isomorphism. Then Theorem 5.4 constructs a natural isomorphism of perverse sheaves on X:

$$\Theta_\Phi : \mathcal{PV}^\bullet_{U,f} \longrightarrow \Phi|_X^*\big(\mathcal{PV}^\bullet_{V,g}\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Phi, \tag{1.1}$$

where $\pi_\Phi : P_\Phi \to X$ is a certain principal $\mathbb{Z}/2\mathbb{Z}$-bundle on X. Writing N_{UV} for the normal bundle of U in V, then the Hessian $\mathrm{Hess}\, g$ induces a nondegenerate quadratic form q_{UV} on $N_{UV}|_X$, and P_Φ parametrizes square roots of $\det(q_{UV}) : K_U^2|_X \to \Phi|_X^*(K_V^2)$.

Theorem 5.4 also shows that the Θ_Φ in (1.1) are functorial in a suitable sense under compositions of embeddings $\Phi : U \hookrightarrow V$, $\Psi : V \hookrightarrow W$.

Here (c) is proved by showing that étale locally there exist equivalences $V \simeq U \times \mathbb{C}^n$ identifying $\Phi(U)$ with $U \times \{0\}$ and $g : V \to \mathbb{C}$ with $f \boxplus z_1^2 + \cdots + z_n^2 : U \times \mathbb{C}^n \to \mathbb{C}$, and applying étale local isomorphisms of perverse sheaves

$$\mathcal{PV}^\bullet_{U,f} \cong \mathcal{PV}^\bullet_{U,f} \overset{L}{\boxtimes} \mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2+\cdots+z_n^2} \cong \mathcal{PV}^\bullet_{U\times\mathbb{C}^n, f\boxplus z_1^2+\cdots+z_n^2} \cong \mathcal{PV}^\bullet_{V,g},$$

using $\mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2+\cdots+z_n^2} \cong A_{\{0\}}$ in the first step, and the Thom–Sebastiani Theorem for perverse sheaves in the second.

Passing from $f : U \to \mathbb{C}$ to $g = f \boxplus z_1^2 + \cdots + z_n^2 : U \times \mathbb{C}^n \to \mathbb{C}$ is an important idea in singularity theory, as in Arnold et al. [1] for instance. It is known as *stabilization*, and f and g are called *stably equivalent*. So, Theorem 5.4 concerns the behaviour of perverse sheaves of vanishing cycles under stabilization.

Our fourth main result, Theorem 6.9, concerns a new class of geometric objects called *d-critical loci*, introduced in Joyce [23], and explained in §6.1. An (algebraic) d-critical locus (X, s) over $\mathbb{C}$ is a $\mathbb{C}$-scheme X with a section s of a certain natural sheaf $\mathcal{S}_X^0$ on X. A d-critical locus (X, s) may be written Zariski locally as a critical locus $\mathrm{Crit}(f : U \to \mathbb{C})$ of a regular function f on a smooth $\mathbb{C}$-scheme U, and s records some information about U, f (in the notation of (b) above, s remembers $f^{(2)}$). There is also a complex analytic version.

Algebraic d-critical loci are classical truncations of the *derived critical loci* (more precisely, -1-*shifted symplectic derived schemes*) introduced in derived algebraic geometry by Pantev, Toën, Vaquié and Vezzosi [42]. Theorem 6.9 roughly says that if (X, s) is an algebraic d-critical locus over $\mathbb{C}$ with an 'orientation', then we may define a natural perverse sheaf $P^\bullet_{X,s}$ on X, such that if (X, s) is locally modelled on $\mathrm{Crit}(f : U \to \mathbb{C})$ then $P^\bullet_{X,s}$ is locally modelled on $\mathcal{PV}^\bullet_{U,f}$. The proof uses Theorem 5.4.

These results have exciting applications in the categorification of Donaldson–Thomas theory on Calabi–Yau 3-folds, and in defining a new kind of 'Fukaya category' of complex Lagrangians in complex symplectic manifold, which we will discuss at length in Remarks 6.14 and 6.15.

Although we have explained our results only for $\mathbb{C}$-schemes and perverse sheaves upon them, the proofs are quite general and work in several contexts:

(i) Perverse sheaves on $\mathbb{C}$-schemes or complex analytic spaces with coefficients in any well-behaved commutative ring A, such as $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{C}$.
(ii) $\mathscr{D}$-modules on $\mathbb{C}$-schemes or complex analytic spaces.
(iii) Saito's mixed Hodge modules on $\mathbb{C}$-schemes or complex analytic spaces.

We discuss all these in §2, before proving our four main results in §3–§6. Appendix A, by Jörg Schürmann, proves two compatibility results between duality and Thom–Sebastiani type isomorphisms needed in the main text.

This is one of six linked papers [6, 9–11, 23], with more to come. The best logical order is that the first is Joyce [23] defining d-critical loci, and the second Bussi, Brav and Joyce [9], which proves Darboux-type theorems for the k-shifted symplectic derived schemes of Pantev et al. [42], and defines a truncation functor from -1-shifted symplectic derived schemes to algebraic d-critical loci.

This paper is the third in the sequence. Combining our results with [23, 42] gives new results on categorifying Donaldson–Thomas invariants of Calabi–Yau 3-folds, as in Remark 6.14. In the fourth paper Bussi, Joyce and Meinhardt [11] will generalize the ideas of this paper to motivic Milnor fibres (we explain the relationship between the motivic and cohomological approaches below in Remark 6.10), and deduce new results on motivic Donaldson–Thomas invariants using [23, 42]. In the fifth, Ben-Bassat, Brav, Bussi and Joyce [6] generalize [9, 11] and this paper from (derived) schemes to (derived) Artin stacks.

Sixthly, Bussi [10] will show that if (S,ω) is a complex symplectic manifold, and L, M are complex Lagrangians in S, then the intersection $X = L \cap M$, as a complex analytic subspace of S, extends naturally to a complex analytic d-critical locus (X, s). If the canonical bundles K_L, K_M have square roots $K_L^{1/2}, K_M^{1/2}$ then (X, s) is oriented, and so Theorem 6.9 below defines a perverse sheaf $P^\bullet_{L,M}$ on X, which Bussi also constructs directly.

As in Remark 6.15, we hope in future work to define a 'Fukaya category' of complex Lagrangians in (X,ω) in which $\mathrm{Hom}(L,M) \cong \mathbb{H}^{-n}(P^\bullet_{L,M})$.

Conventions. All $\mathbb{C}$-schemes are assumed separated and of finite type. All complex analytic spaces are Hausdorff and locally of finite type.

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2. Background on perverse sheaves

Perverse sheaves, and the related theories of $\mathscr{D}$-modules and mixed Hodge modules, make sense in several contexts, both algebraic and complex analytic:

(a) Perverse sheaves on $\mathbb{C}$-schemes with coefficients in a ring A (usually $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{C}$), as in Beilinson, Bernstein and Deligne [5] and Dimca [14].
(b) Perverse sheaves on complex analytic spaces with coefficients in a ring A (usually $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{C}$), as in Dimca [14].
(c) $\mathscr{D}$-modules on $\mathbb{C}$-schemes, as in Borel [8] in the smooth case, and Saito [48] in general.
(d) $\mathscr{D}$-modules on complex manifolds as in Björk [7], and on complex analytic spaces as in Saito [48].
(e) Mixed Hodge modules on $\mathbb{C}$-schemes, as in Saito [45, 47].
(f) Mixed Hodge modules on complex analytic spaces, as in Saito [45, 47].

All our main results and proofs work, with minor modifications, in all six settings (a)–(f). As (a) is arguably the simplest and most complete theory, we begin in §2.1–§2.4 with a general introduction to constructible complexes and perverse sheaves on $\mathbb{C}$-schemes, the nearby and vanishing cycle functors, and perverse sheaves of vanishing cycles $PV^\bullet_{U,f}$ on $\mathbb{C}$-schemes, following Dimca [14].

Several important properties of perverse sheaves in (a) either do not work, or become more complicated, in settings (b)–(f). Section 2.5 lists the parts of §2.1–§2.4 that we will use in proofs in this paper, so the reader can check that they do work in (b)–(f). Then §2.6–§2.10 give brief discussions of settings (b)–(f), focussing on the differences with (a) in §2.1–§2.4.

A good introductory reference on perverse sheaves on $\mathbb{C}$-schemes and complex analytic spaces is Dimca [14]. Three other books are Kashiwara and Schapira [27], Schürmann [50], and Hotta, Tanisaki and Takeuchi [21]. Massey [36] and Rietsch [43] are surveys on perverse sheaves, and Beilinson, Bernstein and Deligne [5] is an important primary source, who cover both $\mathbb{Q}$-perverse sheaves on $\mathbb{C}$-schemes as in (a), and $\mathbb{Q}_l$-perverse sheaves on $\mathbb{K}$-schemes as in (g) below.

Remark 2.1. Two further possible settings, in which not all the results we need are available in the literature, are the following.

(g) Perverse sheaves on $\mathbb{K}$-schemes with coefficients in $\mathbb{Z}/l^n\mathbb{Z}$, $\mathbb{Z}_l$, $\mathbb{Q}_l$, or $\bar{\mathbb{Q}}_l$ for $l \neq \operatorname{char}\mathbb{K} \neq 2$ a prime, as in Beilinson et al. [5].

(h) $\mathscr{D}$-modules on $\mathbb{K}$-schemes for $\mathbb{K}$ an algebraically closed field, as in Borel [8].

The issue is that the Thom–Sebastiani theorem is not available in these contexts in the generality we need it. Once an appropriate form of this result becomes available, our main theorems will hold also in these two contexts, sometimes under the further assumption that char $\mathbb{K} = 0$, needed for the results quoted from [9, 42]. We leave the details to the interested reader.

2.1. **Constructible complexes on $\mathbb{C}$-schemes.** We begin by discussing constructible complexes, following Dimca [14, §2–§4].

Definition 2.2. Fix a well-behaved commutative base ring A (where 'well-behaved' means that we need assumptions on A such as A is regular noetherian, of finite global dimension or finite Krull dimension, a principal ideal domain, or a Dedekind domain, at various points in the theory), to study sheaves of A-modules. For some results A must be a field. Usually we take $A = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{C}$.

Let X be a $\mathbb{C}$-scheme, always assumed of finite type. Write X^{an} for the set of $\mathbb{C}$-points of X with the complex analytic topology. Consider sheaves of A-modules $\mathcal{S}$ on X^{an}. A sheaf $\mathcal{S}$ is called (*algebraically*) *constructible* if all the stalks $\mathcal{S}_x$ for $x \in X^{\mathrm{an}}$ are finite type A-modules, and there is a finite stratification $X^{\mathrm{an}} = \coprod_{j\in J} X_j^{\mathrm{an}}$ of X^{an}, where $X_j \subseteq X$ for $j \in J$ are $\mathbb{C}$-subschemes of X and $X_j^{\mathrm{an}} \subseteq X^{\mathrm{an}}$ the corresponding subsets of $\mathbb{C}$-points, such that $\mathcal{S}|_{X_j^{\mathrm{an}}}$ is an A-local system for all $j \in J$.

Write $D(X)$ for the derived category of complexes $\mathcal{C}^\bullet$ of sheaves of A-modules on X^{an}. Write $D^b_c(X)$ for the full subcategory of bounded complexes $\mathcal{C}^\bullet$ in $D(X)$ whose cohomology sheaves $\mathcal{H}^m(\mathcal{C}^\bullet)$ are constructible for all $m \in \mathbb{Z}$. Then $D(X), D^b_c(X)$ are triangulated categories. An example of a constructible complex on X is the *constant sheaf* A_X on X with fibre A at each point.

Grothendieck's "six operations on sheaves" $f^*, f^!, Rf_*, Rf_!, \mathcal{R}Hom, \overset{L}{\otimes}$ act on $D(X)$ preserving the subcategory $D^b_c(X)$. That is, if $f : X \to Y$ is a morphism of $\mathbb{C}$-schemes, then we have two different pullback functors $f^*, f^! : D(Y) \to D(X)$, which also map $D^b_c(Y) \to D^b_c(X)$. Here f^* is called the *inverse image* [14, §2.3], and $f^!$ the *exceptional inverse image* [14, §3.2].

We also have two different pushforward functors

$$Rf_*, Rf_! : D(X) \longrightarrow D(Y)$$

mapping $D^b_c(X) \to D^b_c(Y)$, where Rf_* is called the *direct image* [14, §2.3] and is right adjoint to $f^* : D(Y) \to D(X)$, and $Rf_!$ is called the *direct image with proper supports* [14, §2.3] and is left adjoint to $f^! : D(Y) \to D(X)$. We need the assumptions from §1 that X, Y are separated and of finite type for $Rf_*, Rf_! : D^b_c(X) \to D^b_c(Y)$ to be defined for arbitrary morphisms $f : X \to Y$.

For $\mathcal{B}^\bullet, \mathcal{C}^\bullet$ in $D^b_c(X)$, we may form their *derived Hom* $\mathcal{R}Hom(\mathcal{B}^\bullet, \mathcal{C}^\bullet)$ [14, §2.1], and *left derived tensor product* $\mathcal{B}^\bullet \overset{L}{\otimes} \mathcal{C}^\bullet$ in $D^b_c(X)$, [14, §2.2]. Given $\mathcal{B}^\bullet \in D^b_c(X)$ and $\mathcal{C}^\bullet \in D^b_c(Y)$, we define $\mathcal{B}^\bullet \overset{L}{\boxtimes} \mathcal{C}^\bullet = \pi_X^*(\mathcal{B}^\bullet) \overset{L}{\otimes} \pi_Y^*(\mathcal{C}^\bullet)$ in $D^b_c(X \times Y)$, where $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$ are the projections.

If X is a $\mathbb{C}$-scheme, there is a functor $\mathbb{D}_X : D^b_c(X) \to D^b_c(X)^{\mathrm{op}}$ with

$$\mathbb{D}_X \circ \mathbb{D}_X \cong \mathrm{id} : D^b_c(X) \longrightarrow D^b_c(X),$$

called *Verdier duality*. It reverses shifts, that is, $\mathbb{D}_X\big(\mathcal{C}^\bullet[k]\big) = \big(\mathbb{D}_X(\mathcal{C}^\bullet)\big)[-k]$ for $\mathcal{C}^\bullet$ in $D^b_c(X)$ and $k \in \mathbb{Z}$.

Remark 2.3. Note how Definition 2.2 mixes the complex analytic and the complex algebraic: we consider sheaves on X^{an} in the *analytic* topology, which are constructible with respect to an *algebraic* stratification $X = \coprod_j X_j$.

Here are some properties of all these:

Theorem 2.4. *In the following, X, Y, Z are $\mathbb{C}$-schemes, and f, g are morphisms, and all isomorphisms '$\cong$' of functors or objects are canonical.*

(i) *For $f : X \to Y$ and $g : Y \to Z$, there are natural isomorphisms of functors*

$$R(g \circ f)_* \cong Rg_* \circ Rf_*, \qquad R(g \circ f)_! \cong Rg_! \circ Rf_!,$$
$$(g \circ f)^* \cong f^* \circ g^*, \qquad (g \circ f)^! \cong f^! \circ g^!.$$

(ii) *If $f : X \to Y$ is proper then $Rf_* \cong Rf_!$.*

(iii) *If $f : X \to Y$ is étale then $f^* \cong f^!$. More generally, if $f : X \to Y$ is smooth of relative (complex) dimension d, then $f^*[d] \cong f^![-d]$, where $f^*[d], f^![-d]$ are the functors $f^*, f^!$ shifted by $\pm d$.*

(iv) *If $f : X \to Y$ then $Rf_! \cong \mathbb{D}_Y \circ Rf_* \circ \mathbb{D}_X$ and $f^! \cong \mathbb{D}_X \circ f^* \circ \mathbb{D}_Y$.*

(v) *If U is a smooth $\mathbb{C}$-scheme then $\mathbb{D}_U(A_U) \cong A_U[2 \dim U]$.*

If X is a $\mathbb{C}$-scheme and $\mathcal{C}^\bullet \in D^b_c(X)$, the *hypercohomology* $\mathbb{H}^*(\mathcal{C}^\bullet)$ and *compactly-supported hypercohomology* $\mathbb{H}^*_{\mathrm{c}}(\mathcal{C}^\bullet)$, both graded A-modules, are

$$\mathbb{H}^k(\mathcal{C}^\bullet) = H^k(R\pi_*(\mathcal{C}^\bullet)) \quad \text{and} \quad \mathbb{H}^k_{\mathrm{c}}(\mathcal{C}^\bullet) = H^k(R\pi_!(\mathcal{C}^\bullet)) \quad \text{for } k \in \mathbb{Z}, \tag{2.1}$$

where $\pi : X \to *$ is projection to a point.

If X is proper then $\mathbb{H}^*(\mathcal{C}^\bullet) \cong \mathbb{H}^*_{\mathrm{c}}(\mathcal{C}^\bullet)$ by Theorem 2.4(ii). They are related to usual cohomology by $\mathbb{H}^k(A_X) \cong H^k(X; A)$ and $\mathbb{H}^k_{\mathrm{c}}(A_X) \cong H^k_{\mathrm{c}}(X; A)$. If A is a field then under Verdier duality we have $\mathbb{H}^k(\mathcal{C}^\bullet) \cong \mathbb{H}^{-k}_{\mathrm{c}}(\mathbb{D}_X(\mathcal{C}^\bullet))^*$.

2.2. **Perverse sheaves on $\mathbb{C}$-schemes.** Next we review perverse sheaves, following Dimca [14, §5].

Definition 2.5. Let X be a $\mathbb{C}$-scheme, and for each $x \in X^{\mathrm{an}}$, let $i_x : * \to X$ map $i_x : * \mapsto x$. If $\mathcal{C}^\bullet \in D^b_c(X)$, then the *support* $\operatorname{supp}^m \mathcal{C}^\bullet$ and *cosupport* $\operatorname{cosupp}^m \mathcal{C}^\bullet$ of $\mathcal{H}^m(\mathcal{C}^\bullet)$ for $m \in \mathbb{Z}$ are

$$\operatorname{supp}^m \mathcal{C}^\bullet = \overline{\big\{x \in X^{\mathrm{an}} : \mathcal{H}^m(i_x^*(\mathcal{C}^\bullet)) \neq 0\big\}},$$
$$\operatorname{cosupp}^m \mathcal{C}^\bullet = \overline{\big\{x \in X^{\mathrm{an}} : \mathcal{H}^m(i_x^!(\mathcal{C}^\bullet)) \neq 0\big\}},$$

where $\overline{\{\cdots\}}$ means the closure in $X^{\rm an}$. If A is a field then $\operatorname{cosupp}^m \mathcal{C}^\bullet = \operatorname{supp}^{-m} \mathbb{D}_X(\mathcal{C}^\bullet)$. We call $\mathcal{C}^\bullet$ *perverse*, or a *perverse sheaf*, if $\dim_{\mathbb{C}} \operatorname{supp}^{-m} \mathcal{C}^\bullet \leqslant m$ and $\dim_{\mathbb{C}} \operatorname{cosupp}^m \mathcal{C}^\bullet \leqslant m$ for all $m \in \mathbb{Z}$, where by convention $\dim_{\mathbb{C}} \emptyset = -\infty$. Write $\operatorname{Perv}(X)$ for the full subcategory of perverse sheaves in $D^b_c(X)$. Then $\operatorname{Perv}(X)$ is an abelian category, the heart of a t-structure on $D^b_c(X)$.

Perverse sheaves have the following properties:

Theorem 2.6. **(a)** *If A is a field then* $\operatorname{Perv}(X)$ *is noetherian and artinian.*

(b) *If A is a field then $\mathbb{D}_X : D^b_c(X) \to D^b_c(X)$ maps* $\operatorname{Perv}(X)$ *to* $\operatorname{Perv}(X)$.

(c) *If $i : X \hookrightarrow Y$ is inclusion of a closed $\mathbb{C}$-subscheme, then $Ri_*, Ri_!$ (which are naturally isomorphic) map* $\operatorname{Perv}(X)$ *to* $\operatorname{Perv}(Y)$.

Write $\operatorname{Perv}(Y)_X$ *for the full subcategory of objects in* $\operatorname{Perv}(Y)$ *supported on X. Then $Ri_* \cong Ri_!$ are equivalences of categories* $\operatorname{Perv}(X) \xrightarrow{\sim} \operatorname{Perv}(Y)_X$. *The restrictions $i^*|_{\operatorname{Perv}(Y)_X}$, $i^!|_{\operatorname{Perv}(Y)_X}$ which map* $\operatorname{Perv}(Y)_X$ *to* $\operatorname{Perv}(X)$, *are naturally isomorphic, and are quasi-inverses for*

$$Ri_*, Ri_! : \operatorname{Perv}(X) \to \operatorname{Perv}(Y)_X.$$

(d) *If $f : X \to Y$ is étale then f^* and $f^!$ (which are naturally isomorphic) map* $\operatorname{Perv}(Y)$ *to* $\operatorname{Perv}(X)$. *More generally, if $f : X \to Y$ is smooth of relative dimension d, then $f^*[d] \cong f^![-d]$ map* $\operatorname{Perv}(Y)$ *to* $\operatorname{Perv}(X)$.

(e) $\overset{L}{\boxtimes} : D^b_c(X) \times D^b_c(Y) \to D^b_c(X \times Y)$ *maps* $\operatorname{Perv}(X) \times \operatorname{Perv}(Y)$ *to* $\operatorname{Perv}(X \times Y)$.

(f) *Let U be a smooth $\mathbb{C}$-scheme. Then $A_U[\dim U]$ is perverse, where A_U is the constant sheaf on U with fibre A, and $[\dim U]$ means shift by $\dim U$ in the triangulated category $D^b_c(X)$. Note that Theorem 2.4*(v) *gives a canonical isomorphism $\mathbb{D}_U\big(A_U[\dim U]\big) \cong A_U[\dim U]$.*

When $A = \mathbb{Q}$, so that $\operatorname{Perv}(X)$ is noetherian and artinian by Theorem 2.6(a), the simple objects in $\operatorname{Perv}(X)$ admit a complete description: they are all isomorphic to intersection cohomology complexes $IC_{\bar V}(\mathcal{L})$ for $V \subseteq X$ a smooth locally closed $\mathbb{C}$-subscheme and $\mathcal{L} \to V$ an irreducible $\mathbb{Q}$-local system, [14, §5.4]. Furthermore, if $f : X \to Y$ is a proper morphism of $\mathbb{C}$-schemes, then the Decomposition Theorem [5, 6.2.5], [14, Th. 5.4.10], [45, Cor. 3] says that, in case $IC_{\bar V}(\mathcal{L})$ is of geometric origin, $Rf_*(IC_{\bar V}(\mathcal{L}))$ is isomorphic to a finite direct sum of shifts of simple objects $IC_{\bar V'}(\mathcal{L}')$ in $\operatorname{Perv}(Y)$.

The next theorem is proved by Beilinson et al. [5, Cor. 2.1.23, §2.2.19, & Th. 3.2.4]. The analogue for $D^b_c(X)$ or $D(X)$ rather than $\operatorname{Perv}(X)$ is false. One moral is that perverse sheaves behave like sheaves, rather than like complexes.

Theorem 2.7(i) will be used throughout §3–§6. Theorem 2.7(ii) will be used only once, in the proof of Theorem 6.9 in §6.3, and we only need Theorem 2.7(ii) to hold in the Zariski topology, rather than the étale topology.

Theorem 2.7. *Let X be a $\mathbb{C}$-scheme. Then perverse sheaves on X form a* ***stack*** *(a kind of sheaf of categories) on X in the étale topology.*

Explicitly, this means the following. Let $\{u_i : U_i \to X\}_{i\in I}$ be an ***étale open cover*** *for X, so that $u_i : U_i \to X$ is an étale morphism of $\mathbb{C}$-schemes for $i \in I$ with $\coprod_i u_i$ surjective. Write $U_{ij} = U_i \times_{u_i,X,u_j} U_j$ for $i,j \in I$ with projections*

$$\pi^i_{ij} : U_{ij} \longrightarrow U_i, \quad \pi^j_{ij} : U_{ij} \longrightarrow U_j, \quad u_{ij} = u_i \circ \pi^i_{ij} = u_j \circ \pi^j_{ij} : U_{ij} \longrightarrow X.$$

Similarly, write $U_{ijk} = U_i \times_X U_j \times_X U_k$ for $i,j,k \in I$ with projections

$$\pi^{ij}_{ijk} : U_{ijk} \longrightarrow U_{ij}, \quad \pi^{ik}_{ijk} : U_{ijk} \longrightarrow U_{ik}, \quad \pi^{jk}_{ijk} : U_{ijk} \longrightarrow U_{jk},$$
$$\pi^i_{ijk} : U_{ijk} \longrightarrow U_i, \;\; \pi^j_{ijk} : U_{ijk} \longrightarrow U_j, \;\; \pi^k_{ijk} : U_{ijk} \longrightarrow U_k, \;\; u_{ijk} : U_{ijk} \longrightarrow X,$$

so that $\pi^i_{ijk} = \pi^i_{ij} \circ \pi^{ij}_{ijk}$, $u_{ijk} = u_{ij} \circ \pi^{ij}_{ijk} = u_i \circ \pi^i_{ijk}$, *and so on. All these morphisms* $u_i, \pi^i_{ij}, \ldots, u_{ijk}$ *are étale, so by Theorem* 2.6(d) $u_i^* \cong u_i^!$ *maps* $\mathrm{Perv}(X) \to \mathrm{Perv}(U_i)$, *and similarly for* $\pi^i_{ij}, \ldots, u_{ijk}$. *With this notation:*

(i) *Suppose* $\mathcal{P}^\bullet, \mathcal{Q}^\bullet \in \mathrm{Perv}(X)$, *and we are given* $\alpha_i : u_i^*(\mathcal{P}^\bullet) \to u_i^*(\mathcal{Q}^\bullet)$ *in* $\mathrm{Perv}(U_i)$ *for all* $i \in I$ *such that for all* $i, j \in I$ *we have*

$$(\pi^i_{ij})^*(\alpha_i) = (\pi^i_{ij})^*(\alpha_j) : u_{ij}^*(\mathcal{P}^\bullet) \longrightarrow u_{ij}^*(\mathcal{Q}^\bullet).$$

Then there is a unique $\alpha : \mathcal{P}^\bullet \to \mathcal{Q}^\bullet$ *in* $\mathrm{Perv}(X)$ *with* $\alpha_i = u_i^*(\alpha)$ *for all* $i \in I$.

(ii) *Suppose we are given objects* $\mathcal{P}_i^\bullet \in \mathrm{Perv}(U_i)$ *for all* $i \in I$ *and isomorphisms*

$$\alpha_{ij} : (\pi^i_{ij})^*(\mathcal{P}_i^\bullet) \longrightarrow (\pi^j_{ij})^*(\mathcal{P}_j^\bullet)$$

in $\mathrm{Perv}(U_{ij})$ *for all* $i, j \in I$ *with* $\alpha_{ii} = \mathrm{id}$ *and*

$$(\pi^{jk}_{ijk})^*(\alpha_{jk}) \circ (\pi^{ij}_{ijk})^*(\alpha_{ij}) = (\pi^{ik}_{ijk})^*(\alpha_{ik}) : (\pi^i_{ijk})^*(\mathcal{P}_i) \longrightarrow (\pi^k_{ijk})^*(\mathcal{P}_k)$$

in $\mathrm{Perv}(U_{ijk})$ *for all* $i, j, k \in I$. *Then there exists* $\mathcal{P}^\bullet$ *in* $\mathrm{Perv}(X)$, *unique up to canonical isomorphism, with isomorphisms* $\beta_i : u_i^*(\mathcal{P}^\bullet) \to \mathcal{P}_i^\bullet$ *for each* $i \in I$, *satisfying*

$$\alpha_{ij} \circ (\pi^i_{ij})^*(\beta_i) = (\pi^j_{ij})^*(\beta_j) : u_{ij}^*(\mathcal{P}^\bullet) \longrightarrow (\pi^j_{ij})^*(\mathcal{P}_j^\bullet),$$

for all $i, j \in I$.

We will need the following proposition in §3.3 to prove Theorem 3.1(b). Most of it is setting up notation, only the last part $\alpha|_{X'} = \beta|_{X'}$ is nontrivial.

Proposition 2.8. *Let* W, X *be* $\mathbb{C}$*-schemes,* $x \in X$, *and* $\pi_{\mathbb{C}} : W \to \mathbb{C}$, $\pi_X : W \to X$, $\iota : \mathbb{C} \to W$ *morphisms, such that* $\pi_{\mathbb{C}} \times \pi_X : W \to \mathbb{C} \times X$ *is étale, and* $\pi_{\mathbb{C}} \circ \iota = \mathrm{id}_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$, *and* $\pi_X \circ \iota(t) = x$ *for all* $t \in \mathbb{C}$. *Write* $W_t = \pi_{\mathbb{C}}^{-1}(t) \subset W$ *for each* $t \in \mathbb{C}$, *and* $j_t : W_t \hookrightarrow W$ *for the inclusion. Then* $\pi_X|_{W_t} = \pi_X \circ j_t : W_t \to X$ *is étale, and* $\iota(t) \in W_t$ *with* $\pi_X|_{W_t}(\iota(t)) = x$, *so we may think of* W_t *for* $t \in \mathbb{C}$ *as a* 1*-parameter family of étale open neighbourhoods of* x *in* X.

Let $\mathcal{P}^\bullet, \mathcal{Q}^\bullet \in \mathrm{Perv}(X)$, *so that by Theorem* 2.6(d) *as* π_X *is smooth of relative dimension* 1 *and* $\pi_X|_{W_t}$ *is étale, we have* $\pi_X^*[1](\mathcal{P}^\bullet) \in \mathrm{Perv}(W)$ *and*

$$\pi_X|_{W_t}^*(\mathcal{P}^\bullet) = j_t^*[-1]\bigl(\pi_X^*[1](\mathcal{P}^\bullet)\bigr) \in \mathrm{Perv}(W_t),$$

and similarly for $\mathcal{Q}^\bullet$.

Suppose $\alpha, \beta : \mathcal{P}^\bullet \to \mathcal{Q}^\bullet$ *in* $\mathrm{Perv}(X)$ *and* $\gamma : \pi_X^*[1](\mathcal{P}^\bullet) \to \pi_X^*[1](\mathcal{Q}^\bullet)$ *in* $\mathrm{Perv}(W)$ *are morphisms such that* $\pi_X|_{W_0}^*(\alpha) = j_0^*[-1](\gamma)$ *in* $\mathrm{Perv}(W_0)$ *and* $\pi_X|_{W_1}^*(\beta) = j_1^*[-1](\gamma)$ *in* $\mathrm{Perv}(W_1)$. *Then there exists a Zariski open neighbourhood* X' *of* x *in* X *such that*

$$\alpha|_{X'} = \beta|_{X'} : \mathcal{P}^\bullet|_{X'} \longrightarrow \mathcal{Q}^\bullet|_{X'}.$$

Here we should think of $j_t^*[-1](\gamma)$ for $t \in \mathbb{C}$ as a family of perverse sheaf morphisms $\mathcal{P}^\bullet \to \mathcal{Q}^\bullet$, defined near x in X locally in the étale topology. But morphisms of perverse sheaves are discrete (to see this, note that we can take $A = \mathbb{Z}$), so as $j_t^*[-1](\gamma)$ depends continuously on t, it should be locally constant in t near x, in a suitable sense. The conclusion $\alpha|_{X'} = \beta|_{X'}$ essentially says that $j_0^*[-1](\gamma) = j_1^*[-1](\gamma)$ near x.

If $P \to X$ is a principal $\mathbb{Z}/2\mathbb{Z}$-bundle on a $\mathbb{C}$-scheme X, and $\mathcal{Q}^\bullet \in \mathrm{Perv}(X)$, we will define a perverse sheaf $\mathcal{Q}^\bullet \otimes_{\mathbb{Z}/2\mathbb{Z}} P$, which will be important in §5–§6.

Definition 2.9. Let X be a $\mathbb{C}$-scheme. A *principal* $\mathbb{Z}/2\mathbb{Z}$*-bundle* $P \to X$ is a proper, surjective, étale morphism of $\mathbb{C}$-schemes $\pi : P \to X$ together with a free involution $\sigma : P \to P$, such that the orbits of $\mathbb{Z}/2\mathbb{Z} = \{1, \sigma\}$ are the fibres of π. We will use the ideas of *isomorphism* of principal bundles $\iota : P \to P'$, *section* $s : X \to P$, *tensor product* $P \otimes_{\mathbb{Z}/2\mathbb{Z}} P'$, and *pullback* $f^*(P) \to W$ under a $\mathbb{C}$-scheme morphism $f : W \to X$, all of which are defined in the obvious ways.

Let $P \to X$ be a principal $\mathbb{Z}/2\mathbb{Z}$-bundle. Write $\mathcal{L}_P \in D^b_c(X)$ for the rank one A-local system on X induced from P by the nontrivial representation of $\mathbb{Z}/2\mathbb{Z} \cong \{\pm 1\}$ on A. It is characterized by $\pi_*(A_P) \cong A_X \oplus \mathcal{L}_P$. For each $\mathcal{Q}^\bullet \in D^b_c(X)$, write $\mathcal{Q}^\bullet \otimes_{\mathbb{Z}/2\mathbb{Z}} P \in D^b_c(X)$ for $\mathcal{Q}^\bullet \overset{L}{\otimes} \mathcal{L}_P$, and call it $\mathcal{Q}^\bullet$ *twisted by* P. If $\mathcal{Q}^\bullet$ is perverse then $\mathcal{Q}^\bullet \otimes_{\mathbb{Z}/2\mathbb{Z}} P$ is perverse.

Perverse sheaves and complexes twisted by principal $\mathbb{Z}/2\mathbb{Z}$-bundles have the obvious functorial behaviour. For example, if $P \to X$, $P' \to X$ are principal $\mathbb{Z}/2\mathbb{Z}$-bundles and $\mathcal{Q}^\bullet \in D^b_c(X)$ there is a canonical isomorphism $(\mathcal{Q}^\bullet \otimes_{\mathbb{Z}/2\mathbb{Z}} P) \otimes_{\mathbb{Z}/2\mathbb{Z}} P' \cong \mathcal{Q}^\bullet \otimes_{\mathbb{Z}/2\mathbb{Z}} (P \otimes_{\mathbb{Z}/2\mathbb{Z}} P')$, and if $f : W \to X$ is a $\mathbb{C}$-scheme morphism there is a canonical isomorphism

$$f^*(\mathcal{Q}^\bullet \otimes_{\mathbb{Z}/2\mathbb{Z}} P) \cong f^*(\mathcal{Q}^\bullet) \otimes_{\mathbb{Z}/2\mathbb{Z}} f^*(P).$$

2.3. **Nearby cycles and vanishing cycles on $\mathbb{C}$-schemes.** We explain nearby cycles and vanishing cycles, as in Dimca [14, §4.2]. The definition is complex analytic, $\widetilde{X^{\mathrm{an}}_*}, \widetilde{\mathbb{C}^*}$ in (2.2) do not come from $\mathbb{C}$-schemes.

Definition 2.10. Let X be a $\mathbb{C}$-scheme, and let $f : X \to \mathbb{C}$ be a regular function. Define $X_0 = f^{-1}(0)$, as a $\mathbb{C}$-subscheme of X, and $X_* = X \setminus X_0$. Consider the commutative diagram of complex analytic spaces:

$$\begin{tikzcd} X_0^{\mathrm{an}} \arrow[r, "i"'] \arrow[d, "f"] & X^{\mathrm{an}} \arrow[d, "f"] & X_*^{\mathrm{an}} \arrow[l, "j"] \arrow[d, "f"] & \widetilde{X_*^{\mathrm{an}}} \arrow[l, "p"] \arrow[d, "\tilde f"] \arrow[lll, bend right, "\pi"'] \\ \{0\} \arrow[r] & \mathbb{C} & \mathbb{C}^* \arrow[l] & \widetilde{\mathbb{C}^*}. \arrow[l, "\rho"'] \end{tikzcd} \tag{2.2}$$

Here $X^{\mathrm{an}}, X_0^{\mathrm{an}}, X_*^{\mathrm{an}}$ are the complex analytic spaces associated to the $\mathbb{C}$-schemes X_0, X, X_*, and $i : X_0^{\mathrm{an}} \hookrightarrow X^{\mathrm{an}}$, $j : X_*^{\mathrm{an}} \hookrightarrow X^{\mathrm{an}}$ are the inclusions, $\rho : \widetilde{\mathbb{C}^*} \to \mathbb{C}^*$ is the universal cover of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and $\widetilde{X_*^{\mathrm{an}}} = X_*^{\mathrm{an}} \times_{f, \mathbb{C}^*, \rho} \widetilde{\mathbb{C}^*}$ the corresponding cover of X_*^{an}, with covering map $p : \widetilde{X_*^{\mathrm{an}}} \to X_*^{\mathrm{an}}$, and $\pi = j \circ p$.

As in §2.6, the triangulated categories $D(X), D^b_c(X)$ and six operations f^*, $f^!$, Rf_*, $Rf_!$, $\mathcal{R}Hom, \overset{L}{\otimes}$ also make sense for complex analytic spaces. So we can define the *nearby cycle functor* $\psi_f : D^b_c(X) \to D^b_c(X_0)$ to be $\psi_f = i^* \circ R\pi_* \circ \pi^*$. Since this definition goes via $\widetilde{X_*^{\mathrm{an}}}$ which is not a $\mathbb{C}$-scheme, it is not obvious that ψ_f maps to (algebraically) constructible complexes $D^b_c(X_0)$ rather than just to $D(X_0)$, but it does [14, p. 103], [27, p. 352].

There is a natural transformation $\Xi : i^* \Rightarrow \psi_f$ between the functors

$$i^*, \psi_f : D^b_c(X) \longrightarrow D^b_c(X_0).$$

The *vanishing cycle functor* $\phi_f : D^b_c(X) \to D^b_c(X_0)$ is a functor such that for every $\mathcal{C}^\bullet$ in $D^b_c(X)$ we have a distinguished triangle

$$i^*(\mathcal{C}^\bullet) \xrightarrow{\Xi(\mathcal{C}^\bullet)} \psi_f(\mathcal{C}^\bullet) \longrightarrow \phi_f(\mathcal{C}^\bullet) \xrightarrow{[+1]} i^*(\mathcal{C}^\bullet)$$

in $D^b_c(X_0)$. Following Dimca [14, p. 108], we write ψ_f^p, ϕ_f^p for the shifted functors $\psi_f[-1]$, $\phi_f[-1] : D^b_c(X) \to D^b_c(X_0)$.

The generator of $\mathbb{Z} = \pi_1(\mathbb{C}^*)$ on $\widetilde{\mathbb{C}^*}$ induces a deck transformation $\delta_{\mathbb{C}^*} : \widetilde{\mathbb{C}^*} \to \widetilde{\mathbb{C}^*}$ which lifts to a deck transformation $\delta_{X^*} : \widetilde{X^*} \to \widetilde{X^*}$ with $p \circ \delta_{X^*} = p$ and $\tilde f \circ \delta_{X^*} = \delta_{\mathbb{C}^*} \circ \tilde f$. As in [14, p. 103,

p. 105], we can use δ_{X^*} to define natural transformations $M_{X,f} : \psi_f^p \Rightarrow \psi_f^p$ and $M_{X,f} : \phi_f^p \Rightarrow \phi_f^p$, called *monodromy*.

Alternative definitions of ψ_f, ϕ_f in terms of specialization and microlocalization functors are given by Kashiwara and Schapira [27, Prop. 8.6.3]. Here are some properties of nearby and vanishing cycles. Parts (i),(ii) can be found in Dimca [14, Th. 5.2.21 & Prop. 4.2.11]. Part (iv) is proved by Massey [37]; compare also Proposition A.1 in the Appendix.

Theorem 2.11. **(i)** *If X is a $\mathbb{C}$-scheme and $f : X \to \mathbb{C}$ is regular, then the functors $\psi_f^p, \phi_f^p : D_c^b(X) \to D_c^b(X_0)$ both map* $\mathrm{Perv}(X)$ *to* $Perv(X_0)$.

(ii) *Let $\Phi : X \to Y$ be a proper morphism of $\mathbb{C}$-schemes, and $g : Y \to \mathbb{C}$ be regular. Write $f = g \circ \Phi : X \to \mathbb{C}$, $X_0 = f^{-1}(0) \subseteq X$, $Y_0 = g^{-1}(0) \subseteq Y$, and $\Phi_0 = \Phi|_{X_0} : X_0 \to Y_0$. Then we have natural isomorphisms*

$$R(\Phi_0)_* \circ \psi_f^p \cong \psi_g^p \circ R\Phi_* \quad \textit{and} \quad R(\Phi_0)_* \circ \phi_f^p \cong \phi_g^p \circ R\Phi_*. \tag{2.3}$$

Note too that $R\Phi_* \cong R\Phi_!$ and $R(\Phi_0)_* \cong R(\Phi_0)_!$, as Φ, Φ_0 are proper.

(iii) *Let $\Phi : X \to Y$ be an étale morphism of $\mathbb{C}$-schemes, and $g : Y \to \mathbb{C}$ be regular. Write $f = g \circ \Phi : X \to \mathbb{C}$, $X_0 = f^{-1}(0) \subseteq X$, $Y_0 = g^{-1}(0) \subseteq Y$, and $\Phi_0 = \Phi|_{X_0} : X_0 \to Y_0$. Then we have natural isomorphisms*

$$\Phi_0^* \circ \psi_f^p \cong \psi_g^p \circ \Phi^* \quad \textit{and} \quad \Phi_0^* \circ \phi_f^p \cong \phi_g^p \circ \Phi^*. \tag{2.4}$$

Note too that $\Phi^* \cong \Phi^!$ and $\Phi_0^* \cong \Phi_0^!$, as Φ, Φ_0 are étale.

More generally, if $\Phi : X \to Y$ is smooth of relative (complex) dimension d and g, f, X_0, Y_0, Φ_0 are as above, then we have natural isomorphisms

$$\Phi_0^*[d] \circ \psi_f^p \cong \psi_g^p \circ \Phi^*[d] \quad \textit{and} \quad \Phi_0^*[d] \circ \phi_f^p \cong \phi_g^p \circ \Phi^*[d]. \tag{2.5}$$

Note too that $\Phi^*[d] \cong \Phi^![-d]$ and $\Phi_0^*[d] \cong \Phi_0^![-d]$.

(iv) *If X is a $\mathbb{C}$-scheme and $f : X \to \mathbb{C}$ is regular, then there are natural isomorphisms $\psi_f^p \circ \mathbb{D}_X \cong \mathbb{D}_{X_0} \circ \psi_f^p$ and $\phi_f^p \circ \mathbb{D}_X \cong \mathbb{D}_{X_0} \circ \phi_f^p$.*

2.4. **Perverse sheaves of vanishing cycles on $\mathbb{C}$-schemes.** We can now define the main subject of this paper, the perverse sheaf of vanishing cycles $\mathcal{PV}_{U,f}^\bullet$ for a regular function $f : U \to \mathbb{C}$.

Definition 2.12. Let U be a smooth $\mathbb{C}$-scheme, and $f : U \to \mathbb{C}$ a regular function. Write $X = \mathrm{Crit}(f)$, as a closed $\mathbb{C}$-subscheme of U.

Then as a map of topological spaces, $f|_X : X \to \mathbb{C}$ is locally constant, with finite image $f(X)$, so we have a decomposition $X = \coprod_{c \in f(X)} X_c$, for $X_c \subseteq X$ the open and closed $\mathbb{C}$-subscheme with $f(x) = c$ for each $\mathbb{C}$-point $x \in X_c$.

(Note that if X is non-reduced, then $f|_X : X \to \mathbb{C}$ need not be locally constant as a morphism of $\mathbb{C}$-schemes, but $f|_{X^{\mathrm{red}}} : X^{\mathrm{red}} \to \mathbb{C}$ is locally constant, where X^{red} is the reduced $\mathbb{C}$-subscheme of X. Since X, X^{red} have the same topological space, $f|_X : X \to \mathbb{C}$ is locally constant on topological spaces.)

For each $c \in \mathbb{C}$, write $U_c = f^{-1}(c) \subseteq U$. Then as in §2.3, we have a vanishing cycle functor $\phi_{f-c}^p : \mathrm{Perv}(U) \to \mathrm{Perv}(U_c)$. So we may form $\phi_{f-c}^p(A_U[\dim U])$ in $\mathrm{Perv}(U_c)$, since $A_U[\dim U] \in \mathrm{Perv}(U)$ by Theorem 2.6(f). One can show $\phi_{f-c}^p(A_U[\dim U])$ is supported on the closed subset $X_c = \mathrm{Crit}(f) \cap U_c$ in U_c, where $X_c = \emptyset$ unless $c \in f(X)$. That is, $\phi_{f-c}^p(A_U[\dim U])$ lies in $\mathrm{Perv}(U_c)_{X_c}$.

But Theorem 2.6(c) says $\mathrm{Perv}(U_c)_{X_c}$ and $\mathrm{Perv}(X_c)$ are equivalent categories, so we may regard $\phi^p_{f-c}(A_U[\dim U])$ as a perverse sheaf on X_c. That is, we can consider

$$\phi^p_{f-c}(A_U[\dim U])|_{X_c} = i^*_{X_c,U_c}\bigl(\phi^p_{f-c}(A_U[\dim U])\bigr)$$

in $\mathrm{Perv}(X_c)$, where $i_{X_c,U_c} : X_c \to U_c$ is the inclusion morphism.

As $X = \coprod_{c\in f(X)} X_c$ with each X_c open and closed in X, we have

$$\mathrm{Perv}(X) = \bigoplus_{c\in f(X)} \mathrm{Perv}(X_c).$$

Define the *perverse sheaf of vanishing cycles* $\mathcal{PV}^\bullet_{U,f}$ *of* U, f in $\mathrm{Perv}(X)$ to be

$$\mathcal{PV}^\bullet_{U,f} = \bigoplus_{c\in f(X)} \phi^p_{f-c}(A_U[\dim U])|_{X_c}.$$

That is, $\mathcal{PV}^\bullet_{U,f}$ is the unique perverse sheaf on $X = \mathrm{Crit}(f)$ with

$$\mathcal{PV}^\bullet_{U,f}|_{X_c} = \phi^p_{f-c}(A_U[\dim U])|_{X_c}$$

for all $c \in f(X)$.

Under Verdier duality, we have $A_U[\dim U] \cong \mathbb{D}_U(A_U[\dim U])$ by Theorem 2.6(f), so

$$\phi^p_{f-c}(A_U[\dim U]) \cong \mathbb{D}_{U_c}\bigl(\phi^p_{f-c}(A_U[\dim U])\bigr)$$

by Theorem 2.11(iv). Applying $i^*_{X_c,U_c}$ and using $\mathbb{D}_{X_c} \circ i^*_{X_c,U_c} \cong i^!_{X_c,U_c} \circ \mathbb{D}_{U_c}$ by Theorem 2.4(iv) and $i^!_{X_c,U_c} \cong i^*_{X_c,U_c}$ on $\mathrm{Perv}(U_c)_{X_c}$ by Theorem 2.6(c) also gives

$$\phi^p_{f-c}(A_U[\dim U])|_{X_c} \cong \mathbb{D}_{X_c}\bigl(\phi^p_{f-c}(A_U[\dim U])|_{X_c}\bigr).$$

Summing over all $c \in f(X)$ yields a canonical isomorphism

$$\sigma_{U,f} : \mathcal{PV}^\bullet_{U,f} \xrightarrow{\cong} \mathbb{D}_X(\mathcal{PV}^\bullet_{U,f}). \tag{2.6}$$

For $c \in f(X)$, we have a monodromy operator

$$M_{U,f-c} : \phi^p_{f-c}(A_U[\dim U]) \longrightarrow \phi^p_{f-c}(A_U[\dim U]),$$

which restricts to $\phi^p_{f-c}(A_U[\dim U])|_{X_c}$. Define the *twisted monodromy operator*

$$\tau_{U,f} : \mathcal{PV}^\bullet_{U,f} \longrightarrow \mathcal{PV}^\bullet_{U,f}$$

by

$$\tau_{U,f}|_{X_c} = (-1)^{\dim U} M_{U,f-c}|_{X_c} : \phi^p_{f-c}(A_U[\dim U])|_{X_c} \longrightarrow \phi^p_{f-c}(A_U[\dim U])|_{X_c}, \tag{2.7}$$

for each $c \in f(X)$.

Here 'twisted' refers to the sign $(-1)^{\dim U}$ in (2.7). We include this sign change as it makes monodromy act naturally under transformations which change dimension — without it, equation (5.15) below would only commute up to a sign $(-1)^{\dim V-\dim U}$, not commute — and it normalizes the monodromy of any nondegenerate quadratic form to be the identity, as in (2.13). The sign $(-1)^{\dim U}$ also corresponds to the twist '$(\frac{1}{2}\dim U)$' in the definition (2.24) of the mixed Hodge module of vanishing cycles $\mathcal{HV}^\bullet_{U,f}$ in §2.10.

The (compactly-supported) hypercohomology $\mathbb{H}^*(\mathcal{PV}^\bullet_{U,f}), \mathbb{H}^*_c(\mathcal{PV}^\bullet_{U,f})$ from (2.1) is an important invariant of U, f. If A is a field then the isomorphism $\sigma_{U,f}$ in (2.6) implies that $\mathbb{H}^k(\mathcal{PV}^\bullet_{U,f}) \cong \mathbb{H}^{-k}_c(\mathcal{PV}^\bullet_{U,f})^*$, a form of Poincaré duality.

We defined $\mathcal{PV}^\bullet_{U,f}$ in perverse sheaves over a base ring A. Writing $\mathcal{PV}^\bullet_{U,f}(A)$ to denote the base ring, one can show that

$$\mathcal{PV}^\bullet_{U,f}(A) \cong \mathcal{PV}^\bullet_{U,f}(\mathbb{Z}) \overset{L}{\otimes}_{\mathbb{Z}} A.$$

Thus, we may as well take $A = \mathbb{Z}$, or $A = \mathbb{Q}$ if we want A to be a field, since the case of general A contains no more information.

There is a Thom–Sebastiani Theorem for perverse sheaves, due to Massey [35] and Schürmann [50, Cor. 1.3.4]. Applied to $\mathcal{PV}^\bullet_{U,f}$, it yields:

Theorem 2.13. *Let U, V be smooth $\mathbb{C}$-schemes and $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ be regular, so that $f \boxplus g : U \times V \to \mathbb{C}$ is regular with $(f \boxplus g)(u, v) := f(u) + g(v)$. Set $X = \mathrm{Crit}(f)$ and $Y = \mathrm{Crit}(g)$ as $\mathbb{C}$-subschemes of U, V, so that $\mathrm{Crit}(f \boxplus g) = X \times Y$. Then there is a natural isomorphism*

$$\mathcal{TS}_{U,f,V,g} : \mathcal{PV}^\bullet_{U\times V,f\boxplus g} \longrightarrow \mathcal{PV}^\bullet_{U,f} \overset{L}{\boxtimes} \mathcal{PV}^\bullet_{V,g} \tag{2.8}$$

in $\mathrm{Perv}(X \times Y)$, such that the following diagrams commute:

$$\begin{tikzcd}[column sep=large]
\mathcal{PV}^\bullet_{U\times V,f\boxplus g} \arrow[rr, "\sigma_{U\times V,f\boxplus g}"'] \arrow[d, "\mathcal{TS}_{U,f,V,g}"] & & \mathbb{D}_{X\times Y}(\mathcal{PV}^\bullet_{U\times V,f\boxplus g}) \\
\mathcal{PV}^\bullet_{U,f} \overset{L}{\boxtimes} \mathcal{PV}^\bullet_{V,g} \arrow[r, "\sigma_{U,f} \overset{L}{\boxtimes} \sigma_{V,g}"] & \mathbb{D}_X(\mathcal{PV}^\bullet_{U,f}) \overset{L}{\boxtimes} \mathbb{D}_Y(\mathcal{PV}^\bullet_{V,g}) \arrow[r, "\cong"] & \mathbb{D}_{X\times Y}\big(\mathcal{PV}^\bullet_{U,f} \overset{L}{\boxtimes} \mathcal{PV}^\bullet_{V,g}\big), \arrow[u, "\mathbb{D}_{X\times Y}(\mathcal{TS}_{U,f,V,g})"']
\end{tikzcd} \tag{2.9}$$

$$\begin{tikzcd}[column sep=huge]
\mathcal{PV}^\bullet_{U\times V,f\boxplus g} \arrow[r, "\tau_{U\times V,f\boxplus g}"'] \arrow[d, "\mathcal{TS}_{U,f,V,g}"] & \mathcal{PV}^\bullet_{U\times V,f\boxplus g} \arrow[d, "\mathcal{TS}_{U,f,V,g}"'] \\
\mathcal{PV}^\bullet_{U,f} \overset{L}{\boxtimes} \mathcal{PV}^\bullet_{V,g} \arrow[r, "\tau_{U,f} \overset{L}{\boxtimes} \tau_{V,g}"] & \mathcal{PV}^\bullet_{U,f} \overset{L}{\boxtimes} \mathcal{PV}^\bullet_{V,g}.
\end{tikzcd} \tag{2.10}$$

The next example will be important later.

Example 2.14. Define $f : \mathbb{C}^n \to \mathbb{C}$ by

$$f(z_1, \ldots, z_n) = z_1^2 + \cdots + z_n^2$$

for $n > 1$. Then $\mathrm{Crit}(f) = \{0\}$, so $\mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2+\cdots+z_n^2} = \phi^p_f(A_{\mathbb{C}^n}[n])|_{\{0\}}$ is a perverse sheaf on the point $\{0\}$. Following Dimca [14, Prop. 4.2.2, Ex. 4.2.3 & Ex. 4.2.6], we find that there is a canonical isomorphism

$$\mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2+\cdots+z_n^2} \cong H^{n-1}\big(MF_f(0); A\big) \otimes_A A_{\{0\}}, \tag{2.11}$$

where $MF_f(0)$ is the *Milnor fibre* of f at 0, as in [14, p. 103]. Since $f(z) = z_1^2 + \cdots + z_n^2$ is homogeneous, we see that

$$MF_f(0) \cong \big\{(z_1, \ldots, z_n) \in \mathbb{C}^n : f(z_1, \ldots, z_n) = 1\big\} \cong T^*\mathcal{S}^{n-1},$$

so that $H^{n-1}\big(MF_f(0); A\big) \cong H^{n-1}\big(\mathcal{S}^{n-1}; A\big) \cong A$. Therefore we have

$$\mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2+\cdots+z_n^2} \cong A_{\{0\}}. \tag{2.12}$$

This isomorphism (2.12) is *natural up to sign* (unless the base ring A has characteristic 2, in which case (2.12) is natural), as it depends on the choice of isomorphism $H^{n-1}(\mathcal{S}^{n-1}, A) \cong A$, which corresponds to an orientation for $\mathcal{S}^{n-1}$. This uncertainty of signs will be important in §5–§6.

We can also use Milnor fibres to compute the monodromy operator on $\mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2+\cdots+z_n^2}$. There is a monodromy map $\mu_f : MF_f(0) \to MF_f(0)$, natural up to isotopy, which is the monodromy in the Milnor fibration of f at 0. Under the identification $MF_f(0) \cong T^*\mathcal{S}^{n-1}$ we may take μ_f to be the map $\mathrm{d}(-1) : T^*\mathcal{S}^{n-1} \to T^*\mathcal{S}^{n-1}$ induced by $-1 : \mathcal{S}^{n-1} \to \mathcal{S}^{n-1}$ mapping

$$-1 : (x_1, \ldots, x_n) \mapsto (-x_1, \ldots, -x_n).$$

This multiplies orientations on $\mathcal{S}^{n-1}$ by $(-1)^n$. Thus, $\mu_{f*} : H^{n-1}(\mathcal{S}^{n-1}, A) \to H^{n-1}(\mathcal{S}^{n-1}, A)$ multiplies by $(-1)^n$.

By [14, Prop. 4.2.2], equation (2.11) identifies the action of the monodromy operator $M_{\mathbb{C}^n,f}|_{\{0\}}$ on $\mathcal{PV}^\bullet_{\mathbb{C}^n,z_1^2+\cdots+z_n^2}$ with the action of μ_{f*} on $H^{n-1}(\mathcal{S}^{n-1}, A)$. So $M_{\mathbb{C}^n,f}|_{\{0\}}$ is multiplication by $(-1)^n$. Combining this with the sign change $(-1)^{\dim U}$ in (2.7) for $U = \mathbb{C}^n$ shows that the twisted monodromy is

$$\tau_{\mathbb{C}^n,z_1^2+\cdots+z_n^2} = \mathrm{id} : \mathcal{PV}^\bullet_{\mathbb{C}^n,z_1^2+\cdots+z_n^2} \longrightarrow \mathcal{PV}^\bullet_{\mathbb{C}^n,z_1^2+\cdots+z_n^2}. \tag{2.13}$$

Equations (2.12)–(2.13) also hold for $n = 0, 1$, though (2.11) does not.

Note also that these results are compatible with the Thom–Sebastiani Theorem 2.13, and can be deduced from it and the case $n = 1$.

We introduce some notation for pullbacks of $\mathcal{PV}^\bullet_{V,g}$ by étale morphisms.

Definition 2.15. Let U, V be smooth $\mathbb{C}$-schemes, $\Phi : U \to V$ an étale morphism, and $g : V \to \mathbb{C}$ a regular function. Write $f = g \circ \Phi : U \to \mathbb{C}$, and $X = \mathrm{Crit}(f)$, $Y = \mathrm{Crit}(g)$ as $\mathbb{C}$-subschemes of U, V. Then $\Phi|_X : X \to Y$ is étale. Define an isomorphism

$$\mathcal{PV}_\Phi : \mathcal{PV}^\bullet_{U,f} \longrightarrow \Phi|_X^*\big(\mathcal{PV}^\bullet_{V,g}\big) \quad \text{in } \mathrm{Perv}(X) \tag{2.14}$$

by the commutative diagram for each $c \in f(X) \subseteq g(Y)$:

$$\begin{tikzcd}
\mathcal{PV}^\bullet_{U,f}|_{X_c} = \phi^p_{f-c}(A_U[\dim U])|_{X_c} \arrow[r, "\alpha"'] \arrow[d, "\mathcal{PV}_\Phi|_{X_c}"] & \phi^p_{f-c} \circ \Phi^*(A_V[\dim V]))|_{X_c} \arrow[d, "\beta"'] \\
\Phi|_{X_c}^*\big(\mathcal{PV}^\bullet_{V,g}\big) \arrow[r, equal] & \Phi_0^* \circ \phi^p_{g-c} \circ (A_V[\dim V]))|_{X_c}.
\end{tikzcd} \tag{2.15}$$

Here α is ϕ^p_{f-c} applied to the canonical isomorphism $A_U \to \Phi^*(A_V)$, noting that, as Φ is étale, $\dim U = \dim V$ and β is induced by (2.4).

By naturality of the isomorphisms α, β in (2.15) we find the following commute, where $\sigma_{U,f}, \tau_{U,f}$ are as in (2.6)–(2.7):

$$\begin{tikzcd}
\mathcal{PV}^\bullet_{U,f} \arrow[rr, "\sigma_{U,f}"'] \arrow[d, "\mathcal{PV}_\Phi"] & & \mathbb{D}_X(\mathcal{PV}^\bullet_{U,f}) \\
\Phi|_X^*\big(\mathcal{PV}^\bullet_{V,g}\big) \arrow[r, "\Phi|_X^*(\sigma_{V,g})"] & \Phi|_X^*\big(\mathbb{D}_Y(\mathcal{PV}^\bullet_{V,g})\big) \arrow[r, "\cong"] & \mathbb{D}_X\big(\Phi|_X^*(\mathcal{PV}^\bullet_{V,g})\big), \arrow[u, "\mathbb{D}_X(\mathcal{PV}_\Phi)"']
\end{tikzcd} \tag{2.16}$$

$$\begin{tikzcd}
\mathcal{PV}^\bullet_{U,f} \arrow[r, "\tau_{U,f}"'] \arrow[d, "\mathcal{PV}_\Phi"] & \mathcal{PV}^\bullet_{U,f} \arrow[d, "\mathcal{PV}_\Phi"'] \\
\Phi|_X^*(\mathcal{PV}^\bullet_{V,g}) \arrow[r, "\Phi|_X^*(\tau_{V,g})"] & \Phi|_X^*(\mathcal{PV}^\bullet_{V,g}).
\end{tikzcd} \tag{2.17}$$

If $U = V$, $f = g$ and $\Phi = \mathrm{id}_U$ then $\mathcal{PV}_{\mathrm{id}_U} = \mathrm{id}_{\mathcal{PV}^\bullet_{U,f}}$.

If W is another smooth $\mathbb{C}$-scheme, $\Psi : V \to W$ is étale, and $h : W \to \mathbb{C}$ is regular with $g = h \circ \Psi : V \to \mathbb{C}$, then composing (2.15) for Φ with $\Phi|_{X_c}^*$ of (2.15) for Ψ shows that

$$\mathcal{PV}_{\Psi\circ\Phi} = \Phi|_X^*(\mathcal{PV}_\Psi) \circ \mathcal{PV}_\Phi : \mathcal{PV}^\bullet_{U,f} \longrightarrow (\Psi \circ \Phi)|_X^*\big(\mathcal{PV}^\bullet_{W,h}\big). \tag{2.18}$$

That is, the isomorphisms $\mathcal{PV}_\Phi$ are functorial.

Example 2.16. In Definition 2.15, set $U = V = \mathbb{C}^n$ and

$$f(z_1, \ldots, z_n) = g(z_1, \ldots, z_n) = z_1^2 + \cdots + z_n^2,$$

so that $Y = Z = \{0\} \subset \mathbb{C}^n$. Let $M \in \mathrm{O}(n, \mathbb{C})$ be an orthogonal matrix, so that $M : \mathbb{C}^n \to \mathbb{C}^n$ is an isomorphism with $f = g \circ M$ and $M|_{\{0\}} = \mathrm{id}_{\{0\}}$. As $M|_Y = \mathrm{id}_Y$, Definition 2.15 defines an isomorphism

$$\mathcal{PV}_M : \mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2+\cdots+z_n^2} \longrightarrow \mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2+\cdots+z_n^2}. \tag{2.19}$$

Equation (2.11) describes $\mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2+\cdots+z_n^2}$ in terms of $MF_f(0) \cong T^*\mathcal{S}^{n-1}$. Now

$$M|_{MF_f(0)} : MF_f(0) \longrightarrow MF_f(0)$$

multiplies orientations on $\mathcal{S}^{n-1}$ by $\det M$, so

$$(M|_{MF_f(0)})_* : H^{n-1}\big(MF_f(0); A\big) \longrightarrow H^{n-1}\big(MF_f(0); A\big)$$

is multiplication by $\det M$. Thus (2.11) implies that $\mathcal{PV}_M$ in (2.19) is multiplication by $\det M = \pm 1$.

2.5. **Summary of the properties we use in this paper.** Since parts of §2.1–§2.4 do not work for the other kinds of perverse sheaves, $\mathscr{D}$-modules and mixed Hodge modules in §2.6–§2.10, we list what we will need for §3–§6, to make it easy to check they are also valid in the settings of §2.6–§2.10.

(i) There should be an A-linear abelian category $\mathcal{P}(X)$ of $\mathcal{P}$-objects defined for each scheme or complex analytic space X, over a fixed, well-behaved base ring A. We do not require A to be a field.

(ii) There should be a Verdier duality functor $\mathbb{D}_X$ with $\mathbb{D}_X \circ \mathbb{D}_X \cong \mathrm{id}$, defined on a suitable subcategory of $\mathcal{P}$-objects on X which includes the objects we are interested in. We do not need $\mathbb{D}_X$ to be defined on all objects in $\mathcal{P}(X)$.

(iii) If U is a smooth scheme or complex manifold, then there should be a canonical object $A_U[\dim U] \in \mathcal{P}(U)$, with a canonical isomorphism

$$\mathbb{D}_U(A_U[\dim U]) \cong A_U[\dim U].$$

(iv) Let $f : X \hookrightarrow Y$ be a closed embedding of schemes or complex analytic spaces; this implies f is proper. Then $f_*, f_! : \mathcal{P}(X) \to \mathcal{P}(Y)$ should exist, inducing an equivalence of categories $\mathcal{P}(X) \xrightarrow{\sim} \mathcal{P}_X(Y)$ as in Theorem 2.6(c), where $\mathcal{P}_X(Y)$ is the full subcategory of objects in $\mathcal{P}(Y)$ supported on X.

(v) Let $f : X \to Y$ be an étale morphism. Then the pullbacks $f^*, f^! : \mathcal{P}(Y) \to \mathcal{P}(X)$ should exist. More generally, if $f : X \to Y$ is smooth of relative dimension d, then there should be pullbacks $f^*[d], f^![-d]$ mapping $\mathcal{P}(Y) \to \mathcal{P}(X)$. If X, Y are smooth, there should be a canonical isomorphism $f^*[d](A_Y[\dim Y]) \cong A_X[\dim X]$. We do not need pullbacks to exist for general morphisms $f : X \to Y$, though see (xi) below.

(vi) An external tensor product $\boxtimes : \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y)$ should exist for all X, Y.

(vii) If X is a scheme or complex analytic space, $P \to X$ a principal $\mathbb{Z}/2\mathbb{Z}$-bundle, and $\mathcal{Q}^\bullet \in \mathcal{P}(X)$, the twisted perverse sheaf $\mathcal{Q}^\bullet \otimes_{\mathbb{Z}/2\mathbb{Z}} P \in \mathcal{P}(X)$ should make sense as in Definition 2.9, and have the obvious functorial properties.

(viii) A vanishing cycle functor $\phi_f^p : \mathcal{P}(U) \to \mathcal{P}(U_0)$ and a monodromy transformation

$$M_{U,f} : \phi_f^p \Rightarrow \phi_f^p$$

in §2.4 should exist for all smooth U and regular/holomorphic $f : U \to \mathbb{A}^1$.

(ix) The functors $\mathbb{D}_X, f^*, f^!, f_*, f_!, \phi_f^p$ should satisfy the natural isomorphisms in Theorems 2.4 and 2.11, provided they exist. They should have the obvious compatibilities with $\overset{L}{\boxtimes}$, and restriction to (Zariski) open sets.

(x) There should be suitable subcategories of $\mathcal{P}$-objects which form a stack in the étale or complex analytic topologies, as in Theorem 2.7. In the algebraic case we only need Theorem 2.7(ii) to hold for Zariski open covers, not étale open covers.

(xi) Proposition 2.8 must hold. This involves pullbacks j_t^* by a morphism $j_t : W_t \hookrightarrow W$ which is not étale or smooth, as in (v) above. But on objects we only consider

$$j_t^*\big(\pi_X^*(\mathcal{P}^\bullet)\big) = \pi_X|_{W_t}^*(\mathcal{P}^\bullet)$$

which exists in $\mathcal{P}(W_t)$ by (v) as $\pi_X|_{W_t}$ is étale, so j_t^* is defined on the objects we need.

(xii) There should be a Thom–Sebastiani Theorem for $\mathcal{P}$-objects, so that the analogue of Theorem 2.13 holds.

Remark 2.17. The existence of a (bounded) derived category of $\mathcal{P}$-objects will not be assumed, or used, in this paper. On the other hand, in all the cases we consider, there will be a realization functor from the category of $\mathcal{P}$-objects to an appropriate category of constructible complexes, and the notation used above reflects this. So in (iii),(v) above, [1] does not stand for a shift in any derived category; the notation means a $\mathcal{P}$-object or morphism whose realization is the appropriate constructible object or morphism. See Remark 2.20 below.

2.6. **Perverse sheaves on complex analytic spaces.** Next we discuss perverse sheaves on complex analytic spaces, as in Dimca [14]. The theory follows §2.1–§2.4, replacing (smooth) $\mathbb{C}$-schemes by complex analytic spaces (complex manifolds), and regular functions by holomorphic functions.

Let X be a complex analytic space, always assumed locally of finite type (that is, locally embeddable in $\mathbb{C}^n$). In the analogue of Definition 2.2, we fix a well-behaved commutative ring A, and consider sheaves of A-modules $\mathcal{S}$ on X in the complex analytic topology. A sheaf $\mathcal{S}$ is called (*analytically*) *constructible* if all the stalks $\mathcal{S}_x$ for $x \in X$ are finite type A-modules, and there is a locally finite stratification $X = \coprod_{j\in J} X_j$ of X, where now $X_j \subseteq X$ for $j \in J$ are complex analytic subspaces of X, such that $\mathcal{S}|_{X_j}$ is an A-local system for all $j \in J$.

Write $D(X)$ for the derived category of complexes $\mathcal{C}^\bullet$ of sheaves of A-modules on X, exactly as in §2.1, and $D^b_c(X)$ for the full subcategory of bounded complexes $\mathcal{C}^\bullet$ in $D(X)$ whose cohomology sheaves $\mathcal{H}^m(\mathcal{C}^\bullet)$ are analytically constructible for all $m \in \mathbb{Z}$. Then $D(X), D^b_c(X)$ are triangulated categories.

When we wish to distinguish the complex algebraic and complex analytic theories, we will write $D^b_c(X)^{\rm alg}, \mathrm{Perv}(X)^{\rm alg}$ for the algebraic versions in §2.1–§2.2 with X a $\mathbb{C}$-scheme, and $D^b_c(X)^{\rm an}, \mathrm{Perv}(X)^{\rm an}$ for the analytic versions.

Here are the main differences between the material of §2.1–§2.4 for perverse sheaves on $\mathbb{C}$-schemes and on complex analytic spaces:

(a) If $f : X \to Y$ is an arbitrary morphism of $\mathbb{C}$-schemes, then as in §2.1 the pushforwards $Rf_*, Rf_! : D(X) \to D(Y)$ also map $D^b_c(X)^{\rm alg} \to D^b_c(Y)^{\rm alg}$.

However, if $f : X \to Y$ is a morphism of complex analytic spaces, then

$$Rf_*, Rf_! : D(X) \longrightarrow D(Y)$$

need not map $D^b_c(X)^{\rm an} \to D^b_c(Y)^{\rm an}$ without extra assumptions on f, for example, if $f : X \to Y$ is proper.

(b) The analogue of Theorem 2.7 says that perverse sheaves on a complex analytic space X form a stack in the complex analytic topology. This is proved in the subanalytic context in [27, Th. 10.2.9]; the analytic case follows upon noting that a sheaf is complex analytically constructible if and only if is locally at all points, as proved in [14, Prop. 4.1.13]. See also [21, Prop. 8.1.26].

The analogues of (i)–(xii) in §2.5 work for complex analytic perverse sheaves, and so our main results hold in this context.

If X is a $\mathbb{C}$-scheme, and X^{an} the corresponding complex analytic space, then $D(X)$ in §2.1 for X a $\mathbb{C}$-scheme coincides with $D(X^{\mathrm{an}})$ for X^{an} a complex analytic space, and

$$D^b_c(X)^{\mathrm{alg}} \subset D^b_c(X^{\mathrm{an}})^{\mathrm{an}}, \quad \mathrm{Perv}(X)^{\mathrm{alg}} \subset \mathrm{Perv}(X^{\mathrm{an}})^{\mathrm{an}}$$

are full subcategories, and the six functors f^*, $f^!$, Rf_*, $Rf_!$, $\mathcal{R}Hom$, $\overset{L}{\otimes}$ for $\mathbb{C}$-scheme morphisms $f : X \to Y$ agree in the algebraic and analytic cases.

2.7. **$\mathscr{D}$-modules on $\mathbb{C}$-schemes and complex analytic spaces.** *$\mathscr{D}$-modules* on a smooth $\mathbb{C}$-scheme or smooth complex analytic space X are sheaves of modules over a certain sheaf of rings of differential operators $\mathscr{D}_X$ on X. Some books on them are Borel et al. [8], Coutinho [12], and Hotta, Takeuchi and Tanisaki [21] in the $\mathbb{C}$-scheme case, and Björk [7] and Kashiwara [26] in the complex analytic case. For a singular complex $\mathbb{C}$-scheme or complex analytic space X, the definition of a well-behaved category of $\mathscr{D}$-modules is given by Saito [48], via locally embedding X into a smooth scheme or space.

The analogue of perverse sheaves on X are called *regular holonomic $\mathscr{D}$-modules*, which form an abelian category $\mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X)$, the heart in the derived category $D^b_{\mathrm{rh}}(\mathrm{Mod}(\mathscr{D}_X))$ of bounded complexes of $\mathscr{D}_X$-modules with regular holonomic cohomology modules. The whole package of §2.1–§2.4 works for $\mathscr{D}$-modules. Our next theorem is known as the *Riemann–Hilbert correspondence* [7, §V.5], [21, Th. 7.2.1], see Borel [8, §14.4] for $\mathbb{C}$-schemes, Kashiwara [25] for complex manifolds, and Saito [48, §6] for complex analytic spaces, and also Maisonobe and Mekhbout [34].

Theorem 2.18. *Let X be a $\mathbb{C}$-scheme or complex analytic space. Then there is a* ***de Rham functor*** $\mathrm{DR} : D^b_{\mathrm{rh}}(\mathrm{Mod}(\mathscr{D}_X)) \xrightarrow{\sim} D^b_c(X, \mathbb{C})$*, which is an equivalence of categories, restricts to an equivalence* $\mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X) \xrightarrow{\sim} \mathrm{Perv}(X, \mathbb{C})$*, and commutes with $f^*, f^!, Rf_*, Rf_!, \mathcal{R}Hom, \overset{L}{\otimes}$, and also with ψ^p_f, ϕ^p_f for X smooth. Here $D^b_c(X, \mathbb{C}), \mathrm{Perv}(X, \mathbb{C})$ are constructible complexes and perverse sheaves over the base ring $A = \mathbb{C}$.*

Because of the Riemann–Hilbert correspondence, all our results on perverse sheaves of vanishing cycles on $\mathbb{C}$-schemes and complex analytic spaces in §3–§6 over a well-behaved base ring A, translate immediately when $A = \mathbb{C}$ to the corresponding results for $\mathscr{D}$-modules of vanishing cycles, with no extra work.

2.8. **Mixed Hodge modules: basics.** We write this section in the minimal generality needed for our applications. The statements made work equally well in the category of (algebraic) $\mathbb{C}$-schemes and the category of complex analytic spaces. By space, we will mean an object in either of these categories. The theory of mixed Hodge modules works with reduced spaces; should a space X be non-reduced, the following constructions are taken by definition on its reduction.

For a space X, let $\mathrm{HM}(X)$ denote Saito's category [45] of polarizable pure Hodge modules, (locally) a direct sum of subcategories $\mathrm{HM}(X)^w$ of pure Hodge modules of fixed weight w. On a smooth X, a pure Hodge module $M^\bullet$ consists of a triple of data: a filtered holonomic $\mathscr{D}$-module (M, F), a $\mathbb{Q}$-perverse sheaf, and a comparison map identifying the former with the complexification of the latter under the Riemann–Hilbert correspondence; see [45, §5.1.1, p. 952] and [47, §4]. This triple has to satisfy many other properties; in particular, the underlying holonomic $\mathscr{D}$-module is automatically regular, and algebraic Hodge modules are asked to be extendable to an algebraic compactification. Thus there is a forgetful functor $\mathrm{HM}(X) \to \mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X)$ from Hodge modules to regular holonomic (algebraic) $\mathscr{D}$-modules. Hodge modules on singular spaces are defined, similarly to $\mathscr{D}$-modules, via embeddings into smooth varieties; see Saito [47] and also Maxim, Saito and Schürmann [39, §1.8].

There is a duality functor $\mathbb{D}_X^H : \mathrm{HM}(X) \to \mathrm{HM}(X)$. Pure Hodge modules also admit a Tate twist functor $M^\bullet \mapsto M^\bullet(1)$, see [45, §5.1.3, p. 952]. This functor shifts the filtration and rotates the rational structure on the underlying perverse sheaf: the $\mathscr{D}$-module filtration (M, F) is shifted to $(M, F[n])$ with $(F[n])_i = F_{i-n}$; the underlying perverse sheaf is tensored by $\mathbb{Z}(n) = (2\pi i)^n\mathbb{Z} \subset \mathbb{C}$, as in [45, (2.0.2), p. 876].

A polarization of weight w on a pure Hodge module $M^\bullet \in \mathrm{HM}(X)^w$ is a morphism of pure Hodge modules

$$\sigma : M^\bullet \longrightarrow \mathbb{D}_X^H(M^\bullet)(-w),$$

satisfying the extra conditions using vanishing cycles described on [45, (5.1.6.2) on p. 956 and (5.2.10.2) on p. 968], as well as the condition that on points it should correspond to the classical notion of a polarization of a pure Hodge structure (including positive definiteness).

Next, let $\mathrm{MHM}(X)$ denote the category of graded polarizable mixed Hodge modules [45, 47]. A graded polarizable mixed Hodge module carries a functorial weight filtration W, with graded pieces being polarizable pure Hodge modules, see [45, §5.2.10, p. 967-8]. The forgetful functor $\mathbf{rat} : \mathrm{MHM}(X) \to \mathrm{Perv}(X)$ to the appropriate category of perverse $\mathbb{Q}$-sheaves on X is faithful and exact; faithfulness in particular means that a morphism in $\mathrm{MHM}(X)$ is uniquely determined by the underlying morphism of perverse sheaves. The Tate twist functor extends to $\mathrm{MHM}(X)$; under this functor, the weight filtration W of the mixed Hodge module is changed to $W[2n]$ with $W[2n]_i = W_{i+2n}$ as on [45, p. 855]. The duality functor $\mathbb{D}_X^H$ also extends to $\mathrm{MHM}(X)$ and is compatible with Verdier duality on the perverse realization. There is also a forgetful functor $\mathrm{MHM}(X) \to \mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X)$ to regular holonomic $\mathscr{D}$-modules, even for singular spaces.

Theorem 2.19. *The categories of graded polarizable mixed Hodge modules have the following properties:*

- **(i)** *By* [47, Th. 3.9, p. 288], *the category of mixed Hodge modules for X a point is canonically equivalent to Deligne's category of graded polarizable mixed Hodge structures.*
- **(ii)** *For a smooth space U, we have a canonical object of weight* $\dim U$

$$\mathbb{Q}_U^H[\dim U] \in \mathrm{HM}(U) \subset \mathrm{MHM}(U),$$

which by [45, Prop. 5.2.16, p. 971] *possesses a canonical polarization*

$$\sigma : \mathbb{Q}_U^H[\dim U] \longrightarrow \mathbb{D}_U^H \mathbb{Q}_U^H[\dim U](-\dim U).$$

- **(iii)** *For an open inclusion $f : Y \hookrightarrow X$ of spaces, there is a pullback functor*

$$f^* = f^! : \mathrm{MHM}(X) \longrightarrow \mathrm{MHM}(Y).$$

More generally, by [47, Prop. 2.19, p. 258], *for an arbitrary morphism $f : Y \to X$, there exist cohomological pullback functors $L^j f^*, L^j f^! : \mathrm{MHM}(X) \to \mathrm{MHM}(Y)$ compatible with (perverse) cohomological pullback on the perverse sheaf level.*

- **(iv)** *For a closed embedding $i : X \hookrightarrow Y$, there is a pushforward functor*

$$i_* = i_! : \mathrm{MHM}(X) \longrightarrow \mathrm{MHM}(Y),$$

whose essential image is the full subcategory $\mathrm{MHM}_X(Y)$ of objects in $\mathrm{MHM}(Y)$ supported on X. Its inverse is $i^* = i^! : \mathrm{MHM}_X(Y) \to \mathrm{MHM}(X)$. More generally, by [45, Th. 5.3.1, p. 977] *and* [47, Th. 2.14, p. 252], *for a* ***projective*** *map $f : X \to Y$ there are cohomological pushforward functors*

$$R^j f_* : \mathrm{MHM}(X) \longrightarrow \mathrm{MHM}(Y).$$

(v) *There is an external tensor product functor*

$$\overset{L}{\boxtimes} : \mathrm{MHM}(X) \times \mathrm{MHM}(Y) \longrightarrow \mathrm{MHM}(X \times Y),$$

which is compatible with duality in the sense that for all $M^\bullet \in \mathrm{MHM}(X)$ *and* $N^\bullet \in \mathrm{MHM}(Y)$, *there is a natural isomorphism*

$$\mathbb{D}^H_X M^\bullet \overset{L}{\boxtimes} \mathbb{D}^H_Y N^\bullet \cong \mathbb{D}^H_{X\times Y}(M^\bullet \overset{L}{\boxtimes} N^\bullet).$$

Remark 2.20. We will not need to use any derived category $D^? \mathrm{MHM}(X)$ of mixed Hodge modules in this paper, which is just as well since on singular analytic X, the appropriate boundedness conditions do not appear to be well understood, and the general pullback and pushforward functors of Theorem 2.19(iii),(iv) are not known to exist as derived functors outside of the algebraic context of [47, §4]. Hence, in part (ii) above, [1] does not stand for a shift in the derived category; $\mathbb{Q}^H_U[\dim U]$ just denotes a mixed Hodge module whose realization is the perverse sheaf $\mathbb{Q}_U[\dim U]$ on U. Compare Remark 2.17 above.

Using the functors above, we can now define the twist of a mixed Hodge module by a principal $\mathbb{Z}/2\mathbb{Z}$-bundle. In the setup of Definition 2.9, given a $\mathbb{Z}/2\mathbb{Z}$-bundle $\pi : P \to X$, and an object $M^\bullet \in \mathrm{MHM}(X)$ on a space X, we have a natural map $M^\bullet \to \pi_*\pi^* M^\bullet$, which is an injection by faithfulness of the realization functor and the fact that it is an injection on the perverse sheaf level. The quotient object will be denoted, by abuse of notation, by $M^\bullet \otimes_{\mathbb{Z}/2\mathbb{Z}} P$ in $\mathrm{MHM}(X)$.

2.9. **Monodromic mixed Hodge modules.** To discuss nearby and vanishing cycle functors in a way consistent with monodromy, we need an extension of the category of mixed Hodge modules. For a space X, following Saito [49, §4.2] denote by $\mathrm{MHM}(X; T_s, N)$ the category of mixed Hodge modules $M^\bullet$ on X with commuting actions of a finite order operator $T_s : M^\bullet \to M^\bullet$ and a locally nilpotent operator $N : M^\bullet \to M^\bullet(-1)$. There is an embedding of categories $\mathrm{MHM}(X) \to \mathrm{MHM}(X; T_s, N)$ defined by setting $T = \mathrm{id}$ and $N = 0$. As proved by [49, (4.6.2)], the category $\mathrm{MHM}(X; T_s, N)$ is equivalent to the category $\mathrm{MHM}(X \times \mathbb{C})_{\mathrm{mon},!}$ of monodromic mixed Hodge modules on $X \times \mathbb{C}^*$ extended by zero to $X \times \mathbb{C}$; compare also [33, §4.2].

Every object $M^\bullet \in \mathrm{MHM}(X; T_s, N)$ decomposes into a direct sum $M^\bullet = M^\bullet_1 \oplus M^\bullet_{\neq 1}$ of the T_s-invariant part and its T_s-equivariant complement. The Tate twist, and appropriate cohomological pullback and pushforward functors continue to exist. There is a duality functor

$$\mathbb{D}^T_X : \mathrm{MHM}(X; T_s, N) \longrightarrow \mathrm{MHM}(X; T_s, N)$$

defined by

$$\mathbb{D}^T_X(M^\bullet) = \mathbb{D}^H_X(M^\bullet_1) \oplus \mathbb{D}^H_T(M^\bullet_{\neq 1})(1),$$

equipped with the finite-order operator $\mathbb{D}_X(T_s)^{-1}$ and the nilpotent operator $-\mathbb{D}_X(N)$. This duality functor still satisfies $\mathbb{D}^T_X \circ \mathbb{D}^T_X = \mathrm{id}$.

Saito [49, §5.1] also defines an external tensor product

$$\overset{T}{\boxtimes} : \mathrm{MHM}(X_1; T_s, N) \times \mathrm{MHM}(X_2; T_s, N) \longrightarrow \mathrm{MHM}(X_1 \times X_2; T_s, N).$$

defined on the monodromic category as follows. The addition map on fibres

$$\pi : (X_1 \times \mathbb{C}) \times (X_2 \times \mathbb{C}) \longrightarrow (X_1 \times X_2) \times \mathbb{C}$$

induces the additive convolution

$$\pi_*(- \boxtimes -) : \mathrm{MHM}(X_1 \times \mathbb{C})_{\mathrm{mon},!} \times \mathrm{MHM}(X_2 \times \mathbb{C})_{\mathrm{mon},!} \longrightarrow \mathrm{MHM}(X_1 \times x_2 \times \mathbb{C})_{\mathrm{mon},!}.$$

One can translate this external tensor product $\overset{T}{\boxtimes}$ to the $\mathrm{MHM}(X; T_s, N)$ defined by concrete data $(M^\bullet, T_s, N)$. On the underlying $\mathscr{D}$-modules and perverse sheaves, it is just the usual product

$\boxtimes$. The operators are defined by $T_s = T_s \boxtimes T_s$ and $N = N \boxtimes \mathrm{id} + \mathrm{id} \boxtimes N$. However, the Hodge and weight filtrations on the underlying $\mathscr{D}$-modules and perverse sheaves are shifted using the finite order endomorphisms T_s; for details, see [49, (5.1.1)–(5.1.2)]. Note that as a consequence of these definitions, the forgetful functors

$$\mathrm{MHM}(-;T_s,N) \longrightarrow \mathrm{MHM}(-)$$

do not map $\overset{T}{\boxtimes}$ to $\overset{L}{\boxtimes}$. Twisted duality and the twisted tensor product commute in the sense that given $M^\bullet \in \mathrm{MHM}(X;T_s,N)$ and $N^\bullet \in \mathrm{MHM}(Y;T_s,N)$, we have a natural isomorphism in $\mathrm{MHM}(X \times Y;T_s,N)$:

$$\mathbb{D}_X^T(M^\bullet) \overset{T}{\boxtimes} \mathbb{D}_Y^T(N^\bullet) \cong \mathbb{D}_{X\times Y}^T(M^\bullet \overset{T}{\boxtimes} N^\bullet). \tag{2.20}$$

For an object $M^\bullet \in \mathrm{MHM}(X;T_s,N)$ whose weight filtration is a (suitable shifted) monodromy filtration of the nilpotent morphism N, there is a stronger notion of polarization which will be useful for us. A *strong polarization of weight* w of such an object $M^\bullet$ is a morphism

$$\sigma : M^\bullet \longrightarrow \mathbb{D}_X^T(M^\bullet)(-w)$$

in $\mathrm{MHM}(X)$, compatible with T_s and N, such that σ defines polarizations on the N-primitive parts of $M^\bullet$, compatible with Hodge filtrations; for precise conditions, see [45, p. 855]. A polarization on a pure Hodge module is a strong polarization (with $N = 0$); a strongly polarized mixed Hodge module is graded polarizable. The partial twist in the definition of $\mathbb{D}_X^T$ implies that $M^\bullet$ is of weight w if and only if $M_1^\bullet$, respectively $M_{\neq 1}^\bullet$ are of weights $w, w-1$ in the sense of [45, p. 855].

Given strongly polarized mixed Hodge modules $M_i^\bullet \in \mathrm{MHM}(X_i;T_s,N)$ of weight w_i for $i = 1,2$, polarized by $\sigma_i : M_i^\bullet \to \mathbb{D}_{X_i}^T(M_i^\bullet)(-w_i)$, there is an induced morphism σ in a commutative diagram

$$\begin{tikzcd}
M_1^\bullet \overset{T}{\boxtimes} M_2^\bullet \arrow[r] \arrow[dr, "\sigma"'] & \mathbb{D}_{X_1}^T(M_1^\bullet)(-w_1) \overset{T}{\boxtimes} \mathbb{D}_{X_2}^T(M_2^\bullet)(-w_2) \arrow[d] \\
& \mathbb{D}_{X_1\times X_2}^T(M_1^\bullet \overset{T}{\boxtimes} M_2^\bullet)(-w_1-w_2),
\end{tikzcd}$$

where the top map is $\sigma_1 \overset{T}{\boxtimes} \sigma_2$ and the right is the isomorphism (2.20). In general, it is *not clear* whether this morphism is a strong polarization of the tensor product $M_1^\bullet \overset{T}{\boxtimes} M_2^\bullet$; this result is not available in the literature. However, in this paper we only use this construction in cases where one of the monodromic mixed Hodge modules is essentially trivial, living on $X_1 = \mathrm{pt}$ with $N = 0$, in which case it is easy to check that the resulting σ is a strong polarization.

Note also that if $M^\bullet$ is strongly polarized by $\sigma : M^\bullet \to \mathbb{D}_X^T(M^\bullet)(-w)$, then its Tate twist is also strongly polarized by the composition

$$M^\bullet(1) \xrightarrow{\sigma(1)} \mathbb{D}_X^T(M^\bullet)(-w+1) \xrightarrow{\sim} \mathbb{D}_X^T(M^\bullet(1))(-w+2). \tag{2.21}$$

The notion of strong polarization leads to gluing, in the following way.

Theorem 2.21. *Let $X = \bigcup_i U_i$ be an open cover of a space X, in any of the Zariski, étale or complex analytic topologies. Then:*

- **(i)** *Suppose we are given mixed Hodge modules $M^\bullet, N^\bullet \in \mathrm{MHM}(X)$, with morphisms $f_i : M^\bullet|_{U_i} \to N^\bullet|_{U_i}$ in $\mathrm{MHM}(U_i)$ which agree on overlaps U_{ij}. Then there is a unique $f \in \mathrm{Hom}_{\mathrm{MHM}(X)}(M^\bullet, N^\bullet)$ with $f|_{U_i} = f_i$.*

(ii) *Suppose we are given mixed Hodge modules $M_i^\bullet \in \mathrm{MHM}(U_i;T_s,N)$, each equipped with a strong polarization σ_i. Suppose also that we are given isomorphisms*

$$\alpha_{ij} : M_i^\bullet|_{U_{ij}} \longrightarrow M_j^\bullet|_{U_{ij}}$$

on intersections, commuting with the restrictions of the maps T_{si}, N_i and σ_i, with

$$\alpha_{jk}|_{U_{ijk}} \circ \alpha_{ij}|_{U_{ijk}} = \alpha_{ik}|_{U_{ijk}}$$

on triple intersections. Then there is a strongly polarized mixed Hodge module

$$M^\bullet \in \mathrm{MHM}(X;T_s,N),$$

restricting to $M_i^\bullet$ on U_i.

Proof. To prove (i), it is enough to note that the f_i glue on the perverse sheaf and $\mathscr{D}$-module levels, respecting filtrations.

To prove (ii), we begin with the case of pure Hodge modules of fixed weight w. The data of a pure Hodge module consists of a pair of a filtered holonomic $\mathscr{D}$-module and a $\mathbb{Q}$-perverse sheaf, with an identification of the former with the complexification of the latter under the Riemann–Hilbert correspondence. Since both filtered holonomic $\mathscr{D}$-modules and $\mathbb{Q}$-perverse sheaves form stacks (both in the algebraic and the analytic case), this data glues over X. As for the (strong) polarization, $T_{si} = \mathrm{id}$ and $N_i = 0$ glue to $T_s = \mathrm{id}$ and $N = 0$, whereas the map σ on the perverse sheaf level glues from the maps σ_i once again from the stack property (now for morphisms) of perverse sheaves.

The conditions [45, §5.1.6, p. 955] which make such a pair a pure Hodge module come from local conditions as well as conditions on vanishing cycles; the latter glue by induction on the dimension. So strongly polarized pure Hodge modules form a stack. The case of mixed Hodge modules is similar: we need to glue filtrations and polarizations, as well as the maps T_{si}, N_i and σ_i, first on the level of perverse sheaves, and then checking the axioms, which are local or follow by induction. □

Remark 2.22. Given a projective $\mathbb{C}$-scheme X, and a polarizable mixed Hodge module $M^\bullet$ in $\mathrm{MHM}(X)$ on it, the second part of Theorem 2.19(iv) applied to $f : X \to \mathrm{pt}$ shows that the hypercohomology $\mathbb{H}^*(X, M^\bullet)$ carries a mixed Hodge structure. In particular, it carries a weight filtration and therefore has a weight polynomial, which will be useful in refinements of Donaldson–Thomas theory, see the discussion in Remark 6.14 below. So we need to glue polarizable objects from local data. On the other hand, graded polarizable mixed Hodge modules may fail to form a stack in the analytic category unless the polarizations glue. This is the reason for using the stronger form of polarization, which allows for gluing as shown above.

2.10. **Mixed Hodge modules of vanishing cycles.** By Saito's work [45–47], for a regular function $f : U \to \mathbb{C}$ on a smooth space U, the perverse nearby and vanishing cycle functors ψ_f^p, ϕ_f^p defined on perverse sheaves in §2.3 lift to functors

$$\psi_f^H, \phi_f^H : \mathrm{MHM}(U) \longrightarrow \mathrm{MHM}(U_0;T_s,N),$$

where $U_0 = f^{-1}(0)$. The actions of the finite order and nilpotent operators T_s, N are given by the semisimple part of the monodromy operator, and the logarithm of its unipotent part. The analogue

$$\phi_f^H \circ \mathbb{D}_U^T \cong \mathbb{D}_{U_0}^T \circ \phi_f^H$$

of Theorem 2.11(iv) is proved in [46]; to make this isomorphism work is the reason for the the twist in the definition of $\mathbb{D}_U^T$. Note also that [46, Th. 1.6] fits with the convention that T_s and N are defined on dual objects as $\mathbb{D}_U(T_s)^{-1}$ and $\mathbb{D}_U(N)$, respectively.

By [45, §5.2], if $M^\bullet \in \mathrm{HM}(U)$ is a pure Hodge module, then a polarization of $M^\bullet$ induces a strong polarization on the (mixed) Hodge module of vanishing cycles $\phi_f^H(M^\bullet)$, of the same weight. In particular, if $M^\bullet = \mathbb{Q}_U^H[\dim U]$ is the canonical object with its canonical polarization from Theorem 2.19(ii), then $\phi_f^H\bigl(\mathbb{Q}_U^H[\dim U]\bigr) \in \mathrm{MHM}(\mathrm{Crit}(f); T_s, N)$ is a strongly polarized mixed Hodge module on the critical locus of f, with polarization

$$\sigma : \phi_f^H\bigl(\mathbb{Q}_U^H[\dim U]\bigr) \longrightarrow \mathbb{D}^T_{\mathrm{Crit}(f)} \circ \phi_f^H\bigl(\mathbb{Q}_U^H[\dim U]\bigr)(-\dim U). \tag{2.22}$$

Example 2.23. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^2$. Then $\mathrm{Crit}(f) = \{0\}$, and we obtain an object $\phi_f^H\bigl(\mathbb{Q}_{\mathbb{C}}^H[1]\bigr)$ in $\mathrm{MHM}(\mathrm{pt}; T_s, N)$, a one-dimensional polarized mixed Hodge structure with monodromy acting by $T_s = -\mathrm{id}$ and $N = 0$. For $g : \mathbb{C}^2 \to \mathbb{C}$ given by $g(z_1, z_2) = z_1^2 + z_2^2$, it is well known that

$$\phi^H_{z_1^2+z_2^2}\bigl(\mathbb{Q}^H_{\mathbb{C}^2}[2]\bigr) \cong \mathbb{Q}(-1),$$

with trivial monodromy action. Applying the Thom–Sebastiani formula for mixed Hodge modules [49, Th. 5.4], we see that

$$\phi^H_{z^2}\bigl(\mathbb{Q}^H_{\mathbb{C}}[1]\bigr) \overset{T}{\boxtimes} \phi^H_{z^2}\bigl(\mathbb{Q}^H_{\mathbb{C}}[1]\bigr) \cong \mathbb{Q}(-1)$$

in the category $\mathrm{MHM}(\mathrm{pt}; T_s, N)$. The objects $\mathbb{Q}(1)$ and $\mathbb{Q}(-1)$ thus admit square roots under $\overset{T}{\boxtimes}$ in this category, which we will denote by $\mathbb{Q}(\frac{1}{2})$ and $\mathbb{Q}(-\frac{1}{2})$, where

$$\phi^H_{z^2}\bigl(\mathbb{Q}^H_{\mathbb{C}}[1]\bigr) = \mathbb{Q}(-\tfrac{1}{2}). \tag{2.23}$$

More explicitly, we have

$$\mathbb{Q}(-\tfrac{1}{2}) = (\mathbb{Q}(0), -\mathrm{id}, 0)$$

and

$$\mathbb{Q}(\tfrac{1}{2}) = (\mathbb{Q}(1), -\mathrm{id}, 0).$$

Define an object $\mathbb{Q}(\frac{n}{2}) \in \mathrm{MHM}(\mathrm{pt}; T_s, N)$ for each $n \in \mathbb{Z}$ by $\mathbb{Q}(\frac{n}{2}) = \mathbb{Q}(\frac{1}{2})^{\overset{T}{\boxtimes} n}$ for $n \geqslant 0$, and $\mathbb{Q}(\frac{n}{2}) = \mathbb{Q}(-\frac{1}{2})^{\overset{T}{\boxtimes} -n}$ for $n < 0$. For any space X with structure morphism $\pi : X \to \mathrm{pt}$, and any $M^\bullet \in D^b\,\mathrm{MHM}(X; T_s, N)$, we define the $\frac{n}{2}$ *twist of* $M^\bullet$ to be $M^\bullet(\frac{n}{2}) = M^\bullet \overset{T}{\boxtimes} \bigl(\mathbb{Q}(\frac{n}{2})\bigr)$. If $M^\bullet$ is strongly polarized, then this tensor product is also strongly polarized by the tensor polarization by our comments above.

Let U be a smooth space, $f : U \to \mathbb{C}$ a regular function, and $X = \mathrm{Crit}(f)$ its critical locus, as a subspace of U. The perverse sheaf of vanishing cycles $\mathcal{PV}^\bullet_{U,f} \in \mathrm{Perv}(X)$ from §2.4 has a lift to a mixed Hodge module $\mathcal{HV}^\bullet_{U,f}$ in $\mathrm{MHM}(X; T_s, N)$, defined for each $c \in f(X)$ by

$$\mathcal{HV}^\bullet_{U,f}|_{X_c} = \phi^H_{f-c}\bigl(\mathbb{Q}^H_U[\dim U]\bigr)\big|_{X_c}\bigl(\tfrac{1}{2}\dim U\bigr) \in \mathrm{MHM}(X_c; T_s, N). \tag{2.24}$$

This mixed Hodge module inherits a strong polarization of weight 0 (compare (2.21) and (2.22))

$$\sigma^H_{U,f} : \mathcal{HV}^\bullet_{U,f} \longrightarrow \mathbb{D}^T_X\bigl(\mathcal{HV}^\bullet_{U,f}\bigr). \tag{2.25}$$

The twist $(\frac{1}{2}\dim U)$ in (2.24), using the notation of Example 2.23, is included for the same reason as the $(-1)^{\dim U}$ in the definition (2.7) of $\tau_{U,f}$. It makes $\mathcal{HV}^\bullet_{U,f}$ act naturally under transformations which change dimension — without it, the mixed Hodge module version of (5.15) below would have to include a twist $(\frac{1}{2}n)$ for $n = \dim V - \dim U$. Then

$$\mathcal{HV}^\bullet_{U,f}, \quad T_s : \mathcal{HV}^\bullet_{U,f} \to \mathcal{HV}^\bullet_{U,f}, \quad N : \mathcal{HV}^\bullet_{U,f} \to \mathcal{HV}^\bullet_{U,f}(-1), \quad \text{and} \quad \sigma^H_{U,f} : \mathcal{HV}^\bullet_{U,f} \to \mathbb{D}^T_X\bigl(\mathcal{HV}^\bullet_{U,f}\bigr)$$

are related to

$$\mathcal{PV}^\bullet_{U,f}, \quad \tau_{U,f} : \mathcal{PV}^\bullet_{U,f} \to \mathcal{PV}^\bullet_{U,f}, \quad \text{and} \quad \sigma_{U,f} : \mathcal{PV}^\bullet_{U,f} \xrightarrow{\cong} \mathbb{D}_X(\mathcal{PV}^\bullet_{U,f})$$

in §2.4 by

$$\mathcal{PV}^\bullet_{U,f} = \mathbf{rat}\big(\mathcal{HV}^\bullet_{U,f}\big), \ \ \tau_{U,f} = \mathbf{rat}(T_s) \circ \exp\big(2\pi i\,\mathbf{rat}(N)\big), \ \ \sigma_{U,f} = \mathbf{rat}(\sigma^H_{U,f});$$

for the last statement, see Proposition A.1 in the Appendix.

The following Thom–Sebastiani type result is the analogue of Theorem 2.13.

Theorem 2.24. *Let U, V be smooth spaces and $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ be regular functions, so that $f \boxplus g : U \times V \to \mathbb{C}$ is given by $(f \boxplus g)(u, v) := f(u) + g(v)$. Set $X = \mathrm{Crit}(f)$ and $Y = \mathrm{Crit}(g)$ as subspaces of U, V, so that $\mathrm{Crit}(f \boxplus g) = X \times Y$. Then there is a natural isomorphism*

$$\mathcal{TS}^H_{U,f,V,g} : \mathcal{HV}^\bullet_{U\times V,f\boxplus g} \xrightarrow{\cong} \mathcal{HV}^\bullet_{U,f} \overset{T}{\boxtimes} \mathcal{HV}^\bullet_{V,g} \ \text{ in } \mathrm{MHM}(X \times Y; T_s, N), \tag{2.26}$$

so that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{HV}^\bullet_{U\times V,f\boxplus g} \arrow[rr, "\sigma_{U\times V,f\boxplus g}"'] \arrow[d, "\mathcal{TS}^H_{U,f,V,g}"] & & \mathbb{D}^T_{X\times Y}(\mathcal{HV}^\bullet_{U\times V,f\boxplus g}) \\
\mathcal{HV}^\bullet_{U,f} \overset{T}{\boxtimes} \mathcal{HV}^\bullet_{V,g} \arrow[r, "\sigma_{U,f} \overset{T}{\boxtimes} \sigma_{V,g}"] & \mathbb{D}^T_X(\mathcal{HV}^\bullet_{U,f}) \overset{T}{\boxtimes} \mathbb{D}^T_Y(\mathcal{HV}^\bullet_{V,g}) \arrow[r, "\cong"] & \mathbb{D}^T_{X\times Y}\big(\mathcal{HV}^\bullet_{U,f} \overset{T}{\boxtimes} \mathcal{HV}^\bullet_{V,g}\big). \arrow[u, "\mathbb{D}^T_{X\times Y}(\mathcal{TS}^H_{U,f,V,g})"']
\end{tikzcd} \tag{2.27}$$

Proof. The existence of the isomorphism (2.26) follows from the Thom–Sebastiani Theorem for mixed Hodge modules due to Saito [49, Th. 5.4], applied to $\mathcal{HV}^\bullet_{U,f}$. The diagram (2.27) exists by (2.20); its commutativity can be checked on the level of the underlying perverse sheaves which is (2.9), in light of Propositions A.1–A.2 in the Appendix. Note that (2.26) also includes the analogue of (2.10) in Theorem 2.13, according to which we have a matching of the monodromy actions

$$\tau_{U\times V,f\boxplus g} \cong \tau_{U,f} \overset{L}{\boxtimes} \tau_{V,g},$$

as (2.26) holds in $\mathrm{MHM}(X \times Y; T_s, N)$ rather than just $\mathrm{MHM}(X \times Y)$. □

In this paper we will only ever apply Theorem 2.24 when $V = \mathbb{C}^n$, $g = z_1^2 + \cdots + z_n^2$ and $Y = \{0\}$. Combining (2.23) and (2.24) shows that

$$\mathcal{HV}^\bullet_{\mathbb{C},z^2} = \big(\mathbb{Q}(-\tfrac{1}{2})\big)(\tfrac{1}{2}) \cong \mathbb{Q}(0) \cong \mathbb{Q}^H_{\{0\}}.$$

Thus, by Theorem 2.24, $\mathbb{Q}^H_{\{0\}} \overset{T}{\boxtimes} \mathbb{Q}^H_{\{0\}} \cong \mathbb{Q}^H_{\{0\}}$, and induction on n, we see that

$$\mathcal{HV}^\bullet_{\mathbb{C}^n,z_1^2+\cdots+z_n^2} \cong \mathbb{Q}^H_{\{0\}}.$$

As for (2.12), this isomorphism is natural up to sign, depending on a choice of orientation for the complex Euclidean space $(\mathbb{C}^n, \mathrm{d}z_1^2 + \cdots + \mathrm{d}z_n^2)$.

3. Action of symmetries on vanishing cycles

Here is our first main result.

Theorem 3.1. *Let U, V be smooth $\mathbb{C}$-schemes, $\Phi, \Psi : U \to V$ étale morphisms, and $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ regular functions with $g \circ \Phi = f = g \circ \Psi$. Write $X = \mathrm{Crit}(f)$ and $Y = \mathrm{Crit}(g)$ as $\mathbb{C}$-subschemes of U, V, so that $\Phi|_X, \Psi|_X : X \to Y$ are étale morphisms. Suppose $\Phi|_X = \Psi|_X$. Then:*

(a) *As Φ,Ψ are étale, $\mathrm{d}\Phi : TU \to \Phi^*(TV)$, $\mathrm{d}\Psi : TU \to \Psi^*(TV)$ are isomorphisms of vector bundles. Restricting to the reduced $\mathbb{C}$-subscheme X^{red} of X, and using $\Phi|_{X^{\mathrm{red}}} = \Psi|_{X^{\mathrm{red}}}$ as $\Phi|_X = \Psi|_X$, gives isomorphisms*

$$\mathrm{d}\Phi|_{X^{\mathrm{red}}}, \mathrm{d}\Psi|_{X^{\mathrm{red}}} : TU|_{X^{\mathrm{red}}} \longrightarrow \Phi|_{X^{\mathrm{red}}}^*(TV),$$
$$\textit{and thus}\quad \mathrm{d}\Psi|_{X^{\mathrm{red}}}^{-1} \circ \mathrm{d}\Phi|_{X^{\mathrm{red}}} : TU|_{X^{\mathrm{red}}} \longrightarrow TU|_{X^{\mathrm{red}}}.$$

Hence

$$\det\bigl(\mathrm{d}\Psi|_{X^{\mathrm{red}}}^{-1} \circ \mathrm{d}\Phi|_{X^{\mathrm{red}}}\bigr) : X^{\mathrm{red}} \longrightarrow \mathbb{C} \setminus \{0\}$$

is a regular function. Then $\det\bigl(\mathrm{d}\Psi|_{X^{\mathrm{red}}}^{-1} \circ \mathrm{d}\Phi|_{X^{\mathrm{red}}}\bigr)$ is a locally constant map

$$X^{\mathrm{red}} \longrightarrow \{\pm 1\} \subset \mathbb{C} \setminus \{0\}.$$

(b) *Definition 2.15 defines isomorphisms $\mathcal{PV}_\Phi, \mathcal{PV}_\Psi : \mathcal{PV}_{U,f}^\bullet \to \Phi|_X^*\bigl(\mathcal{PV}_{V,g}^\bullet\bigr)$ in $\mathrm{Perv}(X)$. These are related by*

$$\mathcal{PV}_\Phi = \det\bigl(\mathrm{d}\Psi|_{X^{\mathrm{red}}}^{-1} \circ \mathrm{d}\Phi|_{X^{\mathrm{red}}}\bigr) \cdot \mathcal{PV}_\Psi, \tag{3.1}$$

regarding $\det\bigl(\mathrm{d}\Psi|_{X^{\mathrm{red}}}^{-1} \circ \mathrm{d}\Phi|_{X^{\mathrm{red}}}\bigr) : X \to \{\pm 1\}$ as a locally constant map of topological spaces, where X, X^{red} have the same topological space.

The analogues of these results also hold for $\mathscr{D}$-modules and mixed Hodge modules on $\mathbb{C}$-schemes, and (with Φ,Ψ local biholomorphisms and f, g analytic functions) for perverse sheaves, $\mathscr{D}$-modules and mixed Hodge modules on complex analytic spaces, as in §2.6–§2.10.

By taking $U = V$, $f = g$, Φ an isomorphism and $\Psi = \mathrm{id}_U$, we deduce a result on the action of symmetries on perverse sheaves of vanishing cycles:

Corollary 3.2. *Let U be a smooth $\mathbb{C}$-scheme, $\Phi : U \to U$ an isomorphism, and $f : U \to \mathbb{C}$ be regular with $f \circ \Phi = f$. Write $X = \mathrm{Crit}(f)$ as a $\mathbb{C}$-subscheme of U and X^{red} for its reduced $\mathbb{C}$-subscheme, and suppose $\Phi|_X = \mathrm{id}_X$. Then $\det\bigl(\mathrm{d}\Phi|_{X^{\mathrm{red}}} : TU|_{X^{\mathrm{red}}} \to TU|_{X^{\mathrm{red}}}\bigr)$ is a locally constant map $X^{\mathrm{red}} \to \{\pm 1\}$, and $\mathcal{PV}_\Phi : \mathcal{PV}_{U,f}^\bullet \xrightarrow{\cong} \mathcal{PV}_{U,f}^\bullet$ in $\mathrm{Perv}(X)$ from Definition 2.15 is multiplication by $\det\bigl(\mathrm{d}\Phi|_{X^{\mathrm{red}}}\bigr) = \pm 1$. The analogues hold in the settings of §2.6–§2.10.*

Example 3.3. Let $U = V = \mathbb{C}^n$ and $f(z_1,\ldots,z_n) = g(z_1,\ldots,z_n) = z_1^2 + \cdots + z_n^2$, so that $X = Y = \{0\} \subset \mathbb{C}^n$. Let $\Phi, \Psi \in \mathrm{O}(n,\mathbb{C})$ be orthogonal matrices, so that $\det\Phi, \det\Psi \in \{\pm 1\}$ and $\Phi, \Psi : \mathbb{C}^n \to \mathbb{C}^n$ are isomorphisms with $f = g \circ \Phi = g \circ \Phi$ and $\Phi|_{\{0\}} = \Psi|_{\{0\}} = \mathrm{id}_{\{0\}}$. In Theorem 3.1(a) we have

$$\mathrm{d}\Psi|_{X^{\mathrm{red}}}^{-1} \circ \mathrm{d}\Phi|_{X^{\mathrm{red}}} = \Psi^{-1} \circ \Phi : \mathbb{C}^n \longrightarrow \mathbb{C}^n,$$

so that $\det\bigl(\mathrm{d}\Psi|_{X^{\mathrm{red}}}^{-1} \circ \mathrm{d}\Phi|_{X^{\mathrm{red}}}\bigr) = \det\Psi^{-1}\det\Phi = \pm 1$.

For Theorem 3.1(b), Example 2.16 shows that $\mathcal{PV}_\Phi, \mathcal{PV}_\Psi : A_{\{0\}} \to A_{\{0\}}$ are multiplication by $\det\Phi, \det\Psi$, so $\mathcal{PV}_\Phi = (\det\Psi^{-1}\det\Phi) \cdot \mathcal{PV}_\Psi$, as in (3.1).

The proof of Theorem 3.1(b) uses the following proposition. To interpret it, pretend for simplicity that the étale morphisms $\pi_U|_{W_t} : W_t \to U$ in (b) are invertible. Then

$$\Theta_t := \pi_V|_{W_t} \circ \pi_U|_{W_t}^{-1}$$

for $t \in \mathbb{C}$ are a 1-parameter family of morphisms $U \to V$, which satisfy $f = g \circ \Theta_t$ and $\Theta_t|_X = \Phi|_X = \Psi|_X$ for $t \in \mathbb{C}$, with $\Theta_0 = \Phi$ and $\Theta_1 = \Psi$. Thus, modulo taking étale covers of U, the family $\{\Theta_t : t \in \mathbb{C}\}$ interpolates between Φ and Ψ.

Proposition 3.4. *Let U, V be smooth $\mathbb{C}$-schemes, let $\Phi, \Psi : U \to V$ be étale morphisms, and let $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ be regular functions with $g \circ \Phi = f = g \circ \Psi$. Write $X = \mathrm{Crit}(f)$ and $Y = \mathrm{Crit}(g)$ as $\mathbb{C}$-subschemes of U, V, so that $\Phi|_X, \Psi|_X : X \to Y$ are étale. Suppose $\Phi|_X = \Psi|_X$, and $x \in X$ such that $\mathrm{d}\Psi|_x^{-1} \circ \mathrm{d}\Phi|_x : T_xU \to T_xU$ satisfies*

$$\big(\mathrm{d}\Psi|_x^{-1} \circ \mathrm{d}\Phi|_x - \mathrm{id}_{T_xU}\big)^2 = 0.$$

Then there exist a smooth $\mathbb{C}$-scheme W and morphisms $\pi_{\mathbb{C}} : W \to \mathbb{C}$, $\pi_U : W \to U$, $\pi_V : W \to V$ and $\iota : \mathbb{C} \to W$ such that:

- **(a)** *$\pi_{\mathbb{C}} \circ \iota(t) = t$, $\pi_U \circ \iota(t) = x$ and $\pi_V \circ \iota(t) = \Phi(x)$ for all $t \in \mathbb{C}$;*
- **(b)** *$\pi_{\mathbb{C}} \times \pi_U : W \to \mathbb{C} \times U$ and $\pi_{\mathbb{C}} \times \pi_V : W \to \mathbb{C} \times V$ are étale. Thus, $W_t := \pi_{\mathbb{C}}^{-1}(t)$ is a smooth $\mathbb{C}$-scheme for each $t \in \mathbb{C}$, and $\pi_U|_{W_t} : W_t \to U$, $\pi_V|_{W_t} : W_t \to V$ are étale, and $\iota(t) \in W_t$ with $\pi_U : \iota(t) \mapsto x$, $\pi_V : \iota(t) \mapsto \Phi(x)$;*
- **(c)** *$h := f \circ \pi_U = g \circ \pi_V : W \to \mathbb{C}$. Thus,*

$$(\pi_{\mathbb{C}} \times \pi_U)|_Z : Z \longrightarrow \mathbb{C} \times X \qquad \text{and} \qquad (\pi_{\mathbb{C}} \times \pi_V)|_Z : Z \longrightarrow \mathbb{C} \times Y$$

  are étale, where $Z := \mathrm{Crit}(h)$;
- **(d)** *$\Phi|_X \circ \pi_U|_Z = \Psi|_X \circ \pi_U|_Z = \pi_V|_Z : Z \to Y \subseteq V$; and*
- **(e)** *$\Phi \circ \pi_U|_{W_0} = \pi_V|_{W_0}$ and $\Psi \circ \pi_U|_{W_1} = \pi_V|_{W_1}$, for W_0, W_1 as in* **(b)**.

We will prove Proposition 3.4 in §3.1, and Theorem 3.1 in §3.2–§3.4.

3.1. **Proof of Proposition 3.4.** Let $U, V, \Phi, \Psi, f, g, X, Y, x$ be as in Proposition 3.4. Choose a Zariski open neighbourhood V' of $\Phi(x) = \Psi(x)$ in V and étale coordinates $(z_1, \ldots, z_n) : V' \to \mathbb{C}^n$ on V', with

$$z_1 = \cdots = z_n = 0$$

at $\Phi(x)$. Let m be the rank of the symmetric matrix $\big(\frac{\partial^2 g}{\partial z_i \partial z_j}|_{\Phi(x)}\big)_{i,j=1}^n$, so that $m \in \{0, \ldots, n\}$. By applying an element of $\mathrm{GL}(n, \mathbb{C})$ to the coordinates $(z_1, \ldots, z_n)$ we can suppose that

$$\frac{\partial^2 g}{\partial z_i \partial z_j}\Big|_{\Phi(x)} = \begin{cases} 1, & i = j \in \{1, \ldots, m\}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

Then $\frac{1}{2}\frac{\partial g}{\partial z_i}$ agrees with z_i to first order at $\Phi(x)$ for $i = 1, \ldots, m$, so replacing z_i by $\frac{1}{2}\frac{\partial g}{\partial z_i}$ for $i = 1, \ldots, m$ and making V' smaller, we can suppose (3.2) holds and $z_1, \ldots, z_m$ lie in the ideal $\big(\frac{\partial g}{\partial z_i},\ i = 1, \ldots, n\big)$ in $\mathcal{O}_{V'}$. Thus we may write

$$z_i = \sum_{j=1}^n A_{ij} \cdot \frac{\partial g}{\partial z_j}, \qquad i = 1, \ldots, m, \tag{3.3}$$

where $A_{ij} : V' \to \mathbb{A}^1$ are regular functions for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Taking $\frac{\partial}{\partial z_j}$ of (3.3) for $j = 1, \ldots, m$ and using (3.2) gives

$$A_{ij}|_{\Phi(x)} = \begin{cases} 1, & i = j, \ \ i, j \in \{1, \ldots, m\}, \\ 0, & i \neq j, \ \ i, j \in \{1, \ldots, m\}. \end{cases} \tag{3.4}$$

Set $U' = \Phi^{-1}(V') \cap \Psi^{-1}(V')$, so that U' is a Zariski open neighbourhood of x in U. Define étale coordinates $(x_1, \ldots, x_n) : U' \to \mathbb{C}^n$ and $(y_1, \ldots, y_n) : U' \to \mathbb{C}^n$ by $x_i = z_i \circ \Phi$ and $y_i = z_i \circ \Psi$, so that $x_1 = \cdots = x_n = y_1 = \cdots = y_n = 0$ at x. Since $f = g \circ \Phi = g \circ \Psi$ we have $\frac{\partial f}{\partial x_j} = \frac{\partial g}{\partial z_j} \circ \Phi$

and $\frac{\partial f}{\partial y_j}=\frac{\partial g}{\partial z_j}\circ\Psi$. Thus (3.2) and (3.3) imply that

$$\frac{\partial^2 f}{\partial x_i\partial x_j}\Big|_x=\frac{\partial^2 f}{\partial y_i\partial y_j}\Big|_x=\begin{cases}1, & i=j\in\{1,\ldots,m\},\\ 0, & \text{otherwise,}\end{cases}\tag{3.5}$$

$$x_i=\sum_{j=1}^n\bigl(A_{ij}\circ\Phi\bigr)\cdot\tfrac{\partial f}{\partial x_j},\quad y_i=\sum_{j=1}^n\bigl(A_{ij}\circ\Psi\bigr)\cdot\tfrac{\partial f}{\partial y_j},\quad i=1,\ldots,m.\tag{3.6}$$

Now $\mathrm{d}\Phi|_x:T_xU\to T_{\Phi(x)}V$ maps $\frac{\partial}{\partial x_j}\mapsto\frac{\partial}{\partial z_j}$, as $x_j=z_j\circ\Phi$, and $\mathrm{d}\Psi|_x:T_xU\to T_{\Phi(x)}V$ maps $\frac{\partial}{\partial y_j}\mapsto\frac{\partial}{\partial z_j}$. Hence $\mathrm{d}\Psi|_x^{-1}\circ\mathrm{d}\Phi|_x:T_xU\to T_xU$ maps $\frac{\partial}{\partial x_j}\mapsto\frac{\partial}{\partial y_j}=\sum_{i=1}^n\frac{\partial x_i}{\partial y_j}\cdot\frac{\partial}{\partial x_i}$. Define $B_{ij}\in\mathbb{C}$ for $i,j=1,\ldots,n$ by

$$\delta_{ij}+B_{ij}=\tfrac{\partial x_i}{\partial y_j}\big|_x.\tag{3.7}$$

Then $\bigl(\delta_{ij}+B_{ij}\bigr)_{i,j=1}^n$ is the matrix of $\mathrm{d}\Psi|_x^{-1}\circ\mathrm{d}\Phi|_x$ w.r.t. the basis $\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}$, and $\bigl(B_{ij}\bigr)_{i,j=1}^n$ is the matrix of $\mathrm{d}\Psi|_x^{-1}\circ\mathrm{d}\Phi|_x-\mathrm{id}_{T_xU}$, so by assumption $\bigl(B_{ij}\bigr)^2=0$. Therefore the inverse matrix of $\bigl(\delta_{ij}+B_{ij}\bigr)$ is $\bigl(\delta_{ij}-B_{ij}\bigr)$, so (3.7) gives

$$\delta_{ij}-B_{ij}=\tfrac{\partial y_i}{\partial x_j}\big|_x.\tag{3.8}$$

More generally, $\bigl(\delta_{ij}+tB_{ij}\bigr)$ is invertible for $t\in\mathbb{C}$, with inverse $\bigl(\delta_{ij}-tB_{ij}\bigr)$.

Now $\Phi|_X=\Psi|_X$ implies that $\mathrm{d}\Psi|_x^{-1}\circ\mathrm{d}\Phi|_x$ is the identity on $T_xX\subseteq T_xU$, and $T_xX=\mathrm{Ker}\bigl(\mathrm{Hess}_x f\bigr)=\langle\frac{\partial}{\partial x_{m+1}},\ldots,\frac{\partial}{\partial x_n}\rangle$ by (3.5), so

$$B_{ij}=0\quad\text{for all } i=1,\ldots,n \text{ and } j=m+1,\ldots,n.\tag{3.9}$$

We have $\frac{\partial^2 f}{\partial y_i\partial y_l}\big|_x=\sum_{j,k}\frac{\partial x_j}{\partial y_i}\frac{\partial^2 f}{\partial x_j\partial x_k}\frac{\partial x_k}{\partial y_l}\big|_x$, $\frac{\partial^2 f}{\partial x_i\partial x_l}\big|_x=\sum_{j,k}\frac{\partial y_j}{\partial x_i}\frac{\partial^2 f}{\partial y_j\partial y_k}\frac{\partial y_k}{\partial x_l}\big|_x$, which by (3.5) and (3.7)–(3.9) give equations equivalent to

$$B_{ij}+B_{ji}=\sum_{k=1}^n B_{ki}B_{kj}=0\quad\text{for all } i,j=1,\ldots,m.\tag{3.10}$$

Define regular $t',x_1',\ldots,x_n',y_1',\ldots,y_n',z_1',\ldots,z_n':\mathbb{C}\times U'\times V'\to\mathbb{C}$ by

$$\begin{aligned}&t'(t,u,v)=t, && x_i'(t,u,v)=x_i(u)=z_i\circ\Phi(u),\\ &y_i'(t,u,v)=y_i(u)=z_i\circ\Psi(u), && z_i'(t,u,v)=z_i(v).\end{aligned}$$

Then $(t',y_1',\ldots,y_n',z_1',\ldots,z_n')$ are étale coordinates on $\mathbb{C}\times U'\times V'$.

Let S be an affine Zariski open neighbourhood of $\mathbb{C}\times(x,\Phi(x))$ in $\mathbb{C}\times U'\times V'$, satisfying a series of smallness conditions we will give during the proof. Regard $(t',y_1',\ldots,y_n',z_1',\ldots,z_n')$ as étale coordinates on S, and write $\pi_{\mathbb{C}}:S\to\mathbb{C}$, $\pi_U:S\to U$, $\pi_V:S\to V$ for the projections. We will work with (sheaves of) ideals in $\mathcal{O}_S$, using notation $\bigl(x_i'-z_i',\ i=1,\ldots,n\bigr)$ to denote the ideal generated by the functions $x_1'-z_1',\ldots,x_n'-z_n'$, and

$$f\circ\pi_U-g\circ\pi_V\in\bigl(x_i'-z_i',\ i=1,\ldots,n\bigr)$$

to mean that $f\circ\pi_U-g\circ\pi_V\in H^0(\mathcal{O}_S)$ is a section of the ideal $\bigl(x_i'-z_i',\ i=1,\ldots,n\bigr)$. Write $I_X\subset\mathcal{O}_U$, $I_Y\subset\mathcal{O}_V$ for the ideals of functions on U,V vanishing on X,Y, and

$$\pi_U^{-1}(I_X),\pi_V^{-1}(I_Y)\subset\mathcal{O}_S$$

for the preimage ideals.

Since $x_i=z_i\circ\Phi$, the functions $x_i'-z_i'$ for $i=1,\ldots,n$ vanish on the smooth, closed $\mathbb{C}$-subscheme $\bigl(\mathbb{C}\times(\mathrm{id}\times\Phi)(U)\bigr)\cap S$ in S, and locally these functions cut out this $\mathbb{C}$-subscheme. So making S smaller we can suppose $\bigl(\mathbb{C}\times(\mathrm{id}\times\Phi)(U)\bigr)\cap S$ is the $\mathbb{C}$-subscheme

$$x_1'-z_1'=\cdots=x_n'-z_n'=0$$

in S. As $f = g\circ\Phi$, the function $f\circ\pi_U - g\circ\pi_V$ is zero on $\bigl(\mathbb{C}\times(\mathrm{id}\times\Phi)(U)\bigr)\cap S$. Hence

$$f\circ\pi_U - g\circ\pi_V \in \bigl(x_i'-z_i',\ i=1,\dots,n\bigr)\subset \mathcal{O}_S. \tag{3.11}$$

Lifting (3.11) from $\bigl(x_i'-z_i',\ i=1,\dots,n\bigr)$ to $\bigl(x_i'-z_i',\ i=1,\dots,n\bigr)^2$, making S smaller if necessary, we may choose regular $C_i : S\to\mathbb{C}$ for $i=1,\dots,n$ with

$$f\circ\pi_U - g\circ\pi_V - \sum_{i=1}^n C_i\cdot(x_i'-z_i') \in \bigl(x_i'-z_i',\ i=1,\dots,n\bigr)^2. \tag{3.12}$$

Apply $\frac{\partial}{\partial z_i'}$ to (3.12), using the étale coordinates $(t',y_1',\dots,y_n',z_1',\dots,z_n')$ on S. Since

$$\frac{\partial}{\partial z_i'}\bigl(g\circ\pi_V\bigr) = \frac{\partial g}{\partial z_i}\circ\pi_V \qquad\text{and}\qquad \frac{\partial}{\partial z_i'}(f\circ\pi_U) = 0 = \frac{\partial x_j'}{\partial z_i'},$$

this gives

$$C_i - \tfrac{\partial g}{\partial z_i}\circ\pi_V \in \bigl(x_i'-z_i',\ i=1,\dots,n\bigr).$$

Combining this with (3.12) yields

$$f\circ\pi_U - g\circ\pi_V - \sum_{i=1}^n\bigl(\tfrac{\partial g}{\partial z_i}\circ\pi_V\bigr)\cdot(x_i'-z_i') \in \bigl(x_i'-z_i',\ i=1,\dots,n\bigr)^2.$$

So making S smaller we can choose regular $D_{ij}:S\to\mathbb{C}$ for $i,j=1,\dots,n$ with $D_{ij}=D_{ji}$ and

$$f\circ\pi_U - g\circ\pi_V = \sum_{i=1}^n\bigl(\tfrac{\partial g}{\partial z_i}\circ\pi_V\bigr)\cdot(x_i'-z_i') + \sum_{i,j=1}^n D_{ij}\cdot(x_i'-z_i')(x_j'-z_j'). \tag{3.13}$$

Similarly, starting from $y_i = z_i\circ\Psi$ and $f = g\circ\Psi$ we may choose regular $E_{ij}:S\to\mathbb{C}$ for $i,j=1,\dots,n$ with $E_{ij}=E_{ji}$ and

$$f\circ\pi_U - g\circ\pi_V = \sum_{i=1}^n\bigl(\tfrac{\partial g}{\partial z_i}\circ\pi_V\bigr)\cdot(y_i'-z_i') + \sum_{i,j=1}^n E_{ij}\cdot(y_i'-z_i')(y_j'-z_j'). \tag{3.14}$$

Applying $\frac{\partial^2}{\partial z_i'\partial z_j'}$ to (3.13) and (3.14), restricting to $(t,x,\Phi(x))$ for $t\in\mathbb{C}$, noting that $x_i'=y_i'=z_i'=0$ at $(t,x,\Phi(x))$, and using (3.2), we deduce that

$$\begin{aligned} D_{ij}(t,x,\Phi(x)) = E_{ij}(t,x,\Phi(x)) &= \tfrac12\,\tfrac{\partial^2 g}{\partial z_i\partial z_j}\Big|_{\Phi(x)} \\ &= \begin{cases}\tfrac12, & i=j\in\{1,\dots,m\},\\ 0, & \text{otherwise.}\end{cases}\end{aligned} \tag{3.15}$$

Summing $1-t'$ times (3.13) with t' times (3.14) and rearranging yields

$$\begin{aligned} f\circ\pi_U - g\circ\pi_V = \sum_{i=1}^n &\Bigl[\tfrac{\partial g}{\partial z_i}\circ\pi_V + 2t'(1-t')\sum_{j=1}^n(D_{ij}-E_{ij})(x_j'-y_j')\Bigr] \\ &\qquad\qquad\cdot\bigl((1-t')x_i'+t'y_i'-z_i'\bigr) \\ +\sum_{i,j=1}^n &\bigl[(1-t')D_{ij}+t'E_{ij}\bigr]\cdot\bigl((1-t')x_i'+t'y_i'-z_i'\bigr)\bigl((1-t')x_j'+t'y_j'-z_j'\bigr) \\ +\sum_{i,j=1}^n &t'(1-t')\bigl[t'D_{ij}+(1-t')E_{ij}\bigr]\cdot(x_i'-y_i')(x_j'-y_j'). \end{aligned} \tag{3.16}$$

Since $x_i'-y_i' = (x_i-y_i)\circ\pi_U$, and $(x_i-y_i)|_X = z_i\circ\Phi|_X - z_i\circ\Psi|_X = 0$ as $\Phi|_X=\Psi|_X$, we see that $x_i'-y_i'\in\pi_U^{-1}(I_X)$. Thus making S smaller if necessary, we may choose regular $F_{ij}:S\to\mathbb{C}$ such that

$$x_i'-y_i' = \textstyle\sum_{j=1}^n F_{ij}\cdot\bigl(\frac{\partial f}{\partial y_j}\circ\pi_U\bigr)\quad\text{for } i=1,\dots,n. \tag{3.17}$$

Furthermore, by (3.6) when $i=1,\ldots,m$ we may take

$$F_{ij}=\Bigl(\sum_{k=1}^n (A_{ik}\circ\Phi)\cdot\tfrac{\partial y_j}{\partial x_k}-A_{ij}\circ\Psi\Bigr)\circ\pi_U.$$

Restricting to $(t,x,\Phi(x))$ and using (3.4), (3.8), (3.9) and $\Phi(x)=\Psi(x)$ gives

$$F_{ij}(t,x,\Phi(x))=-B_{ji}\quad\text{for } i=1,\ldots,m \text{ and } j=1,\ldots,n. \tag{3.18}$$

Applying $\frac{\partial}{\partial x_i'}$ to equation (3.13) shows that

$$\tfrac{\partial g}{\partial z_i}\circ\pi_V-\tfrac{\partial f}{\partial x_i}\circ\pi_U\in\bigl(x_j'-z_j',\ j=1,\ldots,n\bigr).$$

Thus we may write

$$\tfrac{\partial g}{\partial z_i}\circ\pi_V=\textstyle\sum_{j=1}^n\bigl(\tfrac{\partial y_j}{\partial x_i}\circ\pi_U\bigr)\cdot\bigl(\tfrac{\partial f}{\partial y_j}\circ\pi_U\bigr)+\sum_{j=1}^n G_{ij}\cdot(x_j'-z_j'), \tag{3.19}$$

where $G_{ij}:S\to\mathbb{C}$ are regular. Applying $\frac{\partial}{\partial z_j'}$ to (3.19), restricting to $(t,x,\Phi(x))$ and using (3.2) yields

$$G_{ij}(t,x,\Phi(x))=\begin{cases}-1, & i=j\in\{1,\ldots,m\},\\ 0, & \text{otherwise.}\end{cases} \tag{3.20}$$

From (3.17) and (3.19) we see that

$$\tfrac{\partial g}{\partial z_i}\circ\pi_V=\sum_{j=1}^n H_{ij}\cdot\bigl(\tfrac{\partial f}{\partial y_j}\circ\pi_U\bigr)+\textstyle\sum_{j=1}^n G_{ij}\cdot\bigl((1-t')x_j'+t'y_j'-z_j'\bigr), \tag{3.21}$$

where $H_{ij}=\frac{\partial y_j}{\partial x_i}\circ\pi_U+t'\sum_{k=1}^n G_{ik}F_{kj}$, so that from equations (3.8), (3.9), (3.18) and (3.20) we deduce that

$$H_{ij}(t,x,\Phi(x))=\delta_{ij}-(1-t)B_{ji}. \tag{3.22}$$

Combining (3.16), (3.17) and (3.21) gives

$$\begin{aligned}&f\circ\pi_U-g\circ\pi_V-\sum_{i=1}^n I_i\cdot\bigl((1-t')x_i'+t'y_i'-z_i'\bigr)\\ &\quad-\sum_{i,j,k,l=1}^n t'(1-t')\bigl[t'D_{ij}+(1-t')E_{ij}\bigr]F_{ik}F_{jl}\cdot\bigl(\tfrac{\partial f}{\partial y_k}\circ\pi_U\bigr)\bigl(\tfrac{\partial f}{\partial y_l}\circ\pi_U\bigr)\\ &\in\bigl((1-t')x_i'+t'y_i'-z_i',\ i=1,\ldots,n\bigr)^2,\end{aligned} \tag{3.23}$$

$$\text{where}\quad I_i=\sum_{j=1}^n\Bigl[H_{ij}+2t'(1-t')\sum_{k=1}^n(D_{ik}-E_{ik})F_{kj}\Bigr]\cdot\bigl(\tfrac{\partial f}{\partial y_j}\circ\pi_U\bigr). \tag{3.24}$$

Consider the matrix of functions $[H_{ij}+\cdots]_{i,j=1}^n$ appearing in (3.24). Equations (3.15) and (3.22) imply that at $(t,x,\Phi(x))$ this reduces to $(\delta_{ij}-(1-t)B_{ji})$, which is invertible from above. Thus, making S smaller, we can suppose that $[H_{ij}+\cdots]_{i,j=1}^n$ in (3.24) is an invertible matrix on S. Write $[J_{ij}]_{i,j=1}^n$ for the inverse matrix. Then we have

$$\begin{aligned}&\sum_{i,j,k,l=1}^n t'(1-t')\bigl[t'D_{ij}+(1-t')E_{ij}\bigr]F_{ik}F_{jl}\cdot\bigl(\tfrac{\partial f}{\partial y_k}\circ\pi_U\bigr)\bigl(\tfrac{\partial f}{\partial y_l}\circ\pi_U\bigr)\\ &\qquad=t'(1-t')\sum_{i,j=1}^n K_{ij}\cdot I_iI_j,\quad\text{where}\\ &K_{ij}=\sum_{k,l,p,q=1}^n t'(1-t')\bigl[t'D_{kl}+(1-t')E_{kl}\bigr]F_{kp}F_{lq}J_{pi}J_{qj}.\end{aligned} \tag{3.25}$$

Using (3.9), (3.10), (3.15) and (3.18) we find that

$$K_{ij}(t,x,\Phi(x))=0\quad\text{for all } t\in\mathbb{C}. \tag{3.26}$$

Combining (3.23) and (3.25), making S smaller if necessary we may write

$$\begin{aligned}f\circ\pi_U-g\circ\pi_V=\sum_{i=1}^n I_i\cdot\bigl((1-t')x_i'+t'y_i'-z_i'\bigr)+t'(1-t')\sum_{i,j=1}^n K_{ij}\cdot I_iI_j\\+\sum_{i,j=1}^n L_{ij}\cdot\bigl((1-t')x_i'+t'y_i'-z_i'\bigr)\bigl((1-t')x_j'+t'y_j'-z_j'\bigr),\end{aligned}\tag{3.27}$$

for regular $L_{ij}:S\to\mathbb{C}$ for $i,j=1,\dots,n$.

Write $(r_{ij})_{i,j=1}^n$ for the coordinates on $\mathbb{C}^{n^2}$. Let T be a Zariski open neighbourhood of

$$\mathbb{C}\times\bigl(x,\Phi(x),(0)_{i,j=1}^n\bigr)$$

in $S\times\mathbb{C}^{n^2}$ to be chosen shortly, and let W be the closed $\mathbb{C}$-subscheme of T defined by

$$\begin{aligned}W=\bigl\{\bigl(t,u,v,(r_{ij})_{i,j=1}^n\bigr)\in T\subseteq S\times\mathbb{C}^{n^2}\subseteq\mathbb{C}\times U\times V\times\mathbb{C}^{n^2}:\\ \bigl((1-t')x_i'+t'y_i'-z_i'\bigr)(t,u,v)=\sum_{j=1}^n r_{ij}\cdot I_j(t,u,v),\ \ i=1,\dots,n,\\ r_{ij}+t(1-t)K_{ij}(t,u,v)+\sum_{k,l=1}^n L_{kl}(t,u,v)\cdot r_{ki}r_{lj}=0,\ \ i,j=1,\dots,n\bigr\}.\end{aligned}\tag{3.28}$$

Define $\mathbb{C}$-scheme morphisms $\pi_{\mathbb{C}}:W\to\mathbb{C}$, $\pi_U:W\to U$, and $\pi_V:W\to V$ to map $\bigl(t,u,v,(r_{ij})_{i,j=1}^n\bigr)$ to t,u,v, respectively.

At $(t,x,\Phi(x))\in S$ for $t\in\mathbb{C}$ we have $x_i'=y_i'=z_i'=0$, and $I_i=0$ by (3.24) as $\frac{\partial f}{\partial y_j}\big|_x=0$, and $K_{ij}=0$ by (3.26). Hence $\bigl(t,x,\Phi(x),(0)_{i,j=1}^n\bigr)$ satisfies the equations of (3.28), and lies in W. Define $\iota:\mathbb{C}\to W$ by

$$\iota(t)=\bigl(t,x,\Phi(x),(0)_{i,j=1}^n\bigr).\tag{3.29}$$

Now $T\subseteq\mathbb{C}\times U\times V\times\mathbb{C}^{n^2}$ is smooth of dimension $1+n+n+n^2$, and in (3.28) we impose $n+n^2$ equations, so the expected dimension of W is $(1+2n+n^2)-(n+n^2)=n+1$. The linearizations of the $n+n^2$ equations in (3.28) at

$$\bigl(t,u,v,(r_{ij})_{i,j=1}^n\bigr)=\bigl(t,x,\Phi(x),(0)_{i,j=1}^n\bigr)=\iota(t)$$

are

$$\begin{aligned}\mathrm{d}y_i|_x(\delta u)-\mathrm{d}z_i|_{\Phi(x)}(\delta v)=0,\quad i=1,\dots,n,\\ \delta r_{ij}+\mathrm{d}K_{ij}|_{(t,x,\Phi(x))}(\delta t\oplus\delta u\oplus\delta v)=0,\quad i,j=1,\dots,n,\end{aligned}\tag{3.30}$$

for $\delta t\in T_t\mathbb{C}$, $\delta u\in T_xU$, $\delta v\in T_{\Phi(x)}V$, and $(\delta r_{ij})_{i,j=1}^n\in T_{(0)_{i,j=1}^n}\mathbb{C}^{n^2}$, where we have used $\mathrm{d}x_i'=\mathrm{d}y_i'$ and $I_j=K_{ij}=0$ at $(t,x,\Phi(x))$. As $\mathrm{d}y_1|_x,\dots,\mathrm{d}y_n|_x$ are a basis for T_x^*U, equations (3.30) are transverse, so W is smooth of dimension $n+1$ near $\iota(t)$. Hence, taking T small enough, we can suppose W is smooth.

It remains to prove Proposition 3.4(a)–(e). Part (a) is immediate from (3.29). For (b), the vector space of solutions $\bigl(\delta t,\delta u,\delta v,(\delta r_{ij})_{i,j=1}^n\bigr)$ to (3.30) is $T_{\iota(t)}W$, where

$$\mathrm{d}(\pi_{\mathbb{C}}\times\pi_U)|_{\iota(t)}:T_{\iota(t)}W\to T_{(t,x)}(\mathbb{C}\times U)$$

and $\mathrm{d}(\pi_{\mathbb{C}}\times\pi_V)|_{\iota(t)}:T_{\iota(t)}W\to T_{(t,\Phi(x))}(\mathbb{C}\times V)$ map $\bigl(\delta t,\delta u,\delta v,(\delta r_{ij})_{i,j=1}^n\bigr)$ to $(\delta t,\delta u)$ and $(\delta t,\delta v)$. By (3.30), these are isomorphisms, so $\pi_{\mathbb{C}}\times\pi_U$ and $\pi_{\mathbb{C}}\times\pi_V$ are étale near $\iota(\mathbb{C})$. Making T,W smaller, we can suppose $\pi_{\mathbb{C}}\times\pi_U$ and $\pi_{\mathbb{C}}\times\pi_V$ are étale.

For (c), we have

$$\begin{aligned}
&\big(f\circ\pi_W - g\circ\pi_U\big)\big(t,u,v,(r_{ij})_{i,j=1}^n\big) = f(u)-g(v) = (f\circ\pi_U - g\circ\pi_V)(t,u,v)\\
&\quad = \sum_{i=1}^n I_i\cdot\big((1-t')x_i'+t'y_i'-z_i'\big)+t'(1-t')\sum_{i,j=1}^n K_{ij}I_iI_j\\
&\quad + \sum_{i,j=1}^n L_{ij}\cdot\big((1-t')x_i'+t'y_i'-z_i'\big)\big((1-t')x_j'+t'y_j'-z_j'\big)\\
&\quad = \sum_{i=1}^n I_i\cdot\Big(\sum_{j=1}^n r_{ij}\cdot I_j\Big)+t'(1-t')\sum_{i,j=1}^n K_{ij}I_iI_j\\
&\quad + \sum_{i,j=1}^n L_{ij}\cdot\Big(\sum_{k=1}^n r_{ik}\cdot I_k\Big)\Big(\sum_{l=1}^n r_{jl}\cdot I_l\Big)\\
&\quad = \sum_{i,j=1}^n I_iI_j\cdot\Big[r_{ij}+t'(1-t')K_{ij}+\sum_{k,l=1}^n L_{kl}\cdot r_{ki}r_{lj}\Big]=0,
\end{aligned}$$

using (3.27) in the third step, the first equation of (3.28) in the fourth, rearranging and exchanging labels i,k and j,l in the fifth, and the second equation of (3.28) in the sixth. Hence $f\circ\pi_U - g\circ\pi_V = 0 : W\to\mathbb{C}$, proving (c).

For (d), from (3.28) we can show that $\big(\mathbb{C}\times(\mathrm{id}\times\Phi)(X)\times\mathbb{C}^{n^2}\big)\cap W$ is open and closed in $Z=\mathrm{Crit}(h)$, and contains $\iota(\mathbb{C})$. So making T,W smaller we can take

$$Z=\big(\mathbb{C}\times(\mathrm{id}\times\Phi)(X)\times\mathbb{C}^{n^2}\big)\cap W,$$

and then (d) follows as $\Phi|_X=\Psi|_X$.

For (e), observe that when $t=0$ in (3.28), the second equation reduces to $r_{ij}=0$ near $\iota(\mathbb{C})$ as $t(1-t)K_{ij}(t,u,v)=0$, so making T,W smaller gives

$$\begin{aligned}
W_0 &= \big\{\big(0,u,v,(0)_{i,j=1}^n\big)\in T : \big(x_i'-z_i'\big)(0,u,v)=0,\ i=1,\ldots,n\big\}\\
&= \big\{\big(0,u,v,(0)_{i,j=1}^n\big)\in T : v=\Phi(u)\big\}.
\end{aligned}$$

Hence $\Phi\circ\pi_W|_{W_0}=\pi_U|_{W_0}$. Similarly, when $t=1$ we have

$$\begin{aligned}
W_1 &= \big\{\big(1,u,v,(0)_{i,j=1}^n\big)\in T : \big(y_i'-z_i'\big)(0,u,v)=0,\ i=1,\ldots,n\big\}\\
&= \big\{\big(0,u,v,(0)_{i,j=1}^n\big)\in T : v=\Psi(u)\big\},
\end{aligned}$$

so that $\Psi\circ\pi_W|_{W_1}=\pi_U|_{W_1}$. This proves (e), and Proposition 3.4.

3.2. **Part (a):** $\det(\mathrm{d}\Psi|_{X^{\mathrm{red}}}^{-1}\circ\mathrm{d}\Phi|_{X^{\mathrm{red}}})=\pm1$**.** We work in the situation of Theorem 3.1. For each $x\in X\subseteq U$, consider the diagram of linear maps of vector spaces:

$$\begin{tikzcd}
0 \arrow[r] & T_xX \arrow[r] \arrow[d, "\mathrm{d}(\Phi|_X)|_x"'] & T_xU \arrow[r, "\mathrm{Hess}_x f"'] \arrow[d, "\mathrm{d}\Phi|_x"'] & T_x^*U \arrow[r] \arrow[d, "(\mathrm{d}\Phi|_x^{-1})^*"] & T_x^*X \arrow[r] \arrow[d, "(\mathrm{d}(\Phi|_X)|_x^{-1})^*"] & 0\\
0 \arrow[r] & T_{\Phi(x)}Y \arrow[r] & T_{\Phi(x)}V \arrow[r, "\mathrm{Hess}_{\Phi(x)} g"] & T_{\Phi(x)}^*V \arrow[r] & T_{\Phi(x)}^*Y \arrow[r] & 0,
\end{tikzcd} \tag{3.31}$$

where T_xX is the Zariski tangent space of X, and $\mathrm{Hess}_x f=(\partial^2 f)|_x$ the Hessian of f at x. The rows of (3.31) are exact, and the columns isomorphisms. The outer squares of (3.31) clearly commute. We can show the central square commutes by taking second derivatives of $f=g\circ\Phi$ to get $\partial^2 f|_x=\partial^2 g|_{\Phi(x)}\circ(\mathrm{d}\Phi|_x\otimes\mathrm{d}\Phi|_x)$, and composing with $\mathrm{id}\otimes\mathrm{d}\Phi|_x^{-1}$. Thus (3.31) is commutative.

There is also an analogue of (3.31) for Ψ. Since $\Psi(x)=\Phi(x)$, we may compose the columns of (3.31) for Φ with the inverses of the columns of (3.31) for Ψ to get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & T_xX \arrow[r] \arrow[d, "{\begin{subarray}{l}\mathrm{d}(\Psi|_X)|_x^{-1}\circ\\ \mathrm{d}(\Phi|_X)|_x\\ =\mathrm{id}_{T_xX}\end{subarray}}"'] & T_xU \arrow[r, "\mathrm{Hess}_x f"'] \arrow[d, "{\begin{subarray}{l}\mathrm{d}\Psi|_x^{-1}\circ\\ \mathrm{d}\Phi|_x\end{subarray}}"'] & T_x^*U \arrow[r] \arrow[d, "{\begin{subarray}{l}\mathrm{d}\Psi|_x^*\circ\\ (\mathrm{d}\Phi|_x^{-1})^*\end{subarray}}"] & T_x^*X \arrow[r] \arrow[d, "{\begin{subarray}{l}(\mathrm{d}(\Psi|_X)|_x)^*\circ\\ (\mathrm{d}(\Phi|_X)|_x^{-1})^*\\ =\mathrm{id}_{T_x^*X}\end{subarray}}"] & 0 \\ 0 \arrow[r] & T_xX \arrow[r] & T_xU \arrow[r, "\mathrm{Hess}_x f"] & T_x^*U \arrow[r] & T_x^*X \arrow[r] & 0, \end{tikzcd} \tag{3.32}$$

where the outer morphisms are identities as $\Phi|_X=\Psi|_X$.

Choose a complementary vector subspace N_x to T_xX in T_xU, which we think of as the normal to X in U at x, so that $T_xU=T_xX\oplus N_x$. Write $\mathrm{Hess}'_x f$ for the restriction of $\mathrm{Hess}_x f$ to a symmetric bilinear form on N_x. Since $T_xX=\mathrm{Ker}(\mathrm{Hess}_x f)$, we see that $\mathrm{Hess}'_x f$ is a nondegenerate symmetric bilinear form on N_x. We may write equation (3.32) as

$$\begin{tikzcd}[column sep=large] 0 \arrow[r] & T_xX \arrow[r, "\begin{pmatrix}\mathrm{id}\\ 0\end{pmatrix}"'] \arrow[d, "\mathrm{id}"] & T_xX\oplus N_x \arrow[r, "\begin{pmatrix}0 & 0\\ 0 & \mathrm{Hess}'_x f\end{pmatrix}"'] \arrow[d, "{\begin{subarray}{l}\mathrm{d}\Psi|_x^{-1}\\ \circ\mathrm{d}\Phi|_x=\end{subarray}}\begin{pmatrix}\mathrm{id} & A\\ 0 & B\end{pmatrix}"] & T_x^*X\oplus N_x^* \arrow[r, "\begin{pmatrix}\mathrm{id} & 0\end{pmatrix}"'] \arrow[d, "{\mathrm{d}\Psi|_x^*\circ(\mathrm{d}\Phi|_x^{-1})^*=}\begin{pmatrix}\mathrm{id} & -AB^{-1}\\ 0 & B^{-1}\end{pmatrix}"] & T_x^*X \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & T_xX \arrow[r, "\begin{pmatrix}\mathrm{id}\\ 0\end{pmatrix}"] & T_xX\oplus N_x \arrow[r, "\begin{pmatrix}0 & 0\\ 0 & \mathrm{Hess}'_x f\end{pmatrix}"] & T_x^*X\oplus N_x^* \arrow[r, "\begin{pmatrix}\mathrm{id} & 0\end{pmatrix}"] & T_x^*X \arrow[r] & 0, \end{tikzcd} \tag{3.33}$$

for some linear $A: N_x\to T_xX$ and $B: N_x\to N_x$. Then (3.33) commuting implies that B preserves the nondegenerate symmetric bilinear form $\mathrm{Hess}'_x f$ on N_x, and $\det B=\pm 1$. So $\det\bigl(\mathrm{d}\Psi|_x^{-1}\circ\mathrm{d}\Phi|_x\bigr)=\det\bigl(\begin{smallmatrix}\mathrm{id} & A\\ 0 & B\end{smallmatrix}\bigr)=\det B=\pm 1$ for $x\in X$.

Thus, as a map of topological spaces, $\det\bigl(\mathrm{d}\Psi|_{X^{\mathrm{red}}}^{-1}\circ\mathrm{d}\Phi|_{X^{\mathrm{red}}}\bigr): X^{\mathrm{red}}\to\mathbb{C}\setminus\{0\}$ actually maps $X^{\mathrm{red}}\to\{\pm 1\}$. Since it is continuous, it is locally constant. Now if $f,g: Y\to Z$ are morphisms of $\mathbb{C}$-schemes with Y reduced, then $f=g$ if and only if $f(y)=g(y)$ for each point $y\in Y$. Applying this to compare $\det\bigl(\mathrm{d}\Psi|_{X^{\mathrm{red}}}^{-1}\circ\mathrm{d}\Phi|_{X^{\mathrm{red}}}\bigr): X^{\mathrm{red}}\to\mathbb{C}\setminus\{0\}$ locally with the constant maps 1 or -1 on X^{red} shows that $\det\bigl(\mathrm{d}\Psi|_{X^{\mathrm{red}}}^{-1}\circ\mathrm{d}\Phi|_{X^{\mathrm{red}}}\bigr)$ is a locally constant map $X^{\mathrm{red}}\to\{\pm 1\}\subset\mathbb{C}\setminus\{0\}$ as a $\mathbb{C}$-scheme morphism. This proves Theorem 3.1(a).

3.3. **Part (b): $\mathcal{PV}_\Phi=\det\bigl(\mathrm{d}\Psi|_{X^{\mathrm{red}}}^{-1}\circ\mathrm{d}\Phi|_{X^{\mathrm{red}}}\bigr)\cdot\mathcal{PV}_\Psi$.** For Theorem 3.1(b), we begin with the following proposition.

Proposition 3.5. *Let U,V,Φ,Ψ,f,g,X,Y be as in Theorem 3.1, and suppose $x\in X$ with $\bigl(\mathrm{d}\Psi|_x^{-1}\circ\mathrm{d}\Phi|_x-\mathrm{id}_{T_xU}\bigr)^2=0$. Then there exists a Zariski open neighbourhood X' of x in X such that $\mathcal{PV}_\Phi|_{X'}=\mathcal{PV}_\Psi|_{X'}$.*

Proof. Apply Proposition 3.4 to get $W,\pi_{\mathbb{C}},\pi_U,\pi_V,\iota,h,Z$. Then apply Proposition 2.8 with $Z,X,x,\pi_{\mathbb{C}}|_Z,\pi_U|_Z,\iota,\mathcal{PV}^\bullet_{U,f},\Phi|_X^*\bigl(\mathcal{PV}^\bullet_{V,g}\bigr),\mathcal{PV}_\Phi,\mathcal{PV}_\Psi$ in place of W, X, x, $\pi_{\mathbb{C}}$, π_X, ι, $\mathcal{P}^\bullet$, $\mathcal{Q}^\bullet$, α,

β, respectively, and with γ defined by the commuting diagram of isomorphisms:

$$
\begin{tikzcd}
\mathcal{PV}^\bullet_{W,h} \arrow[rr, "\mathcal{PV}_{\pi_{\mathbb{C}}\times\pi_V}"'] \arrow[d, "\mathcal{PV}_{\pi_{\mathbb{C}}\times\pi_U}"] & & (\pi_{\mathbb{C}}\times\pi_V)|_Z^*\bigl(\mathcal{PV}^\bullet_{\mathbb{C}\times V,0\boxplus g}\bigr) \arrow[d, "(\pi_{\mathbb{C}}\times\pi_V)|_Z^*(\mathcal{TS}_{\mathbb{C},0,V,g})"'] \\
(\pi_{\mathbb{C}}\times\pi_U)|_Z^*\bigl(\mathcal{PV}^\bullet_{\mathbb{C}\times U,0\boxplus f}\bigr) \arrow[d, "(\pi_{\mathbb{C}}\times\pi_V)|_Z^*(\mathcal{TS}_{\mathbb{C},0,U,f})"] & & (\pi_{\mathbb{C}}\times\pi_V)|_Z^*\bigl(\mathcal{PV}^\bullet_{\mathbb{C},0}\overset{L}{\boxtimes}\mathcal{PV}^\bullet_{V,g}\bigr) \arrow[d, "\delta'"'] \\
(\pi_{\mathbb{C}}\times\pi_U)|_Z^*\bigl(\mathcal{PV}^\bullet_{\mathbb{C},0}\overset{L}{\boxtimes}\mathcal{PV}^\bullet_{U,f}\bigr) \arrow[d, "\delta"] & & \bigl[\pi_{\mathbb{C}}|_Z^*\bigl(A_{\mathbb{C}}[1]\bigr)\bigr]\overset{L}{\otimes}\bigl[\pi_V|_Z^*\bigl(\mathcal{PV}^\bullet_{V,g}\bigr)\bigr] \arrow[d, "\epsilon'"'] \\
\bigl[\pi_{\mathbb{C}}|_Z^*\bigl(A_{\mathbb{C}}[1]\bigr)\bigr]\overset{L}{\otimes}\bigl[\pi_U|_Z^*\bigl(\mathcal{PV}^\bullet_{U,f}\bigr)\bigr] \arrow[d, "\epsilon"] & & \pi_V|_Z^*[1]\bigl(\mathcal{PV}^\bullet_{V,g}\bigr) \arrow[d, equal] \\
\pi_U|_Z^*[1]\bigl(\mathcal{PV}^\bullet_{U,f}\bigr) \arrow[rr, "\gamma"] & & \pi_U|_Z^*[1]\bigl(\Phi|_X^*\bigl(\mathcal{PV}^\bullet_{V,g}\bigr)\bigr).
\end{tikzcd}
\tag{3.34}
$$

Here $\mathcal{TS}_{\mathbb{C},0,U,f},\mathcal{TS}_{\mathbb{C},0,V,g}$ are as in (2.8), δ,δ' come from $\mathcal{PV}_{\mathbb{C},0}\cong A_{\mathbb{C}}[1]$, and ϵ,ϵ' come from $\pi_{\mathbb{C}}|_Z^*(A_{\mathbb{C}})\cong A_Z$ and $A_Z\overset{L}{\otimes}\mathcal{P}^\bullet\cong\mathcal{P}^\bullet$ for $\mathcal{P}^\bullet\in\mathrm{Perv}(Z)$.

Then the hypothesis $\pi_X|_{W_0}^*(\alpha)=j_0^*[-1](\gamma)$ in Proposition 2.8 follows from comparing $j_0^*[-1]$ applied to (3.34) with the commuting diagram

$$
\begin{tikzcd}
j_0^*[-1]\bigl(\mathcal{PV}^\bullet_{W,h}\bigr) \arrow[r, "j_0^*{[-1]}(\mathcal{PV}_{\pi_{\mathbb{C}}\times\pi_V})"'] \arrow[d, "\cong"] & j_0^*[-1]\circ(\pi_{\mathbb{C}}\times\pi_V)|_Z^*\bigl(\mathcal{PV}^\bullet_{\mathbb{C}\times V,0\boxplus g}\bigr) \arrow[d, "\cong"'] \\
\mathcal{PV}^\bullet_{W_0,h|_{W_0}} \arrow[r, "\mathcal{PV}_{\pi_V|_{W_0}}"] \arrow[d, "\mathcal{PV}_{\pi_U|_{W_0}}"] & \pi_V|_{Z\cap W_0}^*\bigl(\mathcal{PV}^\bullet_{V,g}\bigr) \arrow[d, equal] \\
\pi_V|_{Z\cap W_0}^*\bigl(\mathcal{PV}^\bullet_{U,f}\bigr) \arrow[r, "\pi_U|_{Z\cap W_0}^*(\mathcal{PV}_\Phi)"] & \pi_U|_{Z\cap W_0}^*\circ\Phi|_X^*\bigl(\mathcal{PV}^\bullet_{V,g}\bigr),
\end{tikzcd}
$$

where $j_0:Z\cap W_0\hookrightarrow Z$ is the inclusion, and the bottom square commutes by Proposition 3.4(e) and (2.18). Similarly $\pi_X|_{W_1}^*(\beta)=j_1^*[-1](\gamma)$. Hence Proposition 2.8 gives Zariski open $x\in X'\subseteq X$ with $\mathcal{PV}_\Phi|_{X'}=\mathcal{PV}_\Psi|_{X'}$. $\square$

Now to prove Theorem 3.1(b), let $x\in X$ be arbitrary. As in §3.2, we can choose a splitting $T_xU=T_xX\oplus N_x$ such that

$$
\mathrm{d}\Psi|_x^{-1}\circ\mathrm{d}\Phi|_x=\begin{pmatrix}\mathrm{id} & A\\ 0 & B\end{pmatrix}:\begin{matrix}T_xX\oplus\\ N_x\end{matrix}\longrightarrow\begin{matrix}T_xX\oplus\\ N_x\end{matrix}
\tag{3.35}
$$

for linear $A:N_x\to T_xX$ and $B:N_x\to N_x$, where B preserves the nondegenerate symmetric bilinear form $\mathrm{Hess}'_x f$ on N_x.

Choose a Zariski open neighbourhood U' of x in U and a splitting $TU'=E\oplus F$ for algebraic vector subbundles $E,F\subseteq TU$ with $E|_x=T_xX$ and $F|_x=N_x$. Then $\mathrm{d}f|_{U'}=\alpha\oplus\beta$ for unique $\alpha\in H^0(E)$ and $\beta\in H^0(F)$, and $X\cap U'$ is defined by $\alpha=\beta=0$.

Since $\mathrm{Hess}_x f=\partial(\mathrm{d}f)|_x$ is nondegenerate on N_x, we see that $\nabla\beta|_x:T_xU\to F|_x$ induces an isomorphism $N_x\to F|_x$, so $\nabla\beta|_x$ is surjective. Therefore $S:=\beta^{-1}(0)$ is a smooth $\mathbb{C}$-subscheme of U' near x, and making U' smaller, we can suppose S is smooth.

Set $e=f|_S:S\to\mathbb{C}$. Then the isomorphism $T^*S\cong E|_S$ identifies $\mathrm{d}e\in H^0(T^*S)$ with $\alpha|_S\in H^0(E|_S)$. Hence $\mathrm{Crit}(e:S\to\mathbb{C})=\mathrm{Crit}(f|_{U'}:U'\to\mathbb{C})=X\cap U'$, as $\mathbb{C}$-subschemes of U.

By [23, Prop. 2.23] quoted in Theorem 5.1(i) below, there exist a smooth $\mathbb{C}$-scheme R, morphisms $\gamma:R\to U'$, $\delta:R\to S$, $\epsilon:R\to\mathbb{C}^n$ where $n=\dim U'-\dim S$, and $r\in R$, such that

$\gamma(r)=x$, $\gamma|_Q=\delta|_Q$, $f\circ\gamma=e\circ\delta+(z_1^2+\cdots+z_n^2)\circ\epsilon:R\to\mathbb{C}$, and the following commutes with horizontal morphisms étale:

$$\begin{tikzcd} S \arrow[d, "\subset"] & Q:=\gamma^{-1}(S) \arrow[l, "\gamma|_Q"'] \arrow[r, "\gamma|_Q"'] \arrow[d, "\subset"] & S \arrow[d, "\mathrm{id}_S\times 0"'] \\ U' & R \arrow[l, "\gamma"'] \arrow[r, "\delta\times\epsilon"] & S\times\mathbb{C}^n. \end{tikzcd} \tag{3.36}$$

Taking derivatives at $r\in Q\subseteq R$ in (3.36) gives a commutative diagram

$$\begin{tikzcd} T_xX=T_xS \arrow[d, "\subset"] & T_rQ \arrow[l, "\cong", "\mathrm{d}(\gamma|_Q)|_r"'] \arrow[r, "\cong", "\mathrm{d}(\gamma|_Q)|_r"'] \arrow[d, "\subset"] & T_xX=T_xS \arrow[d, "\mathrm{id}\times 0"'] \\ T_xX\oplus N_x=T_xU & T_rR \arrow[l, "\mathrm{d}\gamma|_r", "\cong"'] \arrow[r, "\mathrm{d}(\delta\times\epsilon)|_r", "\cong"'] & T_xX\oplus T_0\mathbb{C}^n. \end{tikzcd}$$

Therefore $\mathrm{d}(\delta\times\epsilon)|_r\circ\mathrm{d}\gamma|_r^{-1}:T_xX\oplus N_x\to T_xX\oplus T_0\mathbb{C}^n$ is the identity on T_xX, and induces an isomorphism $N_x\to T_0\mathbb{C}^n$, which as $f\circ\gamma=e\circ\delta+(z_1^2+\cdots+z_n^2)\circ\epsilon$ identifies $\mathrm{Hess}'_x f$ on N_x with $\mathrm{Hess}_0(z_1^2+\cdots+z_n^2)=\mathrm{d}z_1\otimes\mathrm{d}z_1+\cdots+\mathrm{d}z_n\otimes\mathrm{d}z_n$ on $T_0\mathbb{C}^n$. Thus, the linear isomorphism $B:N_x\to N_x$ above preserving $\mathrm{Hess}'_x f$ is identified with a linear isomorphism $M:\mathbb{C}^n\to\mathbb{C}^n$ preserving $\mathrm{d}z_1\otimes\mathrm{d}z_1+\cdots+\mathrm{d}z_n\otimes\mathrm{d}z_n$, that is, $M\in\mathrm{O}(n,\mathbb{C})$ satisfies

$$\begin{pmatrix}\mathrm{id}&0\\0&B\end{pmatrix}\circ\mathrm{d}\gamma|_r\circ\mathrm{d}(\delta\times\epsilon)|_r^{-1}=\mathrm{d}\gamma|_r\circ\mathrm{d}(\delta\times\epsilon)|_r^{-1}\circ\begin{pmatrix}\mathrm{id}&0\\0&M\end{pmatrix}. \tag{3.37}$$

Define P to be the $\mathbb{C}$-scheme fibre product $P=R\times_{\delta\times(M\circ\epsilon),S\times\mathbb{C}^n,\delta\times\epsilon}R$, with projections $\pi_1,\pi_2:P\to R$. Then P is smooth and π_1,π_2 are étale, as $R,S\times\mathbb{C}^n$ are smooth and

$$\delta\times(M\circ\epsilon),\delta\times\epsilon:R\to S\times\mathbb{C}^n$$

are étale. As $r\in R$ with $(\delta\times(M\circ\epsilon))(r)=(x,0)=(\delta\times\epsilon)(r)$, we have a point $p\in P$ with $\pi_1(p)=\pi_2(p)=r$. Define $d=f\circ\gamma\circ\pi_1:P\to\mathbb{C}$ and $Z=\mathrm{Crit}(d)$. Then

$$\begin{aligned} d=f\circ\gamma\circ\pi_1&=(e\boxplus z_1^2+\cdots+z_n^2)\circ(\delta\times\epsilon)\circ\pi_1\\ &=(e\boxplus z_1^2+\cdots+z_n^2)\circ(\delta\times(M\circ\epsilon))\circ\pi_1\\ &=(e\boxplus z_1^2+\cdots+z_n^2)\circ(\delta\times\epsilon)\circ\pi_2=f\circ\gamma\circ\pi_2. \end{aligned} \tag{3.38}$$

Consider the étale morphisms $\Phi\circ\gamma\circ\pi_1,\Psi\circ\gamma\circ\pi_2:P\to V$. Both map $p\mapsto\Phi(x)$, and satisfy $g\circ(\Phi\circ\gamma\circ\pi_1)=d=g\circ(\Psi\circ\gamma\circ\pi_2)$ by (3.38) and $g\circ\Phi=f=g\circ\Psi$. Taking derivatives at p to get linear maps $T_pP\to T_{\Phi(x)}V$, we find that

$$\begin{aligned} \mathrm{d}(\Psi\circ\gamma\circ\pi_2)|_p&=\mathrm{d}\Psi|_x\circ\mathrm{d}\gamma|_r\circ\mathrm{d}(\delta\times\epsilon)|_r^{-1}\circ\mathrm{d}((\delta\times\epsilon)\circ\pi_2)|_p\\ &=\mathrm{d}\Psi|_x\circ\mathrm{d}\gamma|_r\circ\mathrm{d}(\delta\times\epsilon)|_r^{-1}\circ\mathrm{d}((\delta\times(M\circ\epsilon))\circ\pi_1)|_p\\ &=\mathrm{d}\Psi|_x\circ\mathrm{d}\gamma|_r\circ\mathrm{d}(\delta\times\epsilon)|_r^{-1}\circ\begin{pmatrix}\mathrm{id}&0\\0&M\end{pmatrix}\circ\mathrm{d}(\delta\times\epsilon)|_r\circ\mathrm{d}\pi_1|_p\\ &=\mathrm{d}\Psi|_x\circ\begin{pmatrix}\mathrm{id}&0\\0&B\end{pmatrix}\circ\mathrm{d}\gamma|_r\circ\mathrm{d}(\delta\times\epsilon)|_r^{-1}\circ\mathrm{d}(\delta\times\epsilon)|_r\circ\mathrm{d}\pi_1|_p\\ &=\mathrm{d}\Psi|_x\circ\begin{pmatrix}\mathrm{id}&-AB^{-1}\\0&\mathrm{id}\end{pmatrix}\begin{pmatrix}\mathrm{id}&A\\0&B\end{pmatrix}\circ\mathrm{d}\gamma|_r\circ\mathrm{d}\pi_1|_p\\ &=\mathrm{d}\Psi|_x\circ\begin{pmatrix}\mathrm{id}&-AB^{-1}\\0&\mathrm{id}\end{pmatrix}\circ\mathrm{d}\Psi|_x^{-1}\circ\mathrm{d}(\Phi\circ\gamma\circ\pi_1)|_p, \end{aligned} \tag{3.39}$$

using $(\delta\times(M\circ\epsilon))\circ\pi_1=(\delta\times\epsilon)\circ\pi_2$ in the second step, (3.37) in the fourth, and (3.35) in the sixth. Since $\bigl[\bigl(\begin{smallmatrix}\mathrm{id} & -AB^{-1}\\ 0 & \mathrm{id}\end{smallmatrix}\bigr)-\mathrm{id}\bigr]^2=0$, equation (3.39) implies that

$$\bigl(\mathrm{d}(\Psi\circ\gamma\circ\pi_2)|_p^{-1}\circ\mathrm{d}(\Phi\circ\gamma\circ\pi_1)|_p-\mathrm{id}_{T_pP}\bigr)^2=0,$$

and thus Proposition 3.5 gives a Zariski open neighbourhood P' of p in P such that

$$\mathcal{PV}_{\Phi\circ\gamma\circ\pi_1}|_{P'}=\mathcal{PV}_{\Psi\circ\gamma\circ\pi_2}|_{P'}:\mathcal{PV}^\bullet_{P,d}|_{P'}\longrightarrow(\Phi\circ\gamma\circ\pi_1)|_Z^*\bigl(\mathcal{PV}^\bullet_{V,g}\bigr)|_{P'}.\tag{3.40}$$

Since $(\delta\times(M\circ\epsilon))\circ\pi_1=(\delta\times\epsilon)\circ\pi_2:P\to S\times\mathbb{C}^n$ are étale with

$$(e\boxplus z_1^2+\cdots+z_n^2)\circ(\delta\times(M\circ\epsilon))\circ\pi_1=d=(e\boxplus z_1^2+\cdots+z_n^2)\circ(\delta\times\epsilon)\circ\pi_2,$$

we see using (2.8) and (2.18) that

$$\begin{aligned}\pi_1|_Z^*\bigl[\mathcal{PV}_\delta\overset{L}{\boxtimes}\bigl(M|_{\{0\}}^*(\mathcal{PV}_\epsilon)\circ\mathcal{PV}_M\bigr)\bigr]\circ\mathcal{PV}_{\pi_1}&=\pi_1|_Z^*\bigl[\mathcal{PV}_\delta\overset{L}{\boxtimes}\mathcal{PV}_{M\circ\epsilon}\bigr]\circ\mathcal{PV}_{\pi_1}\\ &\cong\pi_1|_Z^*(\mathcal{PV}_{\delta\times(M\circ\epsilon)})\circ\mathcal{PV}_{\pi_1}=\mathcal{PV}_{(\delta\times(M\circ\epsilon))\circ\pi_1}=\mathcal{PV}_{(\delta\times\epsilon)\circ\pi_2}\\ &=\pi_2|_Z^*(\mathcal{PV}_{\delta\times\epsilon})\circ\mathcal{PV}_{\pi_2}\cong\pi_2|_Z^*\bigl[\mathcal{PV}_\delta\overset{L}{\boxtimes}\mathcal{PV}_\epsilon\bigr]\circ\mathcal{PV}_{\pi_2},\end{aligned}\tag{3.41}$$

where '$\cong$' are equalities after identifying both sides of (2.8). Since $\pi_1|_Z=\pi_2|_Z$, and $M|_{\{0\}}=\mathrm{id}_{\{0\}}$, and Example 2.16 shows that $\mathcal{PV}_M$ in (2.19) is multiplication by $\det M$, equation (3.41) implies that

$$\det M\cdot\pi_1|_Z^*\bigl[\mathcal{PV}_\delta\overset{L}{\boxtimes}\mathcal{PV}_\epsilon\bigr]\circ\mathcal{PV}_{\pi_1}=\pi_1|_Z^*\bigl[\mathcal{PV}_\delta\overset{L}{\boxtimes}\mathcal{PV}_\epsilon\bigr]\circ\mathcal{PV}_{\pi_2}.$$

As $\pi_1|_Z^*\bigl[\mathcal{PV}_\delta\overset{L}{\boxtimes}\mathcal{PV}_\epsilon\bigr]$ is an isomorphism, this gives

$$\det M\cdot\mathcal{PV}_{\pi_1}=\mathcal{PV}_{\pi_2}:\mathcal{PV}^\bullet_{P,d}\longrightarrow\pi_1|_Z^*\bigl(\mathcal{PV}^\bullet_{R,e}\bigr).\tag{3.42}$$

Writing $Z'=Z\cap P'$, we now have

$$\begin{aligned}(\gamma\circ\pi_1)|_{Z'}^*(\mathcal{PV}_\Phi)\circ\mathcal{PV}_{\gamma\circ\pi_1}|_{P'}&=\mathcal{PV}_{\Phi\circ\gamma\circ\pi_1}|_{P'}=\mathcal{PV}_{\Psi\circ\gamma\circ\pi_2}|_{P'}\\ &=\pi_2|_{Z'}^*(\mathcal{PV}_{\Psi\circ\gamma})\circ\mathcal{PV}_{\pi_2}|_{P'}\\ &=\det M\cdot\pi_1|_{Z'}^*(\mathcal{PV}_{\Psi\circ\gamma})\circ\mathcal{PV}_{\pi_1}|_{P'}=\det M\cdot\mathcal{PV}_{\Psi\circ\gamma\circ\pi_1}|_{P'}\\ &=\det M\cdot(\gamma\circ\pi_1)|_{Z'}^*(\mathcal{PV}_\Psi)\circ\mathcal{PV}_{\gamma\circ\pi_1}|_{P'},\end{aligned}\tag{3.43}$$

using (3.40) in the second step, (3.42) and $\pi_1|_{Z'}=\pi_2|_{Z'}$ in the fourth, and (2.18) in the rest. As $\mathcal{PV}_{\gamma\circ\pi_1}|_{P'}$ is an isomorphism, (3.43) implies that

$$(\gamma\circ\pi_1)|_{Z'}^*(\mathcal{PV}_\Phi)=\det M\cdot(\gamma\circ\pi_1)|_{Z'}^*(\mathcal{PV}_\Psi),$$

and by Theorem 2.7(i) this implies that

$$\mathcal{PV}_\Phi|_{X'}=\det M\cdot\mathcal{PV}_\Phi|_{X'},\tag{3.44}$$

where $X'=(\gamma\circ\pi_1)(Z')$ is a Zariski open neighbourhood of x in X, since $(\gamma\circ\pi_1)|_{Z'}:Z'\to X$ is étale with $\gamma\circ\pi_1(p)=x$. Now (3.35) and (3.37) give

$$\det\bigl(\mathrm{d}\Psi|_x^{-1}\circ\mathrm{d}\Phi|_x\bigr)=\det\bigl(\begin{smallmatrix}\mathrm{id} & A\\ 0 & B\end{smallmatrix}\bigr)=\det B=\det M.$$

So (3.44) proves that (3.1) holds near x in X. As this is true for all $x\in X$, Theorem 3.1(b) follows.

3.4. **$\mathscr{D}$-modules and mixed Hodge modules.** The proof of Proposition 3.4 applies verbatim also in the analytic context. Theorem 3.1(a),(b) then follow from Proposition 3.4 and the argument given above, using §2.5, including the Sheaf Property (x) for morphisms. Hence all these results carry over to our other contexts §2.6–§2.10.

4. Dependence of $\mathcal{PV}^\bullet_{U,f}$ on f

We will use the following notation:

Definition 4.1. Let U be a smooth $\mathbb{C}$-scheme, let $f : U \to \mathbb{C}$ be a regular function, and let X equal $\mathrm{Crit}(f)$ as a closed $\mathbb{C}$-subscheme of U. Write $I_X \subseteq \mathcal{O}_U$ for the sheaf of ideals of regular functions $U \to \mathbb{C}$ vanishing on X, so that $I_X = I_{\mathrm{d}f}$. For each $k = 1, 2, \ldots$, write $X^{(k)}$ for the k^{th} *order thickening of* X *in* U, that is, $X^{(k)}$ is the closed $\mathbb{C}$-subscheme of U defined by the sheaf of ideals I_X^k in $\mathcal{O}_U$. Also write X^{red} for the reduced $\mathbb{C}$-subscheme of U.

Then we have a chain of inclusions of closed $\mathbb{C}$-subschemes

$$X^{\mathrm{red}} \subseteq X = X^{(1)} \subseteq X^{(2)} \subseteq X^{(3)} \subseteq \cdots \subseteq U. \tag{4.1}$$

Write $f^{(k)} := f|_{X^{(k)}} : X^{(k)} \to \mathbb{C}$, and $f^{\mathrm{red}} := f|_{X^{\mathrm{red}}} : X^{\mathrm{red}} \to \mathbb{C}$, so that $f^{(k)}, f^{\mathrm{red}}$ are regular functions on the $\mathbb{C}$-schemes $X^{(k)}, X^{\mathrm{red}}$. Note that $f^{\mathrm{red}} : X^{\mathrm{red}} \to \mathbb{C}$ is locally constant, since $X = \mathrm{Crit}(f)$.

We also use the same notation for complex analytic spaces.

In §2.4 we defined the perverse sheaf of vanishing cycles $\mathcal{PV}^\bullet_{U,f}$ in $\mathrm{Perv}(X)$. So we can ask: how much of the sequence (4.1) does $\mathcal{PV}^\bullet_{U,f}$ depend on? That is, is $\mathcal{PV}^\bullet_{U,f}$ (canonically?) determined by $(X^{\mathrm{red}}, f^{\mathrm{red}})$, or by $(X^{(k)}, f^{(k)})$ for some $k \geq 1$, as well as by (U, f)? Our next theorem shows that $\mathcal{PV}^\bullet_{U,f}$ is determined up to canonical isomorphism by $(X^{(3)}, f^{(3)})$, and hence a fortiori also by $(X^{(k)}, f^{(k)})$ for $k > 3$:

Theorem 4.2. *Let U, V be smooth $\mathbb{C}$-schemes, $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ be regular functions, and $X = \mathrm{Crit}(f)$, $Y = \mathrm{Crit}(g)$ as closed $\mathbb{C}$-subschemes of U, V, so that §2.4 defines perverse sheaves $\mathcal{PV}^\bullet_{U,f}, \mathcal{PV}^\bullet_{V,g}$ on X, Y. Define $X^{(3)}, f^{(3)}$ and $Y^{(3)}, g^{(3)}$ as in Definition 4.1, and suppose $\Phi : X^{(3)} \to Y^{(3)}$ is an isomorphism with $g^{(3)} \circ \Phi = f^{(3)}$, so that $\Phi|_X : X \to Y \subseteq Y^{(3)}$ is an isomorphism.*

Then there is a canonical isomorphism in $\mathrm{Perv}(X)$

$$\Omega_\Phi : \mathcal{PV}^\bullet_{U,f} \longrightarrow \Phi|_X^*(\mathcal{PV}^\bullet_{V,g}), \tag{4.2}$$

which is characterized by the property that if T is a smooth $\mathbb{C}$-scheme and $\pi_U : T \to U$, $\pi_V : T \to V$ are étale morphisms with $e := f \circ \pi_U = g \circ \pi_V : T \to \mathbb{C}$, so that $\pi_U|_Q : Q \to X$, $\pi_V|_Q : Q \to Y$ are étale for $Q := \mathrm{Crit}(e)$, and $\Phi \circ \pi_U|_{Q^{(2)}} = \pi_V|_{Q^{(2)}} : Q^{(2)} \to Y^{(2)}$, then

$$\pi_U|_Q^*(\Omega_\Phi) \circ \mathcal{PV}_{\pi_U} = \mathcal{PV}_{\pi_V} : \mathcal{PV}^\bullet_{T,e} \longrightarrow \pi_V|_Q^*(\mathcal{PV}^\bullet_{U,f}). \tag{4.3}$$

Also the following commute, where $\sigma_{U,f}, \sigma_{V,g}, \tau_{U,f}, \tau_{V,g}$ are as in (2.6)–(2.7):

$$\begin{tikzcd}
\mathcal{PV}^\bullet_{U,f} \arrow[rr, "\sigma_{U,f}"'] \arrow[d, "\Omega_\Phi"] & & \mathbb{D}_X(\mathcal{PV}^\bullet_{U,f}) \\
\Phi|_X^*(\mathcal{PV}^\bullet_{V,g}) \arrow[r, "\Phi|_X^*(\sigma_{V,g})"] & \Phi|_X^*\big(\mathbb{D}_Y(\mathcal{PV}^\bullet_{V,g})\big) \arrow[r, "\cong"] & \mathbb{D}_X\big(\Phi|_X^*(\mathcal{PV}^\bullet_{V,g})\big), \arrow[u, "\mathbb{D}_X(\Omega_\Phi)"']
\end{tikzcd} \tag{4.4}$$

$$\begin{tikzcd}
\mathcal{PV}^\bullet_{U,f} \arrow[r, "\tau_{U,f}"'] \arrow[d, "\Omega_\Phi"] & \mathcal{PV}^\bullet_{U,f} \arrow[d, "\Omega_\Phi"'] \\
\Phi|_X^*(\mathcal{PV}^\bullet_{V,g}) \arrow[r, "\Phi|_X^*(\tau_{V,g})"] & \Phi|_X^*(\mathcal{PV}^\bullet_{V,g}).
\end{tikzcd} \tag{4.5}$$

If there exists an étale morphism $\Xi : U \to V$ with

$$g \circ \Xi = f : U \to \mathbb{C} \qquad \text{and} \qquad \Xi|_{X^{(3)}} = \Phi : X^{(3)} \to Y^{(3)},$$

then $\Omega_\Phi = \mathcal{PV}_\Xi$, for $\mathcal{PV}_\Xi$ as in (2.14).

If W is another smooth $\mathbb{C}$-scheme, $h : W \to \mathbb{C}$ is a regular function, $Z = \mathrm{Crit}(h)$, and $\Psi : Y^{(3)} \to Z^{(3)}$ is an isomorphism with $h^{(3)} \circ \Psi = g^{(3)}$, then

$$\Omega_{\Psi\circ\Phi} = \Phi|_X^*(\Omega_\Psi) \circ \Omega_\Phi : \mathcal{PV}_{U,f}^\bullet \longrightarrow (\Psi \circ \Phi)|_X^*(\mathcal{PV}_{W,h}^\bullet). \tag{4.6}$$

If $U = V$, $f = g$, $X = Y$ and $\Phi = \mathrm{id}_{X^{(3)}}$ then $\Omega_{\mathrm{id}_{X^{(3)}}} = \mathrm{id}_{\mathcal{PV}_{U,f}^\bullet}$.

The analogues of all the above also hold with appropriate modifications for $\mathscr{D}$-modules on $\mathbb{C}$-schemes, for perverse sheaves and $\mathscr{D}$-modules on complex analytic spaces, and for mixed Hodge modules on $\mathbb{C}$-schemes and complex analytic spaces, as in §2.6–§2.10.

We will prove Theorem 4.2 in §4.2–§4.3. The proof for $\mathbb{C}$-schemes depends on the case $k = 2$ of the following proposition, proved in §4.1:

Proposition 4.3. *Let U, V be smooth $\mathbb{C}$-schemes, $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ be regular functions, and $X = \mathrm{Crit}(f) \subseteq U$, $Y = \mathrm{Crit}(g) \subseteq V$. Using the notation of Definition 4.1, suppose $\Phi : X^{(k+1)} \to Y^{(k+1)}$ is an isomorphism with $g^{(k+1)} \circ \Phi = f^{(k+1)}$ for some $k \geqslant 2$. Then for each $x \in X$ we can choose a smooth $\mathbb{C}$-scheme T and étale morphisms $\pi_U : T \to U$, $\pi_V : T \to V$ such that*

- **(a)** *$e := f \circ \pi_U = g \circ \pi_V : T \to \mathbb{C}$;*
- **(b)** *setting $Q = \mathrm{Crit}(e)$, then $\pi_U|_{Q^{(k)}} : Q^{(k)} \to X^{(k)} \subseteq U$ is an isomorphism with a Zariski open neighbourhood $\tilde{X}^{(k)}$ of x in $X^{(k)}$; and*
- **(c)** *$\Phi \circ \pi_U|_{Q^{(k)}} = \pi_V|_{Q^{(k)}} : Q^{(k)} \to Y^{(k)}$.*

The proof of Proposition 4.3 is similar to that of Proposition 3.5 in §3.1. One can also prove an analogue of Proposition 4.3 when $k = 1$, but in part (b) $\pi_U|_{Q^{(1)}} : Q^{(1)} \to X^{(1)}$ must be étale rather than a Zariski open inclusion.

In Proposition 4.3, we start with $\Phi : X^{(k+1)} \xrightarrow{\cong} Y^{(k+1)}$, but we construct T, π_U, π_V with $\Phi \circ \pi_U|_{Q^{(k)}} = \pi_V|_{Q^{(k)}}$. One might expect to find T, π_U, π_V with $\Phi \circ \pi_U|_{Q^{(k+1)}} = \pi_V|_{Q^{(k+1)}}$, but the next example shows this is not possible.

Example 4.4. Let U, V be open neighbourhoods of 0 in $\mathbb{C}$, and $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ be regular functions given as power series by $f(x) = x^{m+1}$ and $g(y) = y^{m+1} + Ay^{(k+1)m} + \cdots$, for $k, m \geqslant 2$ and $0 \neq A \in \mathbb{C}$, where 0 is the only critical point of g.

Then $X := \mathrm{Crit}(f) = \mathrm{Spec}\big(\mathbb{C}[x]/(x^m)\big)$ and $Y := \mathrm{Crit}(g) = \mathrm{Spec}\big(\mathbb{C}[y]/(y^m)\big)$, so $X^{(k+1)} = \mathrm{Spec}\big(\mathbb{C}[x]/(x^{(k+1)m})\big)$, $f^{(k+1)} = x^{m+1} + (x^{(k+1)m})$, $Y^{(k+1)} = \mathrm{Spec}\big(\mathbb{C}[y]/(y^{(k+1)m})\big)$, and $g^{(k+1)} = y^{m+1} + (y^{(k+1)m})$.

Thus $\Phi : X^{(k+1)} \to Y^{(k+1)}$ acting on functions by $y + (y^{(k+1)m}) \mapsto x + (x^{(k+1)m})$ is an isomorphism with $f^{(k+1)} = g^{(k+1)} \circ \Phi$.

Suppose T, π_U, π_V are as in Proposition 4.3, and use $w = x \circ \pi_U$ as a coordinate on T. Then $e(w) = w^{m+1}$, and $Q = \mathrm{Crit}(e) = \mathrm{Spec}\big(\mathbb{C}[w]/(w^m)\big)$. We have $\pi_U(w) = w$, so $\Phi \circ \pi_U(w) = w$, but $w^{m+1} = \pi_V(w)^{m+1} + A\pi_V(w)^{(k+1)m} + \cdots$, so that $\pi_V(w) = w - \frac{1}{m+1}Aw^{km} + \cdots$. Thus, $\Phi \circ \pi_U : T \to V$ and $\pi_V : T \to V$ differ by $-\frac{1}{m+1}Aw^{km} + \cdots$, which is zero on $Q^{(k)}$ but not on $Q^{(k+1)}$. Hence in this example there do not exist T, π_U, π_V with $\Phi \circ \pi_U|_{Q^{(k+1)}} = \pi_V|_{Q^{(k+1)}}$.

Remark 4.5. We can also ask: can we improve $(X^{(3)}, f^{(3)})$ in Theorem 4.2 to $(X^{(2)}, f^{(2)})$ or $(X^{(1)}, f^{(1)})$ or $(X^{\mathrm{red}}, f^{\mathrm{red}})$? Here are some thoughts on this.

(a) The analogue of Proposition 4.3 for $k = 1$ mentioned above implies that étale or complex analytically locally on X, (U, f) and hence $\mathcal{PV}_{U,f}^\bullet$ are determined up to *non-canonical* isomorphism by $(X^{(2)}, f^{(2)})$. Using the ideas of §5–§6, one can show that these non-canonical isomorphisms of $\mathcal{PV}_{U,f}^\bullet$ are unique *up to sign*.

(b) Consider the following example: let $U = (\mathbb{C} \setminus \{0\}) \times \mathbb{C} = V$, and define $f : U \to \mathbb{C}$ and $g : V \to \mathbb{C}$ by $f(x, y) = y^2$ and $g(x, y) = xy^2$. Then $X := \mathrm{Crit}(f) = \{y = 0\} = \mathrm{Crit}(g) =: Y$, and $f^{(2)} = g^{(2)} = 0$, so that $(X^{(2)}, f^{(2)}) = (Y^{(2)}, g^{(2)})$. However, as in Example 5.5 below, $\mathcal{PV}^\bullet_{U,f} \not\cong \mathcal{PV}^\bullet_{V,g}$. Thus, globally, $\mathcal{PV}^\bullet_{U,f}$ is not determined up to isomorphism by $(X^{(2)}, f^{(2)})$.

(c) Suppose U is a complex manifold and $f : U \to \mathbb{C}$ is holomorphic, with $\mathrm{Crit}(f)$ a single (not necessarily reduced) point x. The *Mather–Yau Theorem* [38] shows that the germ of (U, f) at x is determined up to non-canonical isomorphism by the complex analytic subspace $f^{(1)} = 0$ in $X^{(1)}$, and hence by the pair $(X^{(1)}, f^{(1)})$. Therefore, for *isolated* singularities, $\mathcal{PV}^\bullet_{U,f}$ is determined up to non-canonical isomorphism by $(X^{(1)}, f^{(1)})$.

(d) Define $f : U \to \mathbb{C}$ by $U = \mathbb{C}$ and $f(z) = cz^n$ for $0 \neq c \in \mathbb{C}$ and $n > 2$. This has an isolated singularity at 0, and $(X^{(1)}, f^{(1)})$ is independent of c. By moving c in a circle round zero, we see that in this example $\mathcal{PV}^\bullet_{U,f}$ is determined up to a $\mathbb{Z}/n\mathbb{Z}$ group of automorphisms. So the non-canonical isomorphisms of $\mathcal{PV}^\bullet_{U,f}$ are not unique up to sign, in contrast to **(a)**.

(e) Parts **(a)**–**(d)** leave open the question of whether $\mathcal{PV}^\bullet_{U,f}$ is determined locally up to non-canonical isomorphism by $(X^{(1)}, f^{(1)})$ for non-isolated singularities. We do not have a counterexample to this.

However, Gaffney and Hauser [19, §4] give examples of complex manifolds U and holomorphic $f : U \to \mathbb{C}$ with $X = \mathrm{Crit}(f)$ non-isolated, such that the germ of (U, f) at $x \in X$ is not determined up to non-canonical isomorphism by the germ of $(X^{(1)}, f^{(1)})$ at x, in contrast to the Mather–Yau Theorem, and continuous families of distinct germs $[U, f, x]$ can have the same germ $[X^{(1)}, f^{(1)}, x]$. It seems likely that in examples of this kind, the mixed Hodge module $\mathcal{HV}^\bullet_{U,f}$ (which contains continuous Hodge-theoretic information) is not locally determined up to non-canonical isomorphism by $(X^{(1)}, f^{(1)})$.

(f) For the example in **(d)**, $\mathcal{PV}^\bullet_{U,f}$ depends on $n = 3, 4, \ldots$, but $(X^{\mathrm{red}}, f^{\mathrm{red}}) = (\{0\}, 0)$ is independent of n. So $\mathcal{PV}^\bullet_{U,f}$ is not determined even locally up to non-canonical isomorphism by $(X^{\mathrm{red}}, f^{\mathrm{red}})$.

4.1. **Proof of Proposition 4.3.** The $\mathbb{C}$-subscheme $X^{(k+1)}$ in U is the zeroes of the ideal $I_X^{k+1} \subset \mathcal{O}_U$, which vanishes to order $k+1 \geqslant 2$ at $x \in X \subseteq X^{(k+1)} \subseteq U$. Hence $T_x X^{(k+1)} = T_x U$. As $\Phi : X^{(k+1)} \to Y^{(k+1)}$ is an isomorphism, it follows that

$$T_x U = T_x X^{(k+1)} \cong T_{\Phi(x)} Y^{(k+1)} = T_{\Phi(x)} V. \tag{4.7}$$

Therefore $n := \dim U = \dim V$.

Choose a Zariski open neighbourhood V' of $\Phi(x)$ in V and étale coordinates

$$(y_1, \ldots, y_n) : V' \to \mathbb{C}^n$$

on V'. Write $g' = g|_{V'}$ and $Y' = \mathrm{Crit}(g') = Y \cap V'$, so that $Y'^{(k+1)} = Y^{(k+1)} \cap V'$. Then $y_a \circ \Phi$ are regular functions on the open neighbourhood $\Phi^{-1}(V') \subseteq X^{(k+1)}$ of x in $X^{(k+1)}$, so they extend Zariski locally from $X^{(k+1)}$ to U. Thus we can choose a Zariski open neighbourhood U' of x in U with $\Phi(X^{(k+1)} \cap U') \subseteq Y^{(k+1)} \cap V'$, and regular functions $x_i : U' \to \mathbb{C}$ with $x_i|_{X^{(k+1)} \cap U'} = y_i \circ \Phi|_{X^{(k+1)} \cap U'}$ for $i = 1, \ldots, n$.

Write $f' = f|_{U'}$ and $X' = \mathrm{Crit}(f') = X \cap U'$, so that $X'^{(k+1)} = X^{(k+1)} \cap X'$. Since $(y_1, \ldots, y_n)$ are étale coordinates,

$$\mathrm{d}y_1|_{\Phi(x)}, \ldots, \mathrm{d}y_n|_{\Phi(x)}$$

are a basis for $T^*_{\Phi(x)} V$, so $\mathrm{d}x_1|_x, \ldots, \mathrm{d}x_n|_x$ are a basis for $T^*_x X$ by (4.7). Hence by making U' smaller, we can suppose $(x_1, \ldots, x_n)$ are étale coordinates on U'.

Consider the $\mathbb{C}$-scheme $U' \times V'$, with projections

$$\pi_{U'} : U' \times V' \longrightarrow U' \qquad \text{and} \qquad \pi_{V'} : U' \times V' \longrightarrow V',$$

and write

$$x_i' = x_i \circ \pi_{U'} : U' \times V' \to \mathbb{C}, \qquad y_i' = y_i \circ \pi_{V'} : U' \times V' \to \mathbb{C},$$

so that $(x_1', \ldots, x_n', y_1', \ldots, y_n')$ are étale coordinates on $U' \times V'$. We have a morphism

$$\mathrm{id} \times \Phi|_{X'^{(k+1)}} : X'^{(k+1)} \longrightarrow U' \times V'$$

which embeds $X'^{(k+1)}$ as a closed $\mathbb{C}$-subscheme of $U' \times V'$. The image $\bigl(\mathrm{id} \times \Phi|_{X'^{(k+1)}}\bigr)(X'^{(k+1)})$ is locally the zeroes of the sheaf of ideals

$$\bigl(x_i' - y_i',\ i = 1, \ldots, n\bigr) + \pi_{U'}^{-1}\bigl(I_X^{k+1}\bigr) \subset \mathcal{O}_{U' \times V'},$$

where $\bigl(x_i' - y_i', i = 1, \ldots, n\bigr)$ denotes the ideal generated by $x_i' - y_i' : U' \times V' \to \mathbb{C}$ for $i = 1, \ldots, n$, and $\pi_{U'}^{-1}\bigl(I_X^{k+1}\bigr) \subset \mathcal{O}_{U' \times V'}$ the preimage ideal of $I_X^{k+1}|_{U'} \subset \mathcal{O}_{U'}$.

Now $\bigl(f \circ \pi_{U'} - g \circ \pi_{V'}\bigr)|_{(\mathrm{id} \times \Phi)(X'^{(k+1)})} = 0$ as $f^{(k+1)} = g^{(k+1)} \circ \Phi$. Hence

$$f \circ \pi_{U'} - g \circ \pi_{V'} \in \bigl(x_i' - y_i',\ i = 1, \ldots, n\bigr) + \pi_{U'}^{-1}\bigl(I_X^{k+1}\bigr). \tag{4.8}$$

Lifting (4.8) from $\bigl(x_i' - y_i',\ i = 1, \ldots, n\bigr)$ to $\bigl(x_i' - y_i',\ i = 1, \ldots, n\bigr)^2$, after making U', V' smaller if necessary, we can choose regular functions $A_i : U' \times V' \to \mathbb{C}$ for $i = 1, \ldots, n$ such that

$$f \circ \pi_{U'} - g \circ \pi_{V'} - \sum_{i=1}^n A_i \cdot (x_i' - y_i') \in \bigl(x_i' - y_i',\ i = 1, \ldots, n\bigr)^2 + \pi_{U'}^{-1}\bigl(I_X^{k+1}\bigr). \tag{4.9}$$

Apply $\frac{\partial}{\partial x_i'}$ to (4.9), using the étale coordinates $(x_1', \ldots, x_n', y_1', \ldots, y_n')$ on $U' \times V'$. Since

$$\frac{\partial}{\partial x_i'}\bigl(f \circ \pi_{U'}\bigr) = \frac{\partial f}{\partial x_i} \circ \pi_{U'}$$

and $\frac{\partial}{\partial x_i'}\bigl(g \circ \pi_{V'}\bigr) = 0$, this gives

$$\tfrac{\partial f}{\partial x_i} \circ \pi_{U'} - A_i \in \bigl(x_i' - y_i',\ i = 1, \ldots, n\bigr) + \pi_{U'}^{-1}\bigl(I_X^k\bigr). \tag{4.10}$$

Changing A_i by an element of $\bigl(x_i' - y_i',\ i = 1, \ldots, n\bigr)$ can be absorbed in the ideal

$$\bigl(x_i' - y_i',\ i = 1, \ldots, n\bigr)^2$$

in (4.9), so we can suppose $\frac{\partial f}{\partial x_i} \circ \pi_{U'} - A_i \in \pi_{U'}^{-1}\bigl(I_X^k\bigr)$. As

$$I_X = \bigl(\frac{\partial f}{\partial x_j},\ j = 1, \ldots, n\bigr),$$

after making U', V' smaller we may write

$$A_i = \tfrac{\partial f}{\partial x_i} \circ \pi_{U'} + \sum_{j=1}^n B_{ij} \cdot \tfrac{\partial f}{\partial x_j} \circ \pi_{U'}, \tag{4.11}$$

with $B_{ij} \in \pi_{U'}^{-1}\bigl(I_X^{k-1}\bigr)$ for $i, j = 1, \ldots, n$. Consider the matrix of functions $\bigl(\delta_{ij} + B_{ij}\bigr)_{i,j=1}^n$ on $U' \times V'$. At the point $(x, \Phi(x))$ in $U' \times V'$ this matrix is the identity, since $B_{ij}(x, \Phi(x)) = 0$ as $B_{ij} \in \pi_{U'}^{-1}(I_X^{k-1})$ with $k \geqslant 2$, so $\bigl(\delta_{ij} + B_{ij}\bigr)_{i,j=1}^n$ is invertible near $(x, \Phi(x))$, and making U', V' smaller we can suppose $\bigl(\delta_{ij} + B_{ij}\bigr)_{i,j=1}^n$ is invertible on $U' \times V'$. But in matrix notation we have

$$\bigl(A_i\bigr)_{i=1}^n = \bigl(\delta_{ij} + B_{ij}\bigr)_{i,j=1}^n \bigl(\tfrac{\partial f}{\partial x_j} \circ \pi_{U'}\bigr)_{j=1}^n.$$

Hence in ideals in $\mathcal{O}_{U' \times V'}$ we have

$$\bigl(A_i,\ i = 1, \ldots, n\bigr) = \bigl(\tfrac{\partial f}{\partial x_j} \circ \pi_{U'},\ j = 1, \ldots, n\bigr) = \pi_{U'}^{-1}\bigl(I_X\bigr) \subset \mathcal{O}_{U' \times V'}. \tag{4.12}$$

Now by (4.9), after making U', V' smaller if necessary, we may write

$$\begin{aligned} f\circ\pi_{U'} - g\circ\pi_{V'} = \sum_{i=1}^{n} A_i\cdot\bigl(x_i'-y_i'\bigr) \\ + \sum_{i,j=1}^{n} C_{ij}\cdot\bigl(x_i'-y_i'\bigr)\bigl(x_j'-y_j'\bigr) + \sum_{i,j=1}^{n} D_{ij}\cdot A_iA_j, \end{aligned} \tag{4.13}$$

for regular functions $C_{ij}, D_{ij} : U'\times V'\to\mathbb{C}$ with $D_{ij}\in H^0\bigl(\pi_{U'}^{-1}(I_X^{k-1})\bigr)$ for $i,j=1,\ldots,n$, where in the last term we have used (4.12) to write two factors of $\pi_{U'}^{-1}(I_X)$ in terms of $A_1,\ldots,A_n$.

Write $(z_{ij})_{i,j=1}^n$ for the coordinates on $\mathbb{C}^{n^2}$. Let W be a Zariski open neighbourhood of $\bigl(x,\Phi(x),(0)_{i,j=1}^n\bigr)$ in $U'\times V'\times\mathbb{C}^{n^2}$ to be chosen shortly, and let T be the $\mathbb{C}$-subscheme of W defined by

$$\begin{aligned} T = \Bigl\{\bigl(u,v,(z_{ij})_{i,j=1}^n\bigr)\in W\subseteq U'\times V'\times\mathbb{C}^{n^2} : \\ x_i(u)-y_i(v) = \sum_{j=1}^{n} z_{ij}\cdot A_j(u,v),\quad i=1,\ldots,n, \\ z_{ij} + \sum_{l,m=1}^{n} C_{lm}(u,v)\cdot z_{li}z_{mj} + D_{ij}(u,v) = 0,\quad i,j=1,\ldots,n\Bigr\}. \end{aligned} \tag{4.14}$$

Define $\mathbb{C}$-scheme morphisms $\pi_U : T\to U$ by $\pi_U : \bigl(u,v,(z_{ij})_{i,j=1}^n\bigr)\mapsto u$ and $\pi_V : T\to V$ by $\pi_V : \bigl(u,v,(z_{ij})_{i,j=1}^n\bigr)\mapsto v$.

Now $W\subseteq U'\times V'\times\mathbb{C}^{n^2}$ is smooth of dimension $n+n+n^2$, and in (4.14) we impose $n+n^2$ equations, so the expected dimension of T is $(2n+n^2)-(n+n^2)=n$. The linearizations of the $n+n^2$ equations in (4.14) at $\bigl(u,v,(z_{ij})_{i,j=1}^n\bigr)=\bigl(x,\Phi(x),(0)_{i,j=1}^n\bigr)$ are

$$\begin{aligned} \mathrm{d}x_i|_x(\delta u) - \mathrm{d}y_i|_{\Phi(x)}(\delta v) = 0,\quad i=1,\ldots,n, \\ \delta z_{ij} + \mathrm{d}D_{ij}|_{(x,\Phi(x))}(\delta u\oplus\delta v) = 0,\quad i,j=1,\ldots,n, \end{aligned} \tag{4.15}$$

for $\delta u\in T_xU'$, $\delta v\in T_{\Phi(x)}V'$, and $(\delta z_{ij})_{i,j=1}^n\in T_{(0)_{i,j=1}^n}\mathbb{C}^{n^2}$. As $\mathrm{d}x_1|_x,\ldots,\mathrm{d}x_n|_x$ are a basis for T_x^*U', the equations (4.15) are transverse, so that T is smooth of dimension n near $\bigl(x,\Phi(x),(0)_{i,j=1}^n\bigr)$.

The vector space of solutions $\bigl(\delta u,\delta v,(\delta z_{ij})_{i,j=1}^n\bigr)$ to (4.15) is $T_{(x,\Phi(x),(0))}T$, where

$$\mathrm{d}\pi_U|_{(x,\Phi(x),(0))} : T_{(x,\Phi(x),(0))}T \longrightarrow T_xU$$

maps $\bigl(\delta u,\delta v,(\delta z_{ij})_{i,j=1}^n\bigr)\mapsto\delta u$, and

$$\mathrm{d}\pi_V|_{(x,\Phi(x),(0))} : T_{(x,\Phi(x),(0))}T \longrightarrow T_{\Phi(x)}V$$

maps $\bigl(\delta u,\delta v,(\delta z_{ij})_{i,j=1}^n\bigr)\mapsto\delta v$. Clearly, $\mathrm{d}\pi_U|_{(x,\Phi(x),(0))}, \mathrm{d}\pi_V|_{(x,\Phi(x),(0))}$ are isomorphisms, so as T is smooth near $\bigl(x,\Phi(x),(0)_{i,j=1}^n\bigr)$ and U,V are smooth, we see that π_U,π_V are étale near $\bigl(x,\Phi(x),(0)\bigr)$. Thus, by choosing the open neighbourhood $\bigl(x,\Phi(x),(0)\bigr)\in W\subseteq U'\times V'\times\mathbb{C}^{n^2}$ sufficiently small, we can suppose that T is smooth of dimension n and $\pi_U : T\to U$ and $\pi_V : T\to V$ are étale.

It remains to prove Proposition 4.3(a)–(c). For (a), we have

$$\begin{aligned}
\big(f\circ\pi_U-g\circ\pi_V\big)\big(u,v,(z_{ij})_{i,j=1}^n\big) &= f(u)\ -g(v)=(f\circ\pi_{U'}\ -\ g\circ\pi_{V'})(u,v)\\
&=\sum_{i=1}^n A_i(u,v)\cdot\big(x_i(u)-y_i(v)\big)+\sum_{i,j=1}^n C_{ij}(u,v)\cdot\big(x_i(u)-y_i(v)\big)\big(x_j(u)-y_j(v)\big)\\
&\qquad+\sum_{i,j=1}^n D_{ij}(u,v)\cdot A_i(u,v)A_j(u,v)\\
&=\sum_{i=1}^n A_i(u,v)\cdot\Big(\sum_{j=1}^n z_{ij}\cdot A_j(u,v)\Big)+\sum_{i,j=1}^n C_{ij}(u,v)\cdot\Big(\sum_{l=1}^n z_{il}\cdot A_l(u,v)\Big)\Big(\sum_{m=1}^n z_{jm}\cdot A_m(u,v)\Big)\\
&\qquad+\sum_{i,j=1}^n D_{ij}(u,v)\cdot A_i(u,v)A_j(u,v)\\
&=\sum_{i,j=1}^n A_i(u,v)A_j(u,v)\Big[z_{ij}+\sum_{l,m=1}^n C_{lm}(u,v)\cdot z_{li}z_{mj}+D_{ij}(u,v)\Big]=0,
\end{aligned}$$

using (4.13) in the third step, the first equation of (4.14) in the fourth, rearranging and exchanging labels i,l and j,m in the fifth, and the second equation of (4.14) in the sixth. Hence $f\circ\pi_U-g\circ\pi_V=0:T\to\mathbb{C}$, proving (a).

For (b), using the morphism $\mathrm{id}\times\Phi|_X\times(0):X\to U\times V\times\mathbb{C}^{n^2}\supseteq W$, define

$$\tilde X=\big(\mathrm{id}\times\Phi|_X\times(0)\big)^{-1}(W),$$

so that $\tilde X$ is a Zariski open neighbourhood of x in X. Then $\big(\mathrm{id}\times\Phi|_X\times(0)\big)(\tilde X)$ is a closed $\mathbb{C}$-subscheme of W. We claim that:

(i) $\big(\mathrm{id}\times\Phi|_X\times(0)\big)(\tilde X)$ is a closed $\mathbb{C}$-subscheme of $T\subseteq W$; and

(ii) $\big(\mathrm{id}\times\Phi|_X\times(0)\big)(\tilde X)$ is open and closed in $Q:=\mathrm{Crit}(e)\subseteq T$,

where $e:=f\circ\pi_U=g\circ\pi_V:T\to\mathbb{C}$. To prove (i), we have to show that the equations of (4.14) hold on $\big(\mathrm{id}\times\Phi|_X\times(0)\big)(\tilde X)$, which is true as $x_i|_{\tilde X}=y_i\circ\Phi|_{\tilde X}$, and $z_{ij}\circ(0)=0$, and $D_{ij}\circ(\mathrm{id}\times\Phi|_{\tilde X})=0$ as $D_{ij}\in H^0\big(\pi_{U'}^{-1}(I_X^{k-1})\big)$ for $k\geqslant 2$.

For (ii), as $\pi_U:T\to U$ is étale with $e=f\circ\pi_U$, we see that $\pi_U|_Q:Q\to X$ is étale. But $\pi_U|_Q\circ\big(\mathrm{id}\times\Phi|_X\times(0)\big)|_{\tilde X}=\mathrm{id}_{\tilde X}$. Hence $\big(\mathrm{id}\times\Phi|_X\times(0)\big)(\tilde X)$ is open in Q, and is also closed in Q as it is closed in T. Thus, by making W,T smaller to delete other components of Q, we can suppose that $Q=\big(\mathrm{id}\times\Phi|_X\times(0)\big)(\tilde X)$. Then $\pi_U|_Q:Q\to\tilde X$ is an isomorphism with the Zariski open neighbourhood $\tilde X$ of x in X. Since $\pi_U:T\to U$ is étale with $e=f\circ\pi_U$, this extends to the k^{th} order thickenings, so $\pi_U|_{Q^{(k)}}:Q^{(k)}\to\tilde X^{(k)}$ is an isomorphism, proving (b).

For (c), first note that $Q=\big(\mathrm{id}\times\Phi|_X\times(0)\big)(\tilde X)$, so $\Phi\circ\pi_U|_Q=\pi_V|_Q$ is immediate. We have to extend this to the thickening $Q^{(k)}$. Write $I_Q\subset\mathcal{O}_T$ for the ideal of functions vanishing on Q. Then $I_Q=\pi_U^{-1}(I_X)$ as π_U identifies Q with $\tilde X\subseteq X$. We have

$$A_i\circ\pi_{U'\times V'}\in I_Q\quad\text{and}\quad D_{ij}\circ\pi_{U'\times V'}\in I_Q^{k-1},$$

as $A_i\in H^0\big(\pi_{U'}^{-1}(I_X)\big)$, $D_{ij}\in H^0\big(\pi_{U'}^{-1}(I_X^{k-1})\big)$. The second equation of (4.14) then shows that

$$z_{ij}\circ\pi_{\mathbb{C}^{n^2}}\in I_Q^{k-1},$$

since $Q=\big(\mathrm{id}\times\Phi|_X\times(0)\big)(\tilde X)$ implies that $z_{ij}\circ\pi_{\mathbb{C}^{n^2}}=0$ on Q, so we can neglect the terms $\sum_{l,m=1}^n C_{lm}(u,v)\cdot z_{li}z_{mj}$. Hence the first equation of (4.14) gives

$$x_i\circ\pi_U-y_i\circ\pi_V\in I_Q^k.$$

As I_Q^k vanishes on $Q^{(k)}$, and $x_i|_{X'} = y_i \circ \Phi|_{X'}$, this gives

$$y_i \circ \big(\Phi \circ \pi_U|_{Q^{(k)}}\big) = x_i \circ \pi_U|_{Q^{(k)}} = y_i \circ \big(\pi_V|_{Q^{(k)}}\big).$$

Thus $\Phi \circ \pi_U|_{Q^{(k)}} = \pi_V|_{Q^{(k)}}$ follows, as $(y_1, \ldots, y_n)$ are étale coordinates on V near $\pi_V(Q)$ and $\Phi \circ \pi_U|_Q = \pi_V|_Q$. This proves (c), and Proposition 4.3.

4.2. **Proof of Theorem 4.2 for $\mathbb{C}$-schemes.** Let U, V, f, g, X, Y and $\Phi : X^{(3)} \to Y^{(3)}$ be as in Theorem 4.2. Pick $x \in X$, and apply Proposition 4.3 with $k = 2$. This gives a smooth $\mathbb{C}$-scheme T and étale morphisms $\pi_U : T \to U$, $\pi_V : T \to V$ with $e := f \circ \pi_U = g \circ \pi_V : T \to \mathbb{C}$ and $Q := \mathrm{Crit}(e)$, such that $\pi_U|_{Q^{(2)}} : Q^{(2)} \to X^{(2)}$ is an étale open neighbourhood of x in $X^{(2)}$, and

$$\Phi \circ \pi_U|_{Q^{(2)}} = \pi_V|_{Q^{(2)}} : Q^{(2)} \longrightarrow Y^{(2)}.$$

Actually Proposition 4.3 proves more, that $\pi_U|_{Q^{(2)}} : Q^{(2)} \to X^{(2)}$ is an isomorphism with a Zariski open set $x \in \tilde{X}^{(2)} \subseteq X^{(2)}$, but we will not use this.

Thus, we can choose $\big\{(T^a, \pi_U^a, \pi_V^a, e^a, Q^a) : a \in A\big\}$, where A is an indexing set, such that $T^a, \pi_U^a, \pi_V^a, e^a, Q^a$ satisfy the conditions above for each $a \in A$, and $\big\{\pi_U^a|_{Q^a} : Q^a \to X\big\}_{a \in A}$ is an étale open cover of X. Then for each $a \in A$, by Definition 2.15 we have isomorphisms

$$\mathcal{PV}_{\pi_U^a} : \mathcal{PV}^\bullet_{T^a, e^a} \longrightarrow \pi_U^a|_{Q^a}^*\big(\mathcal{PV}^\bullet_{U,f}\big), \quad \mathcal{PV}_{\pi_V^a} : \mathcal{PV}^\bullet_{T^a, e^a} \longrightarrow \pi_V^a|_{Q^a}^*\big(\mathcal{PV}^\bullet_{V,f}\big).$$

Noting that $\pi_V^a|_{Q^a} = \Phi|_X \circ \pi_U^a|_{Q^a}$, we may define an isomorphism

$$\Omega^a = \mathcal{PV}_{\pi_V^a} \circ \mathcal{PV}_{\pi_U^a}^{-1} : \pi_U^a|_{Q^a}^*\big(\mathcal{PV}^\bullet_{U,f}\big) \longrightarrow \pi_U^a|_{Q^a}^*\big(\Phi|_X^*\big(\mathcal{PV}^\bullet_{V,f}\big)\big). \tag{4.16}$$

For $a, b \in A$, define $T^{ab} = T^a \times_{\pi_U^a, U, \pi_U^b} T^b$ to be the $\mathbb{C}$-scheme fibre product, so that T^{ab} is a smooth $\mathbb{C}$-scheme and the projections $\Pi_{T^a} : T^{ab} \to T^a$, $\Pi_{T^b} : T^{ab} \to T^b$ are étale. Define $e^{ab} = e^a \circ \Pi_{T^a} : T^a \to \mathbb{C}$. Then

$$\begin{aligned} e^{ab} &= e^a \circ \Pi_{T^a} = g \circ \pi_V^a \circ \Pi_{T^a} = f \circ \pi_U^a \circ \Pi_{T^a} \\ &= f \circ \pi_U^b \circ \Pi_{T^b} = g \circ \pi_V^b \circ \Pi_{T^b} = e^b \circ \Pi_{T^b}. \end{aligned} \tag{4.17}$$

Write $Q^{ab} = \mathrm{Crit}(e^{ab})$. Then $\Pi_{T^a}|_{Q^{ab}} : Q^{ab} \to Q^a$ and $\Pi_{T^b}|_{Q^{ab}} : Q^{ab} \to Q^b$ are étale. Now $\pi_U^a \circ \Pi_{T^a} = \pi_U^b \circ \Pi_{T^b}$ and $\Phi \circ \pi_U^a|_{Q^{a\,(2)}} = \pi_V^a|_{Q^{a\,(2)}}$ imply that

$$\begin{aligned} \big(\pi_V^a \circ \Pi_{T^a}\big)|_{Q^{ab\,(2)}} &= \pi_V^a|_{Q^{a\,(2)}} \circ \Pi_{T^a}|_{Q^{ab\,(2)}} \\ &= \Phi|_{X^{(2)}} \circ \pi_U^a|_{Q^{a\,(2)}} \circ \Pi_{T^a}|_{Q^{ab\,(2)}} = \Phi|_{X^{(2)}} \circ (\pi_U^a \circ \Pi_{T^a})|_{Q^{ab\,(2)}} \\ &= \Phi|_{X^{(2)}} \circ (\pi_U^b \circ \Pi_{T^b})|_{Q^{ab\,(2)}} = \Phi|_{X^{(2)}} \circ \pi_U^a|_{Q^{b\,(2)}} \circ \Pi_{T^b}|_{Q^{ab\,(2)}} \\ &= \pi_V^a|_{Q^{b\,(2)}} \circ \Pi_{T^b}|_{Q^{ab\,(2)}} = \big(\pi_V^b \circ \Pi_{T^b}\big)|_{Q^{ab\,(2)}}. \end{aligned} \tag{4.18}$$

Hence $(\pi_V^a \circ \Pi_{T^a})|_{Q^{ab}} = (\pi_V^b \circ \Pi_{T^b})|_{Q^{ab}}$. Moreover, as $TQ^{ab\,(2)}|_{Q^{ab}} = T(T^{ab})|_{Q^{ab}}$, we see that $\mathrm{d}(\pi_V^a \circ \Pi_{T^a})|_{Q^{ab}} = \mathrm{d}(\pi_V^b \circ \Pi_{T^b})|_{Q^{ab}}$, so that

$$\mathrm{d}(\pi_V^b \circ \Pi_{T^b})|_{Q^{ab}}^{-1} \circ \mathrm{d}(\pi_V^a \circ \Pi_{T^a})|_{Q^{ab}} = \mathrm{id} : T(T^{ab})|_{Q^{ab}} \longrightarrow T(T^{ab})|_{Q^{ab}}.$$

So $\det\big(\mathrm{d}(\pi_V^b \circ \Pi_{T^b})|_{Q^{ab}}^{-1} \circ \mathrm{d}(\pi_V^a \circ \Pi_{T^a})|_{Q^{ab}}\big) = 1$. Thus, applying Theorem 3.1 with T^{ab}, V, Q^{ab}, $\pi_V^a \circ \Pi_{T^a}$, $\pi_V^b \circ \Pi_{T^b}$, e^{ab}, f in place of V, W, X, Φ, Ψ, f, g gives

$$\mathcal{PV}_{\pi_V^a \circ \Pi_{T^a}} = \mathcal{PV}_{\pi_V^b \circ \Pi_{T^b}} : \mathcal{PV}^\bullet_{T^{ab}, e^{ab}} \longrightarrow (\pi_V^a \circ \Pi_{T^a})|_{Q^{ab}}^*\big(\mathcal{PV}^\bullet_{V,g}\big). \tag{4.19}$$

Now

$$\begin{aligned}\Pi_{T^a}|_{Q^{ab}}^*(\Omega^a) &= \Pi_{T^a}|_{Q^{ab}}^*(\mathcal{PV}_{\pi_V^a}) \circ \Pi_{T^a}|_{Q^{ab}}^*(\mathcal{PV}_{\pi_U^a})^{-1} \\ &= \big[\Pi_{T^a}|_{Q^{ab}}^*(\mathcal{PV}_{\pi_V^a}) \circ \mathcal{PV}_{\Pi_{T^a}}\big] \circ \big[\Pi_{T^a}|_{Q^{ab}}^*(\mathcal{PV}_{\pi_U^a}) \circ \mathcal{PV}_{\Pi_{T^a}}\big]^{-1} \\ &= \mathcal{PV}_{\pi_V^a\circ\Pi_{T^a}} \circ \mathcal{PV}^{-1}_{\pi_U^a\circ\Pi_{T^a}} = \mathcal{PV}_{\pi_V^b\circ\Pi_{T^b}} \circ \mathcal{PV}^{-1}_{\pi_U^b\circ\Pi_{T^b}} \\ &= \big[\Pi_{T^b}|_{Q^{ab}}^*(\mathcal{PV}_{\pi_V^a}) \circ \mathcal{PV}_{\Pi_{T^b}}\big] \circ \big[\Pi_{T^b}|_{Q^{ab}}^*(\mathcal{PV}_{\pi_U^b}) \circ \mathcal{PV}_{\Pi_{T^b}}\big]^{-1} \\ &= \Pi_{T^b}|_{Q^{ab}}^*(\mathcal{PV}_{\pi_V^b}) \circ \Pi_{T^b}|_{Q^{ab}}^*(\mathcal{PV}_{\pi_U^b})^{-1} = \Pi_{T^b}|_{Q^{ab}}^*(\Omega^b),\end{aligned} \tag{4.20}$$

using (4.16) in the first and seventh steps, (2.18) in the third and fifth, and (4.19) and

$$\pi_U^a \circ \Pi_{T^a} = \pi_U^b \circ \Pi_{T^b}$$

in the fourth. Therefore Theorem 2.7(i) applied to the étale open cover $\{\pi_U^a|_{Q^a} : Q^a \to X\}_{a\in A}$ of X shows that there is a unique isomorphism Ω_Φ in (4.2) with $\pi_U^a|_{Q^a}^*(\Omega_\Phi) = \Omega^a$ for all $a \in A$.

Suppose $\{(T^a, \ldots, Q^a) : a \in A\}$ and $\{(T'^a, \ldots, Q'^a) : a \in A'\}$ are alternative choices above, yielding morphisms Ω_Φ and Ω'_Φ in (4.2). By running the same construction using the family $\{(T^a, \ldots, Q^a) : a \in A\} \amalg \{(T'^a, \ldots, Q'^a) : a \in A'\}$, we get a third morphism Ω''_Φ in (4.2), such that $\pi_U^a|_{Q^a}^*(\Omega_\Phi) = \Omega^a = \pi_U^a|_{Q^a}^*(\Omega''_\Phi)$ for $a \in A$, giving $\Omega_\Phi = \Omega''_\Phi$, and

$$\pi'^a_U|_{Q'^a}^*(\Omega'_\Phi) = \Omega'^a = \pi'^a_U|_{Q'^a}^*(\Omega''_\Phi)$$

for $a \in A'$, which forces $\Omega'_\Phi = \Omega''_\Phi$. Thus $\Omega_\Phi = \Omega'_\Phi$, so Ω_Φ is independent of the choice of $\{(T^a, \ldots, Q^a) : a \in A\}$ above.

Let T, π_U, π_V, e, Q be as in Theorem 4.2. Applying the argument above using the family $\{(T^a, \ldots, Q^a) : a \in A\} \amalg \{(T, \pi_U, \pi_V, e, Q)\}$ shows that Ω_Φ satisfies

$$\pi_U|_Q^*(\Omega_\Phi) = \mathcal{PV}_{\pi_V} \circ \mathcal{PV}^{-1}_{\pi_U},$$

by (4.16). Thus (4.3) holds.

To show that (4.4)–(4.5) commute, we can combine equations (2.16)–(2.17), (4.16) and $\pi_U^a|_{Q^a}^*(\Omega_\Phi) = \Omega^a$ to show that $\pi_U^a|_{Q^a}^*$ applied to (4.4)–(4.5) commute in $\mathrm{Perv}(Q^a)$ for each $a \in A$, so (4.4)–(4.5) commute by Theorem 2.7(i).

Suppose there exists an étale morphism $\Xi : U \to V$ with $f = g \circ \Xi : U \to \mathbb{C}$ and

$$\Xi|_{X^{(3)}} = \Phi : X^{(3)} \to Y^{(3)}.$$

Then as we have to prove, we have

$$\mathcal{PV}_\Xi = \mathrm{id}_X^*(\Omega_\Phi) \circ \mathcal{PV}_{\mathrm{id}_U} = \Omega_\Phi \circ \mathrm{id}_{\mathcal{PV}^\bullet_{U,f}} = \Omega_\Phi,$$

where in the first step we use (4.3) with $T = U$, $\pi_U = \mathrm{id}_U$, $\pi_V = \Xi$, $e = f$, and $Q = X$, and in the second we use $\mathcal{PV}_{\mathrm{id}_U} = \mathrm{id}_{\mathcal{PV}^\bullet_{U,f}}$ from Definition 2.15.

Suppose W is another smooth $\mathbb{C}$-scheme, $h : W \to \mathbb{C}$ is regular, $Z = \mathrm{Crit}(h)$, and $\Psi : Y^{(3)} \to Z^{(3)}$ is an isomorphism with $h^{(3)} \circ \Psi = g^{(3)}$. Let $x \in X$, and set $y = \Phi(x) \in Y$. Proposition 4.3 for x, Φ gives a smooth T and étale $\pi_U : T \to U$, $\pi_V : T \to V$ with $e := f \circ \pi_U = g \circ \pi_V$ and $Q := \mathrm{Crit}(e)$, such that $\pi_U|_{Q^{(2)}} : Q^{(2)} \to X^{(2)}$ is an étale open neighbourhood of x, and $\Phi \circ \pi_U|_{Q^{(2)}} = \pi_V|_{Q^{(2)}}$. Proposition 4.3 for y, Ψ gives smooth $\tilde T$ and étale $\tilde\pi_V : \tilde T \to V$, $\tilde\pi_W : \tilde T \to W$ with $\tilde e := g \circ \tilde\pi_V = h \circ \tilde\pi_W$ and $\tilde Q := \mathrm{Crit}(\tilde e)$, such that $\tilde\pi_V|_{\tilde Q^{(2)}} : \tilde Q^{(2)} \to Y^{(2)}$ is an étale open neighbourhood of y, and $\Psi \circ \tilde\pi_V|_{\tilde Q^{(2)}} = \tilde\pi_W|_{\tilde Q^{(2)}}$.

Define $\hat T = T \times_{\pi_V, V, \tilde\pi_V} \tilde T$ with projections $\Pi_T : \hat T \to T$, $\Pi_{\tilde T} : \hat T \to \tilde T$. Then $\hat T$ is smooth and $\Pi_T, \Pi_{\tilde T}$ are étale, as $T, \tilde T, V$ are smooth and $\pi_V, \tilde\pi_V$ étale. Define $\hat\pi_U = \pi_U \circ \Pi_T : \hat T \to U$

and $\hat\pi_W = \tilde\pi_W \circ \Pi_{\tilde T} : \hat T \to W$. Then $\hat\pi_U, \hat\pi_W$ are étale. Set $\hat e = f \circ \hat\pi_U : \hat T \to \mathbb{C}$, and write $\hat Q = \mathrm{Crit}(\hat e)$. Then

$$\hat e = f\circ\hat\pi_U = f\circ\pi_U\circ\Pi_T = g\circ\pi_V\circ\Pi_T = g\circ\tilde\pi_V\circ\Pi_{\tilde T} = h\circ\tilde\pi_W\circ\Pi_{\tilde T} = h\circ\hat\pi_W.$$

Also $\hat\pi_U|_{\hat Q^{(2)}} : \hat Q^{(2)} \to X^{(2)}$ is an étale open neighbourhood of x, and

$$\begin{aligned}(\Psi\circ\Phi)\circ\hat\pi_U|_{\hat Q^{(2)}} &= \Psi\circ\Phi\circ\pi_U|_{Q^{(2)}}\circ\Pi_T|_{\hat Q^{(2)}} = \Psi\circ\pi_V|_{Q^{(2)}}\circ\Pi_T|_{\hat Q^{(2)}}\\ &= \Psi\circ\tilde\pi_V|_{\tilde Q^{(2)}}\circ\Pi_{\tilde T}|_{\hat Q^{(2)}} = \tilde\pi_W|_{\tilde Q^{(2)}}\circ\Pi_{\tilde T}|_{\hat Q^{(2)}} = \hat\pi_W|_{\hat Q^{(2)}}.\end{aligned}$$

Thus we may apply (4.3) for Ω_Φ with $T, \pi_U, \pi_V, \ldots, Q$, and for Ω_Ψ with $\tilde T, \tilde\pi_V, \tilde\pi_W, \ldots, \tilde Q$, and for $\Omega_{\Psi\circ\Phi}$ with $\hat T, \hat\pi_U, \hat\pi_W, \ldots, \hat Q$. This yields

$$\begin{gathered}\pi_U|_Q^*(\Omega_\Phi) = \mathcal{PV}_{\pi_V}\circ\mathcal{PV}_{\pi_U}^{-1}, \quad \tilde\pi_V|_{\tilde Q}^*(\Omega_\Psi) = \mathcal{PV}_{\tilde\pi_W}\circ\mathcal{PV}_{\tilde\pi_V}^{-1},\\ \hat\pi_U|_{\hat Q}^*(\Omega_{\Psi\circ\Phi}) = \mathcal{PV}_{\hat\pi_W}\circ\mathcal{PV}_{\hat\pi_U}^{-1}.\end{gathered} \tag{4.21}$$

Now

$$\begin{aligned}\hat\pi_U|_{\hat Q}^*\big(\Omega_{\Psi\circ\Phi}\big) &= \mathcal{PV}_{\hat\pi_W}\circ\mathcal{PV}_{\hat\pi_U}^{-1} = \mathcal{PV}_{\tilde\pi_W\circ\Pi_{\tilde T}}\circ\mathcal{PV}_{\pi_U\circ\Pi_T}^{-1}\\ &= \big[\Pi_{\tilde T}|_{\hat Q}^*(\mathcal{PV}_{\tilde\pi_W})\circ\mathcal{PV}_{\Pi_{\tilde T}}\big]\circ\big[\Pi_T|_{\hat Q}^*(\mathcal{PV}_{\pi_U})\circ\mathcal{PV}_{\Pi_T}\big]^{-1}\\ &= \Pi_{\tilde T}|_{\hat Q}^*\big(\mathcal{PV}_{\tilde\pi_W}\circ\mathcal{PV}_{\tilde\pi_V}^{-1}\big)\circ\Pi_{\tilde T}|_{\hat Q}^*(\mathcal{PV}_{\tilde\pi_V})\circ\mathcal{PV}_{\Pi_{\tilde T}}\circ\mathcal{PV}_{\Pi_T}^{-1}\circ\Pi_T|_{\hat Q}^*(\mathcal{PV}_{\pi_U}^{-1})\\ &= \Pi_{\tilde T}|_{\hat Q}^*\big(\tilde\pi_V|_{\tilde Q}^*(\Omega_\Psi)\big)\circ\mathcal{PV}_{\tilde\pi_V\circ\Pi_{\tilde T}}\circ\mathcal{PV}_{\Pi_T}^{-1}\circ\Pi_T|_{\hat Q}^*(\mathcal{PV}_{\pi_U}^{-1})\\ &= \big[\tilde\pi_V|_{\tilde Q}\circ\Pi_{\tilde T}|_{\hat Q}\big]^*(\Omega_\Psi)\circ\mathcal{PV}_{\pi_V\circ\Pi_T}\circ\mathcal{PV}_{\Pi_T}^{-1}\circ\Pi_T|_{\hat Q}^*(\mathcal{PV}_{\pi_U}^{-1})\\ &= \big[\pi_V|_Q\circ\Pi_T|_{\hat Q}\big]^*(\Omega_\Psi)\circ\Pi_T|_{\hat Q}^*(\mathcal{PV}_{\pi_V})\circ\mathcal{PV}_{\Pi_T}\circ\mathcal{PV}_{\Pi_T}^{-1}\circ\Pi_T|_{\hat Q}^*(\mathcal{PV}_{\pi_U}^{-1})\\ &= \big[\Phi|_X\circ\pi_U|_Q\circ\Pi_T|_{\hat Q}\big]^*(\Omega_\Psi)\circ\Pi_T|_{\hat Q}^*(\mathcal{PV}_{\pi_V}\circ\mathcal{PV}_{\pi_U}^{-1})\\ &= (\pi_U\circ\Pi_T)|_{\hat Q}^*(\Phi|_X^*(\Omega_\Psi))\circ\Pi_T|_{\hat Q}^*(\pi_U|_Q^*(\Omega_\Phi)) = \hat\pi_U|_{\hat Q}^*\big(\Phi|_X^*(\Omega_\Psi)\circ\Omega_\Phi\big),\end{aligned}$$

using (4.21) in the first, fifth and ninth steps, (2.18) in the third, fifth and seventh steps, $\pi_V\circ\Pi_T = \tilde\pi_V\circ\Pi_{\tilde T}$ in the sixth and seventh, and $\Phi\circ\pi_U|_Q = \pi_V|_Q$ in the eighth. Thus, for each $x \in X$, we have constructed an étale open neighbourhood $\hat\pi_U|_{\hat Q} : \hat Q \to X$ such that $\hat\pi_U|_{\hat Q}^*$ applied to (4.6) holds. Equation (4.6) follows by Theorem 2.7(i). Finally, if $U = V$, $f = g$, $X = Y$ and $\Phi = \mathrm{id}_{X^{(3)}}$ then $\Omega_{\mathrm{id}_{X^{(3)}}} = \mathrm{id}_{\mathcal{PV}^\bullet_{U,f}}$ follows by taking $\Xi = \mathrm{id}_U$ in the fourth paragraph of the theorem. This proves Theorem 4.2 for perverse sheaves on $\mathbb{C}$-schemes.

4.3. **$\mathscr{D}$-modules and mixed Hodge modules.** Once again, the proof of Proposition 4.3 is completely algebraic, so applies in the other contexts of §2.6–§2.10. Theorem 4.2 then follows for our other contexts from that and the general framework of §2.5.

5. Stabilizing vanishing cycles

To set up notation for our main result, which is Theorem 5.4 below, we need the following theorem, which is proved in Joyce [23, Prop.s 2.22, 2.23 & 2.25].

Theorem 5.1 (Joyce [23])**.** *Let U, V be smooth $\mathbb{C}$-schemes, $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ be regular, and $X = \mathrm{Crit}(f)$, $Y = \mathrm{Crit}(g)$ as $\mathbb{C}$-subschemes of U, V. Let $\Phi : U \hookrightarrow V$ be a closed embedding of $\mathbb{C}$-schemes with $f = g\circ\Phi : U \to \mathbb{C}$, and suppose $\Phi|_X : X \to V \supseteq Y$ is an isomorphism $\Phi|_X : X \to Y$. Then:*

(i) *For each $x \in X \subseteq U$ there exist smooth $\mathbb{C}$-schemes U', V', a point $x' \in U'$ and morphisms $\iota : U' \to U$, $\jmath : V' \to V$, $\Phi' : U' \to V'$, $\alpha : V' \to U$ and $\beta : V' \to \mathbb{C}^n$, where $n = \dim V - \dim U$,* *such that $\iota(x') = x$, and $\iota, \jmath$ and $\alpha\times\beta : V \to U\times\mathbb{C}^n$ are étale, and the following diagram commutes*

$$\begin{tikzcd} U \arrow[d, "\Phi"] & U' \arrow[l, "\iota"'] \arrow[r, "\iota"] \arrow[d, "\Phi'"] & U \arrow[d, "\mathrm{id}_U\times 0"'] \\ V & V' \arrow[l, "\jmath"'] \arrow[r, "\alpha\times\beta"] & U\times\mathbb{C}^n, \end{tikzcd} \tag{5.1}$$

and $g\circ\jmath = f\circ\alpha + (z_1^2+\cdots+z_n^2)\circ\beta : V'\to\mathbb{C}$. Thus, setting $f' := f\circ\iota : U'\to\mathbb{C}$, $g' := g\circ\jmath : V'\to\mathbb{C}$, $X' := \mathrm{Crit}(f')\subseteq U'$, and $Y' := \mathrm{Crit}(g')\subseteq V'$, then $f' = g'\circ\Phi' : U'\to\mathbb{C}$, and $\Phi'|_{X'} : X'\to Y'$, $\iota|_{X'} : X'\to X$, $\jmath|_{Y'} : Y'\to Y$, $\alpha|_{Y'} : Y'\to X$ are étale. We also require that $\Phi\circ\alpha|_{Y'} = \jmath|_{Y'} : Y'\to Y$.

(ii) *Write N_{UV} for the normal bundle of $\Phi(U)$ in V, regarded as an algebraic vector bundle on U in the exact sequence of vector bundles on U:*

$$0 \longrightarrow TU \xrightarrow{\mathrm{d}\Phi} \Phi^*(TU) \xrightarrow{\Pi_{UV}} N_{UV} \longrightarrow 0. \tag{5.2}$$

Then there exists a unique $q_{UV}\in H^0(S^2N_{UV}^*|_X)$ which is a nondegenerate quadratic form on $N_{UV}|_X$, such that whenever $U', V', \iota, \jmath, \Phi', \beta, n, X'$ are as in **(i)**, *writing $\langle \mathrm{d}z_1,\ldots,\mathrm{d}z_n\rangle_{U'}$ for the trivial vector bundle on U' with basis $\mathrm{d}z_1,\ldots,\mathrm{d}z_n$, there is a natural isomorphism $\hat\beta : \langle \mathrm{d}z_1,\ldots,\mathrm{d}z_n\rangle_{U'}\to\iota^*(N_{UV}^*)$ making the following diagram commute:*

$$\begin{tikzcd} \iota^*(N_{UV}^*) \arrow[rr, "\iota^*(\Pi_{UV}^*)"'] & & \iota^*\circ\Phi^*(T^*V) = \Phi'^*\circ\jmath^*(T^*V) \arrow[d, "\Phi'^*(\mathrm{d}\jmath^*)"'] \\ \langle \mathrm{d}z_1,\ldots,\mathrm{d}z_n\rangle_{U'} = \Phi'^*\circ\beta^*(T_0^*\mathbb{C}^n) \arrow[u, dotted, "\hat\beta"] \arrow[rr, "\Phi'^*(\mathrm{d}\beta^*)"] & & \Phi'^*(T^*V'), \end{tikzcd} \tag{5.3}$$

$$\text{and} \qquad \iota|_{X'}^*(q_{UV}) = (S^2\hat\beta)|_{X'}(\mathrm{d}z_1\otimes\mathrm{d}z_1+\cdots+\mathrm{d}z_n\otimes\mathrm{d}z_n). \tag{5.4}$$

(iii) *Now suppose W is another smooth $\mathbb{C}$-scheme, $h : W\to\mathbb{C}$ is regular, $Z = \mathrm{Crit}(h)$ as a $\mathbb{C}$-subscheme of W, and $\Psi : V\hookrightarrow W$ is a closed embedding of $\mathbb{C}$-schemes with $g = h\circ\Psi : V\to\mathbb{C}$ and $\Psi|_Y : Y\to Z$ an isomorphism. Define N_{VW}, q_{VW} and N_{UW}, q_{UW} using $\Psi : V\hookrightarrow W$ and $\Psi\circ\Phi : U\hookrightarrow W$ as in* **(ii)** *above. Then there are unique morphisms $\gamma_{UVW}, \delta_{UVW}$ which make the following diagram of vector bundles on U commute, with straight lines exact:*

$$\begin{tikzcd} & & & & & 0 \arrow[d] & \\ & & & & & TU \arrow[dll, "\mathrm{d}\Phi"'] \arrow[ddl, "\mathrm{d}(\Psi\circ\Phi)"] & 0 \arrow[l] \\ & & 0 \arrow[r] & \Phi^*(TV) \arrow[dll, "\Pi_{UV}"'] \arrow[dr, "\Phi^*(\mathrm{d}\Psi)"] & & & \\ 0 \arrow[r] & N_{UV} \arrow[dl] \arrow[drr, dotted, "\gamma_{UVW}"'] & & & (\Psi\circ\Phi)^*(TW) \arrow[dl, "\Pi_{UW}"] \arrow[ddr, "\Phi^*(\Pi_{VW})"] & & \\ 0 & & & N_{UW} \arrow[dl] \arrow[drr, dotted, "\delta_{UVW}"'] & & & \\ & & 0 & & & \Phi^*(N_{VW}) \arrow[r] \arrow[d] & 0 \\ & & & & & 0. & \end{tikzcd} \tag{5.5}$$

Restricting to X gives an exact sequence of vector bundles:

$$0 \longrightarrow N_{UV}|_X \xrightarrow{\gamma_{UVW}|_X} N_{UW}|_X \xrightarrow{\delta_{UVW}|_X} \Phi|_X^*(N_{VW}) \longrightarrow 0. \tag{5.6}$$

Then there is a natural isomorphism of vector bundles on X

$$N_{UW}|_X \cong N_{UV}|_X \oplus \Phi|_X^*(N_{VW}), \tag{5.7}$$

compatible with the exact sequence (5.6), *which identifies*

$$\begin{aligned} q_{UW} &\cong q_{UV} \oplus \Phi|_X^*(q_{VW}) \oplus 0 \qquad \textit{under the splitting} \\ S^2 N_{UW}|_X^* &\cong S^2 N_{UV}|_X^* \oplus \Phi|_X^*\big(S^2 N_{VW}^*|_Y\big) \oplus \big(N_{UV}^*|_X \otimes \Phi|_X^*(N_{VW}^*)\big). \end{aligned} \tag{5.8}$$

(iv) *Analogues of* **(i)**–**(iii)** *hold for complex analytic spaces, replacing the smooth* $\mathbb{C}$*-schemes* U, V, W *by complex manifolds, the regular functions* f, g, h *by holomorphic functions, the* $\mathbb{C}$*-schemes* X, Y, Z *by complex analytic spaces, the étale open sets* $\iota : U' \to U$, $\jmath : V' \to V$ *by complex analytic open sets* $U' \subseteq U$, $V' \subseteq V$, *and with* $\alpha \times \beta : V' \to U \times \mathbb{C}^n$ *a biholomorphism with a complex analytic open neighbourhood of* $(x, 0)$ *in* $U \times \mathbb{C}^n$.

Following [23, Def.s 2.26 & 2.34], we define:

Definition 5.2. Let U, V be smooth $\mathbb{C}$-schemes, $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ be regular, and $X = \mathrm{Crit}(f)$, $Y = \mathrm{Crit}(g)$ as $\mathbb{C}$-subschemes of U, V. Suppose $\Phi : U \hookrightarrow V$ is a closed embedding of $\mathbb{C}$-schemes with $f = g \circ \Phi : U \to \mathbb{C}$ and $\Phi|_X : X \to Y$ an isomorphism. Then Theorem 5.1(ii) defines the normal bundle N_{UV} of U in V, a vector bundle on U of rank $n = \dim V - \dim U$, and a nondegenerate quadratic form $q_{UV} \in H^0(S^2 N_{UV}^*|_X)$. Taking top exterior powers in the dual of (5.2) gives an isomorphism of line bundles on U

$$\rho_{UV} : K_U \otimes \Lambda^n N_{UV}^* \xrightarrow{\cong} \Phi^*(K_V),$$

where K_U, K_V are the canonical bundles of U, V.

Write X^{red} for the reduced $\mathbb{C}$-subscheme of X. As q_{UV} is a nondegenerate quadratic form on $N_{VW}|_X$, its determinant $\det(q_{VW})$ is a nonzero section of $(\Lambda^n N_{VW}^*)|_X^{\otimes^2}$. Define an isomorphism of line bundles on X^{red}:

$$J_\Phi = \rho_{UV}^{\otimes^2} \circ \big(\mathrm{id}_{K_U^2|_{X^{\mathrm{red}}}} \otimes \det(q_{UV})|_{X^{\mathrm{red}}}\big) : K_U^{\otimes^2}\big|_{X^{\mathrm{red}}} \xrightarrow{\cong} \Phi|_{X^{\mathrm{red}}}^*\big(K_V^{\otimes^2}\big). \tag{5.9}$$

Since principal $\mathbb{Z}/2\mathbb{Z}$-bundles $\pi : P \to X$ in the sense of Definition 2.9 are an (étale or complex analytic) topological notion, and X^{red} and X have the same topological space (even in the étale or complex analytic topology), principal $\mathbb{Z}/2\mathbb{Z}$-bundles on X^{red} and on X are equivalent. Define $\pi_\Phi : P_\Phi \to X$ to be the principal $\mathbb{Z}/2\mathbb{Z}$-bundle which parametrizes square roots of J_Φ on X^{red}. That is, (étale or complex analytic) local sections $s_\alpha : X \to P_\Phi$ of P_Φ correspond to local isomorphisms $\alpha : K_U|_{X^{\mathrm{red}}} \to \Phi|_{X^{\mathrm{red}}}^*(K_V)$ on X^{red} with $\alpha \otimes \alpha = J_\Phi$.

Now suppose W is another smooth $\mathbb{C}$-scheme, $h : W \to \mathbb{C}$ is regular, $Z = \mathrm{Crit}(h)$ as a $\mathbb{C}$-subscheme of W, and $\Psi : V \hookrightarrow W$ is a closed embedding of $\mathbb{C}$-schemes with $g = h \circ \Psi : V \to \mathbb{C}$ and $\Psi|_Y : Y \to Z$ an isomorphism. Then Theorem 5.1(iii) applies, and from (5.7)–(5.8) we can deduce that

$$\begin{aligned} J_{\Psi\circ\Phi} = \Phi|_{X^{\mathrm{red}}}^*(J_\Psi) \circ J_\Phi : K_U^{\otimes^2}\big|_{X^{\mathrm{red}}} &\xrightarrow{\cong} (\Psi \circ \Phi)|_{X^{\mathrm{red}}}^*\big(K_W^{\otimes^2}\big) \\ &= \Phi|_{X^{\mathrm{red}}}^*\big[\Psi|_{Y^{\mathrm{red}}}^*\big(K_W^{\otimes^2}\big)\big]. \end{aligned} \tag{5.10}$$

For the principal $\mathbb{Z}/2\mathbb{Z}$-bundles $\pi_\Phi : P_\Phi \to X$, $\pi_\Psi : P_\Psi \to Y$, $\pi_{\Psi\circ\Phi} : P_{\Psi\circ\Phi} \to X$, equation (5.10) implies that there is a canonical isomorphism

$$\Xi_{\Psi,\Phi} : P_{\Psi\circ\Phi} \xrightarrow{\cong} \Phi|_X^*(P_\Psi) \otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Phi. \tag{5.11}$$

It is also easy to see that these $\Xi_{\Psi,\Phi}$ have an associativity property under triple compositions, that is, given another smooth $\mathbb{C}$-scheme T, regular $e : T \to \mathbb{C}$ with $Q := \mathrm{Crit}(e)$, and $\Upsilon : T \hookrightarrow U$

a closed embedding with $e = f \circ \Upsilon : T \to \mathbb{C}$ and $\Upsilon|_Q : Q \to X$ an isomorphism, then

$$\begin{aligned}\bigl(\mathrm{id}_{(\Phi\circ\Upsilon)|_Q^*(P_\Psi)} \otimes \Xi_{\Phi,\Upsilon}\bigr) \circ \Xi_{\Psi,\Phi\circ\Upsilon} = \bigl(\Upsilon|_Q^*(\Xi_{\Psi,\Phi}) \otimes \mathrm{id}_{P_\Upsilon}\bigr) \circ \Xi_{\Psi\circ\Phi,\Upsilon} :\\ P_{\Psi\circ\Phi\circ\Upsilon} \longrightarrow (\Phi\circ\Upsilon)|_Q^*(P_\Psi) \otimes_{\mathbb{Z}/2\mathbb{Z}} \Upsilon|_Q^*(P_\Phi) \otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Upsilon.\end{aligned} \tag{5.12}$$

Analogues of all the above also work for complex manifolds and complex analytic spaces, as in Theorem 5.1(v).

The reason for restricting to X^{red} above is the following [23, Prop. 2.27], whose proof uses the fact that X^{red} is reduced in an essential way.

Lemma 5.3. *In Definition 5.2, the isomorphism J_Φ in (5.9) and the principal $\mathbb{Z}/2\mathbb{Z}$-bundle $\pi_\Phi : P_\Phi \to X$ depend only on U, V, X, Y, f, g and $\Phi|_X : X \to Y$. That is, they do not depend on $\Phi : U \to V$ apart from $\Phi|_X : X \to Y$.*

Using the notation of Definition 5.2, we can state our main result:

Theorem 5.4. **(a)** *Let U, V be smooth $\mathbb{C}$-schemes, $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ be regular, and $X = \mathrm{Crit}(f)$, $Y = \mathrm{Crit}(g)$ as $\mathbb{C}$-subschemes of U, V. Let $\Phi : U \hookrightarrow V$ be a closed embedding of $\mathbb{C}$-schemes with $f = g \circ \Phi : U \to \mathbb{C}$, and suppose $\Phi|_X : X \to V \supseteq Y$ is an isomorphism $\Phi|_X : X \to Y$. Then there is a natural isomorphism of perverse sheaves on X:*

$$\Theta_\Phi : \mathcal{PV}_{U,f}^\bullet \longrightarrow \Phi|_X^*\bigl(\mathcal{PV}_{V,g}^\bullet\bigr) \otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Phi, \tag{5.13}$$

where $\mathcal{PV}_{U,f}^\bullet, \mathcal{PV}_{V,g}^\bullet$ are the perverse sheaves of vanishing cycles from §2.4, and P_Φ the principal $\mathbb{Z}/2\mathbb{Z}$-bundle from Definition 5.2, and if $\mathcal{Q}^\bullet$ is a perverse sheaf on X then $\mathcal{Q}^\bullet \otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Phi$ is as in Definition 2.9. Also the following diagrams commute, where $\sigma_{U,f}, \sigma_{V,g}, \tau_{U,f}, \tau_{V,g}$ are as in (2.6)–(2.7):

$$\begin{tikzcd}
\mathcal{PV}_{U,f}^\bullet \arrow[r, "\Theta_\Phi"'] \arrow[d, "\sigma_{U,f}"] & \Phi|_X^*\bigl(\mathcal{PV}_{V,g}^\bullet\bigr) \otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Phi \arrow[r, "\Phi|_X^*(\sigma_{V,g})\otimes\mathrm{id}"'] & \Phi|_X^*\bigl(\mathbb{D}_Y(\mathcal{PV}_{V,g}^\bullet)\bigr) \otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Phi \arrow[d, "\cong"'] \\
\mathbb{D}_X(\mathcal{PV}_{U,f}^\bullet) & & \mathbb{D}_X\bigl(\Phi|_X^*(\mathcal{PV}_{V,g}^\bullet) \otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Phi\bigr), \arrow[ll, "\mathbb{D}_X(\Theta_\Phi)"']
\end{tikzcd} \tag{5.14}$$

$$\begin{tikzcd}
\mathcal{PV}_{U,f}^\bullet \arrow[r, "\Theta_\Phi"'] \arrow[d, "\tau_{U,f}"] & \Phi|_X^*\bigl(\mathcal{PV}_{V,g}^\bullet\bigr) \otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Phi \arrow[d, "\Phi|_X^*(\tau_{V,g})\otimes\mathrm{id}"'] \\
\mathcal{PV}_{U,f}^\bullet \arrow[r, "\Theta_\Phi"] & \Phi|_X^*\bigl(\mathcal{PV}_{V,g}^\bullet\bigr) \otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Phi.
\end{tikzcd} \tag{5.15}$$

If $U = V$, $f = g$, $\Phi = \mathrm{id}_U$ then $\pi_\Phi : P_\Phi \to X$ is trivial, and Θ_Φ corresponds to $\mathrm{id}_{\mathcal{PV}_{U,f}^\bullet}$ under the natural isomorphism $\mathrm{id}_X^*(\mathcal{PV}_{U,f}^\bullet) \otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Phi \cong \mathcal{PV}_{U,f}^\bullet$.

(b) *The isomorphism Θ_Φ in (5.13) depends only on U, V, X, Y, f, g and $\Phi|_X : X \to Y$. That is, if $\tilde\Phi : U \to V$ is an alternative choice for Φ with $\Phi|_X = \tilde\Phi|_X : X \to Y$, then $\Theta_\Phi = \Theta_{\tilde\Phi}$, noting that $P_\Phi = P_{\tilde\Phi}$ by Lemma 5.3.*

(c) *Now suppose W is another smooth $\mathbb{C}$-scheme, $h : W \to \mathbb{C}$ is a regular function, $Z = \mathrm{Crit}(h)$, and $\Psi : V \hookrightarrow W$ is a closed embedding with $g = h \circ \Psi : V \to \mathbb{C}$ and $\Psi|_Y : Y \to Z$ an isomorphism. Then Definition 5.2 defines principal $\mathbb{Z}/2\mathbb{Z}$-bundles*

$$\pi_\Phi : P_\Phi \longrightarrow X, \qquad \pi_\Psi : P_\Psi \longrightarrow Y, \qquad \pi_{\Psi\circ\Phi} : P_{\Psi\circ\Phi} \longrightarrow X$$

and an isomorphism $\Xi_{\Psi,\Phi}$ *in* (5.11), *and part* **(a)** *defines isomorphisms of perverse sheaves* $\Theta_\Phi, \Theta_{\Psi\circ\Phi}$ *on* X *and* Θ_Ψ *on* Y*. Then the following commutes in* $\mathrm{Perv}(X)$*:*

$$\begin{tikzcd}
\mathcal{PV}^\bullet_{U,f} \arrow[rr, "\Theta_{\Psi\circ\Phi}"'] \arrow[d, "\Theta_\Phi"] && (\Psi\circ\Phi)|_X^*\bigl(\mathcal{PV}^\bullet_{W,h}\bigr)\otimes_{\mathbb{Z}/2\mathbb{Z}} P_{\Psi\circ\Phi} \arrow[d, "\mathrm{id}\otimes\Xi_{\Psi,\Phi}"'] \\
\Phi|_X^*\bigl(\mathcal{PV}^\bullet_{V,g}\bigr)\otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Phi \arrow[rr, "\Phi|_X^*(\Theta_\Psi)\otimes\mathrm{id}"] && \Phi|_X^*\circ\Psi|_Y^*\bigl(\mathcal{PV}^\bullet_{W,h}\bigr)\otimes_{\mathbb{Z}/2\mathbb{Z}}\Phi|_X^*(P_\Psi)\otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Phi.
\end{tikzcd} \tag{5.16}$$

(d) *The analogues of* **(a)**–**(c)** *also hold for* $\mathscr{D}$*-modules on* $\mathbb{C}$*-schemes, for perverse sheaves and* $\mathscr{D}$*-modules on complex analytic spaces, and for mixed Hodge modules on* $\mathbb{C}$*-schemes and complex analytic spaces, as in* §2.6–§2.10.

Example 5.5. Let $U = \mathbb{C}\setminus\{0\}$ and $V = (\mathbb{C}\setminus\{0\})\times\mathbb{C}$ as smooth $\mathbb{C}$-schemes, define regular $f : U \to \mathbb{C}$ and $g : V \to \mathbb{C}$ by $f(x) = 0$ and $g(x,y) = x^k y^2$ for fixed $k \in \mathbb{Z}$, and define $\Phi : U \to V$ by $\Phi : x \mapsto (x,0)$, so that $f = g\circ\Phi : U \to \mathbb{C}$. Then $X := \mathrm{Crit}(f) = U$, and $Y := \mathrm{Crit}(g) = \bigl\{(x,y)\in V : kx^{k-1}y^2 = 2x^k y = 0\bigr\} = \bigl\{(x,y)\in V : y = 0\bigr\}$, as $x \neq 0$. Thus $\Phi|_X : X \to Y$ is an isomorphism.

In Theorem 5.1(ii), N_{UV}^* is the trivial line bundle on U with basis $\mathrm{d}y$, and $q_{UV} = x^k\mathrm{d}y\otimes\mathrm{d}y$. In Definition 5.2, $K_U|_X$ and $\Phi|_X^*(K_V)$ are the trivial line bundles on $X = X^{\mathrm{red}} = U$ with bases $\mathrm{d}x$ and $\mathrm{d}x\wedge\mathrm{d}y$, and J_Φ in (5.9) maps

$$J_\Phi : \mathrm{d}x\otimes\mathrm{d}x \longmapsto x^k\,(\mathrm{d}x\wedge\mathrm{d}y)\otimes(\mathrm{d}x\wedge\mathrm{d}y).$$

The principal $\mathbb{Z}/2\mathbb{Z}$-bundle $\pi_\Phi : P_\Phi \to X$ in Definition 5.2 parametrizes $\alpha : K_U|_X \to \Phi|_X^*(K_V)$ with $\alpha\otimes\alpha = J_\Phi$. Writing $\alpha : \mathrm{d}x \mapsto p\,\mathrm{d}x\wedge\mathrm{d}y$ for p a local function on $X = \mathbb{C}\setminus\{0\}$, $\alpha\otimes\alpha = J_\Phi$ reduces to $p^2 = x^k$. Thus, P_Φ parametrizes (étale local) square roots p of $x^k : \mathbb{C}\setminus\{0\}\to\mathbb{C}\setminus\{0\}$.

If k is even then x^k has a global square root $p = x^{k/2}$, so the principal $\mathbb{Z}/2\mathbb{Z}$-bundle P_Φ has a global section, and is trivial. If k is odd then x^k has no global square root on $X = \mathbb{C}\setminus\{0\}$, so P_Φ has no global section, and is nontrivial.

Thus, Theorem 5.4 implies that if k is even then $\mathcal{PV}^\bullet_{V,g} \cong A_Y[1]$ is the constant perverse sheaf on Y, but if k is odd then $\mathcal{PV}^\bullet_{V,g}$ is the twist of $A_Y[1]$ by the unique nontrivial principal $\mathbb{Z}/2\mathbb{Z}$-bundle on $Y \cong \mathbb{C}\setminus\{0\}$.

5.1. **Theorem 5.4(a): the isomorphism** Θ_Φ**.** Let U, V, f, g, X, Y, Φ be as in Theorem 5.4(a), and use the notation N_{UV}, q_{UV} from Theorem 5.1(ii) and J_Φ, P_Φ from Definition 5.2. We will show that there exists a unique perverse sheaf morphism Θ_Φ in (5.13) which is characterized by the property that whenever $U', V', \iota, \jmath, \Phi', \alpha, \beta, X', Y', f', g'$ are as in Theorem 5.1(i) then the following diagram of isomorphisms in $\mathrm{Perv}(X')$ commutes:

$$\begin{tikzcd}
\iota|_{X'}^*\bigl(\mathcal{PV}^\bullet_{U,f}\bigr) \arrow[r, "\iota|_{X'}^*(\gamma)"'] \arrow[d, "\iota|_{X'}^*(\Theta_\Phi)"] & \iota|_{X'}^*\circ(\mathrm{id}_X\times 0)^*\bigl(\mathcal{PV}^\bullet_{U,f}\overset{L}{\boxtimes}\mathcal{PV}^\bullet_{\mathbb{C}^n,z_1^2+\cdots+z_n^2}\bigr) \arrow[d, "\iota|_{X'}^*\circ(\mathrm{id}_X\times 0)^*(\mathcal{TS}^{-1}_{U,f,\mathbb{C}^n,z_1^2+\cdots+z_n^2})"'] \\
\iota|_{X'}^*\circ\Phi|_X^*\bigl(\mathcal{PV}^\bullet_{V,g}\bigr)\otimes_{\mathbb{Z}/2\mathbb{Z}}\iota|_{X'}^*(P_\Phi) \arrow[d, "\delta"] & \begin{array}{c}\iota|_{X'}^*\circ(\mathrm{id}_X\times 0)^*\bigl(\mathcal{PV}^\bullet_{U\times\mathbb{C}^n,f\boxplus z_1^2+\cdots+z_n^2}\bigr) = \\ \Phi'|_{X'}^*\circ(\alpha\times\beta)|_{Y'}^*\bigl(\mathcal{PV}^\bullet_{U\times\mathbb{C}^n,f\boxplus z_1^2+\cdots+z_n^2}\bigr)\end{array} \arrow[d, "\Phi'|_{X'}^*(\mathcal{PV}^{-1}_{\alpha\times\beta})"'] \\
\begin{array}{c}\iota|_{X'}^*\circ\Phi|_X^*\bigl(\mathcal{PV}^\bullet_{V,g}\bigr) = \\ \Phi'|_{X'}^*\circ\jmath|_{Y'}^*\bigl(\mathcal{PV}^\bullet_{V,g}\bigr)\end{array} & \Phi'|_{X'}^*\bigl(\mathcal{PV}^\bullet_{V',g'}\bigr), \arrow[l, "\Phi'|_{X'}^*(\mathcal{PV}_\jmath)"']
\end{tikzcd} \tag{5.17}$$

where $\mathcal{TS}_{U,f,\mathbb{C}^n,z_1^2+\cdots+z_n^2}$ is as in (2.8), and γ, δ are defined as follows:

(A) $\gamma : \mathcal{PV}^\bullet_{U,f} \to (\mathrm{id}_X \times 0)^*\bigl(\mathcal{PV}^\bullet_{U,f} \overset{L}{\boxtimes} \mathcal{PV}^\bullet_{\mathbb{C}^n,z_1^2+\cdots+z_n^2}\bigr)$ in $\mathrm{Perv}(X)$ comes from the isomorphism $\mathcal{PV}^\bullet_{\mathbb{C}^n,z_1^2+\cdots+z_n^2} \cong A_{\{0\}}$ in (2.12).

(B) The principal $\mathbb{Z}/2\mathbb{Z}$-bundle $P_\Phi \to X$ comes from $(N_{UV}|_{X^{\mathrm{red}}}, q_{UV}|_{X^{\mathrm{red}}})$, as the bundle of square roots of $\det(q_{UV}|_{X^{\mathrm{red}}})$. Thus, the pullback $\iota|_{X'}^*(P_\Phi) \to X'$ comes from $\bigl(\iota|_{X'^{\mathrm{red}}}^*(N_{UV}), \iota|_{X'^{\mathrm{red}}}^*(q_{UV})\bigr)$. Now Theorem 5.1(ii) defines

$$\hat\beta|_{X'^{\mathrm{red}}} : \langle \mathrm{d}z_1,\ldots,\mathrm{d}z_n\rangle_{X'^{\mathrm{red}}} \xrightarrow{\cong} \iota|_{X'^{\mathrm{red}}}^*(N_{UV}^*)$$

identifying $\sum_{j=1}^n \mathrm{d}z_j^2$ with $\iota|_{X'^{\mathrm{red}}}^*(q_{UV})$. Thus, $\hat\beta|_{X'^{\mathrm{red}}}$ induces a trivialization of

$$\iota|_{X'}^*(P_\Phi) \longrightarrow X'.$$

Then $\delta : \iota|_{X'}^* \circ \Phi|_X^*\bigl(\mathcal{PV}^\bullet_{V,g}\bigr) \otimes_{\mathbb{Z}/2\mathbb{Z}} \iota|_{X'}^*(P_\Phi) \to \iota|_{X'}^* \circ \Phi|_X^*\bigl(\mathcal{PV}^\bullet_{V,g}\bigr)$ in $\mathrm{Perv}(X')$ comes from this trivialization of the principal $\mathbb{Z}/2\mathbb{Z}$-bundle $\iota|_{X'}^*(P_\Phi) \to X'$.

Since Theorem 5.1(i) holds for each $x \in X$, we may choose a family

$$\bigl\{(U_a', V_a', \iota_a, \jmath_a, \Phi_a', \alpha_a, \beta_a, f_a', g_a', X_a', Y_a') : a \in A\bigr\}$$

such that $U_a', V_a', \ldots, Y_a'$ satisfy Theorem 5.1(i) for each $a \in A$, and $\bigl\{\iota_a'|_{X_a'} : X_a' \to X\bigr\}_{a\in A}$ is an étale open cover of X. For each $a \in A$, define an isomorphism

$$\Theta_a : \iota_a|_{X_a'}^*\bigl(\mathcal{PV}^\bullet_{U,f}\bigr) \longrightarrow \iota_a|_{X_a'}^* \circ \Phi|_X^*\bigl(\mathcal{PV}^\bullet_{V,g}\bigr)$$

to make the following diagram of isomorphisms commute:

$$\begin{tikzcd}
\iota_a|_{X_a'}^*\bigl(\mathcal{PV}^\bullet_{U,f}\bigr) \arrow[r, "\iota_a|_{X_a'}^*(\gamma)"'] \arrow[d, "\Theta_a"] & \iota_a|_{X_a'}^* \circ (\mathrm{id}_X \times 0)^*\bigl(\mathcal{PV}^\bullet_{U,f} \overset{L}{\boxtimes} \mathcal{PV}^\bullet_{\mathbb{C}^n,z_1^2+\cdots+z_n^2}\bigr) \arrow[d, "\iota_a|_{X_a'}^* \circ (\mathrm{id}_X \times 0)^*(\mathcal{TS}^{-1}_{U,f,\mathbb{C}^n,z_1^2+\cdots+z_n^2})"'] \\
\begin{array}{l}\iota_a|_{X_a'}^* \circ \Phi|_X^*\bigl(\mathcal{PV}^\bullet_{V,g}\bigr) \\ \otimes_{\mathbb{Z}/2\mathbb{Z}} \iota_a|_{X_a'}^*(P_\Phi)\end{array} \arrow[d, "\delta_a"] & \begin{array}{l}\iota_a|_{X_a'}^* \circ (\mathrm{id}_X \times 0)^*\bigl(\mathcal{PV}^\bullet_{U\times\mathbb{C}^n,f\boxplus z_1^2+\cdots+z_n^2}\bigr) = \\ \Phi_a'|_{X_a'}^* \circ (\alpha_a \times \beta_a)|_{Y_a'}^*\bigl(\mathcal{PV}^\bullet_{U\times\mathbb{C}^n,f\boxplus z_1^2+\cdots+z_n^2}\bigr)\end{array} \arrow[d, "\Phi_a'|_{X_a'}^*(\mathcal{PV}^{-1}_{\alpha_a\times\beta_a})"'] \\
\begin{array}{l}\iota_a|_{X_a'}^* \circ \Phi|_X^*\bigl(\mathcal{PV}^\bullet_{V,g}\bigr) = \\ \Phi_a'|_{X_a'}^* \circ \jmath_a|_{Y_a'}^*\bigl(\mathcal{PV}^\bullet_{V,g}\bigr)\end{array} & \Phi_a'|_{X_a'}^*\bigl(\mathcal{PV}^\bullet_{V_a',g_a'}\bigr), \arrow[l, "\Phi_a'|_{X_a'}^*(\mathcal{PV}_{\jmath_a})"']
\end{tikzcd} \tag{5.18}$$

where γ is as in (A), and δ_a defined as in (B) above.

For $a, b \in A$, define

$$U_{ab}' = U_a' \times_{\iota_a,U,\iota_b} U_b' \qquad \text{and} \qquad V_{ab}' = V_a' \times_{\jmath_a,V,\jmath_b} V_b',$$

with projections $\Pi_{U_a'} : U_{ab}' \to U_a'$, $\Pi_{U_b'} : U_{ab}' \to U_b'$, $\Pi_{V_a'} : V_{ab}' \to V_a'$, $\Pi_{V_b'} : V_{ab}' \to V_b'$. Then U_{ab}', V_{ab}' are smooth and $\Pi_{U_a'}, \Pi_{U_b'}, \Pi_{V_a'}, \Pi_{V_b'}$ étale. The universal property of $V_a' \times_{\jmath_a,V,\jmath_b} V_b'$ gives a unique morphism $\Phi_{ab}' : U_{ab}' \to V_{ab}'$ with

$$\Pi_{V_a'} \circ \Phi_{ab}' = \Phi_a' \circ \Pi_{U_a'} \quad \text{and} \quad \Pi_{V_b'} \circ \Phi_{ab}' = \Phi_b' \circ \Pi_{U_b'}. \tag{5.19}$$

Set $f'_{ab} = f'_a \circ \Pi_{U'_a} : U'_{ab} \to \mathbb{C}$, $g'_{ab} = g'_a \circ \Pi_{V'_a} : V'_{ab} \to \mathbb{C}$ and $X'_{ab} = \mathrm{Crit}(f'_{ab}) \subseteq U'_{ab}$, $Y'_{ab} = \mathrm{Crit}(g'_{ab}) \subseteq V'_{ab}$. As for (4.17) we have

$$\begin{aligned} f'_{ab} &= f'_a \circ \Pi_{U'_a} = f \circ \iota_a \circ \Pi_{U'_a} = f \circ \iota_b \circ \Pi_{U'_b} = f'_b \circ \Pi_{U'_b}, \\ g'_{ab} &= g'_a \circ \Pi_{V'_a} = g \circ \jmath_a \circ \Pi_{V'_a} = g \circ \jmath_b \circ \Pi_{V'_b} = g'_b \circ \Pi_{V'_b} \\ &= (f \boxplus z_1^2 + \cdots + z_n^2) \circ (\alpha_a \times \beta_a) \circ \Pi_{V'_a} \\ &= (f \boxplus z_1^2 + \cdots + z_n^2) \circ (\alpha_b \times \beta_b) \circ \Pi_{V'_b}. \end{aligned}$$

Apply Theorem 3.1 with $V'_{ab}, U \times \mathbb{C}^n, (\alpha_a \times \beta_a) \circ \Pi_{V'_a}, (\alpha_b \times \beta_b) \circ \Pi_{V'_b}, g'_{ab}$, and $f \boxplus z_1^2 + \cdots + z_n^2$ in place of V, W, Φ, Ψ, f, g. The analogue of $\Phi|_X = \Psi|_X$ is

$$\begin{aligned} (\alpha_a \times \beta_a) \circ \Pi_{V'_a}|_{Y'_{ab}} &= ((\Phi|_X^{-1} \circ \Phi|_X \circ \alpha_a)|_{Y'_a} \times 0) \circ \Pi_{V'_a}|_{Y'_{ab}} \\ &= (\Phi|_X^{-1} \circ \jmath_a \circ \Pi_{V'_a}|_{Y'_{ab}}) \times 0 = (\Phi|_X^{-1} \circ \jmath_b \circ \Pi_{V'_b}|_{Y'_{ab}}) \times 0 \\ &= ((\Phi|_X^{-1} \circ \Phi|_X \circ \alpha_b)|_{Y'_b} \times 0) \circ \Pi_{V'_b}|_{Y'_{ab}} = (\alpha_b \times \beta_b) \circ \Pi_{V'_b}|_{Y'_{ab}}, \end{aligned} \tag{5.20}$$

using $\Phi|_X : X \to Y$ an isomorphism and $\beta_a|_{Y'_a} = 0$ in the first step, $\jmath_a|_{Y'_a} = \Phi|_X \circ \alpha_a|_{Y'_a}$ in the second, $\jmath_a \circ \Pi_{V'_a} = \jmath_b \circ \Pi_{V'_b}$ in the third, $\jmath_b|_{Y'_b} = \Phi|_X \circ \alpha_b|_{Y'_b}$ in the fourth, and $\beta_b|_{Y'_b} = 0$ in the fifth. Thus Theorem 3.1 gives

$$\begin{aligned} &\mathcal{PV}_{(\alpha_a \times \beta_a) \circ \Pi_{V'_a}} = \det\bigl[\mathrm{d}\bigl((\alpha_b \times \beta_b) \circ \Pi_{V'_b}\bigr)|_{Y'^{\mathrm{red}}_{ab}}^{-1} \circ \mathrm{d}\bigl((\alpha_a \times \beta_a) \circ \Pi_{V'_a}\bigr)|_{Y'^{\mathrm{red}}_{ab}}\bigr] \cdot \\ &\mathcal{PV}_{(\alpha_b \times \beta_b) \circ \Pi_{V'_b}} : \mathcal{PV}_{V'_{ab}, g'_{ab}} \longrightarrow (\alpha_a \times \beta_a) \circ \Pi_{V'_a}|_{Y'_{ab}}^* \bigl(\mathcal{PV}^\bullet_{U \times \mathbb{C}^n, f \boxplus z_1^2 + \cdots + z_n^2}\bigr) \end{aligned} \tag{5.21}$$

in $\mathrm{Perv}(Y'_{ab})$, where $\det[\cdots]$ maps $Y'^{\mathrm{red}}_{ab} \to \{\pm 1\}$.

Consider the morphisms

$$\begin{aligned} \Pi_{U'_a}|_{X'_{ab}}^*(\delta_a), \Pi_{U'_b}|_{X'_{ab}}^*(\delta_b) : (\Phi \circ \iota_a \circ \Pi_{U'_a})|_{X'_{ab}}^* \bigl(\mathcal{PV}^\bullet_{V,g}\bigr) \otimes_{\mathbb{Z}/2\mathbb{Z}} (\iota_a \circ \Pi_{U'_a})|_{X'_{ab}}^*(P_\Phi) \\ \longrightarrow (\Phi \circ \iota_a \circ \Pi_{U'_a})|_{X'_{ab}}^* \bigl(\mathcal{PV}^\bullet_{V,g}\bigr). \end{aligned} \tag{5.22}$$

As in (B) above, these are defined using two different trivializations of the principal $\mathbb{Z}/2\mathbb{Z}$-bundle $(\iota_a \circ \Pi_{U'_a})|_{X'_{ab}}^*(P_\Phi) \to X'_{ab}$, defined using

$$\Pi_{U'_a}|_{X'^{\mathrm{red}}_{ab}}^*(\hat\beta_a), \Pi_{U'_b}|_{X'^{\mathrm{red}}_{ab}}^*(\hat\beta_b) : \langle \mathrm{d}z_1, \ldots, \mathrm{d}z_n \rangle_{X'^{\mathrm{red}}_{ab}} \longrightarrow (\iota_a \circ \Pi_{U'_a})|_{X'^{\mathrm{red}}_{ab}}^* \bigl(N^*_{UV}\bigr),$$

which are isomorphisms of vector bundles on X'^{red}_{ab} identifying the nondegenerate quadratic forms $\sum_{j=1}^n \mathrm{d}z_j^2$ on $\langle \mathrm{d}z_1, \ldots, \mathrm{d}z_n \rangle_{X'^{\mathrm{red}}_{ab}}$ and $(\iota_a \circ \Pi_{U'_a})|_{X'^{\mathrm{red}}_{ab}}^*(q_{UV})$ on $(\iota_a \circ \Pi_{U'_a})|_{X'^{\mathrm{red}}_{ab}}^*\bigl(N^*_{UV}\bigr)$, for $\hat\beta_a, \hat\beta_b$ as in (5.3). Thus we see that

$$\Pi_{U'_a}|_{X'_{ab}}^*(\delta_a) = \det\bigl[\Pi_{U'_a}|_{X'^{\mathrm{red}}_{ab}}^*(\hat\beta_a) \circ \Pi_{U'_b}|_{X'^{\mathrm{red}}_{ab}}^*(\hat\beta_b)^{-1}\bigr] \cdot \Pi_{U'_b}|_{X'_{ab}}^*(\delta_b), \tag{5.23}$$

where $\det[\cdots]$ maps $X'^{\mathrm{red}}_{ab} \to \{\pm 1\}$ since both isomorphisms in (5.22) identify the same nondegenerate quadratic forms.

We have an exact sequence of vector bundles on X'^{red}_{ab}:

$$0 \longrightarrow TU'_{ab}|_{X'^{\mathrm{red}}_{ab}} \longrightarrow \Phi'_{ab}|_{X'^{\mathrm{red}}_{ab}}^*(TV'_{ab}) \longrightarrow (\iota_a \circ \Pi_{U'_a})|_{X'^{\mathrm{red}}_{ab}}^*(N_{UV}) \longrightarrow 0.$$

Choosing a local splitting of this sequence, we may identify

$$\begin{aligned} &\Phi'_{ab}|_{X'^{\mathrm{red}}_{ab}}^* \bigl[\mathrm{d}\bigl((\alpha_b \times \beta_b) \circ \Pi_{V'_b}\bigr)|_{Y'^{\mathrm{red}}_{ab}}^{-1} \circ \mathrm{d}\bigl((\alpha_a \times \beta_a) \circ \Pi_{V'_a}\bigr)|_{Y'^{\mathrm{red}}_{ab}}\bigr] \\ &\qquad \cong \begin{pmatrix} \mathrm{id}_{TU'_{ab}|_{X'^{\mathrm{red}}_{ab}}} & * \\ 0 & \bigl(\Pi_{U'_a}|_{X'^{\mathrm{red}}_{ab}}^*(\hat\beta_a) \circ \Pi_{U'_b}|_{X'^{\mathrm{red}}_{ab}}^*(\hat\beta_b)^{-1}\bigr)^* \end{pmatrix}. \end{aligned}$$

Therefore

$$\begin{aligned}&\Phi'_{ab}|^*_{X'^{\rm red}_{ab}}\bigl(\det\bigl[{\rm d}\bigl((\alpha_b\times\beta_b)\circ\Pi_{V'_b}\bigr)|^{-1}_{Y'^{\rm red}_{ab}}\circ{\rm d}\bigl((\alpha_a\times\beta_a)\circ\Pi_{V'_a}\bigr)|_{Y'^{\rm red}_{ab}}\bigr]\bigr)\\&\quad=\det\bigl[\Pi_{U'_a}|^*_{X'^{\rm red}_{ab}}(\hat\beta_a)\circ\Pi_{U'_b}|^*_{X'^{\rm red}_{ab}}(\hat\beta_b)^{-1}\bigr]:X'^{\rm red}_{ab}\longrightarrow\{\pm1\}.\end{aligned}\tag{5.24}$$

Now

$$\begin{aligned}&\Pi_{U'_a}|^*_{X'_{ab}}(\Theta_a)=\Pi_{U'_a}|^*_{X'_{ab}}(\delta_a^{-1})\circ(\Phi'_a\circ\Pi_{U'_a})|^*_{X'_{ab}}(\mathcal{PV}_{\jmath_a})\circ(\Phi'_a\circ\Pi_{U'_a})|^*_{X'_{ab}}(\mathcal{PV}^{-1}_{\alpha_a\times\beta_a})\\
&\circ((\mathrm{id}_X\times0)\circ\iota_a\circ\Pi_{U'_a})|^*_{X'_{ab}}(\mathcal{TS}^{-1}_{U,f,\mathbb{C}^n,\Sigma_jz_j^2})\circ(\iota_a\circ\Pi_{U'_a})|^*_{X'_{ab}}(\gamma)\\
&=\Pi_{U'_a}|^*_{X'_{ab}}(\delta_a^{-1})\circ(\Pi_{V'_a}\circ\Phi'_{ab})|^*_{X'_{ab}}(\mathcal{PV}_{\jmath_a})\circ\Phi'_{ab}|^*_{X'_{ab}}(\mathcal{PV}_{\Pi_{V'_a}})\circ\Phi'_{ab}|^*_{X'_{ab}}(\mathcal{PV}^{-1}_{\Pi_{V'_a}})\\
&\circ(\Pi_{V'_a}\circ\Phi'_{ab})|^*_{X'_{ab}}(\mathcal{PV}^{-1}_{\alpha_a\times\beta_a})\circ((\mathrm{id}_X\times0)\circ\iota_a\circ\Pi_{U'_a})|^*_{X'_{ab}}(\mathcal{TS}^{-1}_{U,f,\mathbb{C}^n,\Sigma_jz_j^2})\circ(\iota_a\circ\Pi_{U'_a})|^*_{X'_{ab}}(\gamma)\\
&=\Pi_{U'_a}|^*_{X'_{ab}}(\delta_a^{-1})\circ\Phi'_{ab}|^*_{X'_{ab}}(\mathcal{PV}_{\jmath_a\circ\Pi_{V'_a}})\circ\Phi'_{ab}|^*_{X'_{ab}}(\mathcal{PV}^{-1}_{(\alpha_a\times\beta_a)\circ\Pi_{V'_a}})\\
&\circ((\mathrm{id}_X\times0)\circ\iota_a\circ\Pi_{U'_a})|^*_{X'_{ab}}(\mathcal{TS}^{-1}_{U,f,\mathbb{C}^n,\Sigma_jz_j^2})\circ(\iota_a\circ\Pi_{U'_a})|^*_{X'_{ab}}(\gamma)\\
&=\det\bigl[\Pi_{U'_a}|^*_{X'^{\rm red}_{ab}}(\hat\beta_a)\circ\Pi_{U'_b}|^*_{X'^{\rm red}_{ab}}(\hat\beta_b)^{-1}\bigr]^{-1}\cdot\\
&\Phi'_{ab}|^*_{X'^{\rm red}_{ab}}\bigl(\det\bigl[{\rm d}\bigl((\alpha_b\times\beta_b)\circ\Pi_{V'_b}\bigr)|^{-1}_{Y'^{\rm red}_{ab}}\circ{\rm d}\bigl((\alpha_a\times\beta_a)\circ\Pi_{V'_a}\bigr)|_{Y'^{\rm red}_{ab}}\bigr]\bigr)\cdot\\
&\Pi_{U'_b}|^*_{X'_{ab}}(\delta_b^{-1})\circ\Phi'_{ab}|^*_{X'_{ab}}(\mathcal{PV}_{\jmath_b\circ\Pi_{V'_b}})\circ\Phi'_{ab}|^*_{X'_{ab}}(\mathcal{PV}^{-1}_{(\alpha_b\times\beta_b)\circ\Pi_{V'_b}})\\
&\circ((\mathrm{id}_X\times0)\circ\iota_b\circ\Pi_{U'_b})|^*_{X'_{ab}}(\mathcal{TS}^{-1}_{U,f,\mathbb{C}^n,\Sigma_jz_j^2})\circ(\iota_b\circ\Pi_{U'_b})|^*_{X'_{ab}}(\gamma)\\
&=\Pi_{U'_b}|^*_{X'_{ab}}(\delta_b^{-1})\circ(\Pi_{V'_b}\circ\Phi'_{ab})|^*_{X'_{ab}}(\mathcal{PV}_{\jmath_b})\circ\Phi'_{ab}|^*_{X'_{ab}}(\mathcal{PV}_{\Pi_{V'_b}})\circ\Phi'_{ab}|^*_{X'_{ab}}(\mathcal{PV}^{-1}_{\Pi_{V'_b}})\\
&\circ(\Pi_{V'_b}\circ\Phi'_{ab})|^*_{X'_{ab}}(\mathcal{PV}^{-1}_{\alpha_b\times\beta_b})\circ((\mathrm{id}_X\times0)\circ\iota_b\circ\Pi_{U'_b})|^*_{X'_{ab}}(\mathcal{TS}^{-1}_{U,f,\mathbb{C}^n,\Sigma_jz_j^2})\circ(\iota_b\circ\Pi_{U'_b})|^*_{X'_{ab}}(\gamma)\\
&=\Pi_{U'_b}|^*_{X'_{ab}}(\delta_b^{-1})\circ(\Phi'_b\circ\Pi_{U'_b})|^*_{X'_{ab}}(\mathcal{PV}_{\jmath_b})\circ(\Phi'_b\circ\Pi_{U'_b})|^*_{X'_{ab}}(\mathcal{PV}^{-1}_{\alpha_b\times\beta_b})\\
&\circ((\mathrm{id}_X\times0)\circ\iota_b\circ\Pi_{U'_b})|^*_{X'_{ab}}(\mathcal{TS}^{-1}_{U,f,\mathbb{C}^n,\Sigma_jz_j^2})\circ(\iota_b\circ\Pi_{U'_b})|^*_{X'_{ab}}(\gamma)=\Pi_{U'_b}|^*_{X'_{ab}}(\Theta_b),\end{aligned}\tag{5.25}$$

using (5.18) in the first and seventh steps, (5.19) in the second and sixth, (2.18) in the third, (5.21), (5.23), $\iota_a\circ\Pi_{U'_a}=\iota_b\circ\Pi_{U'_b}$ and $\jmath_a\circ\Pi_{V'_a}=\jmath_b\circ\Pi_{V'_b}$ in the fourth, and (2.18) and (5.24) in the fifth. Therefore Theorem 2.7(i) applied to the étale open cover $\bigl\{\iota_a|_{X'_a}:X'_a\to X\bigr\}_{a\in A}$ of X shows that there is a unique isomorphism Θ_Φ in (5.13) with $\iota_a|^*_{X'_a}(\Theta_\Phi)=\Theta_a$ for all $a\in A$.

Suppose $\bigl\{(U'_a,\ldots,Y'_a):a\in A\bigr\}$ and $\bigl\{(\tilde U'_a,\ldots,\tilde Y'_a):a\in\tilde A\bigr\}$ are alternative choices above, yielding morphisms Θ_Φ and $\tilde\Theta_\Phi$ in (5.13). By running the same construction using the family $\bigl\{(U'_a,\ldots,Y'_a):a\in A\bigr\}\amalg\bigl\{(\tilde U'_a,\ldots,\tilde Y'_a):a\in\tilde A\bigr\}$, we can show that $\Theta_\Phi=\tilde\Theta_\Phi$, so Θ_Φ is independent of the choice of $\bigl\{(U'_a,\ldots,Y'_a):a\in A\bigr\}$ above. Let $U',V',\iota,\jmath,\Phi',\alpha,\beta,X',Y',f',g'$ be as in Theorem 5.1(i). Constructing Θ_Φ using $\bigl\{(U'_a,\ldots,Y'_a):a\in A\bigr\}\amalg\bigl\{(U',\ldots,Y')\bigr\}$, we see from (5.18) that (5.17) commutes. This completes the construction of Θ_Φ.

To see that (5.14)–(5.15) commute, in the situation of (5.17) we show that Verdier duality and monodromy operators commute with each morphism in (5.17). Going clockwise from the top left corner, $\iota|^*_{X'}(\gamma)$ is compatible with Verdier duality and monodromy because of the commutative diagrams

$$\begin{tikzcd}A_{\{0\}} \arrow[r, "\cong"] \arrow[d, "\Gamma"] & \mathbb{D}_{\{0\}}\bigl(A_{\{0\}}\bigr) \\ \mathcal{PV}^\bullet_{\mathbb{C}^n,\Sigma_jz_j^2} \arrow[r, "\sigma_{\mathbb{C}^n,\Sigma_jz_j^2}"] & \mathbb{D}_{\{0\}}\bigl(\mathcal{PV}^\bullet_{\mathbb{C}^n,\Sigma_jz_j^2}\bigr), \arrow[u, "\mathbb{D}(\Gamma)"']\end{tikzcd}\qquad\begin{tikzcd}A_{\{0\}} \arrow[r, "\mathrm{id}"] \arrow[d, "\Gamma"] & A_{\{0\}} \arrow[d, "\Gamma"'] \\ \mathcal{PV}^\bullet_{\mathbb{C}^n,\Sigma_jz_j^2} \arrow[r, "\tau_{\mathbb{C}^n,\Sigma_jz_j^2}"] & \mathcal{PV}^\bullet_{\mathbb{C}^n,\Sigma_jz_j^2},\end{tikzcd}$$

where $\Gamma : A_{\{0\}} \to \mathcal{PV}^\bullet_{\mathbb{C}^n,\Sigma_j z_j^2}$ is the isomorphism used to define γ in (A) above. Equations (2.9)–(2.10) imply that $\iota|_{X'}^* \circ (\mathrm{id}_X \times 0)^*(\mathcal{TS}^{-1}_{U,f,\mathbb{C}^n,\Sigma_j z_j^2})$ is compatible with Verdier duality and monodromy, and (2.16)–(2.17) imply that $\Phi'|_{X'}^*(\mathcal{PV}_\jmath), \Phi'|_{X'}^*(\mathcal{PV}^{-1}_{\alpha\times\beta})$ are. Also δ is compatible with Verdier duality and monodromy, since these do not affect the trivialization of $\iota|_{X'}^*(P_\Phi) \to X'$ used to define δ in (B) above.

Thus by (5.17) we see that $\iota|_{X'}^*(\Theta_\Phi)$ is compatible with Verdier duality and monodromy, that is, $\iota|_{X'}^*$ applied to (5.14)–(5.15) commute. Since we can form an étale open cover of X by such $\iota|_{X'} : X' \to X$, Theorem 2.7(i) implies that (5.14)–(5.15) commute.

Finally, if $U = V$, $f = g$ and $\Phi = \mathrm{id}_U$ then $J_\Phi = \mathrm{id} : K_U^2|_{X^{\mathrm{red}}} \to K_U^2|_{X^{\mathrm{red}}}$ in (5.9), which has a natural square root $\alpha = \mathrm{id} : K_U|_{X^{\mathrm{red}}} \to K_U|_{X^{\mathrm{red}}}$, so $\pi_\Phi : P_\Phi \to X$ is trivial in Definition 5.2. In (5.17) we may put $U' = V' = U$, $\iota = \jmath = \alpha = \mathrm{id}_U$, $n = 0$, $\beta = 0$, $X' = Y' = X$, $f' = g' = f$, and then each morphism in (5.17) is essentially the identity on $\mathcal{PV}^\bullet_{U,f}$, so $\Theta_\Phi = \mathrm{id}_X^*(\Theta_\Phi) = \mathrm{id}_{\mathcal{PV}^\bullet_{U,f}}$. This proves Theorem 5.4(a).

5.2. **Theorem 5.4(b): Θ_Φ depends only on $\Phi|_X : X \to Y$.** Suppose $\Phi, \tilde\Phi : U \to V$ are alternative choices in Theorem 5.4(a) with

$$\Phi|_X = \tilde\Phi|_X : X \longrightarrow Y,$$

so that $P_\Phi = P_{\tilde\Phi}$ by Lemma 5.3. Fix $x \in X$, let $a \neq b$ be labels, and let U'_a, V'_a, ι_a, $\jmath_a$, Φ'_a, α_a, β_a, X'_a, Y'_a, f'_a, g'_a be as in Theorem 5.1(i) for x, Φ and $U'_b, V'_b, \ldots, g'_b$ as in Theorem 5.1(i) for $x, \tilde\Phi$. As in §5.1, define Θ_a, Θ_b and $U'_{ab}, V'_{ab}, \Pi_{U'_a}, \Pi_{U'_b}, \Pi_{V'_a}, \Pi_{V'_b}, \Phi'_{ab}, f'_{ab}, g'_{ab}, X'_{ab}, Y'_{ab}$, and follow the proof in §5.1 from (5.19) as far as (5.25).

This proof does not actually need $U'_a, \ldots, g_a, \Theta_a$ and $U'_b, \ldots, g_b, \Theta_b$ to be defined using the same $\Phi : U \to V$, it only uses in (5.20)–(5.22) that $\Phi|_X : X \to Y$ is the same for $U'_a, \ldots, \Theta_a$ and $U'_b, \ldots, \Theta_b$. Thus we can apply it with $U'_a, \ldots, \Theta_a$ defined using Φ, and $U'_b, \ldots, \Theta_b$ defined using $\tilde\Phi$. Hence

$$\begin{aligned}(\iota_a \circ \Pi_{U'_a})|_{X'_{ab}}^*(\Theta_\Phi) &= \Pi_{U'_a}|_{X'_{ab}}^*(\Theta_a) = \Pi_{U'_b}|_{X'_{ab}}^*(\Theta_b)\\ &= (\iota_b \circ \Pi_{U'_b})|_{X'_{ab}}^*(\Theta_{\tilde\Phi}) = (\iota_a \circ \Pi_{U'_a})|_{X'_{ab}}^*(\Theta_{\tilde\Phi}),\end{aligned}$$

using $\iota_a|_{X'_a}^*(\Theta_\Phi) = \Theta_a$ in the first step, (5.25) in the second, $\iota_b|_{X'_b}^*(\Theta_{\tilde\Phi}) = \Theta_b$ in the third, and $\iota_a \circ \Pi_{U'_a} = \iota_b \circ \Pi_{U'_b}$ in the fourth. As such $\iota_a \circ \Pi_{U'_a}|_{X'_{ab}} : X'_{ab} \to X$ form an étale open cover of X, this implies that $\Theta_\Phi = \Theta_{\tilde\Phi}$ by Theorem 2.7(i).

5.3. **Theorem 5.4(c): composition of the Θ_Φ.** Let U, V, W, f, g, h, X, Y, Z, Φ, Ψ be as in Theorem 5.4(c). Let $x \in X$, and set $y = \Phi(x) \in Y$. Apply Theorem 5.1(i) to $U, V, f, g, X, Y, \Phi, x$ to get $\mathbb{C}$-schemes U', V', a point $x' \in U'$, morphisms $\iota : U' \to U$, $\jmath : V' \to V$, $\Phi' : U' \to V'$, $\alpha : V' \to U$ and $\beta : V' \to \mathbb{C}^m$ where $m = \dim V - \dim U$, and $f' := f \circ \iota : U' \to \mathbb{C}$, $g' := g \circ \jmath : V' \to \mathbb{C}$, $X' := \mathrm{Crit}(f') \subseteq U'$, $Y' := \mathrm{Crit}(g') \subseteq V'$, satisfying conditions including $\iota, \jmath, \alpha \times \beta$ étale, (5.1) commutes, and $\iota(x') = x$.

Similarly, apply Theorem 5.1(i) to $V, W, g, h, Y, Z, \Psi, y$ to get $\mathbb{C}$-schemes $\tilde V, \tilde W$, a point $\tilde y \in \tilde V$, morphisms $\tilde\iota : \tilde V \to V$, $\tilde\jmath : \tilde W \to W$, $\tilde\Psi : \tilde V \to \tilde W$, $\tilde\alpha : \tilde W \to V$ and $\tilde\beta : \tilde W \to \mathbb{C}^n$ where $n = \dim W - \dim V$, and $\tilde g := g \circ \tilde\iota : \tilde V \to \mathbb{C}$, $\tilde h := h \circ \tilde\jmath : \tilde W \to \mathbb{C}$, $\tilde Y := \mathrm{Crit}(\tilde g) \subseteq \tilde V$, $\tilde Z := \mathrm{Crit}(\tilde h) \subseteq \tilde W$, satisfying conditions.

Define $\hat U = U' \times_{\Phi\circ\iota, V, \tilde\iota} \tilde V$ and $\hat W = V' \times_{\jmath, V, \tilde\alpha} \tilde W$, with projections $\Pi_{U'} : \hat U \to U'$, $\Pi_{\tilde V} : \hat U \to \tilde V$, $\Pi_{V'} : \hat W \to V'$, $\Pi_{\tilde W} : \hat W \to \tilde W$. As $x' \in U'$ and $\tilde y \in \tilde V$ with $\Phi \circ \iota(x') = y = \tilde\iota(\tilde y)$, there exists $\hat x \in \hat U$ with $\Pi_{U'}(\hat x) = x'$ and $\Pi_{\tilde V}(\hat x) = \tilde y$. Set $\hat f := f' \circ \Pi_{U'} : \hat U \to \mathbb{C}$ and $\hat h := \tilde h \circ \Pi_{\tilde W} : \hat W \to \mathbb{C}$, and $\hat{X} := \mathrm{Crit}(\hat{f}) \subseteq \hat{U}$, $\hat{Z} := \mathrm{Crit}(\hat{h}) \subseteq \hat{W}$. The morphisms $\Phi' \circ \Pi_{U'} : \hat{U} \to V'$, $\tilde{\Psi} \circ \Pi_{\tilde{V}} : \hat{U} \to \tilde{W}$ satisfy

$$\jmath \circ (\Phi' \circ \Pi_{U'}) = \Phi \circ \iota \circ \Pi_{U'} = \tilde{\iota} \circ \Pi_{\tilde{V}} = \tilde{\alpha} \circ (\tilde{\Psi} \circ \Pi_{\tilde{V}}).$$

Hence there exists a unique morphism $\widehat{\Psi \circ \Phi} : \hat{U} \to \hat{W}$ such that $\Pi_{V'} \circ \widehat{\Psi \circ \Phi} = \Phi' \circ \Pi_{U'}$ and $\Pi_{\tilde{W}} \circ \widehat{\Psi \circ \Phi} = \tilde{\Psi} \circ \Pi_{\tilde{V}}$. Then the following diagram

$$\begin{tikzcd}
U \arrow[d, "\Psi\circ\Phi"] & \hat{U} \arrow[l, "\hat{\imath}=\iota\circ\Pi_{U'}"'] \arrow[rr, "\hat{\imath}=\iota\circ\Pi_{U'}"'] \arrow[d, "\widehat{\Psi\circ\Phi}"'] & & U \arrow[d, "\mathrm{id}_U\times 0\times 0"'] \\
W & \hat{W} \arrow[l, "\hat{\jmath}=\tilde{\jmath}\circ\Pi_{\tilde{W}}"'] \arrow[r, "\hat{\alpha}\times\hat{\beta}=(\alpha\circ\Pi_{V'})\times(\beta\circ\Pi_{V'})\times(\tilde{\beta}\circ\Pi_{\tilde{W}})"] & U\times(\mathbb{C}^m\times\mathbb{C}^n) &
\end{tikzcd}$$

is the analogue of (5.1) for $U, W, f, h, X, Z, \Psi \circ \Phi, x$, and the conclusions of Theorem 5.1(i) hold. Thus (5.17) holds for Θ_Φ using $U', V', X', Y' \iota, \jmath, \Phi', \alpha, \beta, m$, and for Θ_Ψ using $\tilde{V}$, $\tilde{W}$, $\tilde{Y}$, $\tilde{Z}$, $\tilde{\iota}$, $\tilde{\jmath}$, $\tilde{\Psi}$, $\tilde{\alpha}$, $\tilde{\beta}$, n, and for $\Theta_{\Psi\circ\Phi}$ using $\hat{U}, \hat{W}, \hat{X}, \hat{Z}, \hat{\imath}, \hat{\jmath}, \widehat{\Psi \circ \Phi}, \hat{\alpha}, \hat{\beta}, m+n$.

We have a commutative diagram in $\mathrm{Perv}(\hat{X})$:

$$\begin{tikzcd}
\hat{\imath}|^*_{\hat{X}}\circ(\Psi\circ\Phi)|^*_X\bigl(\mathcal{PV}^\bullet_{W,h}\bigr) & & & \hat{\imath}|^*_{\hat{X}}\circ(\mathrm{id}_X\times 0)^*\bigl(\mathcal{PV}^\bullet_{U\times\mathbb{C}^{m+n},f\boxplus\Sigma_j y_j^2+\Sigma_j z_j^2}\bigr) \arrow[lll, "\widehat{\Psi\circ\Phi}|^*_{\hat{X}}(\mathcal{PV}_{\hat{\jmath}}\circ\mathcal{PV}^{-1}_{\hat{\alpha}\times\hat{\beta}})"] \\
\hat{\imath}|^*_{\hat{X}}\circ\Phi|^*_X\circ(\mathrm{id}_Y\times 0)^*\bigl(\mathcal{PV}^\bullet_{V\times\mathbb{C}^n,g\boxplus\Sigma_j z_j^2}\bigr) \arrow[u, "(\Psi'\circ\Pi_{\tilde{V}})|^*_{\hat{X}}(\mathcal{PV}_{\tilde{\jmath}}\circ\mathcal{PV}^{-1}_{\tilde{\alpha}\times\tilde{\beta}})"] & & & \hat{\imath}|^*_{\hat{X}}\circ(\mathrm{id}_X\times 0)^*\bigl(\mathcal{PV}^\bullet_{U,f}\overset{L}{\boxtimes}\mathcal{PV}^\bullet_{\mathbb{C}^{m+n},\Sigma_j y_j^2+\Sigma_j z_j^2}\bigr) \arrow[u, "\hat{\imath}|^*_{\hat{X}}\circ(\mathrm{id}_X\times 0)^*(\mathcal{TS}^{-1}_{U,f,\mathbb{C}^{m+n},\Sigma_j y_j^2+\Sigma_j z_j^2})"'] \\
& \hat{\imath}|^*_{\hat{X}}\circ(\mathrm{id}_X\times 0\times 0)^*\bigl(\mathcal{PV}^\bullet_{U\times\mathbb{C}^m,f\boxplus\Sigma_j y_j^2}\overset{L}{\boxtimes}\mathcal{PV}^\bullet_{\mathbb{C}^n,\Sigma_j z_j^2}\bigr) \arrow[uurr, "\hat{\imath}|^*_{\hat{X}}\circ(\mathrm{id}_X\times 0)^*(\mathcal{TS}^{-1}_{U\times\mathbb{C}^m,f\boxplus\Sigma_j y_j^2,\mathbb{C}^n,\Sigma_j z_j^2})"'] \arrow[dl, "(\Phi'\circ\Pi_{U'})|^*_{\hat{X}}(\mathcal{PV}_\jmath\circ\mathcal{PV}^{-1}_{\alpha\times\beta})\overset{L}{\boxtimes}\mathrm{id}"'] & \hat{\imath}|^*_{\hat{X}}\circ(\mathrm{id}_X\times 0\times 0)^*\bigl(\mathcal{PV}^\bullet_{U,f}\overset{L}{\boxtimes}\mathcal{PV}^\bullet_{\mathbb{C}^m,\Sigma_j y_j^2}\overset{L}{\boxtimes}\mathcal{PV}^\bullet_{\mathbb{C}^n,\Sigma_j z_j^2}\bigr) \arrow[l, "\hat{\imath}|^*_{\hat{X}}\circ(\mathrm{id}_X\times 0)^*(\mathcal{TS}^{-1}_{U,f,\mathbb{C}^m,\Sigma_j y_j^2})\overset{L}{\boxtimes}\mathrm{id}"] \arrow[ur, "\hat{\imath}|^*_{\hat{X}}\circ(\mathrm{id}_X\times 0)^*(\mathrm{id}\overset{L}{\boxtimes}\mathcal{TS}^{-1}_{\mathbb{C}^m,\Sigma_j y_j^2,\mathbb{C}^n,\Sigma_j z_j^2})"'] & \\
\hat{\imath}|^*_{\hat{X}}\circ\Phi|^*_X\circ(\mathrm{id}_Y\times 0)^*\bigl(\mathcal{PV}^\bullet_{V,g}\overset{L}{\boxtimes}\mathcal{PV}^\bullet_{\mathbb{C}^n,\Sigma_j z_j^2}\bigr) \arrow[uu, "\hat{\imath}|^*_{\hat{X}}\circ\Phi|^*_X\circ(\mathrm{id}_Y\times 0)^*(\mathcal{TS}^{-1}_{V,g,\mathbb{C}^n,\Sigma_j z_j^2})"] & & & \hat{\imath}|^*_{\hat{X}}\bigl(\mathcal{PV}^\bullet_{U,f}\bigr) \arrow[uu, "\hat{\imath}|^*_{\hat{X}}(\hat{\gamma})"'] \arrow[ul] \arrow[d, "\hat{\imath}|^*_{\hat{X}}(\gamma)"'] \\
\hat{\imath}|^*_{\hat{X}}\circ\Phi|^*_X\bigl(\mathcal{PV}^\bullet_{V,g}\bigr) \arrow[u, "\hat{\imath}|^*_{\hat{X}}\circ\Phi|^*_X(\tilde{\gamma})"] & \hat{\imath}|^*_{\hat{X}}\circ(\mathrm{id}_X\times 0)^*\bigl(\mathcal{PV}^\bullet_{U\times\mathbb{C}^m,f\boxplus\Sigma_j y_j^2}\bigr) \arrow[l, "(\Phi'\circ\Pi_{U'})|^*_{\hat{X}}(\mathcal{PV}_\jmath\circ\mathcal{PV}^{-1}_{\alpha\times\beta})"'] \arrow[uu] & & \hat{\imath}|^*_{\hat{X}}\circ(\mathrm{id}_X\times 0)^*\bigl(\mathcal{PV}^\bullet_{U,f}\overset{L}{\boxtimes}\mathcal{PV}^\bullet_{\mathbb{C}^m,\Sigma_j y_j^2}\bigr), \arrow[ll, "\hat{\imath}|^*_{\hat{X}}\circ(\mathrm{id}_X\times 0)^*(\mathcal{TS}^{-1}_{U,f,\mathbb{C}^m,\Sigma_j y_j^2})"']
\end{tikzcd} \tag{5.26}$$

where the top right quadrilateral commutes because of associativity in the Thom–Sebastiani Theorem for $\mathcal{PV}^\bullet_{V,f}$, Theorem 2.13.

Also we have

$$
\begin{tikzcd}
(\Psi\circ\Phi\circ\hat\imath)|^*_{\hat X}\bigl(\mathcal{PV}^\bullet_{W,h}\bigr) & & (\Psi\circ\Phi\circ\hat\imath)|^*_{\hat X}\bigl(\mathcal{PV}^\bullet_{W,h}\bigr)\otimes_{\mathbb{Z}/2\mathbb{Z}}\hat\imath|^*_{\hat X}(P_{\Psi\circ\Phi}) \arrow[ll, "\hat\delta"'] \arrow[d, "\mathrm{id}\otimes\hat\imath|^*_{\hat X}(\Xi_{\Psi,\Phi})"] \\
\begin{array}{l}(\Psi\circ\Phi\circ\hat\imath)|^*_{\hat X}\bigl(\mathcal{PV}^\bullet_{W,h}\bigr)\\ \otimes_{\mathbb{Z}/2\mathbb{Z}}(\tilde\imath\circ\Pi_{\tilde V})|^*_{\hat X}(P_\Psi)\end{array} \arrow[u, "\Pi_{\tilde V}|^*_{\hat X}(\tilde\delta)"] & & \begin{array}{l}(\Psi\circ\Phi\circ\hat\imath)|^*_{\hat X}\bigl(\mathcal{PV}^\bullet_{W,h}\bigr)\\ \otimes_{\mathbb{Z}/2\mathbb{Z}}(\tilde\imath\circ\Pi_{\tilde V})|^*_{\hat X}(P_\Psi)\otimes_{\mathbb{Z}/2\mathbb{Z}}\hat\imath|^*_{\hat X}(P_\Phi)\end{array} \arrow[d, "(\Phi\circ\hat\imath)|^*_{\hat X}(\Theta_\Psi^{-1})\otimes\mathrm{id}"] \\
(\Phi\circ\hat\imath)|^*_{\hat X}\bigl(\mathcal{PV}^\bullet_{V,g}\bigr) \arrow[u, "(\Phi\circ\hat\imath)|^*_{\hat X}(\Theta_\Psi)"] & & (\Phi\circ\hat\imath)|^*_{\hat X}\bigl(\mathcal{PV}^\bullet_{V,g}\bigr)\otimes_{\mathbb{Z}/2\mathbb{Z}}\hat\imath|^*_{\hat X}(P_\Phi), \arrow[ll, "\Pi_{U'}|^*_{\hat X}(\delta)"']
\end{tikzcd}
\tag{5.27}
$$

which commutes because the trivializations of $\iota|_{X'}(P_\Phi),\tilde\imath|_{\tilde X}(P_\Psi),\hat\imath|_{\hat X}(P_{\Psi\circ\Phi})$ used to define $\delta,\tilde\delta,\hat\delta$ are compatible with $\Xi_{\Psi,\Phi}$.

Combining (5.26) and (5.27) with $\Pi_{U'}|^*_{\hat X}$ applied to (5.17) for Θ_Φ, and $\Pi_{\tilde V}|^*_{\hat X}$ applied to (5.17) for Θ_Ψ, and (5.17) for $\Theta_{\Psi\circ\Phi}$, we can show that the following diagram commutes in $\mathrm{Perv}(\hat X)$:

$$
\begin{tikzcd}
\hat\imath|^*_{\hat X}\bigl(\mathcal{PV}^\bullet_{U,f}\bigr) \arrow[r, "\hat\imath|^*_{\hat X}(\Theta_{\Psi\circ\Phi})"] \arrow[d, "\hat\imath|^*_{\hat X}(\Theta_\Phi)"] & (\Psi\circ\Phi\circ\hat\imath)|^*_{\hat X}\bigl(\mathcal{PV}^\bullet_{W,h}\bigr)\otimes_{\mathbb{Z}/2\mathbb{Z}}\hat\imath|^*_{\hat X}(P_{\Psi\circ\Phi}) \arrow[d, "\mathrm{id}\otimes\hat\imath|^*_{\hat X}(\Xi_{\Psi,\Phi})"] \\
(\Phi\circ\hat\imath)|^*_{\hat X}\bigl(\mathcal{PV}^\bullet_{V,g}\bigr)\otimes_{\mathbb{Z}/2\mathbb{Z}}\hat\imath|^*_{\hat X}(P_\Phi) \arrow[r, "(\Phi\circ\hat\imath)|^*_{\hat X}(\Theta_\Psi)\otimes\mathrm{id}"] & \begin{array}{c}(\Psi\circ\Phi\circ\hat\imath)|^*_{\hat X}\bigl(\mathcal{PV}^\bullet_{W,h}\bigr)\\ \otimes_{\mathbb{Z}/2\mathbb{Z}}(\tilde\imath\circ\Pi_{\tilde V})|^*_{\hat X}(P_\Psi)\otimes_{\mathbb{Z}/2\mathbb{Z}}\hat\imath|^*_{\hat X}(P_\Phi),\end{array}
\end{tikzcd}
$$

which is $\hat\imath|^*_{\hat X}$ applied to (5.16). Since such $\hat\imath|_{\hat X}:\hat X\to X$ form an étale cover of X, equation (5.16) commutes by Theorem 2.7(i). This proves Theorem 5.4(c).

5.4. **$\mathscr{D}$-modules and mixed Hodge modules.** By Theorem 5.1(iv),(v), the earlier parts of that result hold for our other contexts in §2.6–§2.10. Once again, the proofs of Theorem 5.4(a)–(c) then carry over to the other contexts using the general framework of §2.5, now also making use of property (vii).

6. Perverse sheaves on oriented d-critical loci

6.1. **Background material on d-critical loci.** Here are some of the main definitions and results on d-critical loci, from Joyce [23, Th.s 2.1, 2.20, 2.28 & Def.s 2.5, 2.18, 2.31]. For the algebraic case we work with $\mathbb{C}$-schemes.

Theorem 6.1. *Let X be a $\mathbb{C}$-scheme. Then there exists a sheaf $\mathcal{S}_X$ of $\mathbb{C}$-vector spaces on X, unique up to canonical isomorphism, which is uniquely characterized by the following two properties:*

(i) *Suppose $R\subseteq X$ is Zariski open, U is a smooth $\mathbb{C}$-scheme, and $i:R\hookrightarrow U$ is a closed embedding. Then we have an exact sequence of sheaves of $\mathbb{C}$-vector spaces on R:*

$$0\longrightarrow I_{R,U}\longrightarrow i^{-1}(\mathcal{O}_U)\xrightarrow{\;i^\sharp\;}\mathcal{O}_X|_R\longrightarrow 0,$$

where $\mathcal{O}_X,\mathcal{O}_U$ are the sheaves of regular functions on X,U, and $i^\sharp$ is the morphism of sheaves of $\mathbb{C}$-algebras on R induced by i.

There is an exact sequence of sheaves of $\mathbb{C}$-vector spaces on R:

$$0\longrightarrow\mathcal{S}_X|_R\xrightarrow{\;\iota_{R,U}\;}\frac{i^{-1}(\mathcal{O}_U)}{I_{R,U}^2}\xrightarrow{\;\mathrm{d}\;}\frac{i^{-1}(T^*U)}{I_{R,U}\cdot i^{-1}(T^*U)},$$

where d *maps* $f + I_{R,U}^2 \mapsto \mathrm{d}f + I_{R,U} \cdot i^{-1}(T^*U)$.

(ii) *Let* $R \subseteq S \subseteq X$ *be Zariski open,* U, V *be smooth* $\mathbb{C}$*-schemes,* $i : R \hookrightarrow U$, $j : S \hookrightarrow V$ *closed embeddings, and* $\Phi : U \to V$ *a morphism with* $\Phi \circ i = j|_R : R \to V$. *Then the following diagram of sheaves on* R *commutes:*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{S}_X|_R \arrow[r, "\iota_{S,V}|_R"] \arrow[d, "\mathrm{id}"] & \frac{j^{-1}(\mathcal{O}_V)}{I_{S,V}^2}\Big|_R \arrow[r, "\mathrm{d}"] \arrow[d, "i^{-1}(\Phi^\sharp)"] & \frac{j^{-1}(T^*V)}{I_{S,V} \cdot j^{-1}(T^*V)}\Big|_R \arrow[d, "i^{-1}(\mathrm{d}\Phi)"] \\ 0 \arrow[r] & \mathcal{S}_X|_R \arrow[r, "\iota_{R,U}"] & \frac{i^{-1}(\mathcal{O}_U)}{I_{R,U}^2} \arrow[r, "\mathrm{d}"] & \frac{i^{-1}(T^*U)}{I_{R,U} \cdot i^{-1}(T^*U)}. \end{tikzcd} \tag{6.1}$$

Here $\Phi : U \to V$ *induces* $\Phi^\sharp : \Phi^{-1}(\mathcal{O}_V) \to \mathcal{O}_U$ *on* U*, so we have*

$$i^{-1}(\Phi^\sharp) : j^{-1}(\mathcal{O}_V)|_R = i^{-1} \circ \Phi^{-1}(\mathcal{O}_V) \longrightarrow i^{-1}(\mathcal{O}_U), \tag{6.2}$$

a morphism of sheaves of $\mathbb{C}$*-algebras on* R.

As $\Phi \circ i = j|_R$, *equation* (6.2) *maps* $I_{S,V}|_R \to I_{R,U}$, *and so maps* $I_{S,V}^2|_R \to I_{R,U}^2$. *Thus* (6.2) *induces the morphism in the second column of* (6.1).

Similarly, $\mathrm{d}\Phi : \Phi^{-1}(T^*V) \to T^*U$ *induces the third column of* (6.1).

There is a natural decomposition $\mathcal{S}_X = \mathcal{S}_X^0 \oplus \mathbb{C}_X$, *where* $\mathbb{C}_X$ *is the constant sheaf on* X *with fibre* $\mathbb{C}$, *and* $\mathcal{S}_X^0 \subset \mathcal{S}_X$ *is the kernel of the composition*

$$\mathcal{S}_X \xrightarrow{\beta_X} \mathcal{O}_X \xrightarrow{i_X^\sharp} \mathcal{O}_{X^{\mathrm{red}}},$$

with X^{red} *the reduced* $\mathbb{C}$*-subscheme of* X, *and* $i_X : X^{\mathrm{red}} \hookrightarrow X$ *the inclusion.*

The analogue of all the above also holds for complex analytic spaces.

Definition 6.2. An *algebraic d-critical locus* over $\mathbb{C}$ is a pair (X, s), where X is a $\mathbb{C}$-scheme, and $s \in H^0(\mathcal{S}_X^0)$ for $\mathcal{S}_X^0$ as in Theorem 6.1, satisfying the condition that for each $x \in X$, there exists a Zariski open neighbourhood R of x in X, a smooth $\mathbb{C}$-scheme U, a regular function $f : U \to \mathbb{A}^1 = \mathbb{C}$, and a closed embedding $i : R \hookrightarrow U$, such that $i(R) = \mathrm{Crit}(f)$ as $\mathbb{C}$-subschemes of U, and $\iota_{R,U}(s|_R) = i^{-1}(f) + I_{R,U}^2$.

Similarly, a *complex analytic d-critical locus* is a pair (X, s), where X is a complex analytic space, and $s \in H^0(\mathcal{S}_X^0)$ for $\mathcal{S}_X$ as in Theorem 6.1, such that each $x \in X$ has an open neighbourhood $R \subset X$ with a closed embedding $i : R \hookrightarrow U$ into a complex manifold U and a holomorphic function $f : U \to \mathbb{C}$, such that $i(R) = \mathrm{Crit}(f)$, and $\iota_{R,U}(s|_R) = i^{-1}(f) + I_{R,U}^2$.

In both cases we call the quadruple (R, U, f, i) a *critical chart* on (X, s).

Let (X, s) be a d-critical locus (either algebraic or complex analytic), and (R, U, f, i) be a critical chart on (X, s). Let $U' \subseteq U$ be (Zariski) open, and set

$$R' = i^{-1}(U') \subseteq R, \qquad i' = i|_{R'} : R' \hookrightarrow U', \qquad \text{and} \qquad f' = f|_{U'}.$$

Then (R', U', f', i') is also a critical chart on (X, s), and we call it a *subchart* of (R, U, f, i). As a shorthand we write $(R', U', f', i') \subseteq (R, U, f, i)$.

Let $(R, U, f, i), (S, V, g, j)$ be critical charts on (X, s), with $R \subseteq S \subseteq X$. An *embedding* of (R, U, f, i) in (S, V, g, j) is a locally closed embedding $\Phi : U \hookrightarrow V$ such that $\Phi \circ i = j|_R$ and $f = g \circ \Phi$. As a shorthand we write $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$.

If $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$ and $\Psi : (S, V, g, j) \hookrightarrow (T, W, h, k)$ are embeddings, then $\Psi \circ \Phi : (R, U, f, i) \hookrightarrow (T, W, h, k)$ is also an embedding.

Theorem 6.3. *Let (X,s) be a d-critical locus (either algebraic or complex analytic), and let $(R,U,f,i),(S,V,g,j)$ be critical charts on (X,s).*

Then, for each $x\in R\cap S\subseteq X$, there exist subcharts $(R',U',f',i')\subseteq(R,U,f,i)$, and $(S',V',g',j')\subseteq(S,V,g,j)$ with $x\in R'\cap S'\subseteq X$, a critical chart (T,W,h,k) on (X,s), and embeddings $\Phi:(R',U',f',i')\hookrightarrow(T,W,h,k)$, and $\Psi:(S',V',g',j')\hookrightarrow(T,W,h,k)$.

Theorem 6.4. *Let (X,s) be a d-critical locus (either algebraic or complex analytic), and $X^{\mathrm{red}}\subseteq X$ the associated reduced $\mathbb{C}$-scheme or reduced complex analytic space. Then there exists an (algebraic or holomorphic) line bundle $K_{X,s}$ on X^{red} which we call the **canonical bundle** of (X,s), which is natural up to canonical isomorphism, and is characterized by the following properties:*

(i) *If (R,U,f,i) is a critical chart on (X,s), there is a natural isomorphism*

$$\iota_{R,U,f,i}:K_{X,s}|_{R^{\mathrm{red}}}\longrightarrow i^*\big(K_U^{\otimes^2}\big)|_{R^{\mathrm{red}}},\tag{6.3}$$

where $K_U=\Lambda^{\dim U}T^*U$ is the canonical bundle of U in the usual sense.

(ii) *Let $\Phi:(R,U,f,i)\hookrightarrow(S,V,g,j)$ be an embedding of critical charts on (X,s). Then (5.9) defines an isomorphism of line bundles on $\mathrm{Crit}(f)^{\mathrm{red}}$:*

$$J_\Phi:K_U^{\otimes^2}|_{\mathrm{Crit}(f)^{\mathrm{red}}}\xrightarrow{\cong}\Phi|^*_{\mathrm{Crit}(f)^{\mathrm{red}}}\big(K_V^{\otimes^2}\big).$$

Since $i:R\to\mathrm{Crit}(f)$ is an isomorphism with $\Phi\circ i=j|_R$, this gives

$$i|^*_{R^{\mathrm{red}}}(J_\Phi):i|^*_{R^{\mathrm{red}}}\big(K_U^{\otimes^2}\big)\xrightarrow{\cong}j|^*_{R^{\mathrm{red}}}\big(K_V^{\otimes^2}\big),$$

and we must have

$$\iota_{S,V,g,j}|_{R^{\mathrm{red}}}=i|^*_{R^{\mathrm{red}}}(J_\Phi)\circ\iota_{R,U,f,i}:K_{X,s}|_{R^{\mathrm{red}}}\longrightarrow j^*\big(K_V^{\otimes^2}\big)\big|_{R^{\mathrm{red}}}.\tag{6.4}$$

Definition 6.5. Let (X,s) be a d-critical locus (either algebraic or complex analytic), and $K_{X,s}$ its canonical bundle from Theorem 6.4. An *orientation* on (X,s) is a choice of square root line bundle $K_{X,s}^{1/2}$ for $K_{X,s}$ on X^{red}. That is, an orientation is an (algebraic or holomorphic) line bundle L on X^{red}, together with an isomorphism $L^{\otimes^2}=L\otimes L\cong K_{X,s}$. A d-critical locus with an orientation will be called an *oriented d-critical locus*.

In [9, Th. 6.6] we show that algebraic d-critical loci are classical truncations of objects in derived algebraic geometry known as -1-*shifted symplectic derived schemes*, introduced by Pantev, Toën, Vaquié and Vezzosi [42].

Theorem 6.6 (Bussi, Brav and Joyce [9])**.** *Suppose $(\boldsymbol{X},\omega)$ is a -1-shifted symplectic derived scheme in the sense of Pantev et al. [42] over $\mathbb{C}$, and let $X=t_0(\boldsymbol{X})$ be the associated classical $\mathbb{C}$-scheme of $\boldsymbol{X}$. Then X extends naturally to an algebraic d-critical locus (X,s). The canonical bundle $K_{X,s}$ from Theorem 6.4 is naturally isomorphic to the determinant line bundle $\det(\mathbb{L}_{\boldsymbol{X}})|_{X^{\mathrm{red}}}$ of the cotangent complex $\mathbb{L}_{\boldsymbol{X}}$ of $\boldsymbol{X}$.*

Now Pantev et al. [42] show that derived moduli schemes of coherent sheaves, or complexes of coherent sheaves, on a Calabi–Yau 3-fold Y have -1-shifted symplectic structures. Using this, in [9, Cor. 6.7] we deduce:

Corollary 6.7. *Suppose Y is a Calabi–Yau 3-fold over $\mathbb{C}$, and $\mathcal{M}$ is a classical moduli $\mathbb{C}$-scheme of simple coherent sheaves in $\mathrm{coh}(Y)$, or simple complexes of coherent sheaves in $D^b\,\mathrm{coh}(Y)$, with (symmetric) obstruction theory $\phi:\mathcal{E}^\bullet\to\mathbb{L}_{\mathcal{M}}$ as in Behrend [2], Thomas [52], or Huybrechts and Thomas [22]. Then $\mathcal{M}$ extends naturally to an algebraic d-critical locus $(\mathcal{M},s)$. The canonical bundle $K_{\mathcal{M},s}$ from Theorem 6.4 is naturally isomorphic to $\det(\mathcal{E}^\bullet)|_{\mathcal{M}^{\mathrm{red}}}$.*

Here we call $F \in \mathrm{coh}(Y)$ *simple* if $\mathrm{Hom}(F,F) = \mathbb{C}$, and we call $F^\bullet \in D^b\mathrm{coh}(Y)$ *simple* if $\mathrm{Hom}(F^\bullet,F^\bullet) = \mathbb{C}$ and $\mathrm{Ext}^{<0}(F^\bullet,F^\bullet) = 0$. Thus, d-critical loci will have applications in Donaldson–Thomas theory for Calabi–Yau 3-folds [24, 32, 33, 52]. Orientations on $(\mathcal{M}, s)$ are closely related to *orientation data* in the work of Kontsevich and Soibelman [32, 33].

Pantev et al. [42] also show that derived intersections $L \cap M$ of algebraic Lagrangians L, M in an algebraic symplectic manifold (S,ω) have -1-shifted symplectic structures, so that Theorem 6.6 gives them the structure of algebraic d-critical loci. Bussi [10, §3] will prove a complex analytic version of this:

Theorem 6.8 (Bussi [10])**.** *Suppose (S,ω) is a complex symplectic manifold, and L, M are complex Lagrangian submanifolds in S. Then the intersection $X = L\cap M$, as a complex analytic subspace of S, extends naturally to a complex analytic d-critical locus (X,s). The canonical bundle $K_{X,s}$ from Theorem 6.4 is naturally isomorphic to $K_L|_{X^{\mathrm{red}}} \otimes K_M|_{X^{\mathrm{red}}}$.*

6.2. **The main result, and applications.** Here is our main result, which will be proved in §6.3–§6.4.

Theorem 6.9. *Let (X,s) be an oriented algebraic d-critical locus over $\mathbb{C}$, with orientation $K_{X,s}^{1/2}$. Then for any well-behaved base ring A, such as $\mathbb{Z},\mathbb{Q}$ or $\mathbb{C}$, there exists a perverse sheaf $P_{X,s}^\bullet$ in $\mathrm{Perv}(X)$ over A, which is natural up to canonical isomorphism, and Verdier duality and monodromy isomorphisms*

$$\Sigma_{X,s} : P_{X,s}^\bullet \longrightarrow \mathbb{D}_X\bigl(P_{X,s}^\bullet\bigr), \qquad \mathrm{T}_{X,s} : P_{X,s}^\bullet \longrightarrow P_{X,s}^\bullet, \tag{6.5}$$

which are characterized by the following properties:

(i) *If (R,U,f,i) is a critical chart on (X,s), there is a natural isomorphism*

$$\omega_{R,U,f,i} : P_{X,s}^\bullet|_R \longrightarrow i^*\bigl(\mathcal{PV}_{U,f}^\bullet\bigr) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R,U,f,i}, \tag{6.6}$$

where $\pi_{R,U,f,i} : Q_{R,U,f,i} \to R$ is the principal $\mathbb{Z}/2\mathbb{Z}$-bundle parametrizing local isomorphisms $\alpha : K_{X,s}^{1/2} \to i^*(K_U)|_{R^{\mathrm{red}}}$ with $\alpha\otimes\alpha = \iota_{R,U,f,i}$, for $\iota_{R,U,f,i}$ as in (6.3). Furthermore the following commute in $\mathrm{Perv}(R)$:

$$\begin{tikzcd}
P_{X,s}^\bullet|_R \arrow[rr, "\omega_{R,U,f,i}"'] \arrow[d, "\Sigma_{X,s}|_R"] && i^*\bigl(\mathcal{PV}_{U,f}^\bullet\bigr) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R,U,f,i} \arrow[d, "i^*(\sigma_{U,f})\otimes \mathrm{id}_{Q_{R,U,f,i}}"'] \\
\mathbb{D}_R\bigl(P_{X,s}^\bullet|_R\bigr) && \begin{array}{c} i^*\bigl(\mathbb{D}_{\mathrm{Crit}(f)}(\mathcal{PV}_{U,f}^\bullet)\bigr) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R,U,f,i} \\ \cong \mathbb{D}_R\bigl(i^*(\mathcal{PV}_{U,f}^\bullet) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R,U,f,i}\bigr), \end{array} \arrow[ll, "\mathbb{D}_R(\omega_{R,U,f,i})"']
\end{tikzcd} \tag{6.7}$$

$$\begin{tikzcd}
P_{X,s}^\bullet|_R \arrow[rr, "\omega_{R,U,f,i}"'] \arrow[d, "\mathrm{T}_{X,s}|_R"] && i^*\bigl(\mathcal{PV}_{U,f}^\bullet\bigr) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R,U,f,i} \arrow[d, "i^*(\tau_{U,f})\otimes \mathrm{id}_{Q_{R,U,f,i}}"'] \\
P_{X,s}^\bullet|_R \arrow[rr, "\omega_{R,U,f,i}"] && i^*\bigl(\mathcal{PV}_{U,f}^\bullet\bigr) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R,U,f,i}.
\end{tikzcd} \tag{6.8}$$

(ii) *Let $\Phi : (R,U,f,i) \hookrightarrow (S,V,g,j)$ be an embedding of critical charts on (X,s). Then there is a natural isomorphism of principal $\mathbb{Z}/2\mathbb{Z}$-bundles*

$$\Lambda_\Phi : Q_{S,V,g,j}|_R \xrightarrow{\cong} i^*(P_\Phi) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R,U,f,i} \tag{6.9}$$

on R, for P_Φ as in Definition 5.2, defined as follows: local isomorphisms

$$\alpha : K_{X,s}^{1/2}|_{R^{\mathrm{red}}} \longrightarrow i^*(K_U)|_{R^{\mathrm{red}}}, \quad \beta : K_{X,s}^{1/2}|_{R^{\mathrm{red}}} \longrightarrow j^*(K_V)|_{R^{\mathrm{red}}},$$
$$\textit{and} \qquad \gamma : i^*(K_U)|_{R^{\mathrm{red}}} \longrightarrow j^*(K_V)|_{R^{\mathrm{red}}}$$

with $\alpha\otimes\alpha=\iota_{R,U,f,i}$, $\beta\otimes\beta=\iota_{S,V,g,j}|_{R^{\mathrm{red}}}$, $\gamma\otimes\gamma=i|_{R^{\mathrm{red}}}^*(J_\Phi)$ *correspond to local sections* $s_\alpha:R\to Q_{R,U,f,i}$, $s_\beta:R\to Q_{S,V,g,j}|_R$, $s_\gamma:R\to i^*(P_\Phi)$*. Equation* (6.4) *shows that* $\beta=\gamma\circ\alpha$ *is a possible solution for* β*, and we define* Λ_Φ *in* (6.9) *such that* $\Lambda_\Phi(s_\beta)=s_\gamma\otimes_{\mathbb{Z}/2\mathbb{Z}}s_\alpha$ *if and only if* $\beta=\gamma\circ\alpha$*.*

Then the following diagram commutes in $\mathrm{Perv}(R)$*, for* Θ_Φ *as in* (5.13)*:*

$$\begin{tikzcd}
P^\bullet_{X,s}|_R \arrow[rr, "\omega_{R,U,f,i}"'] \arrow[d, "\omega_{S,V,g,j}|_R"] & & i^*\bigl(\mathcal{PV}^\bullet_{U,f}\bigr)\otimes_{\mathbb{Z}/2\mathbb{Z}}Q_{R,U,f,i} \arrow[d, "i^*(\Theta_\Phi)\otimes\mathrm{id}_{Q_{R,U,f,i}}"'] \\
j^*\bigl(\mathcal{PV}^\bullet_{V,g}\bigr)|_R \otimes_{\mathbb{Z}/2\mathbb{Z}}Q_{S,V,g,j}|_R \arrow[rr, "\mathrm{id}_{j^*(\mathcal{PV}^\bullet_{V,g})}\otimes\Lambda_\Phi"] & & i^*\bigl(\Phi^*(\mathcal{PV}^\bullet_{V,g})\otimes_{\mathbb{Z}/2\mathbb{Z}}P_\Phi\bigr) \otimes_{\mathbb{Z}/2\mathbb{Z}}Q_{R,U,f,i}.
\end{tikzcd} \tag{6.10}$$

The analogues of all the above also hold for $\mathscr{D}$*-modules on oriented algebraic d-critical loci over* $\mathbb{C}$*, for perverse sheaves and* $\mathscr{D}$*-modules on oriented complex analytic d-critical loci, and for mixed Hodge modules on oriented algebraic d-critical loci over* $\mathbb{C}$ *and oriented complex analytic d-critical loci, as in* §2.6–§2.10*.*

Remark 6.10. This sheaf-theoretic result is compatible with the motivic result of Bussi, Joyce and Meinhardt in [11]. Given (X,s) an oriented algebraic d-critical locus over $\mathbb{C}$, [11] proves the existence of a natural motivic element $MF_{X,s}\in\overline{\mathcal{M}}^{\hat\mu}_X$ in a version of the relative Grothendieck ring of varieties over X, equivariant with respect to suitable actions of the group $\hat\mu$ of all roots of unity (for detailed definitions, see [11]). Since the mixed Hodge module realization factorizes over the additional relation one has to impose in [11] on the Grothendieck group, the ring $\mathcal{M}^{\hat\mu}_X$ has a map to $K_0(\mathrm{MHM}(X;T_s))$, the K-group of algebraic mixed Hodge modules on X with a finite order automorphism (note that the Grothendieck group only sees the semisimple part T_s of the monodromy and not the nilpotent part N). By a Čech-type argument using the corresponding comparison result of [20, Prop. 3.17], the image of $MF_{X,s}$ in $K_0(\mathrm{MHM}(X;T_s))$ agrees with the image of the mixed Hodge module realization of $P^\bullet_{X,s}$, since both sides are Zariski locally modelled by the same vanishing cycles. Thus, for example, they give the same weight polynomial for global cohomology with compact support.

From Theorem 6.6, Corollary 6.7 and Theorem 6.8 we deduce:

Corollary 6.11. *Let* $(\boldsymbol{X},\omega)$ *be a* -1*-shifted symplectic derived scheme over* $\mathbb{C}$ *in the sense of Pantev et al.* [42]*, and* $X=t_0(\boldsymbol{X})$ *the associated classical* $\mathbb{C}$*-scheme. Suppose we are given a square root* $\det(\mathbb{L}_{\boldsymbol{X}})|_X^{1/2}$ *for* $\det(\mathbb{L}_{\boldsymbol{X}})|_X$*. Then we may define* $P^\bullet_{\boldsymbol{X},\omega}\in\mathrm{Perv}(X)$*, uniquely up to canonical isomorphism, and isomorphisms* $\Sigma_{\boldsymbol{X},\omega}:P^\bullet_{\boldsymbol{X},\omega}\to\mathbb{D}_X(P^\bullet_{\boldsymbol{X},\omega})$*,* $\mathrm{T}_{\boldsymbol{X},\omega}:P^\bullet_{\boldsymbol{X},\omega}\to P^\bullet_{\boldsymbol{X},\omega}$*.*

The same applies for $\mathscr{D}$*-modules and mixed Hodge modules on* X*.*

Corollary 6.12. *Let* Y *be a Calabi–Yau* 3*-fold over* $\mathbb{C}$*, and* $\mathcal{M}$ *a classical moduli* $\mathbb{C}$*-scheme of simple coherent sheaves in* $\mathrm{coh}(Y)$*, or simple complexes of coherent sheaves in* $D^b\,\mathrm{coh}(Y)$*, with natural (symmetric) obstruction theory* $\phi:\mathcal{E}^\bullet\to\mathbb{L}_{\mathcal{M}}$ *as in Behrend* [2]*, Thomas* [52]*, or Huybrechts and Thomas* [22]*. Suppose we are given a square root* $\det(\mathcal{E}^\bullet)^{1/2}$ *for* $\det(\mathcal{E}^\bullet)$*. Then we may define* $P^\bullet_{\mathcal{M}}\in\mathrm{Perv}(\mathcal{M})$*, uniquely up to canonical isomorphism, and isomorphisms* $\Sigma_{\mathcal{M}}:P^\bullet_{\mathcal{M}}\to\mathbb{D}_{\mathcal{M}}(P^\bullet_{\mathcal{M}})$*,* $\mathrm{T}_{\mathcal{M}}:P^\bullet_{\mathcal{M}}\to P^\bullet_{\mathcal{M}}$*.*

The same applies for $\mathscr{D}$*-modules and mixed Hodge modules on* $\mathcal{M}$*.*

Corollary 6.13. *Let* (S,ω) *be a complex symplectic manifold and* L,M *complex Lagrangian submanifolds in* S*, and write* $X=L\cap M$*, as a complex analytic subspace of* S*. Suppose we are given square roots* $K_L^{1/2},K_M^{1/2}$ *for* K_L,K_M*. Then we may define* $P^\bullet_{L,M}\in\mathrm{Perv}(X)$*,* *uniquely up to canonical isomorphism, and isomorphisms* $\Sigma_{L,M} : P^\bullet_{L,M} \to \mathbb{D}_X(P^\bullet_{L,M})$, *and* $\mathrm{T}_{L,M} : P^\bullet_{L,M} \to P^\bullet_{L,M}$.

The same applies for $\mathscr{D}$*-modules and mixed Hodge modules on* X.

The next two remarks discuss applications of Corollaries 6.12 and 6.13 to Donaldson–Thomas theory, and to Lagrangian Floer cohomology.

Remark 6.14. If Y is a Calabi–Yau 3-fold over $\mathbb{C}$ and τ a suitable stability condition on coherent sheaves on Y, the *Donaldson–Thomas invariants* $DT^\alpha(\tau)$ are integers which 'count' the moduli schemes $\mathcal{M}^\alpha_{\mathrm{st}}(\tau)$ of τ-stable coherent sheaves on Y with Chern character $\alpha \in H^{\mathrm{even}}(Y;\mathbb{Q})$, provided there are no strictly τ-semistable sheaves in class α on Y. They were defined by Thomas [52], who showed they are unchanged under deformations of Y, following a suggestion of Donaldson and Thomas [16].

Behrend [2] showed that $DT^\alpha(\tau)$ may be written as a weighted Euler characteristic $\chi(\mathcal{M}^\alpha_{\mathrm{st}}(\tau),\nu)$, where $\nu : \mathcal{M}^\alpha_{\mathrm{st}}(\tau) \to \mathbb{Z}$ is a certain constructible function called the *Behrend function.* Joyce and Song [24] extended the definition of $DT^\alpha(\tau)$ to classes α including τ-semistable sheaves (with $DT^\alpha(\tau) \in \mathbb{Q}$), and proved a wall-crossing formula for $DT^\alpha(\tau)$ under change of stability condition τ. Kontsevich and Soibelman [32] gave a (partly conjectural) motivic generalization of Donaldson–Thomas invariants, also with a wall-crossing formula.

Corollary 6.12 is relevant to the *categorification* of Donaldson–Thomas theory. As in [2, §1.2], the perverse sheaf $P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}$ has pointwise Euler characteristic $\chi\big(P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}\big) = \nu$. This implies that when A is a field, say $A = \mathbb{Q}$, the (compactly-supported) hypercohomologies $\mathbb{H}^*\big(P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}\big), \mathbb{H}^*_{\mathrm{c}}\big(P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}\big)$ from (2.1) satisfy

$$\sum_{k\in\mathbb{Z}}(-1)^k \dim \mathbb{H}^k\big(P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}\big) = \sum_{k\in\mathbb{Z}}(-1)^k \dim \mathbb{H}^k_{\mathrm{c}}\big(P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}\big) = \chi\big(\mathcal{M}^\alpha_{\mathrm{st}}(\tau),\nu\big) = DT^\alpha(\tau),$$

where $\mathbb{H}^k\big(P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}\big) \cong \mathbb{H}^{-k}_{\mathrm{c}}\big(P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}\big)^*$ by Verdier duality. That is, we have produced a natural graded $\mathbb{Q}$-vector space $\mathbb{H}^*\big(P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}\big)$, thought of as some kind of generalized cohomology of $\mathcal{M}^\alpha_{\mathrm{st}}(\tau)$, whose graded dimension is $DT^\alpha(\tau)$. This gives a new interpretation of the Donaldson–Thomas invariant $DT^\alpha(\tau)$.

In fact, as discussed at length in [51, §3], the first natural "refinement" or "quantization" direction of a Donaldson–Thomas invariant $DT^\alpha(\tau) \in \mathbb{Z}$ is not the Poincaré polynomial of this cohomology, but its weight polynomial

$$w\big(\mathbb{H}^*(P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}),t\big) \in \mathbb{Z}\big[t^{\pm\frac{1}{2}}\big],$$

defined using the mixed Hodge structure on the cohomology of the mixed Hodge module version of $P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}$ (which exists assuming that $\mathcal{M}^\alpha_{\mathrm{st}}(\tau)$ is projective, for example, see Remark 2.22).

The material above is related to work by other authors. The idea of categorifying Donaldson–Thomas invariants using perverse sheaves or $\mathscr{D}$-modules is probably first due to Behrend [2], and for Hilbert schemes $\mathrm{Hilb}^n(Y)$ of a Calabi–Yau 3-fold Y is discussed by Dimca and Szendrői [15] and Behrend, Bryan and Szendrői [3, §3.4], using mixed Hodge modules. Corollary 6.12 answers a question of Joyce and Song [24, Question 5.7(a)].

As in [24, 32] representations of *quivers with superpotentials* (Q,W) give 3-Calabi–Yau triangulated categories, and one can define Donaldson–Thomas type invariants $DT^\alpha_{Q,W}(\tau)$ 'counting' such representations, which are simple algebraic 'toy models' for Donaldson–Thomas invariants of Calabi–Yau 3-folds. Kontsevich and Soibelman [33] explain how to categorify these quiver invariants $DT^\alpha_{Q,W}(\tau)$, and define an associative multiplication on the categorification to make a *Cohomological Hall Algebra*. This paper was motivated by the aim of extending [33] to define Cohomological Hall Algebras for Calabi–Yau 3-folds.

The square root $\det(\mathcal{E}^\bullet)^{1/2}$ required in Corollary 6.12 corresponds roughly to *orientation data* in the work of Kontsevich and Soibelman [32, §5], [33].

In a paper written independently of our programme [9,11,23], Kiem and Li [31] have recently proved an analogue of Corollary 6.12 by complex analytic methods, beginning from Joyce and Song's result [24, Th. 5.4], proved using gauge theory, that $\mathcal{M}_{\rm st}^\alpha(\tau)$ is locally isomorphic to $\mathrm{Crit}(f)$ as a complex analytic space, for V a complex manifold and $f : V \to \mathbb{C}$ holomorphic.

Remark 6.15. In the situation of Corollary 6.13, with $\dim_{\mathbb{C}} S = 2n$, we claim that there ought morally to be some kind of approximate comparison

$$\mathbb{H}^k(P^\bullet_{L,M}) \approx HF^{k+n}(L,M), \tag{6.11}$$

where $HF^*(L,M)$ is the *Lagrangian Floer cohomology* of Fukaya, Oh, Ohta and Ono [18]. We can compare and contrast the two sides of (6.11) as follows:

(a) $\mathbb{H}^*(P^\bullet_{L,M})$ is defined over any well-behaved base ring A, e.g. $A = \mathbb{Z}$ or $\mathbb{Q}$, but $HF^*(L,M)$ is defined over a Novikov ring of power series $\Lambda_{\rm nov}$.

(b) $\mathbb{H}^*(P^\bullet_{L,M})$ has extra structure not visible in $HF^*(L,M)$, from Verdier duality and monodromy operators $\Sigma_{L,M}, \mathrm{T}_{L,M}$, plus the mixed Hodge module version has a mixed Hodge structure.

(c) $\mathbb{H}^*(P^\bullet_{L,M})$ is defined for arbitrary complex Lagrangians L,M, not necessarily compact or closed in S, but $HF^*(L,M)$ is only defined for L,M compact, or at least for L,M closed and well-behaved at infinity.

(d) To define $HF^*(L,M)$ one generally assumes L,M intersect transversely, or at least cleanly. But $\mathbb{H}^*(P^\bullet_{L,M})$ is defined when $L\cap M$ is arbitrarily singular, and the construction is only really interesting for singular $L \cap M$.

(e) To define $HF^*(L,M)$ we need L,M to be oriented and spin, to orient moduli spaces of J-holomorphic curves. When L,M are complex Lagrangians they are automatically oriented, and spin structures on L,M correspond to choices of square roots $K_L^{1/2}, K_M^{1/2}$, as used in Corollary 6.13.

Some of the authors are working on defining a 'Fukaya category' of complex Lagrangians in a complex symplectic manifold, using $\mathbb{H}^*(P^\bullet_{L,M})$ as morphisms.

We now discuss related work. Nadler and Zaslow [40, 41] show that if X is a real analytic manifold (for instance, a complex manifold), then the derived category $D^b_c(X)$ of constructible sheaves on X is equivalent to a certain derived Fukaya category $D^b\mathcal{F}(T^*X)$ of exact Lagrangians in T^*X.

Let L,M be complex Lagrangians in a complex symplectic manifold (S,ω). Regarding $\mathcal{O}_L,\mathcal{O}_M$ as coherent sheaves on S, Behrend and Fantechi [4, Th.s 4.3 & 5.2] claim to construct canonical $\mathbb{C}$-linear (not $\mathcal{O}_S$-linear) differentials

$$\mathrm{d} : \mathcal{E}xt^i_{\mathcal{O}_S}(\mathcal{O}_L,\mathcal{O}_M) \longrightarrow \mathcal{E}xt^{i+1}_{\mathcal{O}_S}(\mathcal{O}_L,\mathcal{O}_M)$$

with $\mathrm{d}^2 = 0$, such that $\bigl(\mathcal{E}xt^*_{\mathcal{O}_S}(\mathcal{O}_L,\mathcal{O}_M),\mathrm{d}\bigr)$ is a constructible complex. There is a mistake in the proof of [4, Th. 4.3]. To fix this one should instead work with $\mathcal{E}xt^*_{\mathcal{O}_S}(K_L^{1/2},K_M^{1/2})$ for square roots $K_L^{1/2}, K_M^{1/2}$ as in Corollary 6.13. Also the proof of the constructibility of $\bigl(\mathcal{E}xt^*_{\mathcal{O}_S}(K_L^{1/2},K_M^{1/2}),\mathrm{d}\bigr)$ in [4, Th. 5.2] depended on a result of Kapranov, which later turned out to be false.

Our $P^\bullet_{L,M}$ over $A = \mathbb{C}$ should be the natural perverse sheaf on $L \cap M$ conjectured by Behrend and Fantechi [4, Conj. 5.16], who also suggest there should be a spectral sequence from $\bigl(\mathcal{E}xt^*_{\mathcal{O}_S}(K_L^{1/2},K_M^{1/2}),\mathrm{d}\bigr)[n]$ to $P^\bullet_{L,M}$. (See Sabbah [44, Th. 1.1] for a related result.) In [4, §5.3], Behrend and Fantechi discuss how to define a 'Fukaya category' using their ideas.

Kashiwara and Schapira [29] develop a theory of *deformation quantization modules*, or *DQ-modules*, on a complex symplectic manifold (S,ω), which roughly may be regarded as symplectic versions of $\mathscr{D}$-modules. *Holonomic* DQ-modules $\mathcal{D}^\bullet$ are supported on (possibly singular) complex Lagrangians L in S. If L is a smooth, closed, complex Lagrangian in S and $K_L^{1/2}$ a square root of K_L, D'Agnolo and Schapira [13] show that there exists a simple holonomic DQ-module $\mathcal{D}^\bullet$ supported on L.

If $\mathcal{D}^\bullet,\mathcal{E}^\bullet$ are simple holonomic DQ-modules on S supported on smooth Lagrangians L,M, then Kashiwara and Schapira [28] show that $R\mathscr{H}om(\mathcal{D}^\bullet,\mathcal{E}^\bullet)[n]$ is a perverse sheaf on S over the field $\mathbb{C}((\hbar))$, supported on $X=L\cap M$. Pierre Schapira explained to the authors how to prove that $R\mathscr{H}om(\mathcal{D}^\bullet,\mathcal{E}^\bullet)[n]\cong P^\bullet_{L,M}$, when $P^\bullet_{L,M}$ is defined over the base ring $A=\mathbb{C}((\hbar))$.

Now let L,M,N be Lagrangians in S, with square roots $K_L^{1/2},K_M^{1/2},K_N^{1/2}$. We have a product $HF^k(L,M)\times HF^l(M,N)\to HF^{k+l}(L,N)$ from composition of morphisms in $D^b\mathcal{F}(S)$. So (6.11) suggests there should be a product

$$\mathbb{H}^k(P^\bullet_{L,M})\times\mathbb{H}^l(P^\bullet_{M,N})\longrightarrow\mathbb{H}^{k+l+n}(P^\bullet_{L,N}),\tag{6.12}$$

which would naturally be induced by a morphism in $D^b_c(S)$

$$\mu_{L,M,N}:P^\bullet_{L,M}\overset{L}{\otimes}P^\bullet_{M,N}\longrightarrow P^\bullet_{L,N}[n].\tag{6.13}$$

Observe that the work of Behrend–Fantechi and Kashiwara–Schapira cited above supports the existence of (6.12)–(6.13): there are natural products

$$\mathcal{E}xt^k_{\mathcal{O}_S}(K_L^{1/2},K_M^{1/2})\otimes_{\mathcal{O}_S}\mathcal{E}xt^l_{\mathcal{O}_S}(K_M^{1/2},K_N^{1/2})\longrightarrow\mathcal{E}xt^{k+l}_{\mathcal{O}_S}(K_L^{1/2},K_N^{1/2}),$$
$$R\mathscr{H}om(\mathcal{D}^\bullet,\mathcal{E}^\bullet)\overset{L}{\otimes}R\mathscr{H}om(\mathcal{E}^\bullet,\mathcal{F}^\bullet)\longrightarrow R\mathscr{H}om(\mathcal{D}^\bullet,\mathcal{F}^\bullet).$$

But since (6.13) is a morphism of complexes, not of perverse sheaves, Theorem 2.7(i) does not apply, so we cannot construct $\mu_{L,M,N}$ by naïvely gluing data on an open cover, as we have been doing in §3–§6.

6.3. **Proof of Theorem 6.9 for $\mathbb{C}$-schemes.** Let (X,s) be an oriented algebraic d-critical locus over $\mathbb{C}$, with orientation $K_{X,s}^{1/2}$. By Definition 6.2 we may choose a family

$$\bigl\{(R_a,U_a,f_a,i_a):a\in A\bigr\}$$

of critical charts (R_a,U_a,f_a,i_a) on (X,s) such that $\{R_a:a\in A\}$ is a Zariski open cover of the $\mathbb{C}$-scheme X. Then for each $a\in A$ we have a perverse sheaf

$$i_a^*\bigl(\mathcal{PV}^\bullet_{U_a,f_a}\bigr)\otimes_{\mathbb{Z}/2\mathbb{Z}}Q_{R_a,U_a,f_a,i_a}\in\mathrm{Perv}(R_a),\tag{6.14}$$

for Q_{R_a,U_a,f_a,i_a} as in Theorem 6.9(i). The idea of the proof is to use Theorem 2.7(ii) to glue the perverse sheaves (6.14) on the Zariski open cover $\{R_a:a\in A\}$ to get a global perverse sheaf $P^\bullet_{X,s}$ on X. Note that Theorem 2.7(ii) is written for étale open covers, but this immediately implies the simpler Zariski version.

To do this, for all $a,b\in A$ we have to construct isomorphisms

$$\begin{aligned}\alpha_{ab}:\bigl[i_a^*\bigl(\mathcal{PV}^\bullet_{U_a,f_a}\bigr)\otimes_{\mathbb{Z}/2\mathbb{Z}}Q_{R_a,U_a,f_a,i_a}\bigr]\big|_{R_a\cap R_b}&\longrightarrow\\ \bigl[i_b^*\bigl(\mathcal{PV}^\bullet_{U_b,f_b}\bigr)\otimes_{\mathbb{Z}/2\mathbb{Z}}Q_{R_b,U_b,f_b,i_b}\bigr]\big|_{R_a\cap R_b}&\in\mathrm{Perv}(R_a\cap R_b),\end{aligned}\tag{6.15}$$

satisfying $\alpha_{aa}=\mathrm{id}$ for all $a\in A$ and

$$\alpha_{bc}|_{R_a\cap R_b\cap R_c}\circ\alpha_{ab}|_{R_a\cap R_b\cap R_c}=\alpha_{ac}|_{R_a\cap R_b\cap R_c}\quad\text{for all }a,b,c\in A.\tag{6.16}$$

Fix $a,b\in A$. By applying Theorem 6.3 to the critical charts (R_a,U_a,f_a,i_a), (R_b,U_b,f_b,i_b) at each $x\in R_a\cap R_b$, we can choose an indexing set D_{ab} and for each $d\in D_{ab}$ subcharts

$(R_a'^d, U_a'^d, f_a'^d, i_a'^d) \subseteq (R_a, U_a, f_a, i_a)$ and $(R_b'^d, U_b'^d, f_b'^d, i_b'^d) \subseteq (R_b, U_b, f_b, i_b)$, a critical chart (S^d, V^d, g^d, j^d) on (X, s), and embeddings

$$\Phi^d : (R_a'^d, U_a'^d, f_a'^d, i_a'^d) \hookrightarrow (S^d, V^d, g^d, j^d) \quad \text{and} \quad \Psi^d : (R_b'^d, U_b'^d, f_b'^d, i_b'^d) \hookrightarrow (S^d, V^d, g^d, j^d)$$

such that $\{R_a'^d \cap R_b'^d : d \in D_{ab}\}$ is a Zariski open cover of $R_a \cap R_b$.

For each $d \in D_{ab}$, define an isomorphism

$$\alpha_{ab}^d : \big[i_a^*(\mathcal{PV}^\bullet_{U_a,f_a}) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R_a,U_a,f_a,i_a}\big]\big|_{R_a'^d \cap R_b'^d} \longrightarrow \big[i_b^*(\mathcal{PV}^\bullet_{U_b,f_b}) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R_b,U_b,f_b,i_b}\big]\big|_{R_a'^d \cap R_b'^d}$$

by the commutative diagram

$$\begin{tikzcd}[column sep=large]
\begin{array}{l}\big[i_a^*(\mathcal{PV}^\bullet_{U_a,f_a}) \otimes_{\mathbb{Z}/2\mathbb{Z}} \\ Q_{R_a,U_a,f_a,i_a}\big]\big|_{R_a'^d \cap R_b'^d}\end{array} \arrow[r, "{i_a|^*_{R_a'^d \cap R_b'^d}(\Theta_{\Phi^d})}", "{\otimes \mathrm{id}_{Q_{R_a,U_a,f_a,i_a}}}"'] \arrow[dd, "\alpha_{ab}^d"] & \begin{array}{r} j^d|^*_{R_a'^d \cap R_b'^d}(\mathcal{PV}^\bullet_{V^d,g^d}) \otimes_{\mathbb{Z}/2\mathbb{Z}} i_a|^*_{R_a'^d \cap R_b'^d}(P_{\Phi^d}) \\ \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R_a,U_a,f_a,i_a}|_{R_a'^d \cap R_b'^d}\end{array} \arrow[d, "{\mathrm{id} \otimes \Lambda_{\Phi^d}|^{-1}_{R_a'^d \cap R_b'^d}}"'] \\
& \big[(j^d)^*(\mathcal{PV}^\bullet_{V^d,g^d}) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{S^d,V^d,g^d,j^d}\big]\big|_{R_a'^d \cap R_b'^d} \arrow[d, "{\mathrm{id} \otimes \Lambda_{\Psi^d}|_{R_a'^d \cap R_b'^d}}"'] \\
\begin{array}{l}\big[i_b^*(\mathcal{PV}^\bullet_{U_b,f_b}) \otimes_{\mathbb{Z}/2\mathbb{Z}} \\ Q_{R_b,U_b,f_b,i_b}\big]\big|_{R_a'^d \cap R_b'^d}\end{array} & \begin{array}{r} j^d|^*_{R_a'^d \cap R_b'^d}(\mathcal{PV}^\bullet_{V^d,g^d}) \otimes_{\mathbb{Z}/2\mathbb{Z}} \\ i_b|^*_{R_a'^d \cap R_b'^d}(P_{\Psi^d}) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R_b,U_b,f_b,i_b}|_{R_a'^d \cap R_b'^d},\end{array} \arrow[l, "{i_b|^*_{R_a'^d \cap R_b'^d}(\Theta^{-1}_{\Psi^d})}"', "{\otimes \mathrm{id}_{Q_{R_b,U_b,f_b,i_b}}}"]
\end{tikzcd} \tag{6.17}$$

where $\Theta_{\Phi^d}, \Theta_{\Psi^d}$ are as in Theorem 5.4, and $\Lambda_{\Phi^d}, \Lambda_{\Psi^d}$ as in (6.9).

We claim that for all $d, e \in D_{ab}$ we have

$$\alpha_{ab}^d|_{R_a'^d \cap R_b'^d \cap R_a'^e \cap R_b'^e} = \alpha_{ab}^e|_{R_a'^d \cap R_b'^d \cap R_a'^e \cap R_b'^e}. \tag{6.18}$$

To see this, let $x \in R_a'^d \cap R_b'^d \cap R_a'^e \cap R_b'^e$, and apply Theorem 6.3 to the critical charts (S^d, V^d, g^d, j^d), (S^e, V^e, g^e, j^e) and point $x \in S^d \cap S^e$. This gives subcharts

$$(S'^d, V'^d, g'^d, j'^d) \subseteq (S^d, V^d, g^d, j^d) \quad \text{and} \quad (S'^e, V'^e, g'^e, j'^e) \subseteq (S^e, V^e, g^e, j^e)$$

with $x \in S'^d \cap S'^e$, a critical chart (T, W, h, k) on (X, s), and embeddings

$$\Omega : (S'^d, V'^d, g'^d, j'^d) \hookrightarrow (T, W, h, k) \quad \text{and} \quad \Upsilon : (S'^e, V'^e, g'^e, j'^e) \hookrightarrow (T, W, h, k).$$

Set $R^{de} = R_a'^d \cap R_b'^d \cap R_a'^e \cap R_b'^e \cap S'^d \cap S'^e$, and consider the diagram:

$$
\begin{tikzcd}[column sep=small]
{\begin{array}{c}\bigl[i_a^*(\mathcal{PV}^\bullet_{U_a,f_a})\otimes_{\mathbb{Z}/2\mathbb{Z}} \\ Q_{R_a,U_a,f_a,i_a}\bigr]\big|_{R^{de}}\end{array}} \arrow[rr, "{(\mathrm{id}\otimes\Lambda_{\Phi^d}^{-1})\circ(i_a^*(\Theta_{\Phi^d})\otimes\mathrm{id})|_{R^{de}}}"'] \arrow[dr, "{\begin{array}{l}(\mathrm{id}\circ\Lambda_{\Omega\circ\Phi^d}^{-1})\circ \\ (i_a^*(\Theta_{\Omega\circ\Phi^d})\otimes\mathrm{id})|_{R^{de}}\end{array}}", "{\begin{array}{l}=(\mathrm{id}\circ\Lambda_{\Upsilon\circ\Phi^e}^{-1})\circ \\ (i_a^*(\Theta_{\Upsilon\circ\Phi^e})\otimes\mathrm{id})|_{R^{de}}\end{array}}"'] \arrow[dd, "{\begin{array}{l}(\mathrm{id}\otimes\Lambda_{\Phi^e}^{-1})\circ \\ (i_a^*(\Theta_{\Phi^e})\otimes\mathrm{id})|_{R^{de}}\end{array}}"] & & {\begin{array}{c}\bigl[(j^d)^*(\mathcal{PV}^\bullet_{V^d,g^d})\otimes_{\mathbb{Z}/2\mathbb{Z}} \\ Q_{S^d,V^d,g^d,j^d}\bigr]|_{R^{de}}\end{array}} \arrow[dl, "{\begin{array}{l}(\mathrm{id}\otimes\Lambda_{\Omega}^{-1})\circ \\ ((j^d)^*(\Theta_{\Omega})\otimes\mathrm{id})|_{R^{de}}\end{array}}"'] \arrow[dd, "{\begin{array}{l}(i_b^*(\Theta_{\Psi^d}^{-1})\otimes\mathrm{id})\circ \\ (\mathrm{id}\otimes\Lambda_{\Psi^d})|_{R^{de}}\end{array}}"'] \\
& {\begin{array}{c}\bigl[k^*(\mathcal{PV}^\bullet_{W,h})\otimes_{\mathbb{Z}/2\mathbb{Z}} \\ Q_{T,W,h,k}\bigr]|_{R^{de}}\end{array}} \arrow[dr, "{\begin{array}{l}(i_b^*(\Theta_{\Omega\circ\Psi^d}^{-1})\otimes\mathrm{id})\circ \\ (\mathrm{id}\otimes\Lambda_{\Omega\circ\Psi^d})|_{R^{de}}\end{array}}", "{\begin{array}{l}=(i_b^*(\Theta_{\Upsilon\circ\Psi^e}^{-1})\otimes\mathrm{id})\circ \\ (\mathrm{id}\otimes\Lambda_{\Upsilon\circ\Psi^e})|_{R^{de}}\end{array}}"'] & \\
{\begin{array}{c}\bigl[(j^e)^*(\mathcal{PV}^\bullet_{V^e,g^e})\otimes_{\mathbb{Z}/2\mathbb{Z}} \\ Q_{S^e,V^e,g^e,j^e}\bigr]|_{R^{de}}\end{array}} \arrow[ur, "{\begin{array}{l}(\mathrm{id}\otimes\Lambda_{\Upsilon}^{-1})\circ \\ ((j^e)^*(\Theta_{\Upsilon})\otimes\mathrm{id})|_{R^{de}}\end{array}}"'] \arrow[rr, "{(i_b^*(\Theta_{\Psi^e}^{-1})\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Lambda_{\Psi^e})|_{R^{de}}}"] & & {\begin{array}{c}\bigl[i_b^*(\mathcal{PV}^\bullet_{U_b,f_b})\otimes_{\mathbb{Z}/2\mathbb{Z}} \\ Q_{R_b,U_b,f_b,i_b}\bigr]\big|_{R^{de}}.\end{array}}
\end{tikzcd}
\tag{6.19}
$$

Here we have given two expressions for the top left diagonal morphism in (6.19). To see these are equal, set $R_a'^{de} = R_a'^d \cap R_a'^e \cap S'^d \cap S'^e$, $U_a'^{de} = (\Phi^d)^{-1}(V'^d) \cap (\Phi^e)^{-1}(V'^e)$, $f_a'^{de} = f_a|_{U_a'^{de}}$, and $i_a'^{de} = i_a|_{R_a'^{de}}$. Then $(R_a'^{de}, U_a'^{de}, f_a'^{de}, i_a'^{de}) \subseteq (R_a, U_a, f_a, i_a)$ is a subchart and

$$\Omega \circ \Phi^d|_{U_a'^{de}}, \Upsilon \circ \Phi^e|_{U_a'^{de}} : (R_a'^{de}, U_a'^{de}, f_a'^{de}, i_a'^{de}) \longrightarrow (T, W, h, k)$$

are embeddings.

As $\Omega \circ \Phi^d \circ i_a'^{de} = k|_{R_a'^{de}} = \Upsilon \circ \Phi^e \circ i_a'^{de}$, Theorem 5.4(b) gives $\Theta_{\Omega\circ\Phi^d}|_{i_a(R_a'^{de})} = \Theta_{\Upsilon\circ\Phi^e}|_{i_a(R_a'^{de})}$, so that $i_a|_{R^{de}}^*(\Theta_{\Omega\circ\Phi^d}) = i_a|_{R^{de}}^*(\Theta_{\Upsilon\circ\Phi^e})$ as $R^{de} \subseteq R_a'^{de}$. Also $\Lambda_{\Omega\circ\Phi^d} = \Lambda_{\Upsilon\circ\Phi^e}$ as these are defined in Theorem 6.9(ii) using $J_{\Omega\circ\Phi^d}, J_{\Upsilon\circ\Phi^e}$, which are equal by Lemma 5.3. So the two expressions are equal, and similarly for the bottom right diagonal morphism.

The upper triangle in (6.19) commutes because (5.16) gives

$$(\mathrm{id} \otimes \Xi_{\Omega,\Phi^d}) \circ \Theta_{\Omega\circ\Phi^d}|_{i_a(R^{de})} = (\Phi^d|_{i_a(R^{de})}^*(\Theta_\Omega) \otimes \mathrm{id}) \circ \Theta_{\Phi^d}|_{i_a(R^{de})},$$

and the definitions of Ξ_{Ω,Φ^d} in (5.11) and $\Lambda_\Omega, \Lambda_{\Phi^d}, \Lambda_{\Omega\circ\Phi^d}$ in (6.9) imply that

$$\begin{aligned}&\bigl(i_a|_{R^{de}}^*(\Xi_{\Omega,\Phi^d}) \otimes \mathrm{id}\bigr) \circ \Lambda_{\Omega\circ\Phi^d}|_{R^{de}} = (\mathrm{id} \otimes \Lambda_{\Phi^d}) \circ \Lambda_\Omega|_{R^{de}} : \\ &Q_{T,W,h,k}|_{R^{de}} \longrightarrow j^d|_{R^{de}}^*(P_\Omega) \otimes_{\mathbb{Z}/2\mathbb{Z}} i_a|_{R^{de}}^*(P_{\Phi^d}) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R_a,U_a,f_a,i_a}|_{R^{de}}.\end{aligned}$$

Similarly, the other three triangles in (6.19) commute, so (6.19) commutes.

By (6.17), the two routes round the outside of (6.19) are $\alpha_{ab}^d|_{R^{de}}$ and $\alpha_{ab}^e|_{R^{de}}$, which are equal as (6.19) commutes. As we can cover $R_a'^d \cap R_b'^d \cap R_a'^e \cap R_b'^e$ by such Zariski open R^{de}, equation (6.18) follows. Therefore by the Zariski open cover version of Theorem 2.7(i), there is a unique isomorphism α_{ab} in (6.15) such that $\alpha_{ab}|_{R_a'^d \cap R_b'^d} = \alpha_{ab}^d$ for all $d \in D_{ab}$.

If $D_{ab}, R_a'^d, \ldots, \Phi^d, \Psi^d$ are used to define α_{ab} and $\tilde D_{ab}, \tilde R_a'^d, \ldots, \tilde\Phi^d, \tilde\Psi^d$ are alternative choices yielding $\tilde\alpha_{ab}$, then by our usual argument using $D_{ab} \amalg \tilde D_{ab}$ and both sets of data we see that $\alpha_{ab} = \tilde\alpha_{ab}$, so α_{ab} is independent of choices.

Because the $\Theta_{\Phi^d}, \Theta_{\Psi^d}$ used to define α_{ab} are compatible with Verdier duality and monodromy by (5.14)–(5.15), and the $\Lambda_{\Phi^d}, \Lambda_{\Psi^d}$ affect only the principal $\mathbb{Z}/2\mathbb{Z}$-bundles rather than the perverse sheaves, we can show α_{ab} is compatible with Verdier duality and monodromy, in that the following commute:

$$
\begin{tikzcd}
\begin{array}{l}\bigl[i_a^*(\mathcal{PV}^\bullet_{U_a,f_a})\otimes_{\mathbb{Z}/2\mathbb{Z}} \\ Q_{R_a,U_a,f_a,i_a}\bigr]\big|_{R_a\cap R_b}\end{array} \arrow[rr, "\alpha_{ab}"] \arrow[d, "i_a^*(\sigma_{U_a,f_a})\otimes\mathrm{id}_{Q_{R_a,U_a,f_a,i_a}}|_{R_a\cap R_b}"] && \begin{array}{l}\bigl[i_b^*(\mathcal{PV}^\bullet_{U_b,f_b})\otimes_{\mathbb{Z}/2\mathbb{Z}} \\ Q_{R_b,U_b,f_b,i_b}\bigr]\big|_{R_a\cap R_b}\end{array} \arrow[d, "i_b^*(\sigma_{U_b,f_b})\otimes\mathrm{id}_{Q_{R_b,U_b,f_b,i_b}}|_{R_a\cap R_b}"'] \\
\begin{array}{l}\bigl[i_a^*(\mathbb{D}_{\mathrm{Crit}(f_a)}(\mathcal{PV}^\bullet_{U_a,f_a})) \\ \otimes_{\mathbb{Z}/2\mathbb{Z}}Q_{R_a,U_a,f_a,i_a}\bigr]\big|_{R_a\cap R_b}\bigr) \\ \cong \mathbb{D}_{R_a\cap R_b}\bigl([i_a^*(\mathcal{PV}^\bullet_{U_a,f_a}) \\ \otimes_{\mathbb{Z}/2\mathbb{Z}}Q_{R_a,U_a,f_a,i_a}]|_{R_a\cap R_b}\bigr)\end{array} && \begin{array}{l}\bigl[i_b^*(\mathbb{D}_{\mathrm{Crit}(f_b)}(\mathcal{PV}^\bullet_{U_b,f_b})) \\ \otimes_{\mathbb{Z}/2\mathbb{Z}}Q_{R_b,U_b,f_b,i_b}\bigr]\big|_{R_a\cap R_b}\bigr) \\ \cong \mathbb{D}_{R_a\cap R_b}\bigl([i_b^*(\mathcal{PV}^\bullet_{U_b,f_b}) \\ \otimes_{\mathbb{Z}/2\mathbb{Z}}Q_{R_b,U_b,f_b,i_b}]|_{R_a\cap R_b}\bigr),\end{array} \arrow[ll, "\mathbb{D}_{R_a\cap R_b}(\alpha_{ab})"']
\end{tikzcd}
\tag{6.20}
$$

$$
\begin{tikzcd}
\begin{array}{l}\bigl[i_a^*(\mathcal{PV}^\bullet_{U_a,f_a})\otimes_{\mathbb{Z}/2\mathbb{Z}} \\ Q_{R_a,U_a,f_a,i_a}\bigr]\big|_{R_a\cap R_b}\end{array} \arrow[rr, "\alpha_{ab}"] \arrow[d, "i_a^*(\tau_{U_a,f_a})\otimes\mathrm{id}_{Q_{R_a,U_a,f_a,i_a}}|_{R_a\cap R_b}"] && \begin{array}{l}\bigl[i_b^*(\mathcal{PV}^\bullet_{U_b,f_b})\otimes_{\mathbb{Z}/2\mathbb{Z}} \\ Q_{R_b,U_b,f_b,i_b}\bigr]\big|_{R_a\cap R_b}\end{array} \arrow[d, "i_b^*(\tau_{U_b,f_b})\otimes\mathrm{id}_{Q_{R_b,U_b,f_b,i_b}}|_{R_a\cap R_b}"'] \\
\begin{array}{l}\bigl[i_a^*(\mathcal{PV}^\bullet_{U_a,f_a})\otimes_{\mathbb{Z}/2\mathbb{Z}} \\ Q_{R_a,U_a,f_a,i_a}\bigr]\big|_{R_a\cap R_b}\end{array} \arrow[rr, "\alpha_{ab}"] && \begin{array}{l}\bigl[i_b^*(\mathcal{PV}^\bullet_{U_b,f_b})\otimes_{\mathbb{Z}/2\mathbb{Z}} \\ Q_{R_b,U_b,f_b,i_b}\bigr]\big|_{R_a\cap R_b}.\end{array}
\end{tikzcd}
\tag{6.21}
$$

When $a=b$ we can take $\Psi^d=\Phi^d$, so (6.17) gives $\alpha^d_{aa}=\mathrm{id}$, and $\alpha_{aa}=\mathrm{id}$.

To prove (6.16), let $a,b,c\in A$, and $x\in R_a\cap R_b\cap R_c$. Applying Theorem 6.3 twice and composing the embeddings, we can construct subcharts $(R'_a,U'_a,f'_a,i'_a)\subseteq(R_a,U_a,f_a,i_a)$, $(R'_b,U'_b,f'_b,i'_b)\subseteq(R_b,U_b,f_b,i_b)$, and $(R'_c,U'_c,f'_c,i'_c)\subseteq(R_c,U_c,f_c,i_c)$ with $x\in R'_a\cap R'_b\cap R'_c$, a critical chart (S,V,g,j) on (X,s), and embeddings $\Phi:(R'_a,U'_a,f'_a,i'_a)\hookrightarrow(S,V,g,j)$, $\Psi:(R'_b,U'_b,f'_b,i'_b)\hookrightarrow(S,V,g,j)$, $\Upsilon:(R'_c,U'_c,f'_c,i'_c)\hookrightarrow(S,V,g,j)$. Then the construction of α_{ab} above yields

$$
\begin{aligned}
\alpha_{ab}|_{R'_a\cap R'_b\cap R'_c}&=\bigl((i_b^*(\Theta_\Psi^{-1})\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Lambda_\Psi)\circ(\mathrm{id}\otimes\Lambda_\Phi^{-1})\circ(i_a^*(\Theta_\Phi)\otimes\mathrm{id})\bigr)|_{R'_a\cap R'_b\cap R'_c},\\
\alpha_{bc}|_{R'_a\cap R'_b\cap R'_c}&=\bigl((i_c^*(\Theta_\Upsilon^{-1})\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Lambda_\Upsilon)\circ(\mathrm{id}\otimes\Lambda_\Psi^{-1})\circ(i_b^*(\Theta_\Psi)\otimes\mathrm{id})\bigr)|_{R'_a\cap R'_b\cap R'_c},\\
\alpha_{ac}|_{R'_a\cap R'_b\cap R'_c}&=\bigl((i_c^*(\Theta_\Upsilon^{-1})\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Lambda_\Upsilon)\circ(\mathrm{id}\otimes\Lambda_\Phi^{-1})\circ(i_a^*(\Theta_\Phi)\otimes\mathrm{id})\bigr)|_{R'_a\cap R'_b\cap R'_c},
\end{aligned}
$$

so that $\alpha_{bc}|_{R'_a\cap R'_b\cap R'_c}\circ\alpha_{ab}|_{R'_a\cap R'_b\cap R'_c}=\alpha_{ac}|_{R'_a\cap R'_b\cap R'_c}$. As we can cover $R_a\cap R_b\cap R_c$ by such Zariski open $R'_a\cap R'_b\cap R'_c$, equation (6.16) follows by Theorem 2.7(i).

The Zariski open cover version of Theorem 2.7(ii) now implies that there exists $P^\bullet_{X,s}$ in $\mathrm{Perv}(X)$, unique up to canonical isomorphism, with isomorphisms

$$\omega_{R_a,U_a,f_a,i_a}:P^\bullet_{X,s}|_{R_a}\longrightarrow i_a^*(\mathcal{PV}^\bullet_{U_a,f_a})\otimes_{\mathbb{Z}/2\mathbb{Z}}Q_{R_a,U_a,f_a,i_a}$$

as in (6.6) for each $a\in A$, with $\alpha_{ab}\circ\omega_{R_a,U_a,f_a,i_a}|_{R_a\cap R_b}=\omega_{R_b,U_b,f_b,i_b}|_{R_a\cap R_b}$ for all $a,b\in A$. Also, (6.7)–(6.8) with (R_a,U_a,f_a,i_a) in place of (R,U,f,i) define isomorphisms $\Sigma_{X,s}|_{R_a}$, $\mathrm{T}_{X,s}|_{R_a}$ for each $a\in A$. Equations (6.20)–(6.21) imply that the prescribed values for $\Sigma_{X,s}|_{R_a},\mathrm{T}_{X,s}|_{R_a}$ and $\Sigma_{X,s}|_{R_b},\mathrm{T}_{X,s}|_{R_b}$ agree when restricted to $R_a\cap R_b$ for all $a,b\in A$. Hence, Theorem 2.7(i) gives unique isomorphisms $\Sigma_{X,s},\mathrm{T}_{X,s}$ in (6.5) such that (6.7)–(6.8) commute with (R_a,U_a,f_a,i_a) in place of (R,U,f,i) for all $a\in A$.

Suppose $\bigl\{(R_a,U_a,f_a,i_a):a\in A\bigr\}$ and $\bigl\{(\tilde R_a,\tilde U_a,\tilde f_a,\tilde\imath_a):a\in\tilde A\bigr\}$ are alternative choices above, yielding $P^\bullet_{X,s},\Sigma_{X,s},\mathrm{T}_{X,s}$ and $\tilde P^\bullet_{X,s},\tilde\Sigma_{X,s},\tilde{\mathrm{T}}_{X,s}$. Then applying the same construction to the family $\bigl\{(R_a,U_a,f_a,i_a):a\in A\bigr\}\amalg\bigl\{(\tilde R_a,\tilde U_a,\tilde f_a,\tilde\imath_a):a\in\tilde A\bigr\}$ to get $\hat P^\bullet_{X,s}$, we have canonical isomorphisms $P^\bullet_{X,s} \cong \hat{P}^\bullet_{X,s} \cong \tilde{P}^\bullet_{X,s}$, which identify $\Sigma_{X,s}, \mathrm{T}_{X,s}$ with $\tilde{\Sigma}_{X,s}, \tilde{\mathrm{T}}_{X,s}$. Thus $P^\bullet_{X,s}, \Sigma_{X,s}, \mathrm{T}_{X,s}$ are independent of choices up to canonical isomorphism.

Now fix $\{(R_a, U_a, f_a, i_a) : a \in A\}$, $P^\bullet_{X,s}, \Sigma_{X,s}, \mathrm{T}_{X,s}$ and ω_{R_a,U_a,f_a,i_a} for $a \in A$ above for the rest of the proof. Suppose (R, U, f, i) is a critical chart on (X, s). Running the construction above with the family $\{(R_a, U_a, f_a, i_a) : a \in A\} \amalg \{(R, U, f, i)\}$, we can suppose it yields the same (not just isomorphic) $P^\bullet_{X,s}, \Sigma_{X,s}, \mathrm{T}_{X,s}$ and ω_{R_a,U_a,f_a,i_a}, but it also yields a unique $\omega_{R,U,f,i}$ in (6.6) which makes (6.7)–(6.8) commute. This proves Theorem 6.9(i).

Let $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$ be an embedding of critical charts on (X, s). The definition of Λ_Φ in Theorem 6.9(ii) is immediate. Run the construction above using the family $\{(R_a, U_a, f_a, i_a) : a \in A\} \amalg \{(R, U, f, i), (S, V, g, j)\}$, and follow the definition of α_{ab} with $(R, U, f, i), (S, V, g, j)$ in place of $(R_a, U_a, f_a, i_a), (R_b, U_b, f_b, i_b)$. We can take

$$D_{ab} = \{d\}, \quad (R'^d_a, U'^d_a, f'^d_a, i'^d_a) = (R, U, f, i), \quad (R'^d_b, U'^d_b, f'^d_b, i'^d_b) = (S^d, V^d, g^d, j^d) = (S, V, g, j),$$

$\Phi^d = \Phi$ and $\Psi^d = \mathrm{id}_V$. Then (6.17) gives $\alpha_{ab} = \alpha^d_{ab} = (\mathrm{id} \otimes \Lambda_\Phi^{-1}) \circ (i^*(\Theta_\Phi) \otimes \mathrm{id})$. Thus,

$$\alpha_{ab} \circ \omega_{R_a,U_a,f_a,i_a}|_{R_a \cap R_b} = \omega_{R_b,U_b,f_b,i_b}|_{R_a \cap R_b}$$

implies that (6.10) commutes, proving Theorem 6.9(ii).

6.4. **$\mathscr{D}$-modules and mixed Hodge modules.** Once again, the proof of Theorem 6.9 carries over to our other contexts in §2.6–§2.10 using the general framework of §2.5, now also making use of the Stack Property (x) for objects. For the case of mixed Hodge modules, we use Theorem 2.21(ii) to glue the $i_a^*\big(\mathcal{HV}^\bullet_{U_a,f_a}\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R_a,U_a,f_a,i_a}$ on $R_a \subseteq X$ for $a \in A$ *with their natural strong polarizations* (2.25), which are preserved by the isomorphisms α_{ab} in §6.3 on overlaps $R_a \cap R_b$.

Appendix A. Compatibility results, by Jörg Schürmann

In the main body of the paper, when comparing results for mixed Hodge modules to those involving perverse sheaves, we rely on the compatibility between duality and Thom–Sebastiani type isomorphisms of perverse sheaves and mixed Hodge modules. These compatibility statements cannot easily be read off from the existing literature, so we provide proofs here.

Proposition A.1. *If X is a $\mathbb{C}$-scheme and $f : X \to \mathbb{C}$ is regular, then Massey's natural isomorphisms from [37] quoted as Theorem 2.11*(iv) *coincide with the image under the realization functor of Saito's analogous isomorphisms [45] between functors on mixed Hodge modules. There is also an analogous compatibility result for X a complex analytic space equipped with an analytic function f.*

Proof. Massey's construction in [37] of the duality isomorphisms uses the definition of the vanishing cycle functor in terms of the local cohomology of suitable real half-spaces, compare also [50]. Using their notation, the compatibility comes down to compatibility of the diagram

$$\begin{tikzcd} \phi^p_f \circ \mathbb{D}_X \arrow[rr, "\sim"] \arrow[d, "\cong"] & & \mathbb{D}_{X_0} \circ \phi^p_f \\ (R\Gamma_{\{\mathrm{Re}(f)\geq 0\}}(-)|_{X_0}) \circ \mathbb{D}_X \arrow[rr, "\sim"] & & \mathbb{D}_{X_0} \circ (R\Gamma_{\{\mathrm{Re}(f)\leq 0\}}(-)|_{X_0}). \arrow[u, "\cong"'] \end{tikzcd} \tag{A.1}$$

Here the upper, respectively lower horizontal isomorphisms are the ones of Saito, respectively Massey, and the vertical isomorphisms follow for example from [50, Lem. 1.3.2, p. 69]. Saito deduces his duality isomorphism in [45, Lem. 5.2.4, p. 965] from a pairing on nearby cycles induced by a pairing $F \otimes G \to a_X^! A$ for $F, G \in D^b_c(X)$, with $a_X : X \to \mathrm{pt}$ the constant map and

$A \subset \mathbb{C}$ a coefficient field. But Massey's duality isomorphism can be also be induced from such a pairing fitting into a commutative diagram, with

$$L_0 = \{\mathrm{Re}(f) = 0\} \quad \text{and} \quad j : \{\mathrm{Re}(f) = 0, f \neq 0\} \to L_0$$

the open inclusion:

$$\begin{tikzcd}
(R\Gamma_{\{\mathrm{Re}(f)<0\}}(F)|_{X_0}) \otimes (R\Gamma_{\{\mathrm{Re}(f)>0\}}(G)|_{X_0}) \arrow[d] & \psi_f(F) \otimes \psi_f(G) \arrow[l, "\sim"'] \arrow[d] \\
(Rj_*j^*(a^!_{L_0}A|_{L_0}))|_{X_0} \arrow[d, "\cong"] & \psi_f(a^!_X A) \arrow[l] \arrow[d] \\
(Rj_*j^*(a^!_{L_0}A))|_{X_0}[1] \arrow[r] & a^!_{X_0}A[2].
\end{tikzcd} \tag{A.2}$$

Here the isomorphism $j^*(a^!_X A|_{L_0}) \cong j^*(a^!_{L_0}A)[1]$ comes from the fact that $\mathrm{Re}(f)$ has no critical points (in a stratified sense) in $X \setminus X_0$, locally near X_0. But then the commutativity of (A.2) implies by [45, Lem. 5.2.4, p. 965] the commutativity of (A.1), concluding the proof. $\square$

A similar compatibility question arises for the Thom–Sebastiani isomorphism. Here the precise statement is the following.

Proposition A.2. *Let $f_i : Y_i \to \mathbb{C}$ be regular functions on smooth $\mathbb{C}$-schemes, for $i = 1, 2$. Let $f = f_1 \boxplus f_2 : Y_1 \times Y_2 \to \mathbb{C}$ be as in Theorem 2.13. Then the isomorphism (2.8) of Massey [35,50] coincides with the image under the realization functor of Saito's analogous isomorphism (2.26) of [49] for mixed Hodge modules.*

Proof. The Thom–Sebastiani isomorphism (2.26) is constructed by Saito [49, Th. 2.6] based on the Verdier specialization [53]. First, let $f : Y \to \mathbb{C}$ be a regular function, with $X = f^{-1}(0)$ of codimension one, so that the normal cone $C_X Y = X \times \mathbb{C}$ becomes a trivial line bundle with $f' : C_X Y \to \mathbb{C}$ given by the projection. Let $p : D_X Y \to \mathbb{C}$ be the deformation to the normal cone with $C_X Y = p^{-1}(0) \subset D_X Y$, with $q : D_X Y \to Y$ the natural map. Then f' extends to a function $g : D_X Y \to \mathbb{C}$, with $g = f/s$ on $p \neq 0 = Y \times \mathbb{C}$ for s the usual coordinate on $\mathbb{C}$. For $F \in D^b_c(Y)$, we get a commutative diagram

$$\begin{tikzcd}
\phi^p_{f'}(sp_X F) \arrow[d, "\cong"] & \phi^p_f(F) \arrow[l, "\sim"'] \arrow[d, "\cong"'] \\
R\Gamma_{\{\mathrm{Re}(f')\geq 0\}}(sp_X F)|_X & R\Gamma_{\{\mathrm{Re}(f)\geq 0\}}(F)|_X. \arrow[l]
\end{tikzcd}$$

Here the monodromical sheaf complex $sp_X F \in D^b_c(C_X Y)_{\mathrm{mon}}$ is the Verdier specialization of F as in [49,53]. The upper horizontal isomorphism is the one of [49, Lem. 2.2], whereas the vertical isomorphisms are those of [50, Lem. 1.3.2, p. 69]. The lower horizontal map is defined by the natural base change morphism

$$R\Gamma_{\{\mathrm{Re}(f')\geq 0\}}(sp_X F)|_X \longleftarrow \psi_p(R\Gamma_{\{\mathrm{Re}(g)\geq 0\}}(q^*F))|_X \cong R\Gamma_{\{\mathrm{Re}(f)\geq 0\}}(F)|_X,$$

where the last isomorphism follows as in [50, Lem. 1.3.3, p. 70-71].

Consider now the situation in the proposition, with

$$f = f_1 \boxplus f_2 : Y = Y_1 \times Y_2 \to \mathbb{C},$$

also $X_i = f_i^{-1}(0)$ and $X = f^{-1}(0)$; finally let $\mu_{f_i} = R\Gamma_{\{\mathrm{Re}(f_i)\geq 0\}}$ to shorten the notation. Let

$$\pi : C_{X_1}Y_1 \times C_{X_2}(Y_2) = (X_1 \times X_2) \times \mathbb{C}^2 \to (X_1 \times X_2) \times \mathbb{C} \subset C_X(Y)$$

be the map induced by addition in the fibres. Then, for $F_i \in D^b_c(Y_i)$, one gets a commutative diagram

$$\begin{tikzcd}
\phi^p_f(F_1 \boxtimes F_2)|_{X_1\times X_2} \arrow[d, "\cong"] & & \phi^p_{f_i}(F_1) \otimes \phi^p_{f_2}(F_2) \arrow[ll, "\sim"'] \arrow[d, "\cong"'] \\
\mu_{f'}(\pi_*(sp_{X_1}F_1 \boxtimes sp_{X_2}F_2))|_{X_1\times X_2} & & \mu_{f_1'}(sp_{X_1}F_1)|_{X_1} \otimes \mu_{f_2'}(sp_{X_2}F_2)|_{X_2} \arrow[ll] \\
\mu_f(F_1 \boxtimes F_2)|_{X_1\times X_2} \arrow[u] & & \mu_{f_1}(F_1)|_{X_1} \otimes \mu_{f_2}(F_2)|_{X_2}. \arrow[ll, "\sim"] \arrow[u, "\cong"']
\end{tikzcd}$$

The upper horizontal and left vertical isomorphisms form the Thom–Sebastiani isomorphism (2.26) of [49], whereas the lower horizontal isomorphism is the Thom–Sebastiani isomorphism (2.8) of [35, 50]. This concludes the proof. □

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THE PUNCTUAL HILBERT SCHEMES FOR THE CURVE SINGULARITIES OF TYPE A_{2d}

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ABSTRACT. Pfister and Steenbrink studied punctual Hilbert schemes for irreducible curve singularities. In particular, they analyzed the structure of certain punctual Hilbert schemes for monomial curve singularities. In this paper, we generalize their results about the curve singularities of type A_{2d} by clarifying the relationships among the punctual Hilbert schemes for the singularities.

1. INTRODUCTION

Let $\mathcal{O}$ be the complete local ring of an irreducible curve singularity over an algebraically closed field k of characteristic 0. We denote by $\overline{\mathcal{O}}$ and δ the normalization of $\mathcal{O}$ and the δ-invariant of $\mathcal{O}$ respectively. Pfister and Steenbrink [3] defined a special subset $\mathcal{M}$ of the Grassmannian $\mathrm{Gr}(\delta, \overline{\mathcal{O}}/I(2\delta))$ where $I(2\delta)$ is the set of all elements in $\mathcal{O}$ whose orders are greater than or equal to 2δ. It is a projective variety which consists of $\mathcal{O}$-sub-modules and we call it the *Pfister-Steenbrink variety* (PS variety) for the given singularity. For any positive integer r, the existence of the punctual Hilbert scheme of degree r was also shown there. It is a projective variety which parametrizes the ideals of codimension r in $\mathcal{O}$ and is realized as a connected component of the PS variety.

A numerical semi-group Γ is called *monomial*, if any curve singularity with it has no moduli. Pfister and Steenbrink determined all monomial semi-groups in [3]. Using the intersections with Schubert cells, they also analyzed the structure of the PS varieties for the curve singularities with monomial semi-groups. The cases for the curve singularities of types A_{2d}, E_6 and E_8 were also involved in their study. The punctual Hilbert schemes for the curve singularity of type A_1 and related topics were discussed by Ran in [4, 5]. Kawai [2] computed the Euler characteristic of the Hilbert scheme $C^{[d]}$ of 0-dimensional length d subschemes of a projective curve C with only the A_1 and A_2 singularities. The structures of punctual Hilbert schemes for the curve singularities of types A_1 and A_2 were used in this computation. The result was also discussed in the context of string theory. Recently, by using computational methods, the authors of this paper studied the structure of all punctual Hilbert schemes for the curve singularities of types E_6 and E_8 in [7]. On the other hand, the PS varieties for curve singularities were studied from another point of view. Rego [6] introduced the *compactified Jacobian* of singular curves. He also constructed the *Jacobi factor* for a curve singularity. The Jacobi factor and the PS variety coincide for a given curve singularity.

In this paper, we consider the curve singularities of type A_{2d} (i.e. the curve singularities whose local rings are isomorphic to $k[[t^2, t^{2d+1}]]$ where $d \in \mathbb{N}$). We denote by $\mathcal{M}_{d,r}$ the punctual Hilbert scheme of degree r for the curve singularity of type A_{2d}. Pfister and Steenbrink showed that: *the PS variety* $\mathcal{M}_{d,2d}$ *is an irreducible rational projective variety of* $\dim \mathcal{M}_{d,2d} = d$. *In particular,* (i) $\mathcal{M}_{1,2} \cong \mathbb{P}^1$, (ii) $\mathcal{M}_{2,4}$ *is a quardratic cone in* $\mathbb{P}^5$, (iii) $\mathcal{M}_{3,6}$ *is a threefold with a singular*

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line with transverse singularity of type A_2. In general, the following fact also holds: *if* $r \geq 2\delta$, *then the punctual Hilbert scheme of degree* r *for an irreducible curve singularity coincides with the PS variety for the same singularity* (Corollary 11). So it is enough to consider their degree r within $1 \leq r \leq 2\delta$ for the analysis of the structure of punctual Hilbert schemes (Remark 12). Since the δ-invariant for a given A_{2d} singularity equals d, the PS variety coincides with $\mathcal{M}_{d,2d}$ by the above fact. Our main theorems are stated as follows:

Theorem 1. *Let* d *and* r *be two integers with* $1 \leq r \leq 2d$. *Putting* $s := [r/2]$, *the punctual Hilbert scheme* $\mathcal{M}_{d,r}$ *is a rational projective variety with* $\dim \mathcal{M}_{d,r} = s$. *If* $r \geq 2$, *then it is isomorphic to the Pfister-Steenbrink variety* $\mathcal{M}_{s,2s}$.

Theorem 2. *We keep the notations* d, r *and* s *as in Theorem 1. The punctual Hilbert schemes for the curve singularity of type* A_{2d} *have the following structures*:

(i): *The punctual Hilbert scheme* $\mathcal{M}_{d,1}$ *consists of one point.*
(ii): *The punctual Hilbert schemes* $\mathcal{M}_{d,2}$ *and* $\mathcal{M}_{d,3}$ *are isomorphic to a projective line* $\mathbb{P}^1$.
(iii): *The punctual Hilbert scheme* $\mathcal{M}_{d,r}$ *with* $4 \leq r \leq 2d$ *is a singular projective variety whose singular locus is given by* $\mathcal{M}_{d,2s-2} \cap \mathcal{M}_{d,2s-1} \cong \mathcal{M}_{d,2s-3}$.

The present paper is organized as follows: In Section 2 below, we briefly recall Pfister-Steenbrink theory introduced in [3]. In Section 3, we study ideals in the local ring $k[[t^2, t^{2d+1}]]$ of the curve singularities of type A_{2d}. From the point of view of Γ-semi-module structure of orders, we determine the sets of ideals in $\mathcal{O}$ with codimension r ($1 \leq r \leq 2\delta$) and their decompositions. These yield affine cell decompositions of the punctual Hilbert schemes for the curve singularities of type A_{2d}. In Section 4, we first show the irreducibility of the punctual Hilbert schemes. We also prove the following proposition:

Proposition 3. *The following relations hold for punctual Hilbert schemes*:

(i): *For integers* d *and* s *with* $1 \leq s \leq d-1$, *we have* $\mathcal{M}_{d,2s} \cong \mathcal{M}_{d,2s+1}$.
(ii): *For integers* d, d' *and* r *with* $1 \leq r \leq \min\{2d, 2d'\}$, *we have* $\mathcal{M}_{d,r} \cong \mathcal{M}_{d',r}$.

Finally, we prove Theorem 1 and 2 by using them.

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2. Pfister-Steenbrink theory for punctual Hilbert schemes

In the present paper, we restrict ourselves to monomial curve singularities defined below. However, the notions in this section hold in more general situations. See [3] for details.

Definition 4. *A monomial curve singularity is an irreducible curve singularity whose local ring is isomorphic to* $k[[t^{a_1}, \ldots, t^{a_m}]]$ *for some* $a_1, \ldots, a_m \in \mathbb{N}$.

Remark 5. *Without loss of generality, we may assume that* $\gcd(a_1, \ldots, a_m) = 1$ *in Definition 4.*

Let $\mathcal{O} = k[[t^{a_1}, \ldots, t^{a_m}]]$ be the local ring of a monomial curve singularity. Its normalization $\overline{\mathcal{O}}$ is isomorphic to $k[[t]]$. We call $\Gamma := \{\mathrm{ord}_t(f) \mid f \in \mathcal{O}\}$ the *semi-group* of $\mathcal{O}$. The positive integer $\delta := \dim_k(\overline{\mathcal{O}}/\mathcal{O})$ is called the *δ-invariant* of $\mathcal{O}$. For $n \in \mathbb{N}$, set $\overline{I}(n) := \{ f \in \overline{\mathcal{O}} | \, \mathrm{ord}_t(f) \geq n\}$ and $I(n) := \overline{I}(n) \cap \mathcal{O}$. Setting $\mathrm{ord}_t(0) = \infty$, we regard $\overline{I}(n)$ (resp. $I(n)$) as an ideal of $\overline{\mathcal{O}}$ (resp. $\mathcal{O}$). For an ideal I of $\mathcal{O}$, we denote by $\Gamma(I) := \{\mathrm{ord}_t(f) | \, f \in I\}$ the set of orders of all elements in I. Put $G(I) := \Gamma \setminus \Gamma(I)$. For $r \in \mathbb{N}$, define

$$\mathcal{I}_r := \{I | \, I \text{ is an ideal of } \mathcal{O} \text{ with } \dim_k \mathcal{O}/I = r\}.$$

A subset $\Delta \subset \mathbb{Z}$ is called a Γ-*semi-module*, if $\Delta + \Gamma \subset \Delta$. Note that if Δ is a Γ-semi-module, then $\Delta - r$ is also a Γ-semi-module for any integer r. We write $\Delta = \langle \alpha_1, \cdots, \alpha_p \rangle_\Gamma$ for a Γ-semi-module Δ which is minimally generated by $\alpha_1, \cdots, \alpha_p$ (i.e $\Delta = \sum_{i=1}^{p}(\alpha_i + \Gamma)$ and $\Delta \supsetneq \sum_{i=1, i\neq j}^{p}(\alpha_i + \Gamma)$ for $\forall j \in \{1, \dots, p\}$). We also denote by $\mathcal{I}(\Delta)$ the set of all ideals of $\mathcal{O}$ whose set of orders are Δ. Note that $\mathcal{I}(\Delta) \neq \emptyset$ if and only if $\Delta \subset \Gamma$.

The following facts are known:

Lemma 6 ([7], Lemma 5)**.** *An ideal I in $\mathcal{O}$ belongs to $\mathcal{I}_r$ if and only if we have $\sharp G(I) = r$.*

Proposition 7 ([7], Proposition 7)**.** *There exists a finite number of distinct Γ-semi-modules $\Delta_{r,1}, \cdots, \Delta_{r,n_r}$ such that*

$$\mathcal{I}_r = \bigcup_{l=1}^{n_r} \mathcal{I}(\Delta_{r,l}). \tag{1}$$

Remark 8. *The set of Γ-semi-modules $\Delta_{r,l}$ in (1) is an invariant for the codimension r.*

Let $\mathrm{Gr}\left(\delta, \overline{\mathcal{O}}/\mathrm{I}(2\delta)\right)$ be the Grassmannian which consists of δ-dimensional linear subspaces of $\overline{\mathcal{O}}/I(2\delta)$. For $V \in \mathrm{Gr}\left(\delta, \overline{\mathcal{O}}/\mathrm{I}(2\delta)\right)$, we define a multiplication by $\mathcal{O} \times V \ni (f, \overline{v}) \mapsto \overline{fv} \in V$. Set

$$\mathcal{M} := \left\{ V \in \mathrm{Gr}\left(\delta, \overline{\mathcal{O}}/\mathrm{I}(2\delta)\right) \,\middle|\, V \text{ is an } \mathcal{O}\text{-sub-module w.r.t. the multiplication} \right\}.$$

Consider the composition map

$$\psi : \mathcal{M} \xrightarrow{\psi_1} \mathrm{M}_{\delta,2\delta}(k)/\sim \xrightarrow{\psi_2} \mathbb{P}^N \tag{2}$$

where $\mathrm{M}_{\delta,2\delta}(k)$ is the set of all $\delta \times 2\delta$ matrices over k and the equivalence relation $\sim$ is the similarity of matrices. For a formal power series $f = \sum_{j=0}^{\infty} a_j t^j$ in $\overline{\mathcal{O}}$, we denote its coset in $\overline{\mathcal{O}}/I(2\delta)$ by $\overline{f} = \sum_{j=0}^{2\delta-1} a_j \tau^j$. The notation τ signifies the coset of t. Define $\mathrm{ord}_\tau(\overline{f})$ by $\mathrm{ord}_t(f)$ (resp. ∞), if $\mathrm{ord}_t(f) \leq 2\delta - 1$ (resp. $\mathrm{ord}_t(f) \geq 2\delta$). In this paper, we use the notation $[v_1, \cdots, v_\delta]_k$ for a k-vector space generated by $v_1, \dots, v_\delta$. Let $V = [\overline{f}_1, \cdots, \overline{f}_\delta]_k$ be an element of $\mathcal{M}$ where $\overline{f}_i = \sum_{j=0}^{2\delta-1} a_{ij}\tau^j$. We identify $\overline{f}_i$ with the point $\boldsymbol{a}_i = (a_{i0}, \cdots, a_{i2\delta-1})$ in $k^{2\delta}$. Let A_V be the $\delta \times 2\delta$ matrix whose ith row is $\boldsymbol{a}_i$. We call it the *representation matrix* of V. The first map ψ_1 in (2) is defined by sending a k-vector space V to the coset of A_V. The second map ψ_2 in (2) is the Plücker embedding with $N = \binom{2\delta}{\delta} - 1$. Note that ψ_1 and ψ_2 are injective.

For $r \in \mathbb{N}$, Pfister and Steenbrink defined a map $\varphi_r : \mathcal{I}_r \to \mathcal{M}$ by $\varphi_r(I) = t^{-r}I/I(2\delta)$.

Proposition 9 ([3], Theorem 3)**.** *The map φ_r is injective for any r. Furthermore, it is bijective for $r \geq 2\delta$. The image $(\psi \circ \varphi_r)(\mathcal{I}_r)$ is Zariski closed in $\psi(\mathcal{M})$.*

Put $\mathcal{M}_r := \varphi_r(\mathcal{I}_r)$. Since ψ is injective, we identify $\psi(\mathcal{M})$ and $\psi(\mathcal{M}_r)$ with $\mathcal{M}$ and $\mathcal{M}_r$ respectively.

Definition 10. *We call $\mathcal{M}$ and $\mathcal{M}_r$ the Pfister-Steenbrink variety (PS variety for short) and the punctual Hilbert scheme of degree r for a given curve singularity respectively.*

The following fact follows from Proposition 9:

Corollary 11. *Any punctual Hilbert scheme $\mathcal{M}_r$ with $r \geq 2\delta$ coincides with the PS variety $\mathcal{M}$.*

Remark 12. *By virtue of Corollary 11, it is enough to consider codimensions r within $1 \leq r \leq 2\delta$ for the analysis of $\mathcal{M}_r$.*

Set $\mathcal{M}(\Delta_{r,l}) := \varphi_r(\mathcal{I}(\Delta_{r,l}))$ for each component $\mathcal{I}(\Delta_{r,l})$ in (1). Since ψ is injective, we also identify $\psi(\mathcal{M}(\Delta_{r,l}))$ with $\mathcal{M}(\Delta_{r,l})$. Namely, we regard $\mathcal{M}(\Delta_{r,l})$ as a subset of the punctual Hilbert scheme $\mathcal{M}_r$ parametrizing ideals in $\mathcal{I}(\Delta_{r,l})$. Set $[a, b] := \{x \in \mathbb{Z}_{\geq 0} \,|\, a \leq x \leq b\}$. For a Γ-semi-module $\Delta_{r,l} = \langle \alpha_1, \cdots, \alpha_{p_l} \rangle_\Gamma$, we have $\Delta_{r,l} - r = \langle \alpha_1 - r, \cdots, \alpha_{p_l} - r \rangle_\Gamma$. Define $A := \{\alpha_1 - r, \cdots, \alpha_{p_l} - r\} \cap [0, 2\delta - 1]$ and $J_\alpha := [\alpha + 1, 2\delta - 1] \setminus \{\Delta_{r,l} - r\}$ for $\alpha \in A$. The following facts are known:

Proposition 13 ([3], Theorem 7)**.** *Let I be an element of $\mathcal{I}(\Delta_{r,l})$. There exist uniquely determined $b_{\alpha j} \in k$ such that the $\mathcal{O}$-sub-module $\varphi_r(I)$ is generated by*

$$\overline{f}_\alpha := \tau^\alpha + \sum_{j \in J_\alpha} b_{\alpha j} \tau^j \quad (\alpha \in A).$$

Corollary 14 ([3], Corollary of Theorem 11)**.** *The component $\mathcal{M}(\Delta_{r,l})$ is isomorphic to the affine space k^N where $N = \sum_{\alpha \in A} \sharp J_\alpha$.*

We obtain an affine cell decomposition of $\mathcal{M}_r$ by Proposition 7 and Corollary 14.

Proposition 15. *The punctual Hilbert scheme $\mathcal{M}_r$ of degree r has an affine cell decomposition*

$$\mathcal{M}_r = \bigcup_{l=1}^{n_r} \mathcal{M}(\Delta_{r,l}). \tag{3}$$

The following fact also follows from Corollary 14:

Proposition 16. *If $\mathcal{M}_r$ is irreducible, then it is a rational projective variety.*

The 2δ-dimensional k-vector space $\overline{\mathcal{O}}/I(2\delta)$ has the canonical flag

$$0 \subset V_1 \subset V_2 \subset \cdots \subset V_{2\delta} = \overline{\mathcal{O}}/I(2\delta)$$

where $V_i = \overline{I}(2\delta - i)/I(2\delta)$ for $1 \leq i \leq 2\delta$. This induces a partition of $\mathrm{Gr}\big(\delta, \overline{\mathcal{O}}/I(2\delta)\big)$ into *Schubert cells* $W_{a_1,\ldots,a_\delta}$ for $\delta \geq a_1 \geq \cdots \geq a_\delta \geq 0$, which is defined by

$$W_{a_1,\ldots,a_\delta} := \left\{ W \in \mathrm{Gr}\big(\delta, \overline{\mathcal{O}}/I(2\delta)\big) \;\middle|\; \begin{array}{l} \dim(W \cap V_{\delta+i-a_i}) = i \text{ for } 1 \leq i \leq \delta \\ \dim(W \cap V_j) < i \text{ for } j < \delta + i - a_i \end{array} \right\}.$$

For an index set $\Lambda = \{a_1, \ldots, a_\delta\}$, we sometimes write W_Λ instead of $W_{a_1,\ldots,a_\delta}$.

Proposition 17. *We have $W_{b_1,\ldots,b_\delta} \subset \overline{W_{a_1,\ldots,a_\delta}}$ if and only if $b_i \geq a_i$ holds for $1 \leq i \leq \delta$.*

For the details about Schubert cells, see [1, p.195].

Lemma 18. *Let $\mathcal{M}(\Delta_{r,l})$ be a component in (3) and write $\{b_1, \ldots, b_\delta\} = (\Delta_{r,l} - r) \cap [0, 2\delta - 1]$ where $0 \leq b_1 < \cdots < b_\delta < 2\delta$. Setting $a_{\delta-i+1} = b_i - i + 1$ for $1 \leq i \leq \delta$, we have*

$$\mathcal{M}(\Delta_{r,l}) = \mathcal{M}_r \cap W_{a_1,\ldots,a_\delta}.$$

Proof. It is known that our assertion is true for $r = 2\delta$ (see Lemma 5 in [3]). So we consider the case where $r < 2\delta$. Since $\mathcal{M}(\Delta_{r,l}) \subset \mathcal{M}_r \subset \mathcal{M}_{2\delta}$, there exists a Γ-semi-module $\Delta_{2\delta}$ such that $\mathcal{M}(\Delta_{2\delta})$ is a component of $\mathcal{M}_{2\delta}$ and $\Delta_{2\delta} - 2\delta = \Delta_{r,l} - r$. It follows from the above fact for $r = 2\delta$ that $\mathcal{M}(\Delta_{r,l}) = \mathcal{M}_r \cap \mathcal{M}(\Delta_{2\delta}) = \mathcal{M}_r \cap (\mathcal{M}_{2\delta} \cap W_{a_1,\ldots,a_\delta}) = \mathcal{M}_r \cap W_{a_1,\ldots,a_\delta}$. □

3. Ideals in the local ring of the singularities of type A_{2d}

From this section, we only consider the curve singularities of type A_{2d} and freely use the notations introduced in the previous section. Let $\mathcal{O}$ be the local ring $k[[t^2, t^{2d+1}]]$ for some $d \in \mathbb{N}$. The semi-group Γ of $\mathcal{O}$ is generated by 2 and $2d+1$. Note that any Γ-semi-module Δ contained in Γ is generated by at at most two elements $\alpha_1 := \min\{\Gamma(I)\}$ and $\alpha_2 := \min\{\Gamma(I) \setminus (\alpha_1 + \Gamma)\}$ as Γ-semi-module. We have $\alpha_2 < \alpha_1 + 2d + 1$. We use the notation $\mathcal{I}(\alpha_1, \alpha_2)$ instead of $\mathcal{I}(\Delta)$.

Lemma 19. *For any element I of $\mathcal{I}(\alpha_1,\alpha_2)$, there exist two generators f_1 and f_2 such that*

$$f_1 = t^{\alpha_1} + \sum_{j\in G(I), j>\alpha_1} c_j t^j, \quad f_2 = t^{\alpha_2}.$$

Proof. Since $\Gamma(I)$ is generated by at most two positive integers as Γ-semi-module, the ideal I is generated by at most two elements. Let $g_1 = t^{\alpha_1} + \sum_{j>\alpha_1} c_j t^j$ and $g_2 = t^{\alpha_2} + \sum_{j>\alpha_2} d_j t^j$ be such generators. For any $j \in \Gamma(I)$, there exists an element $h_j = t^j +$ terms of higher order in I. Reducing g_1 and g_2 by h_j's successively, we obtain the desired generators. $\square$

By Lemma 19, normal forms of all ideals in $\mathcal{I}(\alpha_1,\alpha_2)$ are described as follows:

Proposition 20. *If $\alpha_1 = 2p$ and $\alpha_2 = 2q+1$, then we have*

$$\mathcal{I}(2p,2q+1) = \begin{cases} \left\{\left(t^{2p}, t^{2d+1}\right)\right\} & \text{for } p \le d = q, \\ \left\{\left(t^{2p} + \sum_{i=d}^{q-1} a_i t^{2i+1}, t^{2q+1}\right) \middle| a_i \in k\right\} & \text{for } p \le d < q, \\ \left\{\left(t^{2p}, t^{2p+1}\right)\right\} & \text{for } d < p = q, \\ \left\{\left(t^{2p} + \sum_{i=p}^{q-1} a_i t^{2i+1}, t^{2q+1}\right) \middle| a_i \in k\right\} & \text{for } d < p < q. \end{cases}$$

On the other hand, if $\alpha_1 = 2p+1$ and $\alpha_2 = 2q$, then we have

$$\mathcal{I}(2p+1,2q) = \begin{cases} \left\{\left(t^{2p+1}, t^{2p+2}\right)\right\} & \text{for } p \ge d,\ q = p+1, \\ \left\{\left(t^{2p+1} + \sum_{i=p+1}^{q-1} a_i t^{2i}, t^{2q}\right) \middle| a_i \in k\right\} & \text{for } p \ge d,\ q > p+1. \end{cases}$$

Proof. Let I be a non-zero ideal in $\mathcal{O}$. For the set $G(I)$, we have the following two cases:
(Case 1): α_1 is even and α_2 is odd. Write $\alpha_1 = 2p$ and $\alpha_2 = 2q+1$. It follows from the definitions of α_1 and α_2 that $p \le q$. We easily see that $\Gamma(I) = \{2i \mid p \le i \le q\} \cup \{n \mid n \ge 2q+1\}$ and $c(I) = 2q$. Since $2d+1 \le \alpha_2 \le \alpha_1 + 2d + 1$ hold, we also obtain $d \le q \le p+d$. In terms of p, q and d, the set $G(I)$ is described as follows:

$$G(I) = \begin{cases} \{2i \mid 0 \le i \le p-1\} & \text{for } p \le d = q, \\ \{2i \mid 0 \le i \le p-1\} \cup \{2j+1 \mid d \le j \le q-1\} & \text{for } p \le d < q, \\ \{2i \mid 0 \le i \le d\} \cup [2d+1, 2p-1] & \text{for } d < p = q, \\ \{2i \mid 0 \le i \le d\} \cup [2d+1, 2p-1] \cup \{2j+1 \mid p \le j \le q-1\} & \text{for } d < p < q. \end{cases} \tag{4}$$

(Case 2): α_1 is odd and α_2 is even. Put $\alpha_1 = 2p+1$ and $\alpha_2 = 2q$. It follows from the relations $2d+1 \le \alpha_1 < \alpha_2 \le 2(p+d+1)$ that $d \le p < q \le p+d+1$. For $I \in \mathcal{I}(\alpha_1,\alpha_2)$, we have $\Gamma(I) = \{2i+1 \mid p \le i \le q-1\} \cup \{n \mid n \ge 2q\}$. For this case, the following four cases occur:

$$G(I) = \begin{cases} \{2i \mid 0 \le i \le d\} & \text{for } p = d,\ q = d+1, \\ \{2i \mid 0 \le i \le q-1\} & \text{for } p = d,\ q > d+1, \\ \{2i \mid 0 \le i \le d\} \cup [2d+1, 2p] & \text{for } p > d,\ q = p+1, \\ \{2i \mid 0 \le i \le d\} \cup [2d+1, 2p] \cup \{2j \mid p+1 \le j \le q-1\} & \text{for } p > d,\ q > p+1. \end{cases} \tag{5}$$

Our assertions follow from Lemma 19 with (4) and (5). □

Lemma 21. *If I belongs to $\mathcal{I}(2p, 2q+1)$ or $\mathcal{I}(2p+1, 2q)$, then its codimension r is given by*

$$r = p + q - d. \tag{6}$$

Proof. This relation follows from Lemma 6 with (4) and (5). □

The decomposition of $\mathcal{I}_r$ is determined in terms of generators of $\Delta_{r,l}$.

Proposition 22. *The sets $\mathcal{I}_r$ for $1 \le r \le 2d$ are decomposed as follows*:

(A): $1 \le r \le d$ *and* $r = 2s+1$. $\mathcal{I}_{2s+1} = \bigcup_{l=0}^{s} \mathcal{I}\left(r+2l+1, r+2(d-l)\right)$.

(B): $2 \le r \le d$ *and* $r = 2s$. $\mathcal{I}_{2s} = \bigcup_{l=0}^{s} \mathcal{I}\left(r+2l, r+2(d-l)+1\right)$.

For the cases where $d+1 \le r \le 2d-1$, the decompositions depend on d.

(C-i): $d+1 \le r \le 2d-1$, $r = 2s+1$ *and* $d = 2h$.

$$\mathcal{I}_{2s+1} = \left\{ \bigcup_{l=0}^{h-1} \mathcal{I}\left(r+2l+1, r+2(d-l)\right) \right\} \cup \left\{ \bigcup_{l=d-s}^{h} \mathcal{I}\left(r+2l, r+2(d-l)+1\right) \right\}.$$

(C-ii): $d+1 \le r \le 2d-1$, $r = 2s+1$ *and* $d = 2h+1$.

$$\mathcal{I}_{2s+1} = \left\{ \bigcup_{l=0}^{h} \mathcal{I}\left(r+2l+1, r+2(d-l)\right) \right\} \cup \left\{ \bigcup_{l=d-s}^{h} \mathcal{I}\left(r+2l, r+2(d-l)+1\right) \right\}.$$

(D-i): $d+1 \le r \le 2d$, $r = 2s$ *and* $d = 2h$.

$$\mathcal{I}_{2s} = \left\{ \bigcup_{l=0}^{h} \mathcal{I}\left(r+2l, r+2(d-l)+1\right) \right\} \cup \left\{ \bigcup_{l=d-s}^{h-1} \mathcal{I}\left(r+2l+1, r+2(d-l)\right) \right\}.$$

(D-ii): $d+1 \le r \le 2d$, $r = 2s$ *and* $d = 2h+1$.

$$\mathcal{I}_{2s} = \left\{ \bigcup_{l=0}^{h} \mathcal{I}\left(r+2l, r+2(d-l)+1\right) \right\} \cup \left\{ \bigcup_{l=d-s}^{h} \mathcal{I}\left(r+2l+1, r+2(d-l)\right) \right\}.$$

Proof. We infer from the relation (6) in Lemma 21 that

$$\alpha_2 = -\alpha_1 + 2r + 2d + 1. \tag{7}$$

We first consider the case where $1 \le r \le d$. In this case, the positive integer α_1 must be even. Indeed, if not, then we have $\sharp G(I) \ge d+1$ since $\alpha_1 \ge d+1$. So we conclude that $r \ge d+1$ by Lemma 6. It is a contradiction. It follows from the definitions of α_1 and α_2 that

$$\alpha_1 \le \alpha_2 \le \alpha_1 + 2d + 1. \tag{8}$$

Note that $\mathcal{I}(\alpha_1, \alpha_2)$ is a component of $\mathcal{I}_r$ if and only if α_1 and α_2 satisfy both of (7) and (8). According to r, the sets of all pairs (α_1, α_2) which satisfy (7) and (8) are determined as follows:
(A) $r = 2s+1$. $\{(r+2l+1, r+2(d-l))\}_{l=0,\dots,s}$
(B) $r = 2s$. $\{(r+2l, r+2(d-l)+1)\}_{l=0,\dots,s}$

Next we consider the case in which $d+1 \leq r \leq 2d$. According to r and d, we obtain the following sets of the pair (α_1, α_2)which satisfy (7) and (8):
(C-i) $r = 2s+1$ and $d = 2h$.

$$\{(r+2l+1, r+2(d-l))\}_{l=0,\dots,h-1} \cup \{(r+2l, r+2(d-l)+1)\}_{d-s,\dots,h}$$

(C-ii) $r = 2s+1$ and $d = 2h+1$.

$$\{(r+2l+1, r+2(d-l))\}_{l=0,\dots,h} \cup \{(r+2l, r+2(d-l)+1)\}_{d-s,\dots,h}$$

(D-i) $r = 2s$ and $d = 2h$.

$$\{(r+2l, r+2(d-l)+1)\}_{l=0,\dots,h} \cup \{(r+2l+1, r+2(d-l))\}_{l=d-s,\dots,h-1}$$

(D-ii) $r = 2s$ and $d = 2h+1$.

$$\{(r+2l, r+2(d-l)+1)\}_{l=0,\dots,h} \cup \{(r+2l+1, r+2(d-l))\}_{l=d-s,\dots,h}$$

Our assertions follow from Proposition 20 with the above datum. □

Let $\Delta_{r,l}$ be a Γ-semi-module in (3) and take an element I from $\mathcal{I}(\Delta_{r,l})$. For the $\mathcal{O}$-sub-module $\varphi_r(I)$, define the *set of orders* of $\varphi_r(I)$ by $\Gamma(\varphi_r(I)) := \{\mathrm{ord}_\tau(\overline{f}) | \, \overline{f} \in \varphi_r(I)\}$. It is clear that $\Gamma(\varphi_r(I))$ has a Γ-semi-module structure. Furthermore, if the Γ-semi-module $\Delta_{r,l}$ is generated by α_1 and α_2, then $\Gamma(\varphi_r(I))$ is generated by $\alpha_1 - r$ and $\alpha_2 - r$. So we write $\mathcal{M}(\alpha_1 - r, \alpha_2 - r)$ instead of $\mathcal{M}(\Delta_{r,l})$ for such case. For each r, the decomposition (3) of $\mathcal{M}_r$ follows from Proposition 6.

Corollary 23. *The punctual Hilbert schemes $\mathcal{M}_r$ $(1 \leq r \leq 2d)$ are decomposed as follows*:

(A): $1 \leq r \leq d$ *and* $r = 2s+1$. $\mathcal{M}_{2s+1} = \displaystyle\bigcup_{l=0}^{s} \mathcal{M}_{2s+1}(2l+1, 2(d-l))$.

(B): $2 \leq r \leq d$ *and* $r = 2s$. $\mathcal{M}_{2s} = \displaystyle\bigcup_{l=0}^{s} \mathcal{M}_{2s}(2l, 2(d-l)+1)$.

(C-i): $d+1 \leq r \leq 2d-1$, $r = 2s+1$ *and* $d = 2h$.

$$\mathcal{M}_{2s+1} = \left\{ \bigcup_{l=0}^{h-1} \mathcal{M}_{2s+1}(2l+1, 2(d-l)) \right\} \cup \left\{ \bigcup_{l=d-s}^{h} \mathcal{M}_{2s+1}(2l, 2(d-l)+1) \right\}.$$

(C-ii): $d+1 \leq r \leq 2d-1$, $r = 2s+1$ *and* $d = 2h+1$.

$$\mathcal{M}_{2s+1} = \left\{ \bigcup_{l=0}^{h} \mathcal{M}_{2s+1}(2l+1, 2(d-l)) \right\} \cup \left\{ \bigcup_{l=d-s}^{h} \mathcal{M}_{2s+1}(2l, 2(d-l)+1) \right\}.$$

(D-i): $d+1 \leq r \leq 2d$, $r = 2s$ *and* $d = 2h$.

$$\mathcal{M}_{2s} = \left\{ \bigcup_{l=0}^{h} \mathcal{M}_{2s}(2l, 2(d-l)+1) \right\} \cup \left\{ \bigcup_{l=d-s}^{h-1} \mathcal{M}_{2s}(2l+1, 2(d-l)) \right\}.$$

(D-ii): $d+1 \leq r \leq 2d$, $r = 2s$ *and* $d = 2h+1$.

$$\mathcal{M}_{2s} = \left\{ \bigcup_{l=0}^{h} \mathcal{M}_{2s}(2l, 2(d-l)+1) \right\} \cup \left\{ \bigcup_{l=d-s}^{h} \mathcal{M}_{2s}(2l+1, 2(d-l)) \right\}.$$

Remark 24. *The punctual Hilbert scheme $\mathcal{M}_1$ consists of one point which corresponds to the maximal ideal of $\mathcal{O}$.*

4. Proof of Main Theorems

In this section, we prove Theorem 1 and 2. To emphasis d, we use notations $\mathcal{M}_{d,r}$ and $\mathcal{M}_{d,r}(\alpha_1 - r, \alpha_2 - r)$ insted of $\mathcal{M}_r$ and $\mathcal{M}_r(\alpha_1 - r, \alpha_2 - r)$.

Lemma 25. *If the codimension r is odd (resp. even), then the component $\mathcal{M}_{d,r}(1, 2d)$ (resp. $\mathcal{M}_{d,r}(0, 2d+1)$) of the decomposition in Corollary 23 is the open dense subset of $\mathcal{M}_{d,r}$.*

Proof. We should check our assertion for each case in Corollary 23. However, we only consider the case (D-ii) in Corollary 23. The other cases are treated in the same way. For $r = 2s$, set

$$\Lambda_1(l) := \{\underbrace{d, \dots, d}_{2l}, \underbrace{d-1, d-2, \dots, 2l+1, 2l}_{d-2l}\} \quad \text{for } l = 0, \dots, h,$$
$$\Lambda_2(l) := \{\underbrace{d, \dots, d}_{2l+1}, \underbrace{d-1, d-2, \dots, 2d-r+l}_{d-2l-1}\} \quad \text{for } l = d-s, \dots, h.$$

By Lemma 18, we see that

$$\begin{aligned} \mathcal{M}_{d,2s}(2l, 2(d-l)+1) &= \mathcal{M}_{d,2s} \cap W_{\Lambda_1(l)} \quad \text{for } l = 0, \dots, h, \\ \mathcal{M}_{d,2s}(2l+1, 2(d-s)) &= \mathcal{M}_{d,2s} \cap W_{\Lambda_2(l)} \quad \text{for } l = d-s, \dots, h. \end{aligned} \tag{9}$$

The following inclusions also follows from Proposition 17:

$$W_{\Lambda_1(l+1)} \subset \overline{W_{\Lambda_1(l)}},\ W_{\Lambda_2(l+1)} \subset \overline{W_{\Lambda_2(l)}},\ W_{\Lambda_2(d-s)} \subset \overline{W_{\Lambda_1(0)}} \tag{10}$$

It follows from (9), (10) and (D-ii) in Corollary 23 that

$$\begin{aligned} \mathcal{M}_{d,2s} &= \left\{ \mathcal{M}_{d,2s} \cap \left(\bigcup_{l=0}^{h} W_{\Lambda_1(l)} \right) \right\} \cup \left\{ \mathcal{M}_{d,2s} \cap \left(\bigcup_{l=d-s}^{h} W_{\Lambda_2(l)} \right) \right\} \\ &\subset \mathcal{M}_{d,2s} \cap \left(\overline{W_{\Lambda_1(0)}} \cup \overline{W_{\Lambda_2(d-s)}} \right) = \mathcal{M}_{d,2s} \cap \overline{W_{\Lambda_1(0)}} \\ &= \overline{\mathcal{M}_{d,2s}(0, 2d+1)} \subset \mathcal{M}_{d,2s}. \end{aligned}$$

Hence we conclude that $\mathcal{M}_{d,2s} = \overline{\mathcal{M}_{d,2s}(0, 2d+1)}$. ∎

Next we prove Proposition 3.

Proof of Proposition 3. We first prove (i) by constructing an isomorphism between $\mathcal{M}_{d,2s}$ and $\mathcal{M}_{d,2s+1}$. We have the following combination of the decomposition types of $\mathcal{M}_{d,2s}$ and $\mathcal{M}_{d,2s+1}$.

Decomposition type of $\mathcal{M}_{d,2s}$	(B)	(B)	(D-i)	(D-ii)
Decomposition type of $\mathcal{M}_{d,2s+1}$	(A)	(C-i)	(C-i)	(C-ii)

We referred to Corollary 23 for the decomposition types. We only prove our assertion for the pair (D-ii) and (C-ii) here. The other cases can be treated in the same way. Since d is odd in this case, we put $d = 2h+1$. It follows from Corollary 23 that

$$\mathcal{M}_{d,2s} = \left\{ \bigcup_{l=0}^{h} \mathcal{M}_{d,2s}\left(2l, 2(d-l)+1\right) \right\} \cup \left\{ \bigcup_{l=d-s}^{h} \mathcal{M}_{d,2s}\left(2l+1, 2(d-l)\right) \right\},$$

$$\mathcal{M}_{d,2s+1} = \left\{ \bigcup_{l=0}^{h} \mathcal{M}_{d,2s+1}\left(2l+1, 2(d-l)\right) \right\} \cup \left\{ \bigcup_{l=d-s}^{h} \mathcal{M}_{d,2s+1}\left(2l, 2(d-l)+1\right) \right\}.$$

By Propositions 20, 22, Corollary 23 and the definition of $\mathcal{M}_{d,r}$, we obtain the following explicit descriptions of the components of $\mathcal{M}_{d,2s}$ and $\mathcal{M}_{d,2s+1}$:

$$\mathcal{M}_{d,2s}(2l,2(d-l)+1)=$$
$$\begin{cases}\left\{\left(\tau^{2l}+\sum_{j=d-s}^{d-l-1} b_j\tau^{2j+1},\tau^{2(d-l)+1}\right)\middle| b_j\in k\right\} \text{ for } l=0\ldots,d-s,\\ \left\{\left(\tau^{2l}+\sum_{j=l}^{d-l-1} b_j\tau^{2j+1},\tau^{2(d-l)+1}\right)\middle| b_j\in k\right\} \text{ for } l=d+1-s\ldots,h,\end{cases}$$

$$\mathcal{M}_{d,2s}(2l+1,2(d-l))=$$
$$\begin{cases}\left\{\left(\tau^{2l+1}+\sum_{j=l+1}^{d-l-1} b_j\tau^{2j},\tau^{2(d-l)}\right)\middle| b_j\in k\right\} \text{ for } l=d-s\ldots,h-1,\\ \left\{(\tau^{d},\tau^{d+1})\right\} \text{ for } l=h,\end{cases}$$

$$\mathcal{M}_{d,2s+1}(2l+1,2(d-l))=$$
$$\begin{cases}\left\{\left(\tau^{2l+1}+\sum_{j=d-s}^{d-l-1} b_j\tau^{2j},\tau^{2(d-l)}\right)\middle| b_j\in k\right\} \text{ for } l=0,\ldots,d-s-1,\\ \left\{\left(\tau^{2l+1}+\sum_{j=l+1}^{d-l-1} b_j\tau^{2j},\tau^{2(d-l)}\right)\middle| b_j\in k\right\} \text{ for } l=d-s\ldots,h-1,\\ \left\{(\tau^{d},\tau^{d+1})\right\} \text{ for } l=h,\end{cases}$$

$$\mathcal{M}_{d,2s+1}(2l,2(d-l)+1)=$$
$$\left\{\left(\tau^{2l}+\sum_{j=l}^{d-l-1} b_j\tau^{2j+1},\tau^{2(d-l)+1}\right)\middle| b_j\in k\right\} \text{ for } l=d-s\ldots,h$$

Both of $\mathcal{M}_{d,2s}$ and $\mathcal{M}_{d,2s+1}$ have $s+1$ components. Furthermore, the numbers of coefficients involved in the elements of their components are $0,1,\ldots,s-1,s$. So, for two components of $\mathcal{M}_{d,2s}$ and $\mathcal{M}_{d,2s+1}$ which have same number of coefficients, we can define a bijection by sending an element of $\mathcal{M}_{d,2s}$ to that of $\mathcal{M}_{d,2s+1}$ which has same coefficients. In this way, we obtain $s+1$ bijections between the components of $\mathcal{M}_{d,2s}$ and $\mathcal{M}_{d,2s+1}$. It is clear that the union of them is an isomorphism from $\mathcal{M}_{d,2s}$ to $\mathcal{M}_{d,2s+1}$.

Next we prove (ii). If $r=1$, then we have $\mathcal{M}_{d,1}=\{\text{one point}\}$ for any $d\in\mathbb{N}$, as mentioned in Remark 24. So we consider the case where $r\geq 2$. For any $d,d'\in\mathbb{N}$, we can construct an isomorphism between $\mathcal{M}_{d,r}$ and $\mathcal{M}_{d',r}$ by the same argument in the proof of (i). □

The following fact is known:

Theorem 26 ([3])**.** *We have* $\dim(\mathcal{M}_{d,2d})=d$ *for any* $d\in\mathbb{N}$.

Proof of Theorem 1. The rationality of $\mathcal{M}_{d,r}$ is an immediate consequence of Lemma 25 and Proposition 16. The relation $\mathcal{M}_{d,r}\cong\mathcal{M}_{s,2s}$ also follows from (i) and (ii) in Proposition 3. Hence, we obtain $\dim\mathcal{M}_{d,r}=\dim\mathcal{M}_{s,2s}=s$ by Theorem 26. □

Next we prove Theorem 2.

Proof of Theorem 2. The statement (i) already was mentioned in Remark 24. For (ii), the relation $\mathcal{M}_{2,2} \cong \mathcal{M}_{2,3} \cong \mathbb{P}^1$ proved in [3]. Hence it follows from (ii) in Proposition 3 that $\mathcal{M}_{d,2} \cong \mathcal{M}_{d,3} \cong \mathbb{P}^1$ for any d. The statement (ii) is proved. Since it was shown that

$$\mathrm{Sing}(\mathcal{M}_{2,4}) = \mathcal{M}_{2,2} \cap \mathcal{M}_{2,3} = \mathcal{M}_{2,1} = \{\text{one point}\}$$

in [3], we only have to consider the cases where $d \geq 3$ to prove (iii). We may assume that r is even by (i) in Proposition 3. Moreover, since $\mathcal{M}_{d,r}$ is isomorphic to $\mathcal{M}_{s,2s}$ by Theorem 1, it is enough to prove our assertion for some PS variety $\mathcal{M}_{d,2d}$ ($d \in \mathbb{N}$). We divide the rest of the proof of (iii) into the following three cases:

$$\text{Case}\,1 : d = 3, \quad \text{Case}\,2 : d \text{ is even and } d \geq 4, \quad \text{Case}\,3 : d \text{ is odd and } d \geq 5$$

Here we only consider Case 3. The other cases can be treated in a similar manner. Put $d = 2h+1$. By using Propositions 20, 22, Corollary 23 and the definition of $\mathcal{M}_{d,r}$, we have

$$\begin{aligned}
\mathcal{M}_{d,2d-3} &= \left\{ \bigcup_{l=0}^{h} \mathcal{M}_{d,2d-3}\,(2l+1, 2(d-l)) \right\} \cup \left\{ \bigcup_{l=2}^{h} \mathcal{M}_{d,2d-3}\,(2l, 2(d-l)+1) \right\} \\
\mathcal{M}_{d,2d-2} &= \left\{ \bigcup_{l=0}^{h} \mathcal{M}_{d,2d-2}\,(2l, 2(d-l)+1) \right\} \cup \left\{ \bigcup_{l=1}^{h} \mathcal{M}_{d,2d-2}\,(2l+1, 2(d-l)) \right\} \\
\mathcal{M}_{d,2d-1} &= \left\{ \bigcup_{l=0}^{h} \mathcal{M}_{d,2d-1}\,(2l+1, 2(d-l)) \right\} \cup \left\{ \bigcup_{l=1}^{h} \mathcal{M}_{d,2d-1}\,(2l, 2(d-l)+1) \right\} \\
\mathcal{M}_{d,2d} &= \left\{ \bigcup_{l=0}^{h} \mathcal{M}_{d,2d}\,(2l, 2(d-l)+1) \right\} \cup \left\{ \bigcup_{l=0}^{h} \mathcal{M}_{d,2d}\,(2l+1, 2(d-l)) \right\}
\end{aligned}$$

where

$$\mathcal{M}_{d,2d-3}(2l+1, 2(d-l)) = \begin{cases} \left\{ \left(\tau + \sum_{j=2}^{d-1} b_j \tau^{2j}, \tau^{2d} \right) \middle| b_j \in k \right\} \text{ for } l = 0, \\ \left\{ \left(\tau^{2l+1} + \sum_{j=l+1}^{d-l-1} b_j \tau^{2j}, \tau^{2(d-l)} \right) \middle| b_j \in k \right\} \text{ for } l = 1 \ldots, h-1, \\ \left\{ (\tau^d, \tau^{d+1}) \right\} \text{ for } l = h, \end{cases}$$

$$\mathcal{M}_{d,2d-3}(2l, 2(d-l)+1) = \left\{ \left(\tau^{2l} + \sum_{j=l}^{d-l-1} b_j \tau^{2j+1}, \tau^{2(d-l)+1} \right) \middle| b_j \in k \right\} \text{ for } l = 2 \ldots, h,$$

$$\mathcal{M}_{d,2d-2}(2l, 2(d-l)+1) = \begin{cases} \left\{ \left(1 + \sum_{j=1}^{d-1} b_j \tau^{2j+1}, \tau^{2d+1} \right) \middle| b_j \in k \right\} & \text{for } l = 0, \\ \left\{ \left(\tau^{2l} + \sum_{j=l}^{d-l-1} b_j \tau^{2j+1}, \tau^{2(d-l)+1} \right) \middle| b_j \in k \right\} & \text{for } l = 1 \dots, h, \end{cases}$$

$$\mathcal{M}_{d,2d-2}(2l+1, 2(d-l)) = \begin{cases} \left\{ \left(\tau^{2l+1} + \sum_{j=l+1}^{d-l-1} b_j \tau^{2j}, \tau^{2(d-l)} \right) \middle| b_j \in k \right\} & \text{for } l = 1 \dots, h-1, \\ \left\{ (\tau^d, \tau^{d+1}) \right\} \ \text{for } l = h, \end{cases}$$

$$\mathcal{M}_{d,2d-1}(2l+1, 2(d-l)) = \begin{cases} \left\{ \left(\tau^{2l+1} + \sum_{j=l+1}^{d-l-1} b_j \tau^{2j}, \tau^{2(d-l)} \right) \middle| b_j \in k \right\} & \text{for } l = 0 \dots, h-1, \\ \left\{ (\tau^d, \tau^{d+1}) \right\} \ \text{for } l = h, \end{cases}$$

$$\mathcal{M}_{d,2d-1}(2l, 2(d-l)+1) = \left\{ \left(\tau^{2l} + \sum_{j=l}^{d-l-1} b_j \tau^{2j+1}, \tau^{2(d-l)+1} \right) \middle| b_j \in k \right\} \text{ for } l = 1 \dots, h,$$

$$\mathcal{M}_{d,2d}(2l, 2(d-l)+1) = \left\{ \left(\tau^{2l} + \sum_{j=l}^{d-l-1} b_j \tau^{2j+1}, \tau^{2(d-l)+1} \right) \middle| b_j \in k \right\} \text{ for } l = 0 \dots, h,$$

$$\mathcal{M}_{d,2d}(2l+1, 2(d-l)) = \begin{cases} \left\{ \left(\tau^{2l+1} + \sum_{j=l+1}^{d-l-1} b_j \tau^{2j}, \tau^{2(d-l)} \right) \middle| b_j \in k \right\} & \text{for } l = 0 \dots, h-1, \\ \left\{ (\tau^d, \tau^{d+1}) \right\} \ \text{for } l = h \end{cases}$$

It follows from the above descriptions of components that

$$\mathcal{M}_{d,2d} = (\mathcal{M}_{d,2d-2} \cap \mathcal{M}_{d,2d-1}) \cup \mathcal{M}_{d,2d}(1, 2d) \cup \mathcal{M}_{d,2d}(0, 2d+1)$$

where

$$(\mathcal{M}_{d,2d-2} \cap \mathcal{M}_{d,2d-1}) \cap \mathcal{M}_{d,2d}(1, 2d) = \emptyset,$$
$$(\mathcal{M}_{d,2d-2} \cap \mathcal{M}_{d,2d-1}) \cap \mathcal{M}_{d,2d}(0, 2d+1) = \emptyset,$$
$$\mathcal{M}_{d,2d}(1, 2d) \cap \mathcal{M}_{d,2d}(0, 2d+1) = \emptyset.$$

Furthermore, $\mathcal{M}_{d,2d}(1, 2d)$ and $\mathcal{M}_{d,2d}(0, 2d+1)$ are affine spaces by Corollary 14, we must have $\mathrm{Sing}(\mathcal{M}_{d,2d}) = \mathrm{Sing}(\mathcal{M}_{d,2d-2} \cap \mathcal{M}_{d,2d-1})$. Since both of

$$\mathcal{M}_{d,2d-2} \setminus (\mathcal{M}_{d,2d-2} \cap \mathcal{M}_{d,2d-1}) = \mathcal{M}_{d,2d-2}(0, 2d+1),$$
$$\mathcal{M}_{d,2d-1} \setminus (\mathcal{M}_{d,2d-2} \cap \mathcal{M}_{d,2d-1}) = \mathcal{M}_{d,2d-1}(1, 2d)$$

are affine spaces again, we conclude that $\mathrm{Sing}(\mathcal{M}_{d,2d}) = \mathcal{M}_{d,2d-2} \cap \mathcal{M}_{d,2d-1}$. Finally, the relation $\mathcal{M}_{d,2d-2} \cap \mathcal{M}_{d,2d-1} \cong \mathcal{M}_{d,2d-3}$ can be shown by constructing an isomorphism. For the construction, refer to the proof of (i) in Proposition 3. □

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FAMILIES OF DISTRIBUTIONS AND PFAFF SYSTEMS UNDER DUALITY

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Abstract. A singular distribution on a non-singular variety X can be defined either by a subsheaf $D \subseteq TX$ of the tangent sheaf, or by the zeros of a subsheaf $D^0 \subseteq \Omega^1_X$ of 1-forms, that is, a Pfaff system. Although both definitions are equivalent under mild conditions on D, they give rise, in general, to non-equivalent notions of flat families of distributions. In this work we investigate conditions under which both notions of flat families are equivalent. In the last sections we focus on the case where the distribution is *integrable*, and we use our results to generalize a theorem of Cukierman and Pereira.

1. Introduction

Among the several motivations for studying moduli problems in geometry, there is the study of the topological and geometrical properties of differential equations on manifolds. Such problems are studied in the works of R. Thom, V. Arnold, and Kodaira-Spencer in the 1950's and were an influence in the work of Zariski about deformation of singularities; we refer to the introduction and Section 3.10 of [7] for a historical account on the subject. Algebraic geometric techniques were used in the study of (integrable) differential equations on non-singular algebraic varieties at least since Jouanolou [13], moduli and deformation problems of integrable differential equations were studied by Gomez-Mont in [9]. Since then, much has been done in the study of the geometry of integrable 1-forms on algebraic varieties, especially describing singularities of the differential equations they define (see, for instance [15]). Also, much has been done in the determination of irreducible components of the moduli space of integrable 1-forms (e.g.: [5, 6, 14] and references therein).

Many of these works were carried out in a 'naïve' way, without concern for representability of functors or formalization of deformation problems. When trying to formalize the moduli problem for integrable differential equations in non-singular algebraic varieties, an issue appears at the very beginning: although its equivalent to describe a distribution by its tangent sheaf or by the sheaf of 1-forms that vanishes on it, these two descriptions give rise to inequivalent notions of a flat family of distributions. Thus we are led to two different moduli problems, one for (involutive) sub-sheaves of the tangent sheaf, and the other for (integrable) sub-sheaves of the sheaf of 1-forms. In the present work, we prove in Proposition 6.3 and Proposition 6.5 that, under suitable assumptions, these two moduli problems are representable and, in Corollary 7.15, that their irreducible components are moreover birationally equivalent. Using this we can generalize previous theorems in the literature such as Theorem 1 of [5] and the theorems of [16, 17].

In order to better explain our results, we now introduce some notation and definitions. Let X be a non-singular projective algebraic variety. A non-singular k-dimensional distribution D on a non-singular variety X consists of a k-dimensional subspace $D_x \subseteq T_xX$ of the tangent space of X at x varying continuously with x. This notion can be formalized by saying that a k-dimensional

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distribution is a rank k sub-bundle of the tangent bundle $D \hookrightarrow TX$, and taking $D_x \subseteq T_xX$ to be $D \otimes k(x)$. Equivalently, we can say that a k-dimensional distribution is determined by a sub-bundle $I_D \hookrightarrow \Omega^1_X$, and take D_x to be $\{v \in T_xX \mid \omega(v) = 0, \forall \omega \in I_D \otimes k(x)\}$.

When X is an algebraic variety, it is often the case that there are no algebraic sub-bundles of TX or Ω^1_X of a given rank, i.e., there are no *non-singular* distributions. Nevertheless, the definition can be readily generalized to allow $D \subseteq TX$ and $I_D \subseteq \Omega^1_X$ to be subsheaves. In this way, we describe distributions D which are k-dimensional on a dense open subset, but may present singularities along proper subvarieties. Again, a singular distribution is equivalently defined either by a subsheaf $D \subseteq TX$ or by its annihilator $I_D \subseteq \Omega^1_X$. However, as was already observed by Pourcin in [16], when studying *families* of distributions parametrized by a base scheme S, it may happen that while a family of distributions $D \subseteq T_S(X \times S)$ is flat (in the sense that the quotient $T_S(X \times S)/D$ is a flat sheaf over S), its annihilator $I_D \subseteq \Omega^1_{X|S}$ may not be flat. This gives us two different notions of flat family of (singular) distributions, and therefore two different moduli problems for them.

In this work, we prove that the above two notions of flatness coincide as long as the singular set of the distribution (endowed with a convenient scheme structure) is also flat over S (Proposition 7.6 and Theorem 7.14). We focus on the case of *integrable* distributions; in Proposition 6.3, we give constructions for the moduli space Inv^X of involutive (in the sense of the Frobenius theorem) subsheaves of TX and, in Proposition 6.5, for the moduli space iPf^X of integrable subsheaf of Ω^1_X and conclude in Corollary 7.15 that taking annihilators defines a rational map between Inv^X and iPf^X that is a birational equivalence in each irreducible component of Inv^X. Moreover we give in Theorem 8.13 a sufficient criterion in terms of the singularity of the foliation to know when an involutive subsheaf $T\mathcal{F} \subseteq TX$ represents a point in the dense open set $U \subseteq \mathrm{Inv}^X$ where taking duals gives an isomorphism with an open set $V \subseteq \mathrm{iPf}^X$. Using this criterion, we can generalize the main theorem of [5] by specializing our results to the case where $X = \mathbb{P}^n$ and $T\mathcal{F} \cong \bigoplus_i \mathcal{O}_{\mathbb{P}^n}(d_i)$.

In Section 2, we show a particular example of a flat family of Pfaff systems being dual to a non-flat family of distributions.

In Section 3, we treat some preliminary general notions on sheaves and criterion for flatness that will be useful later.

In Section 4, we study the effect of applying the functor $\mathcal{H}om(-, \mathcal{O}_X)$ to a short exact sequence of sheaves. We also include in this section some observations on exterior powers, which are relevant for the study of distributions of codimension higher than 1.

Section 5 consists mainly of definitions of families of distributions and Pfaff systems, and related notions.

In Section 6, the construction of the moduli spaces of involutive distributions and integrable Pfaff systems is given as subschemes of certain Quot schemes.

In Section 7, the main results of the paper are proved. First the singular scheme of a family of distributions is defined, as well as the analogous notion for family of Pfaff systems and it is proved that the singular scheme of a family of distributions is the same as the singular scheme of its dual family (which is a family of Pfaff systems). In Section 7.1, the codimension 1 version of the main result is proved: if the singular scheme of a flat family of codimension 1 Pfaff systems is itself flat, then the dual family is flat as well. In Section 7.2, an analogous statement is proved for arbitrary codimension. In this case, however, flatness of the singular scheme is not enough to assure flatness of the dual family. To obtain a valid criterion, we define a stratification of the singular scheme, if each stratum is flat over the base, we can assure the dual family will be flat as well.

In Section 8, we give a sufficient condition to know when the singular scheme of a family of codimension 1 foliations is flat over the base. This condition is related to two of the better-studied types of singularities of foliations, the Kupka singularities and the Reeb singularities. We prove that, if the singularities of a foliation given by a distribution $T\mathcal{F} \hookrightarrow TX$ are only of these two types, then every flat family

$$0 \to T\mathcal{F}_S \to T_S(X \times S) \to N_{\mathcal{F}} \to 0,$$

such that there is an $s \in S$ with $T\mathcal{F}_s = T\mathcal{F}$, is such that $\operatorname{sing}(\mathcal{F})$ is flat in a neighborhood of s.

In Section 9, we apply the theorem of Section 8 to recover the main result of [5] as a special case of Theorem 7.8, where $X = \mathbb{P}^n$ and $T\mathcal{F}$ splits as direct sum of line bundles.

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2. An example

Set $X = \mathbb{C}^4$, $S = \mathbb{C}^2$ and consider the family of 1-forms on X

$$\omega_{(t,s)} = xdx + ydy + t \cdot zdz + s \cdot wdw,$$

parametrized by $(t, s) \in S = \mathbb{C}^2$, and the related family

$$I = I_{(t,s)} = \mathcal{O}_{X\times S} \cdot (\omega_{(t,s)}) \subseteq \Omega^1_{X\times S|S}$$

of Pfaff systems on $X = \mathbb{C}^4$.

It is easy to see that the sheaf I is flat over $S = \mathbb{C}^2$. On the other hand, if we look at the family of distributions this Pfaff systems define, we will find that it is not flat. Indeed, let D denote the annihilator of I:

$$D = \operatorname{Ann}(I) \subseteq T(X \times S).$$

To see that D is *not* flat over S, we can use Artin's criterion for flatness [2, Corollary to Proposition 3.9].

Proposition 2.1 (Artin's criterion for flatness.)**.** *Let $(A, \mathscr{M})$ and $(B, \mathscr{N})$ be local rings, $B \to A$ a flat morphism of local rings, and M a finitely generated A-module. Suppose we have generators of M, $M = (f_1, \ldots, f_r)$. Let $\overline{f_i}$ be the class of f_i on $M \otimes_B B/\mathscr{N}$. Then M is flat over B if and only if every relation among $(\overline{f_1}, \ldots, \overline{f_r})$ lifts to a relation among $(f_1, \ldots, f_r)$.*

Note that, while this is not exactly the same context in which Artin states the proposition, his proof extends mutatis mutandi to this setting.

Now observe that $D = \operatorname{Ann}(I)$ has 6 generators:

$$f_1 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \quad f_2 = tz\tfrac{\partial}{\partial x} - x\tfrac{\partial}{\partial z}, \quad f_3 = sw\frac{\partial}{\partial x} - x\frac{\partial}{\partial w},$$
$$f_4 = tz\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad f_5 = sw\tfrac{\partial}{\partial y} - y\tfrac{\partial}{\partial w}, \quad f_6 = sw\frac{\partial}{\partial z} - tz\frac{\partial}{\partial w}.$$

These generators have, in turn, numerous relations between them, too many to list completely here. We note, however, that all relations are generated by relations of the form

$$af_i + bf_j + cf_k = 0,$$

where a, b and c are in the ideal $(x, y, tz, sw) \subseteq \mathcal{O}_{X\times S}$.

We apply Artin's criterion with $A = \mathcal{O}_{X\times S,(0,0)}$, $B = \mathcal{O}_{S,0}$ and $M = D_{(0,0)}$. In this case, the $\overline{f_i}$'s consist of evaluating the f_i on $(x, y, z, w, 0, 0) \in \mathbb{C}^4 \times (0, 0)$. We get in this way a particular relation on $M \otimes_B B/\mathscr{N}$, namely $\overline{f_6} = 0$. It is then easy to see that this relation does not lift to a relation on $(f_1, \ldots, f_6)$.

This example shows that there is, in general, no morphism between the moduli spaces $\mathrm{Quot}(X, TX)$ and $\mathrm{Quot}(X, \Omega^1_X)$ induced by taking annihilators. We will show later that taking annihilators does define a *birational* equivalence between irreducible components of $\mathrm{Quot}_X(TX)$ and those of $\mathrm{Quot}_X(\Omega^1_X)$.

The example also hints that a key aspect to understand whether flatness is preserved or not under duality is to look at how the singular set varies. Indeed, we can see that D is flat over $S \setminus \{(0, 0)\}$. For each point $(t, s) \in S \setminus \{(0, 0)\}$, the singular set of $I|_{(t,s)}$ (i.e., the set of points in $p \in X$ such that $\omega_p = 0$) is just the origin of $\mathbb{C}^4 = X$. On the other hand, the singular set of $I|_{(0,0)}$ is the plane $\{(0, 0, z, w) : z, w \in \mathbb{C}\} \subset \mathbb{C}^4$. So D ceases to be flat exactly when the singular set of I ceases to be flat as well. This illustrates the typical behavior of singular codimension 1 Pfaff systems and the distributions they determine. In the arbitrary codimension case, the situation is a bit more subtle, but the singular set still plays a decisive role. Of course, to make sense out of this, we have to define a scheme structure on the singular set of a Pfaff system. This will be done in the course of the present work.

3. Preliminaries

Here we gather known facts of algebraic geometry that will be used later. We include proofs of some of these facts for lack (to the author's best knowledge) of a better reference.

3.1. **Reflexive sheaves and Serre's property S_2.** Property S_2 can be viewed as an algebraic analog of Hartog's theorem on complex holomorphic functions. For this reason, it will be extremely useful to us, for it will allow us to conclude global statements on sheaves that hold, a priori, for the restriction of these sheaves to (suitably large) open sets. Here we recall some known facts about sheaves with the S_2 property, and sheaves with the *relative* S_2 property as defined in [3].

Definition 3.1. Let $p : X \to T$ be a morphism of schemes and, for each point $x \in X$, let $d_T(x)$ equal the codimension of x in its fiber over T. We say that a sheaf $\mathscr{F}$ on X satisfies the *relative Serre condition S_k* with respect to p if and only if

$$\mathrm{depth}\mathscr{F}_x \geq \min(k, d_T(x)),$$

for all $x \in X$.

The proof of the next proposition works exactly as in the non-relative version.

Proposition 3.2. *Let $p : X \to T$ be a morphism of noetherian schemes and $\mathscr{F}$ a torsion-free coherent sheaf with relative property S_2 with respect to p. Let $Y \subset X$ be a closed subset such that $d_T(Y) \geq 2$. Then the restriction map $\rho : \Gamma(X, \mathscr{F}) \to \Gamma(X \setminus Y, \mathscr{F})$ is an isomorphism.*

Corollary 3.3. *Let $p : X \to T$ be a morphism of noetherian schemes and $\mathscr{F}$ a torsion-free coherent sheaf with property S_2 with respect to p. Let $Y \subset X$ be a closed subset such that $d_T(Y) \geq 2$. Let $U = X \setminus Y$ and let $j : U \to X$ be the inclusion. Then $\mathscr{F} = j_*(\mathscr{F}|_U)$.*

This corollary motivates the following definition, which is the relative analogue of a notion due to Grothendieck [10, 5.10].

Definition 3.4. Let $p : X \to T$ be a morphism of noetherian schemes and $\mathscr{F}$ a coherent sheaf. If, for each closed subset $Y \subset X$ such that $d_T(Y) \geq 2$, with $U = X \setminus Y$ and $j : U \to X$ the inclusion, the natural map

$$\rho_U : \mathscr{F} \to j_*(\mathscr{F}|_U)$$

is an epimorphism, then we say $\mathscr{F}$ is $Z^{(2)}$*-closed relative to* p; if ρ_U is an isomorphism, we say that it is $Z^{(2)}$*-pure relative to* p.

Proposition 3.5 ([12, Proposition 1.7])**.** *Let X be a quasi-projective integral scheme. A coherent sheaf $\mathscr{F}$ is reflexive if and only if it can be included in an exact sequence*

$$0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{G} \to 0,$$

where $\mathscr{E}$ is locally free and $\mathscr{G}$ is torsion-free.

Corollary 3.6. *Under the above circumstances, the dual of a coherent sheaf is always reflexive.*

Proposition 3.7 (c.f.:[12, Theorem 1.9])**.** *Let $p : X \to T$ be a morphism of noetherian schemes with normal integral fibers, and $\mathscr{F}$ a coherent sheaf on X. Then, if $\mathscr{F}$ is reflexive, it has relative property S_2 with respect to p.*

Proof. The statement being local, we can assume X is quasi-projective. Given a reflexive sheaf $\mathscr{F}$, we take an exact sequence

$$0 \to \mathscr{F} \to \mathscr{L} \to \mathscr{G} \to 0,$$

with $\mathscr{L}$ locally free and $\mathscr{G}$ torsion-free. Since p have normal fibers, $\mathcal{O}_X$ satisfies relative S_2 with respect to p, and so does $\mathscr{L}$, being locally free. Let $x \in X$ be a point of relative dimension ≥ 2 with respect to its fiber $X_{p(x)}$. Then $\operatorname{depth}\mathscr{L}_x \geq 2$ by S_2 and, as $\mathscr{G}$ is torsion-free, $\operatorname{depth}\mathscr{G}_x \geq 1$. This in turn implies $\operatorname{depth}\mathscr{F}_x \geq 2$. $\square$

3.2. **Support of a sheaf, zeros of a section.** Recall that, given a quasi-coherent sheaf $\mathscr{F}$ on a scheme X, we define the *support* of $\mathscr{F}$, $\operatorname{supp}(\mathscr{F})$ as the closed sub-scheme defined by the ideal sheaf given locally by

$$\mathcal{I}(\mathscr{F})_x := \operatorname{Ann}(\mathscr{F}_x) \subset \mathcal{O}_{X,x}.$$

We have the following useful characterization of the support of a sheaf in terms of a universal property:

Proposition 3.8. *The support of a sheaf $\mathscr{F}$ represents the functor*

$$\begin{array}{rcl} \mathrm{S}_{\mathscr{F}} : Sch & \longrightarrow & Sets \\ T & \mapsto & \{f \in \hom(T, X) \colon \operatorname{Ann}(f^*\mathscr{F}) = 0 \subset \mathcal{O}_T\}. \end{array}$$

Proof. A morphism $f : T \to X$ factors through $\operatorname{supp}(\mathscr{F})$ if and only if the map

$$f^\sharp : f^{-1}\mathcal{O}_X \to \mathcal{O}_T$$

factors through $f^{-1}(\mathcal{O}_X/\operatorname{Ann}(\mathscr{F}))$. But this happens if and only if $f^{-1}(\operatorname{Ann}(\mathscr{F})) = 0$.

On the other hand, we have the equality

$$\operatorname{Ann}(f^*\mathscr{F}) = \mathcal{O}_T \cdot f^{-1}(\operatorname{Ann}(\mathscr{F}));$$

indeed we may check this in every localization at any point $p \in T$, so if $t \in \operatorname{Ann}(f^*\mathscr{F})_p$ in particular t annihilates every element of the form $m \otimes 1 \in \mathscr{F}_x$, so $t = f^{-1}(x)t'$ where $x \in \operatorname{Ann}(\mathscr{F})_{f(p)}$.

So $\operatorname{Ann}(f^*\mathscr{F}) = 0$ if and only if $f^{-1}(\operatorname{Ann}(\mathscr{F})) = 0$ and we are done. $\square$

In other words, we just proved that $\mathrm{supp}(\mathscr{F})$ is the universal scheme with the property that $f^*\mathscr{F}$ is not a torsion module. This simple observation will be very useful when discussing the scheme structure on the singular set of a foliation.

A special case of support of a sheaf is the scheme-theoretic image of a morphism. Remember that the *scheme-theoretic image* of a morphism $f : X \to Y$ is the sub-scheme $\mathrm{supp}(f_*\mathcal{O}_X) \subseteq Y$.

Now we turn our attention to sections and their zeros. So let X be a scheme and $\mathscr{E}$ a locally free sheaf. Having a global section $s \in \Gamma(X, \mathscr{E})$ is the same as having a morphism (that, by abuse of notation, we also call s)

$$s : \mathcal{O}_X \longrightarrow \mathscr{E}.$$

Now, $s : \mathcal{O}_X \to \mathscr{E}$ defines a dual morphism

$$s^\vee : \mathscr{E}^\vee \longrightarrow \mathcal{O}_X^\vee = \mathcal{O}_X.$$

Definition 3.9. We define the *zero scheme* $Z(s)$ of the section s as the closed sub-scheme of X defined by the ideal sheaf $\mathrm{Im}(s^\vee) \subseteq \mathcal{O}_X$.

We'll apply this definition in the well-behaved situation where $\mathcal{O}_X$ (and therefore $\mathscr{E}$) is torsion-free.

Proposition 3.10. *Let $\mathscr{E}$ be a locally free sheaf on X and $s \in \Gamma(X, \mathscr{E})$ a global section. The scheme $Z(s)$ represents the functor*

$$\begin{array}{rcl} \mathrm{Z}_s : Sch & \longrightarrow & Sets \\ T & \mapsto & \{f \in \hom(T, X) \colon s \otimes 1 = 0 \in \Gamma(T, f^*\mathscr{E})\}. \end{array}$$

Proof. A morphism $f : T \to X$ factors through $Z(s)$ if and only if the map

$$(\mathscr{E}^\vee) \otimes \mathcal{O}_T \xrightarrow{s^\vee \otimes 1} \mathcal{O}_T$$

is identically 0. Beign locally free, we have

$$\mathscr{E}^\vee \otimes \mathcal{O}_T = \mathscr{H}om(\mathscr{E}, \mathcal{O}_X) \otimes \mathcal{O}_T \cong \mathscr{H}om(\mathscr{E} \otimes \mathcal{O}_T, \mathcal{O}_T).$$

So then we have

$$(f^*\mathscr{E})^\vee \xrightarrow{s^\vee \otimes 1} \mathcal{O}_T$$

is identically 0, as $f^*\mathscr{E}$ is locally free over T; this means $s \otimes 1 = 0 \in \Gamma(T, f^*\mathscr{E})$. □

3.3. **A criterion for flatness.** For lack of a better reference, we provide here a criterion that will become handy when dealing with both reduced and non-reduced base schemes over an algebraically closed field.

Lemma 3.11. *Let A be a ring of finite type over an algebraically closed field k, $\mathcal{M}$ a maximal ideal in A, and $f \in \mathcal{M}^n \setminus \mathcal{M}^{n+1}$. Then there is a morphism $\psi : A \to k[T]/(T^{n+1})$ such that $\psi^{-1}((T)) = \mathcal{M}$ and $\psi(f) \neq 0$.*

Proof. Fix a presentation $A \cong k[y_1, \ldots, y_r]/I$. By the Nullstelensatz, we can assume that $\mathcal{M} = (x_1, \ldots, x_r)$, where x_i is the class of $y_i \mathrm{mod} I$. Write the class of f in $\mathcal{M}^n/\mathcal{M}^{n+1}$ as

$$\overline{f} = \sum_{|\alpha|=n} a_\alpha \overline{x}^\alpha \in \mathcal{M}^n/\mathcal{M}^{n+1},$$

where $\alpha = (\alpha_1, \ldots, \alpha_r)$ and $\overline{x}^\alpha = (\overline{x_1}^{\alpha_1}, \ldots, \overline{x_r}^{\alpha_r})$.

As $f \notin \mathcal{M}^{n+1}$, the polynomial $q(y_1, \ldots, y_r) := \sum_{|\alpha|=n} a_\alpha y^\alpha$ is not in I. Now, k being algebraically closed, there is an r-tuple $(\lambda_1, \ldots, \lambda_r) \in k^r$ such that $p(\lambda_1, ..., \lambda_r) = 0$ for every $p \in I$ and $q(\lambda_1, ..., \lambda_r) \neq 0$.

Finally we can define $\psi : A \to k[T]/(T^{n+1})$ as follows:

$$\psi(x_i) = \lambda_i T.$$

The morphism is well-defined because $p(\lambda_1, ..., \lambda_r) = 0$ for every $p \in I$, moreover $\psi^{-1}(T) = \mathcal{M}$, and $\psi(f) = q(\lambda_1, ..., \lambda_r)T^n \neq 0$. □

Proposition 3.12. *Let $f : X \to Y$ a projective morphism between schemes of finite type over an algebraically closed field, $\mathscr{F}$ a coherent sheaf over X, $x \in X$ a point, and $y = f(x)$. Then $\mathscr{F}_x$ is f-flat if and only if the following conditions hold:*

(1) For every discrete valuation ring A' and every morphism $\mathcal{O}_{Y,y} \to A'$, the following holds: Taking the pull-back diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' = \mathrm{Spec}(A') \arrow[r] & Y \end{tikzcd}$$

the $\mathcal{O}_{X'}$-module $\mathscr{F}' = \mathscr{F} \otimes \mathcal{O}_{Y'}$ is f'-flat at every point $x' \in X'$ lying over x.

(2) For every $n \in \mathbb{N}$ and every morphism $\mathcal{O}_{Y,y} \to k[T]/(T^{n+1})$, if we take the diagram analogous to the one above (with $k[T]/(T^{n+1})$ instead of A'), then the $\mathcal{O}_{X'}$-module $\mathscr{F}' = \mathscr{F} \otimes \mathcal{O}_{Y'}$ is f'-flat at every point $x' \in X'$ lying over x.

Proof. Clearly Conditions 1 and 2 are necessary. Suppose then that 1 and 2 are satisfied.

Take the flattening stratification (see [8, Section 5.4.2]) of Y with respect to $\mathscr{F}$, $Y = \coprod_P Y_P$. As Condition 1 is satisfied for $\mathscr{F}$ over Y, so it is satisfied for $\iota^*\mathscr{F}$ over Y_{red}, where $\iota : Y_{\mathrm{red}} \to Y$ is the closed immersion of the reduced structure. Then, by the valuative criterion for flatness of [10, 11.8], $\iota^*\mathscr{F}$ is flat over Y_{red}, so by the universal property of the flattening stratification, there is a factorization

$$\begin{tikzcd} Y_{\mathrm{red}} \arrow[r, "\iota"] \arrow[d] & Y . \\ \coprod_P Y_P \arrow[ur] & \end{tikzcd}$$

As Y_{red} and Y share the same underlying topological set, the above factorization is telling us that the flattening factorization consist on a single stratum Y_P and that $Y_{\mathrm{red}} \to Y_P$ is a closed immersion.

Assume, by way of contradiction, $Y_P \subsetneq Y$; then there is an affine open sub-scheme $U \subseteq Y$ such that $V = Y_P \cap U \neq U$. Now take the coordinate rings $k[U]$ and $k[V]$ and the morphism between them induced by the inclusion $\phi : k[U] \twoheadrightarrow k[V]$. Let's take $f \in k[U]$ such that $\phi(f) = 0$. By Lemma 3.11, there exists, for some $n \in \mathbb{N}$, a morphism $\psi : k[U] \to k[T]/(T^{n+1})$ such that $\psi(f) \neq 0$; so ψ doesn't factorize through ϕ.

On the other hand, let $Z = \mathrm{Spec}(k[T]/(T^{n+1}))$ and $g : Z \to Y$ be the morphism induced by ψ. As Condition 2 is satisfied, the pull-back $g^*\mathscr{F}$ is flat over $Z = \mathrm{Spec}(k[T]/(T^{n+1}))$. So, by the universal property of the flattening stratification, g factors as

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d] & Y , \\ Y_P \arrow[ur] & \end{tikzcd}$$

contradicting the statement of the above paragraph, thus proving the proposition. □

Note that the hypotheses of this property on X and Y (aside from reducedness) are quite stronger than the ones of the original theorem of Grothendieck (the valuative criterion for flatness in [10]); such is the price we have paid to allow a criterion for possibly non-reduced schemes. The price paid is okay with us anyway, considering that we will work mostly with schemes of finite type over $\mathbb{C}$.

Next we provide a criterion for a $k[T]/(T^{n+1})$-module to be flat.

Proposition 3.13. *Let $A = k[T]/(T^{n+1})$ and M an A-module. Then M is flat if and only if, for every $m \in M$ such that $T^n \cdot m = 0$, there exists $m' \in M$ such that $m = T \cdot m'$.*

Proof. Flatness of M is equivalent to the injectivity of the map $M \otimes I \to M$ for every ideal $I \subset A$ (see e.g.:[11, IV.1]). In this case, there are finitely many ideals:

$$\mathcal{M} = (T),\ \mathcal{M}^2, \dots,\ \mathcal{M}^n.$$

If M is flat, it is easy to see that the second condition in our statement hold.

Suppose that, for every $m \in M$ such that $T^n \cdot m = 0$, there exists $m' \in M$ such that $m = T \cdot m'$. Let $a \in M \otimes \mathcal{M}^{n-i}$ be in the kernel of $M \otimes \mathcal{M}^{n-i} \to M$. When $i = 0$, we have $a = m \otimes T^n$, and m is such that $T^n \cdot m = 0$; so, by hypothesis, $m = T \cdot m'$ and then $m \otimes T^n = m' \otimes T^{n+1} = 0$.

When $i > 0$, we have $a = \sum_{j=n-i}^{i} m_j \otimes T^j$, so $T^i \cdot a = m_{n-i} \otimes T^n \in M \otimes \mathcal{M}^n$. By hypothesis, $m_{n-i} = T \cdot m'$. So $a \in M \otimes \mathcal{M}^{n-i+1}$ and we are done by induction. ☐

The following will be useful in the study of foliations of codimension greater than 1.

Proposition 3.14. *Let $p : X \to S$ a projective morphism between schemes of finite type over an algebraically closed field k, $f : S \to Y$ another morphism, with Y of finite type over k, and $\mathscr{F}$ a coherent sheaf over X. Take a stratification $\coprod_i S_i \subseteq S$ of S such that $\mathscr{F}|_{S_i} := \mathscr{F} \otimes_S \mathcal{O}_{S_i}$ is flat for all i. If the composition $\coprod_i S_i \hookrightarrow S \xrightarrow{f} Y$ is a flat morphism, then $\mathscr{F}$ is flat over Y.*

Proof. Invoking Proposition 3.12 we can, after applying base change, reduce to the case where Y is either the spectrum of a DVR or $Y = \operatorname{Spec}(k[T]/(T^{n+1}))$.

(i) Case $Y = \operatorname{Spec}(A)$ with A a DVR. Suppose there is, for some point $x \in X$, a section $s \in \mathscr{F}_x$ that is of torsion over A. Consider $Z = \operatorname{supp}_S(s) \subseteq S$, the support of s over S, that is the support of s as an element of $\mathscr{F}_x$ considered as an $\mathcal{O}_{S,p(x)}$-module. Now take any stratum S_i and suppose $Z \cap S_i \neq \emptyset$. Then there is a section of the pullback $\mathscr{F}_{S_i}$ that is of torsion over A. But $\mathscr{F}_{S_i}$ is flat over S_i which is in turn flat over A, so $\mathscr{F}_{S_i}$ is flat and $Z \cap S_i$ must be empty for every stratum S_i, i.e., $s = 0$.

(ii) Case $Y = \operatorname{Spec}(k[T]/(T^{n+1}))$. One can essentially repeat the argument above, now taking the section s to be such that $T^n s = 0$ but $s \notin T \cdot \mathscr{F}_x$. ☐

Corollary 3.15. *Take the flattening stratification $\coprod_P S_P \subseteq S$, of S with respect to $\mathscr{F}$. If the composition $\coprod_P S_P \hookrightarrow S \xrightarrow{f} Y$ is a flat morphism, then $\mathscr{F}$ is flat over Y.*

4. Families of sub-sheaves and their dual families

Definition 4.1. Given a short exact sequence of sheaves

$$0 \to \mathscr{G} \xrightarrow{\iota} \mathscr{F} \to \mathscr{H} \to 0,$$

we apply to it the left-exact contravariant functor $\mathscr{F} \mapsto \mathscr{F}^\vee := \mathcal{H}om_X(\mathscr{F}, \mathcal{O}_X)$ to obtain exact sequences:

$$0 \to \mathscr{H}^\vee \to \mathscr{F}^\vee \to \operatorname{Im}(\iota^\vee) \to 0, \tag{1}$$
$$0 \to \operatorname{Im}(\iota^\vee) \to \mathscr{G}^\vee \to \mathcal{E}xt^1_X(\mathscr{H}, \mathcal{O}_X) \to \mathcal{E}xt^1_X(\mathscr{F}, \mathcal{O}_X).$$

We say that the exact sequence (1), is the *dual exact sequence* of $0 \to \mathscr{G} \xrightarrow{\iota} \mathscr{F} \to \mathscr{H} \to 0$.

Lemma 4.2. *Let $0 \to N \xrightarrow{\iota} T \xrightarrow{\pi} M \to 0$ be a short exact sequence of R-modules such that T is reflexive and M is torsion free. Then $\mathrm{Im}(\iota^\vee)^\vee = N$ and $M = \mathrm{Im}(\pi^{\vee\vee})$.*

Proof. First we take the duals in the short exact sequence to get a sequence

$$0 \to \hom_R(M, R) \xrightarrow{\pi^\vee} \hom_R(T, R) \xrightarrow{\iota^\vee} \mathrm{Im}(\iota^\vee) \to 0.$$

Then we take duals one more time and, given that T is reflexive and that M is torsion-free, we get the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Im}(\iota^\vee)^\vee \arrow[r] & T^{\vee\vee} \arrow[r, "\pi^{\vee\vee}"] & M^{\vee\vee} \arrow[r] & \mathrm{ext}^1_R(\mathrm{Im}(\iota^\vee), R) \\
0 \arrow[r] & N \arrow[r] \arrow[u] & T \arrow[r] \arrow[u, equal] & M \arrow[r] \arrow[u, hook] & 0,
\end{tikzcd}$$

whose rows are exact.

Chasing arrows, we readily see that the leftmost vertical arrow must be an isomorphism. Indeed, since the monomorphism $N \to T^{\vee\vee}$ factors as

$$N \to \mathrm{Im}(\iota^\vee)^\vee \to T^{\vee\vee},$$

the second arrow being a monomorphism, so must $N \to \mathrm{Im}(\iota^\vee)^\vee$ be. On the other hand, given $a \in \mathrm{Im}(\iota^\vee)^\vee$, we can regard it, via the inclusion, as an element in $T^{\vee\vee} = T$, so we can compute $\pi(a)$. As the canonical map $\theta : M \to M^{\vee\vee}$ is an inclusion we have that, $\theta \circ \pi(a) = \pi^{\vee\vee}(a) = 0$, then $\pi(a) = 0$, so $a \in N$. From this, we have $N \cong \mathrm{Im}(\iota^\vee)^\vee$, wich implies $M = \mathrm{Im}(\pi^{\vee\vee})$. $\square$

4.1. **Exterior Powers.** When dealing with foliations of codimension/dimension greater than 1 is usually convenient to work with p-forms. We'll need then to compare sub-sheaves $I \subset \Omega^1_X$ with their exterior powers $\wedge^p I \subset \Omega^p_X$. In order to do that we include the following statements, valid in a wider context.

We'll concentrate on flat modules and their exterior powers. This will be important when dealing with flat families of Pfaff systems of codimension higher than 1 (see Remark 5.5).

Lemma 4.3. *Let A be a ring containing the field $\mathbb{Q}$ of rational numbers, and let M be a flat A-module. Then, for every p, $\wedge^p M$ is also flat.*

Proof. If tensoring with M is an exact functor, so is its iterate $- \otimes M \otimes \cdots \otimes M$. So $M^{\otimes p}$ is flat. As A contains $\mathbb{Q}$, there is an anti-symmetrization operator

$$M^{\otimes p} \to \wedge^p M,$$

which is a retraction of the canonical inclusion $\wedge^p M \subset M^{\otimes p}$. This makes $\wedge^p M$ a direct summand of $M^{\otimes p}$; set $M^{\otimes p} = \wedge^p M \oplus R$ for some module R. As the tensor power distributes over direct sums (i.e., $(\wedge^p M \oplus R) \otimes N \cong (\wedge^p M \otimes N) \oplus (R \otimes N)$), so does its derived functors. In particular we have, for every module N,

$$0 = \mathrm{Tor}_1(M^{\otimes p}, N) = \mathrm{Tor}_1(\wedge^p M, N) \oplus \mathrm{Tor}_1(R, N).$$

So $\wedge^p M$ is flat. $\square$

Finally, we draw some conclusions regarding flat quotients. When dealing with Pfaff systems, we'll be interested in short exact sequences of the form

$$0 \to \wedge^p I \to \Omega^p_X \to \mathcal{G} \to 0,$$

arising from short exact sequences of flat modules

$$0 \to I \to \Omega^1_X \to \Omega \to 0.$$

Note that, in general $\mathcal{G} \neq \wedge^p \Omega$. Nevertheless, we can state:

Proposition 4.4. *Let A be a ring containing $\mathbb{Q}$. Given an exact sequence*

$$0 \to M \to P \to N \to 0$$

of flat A-modules, we have an associated exact sequence

$$0 \to \wedge^p M \to \wedge^p P \to Q \to 0.$$

Then Q is also flat.

Proof. Q inherits a filtration from $\wedge^p P$:

$$Q = \wedge^p P / \wedge^p M = \overline{F}^0 \supseteq \overline{F}^1 \supseteq \cdots \supseteq \overline{F}^p = 0,$$

with quotients

$$F^i / F^{i+1} \cong \wedge^i M \otimes \wedge^{p-i} N.$$

Then Q has a filtration all of whose quotients are flat, so Q itself is flat. $\square$

5. Families of distributions and Pfaff systems

We will consider subsheaves of the relative tangent sheaf $T_S X$ and the relative differentials $\Omega^1_{X|S}$.

Definition 5.1. A *family of distributions* is a short exact sequence

$$0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0.$$

The family is called flat if $N_{\mathcal{F}}$ is flat over the base S. A family of distributions is called *involutive* if it's closed under the Lie bracket operation, that is, if for every pair of local sections X, $Y \in T\mathcal{F}(V)$, we have $[X, Y] \in T\mathcal{F}(V)$, where $[-, -]$ is the Lie bracket in $T_S X(V)$.

Likewise, a *family of Pfaff systems* is just a short exact sequence

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0.$$

It's called flat if $\Omega^1_{\mathcal{F}}$ is flat.

We will say that a family of Pfaff systems is *integrable* if

$$d(I(\mathcal{F})) \wedge \bigwedge^r I(\mathcal{F}) = 0 \subset \Omega^{r+2}_{X|S};$$

where $d : \Omega^j_{X|S} \to \Omega^{j+1}_{X|S}$ is the <u>relative</u> de Rham differential, and r is the generic rank of the sheaf $\Omega^1_{\mathcal{F}}$.

Remark 5.2. Observe that the relative differential $d : \Omega^j_{X|S} \to \Omega^{j+1}_{X|S}$ is not an $\mathcal{O}_X$-linear morphism. It is, however, $f^{-1}\mathcal{O}_S$-linear, so the sheaf $d(I(\mathcal{F})) \wedge \bigwedge^r I(\mathcal{F})$, whose annihilation encodes the integrability of the Pfaff system, is actually a sheaf of $f^{-1}\mathcal{O}_S$-modules.

In particular, the dual to a family of distributions is a family of Pfaff systems and vice-versa.

Remark 5.3. The dual of an involutive family of distributions is an integrable family of Pfaff systems. Reciprocally, the dual of an integrable family of Pfaff systems is a family of involutive distributions. This is just a consequence of the Cartan-Eilenberg formula for the de Rham differential of a 1-form applied to vector fields

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]).$$

Indeed, as involutiveness and integrability can be checked locally over sections, we can proceed as in [19, Prop. 2.30].

Definition 5.4. The *dimension* of a family of distributions is the generic rank of $T\mathcal{F}$. Likewise, the *dimension* of a family of Pfaff systems is the generic rank of $\Omega^1_{\mathcal{F}}$.

If $p : X \to S$ is moreover projective, S is connected, and the family is flat, so $T\mathcal{F}$ is a flat sheaf over S. Then for every $s \in S$ the Hilbert polynomial of $T\mathcal{F}_s$ is the same, and so is its generic rank (being encoded in the principal coeficient of the polynomial). The same occurs with families of Pfaff systems.

Remark 5.5. Frequently, in the study of foliations of codimension higher than 1, it is more convenient and better adapted to calculations to work with an alternative description of foliations. Namely, one can define a codimension q foliation on a variety X as in [15], with a global section ω of $\Omega^q_X \otimes \mathcal{L}$ such that:

- ω is locally decomposable, i.e., there is, for all $x \in X$, an open set such that
$$\omega = \eta_1 \wedge \cdots \wedge \eta_q,$$
with $\eta_i \in \Omega^1_X$.
- ω is integrable, i.e., $\omega \wedge d\eta_i = 0,\ 1 \leq i \leq q$.

With this setting, studying flat families of codimension q foliations (meaning here families of integrable Pfaff systems) as in [5] and [6], parametrized by a scheme S, amounts to studying short exact sequences of flat sheaves

$$0 \to \mathcal{L}^{-1} \to \Omega^q_{X|S} \to \mathcal{G} \to 0$$

that are locally decomposable and integrable. By the results of Section 4.1, a flat family of codimension q Pfaff systems given as a sub-sheaf of $\Omega^1_{X|S}$ gives rise to a flat family in the above sense.

6. Universal families

Now let's take a non-singular projective scheme X, a polynomial $P \in \mathbb{Q}[t]$, and consider the following functor

$$\begin{array}{rcl} \mathfrak{Inv}^P(X) : Sch & \longrightarrow & Sets \\ S & \mapsto & \left\{ \begin{array}{l} \text{flat families } 0 \to T\mathcal{F} \to T_S(X\times S) \to N_{\mathcal{F}} \to 0 \text{ of in-} \\ \text{volutive distributions such that } N_{\mathcal{F}} \text{ have Hilbert} \\ \text{polynomial } P(t). \end{array} \right\}. \end{array}$$

Say $p : X \times S \to X$ is the projection, so $T_S(X \times S) = p^*TX$. Clearly one has $\mathfrak{Inv}^P(X)$ is a sub-functor of $\mathfrak{Quot}^P(X,TX)$. We are going to show that $\mathfrak{Inv}^P(X)$ is actually a *closed* sub-functor of $\mathfrak{Quot}^P(X,TX)$ and therefore also representable.

So take the smooth morphism given by the projection

$$p_1 : \mathrm{Quot}_P(X,TX) \times X \to \mathrm{Quot}_P(X,TX).$$

Here we are taking as base scheme $S = \mathrm{Quot}_P(X, TX)$, then on the total space,

$$S \times X = \mathrm{Quot}_P(X, TX) \times X,$$

we have the natural short exact sequence

$$0 \to \mathscr{F} \to p_2^* TX = T_S(S \times X) \to \mathscr{Q} \to 0.$$

Now we consider the push-forward of these sheaves by p_1; as X is proper, these push-forwards are coherent sheaves over S. In particular, we have maps of coherent sheaves over $\mathrm{Quot}^P(X, TX)$

$$p_{1*}\mathscr{F} \otimes_S p_{1*}\mathscr{F} \xrightarrow{[-,-]} p_{1*}T_S(S \times X) \to p_{1*}\mathscr{Q}$$

induced by the maps over $S \times X$. Note that while the Lie bracket on $T_S(S \times X)$ is only $p_1^{-1}\mathcal{O}_S$-linear, the map induced on the push-forwards is $\mathcal{O}_S$-linear, and so is a morphism of coherent sheaves. We then also have for all $m, n \in \mathbb{Z}$ the twisted morphisms

$$p_{1*}\mathscr{F}(m) \otimes_S p_{1*}\mathscr{F}(n) \xrightarrow{[-,-]} p_{1*}T_S(S \times X)(m+n) \to p_{1*}\mathscr{Q}(m+n).$$

Note also that, as p_1 is a projective morphism, then there exists an $n \in \mathbb{Z}$ such that, for every $m \geq n$, the natural sheaves morphism over $S \times X$, $p_1^* p_{1*}(\mathscr{F})(m) \to \mathscr{F}(m)$ is an epimorphism. So, if for some $f : Z \to S$ and some $m \geq n$, one has that the composition

$$f^* p_{1*}\mathscr{F}(m) \otimes_Z f^* p_{1*}\mathscr{F}(m) \xrightarrow{[-,-]} f^* p_{1*}T_S(S \times X)(2m) \to f^* p_{1*}\mathscr{Q}(2m)$$

is zero, then the map

$$(f \times id)^*\mathscr{F}(m) \otimes_{\pi_1^{-1}\mathcal{O}_Z} (f \times id)^*\mathscr{F}(m) \xrightarrow{[-,-]} T_Z(Z \times X)(2m) \to (f \times id)^*\mathscr{Q}(2m)$$

is zero as well. Here $\pi_1 : Z \times X \to Z$ is the projection, which is by the way the pull-back of p_1.

Now to conclude the representability of $\mathfrak{Inv}^P(X)$ we need one important lemma.

Lemma 6.1. *Let S be a noetherian scheme, $p : X \to S$ a projective morphism, and $\mathscr{F}$ a coherent sheaf on X. Then $\mathscr{F}$ is flat over S if and only if there exists some integer N such that, for all $m \geq N$, the push-forwards $p_*\mathscr{F}(m)$ are locally free.*

Proof. This is [8, Lemma 5.5]. □

We can then take $m \in Z$ big enough so $p_{1*}\mathscr{F}(m)$ and $p_{1*}\mathscr{Q}(2m)$ are locally free and the morphism $p_1^* p_{1*}(\mathscr{F})(m) \to \mathscr{F}(m)$ is epimorphism. Then we can regard the composition

$$p_{1*}\mathscr{F}(m) \otimes_S p_{1*}\mathscr{F}(m) \xrightarrow{[-,-]} p_{1*}T_S(S \times X)(2m) \to p_{1*}\mathscr{Q}(2m)$$

as a global section σ of the locally free sheaf $\mathcal{H}om_S(p_{1*}\mathscr{F}(m) \otimes_S p_{1*}\mathscr{F}(m), \mathscr{Q}(2m))$. We can then make the following definition.

Definition 6.2. We define the scheme $\mathrm{Inv}^P(X)$ to be the zero scheme $Z(\sigma)$ (cf.: Definition 3.9) of the section σ defined above.

A direct application of Proposition 3.10 to this definition together with the discussion so far immediately gives us the following.

Proposition 6.3. *The subscheme* $\mathrm{Inv}^P(X) \subseteq \mathrm{Quot}^P(X, TX)$ *represents the functor* $\mathfrak{Inv}^P(X)$.

Similarly we can consider the sub-functor $\mathfrak{iPf}^P(X)$ of $\mathfrak{Quot}^P(X, \Omega_X^1)$.

$$\begin{array}{rcl} \mathfrak{iPf}^P(X) : Sch & \longrightarrow & Sets \\ S & \mapsto & \left\{\begin{array}{l}\text{flat families } 0 \to I(\mathcal{F}) \to \Omega^1 X|S \to \Omega^1_{\mathcal{F}} \to 0 \text{ of inte-}\\ \text{grable Pfaff systems such that } \Omega^1_{\mathcal{F}} \text{ have Hilbert poly-}\\ \text{nomial } P(t).\end{array}\right\}. \end{array}$$

Then as before we take $S = \mathrm{Quot}^P(X, \Omega^1_X)$ and consider the map

$$p_{1*}\big(d(\mathscr{I}) \wedge \bigwedge^r \mathscr{I}\big)(m) \longrightarrow p_{1*}\Omega^{r+2}_{S\times X|S}(m),$$

which is, for large enough m, a morphism between locally free sheaves on S.

Definition 6.4. We define the scheme $\mathrm{iPf}^P(X)$ to be the zero scheme of the above morphism, viewed as a global section of the locally free sheaf $\mathcal{H}om(p_{1*}(d(\mathscr{I}) \wedge \bigwedge^r \mathscr{I})(m), p_{1*}\Omega^{r+2}_{S\times X|S}(m))$.

And then, by Proposition 3.10, we have representability.

Proposition 6.5. *The subscheme* $\mathrm{iPf}^P(X) \subset \mathrm{Quot}^P(X, \Omega^1_X)$ *represents the functor* $\mathfrak{iPf}^P(X)$.

7. Duality

Definition 7.1. The *singular locus* of a family of distributions $0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0$ is the (scheme-theoretic) support of $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)$. Intuitively, its points are the points where $N_{\mathcal{F}}$ fails to be a fiber bundle.

Similarly, for a family of Pfaff systems $0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0$, its singular locus is $\mathrm{supp}(\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X))$.

Remark 7.2. Call $i : T\mathcal{F} \to T_S X$ the inclusion. We have an open non-empty set U where, for every $x \in U$, $\dim(Im(i \otimes k(x)))$ is maximal. More precisely, U is the open set where $Tor^X_1(N_{\mathcal{F}}, k(x)) = 0$, which is the maximal open set such that $N_{\mathcal{F}}|_U$ is locally free, and therefore so is $T\mathcal{F}$. Then, when restricted to U, $T\mathcal{F}$ can be given locally as the subsheaf of $T_S X$ generated by k linearly independent relative vector fields, i.e., $T\mathcal{F}$ defines a family of non-singular foliations. In U, one has that $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X) = 0$. Then, the underlying topological space of the singular locus of the family given by $T\mathcal{F}$ is the singular set of the foliation in a classical (topologial space) sense.

The above discussion translates verbatim to families of Pfaff systems.

Proposition 7.3. *Let*

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0$$

be a family of Pfaff systems such that $\Omega^1_{\mathcal{F}}$ *is torsion-free. Its singular locus and the singular locus of the dual family*

$$0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0$$

are the same sub-scheme of X*. We denote this sub-scheme by* $\mathrm{sing}(\mathcal{F})$

Proof. We are going to show that the immersions

$$Y_1 := \mathrm{supp}(\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X)) \subseteq X \quad \text{and} \quad Y_2 := \mathrm{supp}(\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)) \subseteq X$$

represent the same sub-functor of $\mathrm{Hom}(-, X)$, thus proving the proposition.

First note that, if $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X) = 0$, then $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X) = 0$. Indeed, if $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X) = 0$, $N_{\mathcal{F}}$ is locally free and then so is $T\mathcal{F}$. Moreover, since $\Omega^1_{\mathcal{F}}$ is torsion free, we can dualize the short exact sequence $0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0$ and, by Lemma 4.2, obtain the equality

$\Omega^1_{\mathcal{F}} = T\mathcal{F}^\vee$. So $\Omega^1_{\mathcal{F}}$ is locally free and $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X) = 0$.

Now, given a quasi-coherent sheaf $\mathscr{G}$ of X, its support $\mathrm{supp}(\mathscr{G}) \subseteq X$ represents the following sub-functor of $\mathrm{Hom}(-, X)$:

$$T \longmapsto \{f : T \to X \quad \text{s.t.: } f^*\mathscr{G} \text{ is not a torsion sheaf}\} \subseteq \mathrm{Hom}(T, X).$$

So, let's take a morphism $f : T \to Y_1 \subseteq X$.

(i) $f : T \to Y_1$ is an immersion: Suppose $f^*\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)$ is a torsion sheaf. Then there's a point $t \in T$ such that

$$\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X) \otimes k(t) = 0.$$

By Nakayama's lemma this implies that there's an open subset $U \subseteq X$ containing t such that $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)|_U = 0$. This in turn implies $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X)|_U = 0$, contradicting the fact that $t \in T \subseteq Y_1$. Then $T \subseteq Y_2$.
Similarly one proves that if $T \subseteq Y_2$ then $T \subseteq Y_1$.

(ii) General case: Taking the scheme-theoretic image of f, we can reduce to the above case where T is a sub-scheme of X.

□

7.1. **The codimension 1 case.** We now treat the case of families of codimension 1 foliations. From now on, we'll suppose that $X \to S$ is a smooth morphism.

Definition 7.4. A family of involutive distributions

$$0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0,$$

is of *codimension* 1 iff $N_{\mathcal{F}}$ is a sheaf of generic rank 1.

Likewise a family of Pfaff systems

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0,$$

is of codimension 1 if the sheaf $I(\mathcal{F})$ have generic rank 1.

Lemma 7.5. *Consider a family of codimension* 1 *Pfaff systems*

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0,$$

over an integral scheme X, such that $\Omega^1_{\mathcal{F}}$ is torsion-free. Then $I(\mathcal{F})$ is a line-bundle over X.

Proof. If $\Omega^1_{\mathcal{F}}$ is torsion-free, by Lemma 4.2 we have $I(\mathcal{F}) \cong N_{\mathcal{F}}^\vee$. In particular $I(\mathcal{F})$ is the dual of a sheaf, and thus is reflexive and observes property S_2. Write $I = I(\mathcal{F})$ and consider now the sheaf $I^\vee \otimes I$ together with the canonical morphism

$$I^\vee \otimes I \to \mathcal{O}_X.$$

The generic rank of $I^\vee \otimes I$ is 1. As I is reflexive, $I^\vee \otimes I$ is self-dual. So the canonical morphism above induces the dual morphism $\mathcal{O}_X \to I^\vee \otimes I$. The composition

$$\mathcal{O}_X \to I^\vee \otimes I \to \mathcal{O}_X$$

must be invertible, otherwise the image of $I^\vee \otimes I$ in $\mathcal{O}_X$ would be a torsion sub-sheaf. Then I is an invertible sheaf. □

Proposition 7.6. *In the case of codimension* 1 *Pfaff systems, if $\Omega^1_{\mathcal{F}}$ is torsion-free over X and the inclusion $I(\mathcal{F}) \to \Omega^1_{X|S}$ is nowhere trivial on S (meaning that $I(\mathcal{F}) \otimes \mathcal{O}_T \to \Omega^1_{X|S} \otimes \mathcal{O}_T$ is never the zero morphism for any $T \to S$), then the family is automatically flat.*

Proof. Indeed, $\Omega^1_{\mathcal{F}}$ being torsion free implies that the rank 1 sheaf $I(\mathcal{F})$ must be a line bundle. Then, if we take any morphism $f : T \to S$ and take pull-backs, we'll have an exact sequence

$$0 \to \operatorname{Tor}_1^S(\Omega^1_{\mathcal{F}}, T) \to f^*I(\mathcal{F}) \to f^*\Omega^1_{X|S} \to f^*\Omega^1 \to 0.$$

Now, as $I(\mathcal{F})$ is a line bundle, the cokernel $f^*I(\mathcal{F})/\operatorname{Tor}_1^S(\Omega^1_{\mathcal{F}}, T)$ must be a torsion sheaf over X_T. But, X being smooth over S, the annihilator $f^*\Omega^1_{X|S}$ is of the form $p^*(J)$, with $J \subset \mathcal{O}_T$, so $f^*I(\mathcal{F}) \to f^*\Omega^1_{X|S}$ must be the zero morphism when restricted to $\mathcal{O}_T/J$, contradicting the nowhere triviality assumption. □

Remark 7.7. In the codimension 1 case, we can calculate $\operatorname{sing}(\mathcal{F})$ by noting that $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X)$ is the cokernel in the exact sequence

$$T_S X \to I(\mathcal{F})^\vee \to \mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X) \to 0.$$

We can then tensor the sequence by $I(\mathcal{F})$ and obtain

$$T_S X \otimes I(\mathcal{F}) \to \mathcal{O}_X \to \mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X) \otimes I(\mathcal{F}) \to 0.$$

Now, $I(\mathcal{F})$ being a line bundle, the support of $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X)$ and that of $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X) \otimes I(\mathcal{F})$ are exactly the same. Note then that, in the second exact sequence, the cokernel is the scheme-theoretic zero locus of the twisted 1-form given by

$$\mathcal{O}_X \xrightarrow{\omega} \Omega^1_{X|S} \otimes I(\mathcal{F})^\vee,$$

as defined in Section 3.2. So, if we have a family of codimension 1 Pfaff systems given locally by a twisted form

$$\omega = \sum_{i=1}^n f_i(x) dx_i,$$

then $\operatorname{sing}(\mathcal{F})$ is the scheme defined by the ideal $(f_1, \dots, f_n)$.

The above proposition and remark tell us that our definition of flat family for Pfaff systems of codimension 1 is essentially the same as the one used in the now classical works of Lins-Neto, Cerveau, et.al.

Theorem 7.8. *Assume we have two families*

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0 \tag{2}$$

$$0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0, \tag{3}$$

satisfying the following conditions:

- *The families 2 and 3 are dual to each other.*
- *$N_{\mathcal{F}}$ is torsion free (or, equivalently, $\Omega^1_{\mathcal{F}}$ is torsion free).*
- *They are codimension 1 families.*
- *$\operatorname{sing}(\mathcal{F})$ is flat over S.*

Then 2 is flat if and only if 3 is flat.

Proof. Let $\Sigma = \operatorname{sing}(\mathcal{F})$.

Let's suppose first that the family

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0$$

is flat. We have to prove that $N_{\mathcal{F}}$ is also flat. To do this we note that applying the functor $\mathcal{H}om_X(-,\mathcal{O}_X)$ to the family of distributions not only gives us the family of Pfaff systems but also the exact sequence

$$0 \to N_{\mathcal{F}} \to I(\mathcal{F})^\vee \to \mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}},\mathcal{O}_X) \to 0.$$

Being $\Omega^1_{\mathcal{F}}$ torsion-free, $I(\mathcal{F})$ must be a line bundle, and so must $I(\mathcal{F})^\vee$; let's call $I(\mathcal{F})^\vee = \mathcal{L}$ to ease the notation. Now $\mathcal{L}$ has $N_{\mathcal{F}}$ as a sub-sheaf generically of rank 1, so $N_{\mathcal{F}} = \mathcal{I}\cdot\mathcal{L}$ for some ideal sheaf $\mathcal{I}$. Then $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}},\mathcal{O}_X) \cong \mathcal{L}\otimes\mathcal{O}_X/\mathcal{I}$. As $\Sigma = \operatorname{supp}(\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}},\mathcal{O}_X))$, one necessarily has $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}},\mathcal{O}_X) \cong \mathcal{L}_\Sigma$. Then $\mathcal{L}_\Sigma$, being a locally free sheaf over Σ which is flat over S, is itself flat over S. Therefore, as $\mathcal{L}$ is also flat over S, flatness for $N_{\mathcal{F}}$ follows.

Let's suppose now that the family

$$0 \to T\mathcal{F} \to T_SX \to N_{\mathcal{F}} \to 0$$

is flat. We have to prove that $\Omega^1_{\mathcal{F}}$ is also flat. By the above proposition, it's enough to show that the morphism $I(\mathcal{F}) \xrightarrow{\iota} \Omega^1_{X|S}$ is nowhere zero. Suppose there is $T \to S$ such that $\iota_T = 0$. Take an open set $U \subset X$ where $\Omega^1_{\mathcal{F}}$ is locally free. In that open set, we can apply base change with respect to the functor $\mathcal{H}om_X(-,\mathcal{O}_X)$ ([1] or [4]); so, restricting everything to U we have $(\iota_T)^\vee \cong (\iota^\vee)_T$. But, in U, $\iota^\vee$ is the morphism $T_SX \to N_{\mathcal{F}}$ and so it cannot become the zero morphism under any base change. □

7.2. **The arbitrary codimension case.** To give an analogous theorem to 7.8 in arbitrary codimension, we'll have to deal with finer invariants than the singular locus of the foliation. In the scheme X, we'll consider a stratification naturally associated with $\mathcal{F}$. This stratification have been already studied and described by Suwa in [18]. To deal with flatness issues, we have to provide a scheme structure to Suwa's stratification; this will be a particular case of a flattening stratification. Before going into that, we begin with some generalities. Remember that we are working over a smooth morphism $X \to S$.

Lemma 7.9. *Let $X \to S$ be a smooth morphism and $\mathscr{F}$ a coherent sheaf on X that is relatively $Z^{(2)}$-closed over S. Then, for all $s \in S$, the sheaf $\mathscr{F}_s = \mathscr{F}\otimes k(s)$ is $Z^{(2)}$-closed over X_s.*

Proof. We have to show that, for every $U \subset X_s$ such that $\operatorname{codim}(X\setminus U) \geq 2$, the restriction

$$\mathscr{F}_s \xrightarrow{\rho_U} \mathscr{F}_s|_U$$

is surjective. As the formal completion $\widehat{\mathcal{O}}_{X_s,x}$ of $\mathcal{O}_{X_s}$ with respect to any closed point x is faithfully flat over $\mathcal{O}_{X_s}$ [11, IV.3.2], we can check surjectivity of ρ_U by looking at every formal completion. As $X \to S$ is smooth formally around a point x, we have $\mathcal{O}_X \cong \mathcal{O}_S\otimes_k k[z_1,\dots,z_d]$ and so we can take an open subset $V \subseteq X$ to be $V = U\times S$. Then, with this choice of V, we have an epimorphism

$$\widehat{\mathscr{F}}|_V \to \widehat{\mathscr{F}_s}|_U \to 0.$$

Then we have a diagram with exact rows and columns

$$\begin{tikzcd}
\widehat{\mathscr{F}} \arrow[r] \arrow[d, "\rho_V"] & \widehat{\mathscr{F}_s} \arrow[r] \arrow[d, "\rho_U"] & 0 . \\
\widehat{\mathscr{F}}|_V \arrow[r] \arrow[d] & \widehat{\mathscr{F}_s}|_U \arrow[r] & 0 \\
0 & &
\end{tikzcd}$$

So ρ_U must be an epimorphism as well.

$\square$

Lemma 7.10. *Let $p : X \to S$ be a smooth morphism. Consider a family of distributions*

$$0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0.$$

If the codimension of $\operatorname{sing}(\mathcal{F})$ *with respect to* $X_{p(\operatorname{sing}(\mathcal{F}))}$ *is greater than* 2*, then, for every map* $T \to S$*, one has*

$$\mathcal{H}om_X(T\mathcal{F}, \mathcal{O}_X) \otimes \mathcal{O}_T \cong \mathcal{H}om_{X_T}(T\mathcal{F}_T, \mathcal{O}_T).$$

The analogous statement is true for $I(\mathcal{F})^\vee$ in a flat family of Pfaff systems.

Proof. By [1, Theorem 1.9], we only have to prove that, for every closed point $s \in S$, the natural map

$$\mathcal{H}om_X(T\mathcal{F}, \mathcal{O}_X) \otimes k(s) \to \mathcal{H}om_{X_s}(T\mathcal{F} \otimes k(s), \mathcal{O}_X \otimes k(s))$$

is surjective. Being the dual of some sheaves, both $\mathcal{H}om_X(T\mathcal{F}, \mathcal{O}_X)$ and

$$\mathcal{H}om_{X_s}(T\mathcal{F} \otimes k(s), \mathcal{O}_X \otimes k(s))$$

possess the relative property S_2 with respect to p (Proposition 3.7), and so are relatively $Z^{(2)}$-closed with respect to p, and so is $\mathcal{H}om_X(T\mathcal{F}, \mathcal{O}_X) \otimes k(s)$ by the above lemma.

Let $U = X \setminus \operatorname{sing}\mathcal{F}$ and $j : U \hookrightarrow X$ be the inclusion. As $T\mathcal{F}|_U$ is locally free over U, so is $T\mathcal{F}^\vee|_U$. Then, in U, we have

$$\mathcal{E}xt^1(T\mathcal{F}|_U, \mathcal{O}_X|_U \otimes_S \mathcal{G}) = 0,$$

for every $\mathcal{G} \in Coh(S)$. Then, from the exchange property for local Ext's [1, Theorem 1.9], we get surjectivity on

$$\mathcal{H}om_X(T\mathcal{F}|_U, \mathcal{O}_X|_U) \otimes k(s) \to \mathcal{H}om_{X_s}(T\mathcal{F}|_U \otimes k(s), \mathcal{O}_X|_U \otimes k(s)).$$

But, as $\operatorname{codim}(\operatorname{sing}(\mathcal{F})) > 1$ and both sheaves are S_2, then surjectivity holds in all of X_s. $\square$

Lemma 7.11. *Suppose that a flat family*

$$0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0$$

is such that the relative codimension $d_S(\operatorname{sing}(\mathcal{F}))$ of $\operatorname{sing}(\mathcal{F})$ over S verifies $d_S(\operatorname{sing}(\mathcal{F})) \geq 2$. Suppose further that the flattening stratification of X over $T\mathcal{F}$ is flat over S (c.f., Proposition 3.14). Then $T\mathcal{F}^\vee$ is also a flat $\mathcal{O}_S$-module.

The analogous statement is true for $I(\mathcal{F})^\vee$ in a flat family of Pfaff systems.

Proof. The proof works exactly the same for distributions of Pfaff systems mutatis mutandi.

Take $\coprod_P X_P$ to be the flattening stratification of X with respect to $T\mathcal{F}$. The restriction $T\mathcal{F}_{X_P}$ (being coherent and flat over X_P) is locally free over X_P, and hence so is its dual $\mathcal{H}om_{X_P}(T\mathcal{F}_{X_P}, \mathcal{O}_{X_P})$. By Lemma 7.10, in each stratum X_P, we have the isomorphism

$$\mathcal{H}om_{X_P}(T\mathcal{F}_{X_P}, \mathcal{O}_{X_P}) \cong \mathcal{H}om_X(T\mathcal{F}, \mathcal{O}_X) \otimes \mathcal{O}_{X_P} = T\mathcal{F}^\vee \otimes \mathcal{O}_{X_P}.$$

So $T\mathcal{F}^\vee$ is flat when restricted to the filtration $\coprod_P X_P$, which is in turn flat over S. Then, by [8, Section 5.4.2], $T\mathcal{F}^\vee$ is flat over S. $\square$

Definition 7.12. For a family of distributions consider the flattening stratification

$$\coprod_{P(\mathcal{F})} X_{P(\mathcal{F})} \subseteq X$$

of X with respect to $T\mathcal{F} \oplus N_{\mathcal{F}}$. We call this the *rank stratification* of X with respect to $T\mathcal{F}$.

Remark 7.13. Note that the flattening stratification of $T\mathcal{F} \oplus N_{\mathcal{F}}$ is the (scheme-theoretic) intersection of the flattening stratification of $T\mathcal{F}$ with that of $N_{\mathcal{F}}$. This is because

$$(T\mathcal{F} \oplus N_{\mathcal{F}}) \otimes \mathcal{O}_Y$$

is flat if and only if both $T\mathcal{F} \otimes \mathcal{O}_Y$ and $N_{\mathcal{F}} \otimes \mathcal{O}_Y$ are.

This tells us, in particular, that each stratum is indexed by two natural numbers r and k such that

$$x \in X_{r,k} \iff \dim(T\mathcal{F} \otimes k(x)) = r \text{ and } \dim(N_{\mathcal{F}} \otimes k(x)) = k.$$

In [18], Suwa studied a related stratification associated to a foliation. Given a distribution $D \subset TM$ on a complex manifold M, he defines the strata $M^{(l)}$ as

$$M^{(l)} = \{x \in M \text{ such that } D_x \subset T_xM \text{ is a sub-space of dimension } l\ \}.$$

Here D is spanned point-wise by vector fields $v_1, \dots, v_r$, and $D_x =< v_i(x) >$. Clearly if D is of generic rank r, the open stratum is $M^{(r)}$.

Note that, in the setting of distribution as sub-sheafs $i : T\mathcal{F} \hookrightarrow TX$ of the tangent sheaf of a variety, the vector space $T_x\mathcal{F}$ is actually the image of the map

$$T\mathcal{F} \otimes k(x) \xrightarrow{i \otimes k(x)} TX \otimes k(x),$$

whose kernel is $\mathrm{Tor}_1^X(N_{\mathcal{F}}, k(x))$. Moreover we have the exact sequence

$$0 \to T_x\mathcal{F} = \mathrm{Im}(i \otimes k(x)) \to TX \otimes k(x) \to N_{\mathcal{F}} \otimes k(x) \to 0.$$

In particular, in a variety X of dimension n, if $\dim(T_x\mathcal{F}) = l$, then $\dim(N_{\mathcal{F}} \otimes k(x)) = n - l$. So what we call the rank stratification of X is actually a refinement of the stratification studied in [18].

Our main motivation for defining this refinement of the stratification of [18] is the following result.

Theorem 7.14. *Assume we have dual families*

$$0 \to T\mathcal{F} \to T_SX \to N_{\mathcal{F}} \to 0, \tag{4}$$

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0, \tag{5}$$

parametrized by a scheme S of finite type over an algebraically closed field, such that

- *$N_{\mathcal{F}}$ is torsion free*
- *The relative codimension of* $\mathrm{sing}(\mathcal{F})$ *over S (that is* $d_S(\mathrm{sing}(\mathcal{F}))$*) verifies* $d_S(\mathrm{sing}(\mathcal{F}) \geq 2$.
- *Each stratum $X_{r,k}$ of the rank stratification is flat over S.*

Then 4 is flat over S if and only if 5 is. Moreover, for each point $s \in S$, we have

$$I(\mathcal{F})_s = (N_{\mathcal{F}s})^\vee,$$

in other terms "the dual family is the family of the duals".

Proof. We prove one of the implications; the proof of the other is identical.

Consider the exact sequence

$$0 \to \Omega^1_{\mathcal{F}} \to T\mathcal{F}^\vee \to \mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X) \to 0.$$

It is clear that, to prove flatness of the dual family, it's enough to show $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)$ is flat over S.

Also, by Lemma 7.10, we have, for every $s \in S$, the diagram with exact rows and columns,

$$\begin{tikzcd} \mathcal{H}om(I(\mathcal{F}), \mathcal{O}_X) \otimes k(s) \arrow[r] \arrow[d] & \mathcal{H}om(I(\mathcal{F})_s, \mathcal{O}_{X_s}) \arrow[r] \arrow[d] & 0 . \\ \mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X) \otimes k(s) \arrow[r] \arrow[d] & \mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}} \otimes k(s), \mathcal{O}_{X_s}) \arrow[d] & \\ 0 & 0 & \end{tikzcd}$$

So $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X) \otimes k(s) \to \mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}} \otimes k(s), \mathcal{O}_{X_s})$ is surjective for every $s \in S$ and so, by [1, Theorem 1.9], the exchange property is valid for $\mathcal{E}xt^1_X(\Omega^1_{\mathcal{F}}, \mathcal{O}_X)$. If moreover $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)$ is flat over S, then, again by [1, Theorem 1.9],

$$I(\mathcal{F})_s = \mathcal{H}om_X(N_{\mathcal{F}}, \mathcal{O}_X) \otimes k(s) \cong \mathcal{H}om_X(N_{\mathcal{F}s}, \mathcal{O}_{X_s}) = (N_{\mathcal{F}s})^{\vee}.$$

By Proposition 3.14, it is enough to show the restriction of $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)$ to every rank stratum is flat over S. So let $Y \subseteq X$ be a rank stratum. If we can show that $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X) \otimes \mathcal{O}_Y$ is locally free, then we're set.

By hypothesis, one has the isomorphism $\mathcal{H}om_X(T\mathcal{F}, \mathcal{O}_X) \otimes \mathcal{O}_Y \cong \mathcal{H}om_Y(T\mathcal{F}_Y, \mathcal{O}_Y)$. So we can express $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X) \otimes \mathcal{O}_Y$ as the cokernel in the $\mathcal{O}_Y$-modules exact sequence

$$\mathcal{H}om_Y(TX_Y, \mathcal{O}_Y) \to \mathcal{H}om_Y(T\mathcal{F}_Y, \mathcal{O}_Y) \to \mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)_Y \to 0.$$

So, localizing at a point $y \in Y$, we can realize the local $\mathcal{O}_{Y,y}$-module $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)_{Y,y}$ as the set of maps $T\mathcal{F}_{Y,y} \to \mathcal{O}_{Y,y}$ modulo the ones that factors as

$$\begin{tikzcd} T\mathcal{F}_{Y,y} \arrow[r, dashed] \arrow[d] & \mathcal{O}_{Y,y} . \\ TX_{Y,y} \arrow[ur] & \end{tikzcd}$$

To study $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)_{Y,y}$ this way, note that we have the following exact sequence.

$$0 \to \mathrm{Tor}_1^X(N_{\mathcal{F}}, \mathcal{O}_{Y,y}) \to T\mathcal{F}_{Y,y} \to TX_{Y,y} \to (N_{\mathcal{F}})_{Y,y} \to 0.$$

which we split into two short exact sequences,

$$0 \to \mathcal{K} \to \quad TX_{Y,y} \quad \to (N_{\mathcal{F}})_{Y,y} \to 0 \quad \text{and} \tag{6}$$

$$0 \to \mathrm{Tor}_1^X(N_{\mathcal{F}}, \mathcal{O}_{Y,y}) \to \quad T\mathcal{F}_{Y,y} \quad \to \mathcal{K} \to 0. \tag{7}$$

Now, as Y is a rank stratum, $\mathcal{Q}_{\mathcal{Y}}$ and $T\mathcal{F}_Y$ are flat over Y, and coherent, and so they are locally free. As a consequence, short exact sequence (6) splits, so $TX_{Y,y} \cong (N_{\mathcal{F}})_{Y,y} \oplus \mathcal{K}$. So

$$\mathcal{H}om_Y(TX_Y, \mathcal{O}_Y)_y \cong \mathcal{H}om_Y(\mathcal{K}, \mathcal{O}_{Y,y}) \oplus \mathcal{H}om_Y((N_{\mathcal{F}})_Y, \mathcal{O}_Y)_y$$

and we get $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)_{Y,y}$ as the cokernel in

$$\mathcal{H}om_Y(\mathcal{K}, \mathcal{O}_{Y,y}) \to \mathcal{H}om_Y(T\mathcal{F}_Y, \mathcal{O}_Y)_y \to \mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)_{Y,y} \to 0. \tag{8}$$

Since $(N_{\mathcal{F}})_Y$ and TX_Y are locally free over Y, so is $\mathcal{K}$. Then short exact sequence (7) splits, so $T\mathcal{F}_{Y,y} \cong \mathrm{Tor}_1^X(N_{\mathcal{F}}, \mathcal{O}_{Y,y}) \oplus \mathcal{K}$. Also, as $T\mathcal{F}_Y$ and $\mathcal{K}$ are locally free over Y, so is $\mathrm{Tor}_1^X(N_{\mathcal{F}}, \mathcal{O}_Y)$. Sequence (8) now reads

$$\mathcal{H}om_Y(\mathcal{K}, \mathcal{O}_{Y,y}) \to \mathcal{H}om_Y(\mathrm{Tor}_1^X(N_{\mathcal{F}}, \mathcal{O}_{Y,y}), \mathcal{O}_Y)_y \oplus \mathcal{H}om_Y(\mathcal{K}, \mathcal{O}_{Y,y}) \to \mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)_{Y,y} \to 0.$$

So we have
$$\mathcal{H}om_Y(\mathrm{Tor}_1^X(N_{\mathcal{F}}, \mathcal{O}_{Y,y}), \mathcal{O}_Y)_y \cong \mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)_{Y,y}.$$
Now, as $\mathrm{Tor}_1^X(N_{\mathcal{F}}, \mathcal{O}_{Y,y})$ is locally free over Y, so is its dual. In other words, we just proved $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)_Y$ is locally free over Y, which settles the theorem. □

Now, by generic flatness, we can conclude the following:

Corollary 7.15. *Every irreducible component of the scheme* Inv_P *is birationally equivalent to an irreducible component of* iPf_P.

Remark 7.16. During the proof of Theorem 7.14, we have actually obtained this result:

Proposition 7.17. $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)$ *is flat over the rank stratification.*

In particular, if $\coprod X_Q$ denotes the flattening stratification of $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)$, there is a morphism
$$\coprod_{P(\mathcal{F})} X_{P(\mathcal{F})} \to \coprod_Q X_Q.$$
Now, by the construction of the flattening stratification, $\coprod X_Q$ consist of an open stratum U such that $\mathcal{E}xt^1_X(N_{\mathcal{F}}, \mathcal{O}_X)|_U = 0$, and closed strata whose closure is $\mathrm{sing}(\mathcal{F})$. So the morphism $\coprod_{P(\mathcal{F})} X_{P(\mathcal{F})} \to \coprod_Q X_Q$ actually defines a stratification of $\mathrm{sing}(\mathcal{F})$.

8. Singularities

Theorem 7.8 gives a condition for a flat family of integrable Pfaff systems to give rise to a flat family of involutive distributions in terms of the flatness of the singular locus. We have then to be able to decide when can we apply the theorem. More precisely, say
$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0$$
is a flat family of codimension 1 integrable Pfaff systems, and let $s \in S$. How do we know when $\mathrm{sing}(\mathcal{F})$ is flat around s? In this section, we address this question and give a sufficient condition for $\mathrm{sing}(\mathcal{F})$ to be flat at s in terms of the classification of singular points of the Pfaff system $0 \to I(\mathcal{F})_s \to \Omega^1_{X_s} \to \Omega^1_{\mathcal{F}_s} \to 0$.

From now on, we will consider only Pfaff systems such that $\Omega^1_{\mathcal{F}}$ is torsion-free.

Remember that, if we have a Pfaff system of codimension 1, $0 \to I(\mathcal{F})_s \to \Omega^1_{X_s} \to \Omega^1_{\mathcal{F}_s} \to 0$, such that $\Omega^1_{\mathcal{F}_s}$ is torsion-free, we can consider, locally on X, that it is given by a single 1-form ω and that it is integrable iff $\omega \wedge d\omega = 0$.

Definition 8.1. Let $0 \to I(\mathcal{F})_s \to \Omega^1_{X_s} \to \Omega^1_{\mathcal{F}_s} \to 0$ be a codimension 1 integrable Pfaff system, and let $x \in X$. We say that x is a *Kupka singularity* of the Pfaff system if, for some 1-form ω locally defining $I(\mathcal{F})$ around x, we have $\omega_x = 0$ and $d\omega_x \neq 0$.

We also say that x is a Kupka singularity of the foliation defined by $I(\mathcal{F})$.

Proposition 8.2. *The set of Kupka singularities of a codimension* 1 *foliation, if non-void, has a natural structure of a codimension* 2 *sub-scheme of* X.

Proof. See [14] □

Definition 8.3. Let $0 \to I(\mathcal{F})_s \to \Omega^1_{X_s} \to \Omega^1_{\mathcal{F}_s} \to 0$ be as before. Say X has dimension n over $\mathbb{C}$. Then we call $x \in X$ a *Reeb singularity* if there exists an analytic neighborhood U of x such that $I(\mathcal{F})$ may be generated by a form ω with the property that ω can be written, locally in U, in the form $\omega = \sum_{i=1}^n f_i dz_i$ with $f_i(x) = 0$ for all i, and $(df_1)_x, \dots, (df_n)_x$ are linearly independent in T_x^*X.

Remark 8.4. Note that Kupka singularities and Reeb singularities are singularities in the sense of 7.1, i.e., they are points in $\operatorname{sing}(\mathcal{F})$.

We now give a version for families of the fundamental result of Kupka.

Proposition 8.5. *Let*
$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0$$
be a flat family of integrable Pfaff systems of codimension 1*, and let* $\Sigma = \operatorname{sing}(\mathcal{F}) \subset X$*. Let* $s \in S$*, and* $x \in \Sigma_s$ *be such that* x *is a Kupka singularity of* $0 \to I(\mathcal{F})_s \to \Omega^1_{X_s} \to \Omega^1_{\mathcal{F}_s} \to 0$*. Then, locally around* x*,* $I(\mathcal{F})$ *can be given by a* relative 1*-form* $\omega(z,s) \in \Omega^1_{X|S}$ *such that*
$$\omega = f_1(z,s)dz_1 + f_2(z,s)dz_2,$$
that is, ω *is locally the pull-back of a relative form* $\eta \in \Omega^1_{Y|S}$*, where* $Y \to S$ *is of relative dimension* 2*.*

The proof is essentially the same as the proof of the classical Kupka theorem, as in [15]. One only needs to note that every ingredient there can be generalized to a relative setup.

For this we note that, as $p : X \to S$ is a smooth morphism, the relative tangent sheaf $T_S X$ is locally free and is the dual sheaf of the locally free sheaf $\Omega^1_{X|S}$. We note also that, if $v \in T_S X(U)$, and $\omega \in \Omega^1_{X|S}(U)$, the relative Lie derivative $L_v(\omega)$ is well-defined by Cartan's formula
$$L_v^S = d_S \iota_v(\omega) + \iota_v(d_S\omega),$$
where $\iota_v(\omega) =< v, \omega >$ is the pairing of dual spaces (and by extension also the map $\Omega^q_{X|S} \to \Omega^{q-1}_{X|S}$ determined by v), and d_S is the *relative* de Rham differential. Also $\Omega^q_{X|S} = \wedge^q \Omega^1_{X|S}$.

Finally we observe that, if $p : X \to S$ is of relative dimension d and X is a regular variety over $\mathbb{C}$ of total dimension n, a family of integrable Pfaff systems gives rise to a foliation on X whose leaves are tangent to the fibers of p. Indeed, we can pull back the subsheaf $I(\mathcal{F}) \subset \Omega^1 X|S$ by the natural epimorphism
$$f^*\Omega^1_S \to \Omega^1_X \to \Omega^1_{X|S} \to 0,$$
and get $J = I(\mathcal{F}) + f^*\Omega^1_S \subset \Omega^1_X$, which is an integrable Pfaff system in X, determining a foliation $\hat{\mathcal{F}}$. As $f^*\Omega^1_S \subset J$, the leaves of the foliation $\hat{\mathcal{F}}$ are contained in the fibers X_s of p.

In the general case, where p is smooth but S and X need not to be regular over $\mathbb{C}$, the Frobenius theorem still gives foliations $\mathcal{F}_s$ in each fiber X_s. Indeed, as $p : X \to S$ is smooth, each fiber X_s is regular over $\mathbb{C}$ and, $\Omega^1_{\mathcal{F}}$ being flat, $I(\mathcal{F})_s \subset \Omega^1_{X_s}$ is an integrable Pfaff system on X_s.

Proposition 8.6. *Let* $p : X \to S$ *a smooth morphism over* $\mathbb{C}$ *and*
$$0 \to I(\mathcal{F}) \to \Omega^1 X|S \to \Omega^1_{\mathcal{F}} \to 0$$
a codimension 1 *flat family of Pfaff systems. Let* $\omega \in \Omega^1_{X|S}(U)$ *be an integrable* 1*-form such that* $I(\mathcal{F})(U) = (\omega)$ *in a neighborhood* U *of a point* $x \in X$*. Then* $d\omega$ *is locally decomposable.*

Proof. As $T_S X = (\Omega^1 X|S)^\vee$ and $\Omega^q_{X|S} = \wedge^q \Omega^1_{X|S}$, we can apply Plücker relations to determine if $d\omega$ is locally decomposable and proceed as in [15, Lemma 2.5]. □

Lemma 8.7. *Suppose that* $d\omega_x \neq 0$*. Consider* $\mathcal{G}_s$ *the codimension* 2 *foliations defined by* $d\omega$ *in* X_s*. In the neighborhood* V *of* $x \in X$ *where* $\mathcal{G}_s$ *is non-singular for every* s*, we have that the leaves of* $\mathcal{G}_s$ *are integral manifolds of* $\omega|_{X_s}$*.*

Proof. We have only to prove, for every $v \in T_S X$ such that $\iota_v(d\omega) = 0$, that $\iota_v(\omega) = 0$. We can do this exactly as in [15, Lemma 2.6]. □

Lemma 8.8. *Assume the same hypothesis as Lemma 8.7. Let v be a vector field tangent to $\mathcal{G}$. Then the* relative *Lie derivative of ω with respect to v is zero.*

Proof. Like the proof of [15, Lemma 2.7]. □

Lemma 8.9. *Under the same hypothesis as Lemma 8.7 and 8.8,* $\operatorname{sing}(\omega)$ *is saturated by leaves of $(\mathcal{G}_s)_{s\in S}$ (i.e., take $y \in V$ a zero of ω such that $p(y) = s$, and L the leaf of $\mathcal{G}_s$ going through y. Then the inclusion $L \to V$ factors through* $\operatorname{sing}(\omega)$*).*

Proof. We can do this entirely on X_s. Then this reduces to [15, Lemma 2.7]. □

Proof of Proposition 8.5. We take an analytic neighborhood V of $x \in X$ such that $V \cong U \times D^d$ with $U \subseteq S$ an open set, D^d a complex polydisk, and

$$\begin{array}{rcl} p|V : V \cong U \times D^d & \longrightarrow & U. \\ (s, z_1, \ldots, z_d) & \mapsto & s \end{array}$$

By the Frobenius theorem, we can choose the coordinates z_i in such a way that

$$v_i = \frac{\partial}{\partial z_i} \in T_S X(V), \qquad 3 \le i \le d,$$

are tangent to $d\omega$. Then, as $L^S_{v_i}\omega = 0$ and $\iota_{v_i}\omega = 0$, we can write ω as

$$\omega = f_1(z, s)dz_1 + f_2(z, s)dz_2.$$

□

Proposition 8.10. *Let*

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0$$

be a flat family of integrable Pfaff systems of codimension 1, and let $\Sigma = \operatorname{sing}(\mathcal{F}) \subset X$. Let $s \in S$, and $x \in \Sigma_s$ be such that x is a Kupka singularity of $0 \to I(\mathcal{F})_s \to \Omega^1_{X_s} \to \Omega^1_{\mathcal{F}_s} \to 0$. Then $\Sigma \to S$ is smooth around x.

Proof. By 8.5 above, we can determine Σ around x as the common zeroes of $f_1(z, s)$ and $f_2(z, s)$. The condition $d\omega \neq 0$ implies Σ is *smooth* over S (remember that we are using the *relative* de Rham differential and that means the variable s counts as a constant). □

Proposition 8.11. *Let*

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0$$

be a flat family of integrable Pfaff systems of codimension 1, $\Sigma = \operatorname{sing}(\mathcal{F}) \subset X$, and $s \in S$. Suppose $x \in \Sigma_s$ is such that x is a Reeb *singularity of $0 \to I(\mathcal{F})_s \to \Omega^1_{X_s} \to \Omega^1_{\mathcal{F}_s} \to 0$. Then $\Sigma \to S$ is étalé around x.*

Proof. The condition on x means we can actually give $I(\mathcal{F})$ locally by a relative 1-form $\omega \in \Omega^1_{X|S}$, $\omega = \sum_{i=1}^n f_i(z, s)dz_i$, with n the relative dimension of X over S and the df_i's linearly independent on x. Then Σ is given by the equations $f_1 = \cdots = f_n = 0$ and is therefore étalé over S. □

With these two propositions, we are almost in a position to state our condition for flatness of the dual family, we just need a general:

Lemma 8.12. *Let $X \xrightarrow{p} S$ be a morphism between schemes of finite type over an algebraically closed field k. Let $U \subseteq X$ be the maximal open sub-scheme such that $U \xrightarrow{p} S$ is flat, and $s \in S$ a point such that X_s is without embedded components. If $U_s \subseteq X_s$ is dense, then $U_s = X_s$.*

Proof. By Proposition 3.12, we must check that, for A either a discrete valuation domain or an Artin ring of the form $k[T]/(T^{n+1})$, and every arrow $\mathrm{Spec}(A) \to S$, the pull-back scheme $X_{\mathrm{Spec}}(A)$ is flat over $\mathrm{Spec}(A)$. In this way, the problem reduces to the case where $S = \mathrm{Spec}(A)$.

(i) Case A is a DVD. In this case, A being a principal domain, flatness of X over $\mathrm{Spec}(A)$ is equivalent to the local rings $\mathcal{O}_{X,x}$ being torsion-free $A_{p(x)}$-modules for every point $x \in X$ ([11, IV.1.3] , so it suffices to consider the case $A_{p(x)} = A$.

Now, let $f \in \mathcal{O}_{X,x}$ and $J = \mathrm{Ann}_A(f) \subseteq A$. Suppose $J \neq (0)$ and consider $V(J) \subseteq \mathrm{Spec}(A)$; clearly $\mathrm{supp}(f) \subseteq p^{-1}(V(J)) \subseteq X$. So $U \cap \mathrm{supp}(f) = \varnothing$. But then the restriction $f|s$ of f to X_s has support disjoint with U_s. On the other hand

$$\mathrm{supp}(f|_s) = \overline{\{\mathfrak{P}_1\}} \cup \cdots \cup \overline{\{\mathfrak{P}_m\}} \subseteq X_s,$$

where $\{\mathfrak{P}_1, \ldots, \mathfrak{P}_m\} = \mathrm{Ass}(\mathcal{O}_{X_s,x}/(f|_s)) \subseteq \mathrm{Ass}(\mathcal{O}_{X_s,x})$.
As X_s is without immersed components, the $\mathfrak{P}_i$'s are all minimal, so $X_k \cap \overline{\mathfrak{P}_i}$ is an irreducible component of X_k. But

$$U_s \cap \overline{\mathfrak{P}_i} = \varnothing,$$

contradicting the hypothesis that U_s is dense in X_s.

(ii) Case $A = k[T]/(T^{n+1})$. Using Proposition 3.13, this works just as the first case taking $f \in \mathcal{O}_{X,x}$ as a section such that $T^n f = 0$ but $f \notin T\mathcal{O}_{X,x}$. □

We have already said that, in a Pfaff system, Kupka singularities, if they exist, form a codimension 2 sub-scheme of X. We will call $\mathcal{K}(\mathcal{F})$ this sub-scheme, and $\overline{\mathcal{K}}(\mathcal{F})$ its closure.

Theorem 8.13. *Let*

$$0 \to I(\mathcal{F}) \to \Omega^1_{X|S} \to \Omega^1_{\mathcal{F}} \to 0$$

be a flat family of integrable Pfaff systems, for $s \in S$. Consider the Pfaff system

$$0 \to I(\mathcal{F})_s \to \Omega^1_{X_s} \to \Omega^1_{\mathcal{F}_s} \to 0.$$

If $\mathrm{sing}(\mathcal{F}_s)$ is without embedded components and $\mathrm{sing}(\mathcal{F}_s) = \overline{\mathcal{K}}(\mathcal{F}_s) \cup \{p_1, \ldots, p_m\}$, where the p_i's are Reeb-type singularities, then $\mathrm{sing}(\mathcal{F}) \to S$ is flat in a neighborhood of $s \in S$.

Proof. Indeed, by Proposition 8.10, $\mathrm{sing}(\mathcal{F})$ is flat in a neighborhood of $\mathcal{K}(\mathcal{F}_s)$, and, as $\mathrm{sing}(\mathcal{F}_s)$ is without embedded components, we can apply Lemma 8.12 to conclude that $\mathrm{sing}(\mathcal{F})$ is flat in a neighborhood of $\overline{\mathcal{K}(\mathcal{F}_s)}$.

Lastly, from Proposition 8.11, it follows that $\mathrm{sing}(\mathcal{F})$ is flat in a neighborhood of $\{p_1, \ldots, p_m\}$. □

9. Applications

Let $X = \mathbb{P}^n(\mathbb{C})$. It's well-known that the class of sheaves $\mathscr{F}$ that split as a direct sum of line bundles $\mathscr{F} \cong \bigoplus_i \mathcal{O}(k_i)$ has no non-trivial deformations. Indeed, as deformation theory teaches us, first-order deformations of $\mathscr{F}$ are parametrized by $\mathrm{Ext}^1(\mathscr{F}, \mathscr{F})$, in this case we have

$$\mathrm{Ext}^1(\mathscr{F}, \mathscr{F}) \cong \bigoplus_{i,j} \mathrm{Ext}^1(\mathcal{O}(k_i), \mathcal{O}(k_j)) \cong$$

$$\bigoplus_{i,j} \mathrm{Ext}^1(\mathcal{O}, \mathcal{O}(k_j - k_i)) \cong \bigoplus_{i,j} H^1(\mathbb{P}^n, \mathcal{O}(k_j - k_i)) = 0$$

In particular, given a flat family of distributions

$$0 \to T\mathcal{F} \to T_S(\mathbb{P}^n \times S) \to N_{\mathcal{F}} \to 0,$$

such that, for some $s \in S$, $T\mathcal{F}_s \cong \bigoplus_i \mathcal{O}(k_i)$, the same decomposition holds true for the rest of the members of the family.

When we deal with codimension 1 foliations, it's more common, however, to work with Pfaff systems or, more concretely, with integrable twisted 1-forms $\omega \in \Omega^1_{\mathbb{P}^n}(d)$, $\omega \wedge d\omega = 0$ (see [14]). It's then than the following question emerged: Given a form $\omega \in \Omega^1_{\mathbb{P}^n}(d)$ such that the vector fields that annihilate ω generate a split sheaf (i.e., a sheaf that decomposes as direct sum of line bundles), will the same feature hold for every deformation of ω? Such a question was addressed by Cukierman and Pereira in [5]. Here, we use our results to recover the theorem of Cukierman-Pereira as a special case.

As was observed before, every time we have a codimension 1 Pfaff system

$$0 \to I(\mathcal{F}) \to \Omega^1_X \to \Omega^1_{\mathcal{F}} \to 0$$

such that $\Omega^1_{\mathcal{F}}$ is torsion-free, it follows that $I(\mathcal{F})$ must be a line bundle. In the case $X = \mathbb{P}^n(\mathbb{C})$, we thus have $I(\mathcal{F}) \cong \mathcal{O}_{\mathbb{P}^n}(-d)$ for some $d \in \mathbb{Z}$. It is then equivalent to give a Pfaff system and to give a morphism $0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \to \Omega^1_{\mathbb{P}^n}$ which is in turn equivalent to $0 \to \mathcal{O}_{\mathbb{P}^n} \to \Omega^1_{\mathbb{P}^n}(d)$, that is, to give a global section ω of the sheaf $\Omega^1_{\mathbb{P}^n}(d)$.

We can explicitly write such a global section as

$$\omega = \sum_{i=0}^{n} f_i(x_0, \ldots, x_n) dx_i,$$

with f_i a homogeneous polynomial of degree $d-1$ and such that $\sum_i x_i f_i = 0$.

Such an expression gives rise to a foliation with split tangent sheaf if and only if there are $n-1$ polynomial vector fields

$$\begin{aligned} X_1 &= g_1^0 \frac{\partial}{\partial x_0} + \cdots + g_1^n \frac{\partial}{\partial x_n}, \\ &\vdots \\ X_{n-1} &= g_{n-1}^0 \frac{\partial}{\partial x_0} + \cdots + g_{n-1}^n \frac{\partial}{\partial x_n} \end{aligned}$$

such that $\omega(X_i) = 0$, for all $1 \leq i \leq n-1$; moreover, on a generic point, the vector fields must be linearly independent.

The singular set of this foliation is given by the ideal $I = (f_0, \ldots, f_n)$. The condition $\omega(X_i) = 0$ means that the ring $\mathbb{C}[x_0, \ldots, x_n]/I$ admits a syzygy of the form

$$0 \to \mathbb{C}[x_0, ..., x_n]^n \xrightarrow{\begin{pmatrix} x_0 & \cdots & x_n \\ g_1^0 & \cdots & g_1^n \\ \vdots & \ddots & \vdots \\ g_{n-1}^0 & \cdots & g_{n-1}^n \end{pmatrix}} \mathbb{C}[x_0, \ldots, x_n]^{n+1}$$

$$\xrightarrow{(f_0, \ldots, f_n)} \mathbb{C}[x_0, \ldots, x_n] \to \mathbb{C}[x_0, \ldots, x_n]/I \to 0.$$

For such rings a theorem of Hilbert and Schaps tells us the following:

Theorem 9.1 (Hilbert, Schaps). *Let $A = k[x_0, ...x_n]/I$ be such that there is a 3-step resolution of A as above by free modules. Then the ring A is Cohen-Macaulay; in particular, it is equidimensional.*

Proof. This is theorem 5.1 in [2] $\square$

We thus recover the theorem of Cukierman and Pereira ([5, Theorem 1]).

Theorem 9.2 ([5]). *Let*

$$0 \to I(\mathcal{F}) \to \Omega^1_{\mathbb{P}^n \times S|S} \to \Omega^1_{\mathcal{F}} \to 0$$

be a flat family of codimension 1 *integrable Pfaff systems, and suppose*

$$0 \to I(\mathcal{F})_s \to \Omega^1_{\mathbb{P}^n} \to \Omega^1_{\mathcal{F}} \to 0$$

defines a foliation with split tangent sheaf. If $\operatorname{sing}(\mathcal{F}_s) \setminus \overline{\mathcal{K}(\mathcal{F}_s)}$ *has codimension greater than* 2*, then every member of the family defines a split tangent sheaf foliation.*

Proof. By the above theorem, $\operatorname{sing}(\mathcal{F}_s)$ is equidimensional. The singular locus of a foliation on $\mathbb{P}^n$ always has an irreducible component of codimension 2 (see [14, Teorema 1.13]). If $\operatorname{sing}(\mathcal{F}_s) \setminus \overline{\mathcal{K}(\mathcal{F}_s)}$ has codimension greater than 2, then it must be empty. So $\operatorname{sing}(\mathcal{F}_s) = \overline{\mathcal{K}(\mathcal{F}_s)}$ and we can then apply Theorem 8.13. So the flat family

$$0 \to I(\mathcal{F}) \to \Omega^1_{\mathbb{P}^n \times S|S} \to \Omega^1_{\mathcal{F}} \to 0$$

gives rise to a flat family

$$0 \to T\mathcal{F} \to T_S X \to N_{\mathcal{F}} \to 0,$$

and so $T\mathcal{F}$ must be flat over S, and then $T\mathcal{F}_s$ splits for every $s \in S$. □

Remark 9.3. In [5], Theorem 2, a similar statement is proved for an arbitrary codimensional distribution (not necessarily involutive). More concretely, they prove:

Let D *be a singular holomorphic distribution on* $\mathbb{P}^n$*. If* $\operatorname{codim} \operatorname{sing}(D) \geq 3$ *and*

$$TD \cong \bigoplus_{i=1}^{d} \mathcal{O}(e_i), \qquad e_i \in \mathbb{Z},$$

then there exists a Zariski-open neighborhood $\mathcal{U}$ *of the space of Pfaff systems such that, for every Pfaff system* $\mathcal{I}' \subset \Omega^1_{\mathbb{P}^n}$ *in* $\mathcal{U}$*, its annihilator* TD' *splits as* $TD' \cong \bigoplus_{i=1}^{d} \mathcal{O}(e_i)$*.*

When we try to arrive at the above statement as a particular case of the the theory hereby exposed, we run into some difficulties, which we will explain now.

Note that generic (non-involutive) singular distributions have isolated singularities. So, in order to apply an analogue of Theorem 8.13 to non-involutive distributions of arbitrary codimension, we should be able to conclude, from the hypothesis $\operatorname{codim} \operatorname{sing}(D) \geq 3$, that D has isolated singularities. The author was unable to do so. It seems that a keener knowledge of the singular set of arbitrary codimensional distributions is necessary to correctly understand and generalize Theorem 2 in [5].

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FREE DIVISORS IN A PENCIL OF CURVES

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Abstract. A plane curve $D \subset \mathbb{P}^2(\boldsymbol{k})$, where $\boldsymbol{k}$ is a field of characteristic zero, is free if its associated sheaf $\mathcal{T}_D$ of vector fields tangent to D is a free $\mathscr{O}_{\mathbb{P}^2(\boldsymbol{k})}$-module (see [6] or [5] for a definition in a more general context). Relatively few free curves are known. Here we prove that the union of all singular members of a pencil of plane projective curves with the same degree and with a smooth base locus is a free divisor.

1. Introduction

Let $\boldsymbol{k}$ be a field of characteristic zero and let $S = \boldsymbol{k}[x, y, z]$ be the graded ring such that $\mathbb{P}^2 = \mathrm{Proj}(S)$. We write $\partial_x := \frac{\partial}{\partial x}$, $\partial_y := \frac{\partial}{\partial y}$, $\partial_z := \frac{\partial}{\partial z}$ and $\nabla F = (\partial_x F, \partial_y F, \partial_z F)$ for a homogenous polynomial $F \in S$.

Let $D = \{F = 0\}$ be a reduced curve of degree n. The kernel $\mathcal{T}_D$ of the map ∇F is a rank two reflexive sheaf, hence a vector bundle on $\mathbb{P}^2$. It is the rank two vector bundle of vector fields tangent along D, defined by the following exact sequence:

$$0 \longrightarrow \mathcal{T}_D \longrightarrow \mathscr{O}^3_{\mathbb{P}^2} \xrightarrow{\nabla F} \mathcal{J}_{\nabla F}(n-1) \longrightarrow 0,$$

where the sheaf $\mathcal{J}_{\nabla F}$ (also denoted $\mathcal{J}_{\nabla D}$ in this text) is the Jacobian ideal of F. Set theoretically $\mathcal{J}_{\nabla F}$ defines the singular points of the divisor D. For instance if D consists of s generic lines then $\mathcal{J}_{\nabla D}$ defines the set of $\binom{s}{2}$ vertices of D.

Remark 1.1. A non zero section $s \in \mathrm{H}^0(\mathcal{T}_D(a))$, for some shift $a \in \mathbb{N}$, corresponds to a derivation $\delta = P_a\partial_x + Q_a\partial_y + R_a\partial_z$ verifying $\delta(F) = 0$, where $(P_a, Q_a, R_a) \in \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^2}(a))^3$.

In some particular cases that can be found in [5], $\mathcal{T}_D$ is a free $\mathscr{O}_{\mathbb{P}^2}$-module; it means that there are two vector fields of degrees a and b that form a basis of $\bigoplus_n \mathrm{H}^0(\mathcal{T}_D(n))$ (D is said to be free with exponents (a, b)); it arises, for instance, when D is the union of the nine inflection lines of a smooth cubic curve. The notion of free divisor was introduced by Saito [6] for reduced divisors and studied by Terao [9] for hyperplane arrangements. Here we recall a definition of freeness for projective curves. For a more general definition we refer to Saito [6].

Definition 1.2. A reduced curve $D \subset \mathbb{P}^2$ is *free with exponents* $(a, b) \in \mathbb{N}^2$ if

$$\mathcal{T}_D \simeq \mathscr{O}_{\mathbb{P}^2}(-a) \oplus \mathscr{O}_{\mathbb{P}^2}(-b).$$

A smooth curve of degree ≥ 2 is not free, an irreducible curve of degree ≥ 3 with only nodes and cusps as singularities is not free (see [1, Example 4.5]). Actually few examples of free curves are known and of course very few families of free curves are known. One such family can be found in [8, Prop. 2.2].

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In a personal communication it was conjectured by E. Artal and J.I. Cogolludo that the union of all the singular members of a pencil of plane curves (assuming that the general one is smooth) should be free. Three different cases occur:

- the base locus is smooth (for instance the union of six lines in a pencil of conics passing through four distinct points);
- the base locus is not smooth but every curve in the pencil is reduced (for instance the four lines in a pencil of conics where two of the base points are infinitely near points);
- the base locus is not smooth and there exists exactly one non reduced curve in the pencil (for instance three lines in a pencil of bitangent conics).

In the third case the divisor of singular members is not reduced but its reduced structure is expected to be free.

We point out that if two distinct curves of the pencil are not reduced then all curves will be singular. Even in this case, we believe that a free divisor can be obtained by chosing a finite number of reduced components through all the singular points.

In this paper we prove that the union of all the singular members of a pencil of degree n plane curves with a smooth base locus (i.e. the base locus consists of n^2 distinct points) is a free divisor and we give its exponents (see theorem 2.7). More generally, we describe the vector bundle of logarithmic vector fields tangent to any union of curves of the pencil (see theorem 2.8) by studying one particular vector field "canonically tangent" to the pencil, that is introduced in the key lemma 2.1.

This gives already a new and easy method to produce free divisors.

I thank J. I. Cogolludo for his useful comments.

2. Pencil of plane curves

2.1. **Generalities and notations.** Let $\{f=0\}$ and $\{g=0\}$ be two reduced curves of degree $n\geq 1$ with no common component. For any $(\alpha,\beta)\in\mathbb{P}^1$ the curve $C_{\alpha,\beta}$ is defined by the equation $\{\alpha f+\beta g=0\}$ and $\mathcal{C}(f,g)=\{C_{\alpha,\beta}|(\alpha,\beta)\in\mathbb{P}^1\}$ is the pencil of all these curves.

In section 2 we will assume that the general member of the pencil $\mathcal{C}(f,g)$ is a smooth curve and that $C_{\alpha,\beta}$ is reduced for every $(\alpha,\beta)\in\mathbb{P}^1$.

Under these assumptions there are finitely many singular curves in $\mathcal{C}(f,g)$ but also finitely many singular points. We recall that the degree of the discriminant variety of degree n curves is $3(n-1)^2$ (it is a particular case of the Boole formula; see [10, Example 6.4]). Since the general curve in the pencil is smooth, the line defined by the pencil $\mathcal{C}(f,g)$ in the space of degree n curves meets the discriminant variety along a finite scheme of length $3(n-1)^2$ (not empty for $n\geq 2$). The number of singular points is of course related to the multiplicity of the singular curves in the pencil as we will see below.

Let us fix some notation. The scheme defined by the ideal sheaf $\mathcal{J}_{\nabla C_{\alpha_i,\beta_i}}$ is denoted by Z_{α_i,β_i}. It is well known that this scheme is locally a complete intersection (for instance, generalize to curves the lemma 2.4 in [7]). The union of all the singular members of the pencil $\mathcal{C}(f,g)$ form a divisor D^{sg}. A union of $k\geq 2$ distinct members of $\mathcal{C}(f,g)$ is denoted by D_k.

2.2. **Derivation tangent to a smooth pencil.** Let us consider the following derivation, associated "canonically" to the pencil:

Lemma 2.1. *For any union D_k of $k\geq 1$ members of the pencil there exists a non zero section $s_{\delta,k}\in \mathrm{H}^0(\mathcal{T}_{D_k}(2n-2))$ induced by the derivation*

$$\delta=(\nabla f\wedge\nabla g).\nabla=(\partial_y f\partial_z g-\partial_z f\partial_y g)\partial_x+(\partial_z f\partial_x g-\partial_x f\partial_z g)\partial_y+(\partial_x f\partial_y g-\partial_y f\partial_x g)\partial_z.$$

Proof. Since $\delta(\alpha f+\beta g)=\det(\nabla f,\nabla g,\nabla(\alpha f+\beta g))=0$ we have for any $k\geq 1$,

$$\delta(f)=\delta(g)=\delta(\alpha f+\beta g)=\delta(\prod_{i=1}^{k}(\alpha_i f+\beta_i g))=0.$$

According to the remark 1.1 it gives the desired section. □

Let us introduce a rank two sheaf $\mathcal{F}$ defined by the following exact sequence:

$$0\longrightarrow \mathscr{O}_{\mathbb{P}^2}(2-2n)\xrightarrow{\nabla f\wedge\nabla g}\mathscr{O}_{\mathbb{P}^2}^3\longrightarrow \mathcal{F}\longrightarrow 0.$$

If we denote by $\mathrm{sg}(\mathcal{F}):=\{p\in\mathbb{P}^2|\mathrm{rank}(\mathcal{F}\otimes\mathscr{O}_p)>2\}$ the set of singular points of $\mathcal{F}$, we have:

Lemma 2.2. *A point $p\in\mathbb{P}^2$ belongs to $\mathrm{sg}(\mathcal{F})$ if and only if two smooth members of the pencil share the same tangent line at p or one curve of the pencil is singular at p. Moreover $\mathrm{sg}(\mathcal{F})$ is a finite closed scheme with length $l(\mathrm{sg}(\mathcal{F}))=3(n-1)^2$.*

Remark 2.3. Let us precise that if two smooth members intersect then all the smooth members of the pencil intersect with the same tangency.

Remark 2.4. If the base locus of $\mathcal{C}(f,g)$ consists of n^2 distinct points then two curves of the pencil meet transversaly at the base points and $p\in\mathrm{sg}(\mathcal{F})$ if and only if p is a singular point for a unique curve $C_{\alpha,\beta}$ in the pencil and does not belong to the base locus. One can assume that $p\in\mathrm{sg}(\mathcal{F})$ is singular for $\{f=0\}$. Then, locally at p, the curve $\{g=0\}$ can be assumed to be smooth and the local ideals $(\nabla f\wedge\nabla g)_p$ and $(\nabla f)_p$ coincide. In other words, when the base locus is smooth, we have

$$\mathrm{sg}(\mathcal{F})=\sqcup_{i=1,\dots,s}Z_{\alpha_i,\beta_i}.$$

Proof. The singular locus of $\mathcal{F}$ is also defined by $\mathrm{sg}(\mathcal{F}):=\{p\in\mathbb{P}^2|(\nabla f\wedge\nabla g)(p)=0\}$. The zero scheme defined by $\nabla f\wedge\nabla g$ and $\nabla(\alpha f+\beta g)\wedge\nabla g$ are clearly the same; it means that the singular points of any member in the pencil is a singular point for $\mathcal{F}$. One can also obtain $(\nabla f\wedge\nabla g)(p)=0$ at a smooth point when the vectors $(\nabla f)(p)$ and $(\nabla g)(p)$ are proportional i.e. when two smooth curves of the pencil share the same tangent line at p.

Since every curve of the pencil is reduced $\mathrm{sg}(\mathcal{F})$ is finite, its length can be computed by writing the resolution of the ideal $\mathcal{J}_{\mathrm{sg}(\mathcal{F})}$ (for a sheaf of ideal $\mathcal{J}_Z$ defining a finite scheme Z of length $l(Z)$, we have $c_2(\mathcal{J}_Z)=l(Z)$). Indeed, if we dualize the following exact sequence

$$0\longrightarrow \mathscr{O}_{\mathbb{P}^2}(2-2n)\xrightarrow{\nabla f\wedge\nabla g}\mathscr{O}_{\mathbb{P}^2}^3\longrightarrow \mathcal{F}\longrightarrow 0$$

we find, according to Hilbert-Burch theorem,

$$0\longrightarrow \mathcal{F}^\vee\xrightarrow{(\nabla f,\nabla g)}\mathscr{O}_{\mathbb{P}^2}^3\xrightarrow{\nabla f\wedge\nabla g}\mathscr{O}_{\mathbb{P}^2}(2n-2).$$

It proves that $\mathcal{F}^\vee=\mathscr{O}_{\mathbb{P}^2}(1-n)^2$ and that the image of the last map is $\mathcal{J}_{\mathrm{sg}(\mathcal{F})}(2n-2)$.

Then $l(\mathrm{sg}(\mathcal{F}))=3(n-1)^2$. We point out that this number is the degree of the discriminant variety of degree n curves. □

Now let us call D_k the divisor defined by $k\geq 2$ members of the pencil and let us consider the section $s_{\delta,k}\in\mathrm{H}^0(\mathcal{T}_{D_k}(2n-2))$ corresponding (see remark 1.1) to the derivation δ. Let $Z_k:=Z(s_{\delta,k})$ be the zero locus of $s_{\delta,k}$.

Lemma 2.5. *The section $s_{\delta,k}$ vanishes in codimension at least two.*

Proof. Let us consider the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{O}_{\mathbb{P}^2}(2-2n) & \xrightarrow{s_{\delta,k}} & \mathcal{T}_{D_k} & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{O}_{\mathbb{P}^2}(2-2n) & \xrightarrow{\nabla f \wedge \nabla g} & \mathscr{O}_{\mathbb{P}^2}^3 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{J}_{\nabla D_k}(nk-1) & = & \mathcal{J}_{\nabla D_k}(nk-1) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where $\mathcal{Q} = \operatorname{coker}(s_{\delta,k})$. Assume that Z_k contains a divisor H. Tensor now the last vertical exact sequence of the above diagram by $\mathscr{O}_p$ for a general point $p \in H$. Since p does not belong to the Jacobian scheme defined by $\mathcal{J}_{\nabla D_k}$ we have $\mathcal{J}_{\nabla D_k} \otimes \mathscr{O}_p = \mathscr{O}_p$ and $\operatorname{Tor}_1(\mathcal{J}_{\nabla D_k}, \mathscr{O}_p) = 0$. Since $p \in H \subset Z_k$ we have $\operatorname{rank}(\mathcal{Q} \otimes \mathscr{O}_p) \geq 2$; it implies $\operatorname{rank}(\mathcal{F} \otimes \mathscr{O}_p) \geq 3$ in other words that $p \in \operatorname{sg}(\mathcal{F})$; this contradicts $\operatorname{codim}(\operatorname{sg}(\mathcal{F}), \mathbb{P}^2) = 2$, proved in lemma 2.2.

Then $\mathcal{Q}$ is the ideal sheaf of the codimension two scheme Z_k, i.e. $\mathcal{Q} = \mathcal{J}_{Z_k}(n(2-k)-1)$ and we have an exact sequence

$$0 \longrightarrow \mathcal{J}_{Z_k}(n(2-k)-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{J}_{\nabla D_k}(nk-1) \longrightarrow 0.$$

□

From this commutative diagram we obtain the following lemma.

Lemma 2.6. *Let D_k be a union of $k \geq 2$ members of $\mathcal{C}(f,g)$. Then*

$$c_2(\mathcal{J}_{\nabla D_k}) + c_2(\mathcal{J}_{Z_k}) = 3(n-1)^2 + n^2(k-1)^2.$$

Proof. According to the above commutative diagram we compute $c_2(\mathcal{F})$ in two different ways. The horizontal exact sequence gives $c_2(\mathcal{F}) = 4(n-1)^2$ when the vertical one gives

$$c_2(\mathcal{F}) = c_2(\mathcal{J}_{\nabla D_k}) + c_2(\mathcal{J}_{Z_k}) + (n-1)^2 - n^2(k-1)^2.$$

The lemma is proved by eliminating $c_2(\mathcal{F})$. □

2.3. **Free divisors in the pencil.** When D_k contains the divisor D^{sg} of all the singular members of the pencil we show now that it is free with exponents $(2n-2, n(k-2)+1)$.

Theorem 2.7. *Assume that the base locus of the pencil $\mathcal{C}(f,g)$ is smooth. Then,*

$$D_k \supseteq D^{\mathrm{sg}} \Leftrightarrow \mathcal{T}_{D_k} = \mathscr{O}_{\mathbb{P}^2}(2-2n) \oplus \mathscr{O}_{\mathbb{P}^2}(n(2-k)-1).$$

Proof. Assume first that $D_k \supseteq D^{\mathrm{sg}}$. Then the singular locus of D_k defined by the Jacobian ideal consists of the base points of the k curves in the pencil and, since it contains all the singular members, of the whole set of singularities of the curves in the pencil. This last set has length $3(n-1)^2$ by lemma 2.2 and remark 2.4. Moreover the subscheme supported by the base points in the scheme defined by $\mathcal{J}_{\nabla D_k}$ has length $n^2(k-1)^2$ since at each point among the n^2 points of the base locus of the pencil, the k curves meet transversaly and define k different directions (i.e. the local ring at the point $(0,0,1)$ is isomorphic to $\boldsymbol{k}[x,y]/(x^{k-1},y^{k-1})$). Then

$$c_2(\mathcal{J}_{\nabla D_k}) = 3(n-1)^2 + n^2(k-1)^2,$$

which implies that $Z_k = \emptyset$. In other words there is an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(2-2n) \xrightarrow{s_{\delta,k}} \mathcal{T}_{D_k} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(n(2-k)-1) \longrightarrow 0.$$

And such an exact sequence splits.

Conversely, assume that there is a singular member C that does not belong to D_k. Let $p \in C$ be one of its singular point. Since $p \notin D_k$ we have $\mathcal{J}_{\nabla D_k} \otimes \mathcal{O}_p = \mathcal{O}_p$ and $\mathrm{Tor}_1(\mathcal{J}_{\nabla D_k}, \mathcal{O}_p) = 0$. Consider the following exact sequence that comes from the commutative diagram above (in the proof of lemma 2.5) when $Z_k = \emptyset$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(n(2-k)-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{J}_{\nabla D_k}(nk-1) \longrightarrow 0.$$

If we tensor this exact sequence by $\mathcal{O}_p$ we find $\mathrm{rank}(\mathcal{F} \otimes \mathcal{O}_p) = 2$. This contradicts $p \in \mathrm{sg}(\mathcal{F})$ that was proved in lemma 2.2. $\square$

2.4. **Singular members ommitted.** When $D_k \supset D^{\mathrm{sg}}$ we have seen in theorem 2.7 that $Z_k = \emptyset$ by computing the length of the scheme defined by the Jacobian ideal of D_k. More generally we can describe, at least when the base locus is smooth, the scheme Z_k for any union of curves of the pencil.

Theorem 2.8. *Assume that the base locus of the pencil $\mathcal{C}(f,g)$ is smooth. Assume also that $D_k \supset D^{\mathrm{sg}} \setminus \bigcup_{i=1,\dots,r} C_{\alpha_i,\beta_i}$, $C_{\alpha_i,\beta_i} \not\subseteq D_k$ and C_{α_i,β_i} is a singular curve for $i = 1,\dots,r$. Then,*

$$\mathcal{J}_{Z_k} = \mathcal{J}_{\nabla C_{\alpha_1,\beta_1}} \otimes \cdots \otimes \mathcal{J}_{\nabla C_{\alpha_r,\beta_r}}.$$

Remark 2.9. When $r = 0$ we obtain the freeness again.

Proof. Since the set of singular points of two disctinct curves are disjoint, it is enough to prove it for $r = 1$ (i.e. $D_k \supset D^{\mathrm{sg}} \setminus C_{\alpha_1,\beta_1}$ and $C_{\alpha_1,\beta_1} \not\subseteq D_k$). Recall that $\mathcal{E}xt^1(\mathcal{J}_Z, \mathcal{O}_{\mathbb{P}^2}) = \omega_Z$ where Z is a finite scheme and ω_Z its dualizing sheaf (see [2, Chapter III, section 7]); it is well known that, since the finite scheme Z is locally complete intersection, $\omega_Z = \mathcal{O}_Z$.

Then the dual exact sequence of

$$0 \longrightarrow \mathcal{J}_{Z_k}(n(2-k)-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{J}_{\nabla D_k}(nk-1) \longrightarrow 0$$

is the long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1-nk) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1-n)^2 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(n(k-2)+1) \longrightarrow \\ \longrightarrow \omega_{\nabla D_k} \longrightarrow \mathcal{O}_{\mathrm{sg}(\mathcal{F})} \longrightarrow \omega_{Z_k} \longrightarrow 0. \end{aligned}$$

The map $\mathcal{F} \longrightarrow \mathcal{J}_{\nabla D_k}(nk-1)$ can be described by composition; indeed it is given by two polynomials (U,V) such that

$$(U,V).(\nabla f, \nabla g) = \nabla(\prod_i (\alpha_i f + \beta_i g)).$$

We find, $U = \sum_i \alpha_i \prod_{j\neq i}(\alpha_j f + \beta_j g)$ and $V = \sum_i \beta_i \prod_{j\neq i}(\alpha_j f + \beta_j g)$. If a point p belongs to one curve C_{α_1,β_1} in D_k and does not belong to the base locus, then $U(p) \neq 0$ and $V(p) \neq 0$. It shows that these two polynomials vanish simultaneously and precisely along the base locus. Then the complete intersection $T = \{U=0\} \cap \{V=0\}$ of length $n^2(k-1)^2$ is supported exactly by the base points. We have

$$0 \longrightarrow \omega_{\nabla D_k}/\mathcal{O}_T \longrightarrow \mathcal{O}_{\mathrm{sg}(\mathcal{F})} \longrightarrow \omega_{Z_k} \longrightarrow 0.$$

We have already seen that the subscheme supported by the base points in the scheme defined by $\mathcal{J}_{\nabla D_k}$ has length $n^2(k-1)^2$. It implies that $\omega_{\nabla D_k}/\mathcal{O}_T = \oplus_{i=2,\dots,s}\mathcal{O}_{Z_{\alpha_i,\beta_i}}$. According to remark 2.4, $\mathcal{O}_{\mathrm{sg}(\mathcal{F})} = \oplus_{i=1,\dots,s}\mathcal{O}_{Z_{\alpha_i,\beta_i}}$. This proves $\omega_{Z_k} = \mathcal{O}_{Z_{\alpha_1,\beta_1}}$.

□

There are exact sequences relating the vector bundles $\mathcal{T}_{D_k}$ and $\mathcal{T}_{D_k\setminus C}$ when $C \subset D_k$.

Proposition 2.10. *We assume that the base locus of the pencil $\mathcal{C}(f,g)$ is smooth and that D_k contains D^{sg}. Let C be a singular member in $\mathcal{C}(f,g)$ and Z its scheme of singular points. Then there is an exact sequence*

$$0 \longrightarrow \mathcal{T}_{D_k} \longrightarrow \mathcal{T}_{D_k\setminus C} \longrightarrow \mathcal{J}_{Z/C}(n(3-k)-1) \longrightarrow 0,$$

where $\mathcal{J}_{Z/C} \subset \mathcal{O}_C$ defines Z into C.

Proof. The derivation $(\nabla f \wedge \nabla g).\nabla$ is tangent to D_k then also to $D_k\setminus C$. It induces the following commutative diagram which proves the proposition:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(2-2n) \arrow[r] \arrow[d, equal] & \mathcal{T}_{D_k} \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^2}(n(2-k)-1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(2-2n) \arrow[r] & \mathcal{T}_{D_k\setminus C} \arrow[r] \arrow[d] & \mathcal{J}_Z(n(3-k)-1) \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{J}_{Z/C}(n(3-k)-1) \arrow[r, equal] \arrow[d] & \mathcal{J}_{Z/C}(n(3-k)-1) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

□

3. The pencil contains a non-reduced curve

When the pencil $\mathcal{C}(f,g)$ contains a non-reduced curve, the arguments used in the previous sections are not valid since the scheme defined by the jacobian ideal contains a divisor. We have to remove this divisor somehow. Remember that if two curves of the pencil are multiple then the general curve is singular. So let us consider that there is only one curve that is not reduced. Let $hh_1^{r_1}\cdots h_s^{r_s} = 0$ be the equation of this unique non-reduced curve where $h=0$ is reduced, $\deg(h_i) = m_i \geq 1$ and $r_i \geq 2$. Since the derivation $\frac{1}{\prod_i h_i^{r_i}}(\nabla f \wedge \nabla g).\nabla$ is still tangent to all curves of the pencil, we believe that the following statement is true:

Conjecture. *Let $hh_1^{r_1}\cdots h_s^{r_s} = 0$ be the equation of the unique non-reduced curve where $\{h=0\}$ is reduced, $\deg(h_i) = m_i \geq 1$ and $r_i \geq 2$. Then,*

$$D_k \supseteq D^{\mathrm{sg}} \Leftrightarrow \mathcal{T}_{D_k} = \mathcal{O}_{\mathbb{P}^2}(2-2n+\sum_{i=1}^{i=s}(r_i-1)m_i) \oplus \mathcal{O}_{\mathbb{P}^2}(n(2-k)-1).$$

4. Examples

Let us call $\Sigma_3 \subset \mathbb{P}^9 = \mathbb{P}(\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^2}(3)))$ the hypersurface of singular cubics. It is well known that its degree is 12 (see [3], for instance).

- **Pappus arrangement freed by nodal cubics**: Let us consider the divisor of the nine lines appearing in the Pappus arrangement; this divisor is the union of three triangles T_1, T_2, T_3 with nine base points. The pencil generated by T_1 and T_2 contains 3 triangles (each one represents a triple point in Σ_3); since $9 < 12$, singular cubics are missing in the pencil. There is no other triangle and no smoth conic+line in the Pappus pencil, when it is general enough. We can conclude that the missing cubics are, in general, nodal cubics C_1, C_2, C_3.

  Let $D = T_1 \cup T_2 \cup T_3 \cup C_1 \cup C_2 \cup C_3$ be the union of all singular fibers in the pencil generated by T_1 and T_2. Then, according to theorem 2.7 we have

$$\mathcal{T}_D = \mathscr{O}_{\mathbb{P}^2}(-4) \oplus \mathscr{O}_{\mathbb{P}^2}(-13).$$

- **Pappus arrangement**: Let $T_1 \cup T_2 \cup T_3$ be the divisor consisting of the nine lines of the projective Pappus arrangement and $D = T_1 \cup T_2 \cup T_3 \cup C_1 \cup C_2 \cup C_3$ be the union of all singular fibers in the pencil generated by two triangles among the T_i's. Let us call $K := C_1 \cup C_2 \cup C_3$ the union of the nodal cubics and z_1, z_2, z_3 their nodes. Then we have, according to theorem 2.8, $\mathcal{T}_D = \mathscr{O}_{\mathbb{P}^2}(-4) \oplus \mathscr{O}_{\mathbb{P}^2}(-13)$ and an exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^2}(-4) \longrightarrow \mathcal{T}_{D\setminus K} \longrightarrow \mathcal{I}_{z_1,z_2,z_3}(-4) \longrightarrow 0.$$

  The logarithmic bundle $\mathcal{T}_{D\setminus K}$ associated to the Pappus configuration is semi-stable and its divisor of jumping lines is the triangle $z_1^\vee \cup z_2^\vee \cup z_3^\vee$ as it is proved by retricting the above exact sequence to any line through one of the zeroes.

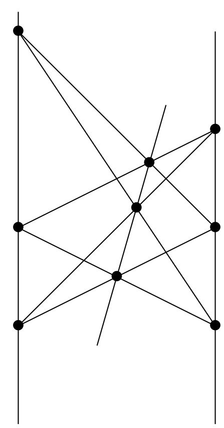

Figure 1. Pappus arrangement

- **Hesse arrangement**: Let us consider the pencil generated by a smooth cubic C and its hessian $\mathrm{Hess}(C)$. The pencil contains 4 triangles T_1, T_2, T_3, T_4 and since the degree of Σ_3 is 12, no other singular cubic can be present. Let us call D the union of these four triangles. Then, according to theorem 2.7 we have

$$\mathcal{T}_D = \mathscr{O}_{\mathbb{P}^2}(-4) \oplus \mathscr{O}_{\mathbb{P}^2}(-7).$$

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