Journal *of* Singularities
Volume 11 (2015), 67-84

received: 25 February 2014
in revised form: 20 April 2015
DOI: 10.5427/jsing.2015.11d

L^2-RIEMANN-ROCH FOR SINGULAR COMPLEX CURVES

JEAN RUPPENTHAL AND MARTIN SERA

ABSTRACT. We present a comprehensive L^2-theory for the $\overline{\partial}$-operator on singular complex curves, including L^2-versions of the Riemann-Roch theorem and some applications.

1. Introduction

The L^2-theory for the $\overline{\partial}$-operator is one of the central tools in complex analysis on complex manifolds, but still not very well developed on singular complex spaces. Just recently, considerable progress has been made in understanding the L^2-cohomology of singular complex spaces with isolated singularities. Let X be a Hermitian complex space of pure dimension n and with isolated singularities only. For simplicity, we assume that X is compact. Let $H^{p,q}_w(X^*)$ be the L^2-Dolbeault cohomology on the level of (p,q)-forms with respect to the $\overline{\partial}$-operator in the sense of distributions, denoted by $\overline{\partial}_w$ in the following, computed on $X^* = \operatorname{Reg} X$. Let $\pi : M \to X$ be a resolution of singularities with snc exceptional divisor, $Z := \pi^{-1}(\operatorname{Sing} X)$ the unreduced exceptional divisor. Then it has been shown by Øvrelid, Vassiliadou [ØV13] and the first author [Rup11, Rup14] by different approaches that there exists an effective divisor $D \geq Z - |Z|$ on M such that there are natural isomorphisms

$$\begin{aligned} H^{n,q}_w(X^*) &\cong H^{n,q}(M)\ , \\ H^{0,q}_w(X^*) &\cong \frac{H^q\big(M, \mathcal{O}(D)\big)}{H^q_{|Z|}\big(M, \mathcal{O}(D)\big)} \end{aligned} \tag{1.1}$$

for all $0 \leq q \leq n$. Here, $H^q_{|Z|}$ denotes the cohomology with support on $|Z|$. If $\dim X \leq 2$, then (1.1) holds with the divisor $D = Z - |Z|$ and $H^q_{|Z|}\big(M, \mathcal{O}(Z - |Z|)\big) = \{0\}$, so that (1.1) gives a nice smooth representation of the L^2-cohomology groups $H^{0,q}_w(X^*)$. In case $\dim X > 2$, it is conjectured that (1.1) holds with $D = Z - |Z|$ (see [Rup11]).

However, the L^2-theory for the $\overline{\partial}$-operator developed in [ØV13] and [Rup11, Rup14] applies only to $\dim X \geq 2$ (for $\dim X = 1$, (1.1) has been known before, see [Par89, PS91]). The purpose of the present paper is to give a complete L^2-theory for the $\overline{\partial}$-operator on a singular complex curve, including L^2-versions of the Riemann-Roch theorem, and to understand the appearance of the divisor $Z - |Z|$ in the case $\dim X = 1$.

Let us explain some of our results in detail. Let X be a Hermitian singular complex space[1] of dimension 1, i.e., a Hermitian complex curve, and $L \to X$ a Hermitian holomorphic line bundle. Let $\overline{\partial}_w : L^{p,q}(X^*, L) \to L^{p,q+1}(X^*, L)$ denote the weak extension of the Cauchy-Riemann operator $\overline{\partial} : \mathscr{D}^{p,q}(X^*, L) \to \mathscr{D}^{p,q+1}(X^*, L)$, i.e., the $\overline{\partial}$-operator in the sense of distributions. Here, $\mathscr{D}^{p,q}(X^*, L) := \mathscr{C}^\infty_{\text{cpt}}(X^*, \Lambda^{p,q}T^*X^* \otimes L)$ denotes the set of smooth differential forms with compact support in X^* and values in L, and $L^{p,q}(X^*, L)$ is the set of square-integrable forms with values in L and respect to the Hermitian metrics on X^* and L.

2010 *Mathematics Subject Classification.* 32W05, 32C36, 14C40.
Key words and phrases. Dolbeault operator, L^2-theory, singular complex curves, Riemann-Roch theorem.

[1] A Hermitian complex space (X, g) is a reduced complex space X with a metric g on the regular part such that the following holds: If $x \in X$ is an arbitrary point there exist a neighborhood $U = U(x)$, a holomorphic embedding of U into a domain G in $\mathbb{C}^N$ and an ordinary smooth Hermitian metric in G whose restriction to U is $g|_U$.

Let $H_w^{p,q}(X^*,L)$ denote the L^2-Dolbeault cohomology on X^* with respect to $\overline{\partial}_w$ and

$$h_w^{p,q}(X^*,L) := \dim H_w^{p,q}(X^*,L).$$

Note that the genus $g = g(X)$ of X and the degree $\deg(L)$ of L are well-defined, even in the presence of singularities (see Section 2.2). For a singular point $x \in \operatorname{Sing} X$, we define its modified multiplicity $\operatorname{mult}'_x X$ as follows: Let X_j, $j = 1,...,m$, be the irreducible components of X in the singular point x. Then

$$\operatorname{mult}'_x X := \sum_{j=1}^{m} (\operatorname{mult}_x X_j - 1)\,.$$

Note that regular irreducible components do not contribute to $\operatorname{mult}'_x X$. In Section 2.2, we recall the definition of the multiplicity $\operatorname{mult}_x X_j$ and present different ways to compute it.

Theorem 1.2 ($\overline{\partial}_w$-Riemann-Roch)**.** *Let X be a compact Hermitian complex curve with m irreducible components and $L \to X$ a holomorphic line bundle. Then*

$$h_w^{0,0}(X^*,L) - h_w^{0,1}(X^*,L) = m - g + \deg(L) + \sum_{x\in\operatorname{Sing} X} \operatorname{mult}'_x X, \tag{1.3}$$

and

$$h_w^{1,1}(X^*,L) - h_w^{1,0}(X^*,L) = m - g - \deg(L).$$

Theorem 1.2 is a corollary of Theorem 4.4 which we prove in Section 4. We also consider an L^2-dual version there, i. e., an L^2-Riemann-Roch theorem for the minimal closed L^2-extension of the $\overline{\partial}$-operator which we denote by $\overline{\partial}_s$ (see Section 2.1).

On singular complex curves, the $\overline{\partial}_s$-operator is of particular importance because of its relation to weakly holomorphic functions. Namely, the weakly holomorphic functions are precisely the $\overline{\partial}_s$-holomorphic L^2_{loc}-functions (for a localized version of the $\overline{\partial}_s$-operator, see Section 5). Let $H^{p,q}_{s,\text{loc}}(X^*)$ denote the L^2_{loc}-Dolbeault cohomology on X^* with respect to $\overline{\partial}_s$, and $\widehat{\mathcal{O}}_X$ the sheaf of germs of weakly holomorphic functions on X. Then:

Theorem 1.4. *Let X be a Hermitian complex curve. Then*

$$\begin{aligned} H^0(X,\widehat{\mathcal{O}}_X) &= H^{0,0}_{s,\text{loc}}(X^*), \\ H^1(X,\widehat{\mathcal{O}}_X) &\cong H^{0,1}_{s,\text{loc}}(X^*). \end{aligned}$$

If X is irreducible and compact, then $\dim H^0(X,\widehat{\mathcal{O}}_X) = 1$, $\dim H^1(X,\widehat{\mathcal{O}}_X) = g(X)$. We prove Theorem 1.4 in Section 5.

To exemplify the use of L^2-theory for the $\overline{\partial}$-operator on a singular complex space, in particular the L^2-Riemann-Roch theorem, we give in Section 6 two applications. There, we use our L^2-theory to give alternative proofs of two well-known facts. First, we show that each compact complex curve can be realized as a ramified covering of $\mathbb{CP}^1$. Second, we show that a positive holomorphic line bundle over a compact complex curve is ample, yielding that any compact complex curve is projective.

Let us clarify the relation to previous work of others. In the case of complex curves, (1.1) was in essence discovered by Pardon [Par89], and one can deduce parts of Theorem 4.4 and the second statement of Corollary 4.8 from Pardon's work by some additional arguments on the regularity of the $\overline{\partial}$-operator. The first part of Corollary 4.8 was discovered by Haskell [Has89], and from that one can deduce the second statement of Theorem 1.2 by use of L^2-Serre duality. Moreover, Theorem 1.2 was proved in essence by Brüning, Peyerimhoff and Schröder in [BPS90] and [Sch89] by computing the indices of the $\overline{\partial}_w$- and the $\overline{\partial}_s$-operator.

The new point in the present work is that we can put all the partial results mentioned above in the general framework of a comprehensive L^2-theory. From that, we draw also a new understanding of weakly holomorphic functions (Theorem 1.4) and of the divisor $Z - |Z|$. Moreover, all the previous work has been done only for forms with values in the trivial bundle (except of [Sch89]), whereas we incorporate line bundles. This is essential for applications as we illustrate by the examples mentioned above.

Acknowledgment. This research was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation), grant RU 1474/2 within DFG's Emmy Noether Programme. The authors thank Eduardo S. Zeron for interesting and fruitful discussions and are grateful to the unknown referee for several suggestions which helped to improve the paper.

2. **Preliminaries**

2.1. **Closed extensions of the Cauchy-Riemann operator.** Let X be a complex curve and $X^* := \operatorname{Reg} X$ the set of regular points. We assume that X is a Hermitian complex space in the sense that X^* carries a Hermitian metric γ which is locally given as the restriction of the metric of the ambient space when X is embedded holomorphically into some complex number space.

We denote by $\mathscr{D}^{p,q}(X^*)$ the smooth differential forms of degree (p,q) with compact support in X^* (test forms) and by $L^{p,q}(X^*)$ the set of square-integrable forms with respect to the metric γ on X^*.

Let $\overline{\partial}_s : L^{p,q}(X^*) \to L^{p,q+1}(X^*)$ be the minimal (strong) closed L^2-extension of the Cauchy-Riemann operator $\overline{\partial} : \mathscr{D}^{p,q}(X^*) \to \mathscr{D}^{p,q+1}(X^*)$, i. e., $\overline{\partial}_s$ is defined by the closure of the graph of $\overline{\partial}$ in $L^{p,q}(X^*) \times L^{p,q+1}(X^*)$. $\overline{\partial}_w : L^{p,q}(X^*) \to L^{p,q+1}(X^*)$ is the maximal (weak) closed L^2-extension of $\overline{\partial}$, i. e., $\overline{\partial}_w$ is defined in sense of distributions. We denote by $H^{p,q}_{w/s}(X^*)$ the Dolbeault cohomology with respect to $\overline{\partial}_w$ or $\overline{\partial}_s$, respectively, and by $h^{p,q}_{w/s}(X^*)$ the dimension of $H^{p,q}_{w/s}(X^*)$.

Let $\vartheta : \mathscr{D}^{p,q+1}(X^*) \to \mathscr{D}^{p,q}(X^*)$ be the formal adjoint of $\overline{\partial}$ and $\vartheta_{s/w} := \overline{\partial}^*_{w/s}$ the Hilbert-space adjoint of $\overline{\partial}_{w/s}$. This notation makes sense as $\vartheta_{w/s}$ is in fact the maximal (weak) or minimal (strong), respectively, L^2-extension of ϑ. Let $\overline{*} : L^{p,q}(X^*) \to L^{1-p,1-q}(X^*)$ be the conjugated Hodge-$*$-operator with respect to the metric γ. Then we have $\vartheta_{w/s} = -\overline{*}\,\overline{\partial}_{w/s}\,\overline{*}$.

Let $L \to X$ be a Hermitian holomorphic line bundle on X with an (arbitrary) metric on L which is smooth on the whole of X. We define $\mathscr{D}^{p,q}(X^*, L) := \mathscr{C}^\infty_{\mathrm{cpt}}(X^*, \Lambda^{p,q}T^*X^* \otimes L)$ as the smooth (p,q)-forms with compact support and values in L, and $L^{p,q}(X^*, L)$ as the Hilbert space of square-integrable forms with values in L. We consider the Cauchy-Riemann operator $\overline{\partial} : \mathscr{D}^{p,q}(X^*, L) \to \mathscr{D}^{p,q+1}(X^*, L)$ locally given by $\overline{\partial} : \mathscr{D}^{p,q}(X^*) \to \mathscr{D}^{p,q+1}(X^*)$. Since $\overline{\partial}$ commutes with the trivializations of the holomorphic line bundle, $\overline{\partial}$ is well defined. We get the weak and strong extensions $\overline{\partial}_w$, $\overline{\partial}_s : L^{p,q}(X^*, L) \to L^{p,q+1}(X^*, L)$ and the cohomology $H^{p,q}_{w/s}(X^*, L)$ as above.

In Section 3, we will study also the following other closed extensions of $\overline{\partial}$ besides the minimal $\overline{\partial}_s$ and the maximal $\overline{\partial}_w$. Let $D \Subset \mathbb{C}^n$ be a domain, and $X \subset D$ an analytic set of dimension one with $\operatorname{Sing} X = \{0\}$. We can interpret $\overline{\partial}_s$ as $\overline{\partial}_w$ with certain boundary conditions. The boundary of X^* consists of two parts, the singular point $\{0\}$ and the boundary at ∂D: $\partial X = \partial X^* \setminus \{0\}$. Therefore, there are two boundary conditions. Let $\overline{\partial}_{s,w}$ denote the closed L^2-extension which satisfies the boundary condition at $\{0\}$, i. e., $f \in \operatorname{dom} \overline{\partial}_{s,w}$ iff $f \in \operatorname{dom} \overline{\partial}_w$ and there is a sequence $\{f_j\}$ in $\operatorname{dom} \overline{\partial}_w$ such that $\operatorname{supp} f_j \cap \{0\} = \varnothing$, $f_j \to f$, and $\overline{\partial}_w f_j \to \overline{\partial}_w f$ in L^2. $\overline{\partial}_{w,s}$ denotes the extension which satisfies the boundary condition at ∂X, i. e., $f \in \operatorname{dom} \overline{\partial}_{w,s}$ iff $f \in \operatorname{dom} \overline{\partial}_w$ and there is a sequence $\{f_j\}$ in $\operatorname{dom} \overline{\partial}_w$ such that $\operatorname{supp} f_j \cap \partial X = \varnothing$, $f_j \to f$, and $\overline{\partial}_w f_j \to \overline{\partial}_w f$ in L^2. We define the adjoint operators

$$\vartheta_{s,w} := -\overline{*}\,\overline{\partial}_{s,w}\,\overline{*} \quad \text{and} \quad \vartheta_{w,s} := -\overline{*}\,\overline{\partial}_{w,s}\,\overline{*},$$

which we can realize as Hilbert-space adjoint operators (see [Rup14, Lem. 5.1]):

Lemma 2.1. *The Hilbert-space adjoints* $\overline{\partial}^*_{s,w}$ *and* $\overline{\partial}^*_{w,s}$ *satisfy the representations*

$$\overline{\partial}^*_{s,w} = \vartheta_{w,s} = -\overline{*}\,\overline{\partial}_{w,s}\,\overline{*} \qquad \text{and} \qquad \overline{\partial}^*_{w,s} = \vartheta_{s,w} = -\overline{*}\,\overline{\partial}_{s,w}\,\overline{*},$$

respectively.

2.2. **Resolution of complex curves, divisors, line bundles.** Every (reduced) complex space X (which is countable at infinity) has a resolution of singularities $\pi : M \to X$, i. e., there are a complex manifold M, a proper complex subspace S of X which contains the singular locus of X and a proper holomorphic map $\pi : M \to X$ such that the restriction $M \setminus \pi^{-1}(S) \to X \setminus S$ of π is biholomorphic, and $\pi^{-1}(S)$ is the locally finite union of smooth hypersurfaces (see [Hir64] and [Hir77, Thm. 7.1]).

If X is a compact complex curve, then such a resolution is given just by the normalization of the curve, and it is unique up to biholomorphism: Let $\pi_1 : M_1 \to X$ and $\pi_2 : M_2 \to X$ be two resolutions of X. Then $\psi := \pi_2^{-1} \circ \pi_1 : M_1 \setminus \pi_1^{-1}(\operatorname{Sing} X) \to M_2 \setminus \pi_2^{-1}(\operatorname{Sing} X)$ is biholomorphic and bounded in the singular locus. Yet, $\pi_i^{-1}(\operatorname{Sing} X)$ consist of isolated points. Therefore, ψ has a (bi-) holomorphic extension.

Let $\pi : M \to X$ be a resolution of a compact complex curve X. We define the *genus* of X by the genus of the resolution

$$g(X) := h^1(M) = \dim H^1(M, \mathcal{O}).$$

If X has more than one irreducible component, then M is not connected and $h^1(M)$ is the sum of the genera of the connected components. Since the resolution is unique up to biholomorphism, this is well-defined.

Throughout the article (except of Section 6.1), we will work with divisors on compact Riemann surfaces only. Therefore, there is no difference between Cartier and Weil divisors, and we can associate to each line bundle a divisor.

Let $L \to X$ be a holomorphic line bundle. Then the pull-back $\pi^*L \to M$ is well-defined by the pull-back of the transition functions of the line bundle. There is a divisor D on M associated to π^*L such that $\mathcal{O}(\pi^*L) \cong \mathcal{O}(D)$[2] and $\deg \pi^*L = \deg D$. The uniqueness of the resolution (up to biholomorphism) implies the independence of $\deg \pi^*L$ from π, so that

$$\deg L := \deg \pi^*L$$

is also well-defined.

For any divisor D on M, there exists a holomorphic line bundle $L_D \to M$ associated to D such that $\mathcal{O}(L_D) \cong \mathcal{O}(D)$. The constant function $f = 1$ induces a meromorphic section s_D of L_D such that $\operatorname{div}(s_D) = D$. One can then identify sections in $\mathcal{O}(D)$ with sections in $\mathcal{O}(L_D)$ by $g \mapsto g \otimes s_D$, and we denote the inverse mapping by $s \mapsto s \cdot s_D^{-1}$. If Y is an effective divisor, then s_Y is a holomorphic section of L_Y and $\mathcal{O} \subset \mathcal{O}(Y)$. Hence, there is the natural inclusion $\mathcal{O}(D) \subset \mathcal{O}(D+Y)$ which induces the inclusion $\mathcal{O}(L_D) \subset \mathcal{O}(L_{D+Y})$ given by $s \mapsto (s \cdot s_D^{-1}) \otimes s_{D+Y}$. For $U \subset M$, we obtain the inclusion

$$L^{p,q}_{\text{loc}}(U, L_D) \hookrightarrow L^{p,q}_{\text{loc}}(U, L_{D+Y}), s \mapsto (s \cdot s_D^{-1}) \otimes s_{D+Y}. \tag{2.2}$$

Here, $L^{p,q}_{\text{loc}}(U, L_D)$ denotes the locally square-integrable forms with values in L_D. This definition is independent of the chosen Hermitian metric on L_D. If M is compact, all metrics are equivalent and we get the inclusion

$$L^{p,q}(M, L_D) \subset L^{p,q}(M, L_{D+Y}). \tag{2.3}$$

Let $Z := \pi^{-1}(\operatorname{Sing} X)$ be the unreduced exceptional divisor and $|Z|$ the underlying reduced divisor. Then $\deg(Z - |Z|)$ is independent of the resolution as well. We will discuss some alternative ways to compute $\deg Z$.

[2] We denote by $\mathcal{O}(D)$ the sheaf of germs of holomorphic functions f such that $\operatorname{div}(f) + D \geq 0$.

Locally, the resolution is given by the Puiseux parametrization: Let A be an analytic set of dimension 1 in $\Omega \Subset \mathbb{C}^n$ with $\operatorname{Sing} A = \{0\}$ which is irreducible at 0. Shrinking Ω, there are coordinates $z, w_1, ..., w_{n-1}$ around 0 such that A is contained in the cone $\|w\| \leq C|z|$, $w = (w_1, .., w_{n-1})$. The projection $\operatorname{pr}_z : A \to \mathbb{C}_z$ on the z-coordinate is a finite ramified covering. Let s be the number of the sheets of pr_z. Generic choice of the coordinates gives the same number of sheets s, called the multiplicity $\operatorname{mult}_0 A$ of A in $\{0\}$. There exists a parametrization $\pi : \Delta \to A, t \mapsto (t^s, w_1(t), ..., w_{n-1}(t))$, where $\Delta := \{t \in \mathbb{C} : |t| < 1\}$; cf. e. g. [Chi89, Sect. 6.1]. π is called the Puiseux parametrization. The unreduced exceptional divisor is just $Z = (\pi^{-1}(z)) = (t^s)$, and so $\deg Z = s$.

The number of sheets of the covering pr_z is also equal to the Lelong number $\nu([A], 0)$ of the positive closed current $[A]$ given by the integration over A (see [Chi89, Prop. 2 in § 3.15], [Dem12, Thm. 7.7] or [GH78, § 3.2]).

The tangent cone gives another way to compute $\operatorname{mult}_0 A$. For a holomorphic function f on Ω, let $f = \sum_{k=k_0}^{\infty} f_k$ be the decomposition in homogeneous polynomials f_k of degree k with $f_{k_0} \neq 0$ (choosing a smaller Ω) and $f^* := f_{k_0} \neq 0$ be the *initial homogeneous polynomial of* f. If A is given by the ideal sheaf $\mathscr{J}_A$, then

$$C_0(A) = \{\alpha \in \mathbb{C}^n : f^*(\alpha) = 0 \ \forall f \in \mathscr{J}_{A,0}\} \subset T_0\mathbb{C}^n$$

is called the *tangent cone* of A in 0 (cf. [Chi89, Sect. 8.4]). The natural projection $\mathbb{C}^n \setminus 0 \to \mathbb{CP}^{n-1}$ maps $C_0(A)$ on a projective variety $\widetilde{C_0}(A)$. The degree $\deg Y$ of a projective variety Y in $\mathbb{CP}^{n-1}$ of dimension p is defined as the class of Y in $H_{2p}(\mathbb{CP}^{n-1}, \mathbb{Z}) \cong \mathbb{Z}$, and $\operatorname{mult}_0 A = \deg \widetilde{C_0}(A)$ (see Sect. 2 of [GH78, § 1.3]). In the case of an irreducible complex curve A, note that $\widetilde{C_0}(A)$ is just a point of multiplicity $\operatorname{mult}_0 A$.

All in all, we have

$$\deg Z = \operatorname{mult}_0 A = \nu([A], 0) = \deg \widetilde{C_0}(A).$$

2.3. **Extension theorems.** We need the following extension theorem. Let Δ be the unit disc in $\mathbb{C}$ and $\Delta^* := \Delta \setminus \{0\}$.

Theorem 2.4 (L^2-extension)**.** *If* $u \in L^{p,0}_{\text{loc}}(\Delta)$ *and* $v \in L^{p,1}_{\text{loc}}(\Delta)$ *satisfy* $\overline{\partial} u = v$ *on* Δ^* *in the sense of distributions, then* $\overline{\partial} u = v$ *on* Δ.

A more general statement is true for domains in $\mathbb{C}^n$ and proper analytic subsets of arbitrary codimension, cf. e. g. [Rup09, Thm. 3.2].

If $A \subset \Omega \Subset \mathbb{C}^n$ is a pure dimensional analytic set, let $\widehat{\mathcal{O}} = \widehat{\mathcal{O}}_A$ be the normalization sheaf of $\mathcal{O}_A$ which is defined stalk-wise by the integral closure of $\mathcal{O}_{A,x}$ in the sheaf $\mathcal{M}_{A,x}$ of meromorphic functions for all $x \in A$. A function in $\widehat{\mathcal{O}}(U)$, $U \subset A$ open, is called *weakly holomorphic*. Weakly holomorphic functions are holomorphic in regular points of A and bounded in singular points. If A is locally irreducible, then weakly holomorphic functions are continuous in $\operatorname{Sing} A$ (cf. e. g. [GR84, § VI.4].)

The classical Riemann extension theorem generalizes to the following result (see e. g. [GR84, Sect. VII.4.1]).

Theorem 2.5 (Riemann extension)**.** *Let* $A \subset \Omega \Subset \mathbb{C}^n$ *be a pure dimensional analytic set. Every holomorphic function on* $A^* := \operatorname{Reg} A$ *which is bounded at* $\operatorname{Sing} A$ *is weakly holomorphic on* A.

3. Local L^2-theory of complex curves

In this section, we study the local L^2-theory of (locally) irreducible analytic curves in $\mathbb{C}^n$. By the remarks on the local structure of singular complex curves in Section 2.2 and Section 4, it follows that the studied situation is general enough. We will compute the L^2-Dolbeault cohomology by use of the Puiseux parametrization and will see why the term $\sum_{x\in \operatorname{Sing} X} \operatorname{mult}'_x X$ occurs in (1.3).

Let A be an irreducible analytic curve in $\Delta^n \subset \mathbb{C}^n_{zw_1\dots w_{n-1}}$ given by the Puiseux parametrization

$$\pi\colon \Delta \to \mathbb{C}^n\ ,\ \pi(t) := (t^s, w(t)),$$

where $w = (w_1, ..., w_{n-1})\colon \Delta \to \Delta^{n-1}$ is a holomorphic map such that each component w_i vanishes at least of the order $s+1$ in the origin. Here, Δ is the unit disk $\{t \in \mathbb{C} : |t| < 1\}$. We can assume that π is bijective, in particular, a resolution/normalization of A such that $\operatorname{mult}_0 A = s$. Further, we can assume that 0 is the only singular point of A.

For a regular point $(z_0, w_0) \in A^* := \operatorname{Reg} A$, let $t_0 \in \Delta^*$ be the preimage under π. Since π is biholomorphic on $\Delta^* := \Delta \setminus \{0\}$, $d\pi_{t_0}(\frac{\partial}{\partial t}) = st_0^{s-1}\frac{\partial}{\partial z} + \sum_{k=1}^{n-1} w'_k(t_0)\frac{\partial}{\partial w_k}$ is a non-vanishing tangent vector of A^* in (z_0, w_0), i. e.,

$$(1 + \|\tfrac{1}{s}t_0^{1-s}w'(t_0)\|^2)^{-1/2}\left(\frac{\partial}{\partial z} + \sum_{k=1}^{n-1}\frac{1}{s}t_0^{1-s}w'_k(t_0)\frac{\partial}{\partial w_k}\right)$$

is a normalized generator of $T_{(z_0,w_0)}A^*$ and $(1 + \|\frac{1}{s}t_0^{1-s}w'(t_0)\|^2)^{1/2}dz$ is a normalized generator of $T^*_{(z_0,w_0)}A^*$. Since w'_k vanishes at least of order s in the origin, we obtain $1 + \|\frac{1}{s}t^{1-s}w'(t)\|^2 \sim 1$ on Δ and $dV_{A^*} \sim i dz \wedge d\bar{z}$, where dV_{A^*} denotes the volume form on A^* induced by the standard Euclidean metric of $\mathbb{C}^n$. Using $\pi^*dz = d(\pi^*z) = dt^s = st^{s-1}dt$ and $\pi^*(dz\wedge d\bar{z}) = s^2|t|^{2(s-1)}dt\wedge d\bar{t}$, we get

$$\pi^*dV_{A^*} \sim |t|^{2(s-1)}dV_\Delta.$$

Let $\iota : A^* \to \Delta^*$ be the inverse of π. Then, $\iota(z, w)$ is the root $t = \sqrt[s]{z}$ with $w = w(t)$. We get $\iota^*(dt) = \frac{1}{s}z^{1/s-1}dz$ and $\iota^*(dt \wedge d\bar{t}) = \frac{1}{s^2}|z|^{2(1/s-1)}dz \wedge d\bar{z}$, i. e.,

$$\iota^*dV_{\Delta^*} \sim |z|^{2(1/s-1)}dV_{A^*}.$$

If g is a measurable function on A^*, we obtain

$$\int_{A^*}|g|^2dV_{A^*} = \int_\Delta |\pi^*g|^2\pi^*dV_{A^*} \sim \int_\Delta |\pi^*g|^2 \cdot |t|^{2(s-1)}dV_\Delta.$$

Hence,

$$g \in L^{0,0}(A^*) \Leftrightarrow t^{s-1}\pi^*g \in L^{0,0}(\Delta).$$

For $(0, 1)$-forms and $(1, 1)$-forms, we have

$$\pi^*(gd\bar{z}) = \pi^*g \cdot \pi^*(d\bar{z}) = \bar{t}^{s-1}\pi^*(g)d\bar{t},$$
$$\pi^*(gdz \wedge d\bar{z}) = |t|^{2(s-1)}\pi^*(g)dt \wedge d\bar{t}.$$

Thus

$$\begin{aligned} &f \in L^{0,0}(A^*) \Leftrightarrow t^{s-1}\cdot \pi^*f \in L^{0,0}(\Delta),\\ &f \in L^{1,0}(A^*) \Leftrightarrow \pi^*f \in L^{1,0}(\Delta),\\ &f \in L^{0,1}(A^*) \Leftrightarrow \pi^*f \in L^{0,1}(\Delta),\ \text{and}\\ &f \in L^{1,1}(A^*) \Leftrightarrow t^{1-s}\cdot \pi^*f \in L^{1,1}(\Delta). \end{aligned} \tag{3.1}$$

On the other hand, if $v \in L^{0,0}(\Delta)$, we get

$$\infty > \int_\Delta |v|^2dV_\Delta = \int_{A^*}|\iota^*v|^2\iota^*dV_\Delta \sim \int_{A^*}|\iota^*v|^2 \cdot |z|^{2(1/s-1)}dV_{A^*}.$$

Thus, $|z|^{1/s-1}\iota^* v$ is square-integrable on A^*. For each $(0,1)$-form $vd\bar{t} \in L^{0,1}(\Delta)$, we get

$$s\iota^*(vd\bar{t}) = \bar{z}^{1/s-1}\iota^*(v)d\bar{z} \in L^{0,1}(A^*),$$

and for each $(1,1)$-form $vdt \wedge d\bar{t} \in L^{1,1}(\Delta)$, we get $|z|^{1-\frac{1}{s}}\iota^*(vdt \wedge d\bar{t}) \in L^{1,1}(A^*)$.

So, if $f \in L^{0,1}(A^*)$, then $u := \pi^* f$ is in L^2, too. Since $\dim \Delta = 1$, there exists $v \in L^{0,0}(\Delta)$ with $\overline{\partial}_w v = u$. We set $g := \iota^* v$. Since $|z|^{1/s-1}g$ is in L^2 and $|z|^{2(1-1/s)}$ is bounded,

$$\|g\|_{L^2}^2 = \int_{A^*} |z^{1/s-1}g|^2 \cdot |z|^{2(1-1/s)} dV_{A^*} \le \|z^{1/s-1}g\|_{L^2} \cdot \|z^{2(1-1/s)}\|_{L^\infty} < \infty.$$

Hence, we get an L^2-solution for $\overline{\partial}_w g = f$ and

$$H_w^{0,1}(A^*) = L^{0,1}(A^*)/\mathcal{R}(\overline{\partial}_w) = 0.$$

In the same way, it is easy to compute

$$H_w^{1,1}(A^*) = 0.$$

We will now determine $H_w^{p,0}(A^*) = \ker(\overline{\partial}_w : L^{p,0} \to L^{p,1})$ by use of the L^2-extension theorem (Theorem 2.4). For this, let $\mathcal{O}_{L^2}(\Delta)$ be the square-integrable holomorphic functions on Δ, and let $\Omega^1_{L^2}(\Delta)$ be the holomorphic 1-forms with square-integrable coefficient. If $g \in L^{0,0}(A^*)$ and $\overline{\partial}_w g = 0$, then $v := \pi^* g \in |t|^{1-s}L^{0,0}(\Delta)$ and $\overline{\partial}_w v = 0$ on Δ^*. Therefore, $\overline{\partial}(t^{s-1}v) = 0$ on Δ^* and $t^{s-1}v \in L^{0,0}(\Delta)$. The extension theorem implies $\overline{\partial}(t^{s-1}v) = 0$ on Δ, i. e., v is a meromorphic function with a pole of order $s-1$ or less at the origin. We say $v \in t^{1-s}\mathcal{O}_{L^2}(\Delta)$. Since, on the other hand, $\iota^*(t^{1-s}\mathcal{O}_{L^2}(\Delta)) \subset \ker \overline{\partial}_w$, we conclude

$$H_w^{0,0}(A^*) \cong t^{1-s}\mathcal{O}_{L^2}(\Delta). \tag{3.2}$$

If $f \in L^{1,0}(A^*)$ and $\overline{\partial}_w f = 0$, then $u := \pi^* f \in L^{1,0}(\Delta)$ and $\overline{\partial}_w u = 0$ on Δ (using the extension theorem again). Hence, u is holomorphic on Δ and

$$H_w^{1,0}(A^*) \cong \Omega^1_{L^2}(\Delta).$$

To compute the cohomology groups $H_s^{*,*}(A^*)$, we use L^2-duality:

Lemma 3.3. *Let $\overline{\partial}_e$ denote either the weak or the strong closed extension of $\overline{\partial}$, and $\overline{\partial}_{e^c}$ the other one. For $p \in \{0,1\}$, let the range $\mathcal{R}(\overline{\partial}_e)$ of $\overline{\partial}_e : L^{p,0} \to L^{p,1}$ be closed. Then*

$$H_e^{p,1}(A^*) \cong H_{e^c}^{1-p,0}(A^*).$$

For the proof see e. g. [Rup14, Thm. 2.3].

Lemma 3.4. *For $p \in \{0,1\}$,*

$$H_s^{p,0}(A^*) \cong H_w^{1-p,1}(A^*) = 0 \text{ and}$$
$$H_s^{p,1}(A^*) \cong H_w^{1-p,0}(A^*).$$

Proof. Recall that $H_w^{1-p,1}(A^*) = 0$. This implies $L^{1-p,1}(A^*) = \mathcal{R}(\overline{\partial}_w)$ and, particularly, that the range of $\overline{\partial}_w : L^{1-p,0} \to L^{1-p,1}$ is closed. As $\vartheta_w = -\bar{*}\,\overline{\partial}_w\,\bar{*}$ and $\bar{*}$ is an isometric isomorphism, we conclude that the range of $\vartheta_w : L^{p,1} \to L^{p,0}$ is closed as well. This is equivalent to the range of $\overline{\partial}_s = \vartheta_w^* : L^{p,0} \to L^{p,1}$ being closed (standard functional analysis). Lemma 3.3 implies both isomorphisms. ☐

To get the complete picture, we also need to understand the Dolbeault cohomology groups of the closed extensions $\overline{\partial}_{s,w}$ and $\overline{\partial}_{w,s}$, respectively.

Lemma 3.5. *For $p \in \{0,1\}$,*

$$H_{w,s}^{p,0}(A^*) = 0.$$

Proof. Let $f \in \ker \overline{\partial}_{w,s} = H^{p,0}_{w,s}(A^*)$. We have showed $\omega \cdot u := \omega \cdot \pi^* f \in L^{p,0}(\Delta)$ with $\omega(t) = t^{s-1}$ if $p = 0$ and $\omega(t) \equiv 1$ if $p = 1$. By the extension theorem, we conclude $\overline{\partial}_s(\omega \cdot u) = 0$ on Δ, where $\overline{\partial}_s$ denotes the (strong) closure of $\overline{\partial}_{\mathrm{cpt}} : \mathscr{C}^\infty_{\mathrm{cpt};p,0}(\Delta) \to \mathscr{C}^\infty_{\mathrm{cpt};p,1}(\Delta)$. The generalized Cauchy condition implies that the trivial extension of ωu to the complex plane is a holomorphic p-form with compact support (cf. [LM02, § V.3]). We deduce that $\omega u = 0$ and, hence, $f = 0$. □

Lemma 3.6.

$$H^{0,0}_{s,w}(A^*) \cong \mathcal{O}_{L^2}(\Delta) \text{ and}$$
$$H^{1,0}_{s,w}(A^*) \cong t^{s-1}\Omega^1_{L^2}(\Delta).$$

As $\mathcal{O}_{L^2}(\Delta) \cong \widehat{\mathcal{O}}_{L^2}(A)$, the first isomorphism implies that the $\overline{\partial}_{s,w}$-holomorphic functions on a singular complex curve are precisely the square-integrable weakly holomorphic functions.

Proof. First, we prove that $\mathcal{O}_{L^2}(\Delta) = \pi^*(\ker \overline{\partial}_{s,w} : L^{0,0}(A^*) \to L^{0,1}(A^*))$.

i) For $v \in \mathcal{O}_{L^2}(\Delta)$, we claim that $g := \iota^* v \in \ker \overline{\partial}_{s,w}$. To see that, choose smooth functions $\tilde{\chi}_k : \mathbb{R} \to [0,1]$ with $\tilde{\chi}_k|_{(-\infty,k]} = 0$, $\tilde{\chi}_k|_{[k+1,\infty)} = 1$ and $|\tilde{\chi}_k'| \le 2$. We get

$$(\tilde{\chi}_k \circ \log \circ |\log|)'(\rho) = \frac{\tilde{\chi}_k'(\log|\log \rho|)}{\rho \log \rho}.$$

We define $\chi_k : A^* \to [0,1], (z,w) \mapsto \tilde{\chi}_k(\log|\log|z||)$ (which is inspired by [PS91, p. 617]) and get $\operatorname{supp} \overline{\partial}\chi_k \subset A^* \cap \Delta^n_{\varepsilon_k}$, where $\varepsilon_k := \exp(-\exp(k)) \to 0$ if $k \to \infty$. As $v \in L^{0,0}(\Delta)$, we have $g \in z^{1-\frac{1}{s}} L^{0,0}(A^*) \subset L^{0,0}(A^*)$. Then $g \cdot \chi_k \to g$ in L^2. As a holomorphic function, v is bounded in a neighborhood of 0. Therefore,

$$\begin{aligned}
\|g\overline{\partial}\chi_k\|^2_{A^*} &= \left\| g \cdot \frac{\tilde{\chi}_k'(\log|\log|z||)}{|z|\log|z|} \overline{\partial}|z| \right\|^2_{A^* \cap \Delta^n_{\varepsilon_k}} \lesssim \left\| g \cdot \frac{1}{|z|\log|z|} \right\|^2_{A^* \cap \Delta^n_{\varepsilon_k}} \\
&\sim \left\| v \cdot \frac{|t|^{s-1}}{|t|^s \log|t|^s} \right\|^2_{\Delta_{\varepsilon_k}} \lesssim \left\| \frac{1}{|t|\log|t|} \right\|^2_{\Delta_{\varepsilon_k}} = \int_{\Delta_{\varepsilon_k}} \frac{1}{|t|^2 \log^2|t|} dV \\
&= 2\pi \int_0^{\varepsilon_k} \frac{\rho}{\rho^2 \log^2 \rho} d\rho \sim \left[-\frac{1}{\log \rho} \right]_0^{\varepsilon_k} \to 0, \text{ if } k \to \infty.
\end{aligned}$$

Hence, $\overline{\partial}(g\chi_k) = g\overline{\partial}\chi_k \to 0 = \overline{\partial}_w g$ in L^2. So, $g \in \operatorname{dom} \overline{\partial}_{s,w}$.

ii) $\pi^*(\ker \overline{\partial}_{s,w}) \subset \mathcal{O}_{L^2}(\Delta)$ (cf. the proof of Lem. 6.2 in [Rup14]): Let g be in $\ker \overline{\partial}_{s,w}$, i. e., there are g_j in $L^2(A^*)$ with $g_j \to g$, $\overline{\partial} g_j \to 0$ in $L^2(A^*)$ and $0 \notin \operatorname{supp} g_j$. Let $\chi \in \mathscr{C}^\infty_{\mathrm{cpt}}(\Delta, [0,1])$ be identically 1 on $\Delta_{1/2}$. We define $u := \chi\pi^* g$ and $u_j := \chi\pi^* g_j$. It follows that $t^{s-1}u_j \to t^{s-1}u$ and $\overline{\partial} u_j \to \overline{\partial} u$ in $L^2(\Delta)$. Let $P : L^2(\Delta) \to L^2(\Delta)$ be the Cauchy-operator on the punctured disc, i. e.,

$$[P(h)](t) := \frac{1}{2\pi \mathrm{i}} \int_{\Delta^*} \frac{h(\zeta)}{\zeta - t} d\zeta \wedge d\overline{\zeta}.$$

Since the support of u_j is away from 0 and $\partial\Delta$, we get $u_j = P\left(\frac{\partial u_j}{\partial \overline{\zeta}}\right)$. The L^2-continuity of P and $\overline{\partial} u_j \to \overline{\partial} u$ in L^2 imply that

$$u_j = P\left(\frac{\partial u_j}{\partial \overline{\zeta}}\right) \to P\left(\frac{\partial u}{\partial \overline{\zeta}}\right)$$

in L^2. Since t^{s-1} is bounded, we obtain $t^{s-1}u_j \to t^{s-1}P\left(\frac{\partial u}{\partial \overline{\zeta}}\right)$ and, hence, $u = P\left(\frac{\partial u}{\partial \overline{\zeta}}\right)$ in L^2. That yields $\pi^* g \in L^2(\Delta)$. With $\pi^* g \in t^{1-s}\mathcal{O}_{L^2}(\Delta)$ and the extension theorem, we conclude $\pi^* g \in \mathcal{O}_{L^2}(\Delta)$.

Second, we claim that $\ker(\overline{\partial}_{s,w} : L^{1,0}(A^*) \to L^{1,1}(A^*)) \cong t^{s-1}\Omega^1_{L^2}(\Delta)$. $f = gdz$ is in $\ker \overline{\partial}_{s,w}$ iff $g \in \ker \overline{\partial}_{s,w}$. This is equivalent to $\pi^* g \in \mathcal{O}_{L^2}(\Delta)$. Since $\pi^*(dz) = t^{s-1}dt$, we infer that $\pi^* : H^{1,0}_{s,w}(A^*) \to t^{s-1}\Omega^1_{L^2}(\Delta), \pi^* f = t^{s-1}\pi^* g dt$ is an isomorphism.

The Riemann extension theorem (Theorem 2.5) implies $\mathcal{O}_{L^2}(\Delta) \cong \widehat{\mathcal{O}}_{L^2}(A)$ and the last statement. □

Lemma 3.7. *For $p \in \{0,1\}$, $\mathcal{R}(\overline{\partial}_{w,s} : L^{p,0}(A^*) \to L^{p,1}(A^*))$ and $\mathcal{R}(\overline{\partial}_{s,w} : L^{p,0}(A^*) \to L^{p,1}(A^*))$ both are closed, and*

$$H^{p,1-q}_{w,s}(A^*) \cong H^{1-p,q}_{s,w}(A^*) \quad \textit{for } q \in \{0,1\}.$$

Proof. Since $\overline{\partial}^*_{w,s} = \vartheta_{s,w}$, $\overline{\partial}^*_{s,w} = \vartheta_{w,s}$, $\vartheta_{s,w} = -\overline{*}\,\overline{\partial}_{s,w}\,\overline{*}$ and $\vartheta_{w,s} = -\overline{*}\,\overline{\partial}_{w,s}\,\overline{*}$ (see Lemma 2.1), it is easy to see that Lemma 3.3 holds for $\overline{\partial}_{w,s}$ and $\overline{\partial}_{s,w}$. We remark (cf. the proof of Lemma 3.4) that

$$\begin{aligned} \mathcal{R}(\overline{\partial}_{s,w}) \text{ is closed } &\Leftrightarrow \mathcal{R}(\vartheta_{w,s}) \text{ is closed} \\ &\Leftrightarrow \mathcal{R}(\overline{\partial}_{w,s}) \text{ is closed.} \end{aligned}$$

Therefore, it is enough to show that $\mathcal{R}(\overline{\partial}_{w,s})$ is closed.

Let $\varphi : \Delta^* \to \mathbb{R}$ be the smooth function defined by $\varphi(t) := (1-s)\log|t|^2$. Then

$$L^{p,q}(\Delta, \varphi) = t^{1-s}L^{p,q}(\Delta)$$

for the L^2-space $L^{p,q}(\Delta,\varphi)$ with weight $e^{-\varphi}$.

We set $T_1 := \pi^*\overline{\partial}_{w,s}\iota^* : L^{0,0}(\Delta,\varphi) \to L^{0,1}(\Delta)$. The extension theorem implies that T_1 is the (strong) closure of $\overline{\partial}_{\text{cpt}} : L^{0,0}(\Delta,\varphi) \to L^{0,1}(\Delta)$. Therefore, T_1^* is the weak closed extension of $\vartheta^\varphi_{\text{cpt}} : L^{0,1}(\Delta) \to L^{0,0}(\Delta,\varphi)$ which is defined by

$$(\overline{\partial}_{\text{cpt}}\alpha, \beta) = (\alpha, \vartheta^\varphi_{\text{cpt}}\beta)_\varphi = \int \langle \alpha, \vartheta^\varphi_{\text{cpt}}\beta\rangle e^{-\varphi} dV.$$

We set $\overline{*}_\varphi := e^{-\varphi}\,\overline{*}$. Then $T_2 := -\overline{*}_\varphi T_1^*\,\overline{*}$ is the weak closed extension of

$$\overline{\partial}_{\text{cpt}} : L^{1,0}(\Delta) \to L^{1,1}(\Delta, -\varphi)$$

because integration by parts implies $\vartheta^\varphi_{\text{cpt}} = -\overline{*}_\varphi\,\overline{\partial}_{\text{cpt}}\,\overline{*}$:

$$\begin{aligned} (\alpha, \overline{*}_{-\varphi}\,\overline{\partial}_{\text{cpt}}\,\overline{*}\,\beta)_\varphi &= \int \alpha \wedge \overline{*}_\varphi\,\overline{*}_{-\varphi}\,\overline{\partial}_{\text{cpt}}\,\overline{*}\,\beta = (-1)^{1-p}\int \alpha \wedge \overline{\partial}_{\text{cpt}}\,\overline{*}\,\beta \\ &= -\int \overline{\partial}_{\text{cpt}}\alpha \wedge \overline{*}\,\beta = -(\overline{\partial}_{\text{cpt}}\alpha, \beta). \end{aligned}$$

Hence, T_2 is $\overline{\partial}_w : L^{1,0}(\Delta) \to L^{1,1}(\Delta,-\varphi)$ in sense of distributions. Since for all

$$u \in L^{1,1}(\Delta, -\varphi) = t^{s-1}L^{1,1}(\Delta) \subset L^{1,1}(\Delta)$$

there is a $v \in L^{1,0}(\Delta)$ with $T_2 v = \overline{\partial}_w v = u$, the range of T_2 is closed. Thus, the range of T_1^* and the range of $\overline{\partial}_{w,s} = \iota^* T_1 \pi^* : L^{0,0}(A^*) \to L^{0,1}(A^*)$ are closed as well.

We set $S_1 := \pi^*\overline{\partial}_{w,s}\iota^* : L^{1,0}(\Delta) \to L^{1,1}(\Delta,-\varphi)$ and $S_2 := -\overline{*}\,S_1^*\,\overline{*}_\varphi$. Then S_2 is the weak closure of $\overline{\partial}_{\text{cpt}} : L^{0,0}(\Delta,\varphi) \to L^{0,1}(\Delta)$.

$$\begin{aligned} \mathcal{R}(S_2) &= \{u \in L^{0,1}(\Delta) : \exists v \in L^{0,0}(\Delta,\varphi) = t^{1-s}L^{0,0}(\Delta) \text{ with } S_2 v = u\} \\ &\supset \{u \in L^{0,1}(\Delta) : \exists v \in L^{0,0}(\Delta) \text{ with } \overline{\partial}_w v = u\} = L^{0,1}(\Delta). \end{aligned}$$

Therefore, $\mathcal{R}(S_2) = L^{0,1}(\Delta)$ is closed. This implies the claim. □

Summarizing, we computed (with $s = \mathrm{mult}_0 A$):

$$\begin{aligned}
H^{0,0}_w(A^*) &\cong H^{1,1}_s(A^*) &&\cong t^{1-s}\mathcal{O}_{L^2}(\Delta),\\
H^{1,0}_w(A^*) &\cong H^{0,1}_s(A^*) &&\cong \Omega^1_{L^2}(\Delta),\\
H^{p,1}_w(A^*) &= H^{1-p,0}_s(A^*) &&= 0,\\
H^{0,0}_{s,w}(A^*) &\cong H^{1,1}_{w,s}(A^*) &&\cong \mathcal{O}_{L^2}(\Delta),\\
H^{1,0}_{s,w}(A^*) &\cong H^{0,1}_{w,s}(A^*) &&\cong t^{s-1}\Omega^1_{L^2}(\Delta), \text{ and}\\
H^{p,1}_{s,w}(A^*) &= H^{1-p,0}_{w,s}(A^*) &&= 0.
\end{aligned} \tag{3.8}$$

4. L^2-cohomology of complex curves

We will prove Theorem 1.2 in this section. As a preparation, we consider the following local situation: Let A be a locally irreducible analytic set of dimension one in a domain $\Omega \Subset \mathbb{C}^n_{zw_1\dots w_{n-1}}$ with $\mathrm{Sing}\, A = \{0\}$, let dV denote the volume form on $A^* := \mathrm{Reg}\, A$ which is induced by the Euclidean metric and let $z : A \to \mathbb{C}_z$ be the projection on the first coordinate. Let us mention (cf. e. g. Prop. in [Chi89, Sect. 8.1]):

Theorem 4.1. *The set of all tangent vectors at a point of a one-dimensional irreducible analytic set in $\mathbb{C}^n$ is a complex line.*

Thus, we can assume that $C_0(A) = \mathbb{C}_z \times \{0\} \subset \mathbb{C}_z \times \mathbb{C}^{n-1}_{w_1\dots w_{n-1}}$, and, therefore, $dV \sim dz \wedge d\overline{z}$ (by shrinking Ω if necessary).

Let $\pi : M \to A$ be a resolution of A, $x_0 := \pi^{-1}(0)$. Then $Z = (\pi^*(z))$ is the unreduced exceptional divisor of the resolution. After shrinking A and M again, we can assume that M is covered by a single chart $\psi : M \to \mathbb{C}$ with $x_0 \in M$ and $\psi(x_0) = 0$. We set $\zeta := \pi^*(z)$ and get $Z = (\zeta)$. $|Z| = (\psi)$ implies $Z - |Z| = (\frac{\zeta}{\psi})$. We obtain

$$\pi^*(dz) = d(\pi^* z) = \frac{\partial \zeta}{\partial \psi} d\psi \sim \frac{\zeta}{\psi} d\psi.$$

Therefore, $\pi^*(dV) \sim \left|\frac{\zeta}{\psi}\right|^2 d\psi \wedge d\overline{\psi}$, and we conclude (recall the definition of line bundles L_D from Section 2.2):

$$\begin{aligned}
f \in L^{p,q}(A^*) \Leftrightarrow \left|\frac{\zeta}{\psi}\right|^{1-p-q} \cdot \pi^* f \in L^{p,q}(M)&\\
\Leftrightarrow \pi^* f \in L^{p,q}(M, L_{(1-p-q)(Z-|Z|)}),&
\end{aligned} \tag{4.2}$$

Nagase stated this equivalence already in Lem. 5.1 of [Nag90]. By use of the extension Theorem 2.4, we get:

$$\begin{aligned}
f \in \mathrm{dom}\,\big(\overline{\partial}_w : L^{p,0}(A^*) \to L^{p,1}(A^*)\big)&\\
\Leftrightarrow \pi^* f \in \mathrm{dom}\,\big(\overline{\partial}_w : L^{p,0}(M, L_{(1-p)(Z-|Z|)}) \to L^{p,1}(M, L_{p(|Z|-Z)})\big).&
\end{aligned} \tag{4.3}$$

The essential observation for the proof of Theorem 1.2 is the following:

Theorem 4.4. *Let X be a compact complex curve and $L \to X$ a holomorphic line bundle. Let $\pi : M \to X$ be a resolution of X with exceptional divisor Z, and D a divisor on M such that $\pi^* L \cong L_D$, i. e., $\mathcal{O}(\pi^* L) \cong \mathcal{O}(D)$. Then*

$$\begin{aligned}
H^{0,0}_w(X^*, L) &\cong H^0(M, \mathcal{O}(Z - |Z| + D)),\\
H^{0,1}_w(X^*, L) &\cong H^1(M, \mathcal{O}(Z - |Z| + D)),\\
H^{1,0}_w(X^*, L) &\cong H^0(M, \Omega^1(D)) \cong H^1(M, \mathcal{O}(-D)), \text{ and}\\
H^{1,1}_w(X^*, L) &\cong H^1(M, \Omega^1(D)) \cong H^0(M, \mathcal{O}(-D)).
\end{aligned}$$

In [Par89, §5], Pardon proved that $H^{0,q}_{(2),\mathrm{sm}}(X^*) \cong H^q(M, \mathcal{O}(Z-|Z|)$, where $H^{p,q}_{(2),\mathrm{sm}}(X^*)$ denotes the $\overline{\partial}$-cohomology with respect to smooth L^2-forms. We will use similar arguments here.

Proof. Let x_0 be in $\operatorname{Sing} X$, and let A be an open neighborhood of $x_0 = 0$ in X embedded locally in $\mathbb{C}^n$. We assume that $A = A_1 \cup ... \cup A_m$ with at x_0 irreducible analytic sets A_i. We obtain resolutions $\pi_i := \pi|_{\pi^{-1}(A_i)} : M_i \to A_i$ of A_i. The sets M_i are pairwise disjoint in M and, also, the support of the exceptional divisors Z_i of the resolution π_i. We get $Z|_{\pi^{-1}(A)} = \sum_{i=1}^m Z_i$ and $|Z||_{\pi^{-1}(A)} = \sum_{i=1}^m |Z_i|$. Therefore, the consideration in the local case (see (4.3)) implies that $\overline{\partial}_w : L^{p,0}(X^*, L) \to L^{p,1}(X^*, L)$ can be identified with

$$\overline{\partial}_w : L^{p,0}(M, L_{(1-p)(Z-|Z|)+D}) \to L^{p,1}(M, L_{p(|Z|-Z)+D}).$$

Hence,

$$H^{0,0}_w(X^*, L) \cong \ker(\overline{\partial}_w : L^{0,0}(M, L_{Z-|Z|+D}) \to L^{0,1}(M, L_D)) \cong H^0(M, \mathcal{O}(Z-|Z|+D))$$

and

$$H^{1,0}_w(X^*, L) \cong \ker(\overline{\partial}_w : L^{1,0}(M, L_D) \to L^{1,1}(M, L_{|Z|-Z+D})) \cong H^0(M, \Omega^1(D)).$$

Serre duality (see Theorem 2 in [Ser55, § 3.10]) implies

$$H^{1,0}_w(X^*, L) \cong H^0(M, \Omega^1(D)) \cong H^1(M, \mathcal{O}(-D)).$$

To prove the other two isomorphisms, consider the following general situation: Let E be a divisor on M, L_E the associated bundle, and let $\mathcal{L}^{p,q}_E$ denote the sheaf on M which is defined by $\mathcal{L}^{p,q}_E(U) := L^{p,q}_{\mathrm{loc}}(U, L_E)$ for each open set $U \subset M$. Let $E' \leq E$ be another divisor. Consider the $\overline{\partial}$-operator in the sense of distributions $\overline{\partial}_w : \mathcal{L}^{p,0}_E \to \mathcal{L}^{p,1}_{E'}$. Let $\mathcal{C}^{p,0}_{E,E'}$ denote the sheaf defined by

$$\mathcal{C}^{p,0}_{E,E'}(U) := \operatorname{dom}\left(\overline{\partial}_w : L^{p,0}_{\mathrm{loc}}(U, L_E) \to L^{p,1}_{\mathrm{loc}}(U, L_{E'})\right).$$

Then $\mathcal{C}^{p,0}_{E,E'}$ is fine and, in particular, $H^1(M, \mathcal{C}^{p,0}_{E,E'}) = 0$. We get the sequence

$$0 \to \Omega^p(E) \to \mathcal{C}^{p,0}_{E,E'} \xrightarrow{\overline{\partial}_w} \mathcal{L}^{p,1}_{E'} \to 0 \tag{4.5}$$

which is exact by the usual Grothendieck-Dolbeault lemma because there is an embedding $\mathcal{L}^{p,q}_{E'} \subset \mathcal{L}^{p,q}_E$ (induced by the natural inclusion $\mathcal{O}(E') \subset \mathcal{O}(E)$, see (2.2)).

This induces the long exact sequence of cohomology groups

$$0 \to \Gamma(M, \Omega^p(E)) \to \mathcal{C}^{p,0}_{E,E'}(M) \xrightarrow{\overline{\partial}_w} \mathcal{L}^{p,1}_{E'}(M) \to H^1(M, \Omega^p(E)) \to H^1(M, \mathcal{C}^{p,0}_{E,E'}) = 0.$$

Hence, $\mathcal{L}^{p,1}_{E'}(M)/\overline{\partial}_w \mathcal{C}^{p,0}_{E,E'}(M) \cong H^1(M, \Omega^p(E))$. We conclude

$$H^{0,1}_w(X^*, L) \cong \mathcal{L}^{0,1}_D(M)/\overline{\partial}_w \mathcal{C}^{0,0}_{Z-|Z|+D,D}(M) \cong H^1(M, \mathcal{O}(Z-|Z|+D))$$

and, using the Serre duality again,

$$H^{1,1}_w(X^*, L) \cong \mathcal{L}^{1,1}_{|Z|-Z+D}(M)/\overline{\partial}_w \mathcal{C}^{1,0}_{D,|Z|-Z+D}(M) \cong H^1(M, \Omega^1(D)) \cong H^0(M, \mathcal{O}(-D)).$$

□

Theorem 1.2 follows now as a simple corollary by use of the classical Riemann-Roch theorem for each connected component of the Riemann surface M, keeping in mind that by definition $g(M) = g(X)$, $\deg L = \deg \pi^* L = \deg D$ and $\operatorname{mult}'_x X = \sum_{p \in \pi^{-1}(x)} \deg_p(Z-|Z|)$.

To deduce also a Riemann-Roch theorem for the $\overline{\partial}_s$-cohomology, we can use the following L^2-version of Serre duality:

Theorem 4.6. *For each $p \in \{0,1\}$, the range of $\overline{\partial}_w : L^{p,0}(X^*, L) \to L^{p,1}(X^*, L)$ is closed. In particular, we get*

$$H^{p,q}_w(X^*, L) \cong H^{1-p,1-q}_s(X^*, L^{-1}).$$

Proof. Recall the following well-known fact. If $P : H_1 \to H_2$ is a densely defined closed operator between Hilbert spaces with range $\mathcal{R}(P)$ of finite codimension, then the range $\mathcal{R}(P)$ is closed in H_2 (see e. g. [HL84], Appendix 2.4).

As M is compact, Theorem 4.4 implies particularly that the range of $\overline{\partial}_w$ is finite codimensional and, therefore, closed. Since $\overline{\partial}_s$ is the adjoint of $-\overline{*}\,\overline{\partial}_w\,\overline{*}$, the range of

$$\overline{\partial}_s : L^{1-p,0}(X^*, L^{-1}) \to L^{1-p,1}(X^*, L^{-1})$$

is closed as well. That both ranges are closed implies the L^2-duality (cf. Lemma 3.3)

$$H_w^{p,q}(X^*, L) \cong \mathscr{H}_w^{p,q}(X^*, L) \cong \mathscr{H}_s^{1-p,1-q}(X^*, L^{-1}) \cong H_s^{1-p,1-q}(X^*, L^{-1}),$$

where $\mathscr{H}_{w/s}^{p,q}(X^*, L) := \ker \overline{\partial}_{w/s} \cap \ker \overline{\partial}^*_{w/s}$ denotes the space of $\overline{\partial}$-harmonic forms with values in L. □

Therefore, Theorem 4.4 yields:

$$\begin{aligned}
H_s^{0,0}(X^*, L) &\cong H_w^{1,1}(X^*, L^{-1}) \cong H^0(M, \mathcal{O}(D)),\\
H_s^{0,1}(X^*, L) &\cong H_w^{1,0}(X^*, L^{-1}) \cong H^1(M, \mathcal{O}(D)),\\
H_s^{1,0}(X^*, L) &\cong H_w^{0,1}(X^*, L^{-1}) \cong H^1(M, \mathcal{O}(Z - |Z| - D)), \text{and}\\
H_s^{1,1}(X^*, L) &\cong H_w^{0,0}(X^*, L^{-1}) \cong H^0(M, \mathcal{O}(Z - |Z| - D)).
\end{aligned} \tag{4.7}$$

Haskell computed $H_{\mathrm{cpt}}^{0,q}(X^*) \cong H^q(M, \mathcal{O}_M)$, where $H_{\mathrm{cpt}}^{p,q}(X^*)$ denotes the $\overline{\partial}$-cohomology with respect to smooth forms with compact support (see Thm. 3.1 in [Has89]). From (4.7), we obtain the dual version of Theorem 1.2, i. e., the Riemann-Roch theorem for the $\overline{\partial}_s$-cohomology:

Corollary 4.8 ($\overline{\partial}_s$-Riemann-Roch)**.** *Let X be a compact complex curve with m irreducible components, $L \to X$ be a holomorphic line bundle, and $\pi : M \to X$ be a resolution of X. Then,*

$$\begin{aligned}
h_s^{0,0}(X^*, L) - h_s^{0,1}(X^*, L) &= m - g + \deg L, \textit{ and}\\
h_s^{1,1}(X^*, L) - h_s^{1,0}(X^*, L) &= m - g + \deg(Z - |Z|) - \deg L,
\end{aligned}$$

where Z is the exceptional divisor of the resolution.

In [BPS90], Brüning, Peyerimhoff and Schröder proved that $h_s^{0,0}(X^*) - h_s^{0,1}(X^*) = m - g$ and $h_w^{0,0}(X^*) - h_w^{0,1}(X^*) = m - g + \deg Z - |Z|$ by computing the indices of the differential operators $\overline{\partial}_s$ and $\overline{\partial}_w$. Schröder generalized this result for vector bundles in [Sch89].

5. Weakly holomorphic functions

In this section, we will prove Theorem 1.4 by studying weakly holomorphic functions and a localized version of the $\overline{\partial}_s$-operator.

Recalling the arguments at the beginning of Section 4, it is easy to see that the results of Section 3 generalize to arbitrary complex curves. In particular, the $\overline{\partial}_{s,w}$-holomorphic functions on a singular complex curve are precisely the square-integrable weakly holomorphic functions (cf. Lemma 3.6), and the $\overline{\partial}_{s,w}$-equation is locally solvable in the L^2-sense (combine Lemma 3.5 and Lemma 3.7).

Let X be a singular complex curve, $\mathcal{L}_X^{p,q}$ the sheaf of locally square-integrable forms, and let

$$\overline{\partial}_w : \mathcal{L}_X^{p,0} \to \mathcal{L}_X^{p,1}$$

be the $\overline{\partial}$-operator in the sense of distributions. For each open set $U \subset X$, we define $\overline{\partial}_{s,\mathrm{loc}}$ on $L_{\mathrm{loc}}^{p,0}(U)$ by $f \in \operatorname{dom} \overline{\partial}_{s,\mathrm{loc}}$ iff $f \in \operatorname{dom} \overline{\partial}_w$ and $f \in \operatorname{dom} \left(\overline{\partial}_{s,w} : L^{p,0}(V) \to L^{p,1}(V)\right)$ for all $V \Subset U$ (for more details, see [Rup14, Sect. 6]). Let $\mathcal{F}_X^{p,0}$ be the sheaf of germs defined by

$\mathcal{F}_X^{p,0}(U) := \operatorname{dom}\left(\overline{\partial}_{s,\mathrm{loc}} : L^{p,0}_{\mathrm{loc}}(U) \to L^{p,1}_{\mathrm{loc}}(U)\right)$, and let $\widehat{\mathcal{O}}_X$ denote the sheaf of germs of weakly holomorphic functions on X. Then our considerations above yield an exact sequence

$$0 \to \widehat{\mathcal{O}}_X = \ker \overline{\partial}_{s,\mathrm{loc}} \hookrightarrow \mathcal{F}_X^{0,0} \xrightarrow{\overline{\partial}_{s,\mathrm{loc}}} \mathcal{L}_X^{0,1} \to 0. \tag{5.1}$$

The sheaves $\mathcal{F}_X^{0,0}$ and $\mathcal{L}_X^{0,1}$ are fine and so (5.1) is a fine resolution of $\widehat{\mathcal{O}}_X$. Let $H^{p,q}_{s,\mathrm{loc}}(X^*)$ denote the L^2_{loc}-Dolbeault cohomology on X^* with respect to the $\overline{\partial}_{s,\mathrm{loc}}$-operator. Using $\widehat{\mathcal{O}}_X = \pi_* \mathcal{O}_M$, we deduce from (5.1):

$$H^0(M, \mathcal{O}_M) \cong H^0(X, \widehat{\mathcal{O}}_X) = H^{0,0}_{s,\mathrm{loc}}(X^*),$$
$$H^1(M, \mathcal{O}_M) \cong H^1(X, \widehat{\mathcal{O}}_X) \cong H^{0,1}_{s,\mathrm{loc}}(X^*),$$

where $\pi : M \to X$ is a resolution of X. That proves Theorem 1.4.

6. **Applications**

There are many applications of the classical Riemann-Roch theorem; we will transfer two of them to our situation to exemplify how the L^2-Riemann-Roch theorem can substitute the classical one on singular spaces.

6.1. **Compact complex curves as covering spaces of $\mathbb{CP}^1$.** Let X be a compact irreducible complex curve with $\operatorname{Sing} X = \{x_1, ..., x_k\}$, let $(h_i)_{x_i} \in \mathcal{O}_{x_i}$ be chosen such that $(h_i)_{x_i} \widehat{\mathcal{O}}_{x_i} \subset \mathcal{O}_{x_i}$ and let $U_i \subset X$ be a (Stein) neighborhood of x_i with $h_i \cdot \widehat{\mathcal{O}}(U_i) \subset \mathcal{O}(U_i)$ (for the existence of the h_i, see e.g. Thm. 6 and its Cor. in [Nar66, § III.2]). Choose an $x_0 \in X^*$ and a (Stein) neighborhood U_0 of x_0. We can assume that U_0, ..., U_k are pairwise disjoint.

We define a line bundle $L \to X$ as follows. Let $U_{k+1} = X^* \setminus \{x_0\}$ and choose $f_0 \in \mathcal{O}(U_0)$ such that f_0 is vanishing to the order $r := \operatorname{ord}_{x_0} f_0 \geq 1$ in x_0, which we will determine later, but has no other zeros. We also set $f_i := 1/h_i$ for $i = 1, ..., k$ and $f_{k+1} = 1$, and consider the Cartier divisor $\{(U_i, f_i)\}_{i=0,...,k+1}$ on X. Let $L \to X$ be the line bundle associated to this divisor. As the f_i have no zeros for $i > 0$, there exists a non-negative integer δ such that $\deg L = r - \delta$. Now choose $r := g(X) + \delta + 1$. It follows that $\deg L = g(X) + 1$. Give L an arbitrary smooth Hermitian metric.

There is a canonical way to identify holomorphic sections of L with meromorphic functions on X. A holomorphic section $s \in \mathcal{O}(L)$ is represented by a tuple $\{s_i\}_i$ where $s_j/f_j = s_l/f_l$ on $U_j \cap U_l$. This gives a meromorphic function $\Psi(s)$ by setting $\Psi(s) := s_j/f_j$ on U_j. Note that $\Psi(s)$ has zeros in the singular points $x_1, ..., x_k$ and may have a pole of order r at $x_0 \notin \operatorname{Sing} X$.

We can now apply our L^2-Riemann-Roch theorem. The $\overline{\partial}_s$-Riemann-Roch theorem, Corollary 4.8, implies $\dim H^{0,0}_s(X^*, L) \geq 1 - g(X) + \deg L = 2$. Therefore, there is a section $\tau \in L^{0,0}(X^*, L)$ with $\overline{\partial}_s \tau = 0$ and τ_{k+1} is non-constant, where $\tau = \{\tau_i\}_{i=0,..,k+1}$ is written in the trivialization as above. This means that $\tau_i \in L^{0,0}(X^* \cap U_i)$, $\overline{\partial}_s \tau_i = 0$, and $\tau_j/f_j = \tau_{k+1}$ is non-constant on $U_j \cap U_{k+1}$. Theorem 1.4 implies that $\tau_i \in \widehat{\mathcal{O}}(U_i)$, $i = 1, ..., k+1$. Now consider $\Psi(\tau)$ as defined above, i.e., $\Psi(\tau) = \tau_i/f_i$ on U_i. We conclude that $\Psi(\tau)/h_i \in \widehat{\mathcal{O}}(U_i)$, thus $\Psi(\tau) \in \mathcal{O}(U_i)$ for $i = 1, ..., k$. Moreover, $\Psi(\tau)$ is non-constant, so it cannot be holomorphic on the whole compact space X, thus must have a pole of some order $\leq r$ in x_0. Thus:

$$\Psi(\tau) : X \setminus \{x_0\} \to \mathbb{C}, \text{and}$$
$$\widetilde{\Psi}(\tau) : X \to \mathbb{CP}^1, x \mapsto \begin{cases} [\Psi(\tau)(x) : 1], & x \neq x_0 \\ [1 : \frac{1}{\Psi(\tau)(x)}], & x \in U_0 \end{cases}$$

are finite, open and, hence, analytic ramified coverings (Covering Lemma, see [GR84, Sect. VII.2.2]). In particular, $X \setminus \{x_0\}$ is Stein (use e.g. Thm. 1 in [GR79, § V.1]).

6.2. **Projectivity of compact complex curves.** A line bundle $L \to X$ on a compact complex space is called very ample if its global holomorphic sections induce a holomorphic embedding into the projective space $\mathbb{CP}^N$, i. e., if $s_0, ..., s_N$ is a basis of the space of holomorphic sections $\Gamma(X, \mathcal{O}(L))$, then the map

$$\Phi : X \to \mathbb{CP}^N \ , \ x \mapsto [s_0(x) : ... : s_N(x)], \tag{6.1}$$

given in local trivializations of the s_i, defines a holomorphic embedding of X in $\mathbb{CP}^N$. If some positive power of the line bundle has this property, then we say that it is ample. A compact complex space is called projective if there is an ample (and, hence, a very ample) line bundle on it.

A classical application of the Riemann-Roch theorem is that any compact Riemann surface is projective, and a line bundle on a Riemann surface is ample if its degree is positive (cf. e. g. [Nar92, Sect. 10]). This generalizes to singular complex curves:

Theorem 6.2. *Let X be a compact locally irreducible complex curve. If $L \to X$ is a holomorphic line bundle with $\deg L \gg 0$, then L is very ample. In particular, X is projective and each holomorphic line bundle on X is ample if its degree is positive.*

Clearly, this result is well-known and follows from more general sheaf-theoretical methods (vanishing theorems) once one knows that L is positive iff $\deg L > 0$ (cf. e. g. Thm. 4.4 in [Pet94, Sect. V.4.3] or Satz 2 in [Gra62, § 3]). Nevertheless, it seems interesting to us to present another proof of Theorem 6.2 which is based on the L^2-Riemann-Roch of singular complex curves. The assumption that X must be locally irreducible in Theorem 6.2 is not necessary. One can prove the result without this assumption easily by the same technique. Yet, to keep the notation simple, we present here only the locally irreducible case.

Let us make some preparations for the proof of Theorem 6.2. Let X be a connected complex curve and $\pi : M \to X$ a resolution of X. We choose a point $x_0 \in \operatorname{Sing} X$ and a small neighborhood $U \subset X$ of x_0 with $U^* := U \setminus \{x_0\} \subset \operatorname{Reg} X$. Assume X is irreducible at x_0. We define $p_0 := \pi^{-1}(x_0)$, $V := \pi^{-1}(U)$, and $V^* := V \setminus \{p_0\}$. We can assume that there is a chart $t : V \to \mathbb{C}$ such that the image of t is bounded.

The Riemann extension theorem implies that $\pi^{-1} : U \to V$ is weakly holomorphic or, briefly, $\tau := t \circ \pi^{-1} \in \widehat{\mathcal{O}}(U)$ (see Theorem 2.5). We show that τ generates the weakly holomorphic functions at x_0 in the following sense: Let $f \in \widehat{\mathcal{O}}(U)$. Then $f \circ \pi$ is holomorphic on V^* and bounded in p_0. This implies that $f \circ \pi$ is holomorphic on V, $f \circ \pi(t) = \sum_{\iota=0}^{\infty} a_\iota t^\iota$, and $f(x) = \sum a_\iota \tau(x)^\iota$ (by shrinking U and V if necessary). This allows to define the order $\operatorname{ord}_{x_0} f$ of vanishing of f in x_0 by $r \in \mathbb{N}_0$ if $a_r \neq 0$ and $a_\iota = 0$ for $\iota < r$. In particular,

$$\operatorname{ord}_{x_0} f = \operatorname{ord}_{p_0}(f \circ \pi).$$

Note that this definition does not depend on the resolution as different resolutions are biholomorphically equivalent.

The L^2-extension theorem (see Theorem 2.4) and (4.3) imply

$$f \in H^{0,0}_w(U) \Leftrightarrow t^{r_0} \cdot \pi^* f \in \mathcal{O}_{L^2}(V) \Leftrightarrow (\tau^{r_0} \cdot f \in \widehat{\mathcal{O}}(U) \text{ and } f \in L^2(U)), \tag{6.3}$$

where Z denotes the exceptional divisor of the resolution and $r_0 := \deg_{p_0}(Z - |Z|)$. In particular, we get the representation $f(x) = \sum_{\iota \geq -r_0} a_\iota \tau(x)^\iota$ and $\operatorname{ord}_{x_0} f := \operatorname{ord}_{p_0} \pi^* f \geq -r_0$ is again well-defined. f is weakly holomorphic iff $\operatorname{ord}_{x_0} f \geq 0$.

We denote by L_{x_0} the holomorphic line bundle on X which is trivial on $X \setminus \{x_0\}$ and is given by τ on U, i. e., the line bundle on X given by the open covering $U_1 := X \setminus \{x_0\}, U_0 := U$ and the transition function $g_{01} := \tau : U_0 \cap U_1 \to \mathbb{C}$. Then $\pi^* L_{x_0} \cong L_{p_0}$, where L_{p_0} is the holomorphic line bundle $L_{p_0} \to M$ associated to the divisor $\{p_0\}$.

Let $L \to X$ be any holomorphic line bundle, $L' := L \otimes L_{x_0}^{-1}$, and let s' be a section in $H^{0,0}_w(X^*, L')$. We can assume that L and L' are given by divisors $\{(U_j, f_j)\}$ and $\{(U_j, f'_j)\}$, respectively, where $\{U_j\}$ is an open covering of X with $U_0 = U$ and $x_0 \notin U_j$ for $j \neq 0$ and where

$f_j, f'_j \in \mathcal{M}(U_j)$ with $g_{jk} := f_j/f_k$ and $g'_{jk} := f'_j/f'_k$ in $\mathcal{O}(U_j \cap U_k)$ ($g_{j,k}$ and g'_{jk} are the transition functions of L and L', respectively).

We get $f_0 = f'_0 \cdot \tau$ and $f_j = f'_j$ for $j \neq 0$. There is a meromorphic function $\tilde{s} := \Psi(s') \in \mathcal{M}(X)$ representing s'. This meromorphic function is defined by $\tilde{s} = s'_j/f'_j$ on U_j, where s'_j is the trivialization of s' on U_j. We can define a section $s = \{s_j\} \in H^{0,0}_w(X^*, L)$ by $s_j = \tilde{s} \cdot f_j$. Thus $s_0 = s'_0 \cdot \tau$ and $s_j = s'_j$ for $j \neq 0$. Hence, $\operatorname{ord}_{x_0} s_0 = \operatorname{ord}_{x_0} s'_0 + 1$. Summarizing, we get an injective linear map

$$T : H^{0,0}_w(X^*, L \otimes L^{-1}_{x_0}) \to H^{0,0}_w(X^*, L),\ s' \mapsto s,$$

which we call the *natural inclusion*. It follows from the construction above and by use of (6.3) that each section $s \in H^{0,0}_w(X^*, L)$ with $\operatorname{ord}_{x_0} s_0 > -r_0$ is in the image of T.

As $H^1(M, \mathcal{O}(D')) = 0$ for a divisor D' with $\deg D' > 2g - 2$ by the classical Riemann-Roch theorem (cf. e. g. [Nar92, Sect. 10]), Theorem 4.4 – more precisely,

$$H^{0,1}_w(X^*, L) \cong H^1(M, \mathcal{O}(Z - |Z| + D))$$

– implies the following vanishing theorem.

Theorem 6.4. *If $L \to X$ is a holomorphic line bundle on an irreducible compact complex curve X with $\deg L > 2g - 2 - \sum_{x \in \operatorname{Sing} X} \operatorname{mult}'_x X$, then $H^{0,1}_w(X^*, L) = 0$.*

As a preparation for the proof of Theorem 6.2, we get our main ingredient:

Lemma 6.5. *Let $L \to X$ be a holomorphic line bundle on a connected compact locally irreducible complex curve X with $\deg L > 2g - 1 - \sum_{x \in \operatorname{Sing} X} \operatorname{mult}'_x X$. Then the natural inclusion*

$$T : H^{0,0}_w(X^*, L \otimes L^{-1}_{x_0}) \to H^{0,0}_w(X^*, L)$$

is not surjective. If $\deg L > 2g + r_0 - 1 - \sum_{x \in \operatorname{Sing} X} \operatorname{mult}'_x X$, then there is a section $s \in H^{0,0}_w(X^*, L)$ which is weakly holomorphic on $U(x_0)$ and does not vanish in x_0.

Recall that $r_0 = \operatorname{mult}_{x_0} X - 1 = \deg_{p_0}(Z - |Z|)$.

Proof. i) As $\pi^*(L \otimes L^{-1}_{x_0}) \cong \pi^* L \otimes L^{-1}_{p_0}$, we get $\deg L \otimes L^{-1}_{x_0} = \deg L - 1 > 2g - 2 - \deg(Z - |Z|)$. The $\overline{\partial}_w$-Riemann-Roch theorem and $h^{0,1}_w(X^*, L) = 0 = h^{0,1}_w(X^*, L \otimes L^{-1}_{x_0})$ (using Theorem 6.4) yield

$$\begin{aligned} h^{0,0}_w(X^*, L \otimes L^{-1}_{x_0}) &= 1 - g + \deg(Z - |Z|) + \deg L \otimes L^{-1}_{x_0} \\ &< 1 - g + \deg(Z - |Z|) + \deg L = h^{0,0}_w(X^*, L). \end{aligned}$$

Therefore, the natural inclusion T cannot be surjective.

ii) The image of $T^{r_0} : H^{0,0}_w(X^*, L \otimes L^{-r_0}_{x_0}) \to H^{0,0}_w(X^*, L)$ are the sections s with $\operatorname{ord}_{x_0} s_0 \geq 0$, where s_0 is the trivialization of s over $U(x_0)$, i. e., the sections where s_0 is weakly holomorphic on $U(x_0)$. As $H^{0,0}_w(X^*, L \otimes L^{-r_0-1}_{x_0}) \to H^{0,0}_w(X^*, L \otimes L^{-r_0}_{x_0})$ is not surjective (use

$$\deg L \otimes L^{-r_0}_{x_0} = \deg L - r_0 > 2g - 1 - \deg(Z - |Z|)$$

and part (i)), there is a section

$$s' \in H^{0,0}_w(X^*, L \otimes L^{-r_0}_{x_0})$$

with $\operatorname{ord}_{x_0} s'_0 = -r_0$ and $\operatorname{ord}_{x_0}(T^{r_0}(s'))_0 = 0$. So, $s := T^{r_0}(s')$ is the section of $H^{0,0}_w(X^*, L)$ we were looking for. □

Proof of Theorem 6.2. Let X be a connected compact locally irreducible complex curve with $\operatorname{Sing} X = \{x_1, ..., x_k\}$, and $L \to X$ a line bundle with $\deg L \gg 0$. Following the classical arguments to show that the map Φ in (6.1) is a well-defined holomorphic embedding (see e. g. [Pet94, V.4, Thm. 4.4]), we have to prove:

(i) Φ is well-defined: For $x \in X$, there exists $s \in \Gamma(X, \mathcal{O}(L))$ such that $s(x) \neq 0$.

(ii) Φ is injective: For $x, y \in X$, $x \neq y$, there exists $s \in \Gamma(X, \mathcal{O}(L))$ such that $s(x) \neq 0$ and $s(y) = 0$.

(iii) Φ is an immersion: For $x \in X$, the differential $T_x\Phi$ is injective.

Since (obviously) Φ is closed, (ii) and (iii) imply that Φ is an embedding (see e. g. Sect. 1.2.7 in [GR84]).

We will prove the statements (i) and (iii) for singular points $x \in \operatorname{Sing} X$. The case of regular points is simpler and follows easily with the natural inclusion and Lemma 6.5. The statement (ii) can be seen just as (i) by imposing the additional condition that $s(y) = 0$ in what we do to prove the statement (i).

Let $\pi : M \to X$ be a resolution of singularities. Set $X^* = \operatorname{Reg} X$, $M^* = \pi^{-1}(X^*)$, $p_j := \pi^{-1}(x_j)$, and $r_j := \deg_{p_j}(Z - |Z|)$, where Z is the unreduced exceptional divisor of the resolution. Fix a $\mu \in \{1, ..., k\}$ and choose a neighborhood U_μ of x_μ such that there exist a resolution of the singularities $\pi : V_\mu \to U_\mu$ and a chart $t : V_\mu \to \mathbb{C}$ with $t \circ \pi^{-1}(x_\mu) = 0$, and set $\tau := t \circ \pi^{-1}$.

For each singularity x_j, we can choose a function $h_j \in \mathcal{O}(U_j)$ such that $h_j \cdot \widehat{\mathcal{O}}(U_j) \subset \mathcal{O}(U_j)$ for a neighborhood U_j of x_j small enough (see [Nar66, § III.2]). The number

$$\eta_j := \operatorname{ord}_{x_j} h_j$$

is important for our considerations because of the following fact. If f is a function on U_j with $\operatorname{ord}_{x_j} f \geq \eta_j$, then f/h_j is bounded at x_j ($\operatorname{ord}_{x_j} f/h_j \geq 0$); this implies $f/h_j \in \widehat{\mathcal{O}}(U_j)$ and, hence, $f \in \mathcal{O}(U_j)$. For the maximal ideal in $\mathcal{O}_{X,x_j}$, we get $\mathfrak{m}_{x_j} = \{f \in \mathcal{O}_{X,x_j} : \operatorname{ord}_{x_j} f > 0\}$ and $\{f \in \mathcal{O}_{X,x_j} : \operatorname{ord}_{x_j} f \geq 2\eta_j\} \subset \mathfrak{m}^2_{x_j}$.

We can choose a weakly holomorphic section $\sigma \in H^{0,0}_w(X^*, L)$ such that σ does not vanish in x_μ and $\operatorname{ord}_{x_j} \sigma \geq \eta_j$ for $j \neq \mu$. This section σ exists as we have the natural inclusion (see the construction above)

$$H^{0,0}_w\left(X^*, L \otimes L_{x_\mu}^{-r_\mu} \otimes \bigotimes_{j\neq\mu} L_{x_j}^{-\eta_j - r_j}\right) \to H^{0,0}_w(X^*, L),$$

and $\deg L \gg 0$ implies by Lemma 6.5 that the natural inclusion

$$H^{0,0}_w\left(X^*, L \otimes L_{x_\mu}^{-r_\mu - 1} \otimes \bigotimes_{j\neq\mu} L_{x_j}^{-\eta_j - r_j}\right) \to H^{0,0}_w\left(X^*, L \otimes L_{x_\mu}^{-r_\mu} \otimes \bigotimes_{j\neq\mu} L_{x_j}^{-\eta_j - r_j}\right)$$

is not surjective.

Note that σ is holomorphic on $X - \{x_\mu\}$ but just weakly holomorphic in x_μ. We will now modify σ so that it becomes holomorphic and non-vanishing in x_μ. Shrink U_μ such that $\sigma = \sum_{\iota\geq 0} a_\iota \tau^\iota$ on U_μ with $a_0 \neq 0$. Let $\sigma' := \sigma/a_0$ so that $\operatorname{ord}_{x_\mu}(\sigma' - 1) \geq 1$, i. e., $\sigma' - 1 = \sum_{\iota\geq 1} a'_\iota \tau^\iota$ on U_μ. Choose as above a $\tilde{\sigma} \in H^{0,0}_w(X^*, L)$ with $\operatorname{ord}_{x_\mu} \tilde{\sigma} = 1$ and $\operatorname{ord}_{x_j} \tilde{\sigma} \geq \eta_j$ for $j \neq \mu$. Let $\tilde{\sigma} = \sum_{\iota\geq 1} \tilde{a}_\iota \tau^\iota$ close to x_μ with $\tilde{a}_1 \neq 0$. We define $\sigma'' := \sigma' - \frac{a'_1}{\tilde{a}_1}\tilde{\sigma}$. Then, $\operatorname{ord}_{x_\mu}(\sigma'' - 1) \geq 2$ and $\operatorname{ord}_{x_j} \sigma'' \geq \eta_j$ for $j \neq \mu$. We repeat this procedure recursively to get a section $\xi = \{\xi_j\} \in H^{0,0}_w(X^*, L)$ with $\operatorname{ord}_{x_\mu}(\xi_\mu - 1) \geq \eta_\mu$ and $\operatorname{ord}_{x_j} \xi_j \geq \eta_j$ for $j \neq \mu$. Thus, ξ is a holomorphic section on X, non-vanishing in x_μ. That shows (i) for $x = x_\mu$.

We will prove (iii) for x_μ. Let $v \in T_{x_\mu}X = (\mathfrak{m}_{x_\mu}/\mathfrak{m}^2_{x_\mu})^*$ satisfy $v \neq 0$, i. e., there exists an $f \in \mathfrak{m}_{x_\mu}$ with $v(f + \mathfrak{m}^2_{x_\mu}) \neq 0$. We claim there exists a $g \in \mathfrak{m}_{\Phi(x_\mu)}$ with $g \circ \Phi - f \in \mathfrak{m}^2_{x_\mu}$. Then $v(g \circ \Phi + \mathfrak{m}^2_{x_\mu}) = v(f + \mathfrak{m}^2_{x_\mu}) \neq 0$, i. e., $T_x\Phi(v) \neq 0$.

Proof of the claim: Replacing 1 with $f = \sum_{\iota\geq 1} f_\iota \tau^\iota$, we can repeat the procedure in (i) to construct a section $\xi = \{\xi_j\} \in H^{0,0}_w(X^*, L)$ with $\operatorname{ord}_{x_\mu}(\xi_\mu - f) \geq 2\eta_\mu$ and $\operatorname{ord}_{x_j} \xi_j \geq \eta_j$ for $j \neq \mu$. We get ξ is holomorphic, $\xi_\mu \in \mathfrak{m}_{x_\mu}$ and $\xi_\mu - f \in \mathfrak{m}^2_{x_\mu}$. Let Φ be defined by $\Phi(x) = [s_0(x): ... :s_N(x)]$ with holomorphic sections $s_i = \{s_{ij}\}$ (see (6.1)). Hence, we can choose a vector $(g_0, ..., g_N) \in \mathbb{C}^{N+1}$

such that $\xi = \sum_i g_i s_i$. Because of (i), there exits an i_0 such that $c := s_{i_0}(x_\mu) \neq 0$ – we can assume $i_0 = 0$. We set $U := \{x \in U_\mu \colon s_{0\mu}(x) \neq 0\}$ and identify $\{[t_0{:}...{:}t_N] \colon t_0 = 1\} \subset \mathbb{CP}^N$ with $\mathbb{C}^N$ such that $\Phi|_U \colon U \to \mathbb{C}^N$ is defined by $\Phi(x) = \left(\frac{s_{1\mu}(x)}{s_{0\mu}(x)}, ..., \frac{s_{N\mu}(x)}{s_{0\mu}(x)}\right)$. Let $g \colon \mathbb{C}^N \to \mathbb{C}$ denote the holomorphic function $g(t_1, ..., t_N) := c \cdot (g_0 + \sum_{i=1}^N g_i t_i)$, i. e.,

$$s_{0\mu} \cdot (g \circ \Phi|_U) = c \sum_{i=0}^N g_i s_{i\mu} = c \cdot \xi_\mu$$

on U. Since $c = s_{0\mu}(x_\mu) \neq 0$ and since f and $\frac{c}{s_{0\mu}} - 1$ are in $\mathfrak{m}_{x_\mu}$, we get $g \in \mathfrak{m}_{\Phi(x_\mu)}$ and

$$g \circ \Phi - f = \frac{c}{s_{0\mu}} (\xi_\mu - f) + f \cdot \left(\frac{c}{s_{0\mu}} - 1\right) \in \mathfrak{m}_{x_\mu}^2 . \qquad \square$$

For this proof, L has to satisfy

$$\deg L > 2g + \max\{\eta_j\} + \sum_{j=1}^k (\eta_j + r_j) - \deg(Z - |Z|) = 2g + k + \max\{\eta_j\} + \sum_{j=1}^k \eta_j .$$

References

[BPS90] Jochen Brüning, Norbert Peyerimhoff and Herbert Schröder, *The $\overline{\partial}$-operator on algebraic curves*, Comm. Math. Phys. **129** (1990), no. 3, 525–534.

[Chi89] Evgeni M. Chirka, *Complex Analytic Sets*, Mathematics and its Applications (Soviet Series), vol. 46, Kluwer Academic Publishers Group, Dordrecht, 1989.

[Dem12] Jean-Pierre Demailly, *Complex Analytic and Differential Geometry*, Institut Fourier, Université de Grenoble I.

[Gra62] Hans Grauert, *Über Modifikationen und Exzeptionelle Analytische Mengen*, Math. Ann. **146** (1962), 331–368. DOI: 10.1007/BF01441136

[GR84] Hans Grauert and Reinhold Remmert, *Coherent Analytic Sheaves*, Grundlehren der Mathematischen Wissenschaften, no. 265, Springer, Berlin, 1984.

[GR79] ———, *Theory of Stein Spaces*, Grundlehren der Mathematischen Wissenschaften, vol. 236, Springer, Berlin, 1979, Translated from the German by Alan Huckleberry.

[GH78] Phillip Griffiths and Joseph Harris, *Principles of Algebraic Geometry*, Wiley-Interscience [John Wiley & Sons], New York, 1978, Pure and Applied Mathematics.

[Has89] Peter Haskell, *L^2-Dolbeault complexes on singular curves and surfaces*, Proc. Amer. Math. Soc. **107** (1989), no. 2, 517–526.

[HL84] Gennadi Henkin and Jürgen Leiterer, *Theory of functions on complex manifolds*, Monographs in Mathematics, vol. 79, Birkhäuser Verlag, Basel, 1984.

[Hir77] Heisuke Hironaka, *Bimeromorphic smoothing of a complex-analytic space*, Acta Math. Vietnam. **2** (1977), no. 2, 103–168.

[Hir64] ———, *Resolution of singularities of an algebraic variety over a field of characteristic zero I, II*, Ann. of Math. (2) **79** (1964), 109–203, 205–326. DOI: 10.2307/1970486

[LM02] Ingo Lieb and Joachim Michel, *The Cauchy-Riemann Complex*, Aspects of Mathematics, no. E 34, Vieweg, Braunschweig , 2002. DOI: 10.1007/978-3-322-91608-2

[Nag90] Masayoshi Nagase, *Remarks on the L^2-Dolbeault cohomology groups of singular algebraic surfaces and curves*, Publ. Res. Inst. Math. Sci. **26** (1990), no. 5, 867–883.

[Nar92] Raghavan Narasimhan, *Compact Riemann Surfaces*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1992.

[Nar66] ———, *Introduction to the Theory of Analytic Spaces*, Lecture Notes in Mathematics, No. 25, Springer-Verlag, Berlin, 1966.

[ØV13] Nils Øvrelid and Sophia Vassiliadou, *L^2-$\overline{\partial}$-cohomology groups of some singular complex spaces*, Invent. Math. **192** (2013), no. 2, 413–458.

[Pet94] Thomas Peternell, *Pseudoconvexity, the Levi problem and vanishing theorems*, Several complex variables, VII, Encyclopaedia Math. Sci., vol. 74, Springer, Berlin, 1994, pp. 221–257. DOI: 10.1007/978-3-662-09873-8_6

[Par89] William L. Pardon, *The L_2-$\overline{\partial}$-cohomology of an algebraic surface*, Topology **28** (1989), no. 2, 171–195.

[PS91] William L. Pardon and Mark A. Stern, *L^2-$\overline{\partial}$-cohomology of complex projective varieties*, J. Amer. Math. Soc. **4** (1991), no. 3, 603–621.

[Rup09] Jean Ruppenthal, *About the $\overline{\partial}$-equation at isolated singularities with regular exceptional set*, Internat. J. Math. **20** (2009), no. 4, 459–489.

[Rup14] ______, *L^2-theory for the $\overline{\partial}$-operator on compact complex spaces*, Duke Math. J. **163** (2014), no. 15, 2887–2934.

[Rup11] ______, *L^2-theory for the $\overline{\partial}$-operator on complex spaces with isolated singularities*, Preprint 2011. arXiv: 1110.2373

[Sch89] Herbert Schröder, *The Riemann-Roch theorem on algebraic curves*, Séminaire de Théorie Spectrale et Géométrie, No. 7, Année 1988–1989, Univ. Grenoble I, Saint-Martin-d'Hères, 1989, pp. 115–121.

[Ser55] Jean-Pierre Serre, *Un théorème de dualité*, Comm. Math. Helv. **29** (1955), 9–26.

Department of Mathematics, University of Wuppertal, Gauss Str. 20, 42119 Wuppertal, Germany.
E-mail address: ruppenthal@uni-wuppertal.de, sera@math.uni-wuppertal.de