Journal *of* Singularities
Volume 11 (2015), 33-51

received: 3 September 2014
in revised form: 3 March 2015
DOI: 10.5427/jsing.2015.11b

RESONANT BANDS, AOMOTO COMPLEX, AND REAL 4-NETS

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Abstract. The resonant band is a useful notion for the computation of the nontrivial monodromy eigenspaces of the Milnor fiber of a real line arrangement. In this article, we develop the resonant band description for the cohomology of the Aomoto complex. As an application, we prove that real 4-nets do not exist.

1. Introduction

Combinatorial decisions of topological invariants are the central problems in the theory of hyperplane arrangements. Milnor fibers and their eigenspace decompositions have received a lot of attention and have been studied by diverse techniques ([23]) (e.g., Alexander polynomials, Hodge theory, nets and multinets, covering spaces, Salvetti complexes, characteristic and resonance varieties etc.) Among others, the authors follow the previous studies using real structures, ([24, 29, 30]) and Aomoto complexes over finite fields, [4, 7, 17, 20].

Concerning the relation between Milnor fibers and Aomoto complexes, two key results were obtained by Papadima and Suciu [20, 21].

$$\text{Monodromy eigenspaces} \overset{(1)}{\longleftrightarrow} \text{Aomoto complex} \overset{(2)}{\longleftrightarrow} \text{Multinets}$$

The first one is an upper bound for the rank of the eigenspaces in terms of the Betti numbers of the Aomoto complexes over finite fields [20]. It was subsequently used by many authors to prove vanishing theorems [1, 2, 8, 17]. The second one is the bijective correspondence between 3-nets and nonzero elements in the cohomology group of the Aomoto complex over $\mathbb{F}_3$. A degree one element of the Orlik-Solomon algebra over the finite field $\mathbb{F}_q$ is bijectively corresponding to the coloring (with q-colors) of the arrangement. Papadima and Suciu succeeded to translate the cocycle condition into combinatorics of coloring [21]. A deep relation between nontrivial eigenspaces and multinet structures had been conjectured. Papadima-Suciu's results provide a beautiful framework to understand the nontrivial eigenspaces via multinets.

If we restrict our attention to real arrangements, the real structure contains a lot of information about the topology of the complexification. The resonant band, introduced in [29, 30], is a useful tool for computing nontrivial eigenspaces of the Milnor fibers and local system cohomology groups. The purpose of this paper is to introduce the notion of resonant bands for the Aomoto complex (over any coefficient ring) of a real arrangement. Then combining resonant bands techniques with the above Papadima-Suciu's picture (over $\mathbb{F}_2$), we prove that real 4-nets do not exist, which is a partial answer to a conjecture that the Hessian arrangement is the only 4-net.

The paper is organized as follows. §2 is a summary of well known facts on multinets and Orlik-Solomon algebras. Especially, we describe in detail the transformation of the Orlik-Solomon algebra when we exchange the hyperplane at infinity, which will be used later. §3 is a summary of the recent work by Papadima-Suciu. The crucial result that we use later is Proposition 3.4. Proposition 3.4 translates the cocycle conditions of the Aomoto complex (over $\mathbb{F}_2$) into combinatorial structures of subarrangements. §4 is the main part of this paper. After recalling a description of the Aomoto complex in terms of chambers in §4.1 (following [27]), we introduce the notion of η-resonant band in §4.2. In the main theorem (Theorem 4.8), we prove that the cohomology of the Aomoto complex is isomorphic to a submodule of the free module generated by resonant bands under a certain non-resonant condition at infinity. When the coefficient ring of the Aomoto complex is $\mathbb{F}_2$, everything can be described in terms of combinatorics of subarrangements. This translation is done in §4.3. In §5, we prove the non-existence of real 4-nets. The key result is the Non-Separation Theorem 5.1 in §5.1 which concerns subarrangements corresponding to cocycles of the Aomoto complex over $\mathbb{F}_2$ with differential given by the diagonal element. The Non-Separation Theorem asserts that at an intersection of multiplicity 4, the subarrangement corresponding to a nontrivial cohomology class has a special ordering. This assertion heavily relies on the real structure. Therefore, at this moment, it seems hopeless to generalize our argument to the complex case. Assuming a real 4-net exists, it is easy to construct a subarrangement which contradicts the Non-Separation Theorem. Hence real 4-nets do not exist (§5.2). (This fact was first proved by Cordovil-Forge [6, Lem. 2.4]. Our arguments prove a little bit stronger version. See Remark 5.3.)

2. Preliminaries

2.1. **Conventions.** In this paper, three types of hyperplane arrangements appear: affine arrangements in $\mathbb{K}^\ell$, arrangements in the projective space $\mathbb{KP}^\ell$ and central arrangements in $\mathbb{K}^{\ell+1}$. It is better to distinguish them by notations ([18, 19]).

- $\mathcal{A} = \{H_1, \dots, H_n\}$ denotes an arrangement of affine hyperplanes in the affine ℓ-space $\mathbb{K}^\ell$.
- $\widetilde{\mathcal{A}} = c\mathcal{A} = \{\widetilde{H}_0, \widetilde{H}_1, \dots, \widetilde{H}_n\}$ denotes the coning of $\mathcal{A}$, which is a central hyperplane arrangement in $\mathbb{K}^{\ell+1}$. The hyperplane $\widetilde{H}_0$ is corresponding to the hyperplane at infinity of $\mathcal{A}$.
- $\overline{\mathcal{A}} = \{\overline{H}_0, \overline{H}_1, \dots, \overline{H}_n\}$ denotes the projectivization of $\widetilde{\mathcal{A}}$, which is a hyperplane arrangement induced by $\widetilde{\mathcal{A}}$ in the projective ℓ-space $\mathbb{KP}^\ell$.
- $\mathsf{d}_{\widetilde{H}_i}\widetilde{\mathcal{A}} = \{\mathsf{d}_{\widetilde{H}_i}\widetilde{H}_0, \dots, \widehat{\mathsf{d}_{\widetilde{H}_i}\widetilde{H}_i}, \dots, \mathsf{d}_{\widetilde{H}_i}\widetilde{H}_n\}$ denotes the deconing of $\widetilde{\mathcal{A}}$ with respect to the hyperplane $\widetilde{H}_i$. Note that $\mathsf{d}_{\widetilde{H}_0}\widetilde{\mathcal{A}} = \mathcal{A}$.

Other frequently used notations are:

- R: a commutative ring (unless stated otherwise),
- $\mathbb{K}$: a field,
- $M(\mathcal{A})$: the complexified complement of $\mathcal{A}$.

2.2. **Multinets.** In this subsection, we recall several facts on multinets.

Definition 2.1. *Let $\overline{\mathcal{A}} = \{\overline{H}_0, \dots, \overline{H}_n\}$ be a projective line arrangement in $\mathbb{CP}^2$. Let $k \geq 3$ and $d \geq 2$ be integers. A* (reduced) (k,d)-multinet *(or k-*multinet *for simplicity) on $\mathcal{A}$ is a pair $(\mathcal{N}, \mathcal{X})$, where $\mathcal{N}$ is a partition of $\mathcal{A}$ into k classes $\overline{\mathcal{A}} = \overline{\mathcal{A}}_1 \sqcup \cdots \sqcup \overline{\mathcal{A}}_k$ and $\mathcal{X} \subset \mathbb{CP}^2$ is a set of multiple points (called the base locus) such that*

- *(i) $|\overline{\mathcal{A}}_1| = \cdots = |\overline{\mathcal{A}}_k| =: d$;*
- *(ii) $\overline{H} \in \overline{\mathcal{A}}_i$ and $\overline{H}' \in \overline{\mathcal{A}}_j$ $(i \neq j)$ imply that $\overline{H} \cap \overline{H}' \in \mathcal{X}$;*
- *(iii) for all $p \in \mathcal{X}$, $n_p := |\{\overline{H} \in \overline{\mathcal{A}}_i \mid \overline{H} \ni p\}|$ is constant and independent of i;*
- *(iv) for any $\overline{H}, \overline{H}' \in \overline{\mathcal{A}}_i$ $(i = 1, \dots, k)$, there is a sequence $\overline{H} = \overline{H}'_0, \overline{H}'_1, \dots, \overline{H}'_r = \overline{H}'$ in $\overline{\mathcal{A}}_i$ such that $\overline{H}'_{j-1} \cap \overline{H}'_j \notin \mathcal{X}$ for $1 \leq j \leq r$.*

If $n_p = 1$ for every $p \in \mathcal{X}$, then $(\mathcal{N}, \mathcal{X})$ is called a net.

Note that in the previous Definition (ii) and (iii) implies (i), and, by [13], $\mathcal{N}$ and $\mathcal{X}$ determine each other. Moreover, note that if $(\mathcal{N}, \mathcal{X})$ is a (k,d)-net, then each $p \in \mathcal{X}$ has multiplicity k.

The next theorem, which combines results of Pereira and Yuzvinsky [22, 33], summarizes what is known about the existence of non-trivial multinets on arrangements (see also [23, 3, 32] for more results).

Theorem 2.2. *Let $\overline{\mathcal{A}}$ be a k-multinet, with base locus $\mathcal{X}$. Then*

1. *If $|\mathcal{X}| > 1$, then $k = 3$ or 4.*
2. *If there is a hyperplane $\overline{H} \in \overline{\mathcal{A}}$ such that $m_H > 1$, then $k = 3$.*
3. *If $k = 4$, then $|\mathcal{X}| = d^2$ and $\overline{\mathcal{A}}$ is a $(4,d)$-net.*

Although several infinite families of multinets with $k = 3$ are known, only one multinet with $k = 4$ is known to exist: the $(4,3)$-net on the Hessian arrangement (which is defined over $\mathbb{Q}(\sqrt{-3})$). It is conjectured that the only $(4,d)$-net is the Hessian arrangement. In [10], it is proved that the Hessian is the unique $(4,d)$-net for $d \leq 6$ (hence for $|\mathcal{A}| \leq 24$). We will later prove that real $(4,d)$-nets do not exist, for any d.

2.3. **Orlik-Solomon algebra and Aomoto complex.** Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement of affine hyperplanes in $\mathbb{K}^\ell$ and R be a commutative ring. Let $E_1 = \bigoplus_{j=1}^n Re_j$ be the free module generated by $e_1, e_2, \ldots, e_n$, where e_i is a symbol corresponding to the hyperplane H_i. Let $E = \wedge E_1$ be the exterior algebra over R. The algebra E is graded via $E = \bigoplus_{p=0}^n E_p$, where $E_p = \wedge^p E_1$. The R-module E_p is free and has the distinguished basis consisting of monomials $e_S := e_{i_1} \wedge \cdots \wedge e_{i_p}$, where $S = \{i_1, \ldots, i_p\}$ is running through all the subsets of $\{1, \ldots, n\}$ of cardinality p and $i_1 < i_2 < \cdots < i_p$. The graded algebra E is a commutative DGA with respect to the differential ∂ of degree -1 uniquely defined by the conditions $\partial e_i = 1$ for all $i = 1, \ldots, n$ and the graded Leibniz formula. Then for every $S \subset \{1, \ldots, n\}$ of cardinality p

$$\partial e_S = \sum_{j=1}^{p} (-1)^{j-1} e_{S_j},$$

where S_j is the complement in S to its j-th element.

For every $S \subset \{1, \ldots, n\}$, put $\cap S = \bigcap_{i \in S} H_i$ (possibly $\cap S = \emptyset$). The set of all intersections $L(\mathcal{A}) := \{\cap S \mid S \subset \{1, \ldots, n\}\}$ is called the intersection poset. The subset $S \subset \{1, \ldots, n\}$ is called *dependent* if $\cap S \neq \emptyset$ and the set of linear polynomials $\{\alpha_i \mid i \in S\}$ with $H_i = \alpha_i^{-1}(0)$, is linearly dependent.

Definition 2.3. *The* Orlik-Solomon ideal *of $\mathcal{A}$ is the ideal $I = I(\mathcal{A})$ of E generated by*

1. *all e_S with $\cap S = \emptyset$ and*
2. *all ∂e_S with S dependent.*

The algebra $A := A_R^\bullet(\mathcal{A}) = E/I(\mathcal{A})$ is called the Orlik-Solomon algebra *of $\mathcal{A}$.*

Clearly I is a homogeneous ideal of E whence A is a graded algebra and we can write $A = \bigoplus_{p \geq 0} A_R^p$, where $A_R^p = E_p/(I \cap E_p)$. If $\mathcal{A}$ is central, then for any $S \subset \mathcal{A}$, we have $\cap S \neq \emptyset$. Therefore, the Orlik-Solomon ideal is generated by the elements of type (2) from Definition 2.3. In this case, the map ∂ induces a well-defined differential $\partial \colon A_R^\bullet(\mathcal{A}) \longrightarrow A_R^{\bullet-1}(\mathcal{A})$.

Recall that [18, Cor. 3.73], for each p, we can write (Brieskorn decomposition)

$$A_R^p(\mathcal{A}) = \bigoplus_{X \in L_p(\mathcal{A})} A_R^p(\mathcal{A}_X), \tag{1}$$

where $L_p(\mathcal{A}) := \{X \in L(\mathcal{A}) \mid \operatorname{codim} X = p\}$ and $\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subset H\}$. See for example [18, Corollary 3.73].

Recall that the coning $\widetilde{\mathcal{A}} = c\mathcal{A} = \{\widetilde{H}_0, \widetilde{H}_1, \ldots, \widetilde{H}_n\}$ of $\mathcal{A}$ is a central arrangement in $\mathbb{K}^{\ell+1}$. We denote the corresponding generators of the Orlik-Solomon algebra $A^\bullet_R(\widetilde{\mathcal{A}})$ by $\widetilde{e}_0, \widetilde{e}_1, \ldots, \widetilde{e}_n$. The map

$$\iota\colon A^1_R(\mathcal{A}) \longrightarrow A^1_R(\widetilde{\mathcal{A}}) : e_i \longmapsto \widetilde{e}_i - \widetilde{e}_0,$$

induces an injective R-algebra homomorphism $\iota\colon A^\bullet_R(\mathcal{A}) \longrightarrow A^\bullet_R(\widetilde{\mathcal{A}})$ ([31]). The image of the embedding ι is equal to the subalgebra

$$A^\bullet_R(\widetilde{\mathcal{A}})_0 := \{\omega \in A^\bullet_R(\widetilde{\mathcal{A}}) \mid \partial(\omega) = 0\}$$

of $A^\bullet_R(\widetilde{\mathcal{A}})$.

Consider the deconing $\mathcal{A}' := \mathsf{d}_{\widetilde{H}_i}\widetilde{\mathcal{A}} = \{H'_0, \ldots, \widehat{H'_i}, \ldots, H'_n\}$ with respect to the hyperplane $\widetilde{H}_i \in \widetilde{\mathcal{A}}$. We denote the generators of the Orlik-Solomon algebra $A^\bullet_R(\mathcal{A}')$ by $e'_0, \ldots, \widehat{e'_i}, \ldots, e'_n$. Then the Orlik-Solomon algebras of $\mathcal{A}$ and $\mathcal{A}'$ are isomorphic. The explicit isomorphism is given by

$$e_j \longmapsto \begin{cases} e'_j - e'_0, & \text{if } 1 \le j \le n, j \ne i, \\ -e'_0, & \text{if } j = i. \end{cases}$$

Let us fix an element $\eta = \sum_{i=1}^n a_i e_i \in A^1_R(\mathcal{A})$. Since $\eta \wedge \eta = 0$,

$$0 \longrightarrow A^1_R(\mathcal{A}) \xrightarrow{\eta} A^2_R(\mathcal{A}) \xrightarrow{\eta} \cdots \xrightarrow{\eta} A^\ell_R(\mathcal{A}) \xrightarrow{\eta} 0$$

forms a cochain complex, $(A^\bullet_R(\mathcal{A}), \eta)$, which is called the *Aomoto complex*. By the above embedding ι, we can identify the Aomoto complex $(A^\bullet_R(\mathcal{A}), \eta)$ with $(A^\bullet_R(\widetilde{\mathcal{A}})_0, \widetilde{\eta})$, where

$$\widetilde{\eta} = \iota(\eta) = \sum_{i=1}^n a_i \widetilde{e}_i - (a_1 + \cdots + a_n)\widetilde{e}_0 \quad ([11]).$$

3. Mod p Aomoto complex and the Papadima-Suciu correspondence

In this section, we recall recent results by Papadima and Suciu [21]. They found a way of constructing a 3-net from a non-trivial element of the cohomology of Aomoto complex over $\mathbb{F}_3$. Let $\overline{\mathcal{A}} = \{\overline{H}_0, \ldots, \overline{H}_n\}$ be a line arrangement on the projective plane $\mathbb{KP}^2$ with $3 | |\overline{\mathcal{A}}|$. Assume that there do not exist multiple points of multiplicity $\{3r \mid r \in \mathbb{Z}, r > 1\}$. Let

$$\widetilde{\eta}_0 := \sum_{i=0}^n \widetilde{e}_i \in A^1_{\mathbb{F}_3}(\widetilde{\mathcal{A}})_0$$

be the diagonal element. Then there is a natural bijective correspondence:

$$(H^1(A^\bullet_{\mathbb{F}_3}(\widetilde{\mathcal{A}})_0, \widetilde{\eta}_0) \setminus \{0\})/\mathbb{F}_3^\times \xrightarrow{\simeq} \left\{ \begin{array}{c} \text{Isomorphism classes of} \\ \text{3-net structures on } \overline{\mathcal{A}} \end{array} \right\}. \tag{2}$$

The correspondence is explicitly given by $H^1(A^\bullet_{\mathbb{F}_3}(\widetilde{\mathcal{A}})_0, \widetilde{\eta}_0) \ni \omega = \sum_{i=0}^n a_i \widetilde{e}_i \longmapsto (\overline{\mathcal{A}}_0, \overline{\mathcal{A}}_1, \overline{\mathcal{A}}_2)$, where $\overline{\mathcal{A}}_m = \{\overline{H}_i \mid a_i = m\}$ $(m = 0, 1, 2)$. The point of the above correspondence is that by using the local structures of the Orlik-Solomon algebra, we can translate the cocycle condition into the combinatorial conditions of $(\overline{\mathcal{A}}_0, \overline{\mathcal{A}}_1, \overline{\mathcal{A}}_2)$, which turn out to be exactly the defining conditions of 3-nets. Later we will employ a similar consideration for the Aomoto complex over $\mathbb{F}_2$ which we summarize in this section.

3.1. **A local lemma.** The next lemma (cf. [16, §3]) is useful for analyzing the map $\eta\colon A^1_R(\mathcal{A}) \longrightarrow A^2_R(\mathcal{A})$ by the Brieskorn decomposition (1).

Lemma 3.1. *Let $\mathcal{C}_s = \{H_1, \dots, H_s\}$ be a central arrangement in $\mathbb{K}^2$ (Figure 1). Let R be a commutative ring and $\eta = a_1e_1 + \cdots + a_se_s \in A^1_R(\mathcal{C}_s)$ be a degree one element of Orlik-Solomon algebra.*

(1) *$\eta \wedge (e_i - e_j) = -(\sum_{i=1}^s a_i) \cdot e_i \wedge e_j$.*
(2) *Let $\omega = b_1e_1 + \cdots + b_se_s \in A^1_R(\mathcal{A})$ be another element. Assume that ω and η are linearly independent (i.e., $c_1\eta + c_2\omega = 0, (c_1, c_2 \in R) \Longrightarrow c_1 = c_2 = 0$). Then $\eta \wedge \omega = 0$ if and only if $\sum_{i=1}^s a_i = \sum_{i=1}^s b_i = 0$.*

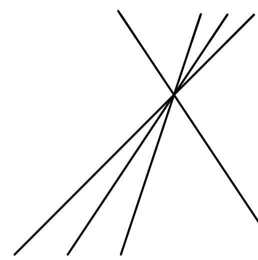

FIGURE 1. Central arrangement $\mathcal{C}_s$

Proof. (1) It is straightforward from the relation $e_{ij} = e_{ik} - e_{jk}$, where $e_{ij} := e_i \wedge e_j$.

(2) (Cf. [34, Proposition 2.1]) If $\sum_{i=1}^s b_i = 0$, then $\omega = b_1(e_1 - e_s) + \cdots + b_{s-1}(e_1 - e_{s-1})$. Then applying (1), we have $\eta \wedge \omega = -(\sum_{i=1}^s a_i) \cdot (b_1e_{1s} + \cdots + b_{s-1}e_{1,s-1})$. This is zero if $\sum_{i=1}^s a_i = 0$. Conversely, suppose $\eta \wedge \omega = 0$. Since $\mathcal{C}_s$ is central, we can apply the differential ∂. We have

$$0 = \partial(\eta \wedge \omega) = (\partial\eta)\omega - (\partial\omega)\eta.$$

By the assumption that η and ω are linearly independent, $\partial\eta = \partial\omega = 0$. $\square$

3.2. **Aomoto complex over $\mathbb{F}_p$.** Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of affine lines in $\mathbb{K}^2$. Choose a prime p such that $p|(n+1)$. Consider the Aomoto complex over $R = \mathbb{F}_p$ and the embedding $\iota\colon A^\bullet_{\mathbb{F}_p}(\mathcal{A}) \xrightarrow{\simeq} A^\bullet_{\mathbb{F}_p}(\widetilde{\mathcal{A}})_0 \subset A^\bullet_{\mathbb{F}_p}(\widetilde{\mathcal{A}})$. Since n is equal to -1 in $\mathbb{F}_p$, the image of the diagonal element $\eta_0 := e_1 + \cdots + e_n \in A^1_{\mathbb{F}_p}(\mathcal{A})$ is

$$\widetilde{\eta}_0 := \iota(\eta_0) = \widetilde{e}_0 + \widetilde{e}_1 + \cdots + \widetilde{e}_n \in A^1_{\mathbb{F}_p}(\widetilde{\mathcal{A}})_0.$$

We consider the first cohomology group of the Aomoto complex $(A^\bullet_{\mathbb{F}_p}(\mathcal{A}), \eta_0) \simeq (A^\bullet_{\mathbb{F}_p}(\widetilde{\mathcal{A}})_0, \widetilde{\eta}_0)$. Let $\widetilde{\omega} = \sum_{i=0}^n a_i\widetilde{e}_i \in A^1_{\mathbb{F}_p}(\widetilde{\mathcal{A}})_0$. Let us translate the relation $\widetilde{\eta} \wedge \widetilde{\omega} = 0$ in terms of the coefficients a_i of $\widetilde{\omega}$ by using the Brieskorn decomposition (1). For an intersection $X \in L_2(\widetilde{\mathcal{A}})$ of codimension two, let us define the localization at X by

$$\widetilde{\omega}|_X := \sum_{\widetilde{H}_i \in \widetilde{\mathcal{A}}_X} a_i\widetilde{e}_i. \tag{3}$$

Proposition 3.2. *With notation as above, $\widetilde{\eta}_0 \wedge \widetilde{\omega} = 0$ if and only if the following (i) and (ii) hold, for any $X \in L_2(\widetilde{\mathcal{A}})$.*

(*i*) *If $|\widetilde{\mathcal{A}}_X|$ is divisible by p, then $\sum_{\widetilde{H}_i \in \widetilde{\mathcal{A}}_X} a_i = 0$ in $\mathbb{F}_p$.*

(*ii*) *If* $|\widetilde{\mathcal{A}}_X|$ *is not divisible by* p, *then*

$$a_{i_1} = a_{i_2} = \cdots = a_{i_t},$$

where $\widetilde{\mathcal{A}}_X = \{\widetilde{H}_{i_1}, \widetilde{H}_{i_2}, \dots, \widetilde{H}_{i_t}\}$. *(This is equivalent to that* $\widetilde{\omega}|_X$ *and* $\widetilde{\eta}_0|_X$ *are linearly dependent.)*

Proof. By the Brieskorn decomposition (1), $\widetilde{\eta}_0 \wedge \widetilde{\omega} = 0$ if and only if $\widetilde{\eta}_0|_X \wedge \omega|_X = 0$ for all $X \in L_2(\widetilde{\mathcal{A}})$. Using the Lemma 3.1 (2), it is equivalent to (*i*) and (*ii*) above. □

3.3. **Aomoto complex over $\mathbb{F}_2$ and subarrangements.** Now we consider the Aomoto complex over $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. Since the coefficient is either 0 or $1 \in \mathbb{F}_2$, elements of $A^1_{\mathbb{F}_2}(\widetilde{\mathcal{A}})$ can be identified with subarrangements of $\widetilde{\mathcal{A}}$.

Definition 3.3. *Let* $\widetilde{\mathcal{S}} \subset \widetilde{\mathcal{A}}$ *be a subset. Let us define an element corresponding to the subset by*

$$\widetilde{e}(\widetilde{\mathcal{S}}) := \sum_{\widetilde{H}_i \in \widetilde{\mathcal{S}}} \widetilde{e}_i \in A^1_{\mathbb{F}_2}(\widetilde{\mathcal{A}}).$$

For an affine arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$ *and a subset* $\mathcal{S} \subset \mathcal{A}$, *similarly we define*

$$e(\mathcal{S}) := \sum_{H_i \in \mathcal{S}} e_i \in A^1_{\mathbb{F}_2}(\mathcal{A}).$$

Obviously the diagonal element is $\widetilde{\eta}_0 = \widetilde{e}(\widetilde{\mathcal{A}})$ and $\widetilde{e}(\widetilde{\mathcal{S}}) + \widetilde{\eta}_0 = \widetilde{e}(\widetilde{\mathcal{A}} \setminus \widetilde{\mathcal{S}})$.

Applying Proposition 3.2 for $p = 2$, we have the following.

Proposition 3.4. *Let* $\widetilde{\mathcal{A}} = \{\widetilde{H}_0, \widetilde{H}_1, \dots, \widetilde{H}_n\}$ *be central arrangement in* $\mathbb{K}^3$. *Let* $\widetilde{\mathcal{S}} \subset \widetilde{\mathcal{A}}$ *be a subset. Then* $\widetilde{\eta}_0 \wedge \widetilde{e}(\widetilde{\mathcal{S}}) = 0$ *if and only if the following* (*i*) *and* (*ii*) *hold, for any* $X \in L_2(\widetilde{\mathcal{A}})$.

(*i*) *If* $|\widetilde{\mathcal{A}}_X|$ *is even, then* $|\widetilde{\mathcal{S}}_X|$ *is also even.*
(*ii*) *If* $|\widetilde{\mathcal{A}}_X|$ *is odd, then either* $\widetilde{\mathcal{A}}_X = \widetilde{\mathcal{S}}_X$ *or* $\widetilde{\mathcal{S}}_X = \emptyset$.

Remark 3.5. *The existence of* $\widetilde{\omega} \in A^1_{\mathbb{F}_2}(\widetilde{\mathcal{A}})$ *such that* $\widetilde{\omega} \neq 0, \widetilde{\omega} \neq \widetilde{\eta}_0$ *and* $\widetilde{\eta}_0 \wedge \widetilde{\omega} = 0$ *is equivalent to the existence of a partition* $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}_1 \sqcup \widetilde{\mathcal{A}}_2$ *such that at each intersection* $X \in L_2(\widetilde{\mathcal{A}})$ *of codimension* 2, *(at least) one of the following is satisfied:*

(1) $\widetilde{\mathcal{A}}_X$ *is included in* $\widetilde{\mathcal{A}}_1$ *or in* $\widetilde{\mathcal{A}}_2$,
(2) $|(\widetilde{\mathcal{A}}_1)_X|$ *and* $|(\widetilde{\mathcal{A}}_2)_X|$ *are both even.*

The authors do not know any real essential arrangement which possesses the above partition. Hence, we do not know any real essential arrangement which satisfies $H^1(A^\bullet_{\mathbb{F}_2}(\widetilde{\mathcal{A}})_0, \widetilde{\eta}_0) \neq 0$.

Example 3.6. *Suppose that* $\overline{\mathcal{A}} = \overline{\mathcal{A}}_1 \sqcup \overline{\mathcal{A}}_2 \sqcup \overline{\mathcal{A}}_3 \sqcup \overline{\mathcal{A}}_4$ *is a 4-net. Then*

$$\begin{aligned}\widetilde{\eta}_0 \wedge \widetilde{e}(\widetilde{\mathcal{A}}_1 \cup \widetilde{\mathcal{A}}_2) &= \widetilde{\eta}_0 \wedge \widetilde{e}(\widetilde{\mathcal{A}}_1 \cup \widetilde{\mathcal{A}}_3) \\ &= \widetilde{\eta}_0 \wedge \widetilde{e}(\widetilde{\mathcal{A}}_1 \cup \widetilde{\mathcal{A}}_4) \\ &= 0\end{aligned}$$

These three elements satisfy a linear relation,

$$\widetilde{e}(\widetilde{\mathcal{A}}_1 \cup \widetilde{\mathcal{A}}_2) + \widetilde{e}(\widetilde{\mathcal{A}}_1 \cup \widetilde{\mathcal{A}}_3) + \widetilde{e}(\widetilde{\mathcal{A}}_1 \cup \widetilde{\mathcal{A}}_4) = \widetilde{\eta}_0,$$

and span a two dimensional subspace in $H^1(A^\bullet_{\mathbb{F}_2}(\widetilde{\mathcal{A}})_0, \widetilde{\eta}_0)$. *We obtain a well-known inequality:*

$$\dim H^1(A^\bullet_{\mathbb{F}_2}(\widetilde{\mathcal{A}})_0, \widetilde{\eta}_0) \geq 2 \quad ([9, 20]).$$

4. Resonant bands description of Aomoto complex

Resonant bands provide effective tools to compute local system cohomology groups and eigenspaces of Milnor monodromies. In this section, we give a description of the cohomology of the Aomoto complex in terms of resonant bands.

4.1. **Aomoto complex via chambers.** We first introduce several notions related to the real structure of line arrangements. (The notions are summarized in Example 4.4 and Figure 2.) Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of affine lines in $\mathbb{R}^2$. A connected component of $\mathbb{R}^2 \setminus \bigcup_{H \in \mathcal{A}} H$ is called a chamber. The set of all chambers is denoted by $\mathsf{ch}(\mathcal{A})$. Let $C, C' \in \mathsf{ch}(\mathcal{A})$. The line $H \in \mathcal{A}$ is said to separate C and C' when they are contained in opposite sides of H. The set of all lines separating C and C' is denoted by $\operatorname{Sep}(C, C')$. The set of chambers $\mathsf{ch}(\mathcal{A})$ is provided with a natural metric, the so-called adjacency distance, $d(C, C') = |\operatorname{Sep}(C, C')|$.

Let us fix a flag

$$\emptyset = \mathcal{F}^{-1} \subset \mathcal{F}^0 \subset \mathcal{F}^1 \subset \mathcal{F}^2 = \mathbb{R}^2,$$

(of affine subspaces with $\dim \mathcal{F}^i = i$, we also fix orientations of subspaces) satisfying the following conditions:

(i) (genericity) $\mathcal{F}^0$ is not contained in $\bigcup_{i=1}^n H_i$, and $\mathcal{F}^1$ intersects with $\bigcup_{i=1}^n H_i$ at distinct n points.

(ii) (near to ∞)
  - $\mathcal{F}^0$ does not separate n points $\mathcal{F}^1 \cap H_i$ $(i = 1, \dots, n)$ in $\mathcal{F}^1$.
  - $\mathcal{F}^1$ does not separate intersections of $\mathcal{A}$ in $\mathbb{R}^2$.

(See Figure 2 for example.) Each line H_i determines two half spaces $H_i^{\pm}$. We choose $H_i^{\pm}$ so that $\mathcal{F}^0 \in H_i^-$ for all $i = 1, \dots, n$. We also fix an orientation of $\mathcal{F}^1$ and after re-numbering the lines, if necessary, we may assume the following

$$\mathcal{F}^0 < H_1 \cap \mathcal{F}^1 < H_2 \cap \mathcal{F}^1 < \cdots < H_n \cap \mathcal{F}^1,$$

with respect to the ordering of $\mathcal{F}^1$.

Associated to such a flag $\mathcal{F} = \{\mathcal{F}^\bullet\}$, we define a subset of $\mathsf{ch}(\mathcal{A})$ as follows.

$$\mathsf{ch}^i_{\mathcal{F}}(\mathcal{A}) = \{C \in \mathsf{ch}(\mathcal{A}) \mid C \cap \mathcal{F}^{i-1} = \emptyset, C \cap \mathcal{F}^i \neq \emptyset\}.$$

We denote by $R[\mathsf{ch}^i_{\mathcal{F}}(\mathcal{A})] = \bigoplus_{C \in \mathsf{ch}^i_{\mathcal{F}}(\mathcal{A})} R \cdot [C]$, the free R-module generated by $C \in \mathsf{ch}^i_{\mathcal{F}}(\mathcal{A})$, where R is a commutative ring. It is known that $\operatorname{rank}_R A^i_R(\mathcal{A}) = |\mathsf{ch}^i_{\mathcal{F}}(\mathcal{A})|$ ([25]). We fix notations as follows.

Assumption 4.1. *Let us set*

$$\mathsf{ch}^0_{\mathcal{F}}(\mathcal{A}) = \{C_0\}, \ \mathsf{ch}^1_{\mathcal{F}}(\mathcal{A}) = \{C_1, \dots, C_n\}, \ \text{ and } \ \mathsf{ch}^2_{\mathcal{F}}(\mathcal{A}) = \{D_1, D_2, \dots, D_b\},$$

where $b = |\mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})|$. We can choose $C_1, \dots, C_n$ such that $\operatorname{Sep}(C_0, C_i) = \{H_1, H_2, \dots, H_i\}$, or equivalently, $C_i = H_1^+ \cap \cdots \cap H_i^+ \cap H_{i+1}^- \cap \cdots \cap H_n^-$, for all $i = 0, 1, \dots, n$ (see Figure 2).

When $1 \leq i < n$, the boundary of $C_i \cap \mathcal{F}^1$ consists of two points, $H_i \cap \mathcal{F}^1$ and $H_{i+1} \cap \mathcal{F}^1$, while $C_n \cap \mathcal{F}^1$ is a half-line and its boundary consists of a point $H_n \cap \mathcal{F}^1$.

Definition 4.2. *We use the notations above. Consider $\eta = \sum_{i=1}^n a_i e_i \in A^1_R(\mathcal{A})$.*

(1) *Define the R-homomorphisms $\nabla_\eta\colon R[\mathsf{ch}^0_{\mathcal{F}}(\mathcal{A})] \longrightarrow R[\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})]$ as follows.*

$$\begin{aligned}\nabla_\eta([C_0]) &= \sum_{C\in \mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})} \left(\sum_{H_i\in \mathrm{Sep}(C_0,C)} a_i\right)\cdot [C]\\ &= \sum_{i=1}^{n}(a_1+\cdots+a_i)\cdot [C_i].\end{aligned}$$

(2) *Define the map*

$$\deg\colon \mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})\times \mathsf{ch}^2_{\mathcal{F}}(\mathcal{A}) \longrightarrow \{\pm 1, 0\},$$

as follows.

(*i*) *If $i<n$, then the segment $C_i\cap\mathcal{F}^1$ has two boundaries, say, $H_i\cap\mathcal{F}^1$ and $H_{i+1}\cap\mathcal{F}^1$.*

$$\deg(C_i,D) = \begin{cases} 1 & \text{if } D\subset H_i^-\cap H_{i+1}^+,\\ -1 & \text{if } D\subset H_i^+\cap H_{i+1}^-,\\ 0 & \text{otherwise.}\end{cases}$$

(*ii*) *If $i=n$,*

$$\deg(C_n,D) = \begin{cases} -1 & \text{if } D\subset H_n^+,\\ 0 & \text{if } D\subset H_n^-.\end{cases}$$

(3) *Define the R-homomorphisms $\nabla_\eta\colon R[\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})] \longrightarrow R[\mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})]$ as follows.*

$$\nabla_\eta([C]) = \sum_{D\in \mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})} \deg(C,D)\left(\sum_{H_i\in \mathrm{Sep}(C,D)} a_i\right)\cdot [D].$$

Proposition 4.3. *([27]) $(R[\mathsf{ch}^\bullet_{\mathcal{F}}(\mathcal{A})],\nabla_\eta)$ is a cochain complex. Furthermore, there is a natural isomorphism of cochain complexes,*

$$\varphi\colon (R[\mathsf{ch}^\bullet_{\mathcal{F}}(\mathcal{A})],\nabla_\eta) \xrightarrow{\simeq} (A^\bullet_R(\mathcal{A}),\eta). \tag{4}$$

In degree 1, the isomorphism is explicitly given by

$$R[\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})] \xrightarrow{\simeq} A^1_R(\mathcal{A}),\ [C_i]\longmapsto \varphi([C_i]) = \begin{cases} e_i - e_{i+1} & \text{if } i<n,\\ e_n & \text{if } i=n.\end{cases} \tag{5}$$

In particular, we have

$$H^1(R[\mathsf{ch}^\bullet_{\mathcal{F}}(\mathcal{A})],\nabla_\eta)\simeq H^1(A^\bullet_R(\mathcal{A}),\eta).$$

The isomorphism (4) is natural in the sense that it respects Borel-Moore homology [27, 14, 28]. Recall that each chamber $C\in \mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})$ (with suitable orientation) determines a Borel-Moore 2-homology cycle $[C]\in H_2^{BM}(M(\mathcal{A}),R)$ of the complexified complement $M(\mathcal{A})$. The isomorphism (4), for $i=2$, is obtained by the composition

$$R[\mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})] \longrightarrow H_2^{BM}(M(\mathcal{A}),R) \xrightarrow{\simeq} H^2(M(\mathcal{A}),R)\simeq A^i_R(\mathcal{A}). \tag{6}$$

Example 4.4. *Let $\mathcal{A}=\{H_1,\dots,H_6\}$ be six affine lines as in Figure 2. We also fix a flag (with orientation) $\mathcal{F}=\{\mathcal{F}^0\subset\mathcal{F}^1\}$ (as in Figure 2). There are 16 chambers. We have*

$$\begin{aligned}\mathsf{ch}^0_{\mathcal{F}}(\mathcal{A}) &= \{C_0\},\\ \mathsf{ch}^1_{\mathcal{F}}(\mathcal{A}) &= \{C_1,C_2,\dots,C_6\},\\ \mathsf{ch}^2_{\mathcal{F}}(\mathcal{A}) &= \{D_1,D_2,\dots,D_9\}.\end{aligned}$$

The degree maps are computed, as follows.

$\deg(C_i, D_j)$	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9
C_1	0	-1	0	0	1	0	1	1	0
C_2	-1	0	0	-1	-1	0	0	-1	0
C_3	1	1	1	1	1	1	0	1	1
C_4	-1	-1	-1	0	0	0	0	0	0
C_5	0	0	0	-1	-1	-1	0	0	0
C_6	0	0	0	0	0	0	-1	-1	-1

Consider $\eta = \sum_{i=1}^{6} a_i e_i \in A^1_R(\mathcal{A})$. *We will compute the map* ∇_η.

The first one $\nabla_\eta \colon R[\mathsf{ch}^0_{\mathcal{F}}(\mathcal{A})] \longrightarrow R[\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})]$ *is, by definition,*

$$\nabla_\eta([C_0]) = a_1 \cdot [C_1] + a_{12} \cdot [C_2] + a_{123} \cdot [C_3] + a_{1234} \cdot [C_4] + a_{12345} \cdot [C_5] + a_{123456} \cdot [C_6],$$

where $a_{ijk} = a_i + a_j + a_k$, *etc. The second one* $\nabla_\eta \colon R[\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})] \longrightarrow R[\mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})]$ *is given as follows.*

$$\nabla_\eta \begin{pmatrix} [C_1] \\ [C_2] \\ \vdots \\ [C_6] \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -a_4 & 0 & 0 & a_{1245} & 0 & a_{123456} & a_{12456} & 0 \\ -a_4 & 0 & 0 & -a_{45} & -a_{145} & 0 & 0 & -a_{1456} & 0 \\ a_{34} & a_{234} & a_{1234} & a_{345} & a_{1345} & a_{12345} & 0 & a_{13456} & a_{123456} \\ -a_3 & -a_{23} & -a_{123} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_3 & -a_{13} & -a_{123} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a_1 & -a_{13} & -a_{123} \end{pmatrix} \begin{pmatrix} [D_1] \\ [D_2] \\ \vdots \\ [D_9] \end{pmatrix}.$$

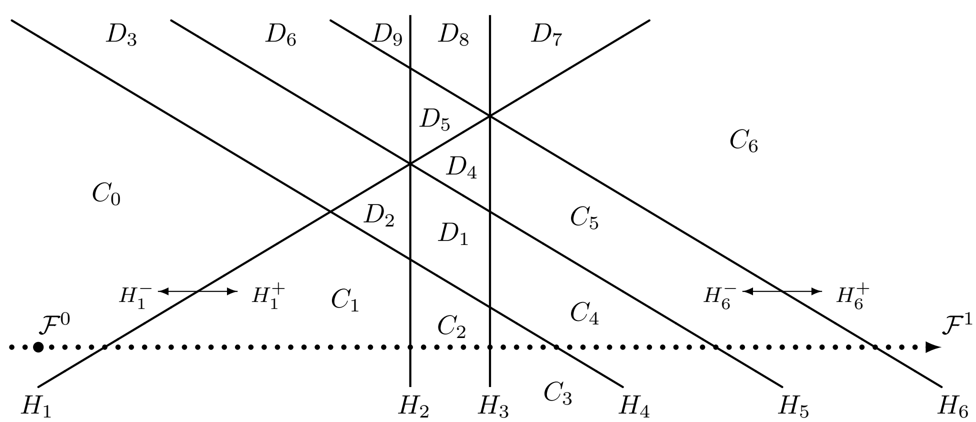

FIGURE 2. Example 4.4

4.2. **Aomoto complex via resonant bands.** Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement of affine lines in $\mathbb{R}^2$. We fix the flag $\mathcal{F}$ as in §4.1. The cohomology of the Aomoto complex can be computed using chambers. In this subsection, we introduce the notion "η-resonant bands" which enables us to simplify the computation of cohomology. This can be regarded as "the Aomoto complex version" of the results in [29, 30].

Definition 4.5. *A* band B *is a region bounded by a pair of consecutive parallel lines* H_i *and* H_{i+1}.

Each band B contains two unbounded chambers $U_1(B), U_2(B) \in \mathsf{ch}(\mathcal{A})$. Since B intersects $\mathcal{F}^1$, we may assume that $B \cap \mathcal{F}^1 = U_1(B) \cap \mathcal{F}^1$ and $U_2(B) \cap \mathcal{F}^1 = \emptyset$. In other words, $U_1(B) \in \mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})$ and $U_2(B) \in \mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})$. The distance $d(U_1(B), U_2(B))$ is called the *length* of the band B.

Definition 4.6. *Let* $\eta = \sum_{i=1}^n a_i e_i \in A^1_R(\mathcal{A})$. *A band* B *is called* η-resonant *if*

$$\sum_{H_i \in \mathrm{Sep}(U_1(B), U_2(B))} a_i = 0.$$

We denote by $\mathrm{RB}_\eta(\mathcal{A})$ *the set of all* η*-resonant bands.*

We can extend U_1 to an injective R-module homomorphism $U_1 \colon R[\mathrm{RB}_\eta(\mathcal{A})] \hookrightarrow R[\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})]$. We denote by

$$\widetilde{\nabla}_\eta := -\nabla_\eta \circ U_1 \colon R[\mathrm{RB}_\eta(\mathcal{A})] \longrightarrow R[\mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})]$$

the composition of U_1 and ∇_η (multiplied by -1).

More precisely, to each η-resonant band $B \in \mathrm{RB}_\eta(\mathcal{A})$, we associate an element

$$\widetilde{\nabla}_\eta(B) \in R[\mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})]$$

as follows:

$$\widetilde{\nabla}_\eta(B) := -\nabla_\eta(U_1(B)) = \sum_{D \in \mathsf{ch}(\mathcal{A}), D \subset B} \left(\sum_{H_i \in \mathrm{Sep}(U_1(B), D)} a_i \right) \cdot [D].$$

Example 4.7. *Let* $\mathcal{A} = \{H_1, \dots, H_6\}$ *be an arrangement of lines as in Figure 2. There are three bands* B_1, B_2, B_3*, i.e., those defined by* (H_2, H_3)*,* (H_4, H_5) *and* (H_5, H_6)*, respectively. We have*

$$U_1(B_1) = C_2, U_2(B_1) = D_8,\ U_1(B_2) = C_4, U_2(B_2) = D_3, \quad \text{and} \quad U_1(B_3) = C_5, U_2(B_3) = D_6.$$

The band B_1 *has length* 4*, while* B_2 *and* B_3 *have length* 3*. Let* $\eta = a_1 e_1 + \cdots + a_6 e_6 \in A^1_R(\mathcal{A})$*. The band* B_1 *is* η*-resonant if and only if* $a_1 + a_4 + a_5 + a_6 = 0$*. Then we have*

$$\widetilde{\nabla}_\eta([B_1]) = a_4[D_1] + (a_4 + a_5)[D_4] + (a_1 + a_4 + a_5)[D_5].$$

Obviously the map U_1 induces $U_1 \colon \mathrm{Ker}(\widetilde{\nabla}_\eta) \longrightarrow \mathrm{Ker}(\nabla_\eta : R[\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})] \longrightarrow R[\mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})])$. Thus we have a natural map

$$\widetilde{U}_1 \colon \mathrm{Ker}(\widetilde{\nabla}_\eta) \longrightarrow H^1(R[\mathsf{ch}^\bullet_{\mathcal{F}}(\mathcal{A})], \nabla_\eta). \tag{7}$$

The above map $\widetilde{U}_1$ is neither injective nor surjective in general. The following is the main result concerning resonant bands which asserts that the map $\widetilde{U}_1$ above (7) is isomorphic under certain non-resonant assumption at infinity. This provides an effective way to compute $H^1(R[\mathsf{ch}^\bullet_{\mathcal{F}}(\mathcal{A})], \nabla_\eta)$. Indeed, normally, $|\mathrm{RB}_\eta(\mathcal{A})|$ is much smaller than $|\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})|$.

Theorem 4.8. *Let* R *be a commutative ring and* $\eta = \sum_{i=1}^n a_i e_i \in A^1_R(\mathcal{A})$.

(i) *Suppose that* $\alpha := \sum_{i=1}^n a_i \in R^\times$ *is invertible. Then the natural map* $\widetilde{U}_1$ *is injective.*
(ii) *We assume that* R *is an integral domain and* $\alpha := \sum_{i=1}^n a_i \in R^\times$*. Then* $\widetilde{U}_1$ *is an isomorphism.*
(iii) *Let* R *be an arbitrary commutative ring. If* $\alpha := \sum_{i=1}^n a_i \in R^\times$ *and all bands are* η*-resonant, then the natural map* $\widetilde{U}_1$ *is an isomorphism.*

Proof. (i) Let

$$\sum r_B \cdot [B] := \sum_{B \in \mathrm{RB}_\eta(\mathcal{A})} r_B \cdot [B] \in R[\mathrm{RB}_\eta(\mathcal{A})], \quad r_B \in R.$$

Suppose $\sum r_B \cdot [B] \in \operatorname{Ker} \widetilde{U}_1$, that is, $U_1(\sum r_B \cdot [B]) \in \operatorname{Im}\big(\nabla_\eta \colon R[\mathsf{ch}^0_{\mathcal{F}}(\mathcal{A})] \longrightarrow R[\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})]\big)$. Since $R[\mathsf{ch}^0_{\mathcal{F}}(\mathcal{A})] = R \cdot [C_0]$, there exists an element $s \in R$ such that

$$\sum r_B \cdot [U_1(B)] = s \cdot \nabla_\eta([C_0]). \tag{8}$$

Note that in the left hand side of (8), the chamber C_n does not appear, because C_n is not bounded by two parallel lines. By Definition 4.2, $\nabla_\eta([C_0]) = \sum_{i=1}^n (a_1 + \cdots + a_i) \cdot [C_i]$. The coefficient of $[C_n]$ is equal to $s \cdot (a_1 + \cdots + a_n) = s \cdot \alpha$. By the assumption that α is invertible, we have $s = 0$. Hence $\sum r_B \cdot U_1(B) = s \cdot \nabla_\eta([C_0]) = 0$, and, since U_1 is injective, we have $\sum r_B \cdot [B] = 0$.

Next we show the surjectivity of (7). Suppose that

$$\beta = \sum_{i=1}^n b_i \cdot [C_i] \in \operatorname{Ker}(\nabla_\eta \colon R[\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})] \longrightarrow R[\mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})]).$$

Consider the following element,

$$\begin{aligned} \beta' &= \beta - \frac{b_n}{\alpha} \cdot \nabla_\eta([C_0]) \\ &= \sum_{i=1}^{n-1} b'_i \cdot [C_i]. \end{aligned} \tag{9}$$

Obviously, β and β' represent the same element in $H^1(R[\mathsf{ch}^\bullet_{\mathcal{F}}(\mathcal{A})], \nabla_\eta)$. It is sufficient to show that $\beta' \in \operatorname{Im} U_1$.

Next we consider a chamber C_i $(i < n)$ such that H_i and H_{i+1} are not parallel. Then there is a unique chamber $D_p \in \mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})$ such that $\operatorname{Sep}(C_i, D_p) = \mathcal{A}$, which is called the "opposite chamber of C_i" in [26, Def. 2.1] and denoted by $D_p = C_i^\vee$. Then we consider the coefficient $c_{C_i^\vee}$ of $[C_i^\vee]$ in $\nabla_\eta(\beta') = \sum c_D \cdot [D]$. Since $C_i^\vee$ appears only in $\nabla_\eta([C_i])$ and $\nabla_\eta([C_n])$, and the coefficient of $[C_n]$ is already zero, we have $c_{C_i^\vee} = \alpha \cdot b'_i$. By the assumption that $\alpha \in R^\times$, $\nabla_\eta(\beta') = 0$, in particular $c_{C_i^\vee} = 0$, implies that $b'_i = 0$. So $\beta' = \sum_{i=1}^{n-1} b'_i \cdot [C_i]$ is a linear combination of C_i's $(i < n)$ such that H_i and H_{i+1} are parallel. So far, we only use the fact $\alpha \in R^\times$. If all bands are η-resonant, then we have already proved that β' is generated by $U_1(B)$ with $B \in \mathrm{RB}_\eta(\mathcal{A})$. Thus (iii) is proved.

Now we assume that R is an integral domain. We will prove (ii). Let C_i be a chamber such that the walls H_i and H_{i+1} are parallel. Let B be the corresponding band defined by H_i and H_{i+1}. Note that $C_i = U_1(B)$ and its opposite chamber is $U_2(B)$. Suppose that B is not an η-resonant band, that is, $\alpha' := \sum_{H_j \in \operatorname{Sep}(U_1(B), U_2(B))} a_j \neq 0$. Again consider the coefficient of $[U_2(B)]$ in $\nabla_\eta(\beta')$. $[U_2(B)]$ appears in $\nabla_\eta([C_i])$ and some other terms $\nabla_\eta([C_k])$ for k such that H_k and H_{k+1} are not parallel. However the coefficients of chambers of the second type in β' are already zero. Therefore the coefficient of $[U_2(B)]$ in $\nabla_\eta(\beta')$ is $-\alpha' \cdot b'_i$, which is zero. Since R is an integral domain, we have $b'_i = 0$. Hence β' is a linear combination of $U_1(B)$'s where $B \in \mathrm{RB}_\eta(\mathcal{A})$. This completes the proof of the surjectivity. $\square$

Remark 4.9. *Equation (7) and Theorem 4.8 are concerning the following homomorphism of cochains.*

$$\begin{tikzcd} & 0 \arrow[r] \arrow[d] & R[\mathrm{RB}_\eta(\mathcal{A})] \arrow[r, "\widetilde{\nabla}_\eta"] \arrow[d, "\varphi_1"] & R[\mathsf{ch}(\mathcal{A})] \arrow[r] \arrow[d, "\varphi_2"] & 0 \\ 0 \arrow[r] & A^0_R(\mathcal{A}) \arrow[r, "\eta"] & A^1_R(\mathcal{A}) \arrow[r, "\eta"] & A^2_R(\mathcal{A}) \arrow[r] & 0 \end{tikzcd}$$

The map $\widetilde{U}_1$ is nothing but the homomorphism $\mathrm{Ker}(\widetilde{\nabla}_\eta) \longrightarrow H^1(A^\bullet_R(\mathcal{A}), \eta)$ induced by φ_1. By Proposition 4.3 (especially, the explicit map (5)), the map φ_1 above is given by

$$[B] \longmapsto e_i - e_{i+1},$$

where B is a η-resonant band bounded by the lines H_i and H_{i+1}.

Example 4.10. *Let $R = \mathbb{F}_2$. Let $\mathcal{A} = \{H_1, \ldots, H_6\}$ be an arrangement of affine lines as in Figure 3 (which is $\mathcal{A}(7,1)$ in [12]). Let $\eta = e_2 + e_3 + e_6 \in A^1_R(\mathcal{A})$ (the supporting lines of η are colored blue). There are three bands B_1 (bounded by H_1 and H_2), B_2 (bounded by H_3 and H_4), and B_3 (bounded by H_5 and H_6). $\mathrm{Sep}(U_1(B_1), U_2(B_1)) = \{H_3, H_4, H_5, H_6\}$ and two of the lines, H_3 and H_6, have non-zero coefficients in η. Hence B_1 is an η-resonant band. Similarly, B_2 and B_3 are η-resonant, and we have $\mathrm{RB}_\eta(\mathcal{A}) = \{B_1, B_2, B_3\}$. By definition, $\widetilde{\nabla}_\eta(B_1) = \widetilde{\nabla}_\eta(B_2) = \widetilde{\nabla}_\eta(B_3) = [D_1]$. Hence the kernel*

$$\mathrm{Ker}(\widetilde{\nabla}_\eta \colon \mathbb{F}_2[\mathrm{RB}_\eta(\mathcal{A})] \longrightarrow \mathbb{F}_2[\mathsf{ch}(\mathcal{A})])$$

is 2-dimensional (generated by $[B_1]-[B_2]$ and $[B_2]-[B_3]$). By Theorem 4.8, $H^1(A^\bullet_{\mathbb{F}_2}(\mathcal{A}), \eta) \simeq \mathbb{F}_2^2$.

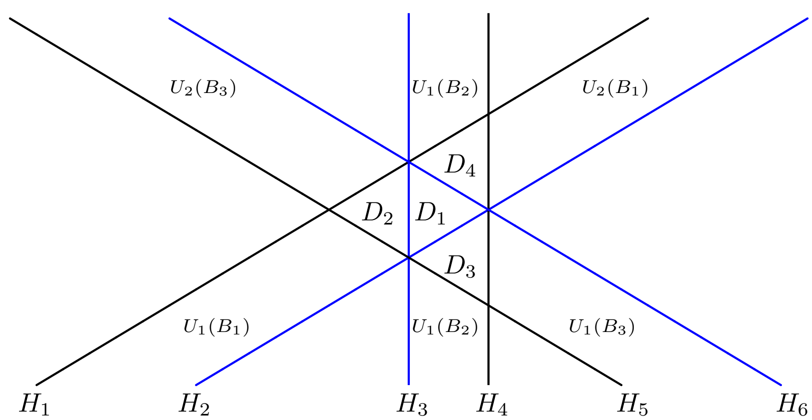

FIGURE 3. Example 4.10

Example 4.11. *We consider $\overline{\mathcal{A}} = \mathcal{A}(16,1) = \{\overline{H}_1, \ldots, \overline{H}_{16}\}$ from the Grünbaum's catalogue [12], see Figure 4. Let us denote by $\mathcal{A} = \{H_2, H_3, \ldots, H_{16}\}$ the deconing $\mathsf{d}_{\widetilde{H}_1}\widetilde{\mathcal{A}}$, the lower-left one in Figure 4. The affine arrangement $\mathcal{A}$ has 7 bands $B_1, \ldots, B_7$. To indicate the choice of $U_1(B)$ and $U_2(B)$, we always put the name B of the band in the unbounded chamber $U_1(B)$.*

Let $R = \mathbb{Z}/8\mathbb{Z}$. Define $\widetilde{\eta}_1, \widetilde{\eta}_2 \in A^1_R(\widetilde{\mathcal{A}})_0$ by

$$\begin{aligned} \widetilde{\eta}_1 &= \widetilde{e}_1 + \widetilde{e}_3 + \widetilde{e}_5 + \widetilde{e}_7 + \widetilde{e}_9 + \widetilde{e}_{11} + \widetilde{e}_{13} + \widetilde{e}_{15}, \\ \widetilde{\eta}_2 &= \widetilde{e}_2 + \widetilde{e}_4 + \widetilde{e}_6 + \widetilde{e}_8 + \widetilde{e}_{10} + \widetilde{e}_{12} + \widetilde{e}_{14} + \widetilde{e}_{16}, \end{aligned}$$

and set $\widetilde{\eta} := \widetilde{\eta}_1 + 6\widetilde{\eta}_2$.

Let $\eta = (e_3+e_5+e_7+\cdots+e_{15})+6(e_2+e_4+\cdots+e_{16}) \in A^1_R(\mathcal{A})$*. Then all* 7 *bands are* η*-resonant. Thus we can apply theorem Theorem 4.8 (iii). The kernel* $\operatorname{Ker}(\widetilde{\nabla}_\eta \colon R[\mathrm{RB}_\eta(\mathcal{A})] \longrightarrow R[\mathsf{ch}(\mathcal{A})])$ *is a free* R*-module generated by*

$$[B_1] + 2[B_2] + 3[B_3] + 4[B_4] + 5[B_5] + 6[B_6] + 7[B_7].$$

The corresponding element (via the correspondence Remark 4.9) in $A^1_R(\widetilde{\mathcal{A}})_0$ *is*

$$4(\widetilde{e}_2 + \widetilde{e}_3) + 3(\widetilde{e}_4 - \widetilde{e}_7 + \widetilde{e}_{13} - \widetilde{e}_{16}) + 2(\widetilde{e}_6 + \widetilde{e}_9 - \widetilde{e}_{11} - \widetilde{e}_{14}) + (\widetilde{e}_5 + \widetilde{e}_8 - \widetilde{e}_{12} - \widetilde{e}_{15}).$$

By Theorem 4.8 (iii), the cohomology of the Aomoto complex

$$H^1(A^1_R(\widetilde{\mathcal{A}})_0, \widetilde{\eta}) \simeq H^1(A^1_R(\mathcal{A}), \eta) \simeq \operatorname{Ker}(\widetilde{\nabla}_\eta) \simeq R \simeq \mathbb{Z}/8\mathbb{Z}$$

is non-vanishing.

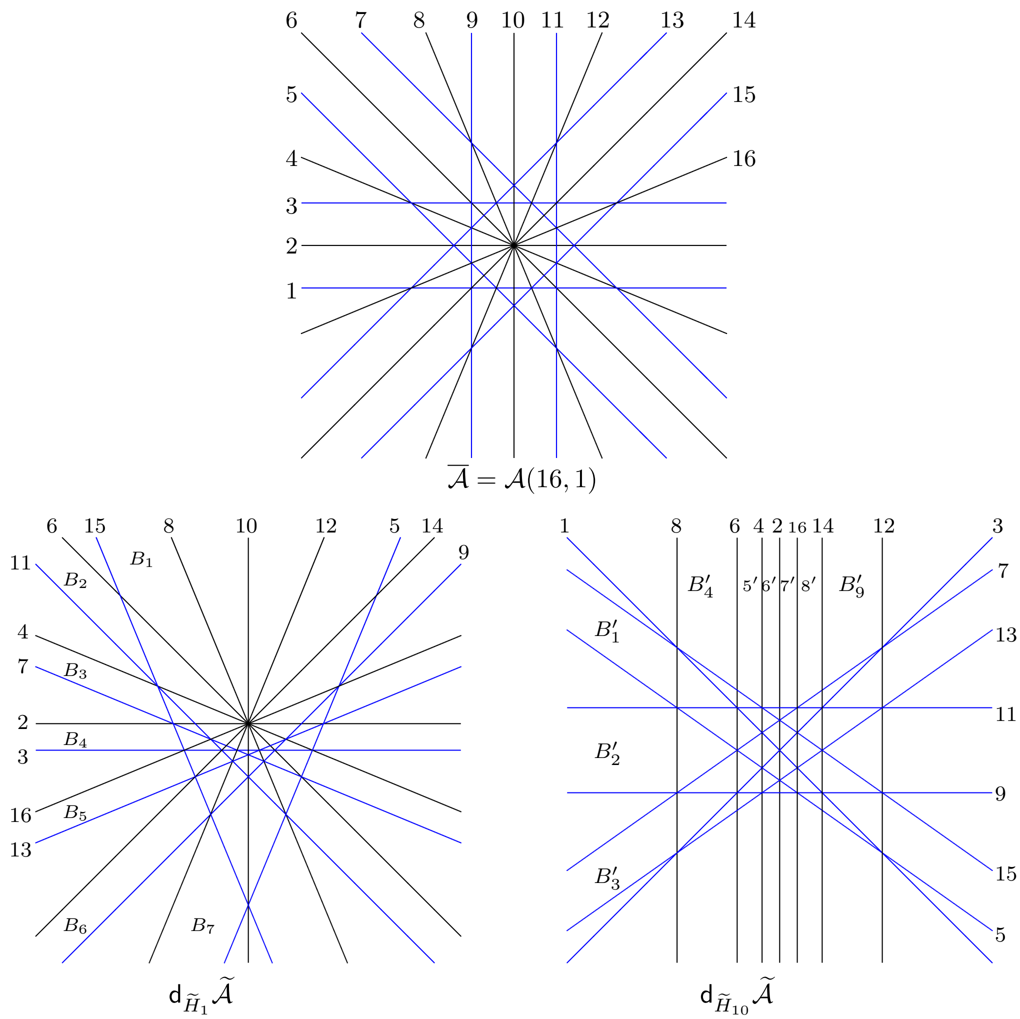

FIGURE 4. $\mathcal{A}(16,1)$ and deconings with respect to H_1 and H_{10}.

Remark 4.12. *Let us point out a possible relation between* $\mathbb{Z}/8\mathbb{Z}$*-resonance in Example 4.11 and isolated torsion points of order* 8 *in the characteristic variety of* $\mathcal{A}(16,1)$.

Let us denote $M = M(\mathcal{A}(16,1)) = \mathbb{CP}^2 \setminus \bigcup_{H\in\mathcal{A}(16,1)} H_{\mathbb{C}}$ the complexified complement. Recall that the character torus of M is

$$\mathbb{T} := \mathrm{Hom}(\pi_1(M), \mathbb{C}^\times) \simeq \{\boldsymbol{t} = (t_1, t_2, \dots, t_{16}) \in (\mathbb{C}^\times)^{16} \mid \prod_{i=1}^{16} t_i = 1\}.$$

We also define the essential open subset of $\mathbb{T}$ by

$$\mathbb{T}^\circ := \{\boldsymbol{t} = (t_1, \dots, t_{16}) \in \mathbb{T} \mid t_i \neq 1, \forall i = 1, \dots, 16\}.$$

The characteristic variety $\mathcal{V}^1(\mathcal{A}(16,1))$ of $\mathcal{A}(16,1)$ is the set of points in the character torus $\mathbb{T}$ such that the associated local system has non-vanishing first cohomology, i.e.,

$$\mathcal{V}^1(\mathcal{A}(16,1)) = \{\boldsymbol{t} \in \mathbb{T} \mid \dim H^1(M, \mathcal{L}_{\boldsymbol{t}}) \geq 1\}.$$

Let $\zeta = e^{2\pi i/8}$ and consider the following point,

$$\rho = (\zeta, \zeta^6, \zeta, \zeta^6, \zeta, \zeta^6, \zeta, \zeta^6, \zeta, \zeta^6, \zeta, \zeta^6, \zeta, \zeta^6, \zeta, \zeta^6) \in \mathbb{T}^\circ.$$

Let us recall quickly the resonant band algorithm for computing local system cohomology groups (see [30] for details). For a given local system $\mathcal{L}_{\boldsymbol{t}}$, we define the set $\mathrm{RB}_{\mathcal{L}_{\boldsymbol{t}}}(\mathcal{A})$ of $\mathcal{L}_{\boldsymbol{t}}$-resonant bands and the map $\nabla_{\mathcal{L}_{\boldsymbol{t}}} \colon \mathbb{C}[\mathrm{RB}_{\mathcal{L}_{\boldsymbol{t}}}(\mathcal{A})] \longrightarrow \mathbb{C}[\mathsf{ch}(\mathcal{A})]$. If $\mathcal{L}_{\boldsymbol{t}}$ has non-trivial monodromy around the line at infinity, then we have the isomorphism $H^1(M, \mathcal{L}_{\boldsymbol{t}}) \simeq \mathrm{Ker}(\nabla_{\mathcal{L}_{\boldsymbol{t}}})$.

Since $\mathcal{L}_\rho$ defined above has non trivial monodromy around any line, we can apply resonant band algorithm to any deconings. Here we exhibit two cases (although the results coincide logically), $\mathsf{d}_{\widetilde{H}_1}\widetilde{\mathcal{A}}$ and $\mathsf{d}_{\widetilde{H}_{10}}\widetilde{\mathcal{A}}$. (See Figure 4.)

- *The affine arrangement $\mathsf{d}_{\widetilde{H}_1}\widetilde{\mathcal{A}}$ has seven bands $B_1, \dots, B_7$, which are all $\mathcal{L}_\rho$-resonant. Then $\mathrm{Ker}(\nabla_{\mathcal{L}_\rho})$ is one dimensional and generated by the following element,*

$$\sin\left(\frac{\pi}{8}\right)[B_1] - \sin\left(\frac{\pi}{4}\right)[B_2] + \sin\left(\frac{3\pi}{8}\right)[B_3] - \sin\left(\frac{\pi}{2}\right)[B_4] + \sin\left(\frac{3\pi}{8}\right)[B_5] - \sin\left(\frac{\pi}{4}\right)[B_6] + \sin\left(\frac{\pi}{8}\right)[B_7].$$

- *The affine arrangement $\mathsf{d}_{\widetilde{H}_{10}}\widetilde{\mathcal{A}}$ has nine bands $B_1', \dots, B_9'$, which are all $\mathcal{L}_\rho$-resonant. Then $\mathrm{Ker}(\nabla_{\mathcal{L}_\rho})$ is one dimensional and generated by the following element,*

$$[B_1] + \sqrt{2}[B_2] + [B_3] - [B_4] + (1+\sqrt{2})[B_5] - (2+\sqrt{2})[B_6] + (2+\sqrt{2})[B_7] + (1+\sqrt{2})[B_8] + [B_9].$$

Hence we have $\dim H^1(M, \mathcal{L}_\rho) = 1$. Furthermore, we can prove that ρ generates the essential part of the characteristic variety. More precisely, we have the following,

$$\mathcal{V}^1(\mathcal{A}(16,1)) \cap \mathbb{T}^\circ = \{\rho, \rho^2, \rho^3, \rho^5, \rho^6, \rho^7\}. \tag{10}$$

4.3. **Resonant bands over $\mathbb{F}_2$ and subarrangements.** Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of affine lines in $\mathbb{R}^2$. Let $\mathcal{S} \subset \mathcal{A}$ be a subset. Denote $e(\mathcal{S}) := \sum_{H_i\in\mathcal{S}} e_i \in A^1_{\mathbb{F}_2}(\mathcal{A})$.

Clearly, $e(\mathcal{S}) + e(\mathcal{A}) = e(\mathcal{A} \setminus \mathcal{S})$. Below is the summary of "subarrangement description of resonant band algorithm":

(a) Let B be a band of $\mathcal{A}$. Then $B \in \mathrm{RB}_{e(\mathcal{S})}(\mathcal{A})$ if and only if the number of lines in $\mathcal{S}$ separating $U_1(B)$ and $U_2(B)$ is even, i.e., $2 \,|\, |\mathcal{S} \cap \mathrm{Sep}(U_1(B), U_2(B))|$.

(b) $\widetilde{\nabla}_{e(\mathcal{S})}\colon \mathbb{F}_2[\mathrm{RB}_{e(\mathcal{S})}(\mathcal{A})] \longrightarrow \mathbb{F}_2[\mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})]$ is given by the following formula.

$$\widetilde{\nabla}_{e(\mathcal{S})}(B) = \sum_{C\in \mathsf{ch}(\mathcal{A}), C\subset B} |\mathcal{S}\cap \mathrm{Sep}(U_1(B), C)|\cdot [C].$$

(See Example 4.10). In particular, if we consider $\eta_0 = e(\mathcal{A}) = e_1 + e_2 + \cdots + e_n$, then we have

$$\widetilde{\nabla}_{\eta_0}(B) = \sum_{C\in \mathsf{ch}(\mathcal{A}), C\subset B} d(U_1(B), C)\cdot [C].$$

(c) Suppose that $|\mathcal{S}|$ is odd. Then we can apply Theorem 4.8, and we have an isomorphism

$$\Psi_{e(S)}\colon \operatorname{Ker}(\widetilde{\nabla}_{e(\mathcal{S})}) \xrightarrow{\simeq} H^1(A^{\bullet}_{\mathbb{F}_2}(\mathcal{A}), e(\mathcal{S})).$$

(d) Using Remark 4.9 (and Proposition 4.3 (especially, the explicit map (5))), the above isomorphism is given by

$$\Psi_{e(S)}([B]) = e_i + e_{i+1} \in A^1_{\mathbb{F}_2}(\mathcal{A}),$$

where B is a $e(\mathcal{S})$-resonant band determined by the lines H_i and H_{i+1}.

5. Non-existence of real 4-nets

5.1. **Aomoto complex for the diagonal element.** Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of affine lines in $\mathbb{R}^2$ with odd n. Let $\widetilde{\mathcal{A}} = \{\widetilde{H}_0, \widetilde{H}_1, \dots, \widetilde{H}_n\}$ be the coning of $\mathcal{A}$ and $\overline{\mathcal{A}} = \{\overline{H}_0, \overline{H}_1, \dots, \overline{H}_n\}$ be the projectivization. Recall that

$$\widetilde{\eta}_0 := \widetilde{e}(\widetilde{\mathcal{A}}) = \widetilde{e}_0 + \widetilde{e}_1 + \cdots + \widetilde{e}_n \in A^1_{\mathbb{F}_2}(\widetilde{\mathcal{A}})_0$$

is the diagonal element and $\eta_0 = e(\mathcal{A}) = e_1 + \cdots + e_n \in A^1_{\mathbb{F}_2}(\mathcal{A})$. Notice that n odd implies that the map Ψ_{η_0} is an isomorphism by §4.3 (c).

Choose a subset $\widetilde{\mathcal{S}} \subset \widetilde{\mathcal{A}}$. In the figures below, the lines in $\widetilde{\mathcal{S}}$ are colored in red. The other lines are black.

As we saw in Proposition 3.4, the relation $\widetilde{\eta}_0 \wedge \widetilde{e}(\widetilde{\mathcal{S}}) = 0$ is equivalent to "$|\overline{\mathcal{A}}_X|$ is even $\Longrightarrow |\overline{\mathcal{S}}_X|$ is even" and "$|\overline{\mathcal{A}}_X|$ is odd $\Longrightarrow$ either $\overline{\mathcal{S}}_X = \emptyset$ or $\overline{\mathcal{S}}_X = \overline{\mathcal{A}}_X$" for $\forall X \in L_2(\overline{\mathcal{A}})$. From this, it is easily seen that if the multiplicity is $|\overline{\mathcal{A}}_X| \le 3$, then $\overline{\mathcal{A}}_X$ is monocolor (either all red $\overline{\mathcal{S}}_X = \overline{\mathcal{A}}_X$ or all black $\overline{\mathcal{S}}_X = \emptyset$). However, when $|\overline{\mathcal{A}}_X| = 4$, there are four cases (Figure 5):

(*i*) $\overline{\mathcal{S}}_X = \emptyset$.
(*ii*) $\overline{\mathcal{S}}_X = \overline{\mathcal{A}}_X$.
(*iii*) $|\overline{\mathcal{S}}_X| = 2$ and the lines in $\overline{\mathcal{S}}_X$ are adjacent.
(*iv*) $|\overline{\mathcal{S}}_X| = 2$ and the lines in $\overline{\mathcal{S}}_X$ are separated by lines in $\overline{\mathcal{A}}_X \setminus \widetilde{\mathcal{S}}_X$.

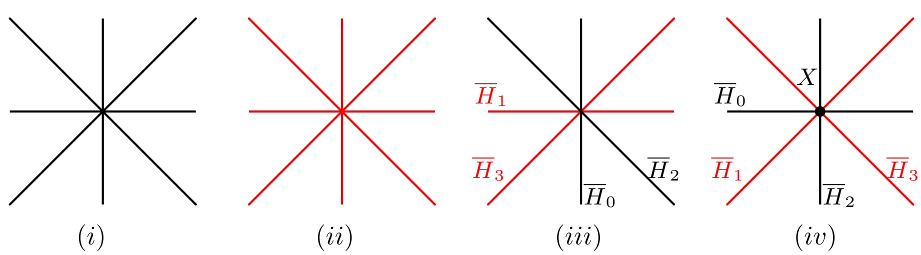

Figure 5. Local structures of $\overline{\mathcal{S}}_X$. (Members of $\overline{\mathcal{S}}_X$ are red, and $\overline{\mathcal{S}}_X$ equals $\{\overline{H}_1, \overline{H}_3\}$ in (*iii*) and (*iv*)).

The cases (iii) and (iv) are combinatorially identical. However, the real structures are different. This difference is crucial, actually, by using resonant bands, we can prove that (iv) can not happen ("Non Separation Theorem").

Theorem 5.1. *Let $\overline{\mathcal{S}} \subset \overline{\mathcal{A}}$. Suppose that $\widetilde{\eta}_0 \wedge \widetilde{e}(\widetilde{\mathcal{S}}) = 0$. Let $X \in \mathbb{RP}^2$ be an intersection of $\overline{\mathcal{A}}$ such that $|\overline{\mathcal{A}}_X| = 4$ and $|\overline{\mathcal{S}}_X| = 2$. Then the two lines of $\overline{\mathcal{S}}_X$ are adjacent as Figure 5 (iii). In particular, (iv) does not happen.*

Proof. Suppose that there exists $X \in \mathbb{RP}^2$ such that $\overline{\mathcal{A}}_X = \{\overline{H}_0, \overline{H}_1, \overline{H}_2, \overline{H}_3\}$ with $\overline{\mathcal{S}}_X = \{\overline{H}_1, \overline{H}_3\}$ arranging as (iv) in Figure 5.

First consider the deconing with respect to $\overline{H}_0$, we have $\mathcal{A} = \mathsf{d}_{\widetilde{H}_0}\widetilde{\mathcal{A}} = \{H_1, \dots, H_n\}$. Then

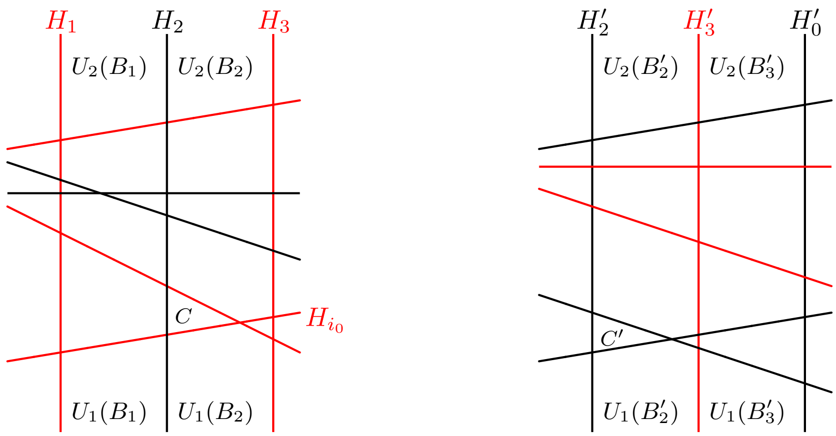

FIGURE 6. Deconings $\mathsf{d}_{\widetilde{H}_0}\widetilde{\mathcal{A}}$ and $\mathsf{d}_{\widetilde{H}_1}\widetilde{\mathcal{A}}$

$$\mathcal{S} = \{H_1, H_3, \dots\} \subset \mathcal{A}.$$

The lines H_1, H_2, H_3 are parallel (the left of Figure 6) and determine two bands B_1 (bounded by H_1 and H_2) and B_2 (bounded by H_2 and H_3). Note that $e(\mathcal{S}) = e_1 + e_3 + \dots \in A^1_{\mathbb{F}_2}(\mathcal{A})$. By the correspondence in §4.3 (d), we have

$$\Psi^{-1}_{\eta_0}(e(\mathcal{S})) = [B_1] + [B_2] + \dots,$$

in particular, both $[B_1]$ and $[B_2]$ appear. (Otherwise, e_1, e_3 can not appear.) On the other hand, we have the following relation

$$\widetilde{\nabla}_{\eta_0}(\Psi^{-1}_{\eta_0}(e(\mathcal{S})) = \widetilde{\nabla}_{\eta_0}([B_1]) + \widetilde{\nabla}_{\eta_0}([B_2]) + \dots = 0. \tag{11}$$

Choose a chamber C such that $C \subset B_2$ and $d(U_1(B_2), C) = 1$. Let $\mathrm{Sep}(U_1(B_2), C) = \{H_{i_0}\}$. The chamber C is adjacent to an unbounded chamber $U_1(B_2)$, hence, C is contained in at most two bands B_2 and B_{j_0}. Since $\widetilde{\nabla}_{\widetilde{\eta}}([B_2]) = [C] + \dots \in \mathbb{F}_2[\mathrm{RB}_{\widetilde{\eta}}(\mathcal{A})]$, by (11), $[C]$ must be cancelled by another resonant band B_{j_0} which appears in $\Psi^{-1}_{\eta_0}(e(\mathcal{S}))$. Thus we have

$$\Psi^{-1}_{\eta_0}(e(\mathcal{S})) = [B_1] + [B_2] + \dots + [B_{j_0}] + \dots.$$

Let H_{i_0} and H_{i_0+1} be walls of B_{j_0}. Then applying Ψ, we have

$$\begin{aligned} e(\mathcal{S}) &= (e_1 + e_2) + (e_2 + e_3) + \dots + (e_{i_0} + e_{i_0+1}) + \dots \\ &= e_1 + e_3 + \dots + e_{i_0} + \dots. \end{aligned}$$

Here note that e_{i_0} survives because B_{j_0} is the only band which has H_{i_0} as a wall. This implies $H_{i_0} \in \mathcal{S}$. Therefore, if $C \subset B_2$ and $d(U_1(B_2), C) = 1$, then $\mathrm{Sep}(U_1(B_2), C) \subset \mathcal{S}$. (Left hand side of Figure 6.) The same assertion holds for the opposite unbounded chamber $U_2(B_2)$.

Next we consider $\overline{\mathcal{S}}' := \overline{\mathcal{A}} \setminus \overline{\mathcal{S}}$. Since $\widetilde{e}(\widetilde{\mathcal{S}}') = \widetilde{\eta}_0 + \widetilde{e}(\widetilde{\mathcal{S}})$, $\widetilde{\eta}_0 \wedge \widetilde{e}(\widetilde{\mathcal{S}}') = 0$. In Figure 5 (iv), the roles of black and red lines exchange. Black lines are the members of $\overline{\mathcal{S}}'$ and red lines are not. We take the deconing with respect to $\widetilde{H}_1$, we have $\mathsf{d}_{\widetilde{H}_1}\widetilde{\mathcal{A}} = \{H_0', H_2', H_3', \dots, H_n'\}$ (Right hand side of Figure 6). Then $\mathcal{S}' = \{H_0', H_2', \dots\} \subset \mathsf{d}_{\widetilde{H}_1}\widetilde{\mathcal{A}}$. The lines H_0', H_2', H_3' are parallel and determine two bands B_2' (bounded by H_2' and H_3') and B_3' (bounded by H_3' and H_0'). By a similar argument to the previous case (deconing with respect to $\widetilde{H}_0$), we can conclude that if $C' \subset B_2'$ and $d(U_1(B_2'), C') = 1$, then $\mathrm{Sep}(U_1(B_2'), C') \subset \mathcal{S}'$. (Right hand side of Figure 6.) The same assertion holds for the opposite unbounded chamber $U_2(B_2')$.

The bands B_2 and B_2' are identical in the projective plane $\mathbb{RP}^2$. However, the colors of boundaries of unbounded chambers are different. This is a contradiction. Thus the case (iv) can not happen. □

5.2. **Real 4-nets do not exist.**

Theorem 5.2. *There does not exist a real arrangement $\overline{\mathcal{A}}$ that supports a 4-net structure.*

Proof. Suppose $\overline{\mathcal{A}}$ supports a 4-net structure with partition $\overline{\mathcal{A}} = \overline{\mathcal{A}}_1 \sqcup \overline{\mathcal{A}}_2 \sqcup \overline{\mathcal{A}}_3 \sqcup \overline{\mathcal{A}}_4$. There exists a multiple point $X \in \mathbb{RP}^2$ of $\overline{\mathcal{A}}$ with multiplicity 4 such that X is the intersection point of 4 lines $H_i \in \mathcal{A}_i$. Suppose that the lines are ordered as in Figure 7.

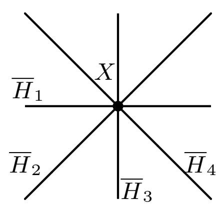

FIGURE 7. Local structure of a 4-net.

We can now define $\widetilde{\mathcal{S}} = \widetilde{\mathcal{A}}_1 \sqcup \widetilde{\mathcal{A}}_3$. Then as in Example 3.6, we have $\widetilde{\eta}_0 \wedge \widetilde{e}(\widetilde{\mathcal{S}}) = 0$. By definition, $\overline{\mathcal{S}}_X = \{\overline{H}_1, \overline{H}_3\}$ consists of two lines separated by the other two lines $\overline{H}_2$ and $\overline{H}_4$. Therefore (iv) in Figure 5 happens. This contradicts the Non-separation Theorem 5.1. □

Remark 5.3. *The non-existence of real 4-nets was proved in* [6, Lem. 2.4]. *Their proof relies on the metric structure of $\mathbb{R}^2$. So it does not apply to oriented matroids. Our arguments actually prove that there do not exist rank 3 oriented matroids (equivalently, pseudo-line arrangements in $\mathbb{RP}^2$) which have 4-net structures. The details are omitted.*

Acknowledgments: The authors thank Alex Suciu for explaining the ideas of the preprint [21], Mike Falk and Mustafa Hakan Gunturkun for telling us of the reference [6]. The authors also thank the referee for the careful reading and many suggestions. During the preparation of this paper, the authors were partially supported by JSPS Postdoctoral Fellowship For Foreign Researchers, the Grant-in-Aid for Scientific Research (C) 25400060, and the Grant-in-aid (No. 23224001 (S)) for Scientific Research, JSPS.

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