Journal *of* Singularities
Volume 11 (2015), 152-163

received: 28 March 2015
in revised form: 10 June 2015
DOI: 10.5427/jsing.2015.11f

THE PUNCTUAL HILBERT SCHEMES FOR THE CURVE SINGULARITIES OF TYPE A_{2d}

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Abstract. Pfister and Steenbrink studied punctual Hilbert schemes for irreducible curve singularities. In particular, they analyzed the structure of certain punctual Hilbert schemes for monomial curve singularities. In this paper, we generalize their results about the curve singularities of type A_{2d} by clarifying the relationships among the punctual Hilbert schemes for the singularities.

1. Introduction

Let $\mathcal{O}$ be the complete local ring of an irreducible curve singularity over an algebraically closed field k of characteristic 0. We denote by $\overline{\mathcal{O}}$ and δ the normalization of $\mathcal{O}$ and the δ-invariant of $\mathcal{O}$ respectively. Pfister and Steenbrink [3] defined a special subset $\mathcal{M}$ of the Grassmannian $\mathrm{Gr}\big(\delta, \overline{\mathcal{O}}/I(2\delta)\big)$ where $I(2\delta)$ is the set of all elements in $\mathcal{O}$ whose orders are greater than or equal to 2δ. It is a projective variety which consists of $\mathcal{O}$-sub-modules and we call it the *Pfister-Steenbrink variety* (PS variety) for the given singularity. For any positive integer r, the existence of the punctual Hilbert scheme of degree r was also shown there. It is a projective variety which parametrizes the ideals of codimension r in $\mathcal{O}$ and is realized as a connected component of the PS variety.

A numerical semi-group Γ is called *monomial*, if any curve singularity with it has no moduli. Pfister and Steenbrink determined all monomial semi-groups in [3]. Using the intersections with Schubert cells, they also analyzed the structure of the PS varieties for the curve singularities with monomial semi-groups. The cases for the curve singularities of types A_{2d}, E_6 and E_8 were also involved in their study. The punctual Hilbert schemes for the curve singularity of type A_1 and related topics were discussed by Ran in [4, 5]. Kawai [2] computed the Euler characteristic of the Hilbert scheme $C^{[d]}$ of 0-dimensional length d subschemes of a projective curve C with only the A_1 and A_2 singularities. The structures of punctual Hilbert schemes for the curve singularities of types A_1 and A_2 were used in this computation. The result was also discussed in the context of string theory. Recently, by using computational methods, the authors of this paper studied the structure of all punctual Hilbert schemes for the curve singularities of types E_6 and E_8 in [7]. On the other hand, the PS varieties for curve singularities were studied from another point of view. Rego [6] introduced the *compactified Jacobian* of singular curves. He also constructed the *Jacobi factor* for a curve singularity. The Jacobi factor and the PS variety coincide for a given curve singularity.

In this paper, we consider the curve singularities of type A_{2d} (i.e. the curve singularities whose local rings are isomorphic to $k[[t^2, t^{2d+1}]]$ where $d \in \mathbb{N}$). We denote by $\mathcal{M}_{d,r}$ the punctual Hilbert scheme of degree r for the curve singularity of type A_{2d}. Pfister and Steenbrink showed that: *the PS variety* $\mathcal{M}_{d,2d}$ *is an irreducible rational projective variety of* $\dim \mathcal{M}_{d,2d} = d$. *In particular,* (i) $\mathcal{M}_{1,2} \cong \mathbb{P}^1$, (ii) $\mathcal{M}_{2,4}$ *is a quardratic cone in* $\mathbb{P}^5$, (iii) $\mathcal{M}_{3,6}$ *is a threefold with a singular*

2000 *Mathematics Subject Classification.* 14C05, 14H20.

Key words and phrases. punctual Hilbert schemes, curve singularities of type A_{2d}.

line with transverse singularity of type A_2. In general, the following fact also holds: *if* $r \geq 2\delta$, *then the punctual Hilbert scheme of degree* r *for an irreducible curve singularity coincides with the PS variety for the same singularity* (Corollary 11). So it is enough to consider their degree r within $1 \leq r \leq 2\delta$ for the analysis of the structure of punctual Hilbert schemes (Remark 12). Since the δ-invariant for a given A_{2d} singularity equals d, the PS variety coincides with $\mathcal{M}_{d,2d}$ by the above fact. Our main theorems are stated as follows:

Theorem 1. *Let* d *and* r *be two integers with* $1 \leq r \leq 2d$. *Putting* $s := [r/2]$, *the punctual Hilbert scheme* $\mathcal{M}_{d,r}$ *is a rational projective variety with* $\dim \mathcal{M}_{d,r} = s$. *If* $r \geq 2$, *then it is isomorphic to the Pfister-Steenbrink variety* $\mathcal{M}_{s,2s}$.

Theorem 2. *We keep the notations* d, r *and* s *as in Theorem 1. The punctual Hilbert schemes for the curve singularity of type* A_{2d} *have the following structures*:

- (i): *The punctual Hilbert scheme* $\mathcal{M}_{d,1}$ *consists of one point.*
- (ii): *The punctual Hilbert schemes* $\mathcal{M}_{d,2}$ *and* $\mathcal{M}_{d,3}$ *are isomorphic to a projective line* $\mathbb{P}^1$.
- (iii): *The punctual Hilbert scheme* $\mathcal{M}_{d,r}$ *with* $4 \leq r \leq 2d$ *is a singular projective variety whose singular locus is given by* $\mathcal{M}_{d,2s-2} \cap \mathcal{M}_{d,2s-1} \cong \mathcal{M}_{d,2s-3}$.

The present paper is organized as follows: In Section 2 below, we briefly recall Pfister-Steenbrink theory introduced in [3]. In Section 3, we study ideals in the local ring $k[[t^2, t^{2d+1}]]$ of the curve singularities of type A_{2d}. From the point of view of Γ-semi-module structure of orders, we determine the sets of ideals in $\mathcal{O}$ with codimension r $(1 \leq r \leq 2\delta)$ and their decompositions. These yield affine cell decompositions of the punctual Hilbert schemes for the curve singularities of type A_{2d}. In Section 4, we first show the irreducibility of the punctual Hilbert schemes. We also prove the following proposition:

Proposition 3. *The following relations hold for punctual Hilbert schemes*:

- (i): *For integers* d *and* s *with* $1 \leq s \leq d-1$, *we have* $\mathcal{M}_{d,2s} \cong \mathcal{M}_{d,2s+1}$.
- (ii): *For integers* d, d' *and* r *with* $1 \leq r \leq \min\{2d, 2d'\}$, *we have* $\mathcal{M}_{d,r} \cong \mathcal{M}_{d',r}$.

Finally, we prove Theorem 1 and 2 by using them.

Acknowledgement The authors would like to express his sincere gratitude to Professor Fumio Sakai for his valuable advices and warm encouragement during the preparation of the present article.

2. Pfister-Steenbrink theory for punctual Hilbert schemes

In the present paper, we restrict ourselves to monomial curve singularities defined below. However, the notions in this section hold in more general situations. See [3] for details.

Definition 4. *A monomial curve singularity is an irreducible curve singularity whose local ring is isomorphic to* $k[[t^{a_1}, \ldots, t^{a_m}]]$ *for some* $a_1, \ldots, a_m \in \mathbb{N}$.

Remark 5. *Without loss of generality, we may assume that* $\gcd(a_1, \ldots, a_m) = 1$ *in Definition 4.*

Let $\mathcal{O} = k[[t^{a_1}, \ldots, t^{a_m}]]$ be the local ring of a monomial curve singularity. Its normalization $\overline{\mathcal{O}}$ is isomorphic to $k[[t]]$. We call $\Gamma := \{\mathrm{ord}_t(f) \mid f \in \mathcal{O}\}$ the *semi-group* of $\mathcal{O}$. The positive integer $\delta := \dim_k(\overline{\mathcal{O}}/\mathcal{O})$ is called the *δ-invariant* of $\mathcal{O}$. For $n \in \mathbb{N}$, set $\overline{I}(n) := \{ f \in \overline{\mathcal{O}} \mid \mathrm{ord}_t(f) \geq n\}$ and $I(n) := \overline{I}(n) \cap \mathcal{O}$. Setting $\mathrm{ord}_t(0) = \infty$, we regard $\overline{I}(n)$ (resp. $I(n)$) as an ideal of $\overline{\mathcal{O}}$ (resp. $\mathcal{O}$). For an ideal I of $\mathcal{O}$, we denote by $\Gamma(I) := \{\mathrm{ord}_t(f) \mid f \in I\}$ the set of orders of all elements in I. Put $G(I) := \Gamma \setminus \Gamma(I)$. For $r \in \mathbb{N}$, define

$$\mathcal{I}_r := \{I \mid I \text{ is an ideal of } \mathcal{O} \text{ with } \dim_k \mathcal{O}/I = r\}.$$

A subset $\Delta \subset \mathbb{Z}$ is called a Γ-*semi-module*, if $\Delta + \Gamma \subset \Delta$. Note that if Δ is a Γ-semi-module, then $\Delta - r$ is also a Γ-semi-module for any integer r. We write $\Delta = \langle \alpha_1, \cdots, \alpha_p \rangle_\Gamma$ for a Γ-semi-module Δ which is minimally generated by $\alpha_1, \cdots, \alpha_p$ (i.e $\Delta = \sum_{i=1}^{p} (\alpha_i + \Gamma)$ and $\Delta \supsetneq \sum_{i=1, i \neq j}^{p} (\alpha_i + \Gamma)$ for $\forall j \in \{1, \ldots, p\}$). We also denote by $\mathcal{I}(\Delta)$ the set of all ideals of $\mathcal{O}$ whose set of orders are Δ. Note that $\mathcal{I}(\Delta) \neq \emptyset$ if and only if $\Delta \subset \Gamma$.

The following facts are known:

Lemma 6 ([7], Lemma 5)**.** *An ideal I in $\mathcal{O}$ belongs to $\mathcal{I}_r$ if and only if we have $\sharp G(I) = r$.*

Proposition 7 ([7], Proposition 7)**.** *There exists a finite number of distinct Γ-semi-modules $\Delta_{r,1}, \cdots, \Delta_{r,n_r}$ such that*

$$\mathcal{I}_r = \bigcup_{l=1}^{n_r} \mathcal{I}(\Delta_{r,l}). \tag{1}$$

Remark 8. *The set of Γ-semi-modules $\Delta_{r,l}$ in (1) is an invariant for the codimension r.*

Let $\mathrm{Gr}\left(\delta, \overline{\mathcal{O}}/\mathrm{I}(2\delta)\right)$ be the Grassmannian which consists of δ-dimensional linear subspaces of $\overline{\mathcal{O}}/I(2\delta)$. For $V \in \mathrm{Gr}\left(\delta, \overline{\mathcal{O}}/\mathrm{I}(2\delta)\right)$, we define a multiplication by $\mathcal{O} \times V \ni (f, \overline{v}) \mapsto \overline{fv} \in V$. Set

$$\mathcal{M} := \left\{ V \in \mathrm{Gr}\left(\delta, \overline{\mathcal{O}}/\mathrm{I}(2\delta)\right) \,\middle|\, V \text{ is an } \mathcal{O}\text{-sub-module w.r.t. the multiplication} \right\}.$$

Consider the composition map

$$\psi : \mathcal{M} \xrightarrow{\psi_1} \mathrm{M}_{\delta,2\delta}(k)/\sim \xrightarrow{\psi_2} \mathbb{P}^N \tag{2}$$

where $\mathrm{M}_{\delta,2\delta}(k)$ is the set of all $\delta \times 2\delta$ matrices over k and the equivalence relation $\sim$ is the similarity of matrices. For a formal power series $f = \sum_{j=0}^{\infty} a_j t^j$ in $\overline{\mathcal{O}}$, we denote its coset in $\overline{\mathcal{O}}/I(2\delta)$ by $\overline{f} = \sum_{j=0}^{2\delta-1} a_j \tau^j$. The notation τ signifies the coset of t. Define $\mathrm{ord}_\tau(\overline{f})$ by $\mathrm{ord}_t(f)$ (resp. ∞), if $\mathrm{ord}_t(f) \leq 2\delta - 1$ (resp. $\mathrm{ord}_t(f) \geq 2\delta$). In this paper, we use the notation $[v_1, \cdots, v_\delta]_k$ for a k-vector space generated by $v_1, \ldots, v_\delta$. Let $V = [\overline{f}_1, \cdots, \overline{f}_\delta]_k$ be an element of $\mathcal{M}$ where $\overline{f}_i = \sum_{j=0}^{2\delta-1} a_{ij} \tau^j$. We identify $\overline{f}_i$ with the point $\boldsymbol{a}_i = (a_{i0}, \cdots, a_{i2\delta-1})$ in $k^{2\delta}$. Let A_V be the $\delta \times 2\delta$ matrix whose ith row is $\boldsymbol{a}_i$. We call it the *representation matrix* of V. The first map ψ_1 in (2) is defined by sending a k-vector space V to the coset of A_V. The second map ψ_2 in (2) is the Plücker embedding with $N = \binom{2\delta}{\delta} - 1$. Note that ψ_1 and ψ_2 are injective.

For $r \in \mathbb{N}$, Pfister and Steenbrink defined a map $\varphi_r : \mathcal{I}_r \to \mathcal{M}$ by $\varphi_r(I) = t^{-r} I / I(2\delta)$.

Proposition 9 ([3], Theorem 3)**.** *The map φ_r is injective for any r. Furthermore, it is bijective for $r \geq 2\delta$. The image $(\psi \circ \varphi_r)(\mathcal{I}_r)$ is Zariski closed in $\psi(\mathcal{M})$.*

Put $\mathcal{M}_r := \varphi_r(\mathcal{I}_r)$. Since ψ is injective, we identify $\psi(\mathcal{M})$ and $\psi(\mathcal{M}_r)$ with $\mathcal{M}$ and $\mathcal{M}_r$ respectively.

Definition 10. *We call $\mathcal{M}$ and $\mathcal{M}_r$ the Pfister-Steenbrink variety* (*PS variety for short*) *and the punctual Hilbert scheme of degree r for a given curve singularity respectively.*

The following fact follows from Proposition 9:

Corollary 11. *Any punctual Hilbert scheme $\mathcal{M}_r$ with $r \geq 2\delta$ coincides with the PS variety $\mathcal{M}$.*

Remark 12. *By virtue of Corollary 11, it is enough to consider codimensions r within $1 \leq r \leq 2\delta$ for the analysis of $\mathcal{M}_r$.*

Set $\mathcal{M}(\Delta_{r,l}) := \varphi_r(\mathcal{I}(\Delta_{r,l}))$ for each component $\mathcal{I}(\Delta_{r,l})$ in (1). Since ψ is injective, we also identify $\psi(\mathcal{M}(\Delta_{r,l}))$ with $\mathcal{M}(\Delta_{r,l})$. Namely, we regard $\mathcal{M}(\Delta_{r,l})$ as a subset of the punctual Hilbert scheme $\mathcal{M}_r$ parametrizing ideals in $\mathcal{I}(\Delta_{r,l})$. Set $[a, b] := \{x \in \mathbb{Z}_{\geq 0} | \, a \leq x \leq b\}$. For a Γ-semi-module $\Delta_{r,l} = \langle \alpha_1, \cdots, \alpha_{p_l} \rangle_\Gamma$, we have $\Delta_{r,l} - r = \langle \alpha_1 - r, \cdots, \alpha_{p_l} - r \rangle_\Gamma$. Define $A := \{\alpha_1 - r, \cdots, \alpha_{p_l} - r\} \cap [0, 2\delta - 1]$ and $J_\alpha := [\alpha + 1, 2\delta - 1] \setminus \{\Delta_{r,l} - r\}$ for $\alpha \in A$. The following facts are known:

Proposition 13 ([3], Theorem 7)**.** *Let I be an element of $\mathcal{I}(\Delta_{r,l})$. There exist uniquely determined $b_{\alpha j} \in k$ such that the $\mathcal{O}$-sub-module $\varphi_r(I)$ is generated by*

$$\overline{f}_\alpha := \tau^\alpha + \sum_{j \in J_\alpha} b_{\alpha j} \tau^j \quad (\alpha \in A).$$

Corollary 14 ([3], Corollary of Theorem 11)**.** *The component $\mathcal{M}(\Delta_{r,l})$ is isomorphic to the affine space k^N where $N = \sum_{\alpha \in A} \sharp J_\alpha$.*

We obtain an affine cell decomposition of $\mathcal{M}_r$ by Proposition 7 and Corollary 14.

Proposition 15. *The punctual Hilbert scheme $\mathcal{M}_r$ of degree r has an affine cell decomposition*

$$\mathcal{M}_r = \bigcup_{l=1}^{n_r} \mathcal{M}(\Delta_{r,l}). \tag{3}$$

The following fact also follows from Corollary 14:

Proposition 16. *If $\mathcal{M}_r$ is irreducible, then it is a rational projective variety.*

The 2δ-dimensional k-vector space $\overline{\mathcal{O}}/I(2\delta)$ has the canonical flag

$$0 \subset V_1 \subset V_2 \subset \cdots \subset V_{2\delta} = \overline{\mathcal{O}}/I(2\delta)$$

where $V_i = \overline{I}(2\delta - i)/I(2\delta)$ for $1 \le i \le 2\delta$. This induces a partition of $\mathrm{Gr}(\delta, \overline{\mathcal{O}}/I(2\delta))$ into *Schubert cells* $W_{a_1,\ldots,a_\delta}$ for $\delta \ge a_1 \ge \cdots \ge a_\delta \ge 0$, which is defined by

$$W_{a_1,\ldots,a_\delta} := \left\{ W \in \mathrm{Gr}(\delta, \overline{\mathcal{O}}/I(2\delta)) \;\middle|\; \begin{array}{l} \dim(W \cap V_{\delta+i-a_i}) = i \text{ for } 1 \le i \le \delta \\ \dim(W \cap V_j) < i \text{ for } j < \delta + i - a_i \end{array} \right\}.$$

For an index set $\Lambda = \{a_1, \ldots, a_\delta\}$, we sometimes write W_Λ instead of $W_{a_1,\ldots,a_\delta}$.

Proposition 17. *We have $W_{b_1,\ldots,b_\delta} \subset \overline{W_{a_1,\ldots,a_\delta}}$ if and only if $b_i \ge a_i$ holds for $1 \le i \le \delta$.*

For the details about Schubert cells, see [1, p.195].

Lemma 18. *Let $\mathcal{M}(\Delta_{r,l})$ be a component in (3) and write $\{b_1, \ldots, b_\delta\} = (\Delta_{r,l} - r) \cap [0, 2\delta - 1]$ where $0 \le b_1 < \cdots < b_\delta < 2\delta$. Setting $a_{\delta-i+1} = b_i - i + 1$ for $1 \le i \le \delta$, we have*

$$\mathcal{M}(\Delta_{r,l}) = \mathcal{M}_r \cap W_{a_1,\ldots,a_\delta}.$$

Proof. It is known that our assertion is true for $r = 2\delta$ (see Lemma 5 in [3]). So we consider the case where $r < 2\delta$. Since $\mathcal{M}(\Delta_{r,l}) \subset \mathcal{M}_r \subset \mathcal{M}_{2\delta}$, there exists a Γ-semi-module $\Delta_{2\delta}$ such that $\mathcal{M}(\Delta_{2\delta})$ is a component of $\mathcal{M}_{2\delta}$ and $\Delta_{2\delta} - 2\delta = \Delta_{r,l} - r$. It follows from the above fact for $r = 2\delta$ that $\mathcal{M}(\Delta_{r,l}) = \mathcal{M}_r \cap \mathcal{M}(\Delta_{2\delta}) = \mathcal{M}_r \cap (\mathcal{M}_{2\delta} \cap W_{a_1,\ldots,a_\delta}) = \mathcal{M}_r \cap W_{a_1,\ldots,a_\delta}$. ∎

3. Ideals in the local ring of the singularities of type A_{2d}

From this section, we only consider the curve singularities of type A_{2d} and freely use the notations introduced in the previous section. Let $\mathcal{O}$ be the local ring $k[[t^2, t^{2d+1}]]$ for some $d \in \mathbb{N}$. The semi-group Γ of $\mathcal{O}$ is generated by 2 and $2d+1$. Note that any Γ-semi-module Δ contained in Γ is generated by at at most two elements $\alpha_1 := \min\{\Gamma(I)\}$ and $\alpha_2 := \min\{\Gamma(I) \setminus (\alpha_1 + \Gamma)\}$ as Γ-semi-module. We have $\alpha_2 < \alpha_1 + 2d + 1$. We use the notation $\mathcal{I}(\alpha_1, \alpha_2)$ instead of $\mathcal{I}(\Delta)$.

Lemma 19. *For any element I of $\mathcal{I}(\alpha_1,\alpha_2)$, there exist two generators f_1 and f_2 such that*

$$f_1 = t^{\alpha_1} + \sum_{j\in G(I), j>\alpha_1} c_j t^j, \quad f_2 = t^{\alpha_2}.$$

Proof. Since $\Gamma(I)$ is generated by at most two positive integers as Γ-semi-module, the ideal I is generated by at most two elements. Let $g_1 = t^{\alpha_1} + \sum_{j>\alpha_1} c_j t^j$ and $g_2 = t^{\alpha_2} + \sum_{j>\alpha_2} d_j t^j$ be such generators. For any $j \in \Gamma(I)$, there exists an element $h_j = t^j + \text{terms of higher order in } I$. Reducing g_1 and g_2 by h_j's successively, we obtain the desired generators. □

By Lemma 19, normal forms of all ideals in $\mathcal{I}(\alpha_1,\alpha_2)$ are described as follows:

Proposition 20. *If $\alpha_1 = 2p$ and $\alpha_2 = 2q+1$, then we have*

$$\mathcal{I}(2p,2q+1) = \begin{cases} \left\{\left(t^{2p}, t^{2d+1}\right)\right\} & \text{for } p \le d = q,\\ \left\{\left(t^{2p} + \sum_{i=d}^{q-1} a_i t^{2i+1}, t^{2q+1}\right) \middle| a_i \in k\right\} & \text{for } p \le d < q,\\ \left\{\left(t^{2p}, t^{2p+1}\right)\right\} & \text{for } d < p = q,\\ \left\{\left(t^{2p} + \sum_{i=p}^{q-1} a_i t^{2i+1}, t^{2q+1}\right) \middle| a_i \in k\right\} & \text{for } d < p < q. \end{cases}$$

On the other hand, if $\alpha_1 = 2p+1$ and $\alpha_2 = 2q$, then we have

$$\mathcal{I}(2p+1,2q) = \begin{cases} \left\{\left(t^{2p+1}, t^{2p+2}\right)\right\} & \text{for } p \ge d,\ q = p+1,\\ \left\{\left(t^{2p+1} + \sum_{i=p+1}^{q-1} a_i t^{2i}, t^{2q}\right) \middle| a_i \in k\right\} & \text{for } p \ge d,\ q > p+1. \end{cases}$$

Proof. Let I be a non-zero ideal in $\mathcal{O}$. For the set $G(I)$, we have the following two cases:
(Case 1): α_1 is even and α_2 is odd. Write $\alpha_1 = 2p$ and $\alpha_2 = 2q+1$. It follows from the definitions of α_1 and α_2 that $p \le q$. We easily see that $\Gamma(I) = \{2i \mid p \le i \le q\} \cup \{n \mid n \ge 2q+1\}$ and $c(I) = 2q$. Since $2d+1 \le \alpha_2 \le \alpha_1 + 2d+1$ hold, we also obtain $d \le q \le p+d$. In terms of p, q and d, the set $G(I)$ is described as follows:

$$G(I) = \begin{cases} \{2i \mid 0 \le i \le p-1\} & \text{for } p \le d = q,\\ \{2i \mid 0 \le i \le p-1\} \cup \{2j+1 \mid d \le j \le q-1\} & \text{for } p \le d < q,\\ \{2i \mid 0 \le i \le d\} \cup [2d+1, 2p-1] & \text{for } d < p = q,\\ \{2i \mid 0 \le i \le d\} \cup [2d+1, 2p-1] \cup \{2j+1 \mid p \le j \le q-1\} & \text{for } d < p < q. \end{cases} \tag{4}$$

(Case 2): α_1 is odd and α_2 is even. Put $\alpha_1 = 2p+1$ and $\alpha_2 = 2q$. It follows from the relations $2d+1 \le \alpha_1 < \alpha_2 \le 2(p+d+1)$ that $d \le p < q \le p+d+1$. For $I \in \mathcal{I}(\alpha_1,\alpha_2)$, we have $\Gamma(I) = \{2i+1 \mid p \le i \le q-1\} \cup \{n \mid n \ge 2q\}$. For this case, the following four cases occur:

$$G(I) = \begin{cases} \{2i \mid 0 \le i \le d\} & \text{for } p = d,\ q = d+1,\\ \{2i \mid 0 \le i \le q-1\} & \text{for } p = d,\ q > d+1,\\ \{2i \mid 0 \le i \le d\} \cup [2d+1, 2p] & \text{for } p > d,\ q = p+1,\\ \{2i \mid 0 \le i \le d\} \cup [2d+1, 2p] \cup \{2j \mid p+1 \le j \le q-1\} & \text{for } p > d,\ q > p+1. \end{cases} \tag{5}$$

Our assertions follow from Lemma 19 with (4) and (5). □

Lemma 21. *If I belongs to $\mathcal{I}(2p, 2q+1)$ or $\mathcal{I}(2p+1, 2q)$, then its codimension r is given by*

$$r = p + q - d. \tag{6}$$

Proof. This relation follows from Lemma 6 with (4) and (5). □

The decomposition of $\mathcal{I}_r$ is determined in terms of generators of $\Delta_{r,l}$.

Proposition 22. *The sets $\mathcal{I}_r$ for $1 \le r \le 2d$ are decomposed as follows*:

(A): $1 \le r \le d$ *and* $r = 2s+1$. $\mathcal{I}_{2s+1} = \bigcup_{l=0}^{s} \mathcal{I}\left(r+2l+1, r+2(d-l)\right)$.

(B): $2 \le r \le d$ *and* $r = 2s$. $\mathcal{I}_{2s} = \bigcup_{l=0}^{s} \mathcal{I}\left(r+2l, r+2(d-l)+1\right)$.

For the cases where $d+1 \le r \le 2d-1$, the decompositions depend on d.

(C-i): $d+1 \le r \le 2d-1$, $r = 2s+1$ *and* $d = 2h$.

$$\mathcal{I}_{2s+1} = \left\{ \bigcup_{l=0}^{h-1} \mathcal{I}\left(r+2l+1, r+2(d-l)\right) \right\} \cup \left\{ \bigcup_{l=d-s}^{h} \mathcal{I}\left(r+2l, r+2(d-l)+1\right) \right\}.$$

(C-ii): $d+1 \le r \le 2d-1$, $r = 2s+1$ *and* $d = 2h+1$.

$$\mathcal{I}_{2s+1} = \left\{ \bigcup_{l=0}^{h} \mathcal{I}\left(r+2l+1, r+2(d-l)\right) \right\} \cup \left\{ \bigcup_{l=d-s}^{h} \mathcal{I}\left(r+2l, r+2(d-l)+1\right) \right\}.$$

(D-i): $d+1 \le r \le 2d$, $r = 2s$ *and* $d = 2h$.

$$\mathcal{I}_{2s} = \left\{ \bigcup_{l=0}^{h} \mathcal{I}\left(r+2l, r+2(d-l)+1\right) \right\} \cup \left\{ \bigcup_{l=d-s}^{h-1} \mathcal{I}\left(r+2l+1, r+2(d-l)\right) \right\}.$$

(D-ii): $d+1 \le r \le 2d$, $r = 2s$ *and* $d = 2h+1$.

$$\mathcal{I}_{2s} = \left\{ \bigcup_{l=0}^{h} \mathcal{I}\left(r+2l, r+2(d-l)+1\right) \right\} \cup \left\{ \bigcup_{l=d-s}^{h} \mathcal{I}\left(r+2l+1, r+2(d-l)\right) \right\}.$$

Proof. We infer from the relation (6) in Lemma 21 that

$$\alpha_2 = -\alpha_1 + 2r + 2d + 1. \tag{7}$$

We first consider the case where $1 \le r \le d$. In this case, the positive integer α_1 must be even. Indeed, if not, then we have $\sharp G(I) \ge d+1$ since $\alpha_1 \ge d+1$. So we conclude that $r \ge d+1$ by Lemma 6. It is a contradiction. It follows from the definitions of α_1 and α_2 that

$$\alpha_1 \le \alpha_2 \le \alpha_1 + 2d + 1. \tag{8}$$

Note that $\mathcal{I}(\alpha_1, \alpha_2)$ is a component of $\mathcal{I}_r$ if and only if α_1 and α_2 satisfy both of (7) and (8). According to r, the sets of all pairs (α_1, α_2) which satisfy (7) and (8) are determined as follows:
(A) $r = 2s+1$. $\{(r+2l+1, r+2(d-l))\}_{l=0,\dots,s}$
(B) $r = 2s$. $\{(r+2l, r+2(d-l)+1)\}_{l=0,\dots,s}$

Next we consider the case in which $d+1 \leq r \leq 2d$. According to r and d, we obtain the following sets of the pair (α_1, α_2)which satisfy (7) and (8):
(C-i) $r = 2s+1$ and $d = 2h$.

$$\{(r+2l+1, r+2(d-l))\}_{l=0,\ldots,h-1} \cup \{(r+2l, r+2(d-l)+1)\}_{d-s,\ldots,h}$$

(C-ii) $r = 2s+1$ and $d = 2h+1$.

$$\{(r+2l+1, r+2(d-l))\}_{l=0,\ldots,h} \cup \{(r+2l, r+2(d-l)+1)\}_{d-s,\ldots,h}$$

(D-i) $r = 2s$ and $d = 2h$.

$$\{(r+2l, r+2(d-l)+1)\}_{l=0,\ldots,h} \cup \{(r+2l+1, r+2(d-l))\}_{l=d-s,\ldots,h-1}$$

(D-ii) $r = 2s$ and $d = 2h+1$.

$$\{(r+2l, r+2(d-l)+1)\}_{l=0,\ldots,h} \cup \{(r+2l+1, r+2(d-l))\}_{l=d-s,\ldots,h}$$

Our assertions follow from Proposition 20 with the above datum. □

Let $\Delta_{r,l}$ be a Γ-semi-module in (3) and take an element I from $\mathcal{I}(\Delta_{r,l})$. For the $\mathcal{O}$-sub-module $\varphi_r(I)$, define the *set of orders* of $\varphi_r(I)$ by $\Gamma(\varphi_r(I)) := \{\mathrm{ord}_\tau(\overline{f}) | \, \overline{f} \in \varphi_r(I)\}$. It is clear that $\Gamma(\varphi_r(I))$ has a Γ-semi-module structure. Furthermore, if the Γ-semi-module $\Delta_{r,l}$ is generated by α_1 and α_2, then $\Gamma(\varphi_r(I))$ is generated by $\alpha_1 - r$ and $\alpha_2 - r$. So we write $\mathcal{M}(\alpha_1 - r, \alpha_2 - r)$ instead of $\mathcal{M}(\Delta_{r,l})$ for such case. For each r, the decomposition (3) of $\mathcal{M}_r$ follows from Proposition 6.

Corollary 23. *The punctual Hilbert schemes $\mathcal{M}_r$ $(1 \leq r \leq 2d)$ are decomposed as follows*:

(A): $1 \leq r \leq d$ *and* $r = 2s+1$. $\mathcal{M}_{2s+1} = \bigcup_{l=0}^{s} \mathcal{M}_{2s+1}(2l+1, 2(d-l))$.

(B): $2 \leq r \leq d$ *and* $r = 2s$. $\mathcal{M}_{2s} = \bigcup_{l=0}^{s} \mathcal{M}_{2s}(2l, 2(d-l)+1)$.

(C-i): $d+1 \leq r \leq 2d-1$, $r = 2s+1$ *and* $d = 2h$.

$$\mathcal{M}_{2s+1} = \left\{ \bigcup_{l=0}^{h-1} \mathcal{M}_{2s+1}(2l+1, 2(d-l)) \right\} \cup \left\{ \bigcup_{l=d-s}^{h} \mathcal{M}_{2s+1}(2l, 2(d-l)+1) \right\}.$$

(C-ii): $d+1 \leq r \leq 2d-1$, $r = 2s+1$ *and* $d = 2h+1$.

$$\mathcal{M}_{2s+1} = \left\{ \bigcup_{l=0}^{h} \mathcal{M}_{2s+1}(2l+1, 2(d-l)) \right\} \cup \left\{ \bigcup_{l=d-s}^{h} \mathcal{M}_{2s+1}(2l, 2(d-l)+1) \right\}.$$

(D-i): $d+1 \leq r \leq 2d$, $r = 2s$ *and* $d = 2h$.

$$\mathcal{M}_{2s} = \left\{ \bigcup_{l=0}^{h} \mathcal{M}_{2s}(2l, 2(d-l)+1) \right\} \cup \left\{ \bigcup_{l=d-s}^{h-1} \mathcal{M}_{2s}(2l+1, 2(d-l)) \right\}.$$

(D-ii): $d+1 \leq r \leq 2d$, $r = 2s$ *and* $d = 2h+1$.

$$\mathcal{M}_{2s} = \left\{ \bigcup_{l=0}^{h} \mathcal{M}_{2s}(2l, 2(d-l)+1) \right\} \cup \left\{ \bigcup_{l=d-s}^{h} \mathcal{M}_{2s}(2l+1, 2(d-l)) \right\}.$$

Remark 24. *The punctual Hilbert scheme $\mathcal{M}_1$ consists of one point which corresponds to the maximal ideal of $\mathcal{O}$.*

4. Proof of Main Theorems

In this section, we prove Theorem 1 and 2. To emphasis d, we use notations $\mathcal{M}_{d,r}$ and $\mathcal{M}_{d,r}(\alpha_1 - r, \alpha_2 - r)$ insted of $\mathcal{M}_r$ and $\mathcal{M}_r(\alpha_1 - r, \alpha_2 - r)$.

Lemma 25. *If the codimension r is odd (resp. even), then the component $\mathcal{M}_{d,r}(1, 2d)$ (resp. $\mathcal{M}_{d,r}(0, 2d+1)$) of the decomposition in Corollary 23 is the open dense subset of $\mathcal{M}_{d,r}$.*

Proof. We should check our assertion for each case in Corollary 23. However, we only consider the case (D-ii) in Corollary 23. The other cases are treated in the same way. For $r = 2s$, set

$$\Lambda_1(l) := \{\underbrace{d, \dots, d}_{2l}, \underbrace{d-1, d-2, \dots, 2l+1, 2l}_{d-2l}\} \quad \text{for } l = 0, \dots, h,$$
$$\Lambda_2(l) := \{\underbrace{d, \dots, d}_{2l+1}, \underbrace{d-1, d-2, \dots, 2d-r+l}_{d-2l-1}\} \quad \text{for } l = d-s, \dots, h.$$

By Lemma 18, we see that

$$\begin{aligned} \mathcal{M}_{d,2s}(2l, 2(d-l)+1) &= \mathcal{M}_{d,2s} \cap W_{\Lambda_1(l)} \quad \text{for } l = 0, \dots, h, \\ \mathcal{M}_{d,2s}(2l+1, 2(d-s)) &= \mathcal{M}_{d,2s} \cap W_{\Lambda_2(l)} \quad \text{for } l = d-s, \dots, h. \end{aligned} \tag{9}$$

The following inclusions also follows from Proposition 17:

$$W_{\Lambda_1(l+1)} \subset \overline{W_{\Lambda_1(l)}},\ W_{\Lambda_2(l+1)} \subset \overline{W_{\Lambda_2(l)}},\ W_{\Lambda_2(d-s)} \subset \overline{W_{\Lambda_1(0)}} \tag{10}$$

It follows from (9), (10) and (D-ii) in Corollary 23 that

$$\begin{aligned} \mathcal{M}_{d,2s} &= \left\{ \mathcal{M}_{d,2s} \cap \left(\bigcup_{l=0}^{h} W_{\Lambda_1(l)} \right) \right\} \cup \left\{ \mathcal{M}_{d,2s} \cap \left(\bigcup_{l=d-s}^{h} W_{\Lambda_2(l)} \right) \right\} \\ &\subset \mathcal{M}_{d,2s} \cap \left(\overline{W_{\Lambda_1(0)}} \cup \overline{W_{\Lambda_2(d-s)}} \right) = \mathcal{M}_{d,2s} \cap \overline{W_{\Lambda_1(0)}} \\ &= \overline{\mathcal{M}_{d,2s}(0, 2d+1)} \subset \mathcal{M}_{d,2s}. \end{aligned}$$

Hence we conclude that $\mathcal{M}_{d,2s} = \overline{\mathcal{M}_{d,2s}(0, 2d+1)}$. □

Next we prove Proposition 3.

Proof of Proposition 3. We first prove (i) by constructing an isomorphism between $\mathcal{M}_{d,2s}$ and $\mathcal{M}_{d,2s+1}$. We have the following combination of the decomposition types of $\mathcal{M}_{d,2s}$ and $\mathcal{M}_{d,2s+1}$.

Decomposition type of $\mathcal{M}_{d,2s}$	(B)	(B)	(D-i)	(D-ii)
Decomposition type of $\mathcal{M}_{d,2s+1}$	(A)	(C-i)	(C-i)	(C-ii)

We referred to Corollary 23 for the decomposition types. We only prove our assertion for the pair (D-ii) and (C-ii) here. The other cases can be treated in the same way. Since d is odd in this case, we put $d = 2h+1$. It follows from Corollary 23 that

$$\mathcal{M}_{d,2s} = \left\{ \bigcup_{l=0}^{h} \mathcal{M}_{d,2s}\left(2l, 2(d-l)+1\right) \right\} \cup \left\{ \bigcup_{l=d-s}^{h} \mathcal{M}_{d,2s}\left(2l+1, 2(d-l)\right) \right\},$$

$$\mathcal{M}_{d,2s+1} = \left\{ \bigcup_{l=0}^{h} \mathcal{M}_{d,2s+1}\left(2l+1, 2(d-l)\right) \right\} \cup \left\{ \bigcup_{l=d-s}^{h} \mathcal{M}_{d,2s+1}\left(2l, 2(d-l)+1\right) \right\}.$$

By Propositions 20, 22, Corollary 23 and the definition of $\mathcal{M}_{d,r}$, we obtain the following explicit descriptions of the components of $\mathcal{M}_{d,2s}$ and $\mathcal{M}_{d,2s+1}$:

$$\mathcal{M}_{d,2s}(2l, 2(d-l)+1) = \begin{cases} \left\{ \left(\tau^{2l} + \sum_{j=d-s}^{d-l-1} b_j \tau^{2j+1}, \tau^{2(d-l)+1} \right) \middle| b_j \in k \right\} \text{ for } l = 0 \ldots, d-s, \\ \left\{ \left(\tau^{2l} + \sum_{j=l}^{d-l-1} b_j \tau^{2j+1}, \tau^{2(d-l)+1} \right) \middle| b_j \in k \right\} \text{ for } l = d+1-s \ldots, h, \end{cases}$$

$$\mathcal{M}_{d,2s}(2l+1, 2(d-l)) = \begin{cases} \left\{ \left(\tau^{2l+1} + \sum_{j=l+1}^{d-l-1} b_j \tau^{2j}, \tau^{2(d-l)} \right) \middle| b_j \in k \right\} \text{ for } l = d-s \ldots, h-1, \\ \left\{ (\tau^d, \tau^{d+1}) \right\} \text{ for } l = h, \end{cases}$$

$$\mathcal{M}_{d,2s+1}(2l+1, 2(d-l)) = \begin{cases} \left\{ \left(\tau^{2l+1} + \sum_{j=d-s}^{d-l-1} b_j \tau^{2j}, \tau^{2(d-l)} \right) \middle| b_j \in k \right\} \text{ for } l = 0, \ldots, d-s-1, \\ \left\{ \left(\tau^{2l+1} + \sum_{j=l+1}^{d-l-1} b_j \tau^{2j}, \tau^{2(d-l)} \right) \middle| b_j \in k \right\} \text{ for } l = d-s \ldots, h-1, \\ \left\{ (\tau^d, \tau^{d+1}) \right\} \text{ for } l = h, \end{cases}$$

$$\mathcal{M}_{d,2s+1}(2l, 2(d-l)+1) = \left\{ \left(\tau^{2l} + \sum_{j=l}^{d-l-1} b_j \tau^{2j+1}, \tau^{2(d-l)+1} \right) \middle| b_j \in k \right\} \text{ for } l = d-s \ldots, h$$

Both of $\mathcal{M}_{d,2s}$ and $\mathcal{M}_{d,2s+1}$ have $s+1$ components. Furthermore, the numbers of coefficients involved in the elements of their components are $0, 1, \ldots, s-1, s$. So, for two components of $\mathcal{M}_{d,2s}$ and $\mathcal{M}_{d,2s+1}$ which have same number of coefficients, we can define a bijection by sending an element of $\mathcal{M}_{d,2s}$ to that of $\mathcal{M}_{d,2s+1}$ which has same coefficients. In this way, we obtain $s+1$ bijections between the components of $\mathcal{M}_{d,2s}$ and $\mathcal{M}_{d,2s+1}$. It is clear that the union of them is an isomorphism from $\mathcal{M}_{d,2s}$ to $\mathcal{M}_{d,2s+1}$.

Next we prove (ii). If $r = 1$, then we have $\mathcal{M}_{d,1} = \{\text{one point}\}$ for any $d \in \mathbb{N}$, as mentioned in Remark 24. So we consider the case where $r \geq 2$. For any $d, d' \in \mathbb{N}$, we can construct an isomorphism between $\mathcal{M}_{d,r}$ and $\mathcal{M}_{d',r}$ by the same argument in the proof of (i). □

The following fact is known:

Theorem 26 ([3]). *We have* $\dim(\mathcal{M}_{d,2d}) = d$ *for any* $d \in \mathbb{N}$.

Proof of Theorem 1. The rationality of $\mathcal{M}_{d,r}$ is an immediate consequence of Lemma 25 and Proposition 16. The relation $\mathcal{M}_{d,r} \cong \mathcal{M}_{s,2s}$ also follows from (i) and (ii) in Proposition 3. Hence, we obtain $\dim \mathcal{M}_{d,r} = \dim \mathcal{M}_{s,2s} = s$ by Theorem 26. □

Next we prove Theorem 2.

Proof of Theorem 2. The statement (i) already was mentioned in Remark 24. For (ii), the relation $\mathcal{M}_{2,2} \cong \mathcal{M}_{2,3} \cong \mathbb{P}^1$ proved in [3]. Hence it follows from (ii) in Proposition 3 that $\mathcal{M}_{d,2} \cong \mathcal{M}_{d,3} \cong \mathbb{P}^1$ for any d. The statement (ii) is proved. Since it was shown that

$$\mathrm{Sing}(\mathcal{M}_{2,4}) = \mathcal{M}_{2,2} \cap \mathcal{M}_{2,3} = \mathcal{M}_{2,1} = \{\text{one point}\}$$

in [3], we only have to consider the cases where $d \geq 3$ to prove (iii). We may assume that r is even by (i) in Proposition 3. Moreover, since $\mathcal{M}_{d,r}$ is isomorphic to $\mathcal{M}_{s,2s}$ by Theorem 1, it is enough to prove our assertion for some PS variety $\mathcal{M}_{d,2d}$ ($d \in \mathbb{N}$). We divide the rest of the proof of (iii) into the following three cases:

$$\text{Case 1}: d = 3, \quad \text{Case 2}: d \text{ is even and } d \geq 4, \quad \text{Case 3}: d \text{ is odd and } d \geq 5$$

Here we only consider Case 3. The other cases can be treated in a similar manner. Put $d = 2h+1$. By using Propositions 20, 22, Corollary 23 and the definition of $\mathcal{M}_{d,r}$, we have

$$\begin{aligned}
\mathcal{M}_{d,2d-3} &= \left\{ \bigcup_{l=0}^{h} \mathcal{M}_{d,2d-3}\,(2l+1, 2(d-l)) \right\} \cup \left\{ \bigcup_{l=2}^{h} \mathcal{M}_{d,2d-3}\,(2l, 2(d-l)+1) \right\} \\
\mathcal{M}_{d,2d-2} &= \left\{ \bigcup_{l=0}^{h} \mathcal{M}_{d,2d-2}\,(2l, 2(d-l)+1) \right\} \cup \left\{ \bigcup_{l=1}^{h} \mathcal{M}_{d,2d-2}\,(2l+1, 2(d-l)) \right\} \\
\mathcal{M}_{d,2d-1} &= \left\{ \bigcup_{l=0}^{h} \mathcal{M}_{d,2d-1}\,(2l+1, 2(d-l)) \right\} \cup \left\{ \bigcup_{l=1}^{h} \mathcal{M}_{d,2d-1}\,(2l, 2(d-l)+1) \right\} \\
\mathcal{M}_{d,2d} &= \left\{ \bigcup_{l=0}^{h} \mathcal{M}_{d,2d}\,(2l, 2(d-l)+1) \right\} \cup \left\{ \bigcup_{l=0}^{h} \mathcal{M}_{d,2d}\,(2l+1, 2(d-l)) \right\}
\end{aligned}$$

where

$$\mathcal{M}_{d,2d-3}(2l+1, 2(d-l)) =
\begin{cases}
\left\{ \left(\tau + \sum_{j=2}^{d-1} b_j \tau^{2j}, \tau^{2d} \right) \middle| b_j \in k \right\} & \text{for } l = 0, \\
\left\{ \left(\tau^{2l+1} + \sum_{j=l+1}^{d-l-1} b_j \tau^{2j}, \tau^{2(d-l)} \right) \middle| b_j \in k \right\} & \text{for } l = 1 \ldots, h-1, \\
\left\{ (\tau^d, \tau^{d+1}) \right\} & \text{for } l = h,
\end{cases}$$

$$\mathcal{M}_{d,2d-3}(2l, 2(d-l)+1) =
\left\{ \left(\tau^{2l} + \sum_{j=l}^{d-l-1} b_j \tau^{2j+1}, \tau^{2(d-l)+1} \right) \middle| b_j \in k \right\} \text{ for } l = 2 \ldots, h,$$

$$\mathcal{M}_{d,2d-2}(2l,2(d-l)+1) = \begin{cases} \left\{ \left(1+\sum_{j=1}^{d-1} b_j\tau^{2j+1}, \tau^{2d+1} \right) \middle| b_j \in k \right\} \text{ for } l=0, \\ \left\{ \left(\tau^{2l}+\sum_{j=l}^{d-l-1} b_j\tau^{2j+1}, \tau^{2(d-l)+1} \right) \middle| b_j \in k \right\} \text{ for } l=1\dots,h, \end{cases}$$

$$\mathcal{M}_{d,2d-2}(2l+1,2(d-l)) = \begin{cases} \left\{ \left(\tau^{2l+1}+\sum_{j=l+1}^{d-l-1} b_j\tau^{2j}, \tau^{2(d-l)} \right) \middle| b_j \in k \right\} \text{ for } l=1\dots,h-1, \\ \left\{ (\tau^d, \tau^{d+1}) \right\} \text{ for } l=h, \end{cases}$$

$$\mathcal{M}_{d,2d-1}(2l+1,2(d-l)) = \begin{cases} \left\{ \left(\tau^{2l+1}+\sum_{j=l+1}^{d-l-1} b_j\tau^{2j}, \tau^{2(d-l)} \right) \middle| b_j \in k \right\} \text{ for } l=0\dots,h-1, \\ \left\{ (\tau^d, \tau^{d+1}) \right\} \text{ for } l=h, \end{cases}$$

$$\mathcal{M}_{d,2d-1}(2l,2(d-l)+1) = \left\{ \left(\tau^{2l}+\sum_{j=l}^{d-l-1} b_j\tau^{2j+1}, \tau^{2(d-l)+1} \right) \middle| b_j \in k \right\} \text{ for } l=1\dots,h,$$

$$\mathcal{M}_{d,2d}(2l,2(d-l)+1) = \left\{ \left(\tau^{2l}+\sum_{j=l}^{d-l-1} b_j\tau^{2j+1}, \tau^{2(d-l)+1} \right) \middle| b_j \in k \right\} \text{ for } l=0\dots,h,$$

$$\mathcal{M}_{d,2d}(2l+1,2(d-l)) = \begin{cases} \left\{ \left(\tau^{2l+1}+\sum_{j=l+1}^{d-l-1} b_j\tau^{2j}, \tau^{2(d-l)} \right) \middle| b_j \in k \right\} \text{ for } l=0\dots,h-1, \\ \left\{ (\tau^d, \tau^{d+1}) \right\} \text{ for } l=h \end{cases}$$

It follows from the above descriptions of components that

$$\mathcal{M}_{d,2d} = (\mathcal{M}_{d,2d-2} \cap \mathcal{M}_{d,2d-1}) \cup \mathcal{M}_{d,2d}(1,2d) \cup \mathcal{M}_{d,2d}(0,2d+1)$$

where

$$(\mathcal{M}_{d,2d-2} \cap \mathcal{M}_{d,2d-1}) \cap \mathcal{M}_{d,2d}(1, 2d) = \emptyset,$$
$$(\mathcal{M}_{d,2d-2} \cap \mathcal{M}_{d,2d-1}) \cap \mathcal{M}_{d,2d}(0, 2d+1) = \emptyset,$$
$$\mathcal{M}_{d,2d}(1, 2d) \cap \mathcal{M}_{d,2d}(0, 2d+1) = \emptyset.$$

Furthermore, $\mathcal{M}_{d,2d}(1, 2d)$ and $\mathcal{M}_{d,2d}(0, 2d+1)$ are affine spaces by Corollary 14, we must have $\mathrm{Sing}(\mathcal{M}_{d,2d}) = \mathrm{Sing}(\mathcal{M}_{d,2d-2} \cap \mathcal{M}_{d,2d-1})$. Since both of

$$\mathcal{M}_{d,2d-2} \setminus (\mathcal{M}_{d,2d-2} \cap \mathcal{M}_{d,2d-1}) = \mathcal{M}_{d,2d-2}(0, 2d+1),$$
$$\mathcal{M}_{d,2d-1} \setminus (\mathcal{M}_{d,2d-2} \cap \mathcal{M}_{d,2d-1}) = \mathcal{M}_{d,2d-1}(1, 2d)$$

are affine spaces again, we conclude that $\mathrm{Sing}(\mathcal{M}_{d,2d}) = \mathcal{M}_{d,2d-2} \cap \mathcal{M}_{d,2d-1}$. Finally, the relation $\mathcal{M}_{d,2d-2} \cap \mathcal{M}_{d,2d-1} \cong \mathcal{M}_{d,2d-3}$ can be shown by constructing an isomorphism. For the construction, refer to the proof of (i) in Proposition 3. $\square$

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