SYMMETRIES AND STABILIZATION FOR SHEAVES
OF VANISHING CYCLES

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WITH AN APPENDIX BY JÖRG SCHÜRMANN

ABSTRACT. We study symmetries and stabilization properties of perverse sheaves of vanishing cycles $\mathcal{PV}^\bullet_{U,f}$ of a regular function $f : U \to \mathbb{C}$ on a smooth $\mathbb{C}$-scheme U, with critical locus $X = \mathrm{Crit}(f)$. We prove four main results:

(a) If $\Phi : U \to U$ is an isomorphism fixing X and compatible with f, then the action of Φ_* on $\mathcal{PV}^\bullet_{U,f}$ is multiplication by $\det\big(\mathrm{d}\Phi|_{X^{\mathrm{red}}}\big) = \pm 1$.

(b) $\mathcal{PV}^\bullet_{U,f}$ depends up to canonical isomorphism only on $(X^{(3)}, f^{(3)})$, for $X^{(3)}$ the third-order thickening of X in U, and $f^{(3)} = f|_{X^{(3)}} : X^{(3)} \to \mathbb{C}$.

(c) If U, V are smooth $\mathbb{C}$-schemes, $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ are regular, $X = \mathrm{Crit}(f)$, $Y = \mathrm{Crit}(g)$, and $\Phi : U \to V$ is an embedding with $f = g \circ \Phi$ and $\Phi|_X : X \to Y$ an isomorphism, there is a natural isomorphism $\Theta_\Phi : \mathcal{PV}^\bullet_{U,f} \to \Phi|_X^*(\mathcal{PV}^\bullet_{V,g}) \otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Phi$, for P_Φ a principal $\mathbb{Z}/2\mathbb{Z}$-bundle on X.

(d) If (X, s) is an *oriented d-critical locus* in the sense of Joyce [23], there is a natural perverse sheaf $P^\bullet_{X,s}$ on X, such that if (X, s) is locally modelled on $\mathrm{Crit}(f : U \to \mathbb{C})$ then $P^\bullet_{X,s}$ is locally modelled on $\mathcal{PV}^\bullet_{U,f}$.

We also generalize our results to replace U, X by complex analytic spaces, and $\mathcal{PV}^\bullet_{U,f}$ by $\mathscr{D}$-modules or mixed Hodge modules.

We discuss applications of (d) to categorifying Donaldson–Thomas invariants of Calabi–Yau 3-folds, and to defining a 'Fukaya category' of Lagrangians in a complex symplectic manifold using perverse sheaves.

CONTENTS

1. Introduction

Let U be a smooth $\mathbb{C}$-scheme and $f : U \to \mathbb{C}$ a regular function, and write $X = \mathrm{Crit}(f)$, as a $\mathbb{C}$-subscheme of U. Then one can define the *perverse sheaf of vanishing cycles* $\mathcal{PV}_{U,f}^\bullet$ on X. Formally, $X = \coprod_{c \in f(X)} X_c$, where $X_c \subseteq X$ is the open and closed $\mathbb{C}$-subscheme of points $x \in X$ with $f(x) = c$, and

$$\mathcal{PV}_{U,f}^\bullet|_{X_c} = \phi_{f-c}^p(A_U[\dim U])|_{X_c}$$

for each $c \in f(X)$, where $A_U[\dim U]$ is the constant perverse sheaf on U over a base ring A, and

$$\phi_{f-c}^p : \mathrm{Perv}(U) \longrightarrow \mathrm{Perv}(f^{-1}(c))$$

is the vanishing cycle functor for $f - c : U \to \mathbb{C}$. See §2 for an introduction to perverse sheaves, and an explanation of this notation.

This paper will prove four main results, Theorems 3.1, 4.2, 5.4 and 6.9. The first three give properties of the $\mathcal{PV}_{U,f}^\bullet$, which we may summarize as follows:

(a) Let U, f, X be as above, and write X^{red} for the reduced $\mathbb{C}$-subscheme of X. Suppose $\Phi : U \to U$ is an isomorphism with $f \circ \Phi = f$ and $\Phi|_X = \mathrm{id}_X$. Then Φ induces a natural isomorphism $\Phi_* : \mathcal{PV}_{U,f}^\bullet \to \mathcal{PV}_{U,f}^\bullet$.

Theorem 3.1 implies that $\mathrm{d}\Phi|_{TU|_{X^{\mathrm{red}}}} : TU|_{X^{\mathrm{red}}} \to TU|_{X^{\mathrm{red}}}$ has determinant

$$\det\big(\mathrm{d}\Phi|_{X^{\mathrm{red}}}\big) : X^{\mathrm{red}} \longrightarrow \mathbb{C} \setminus \{0\},$$

which is a locally constant map $X^{\mathrm{red}} \to \{\pm 1\}$, and $\Phi_* : \mathcal{PV}_{U,f}^\bullet \to \mathcal{PV}_{U,f}^\bullet$ is multiplication by $\det\big(\mathrm{d}\Phi|_{X^{\mathrm{red}}}\big)$.

In fact Theorem 3.1 proves a more complicated statement, which only requires Φ to be defined étale locally on U.

(b) Let U, f, X be as above, and write $I_X \subseteq \mathcal{O}_U$ for the sheaf of ideals of regular functions $U \to \mathbb{C}$ vanishing on X. For each $k = 1, 2, \ldots$, write $X^{(k)}$ for the kth *order thickening of X in U*, that is, $X^{(k)}$ is the closed $\mathbb{C}$-subscheme of U defined by the vanishing of the sheaf of ideals I_X^k in $\mathcal{O}_U$. Write $f^{(k)} := f|_{X^{(k)}} : X^{(k)} \to \mathbb{C}$.

Theorem 4.2 says that the perverse sheaf $\mathcal{PV}_{U,f}^\bullet$ depends only on the third-order thickenings $(X^{(3)}, f^{(3)})$ up to canonical isomorphism.

As in Remark 4.5, étale locally, $\mathcal{PV}_{U,f}^\bullet$ depends only on $(X^{(2)}, f^{(2)})$ up to non-canonical isomorphism, with isomorphisms natural up to sign.

(c) Let U, V be smooth $\mathbb{C}$-schemes, $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ be regular, and $X = \mathrm{Crit}(f)$, $Y = \mathrm{Crit}(g)$ as $\mathbb{C}$-subschemes of U, V. Let $\Phi : U \hookrightarrow V$ be a closed embedding of $\mathbb{C}$-schemes with $f = g \circ \Phi : U \to \mathbb{C}$, and suppose $\Phi|_X : X \to Y$ is an isomorphism. Then Theorem 5.4 constructs a natural isomorphism of perverse sheaves on X:

$$\Theta_\Phi : \mathcal{PV}_{U,f}^\bullet \longrightarrow \Phi|_X^* (\mathcal{PV}_{V,g}^\bullet) \otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Phi, \tag{1.1}$$

where $\pi_\Phi : P_\Phi \to X$ is a certain principal $\mathbb{Z}/2\mathbb{Z}$-bundle on X. Writing N_{UV} for the normal bundle of U in V, then the Hessian $\mathrm{Hess}\, g$ induces a nondegenerate quadratic form q_{UV} on $N_{UV}|_X$, and P_Φ parametrizes square roots of $\det(q_{UV}) : K_U^2|_X \to \Phi|_X^* (K_V^2)$.

Theorem 5.4 also shows that the Θ_Φ in (1.1) are functorial in a suitable sense under compositions of embeddings $\Phi : U \hookrightarrow V$, $\Psi : V \hookrightarrow W$.

Here (c) is proved by showing that étale locally there exist equivalences $V \simeq U \times \mathbb{C}^n$ identifying $\Phi(U)$ with $U \times \{0\}$ and $g : V \to \mathbb{C}$ with $f \boxplus z_1^2 + \cdots + z_n^2 : U \times \mathbb{C}^n \to \mathbb{C}$, and applying étale local isomorphisms of perverse sheaves

$$\mathcal{PV}_{U,f}^\bullet \cong \mathcal{PV}_{U,f}^\bullet \overset{L}{\boxtimes} \mathcal{PV}_{\mathbb{C}^n, z_1^2 + \cdots + z_n^2}^\bullet \cong \mathcal{PV}_{U \times \mathbb{C}^n, f \boxplus z_1^2 + \cdots + z_n^2}^\bullet \cong \mathcal{PV}_{V,g}^\bullet,$$

using $\mathcal{PV}_{\mathbb{C}^n, z_1^2 + \cdots + z_n^2}^\bullet \cong A_{\{0\}}$ in the first step, and the Thom–Sebastiani Theorem for perverse sheaves in the second.

Passing from $f : U \to \mathbb{C}$ to $g = f \boxplus z_1^2 + \cdots + z_n^2 : U \times \mathbb{C}^n \to \mathbb{C}$ is an important idea in singularity theory, as in Arnold et al. [1] for instance. It is known as *stabilization*, and f and g are called *stably equivalent*. So, Theorem 5.4 concerns the behaviour of perverse sheaves of vanishing cycles under stabilization.

Our fourth main result, Theorem 6.9, concerns a new class of geometric objects called *d-critical loci*, introduced in Joyce [23], and explained in §6.1. An (algebraic) d-critical locus (X, s) over $\mathbb{C}$ is a $\mathbb{C}$-scheme X with a section s of a certain natural sheaf $\mathcal{S}_X^0$ on X. A d-critical locus (X, s) may be written Zariski locally as a critical locus $\mathrm{Crit}(f : U \to \mathbb{C})$ of a regular function f on a smooth $\mathbb{C}$-scheme U, and s records some information about U, f (in the notation of (b) above, s remembers $f^{(2)}$). There is also a complex analytic version.

Algebraic d-critical loci are classical truncations of the *derived critical loci* (more precisely, -1-*shifted symplectic derived schemes*) introduced in derived algebraic geometry by Pantev, Toën, Vaquié and Vezzosi [42]. Theorem 6.9 roughly says that if (X, s) is an algebraic d-critical locus over $\mathbb{C}$ with an 'orientation', then we may define a natural perverse sheaf $P_{X,s}^\bullet$ on X, such that if (X, s) is locally modelled on $\mathrm{Crit}(f : U \to \mathbb{C})$ then $P_{X,s}^\bullet$ is locally modelled on $\mathcal{PV}_{U,f}^\bullet$. The proof uses Theorem 5.4.

These results have exciting applications in the categorification of Donaldson–Thomas theory on Calabi–Yau 3-folds, and in defining a new kind of 'Fukaya category' of complex Lagrangians in complex symplectic manifold, which we will discuss at length in Remarks 6.14 and 6.15.

Although we have explained our results only for $\mathbb{C}$-schemes and perverse sheaves upon them, the proofs are quite general and work in several contexts:

(i) Perverse sheaves on $\mathbb{C}$-schemes or complex analytic spaces with coefficients in any well-behaved commutative ring A, such as $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{C}$.

(ii) $\mathscr{D}$-modules on $\mathbb{C}$-schemes or complex analytic spaces.

(iii) Saito's mixed Hodge modules on $\mathbb{C}$-schemes or complex analytic spaces.

We discuss all these in §2, before proving our four main results in §3–§6. Appendix A, by Jörg Schürmann, proves two compatibility results between duality and Thom–Sebastiani type isomorphisms needed in the main text.

This is one of six linked papers [6, 9–11, 23], with more to come. The best logical order is that the first is Joyce [23] defining d-critical loci, and the second Bussi, Brav and Joyce [9], which proves Darboux-type theorems for the k-shifted symplectic derived schemes of Pantev et al. [42], and defines a truncation functor from -1-shifted symplectic derived schemes to algebraic d-critical loci.

This paper is the third in the sequence. Combining our results with [23, 42] gives new results on categorifying Donaldson–Thomas invariants of Calabi–Yau 3-folds, as in Remark 6.14. In the fourth paper Bussi, Joyce and Meinhardt [11] will generalize the ideas of this paper to motivic Milnor fibres (we explain the relationship between the motivic and cohomological approaches below in Remark 6.10), and deduce new results on motivic Donaldson–Thomas invariants using [23, 42]. In the fifth, Ben-Bassat, Brav, Bussi and Joyce [6] generalize [9, 11] and this paper from (derived) schemes to (derived) Artin stacks.

Sixthly, Bussi [10] will show that if (S, ω) is a complex symplectic manifold, and L, M are complex Lagrangians in S, then the intersection $X = L \cap M$, as a complex analytic subspace of S, extends naturally to a complex analytic d-critical locus (X, s). If the canonical bundles K_L, K_M have square roots $K_L^{1/2}, K_M^{1/2}$ then (X, s) is oriented, and so Theorem 6.9 below defines a perverse sheaf $P_{L,M}^\bullet$ on X, which Bussi also constructs directly.

As in Remark 6.15, we hope in future work to define a 'Fukaya category' of complex Lagrangians in (X, ω) in which $\mathrm{Hom}(L, M) \cong \mathbb{H}^{-n}(P_{L,M}^\bullet)$.

Conventions. All $\mathbb{C}$-schemes are assumed separated and of finite type. All complex analytic spaces are Hausdorff and locally of finite type.

2. BACKGROUND ON PERVERSE SHEAVES

Perverse sheaves, and the related theories of $\mathscr{D}$-modules and mixed Hodge modules, make sense in several contexts, both algebraic and complex analytic:

(a) Perverse sheaves on $\mathbb{C}$-schemes with coefficients in a ring A (usually $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{C}$), as in Beilinson, Bernstein and Deligne [5] and Dimca [14].
(b) Perverse sheaves on complex analytic spaces with coefficients in a ring A (usually $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{C}$), as in Dimca [14].
(c) $\mathscr{D}$-modules on $\mathbb{C}$-schemes, as in Borel [8] in the smooth case, and Saito [48] in general.
(d) $\mathscr{D}$-modules on complex manifolds as in Björk [7], and on complex analytic spaces as in Saito [48].
(e) Mixed Hodge modules on $\mathbb{C}$-schemes, as in Saito [45, 47].
(f) Mixed Hodge modules on complex analytic spaces, as in Saito [45, 47].

All our main results and proofs work, with minor modifications, in all six settings (a)–(f). As (a) is arguably the simplest and most complete theory, we begin in §2.1–§2.4 with a general introduction to constructible complexes and perverse sheaves on $\mathbb{C}$-schemes, the nearby and

vanishing cycle functors, and perverse sheaves of vanishing cycles $PV_{U,f}^\bullet$ on $\mathbb{C}$-schemes, following Dimca [14].

Several important properties of perverse sheaves in (a) either do not work, or become more complicated, in settings (b)–(f). Section 2.5 lists the parts of §2.1–§2.4 that we will use in proofs in this paper, so the reader can check that they do work in (b)–(f). Then §2.6–§2.10 give brief discussions of settings (b)–(f), focussing on the differences with (a) in §2.1–§2.4.

A good introductory reference on perverse sheaves on $\mathbb{C}$-schemes and complex analytic spaces is Dimca [14]. Three other books are Kashiwara and Schapira [27], Schürmann [50], and Hotta, Tanisaki and Takeuchi [21]. Massey [36] and Rietsch [43] are surveys on perverse sheaves, and Beilinson, Bernstein and Deligne [5] is an important primary source, who cover both $\mathbb{Q}$-perverse sheaves on $\mathbb{C}$-schemes as in (a), and $\mathbb{Q}_l$-perverse sheaves on $\mathbb{K}$-schemes as in (g) below.

Remark 2.1. Two further possible settings, in which not all the results we need are available in the literature, are the following.

   (g) Perverse sheaves on $\mathbb{K}$-schemes with coefficients in $\mathbb{Z}/l^n\mathbb{Z}$, $\mathbb{Z}_l$, $\mathbb{Q}_l$, or $\bar{\mathbb{Q}}_l$ for $l \neq \operatorname{char} \mathbb{K} \neq 2$ a prime, as in Beilinson et al. [5].
   (h) $\mathscr{D}$-modules on $\mathbb{K}$-schemes for $\mathbb{K}$ an algebraically closed field, as in Borel [8].

The issue is that the Thom–Sebastiani theorem is not available in these contexts in the generality we need it. Once an appropriate form of this result becomes available, our main theorems will hold also in these two contexts, sometimes under the further assumption that char $\mathbb{K} = 0$, needed for the results quoted from [9, 42]. We leave the details to the interested reader.

2.1. **Constructible complexes on $\mathbb{C}$-schemes.** We begin by discussing constructible complexes, following Dimca [14, §2–§4].

Definition 2.2. Fix a well-behaved commutative base ring A (where 'well-behaved' means that we need assumptions on A such as A is regular noetherian, of finite global dimension or finite Krull dimension, a principal ideal domain, or a Dedekind domain, at various points in the theory), to study sheaves of A-modules. For some results A must be a field. Usually we take $A = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{C}$.

Let X be a $\mathbb{C}$-scheme, always assumed of finite type. Write X^{an} for the set of $\mathbb{C}$-points of X with the complex analytic topology. Consider sheaves of A-modules $\mathcal{S}$ on X^{an}. A sheaf $\mathcal{S}$ is called (*algebraically*) *constructible* if all the stalks $\mathcal{S}_x$ for $x \in X^{\mathrm{an}}$ are finite type A-modules, and there is a finite stratification $X^{\mathrm{an}} = \coprod_{j \in J} X_j^{\mathrm{an}}$ of X^{an}, where $X_j \subseteq X$ for $j \in J$ are $\mathbb{C}$-subschemes of X and $X_j^{\mathrm{an}} \subseteq X^{\mathrm{an}}$ the corresponding subsets of $\mathbb{C}$-points, such that $\mathcal{S}|_{X_j^{\mathrm{an}}}$ is an A-local system for all $j \in J$.

Write $D(X)$ for the derived category of complexes $\mathcal{C}^\bullet$ of sheaves of A-modules on X^{an}. Write $D_c^b(X)$ for the full subcategory of bounded complexes $\mathcal{C}^\bullet$ in $D(X)$ whose cohomology sheaves $\mathcal{H}^m(\mathcal{C}^\bullet)$ are constructible for all $m \in \mathbb{Z}$. Then $D(X), D_c^b(X)$ are triangulated categories. An example of a constructible complex on X is the *constant sheaf* A_X on X with fibre A at each point.

Grothendieck's "six operations on sheaves" $f^*, f^!, Rf_*, Rf_!, \mathcal{R}\mathcal{H}om, \overset{L}{\otimes}$ act on $D(X)$ preserving the subcategory $D_c^b(X)$. That is, if $f : X \to Y$ is a morphism of $\mathbb{C}$-schemes, then we have two different pullback functors $f^*, f^! : D(Y) \to D(X)$, which also map $D_c^b(Y) \to D_c^b(X)$. Here f^* is called the *inverse image* [14, §2.3], and $f^!$ the *exceptional inverse image* [14, §3.2].

We also have two different pushforward functors

$$Rf_*, Rf_! : D(X) \longrightarrow D(Y)$$

mapping $D_c^b(X) \to D_c^b(Y)$, where Rf_* is called the *direct image* [14, §2.3] and is right adjoint to $f^* : D(Y) \to D(X)$, and $Rf_!$ is called the *direct image with proper supports* [14, §2.3] and is left adjoint to $f^! : D(Y) \to D(X)$. We need the assumptions from §1 that X, Y are separated and of finite type for $Rf_*, Rf_! : D_c^b(X) \to D_c^b(Y)$ to be defined for arbitrary morphisms $f : X \to Y$.

For $\mathcal{B}^\bullet, \mathcal{C}^\bullet$ in $D_c^b(X)$, we may form their *derived Hom* $\mathcal{R}Hom(\mathcal{B}^\bullet, \mathcal{C}^\bullet)$ [14, §2.1], and *left derived tensor product* $\mathcal{B}^\bullet \overset{L}{\otimes} \mathcal{C}^\bullet$ in $D_c^b(X)$, [14, §2.2]. Given $\mathcal{B}^\bullet \in D_c^b(X)$ and $\mathcal{C}^\bullet \in D_c^b(Y)$, we define $\mathcal{B}^\bullet \overset{L}{\boxtimes} \mathcal{C}^\bullet = \pi_X^*(\mathcal{B}^\bullet) \overset{L}{\otimes} \pi_Y^*(\mathcal{C}^\bullet)$ in $D_c^b(X \times Y)$, where $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$ are the projections.

If X is a $\mathbb{C}$-scheme, there is a functor $\mathbb{D}_X : D_c^b(X) \to D_c^b(X)^{\mathrm{op}}$ with

$$\mathbb{D}_X \circ \mathbb{D}_X \cong \mathrm{id} : D_c^b(X) \longrightarrow D_c^b(X),$$

called *Verdier duality.* It reverses shifts, that is, $\mathbb{D}_X(\mathcal{C}^\bullet[k]) = (\mathbb{D}_X(\mathcal{C}^\bullet))[-k]$ for $\mathcal{C}^\bullet$ in $D_c^b(X)$ and $k \in \mathbb{Z}$.

Remark 2.3. Note how Definition 2.2 mixes the complex analytic and the complex algebraic: we consider sheaves on X^{an} in the *analytic* topology, which are constructible with respect to an *algebraic* stratification $X = \coprod_j X_j$.

Here are some properties of all these:

Theorem 2.4. *In the following, X, Y, Z are $\mathbb{C}$-schemes, and f, g are morphisms, and all isomorphisms '$\cong$' of functors or objects are canonical.*

(i) *For $f : X \to Y$ and $g : Y \to Z$, there are natural isomorphisms of functors*

$$R(g \circ f)_* \cong Rg_* \circ Rf_*, \qquad\qquad R(g \circ f)_! \cong Rg_! \circ Rf_!,$$
$$(g \circ f)^* \cong f^* \circ g^*, \qquad\qquad (g \circ f)^! \cong f^! \circ g^!.$$

(ii) *If $f : X \to Y$ is proper then $Rf_* \cong Rf_!$.*

(iii) *If $f : X \to Y$ is étale then $f^* \cong f^!$. More generally, if $f : X \to Y$ is smooth of relative (complex) dimension d, then $f^*[d] \cong f^![-d]$, where $f^*[d], f^![-d]$ are the functors $f^*, f^!$ shifted by $\pm d$.*

(iv) *If $f : X \to Y$ then $Rf_! \cong \mathbb{D}_Y \circ Rf_* \circ \mathbb{D}_X$ and $f^! \cong \mathbb{D}_X \circ f^* \circ \mathbb{D}_Y$.*

(v) *If U is a smooth $\mathbb{C}$-scheme then $\mathbb{D}_U(A_U) \cong A_U[2 \dim U]$.*

If X is a $\mathbb{C}$-scheme and $\mathcal{C}^\bullet \in D_c^b(X)$, the *hypercohomology* $\mathbb{H}^*(\mathcal{C}^\bullet)$ and *compactly-supported hypercohomology* $\mathbb{H}_c^*(\mathcal{C}^\bullet)$, both graded A-modules, are

$$\mathbb{H}^k(\mathcal{C}^\bullet) = H^k(R\pi_*(\mathcal{C}^\bullet)) \quad \text{and} \quad \mathbb{H}_c^k(\mathcal{C}^\bullet) = H^k(R\pi_!(\mathcal{C}^\bullet)) \quad \text{for } k \in \mathbb{Z}, \tag{2.1}$$

where $\pi : X \to *$ is projection to a point.

If X is proper then $\mathbb{H}^*(\mathcal{C}^\bullet) \cong \mathbb{H}_c^*(\mathcal{C}^\bullet)$ by Theorem 2.4(ii). They are related to usual cohomology by $\mathbb{H}^k(A_X) \cong H^k(X; A)$ and $\mathbb{H}_c^k(A_X) \cong H_c^k(X; A)$. If A is a field then under Verdier duality we have $\mathbb{H}^k(\mathcal{C}^\bullet) \cong \mathbb{H}_c^{-k}(\mathbb{D}_X(\mathcal{C}^\bullet))^*$.

2.2. **Perverse sheaves on $\mathbb{C}$-schemes.** Next we review perverse sheaves, following Dimca [14, §5].

Definition 2.5. Let X be a $\mathbb{C}$-scheme, and for each $x \in X^{\mathrm{an}}$, let $i_x : * \to X$ map $i_x : * \mapsto x$. If $\mathcal{C}^\bullet \in D_c^b(X)$, then the *support* $\mathrm{supp}^m \mathcal{C}^\bullet$ and *cosupport* $\mathrm{cosupp}^m \mathcal{C}^\bullet$ of $\mathcal{H}^m(\mathcal{C}^\bullet)$ for $m \in \mathbb{Z}$ are

$$\mathrm{supp}^m \mathcal{C}^\bullet = \overline{\{x \in X^{\mathrm{an}} : \mathcal{H}^m(i_x^*(\mathcal{C}^\bullet)) \neq 0\}},$$
$$\mathrm{cosupp}^m \mathcal{C}^\bullet = \overline{\{x \in X^{\mathrm{an}} : \mathcal{H}^m(i_x^!(\mathcal{C}^\bullet)) \neq 0\}},$$

where $\overline{\{\cdots\}}$ means the closure in X^{an}. If A is a field then $\mathrm{cosupp}^m\,\mathcal{C}^\bullet = \mathrm{supp}^{-m}\,\mathbb{D}_X(\mathcal{C}^\bullet)$. We call $\mathcal{C}^\bullet$ *perverse*, or a *perverse sheaf*, if $\dim_{\mathbb{C}}\mathrm{supp}^{-m}\,\mathcal{C}^\bullet \leqslant m$ and $\dim_{\mathbb{C}}\mathrm{cosupp}^m\,\mathcal{C}^\bullet \leqslant m$ for all $m \in \mathbb{Z}$, where by convention $\dim_{\mathbb{C}}\emptyset = -\infty$. Write $\mathrm{Perv}(X)$ for the full subcategory of perverse sheaves in $D_c^b(X)$. Then $\mathrm{Perv}(X)$ is an abelian category, the heart of a t-structure on $D_c^b(X)$.

Perverse sheaves have the following properties:

Theorem 2.6. (a) *If A is a field then $\mathrm{Perv}(X)$ is noetherian and artinian.*

(b) *If A is a field then $\mathbb{D}_X : D_c^b(X) \to D_c^b(X)$ maps $\mathrm{Perv}(X)$ to $\mathrm{Perv}(X)$.*

(c) *If $i : X \hookrightarrow Y$ is inclusion of a closed $\mathbb{C}$-subscheme, then $Ri_*, Ri_!$ (which are naturally isomorphic) map $\mathrm{Perv}(X)$ to $\mathrm{Perv}(Y)$.*

Write $\mathrm{Perv}(Y)_X$ for the full subcategory of objects in $\mathrm{Perv}(Y)$ supported on X. Then $Ri_* \cong Ri_!$ are equivalences of categories $\mathrm{Perv}(X) \xrightarrow{\sim} \mathrm{Perv}(Y)_X$. The restrictions $i^*|_{\mathrm{Perv}(Y)_X}$, $i^!|_{\mathrm{Perv}(Y)_X}$ which map $\mathrm{Perv}(Y)_X$ to $\mathrm{Perv}(X)$, are naturally isomorphic, and are quasi-inverses for

$$Ri_*, Ri_! : \mathrm{Perv}(X) \to \mathrm{Perv}(Y)_X.$$

(d) *If $f : X \to Y$ is étale then f^* and $f^!$ (which are naturally isomorphic) map $\mathrm{Perv}(Y)$ to $\mathrm{Perv}(X)$. More generally, if $f : X \to Y$ is smooth of relative dimension d, then $f^*[d] \cong f^![-d]$ map $\mathrm{Perv}(Y)$ to $\mathrm{Perv}(X)$.*

(e) $\overset{L}{\boxtimes} : D_c^b(X) \times D_c^b(Y) \to D_c^b(X \times Y)$ *maps $\mathrm{Perv}(X) \times \mathrm{Perv}(Y)$ to $\mathrm{Perv}(X \times Y)$.*

(f) *Let U be a smooth $\mathbb{C}$-scheme. Then $A_U[\dim U]$ is perverse, where A_U is the constant sheaf on U with fibre A, and $[\dim U]$ means shift by $\dim U$ in the triangulated category $D_c^b(X)$. Note that Theorem 2.4(v) gives a canonical isomorphism $\mathbb{D}_U\big(A_U[\dim U]\big) \cong A_U[\dim U]$.*

When $A = \mathbb{Q}$, so that $\mathrm{Perv}(X)$ is noetherian and artinian by Theorem 2.6(a), the simple objects in $\mathrm{Perv}(X)$ admit a complete description: they are all isomorphic to intersection cohomology complexes $IC_{\bar{V}}(\mathcal{L})$ for $V \subseteq X$ a smooth locally closed $\mathbb{C}$-subscheme and $\mathcal{L} \to V$ an irreducible $\mathbb{Q}$-local system, [14, §5.4]. Furthermore, if $f : X \to Y$ is a proper morphism of $\mathbb{C}$-schemes, then the Decomposition Theorem [5, 6.2.5], [14, Th. 5.4.10], [45, Cor. 3] says that, in case $IC_{\bar{V}}(\mathcal{L})$ is of geometric origin, $Rf_*(IC_{\bar{V}}(\mathcal{L}))$ is isomorphic to a finite direct sum of shifts of simple objects $IC_{\bar{V}'}(\mathcal{L}')$ in $\mathrm{Perv}(Y)$.

The next theorem is proved by Beilinson et al. [5, Cor. 2.1.23, §2.2.19, & Th. 3.2.4]. The analogue for $D_c^b(X)$ or $D(X)$ rather than $\mathrm{Perv}(X)$ is false. One moral is that perverse sheaves behave like sheaves, rather than like complexes.

Theorem 2.7(i) will be used throughout §3–§6. Theorem 2.7(ii) will be used only once, in the proof of Theorem 6.9 in §6.3, and we only need Theorem 2.7(ii) to hold in the Zariski topology, rather than the étale topology.

Theorem 2.7. *Let X be a $\mathbb{C}$-scheme. Then perverse sheaves on X form a **stack** (a kind of sheaf of categories) on X in the étale topology.*

*Explicitly, this means the following. Let $\{u_i : U_i \to X\}_{i \in I}$ be an **étale open cover** for X, so that $u_i : U_i \to X$ is an étale morphism of $\mathbb{C}$-schemes for $i \in I$ with $\coprod_i u_i$ surjective. Write $U_{ij} = U_i \times_{u_i, X, u_j} U_j$ for $i, j \in I$ with projections*

$$\pi_{ij}^i : U_{ij} \longrightarrow U_i, \quad \pi_{ij}^j : U_{ij} \longrightarrow U_j, \quad u_{ij} = u_i \circ \pi_{ij}^i = u_j \circ \pi_{ij}^j : U_{ij} \longrightarrow X.$$

Similarly, write $U_{ijk} = U_i \times_X U_j \times_X U_k$ for $i, j, k \in I$ with projections

$$\pi_{ijk}^{ij} : U_{ijk} \longrightarrow U_{ij}, \quad \pi_{ijk}^{ik} : U_{ijk} \longrightarrow U_{ik}, \quad \pi_{ijk}^{jk} : U_{ijk} \longrightarrow U_{jk},$$

$$\pi_{ijk}^i : U_{ijk} \longrightarrow U_i, \ \pi_{ijk}^j : U_{ijk} \longrightarrow U_j, \ \pi_{ijk}^k : U_{ijk} \longrightarrow U_k, \ u_{ijk} : U_{ijk} \longrightarrow X,$$

so that $\pi^i_{ijk} = \pi^i_{ij} \circ \pi^{ij}_{ijk}$, $u_{ijk} = u_{ij} \circ \pi^{ij}_{ijk} = u_i \circ \pi^i_{ijk}$, and so on. All these morphisms $u_i, \pi^i_{ij}, \ldots, u_{ijk}$ are étale, so by Theorem 2.6(d) $u_i^* \cong u_i^!$ maps $\mathrm{Perv}(X) \to \mathrm{Perv}(U_i)$, and similarly for $\pi^i_{ij}, \ldots, u_{ijk}$. With this notation:

(i) Suppose $\mathcal{P}^\bullet, \mathcal{Q}^\bullet \in \mathrm{Perv}(X)$, and we are given $\alpha_i : u_i^*(\mathcal{P}^\bullet) \to u_i^*(\mathcal{Q}^\bullet)$ in $\mathrm{Perv}(U_i)$ for all $i \in I$ such that for all $i, j \in I$ we have

$$(\pi^i_{ij})^*(\alpha_i) = (\pi^i_{ij})^*(\alpha_j) : u_{ij}^*(\mathcal{P}^\bullet) \longrightarrow u_{ij}^*(\mathcal{Q}^\bullet).$$

Then there is a unique $\alpha : \mathcal{P}^\bullet \to \mathcal{Q}^\bullet$ in $\mathrm{Perv}(X)$ with $\alpha_i = u_i^*(\alpha)$ for all $i \in I$.

(ii) Suppose we are given objects $\mathcal{P}_i^\bullet \in \mathrm{Perv}(U_i)$ for all $i \in I$ and isomorphisms

$$\alpha_{ij} : (\pi^i_{ij})^*(\mathcal{P}_i^\bullet) \longrightarrow (\pi^j_{ij})^*(\mathcal{P}_j^\bullet)$$

in $\mathrm{Perv}(U_{ij})$ for all $i, j \in I$ with $\alpha_{ii} = \mathrm{id}$ and

$$(\pi^{jk}_{ijk})^*(\alpha_{jk}) \circ (\pi^{ij}_{ijk})^*(\alpha_{ij}) = (\pi^{ik}_{ijk})^*(\alpha_{ik}) : (\pi^i_{ijk})^*(\mathcal{P}_i) \longrightarrow (\pi^k_{ijk})^*(\mathcal{P}_k)$$

in $\mathrm{Perv}(U_{ijk})$ for all $i, j, k \in I$. Then there exists $\mathcal{P}^\bullet$ in $\mathrm{Perv}(X)$, unique up to canonical isomorphism, with isomorphisms $\beta_i : u_i^*(\mathcal{P}^\bullet) \to \mathcal{P}_i^\bullet$ for each $i \in I$, satisfying

$$\alpha_{ij} \circ (\pi^i_{ij})^*(\beta_i) = (\pi^j_{ij})^*(\beta_j) : u_{ij}^*(\mathcal{P}^\bullet) \longrightarrow (\pi^j_{ij})^*(\mathcal{P}_j^\bullet),$$

for all $i, j \in I$.

We will need the following proposition in §3.3 to prove Theorem 3.1(b). Most of it is setting up notation, only the last part $\alpha|_{X'} = \beta|_{X'}$ is nontrivial.

Proposition 2.8. Let W, X be $\mathbb{C}$-schemes, $x \in X$, and $\pi_\mathbb{C} : W \to \mathbb{C}$, $\pi_X : W \to X$, $\iota : \mathbb{C} \to W$ morphisms, such that $\pi_\mathbb{C} \times \pi_X : W \to \mathbb{C} \times X$ is étale, and $\pi_\mathbb{C} \circ \iota = \mathrm{id}_\mathbb{C} : \mathbb{C} \to \mathbb{C}$, and $\pi_X \circ \iota(t) = x$ for all $t \in \mathbb{C}$. Write $W_t = \pi_\mathbb{C}^{-1}(t) \subset W$ for each $t \in \mathbb{C}$, and $j_t : W_t \hookrightarrow W$ for the inclusion. Then $\pi_X|_{W_t} = \pi_X \circ j_t : W_t \to X$ is étale, and $\iota(t) \in W_t$ with $\pi_X|_{W_t}(\iota(t)) = x$, so we may think of W_t for $t \in \mathbb{C}$ as a 1-parameter family of étale open neighbourhoods of x in X.

Let $\mathcal{P}^\bullet, \mathcal{Q}^\bullet \in \mathrm{Perv}(X)$, so that by Theorem 2.6(d) as π_X is smooth of relative dimension 1 and $\pi_X|_{W_t}$ is étale, we have $\pi_X^*[1](\mathcal{P}^\bullet) \in \mathrm{Perv}(W)$ and

$$\pi_X|_{W_t}^*(\mathcal{P}^\bullet) = j_t^*[-1]\big(\pi_X^*[1](\mathcal{P}^\bullet)\big) \in \mathrm{Perv}(W_t),$$

and similarly for $\mathcal{Q}^\bullet$.

Suppose $\alpha, \beta : \mathcal{P}^\bullet \to \mathcal{Q}^\bullet$ in $\mathrm{Perv}(X)$ and $\gamma : \pi_X^*[1](\mathcal{P}^\bullet) \to \pi_X^*[1](\mathcal{Q}^\bullet)$ in $\mathrm{Perv}(W)$ are morphisms such that $\pi_X|_{W_0}^*(\alpha) = j_0^*[-1](\gamma)$ in $\mathrm{Perv}(W_0)$ and $\pi_X|_{W_1}^*(\beta) = j_1^*[-1](\gamma)$ in $\mathrm{Perv}(W_1)$. Then there exists a Zariski open neighbourhood X' of x in X such that

$$\alpha|_{X'} = \beta|_{X'} : \mathcal{P}^\bullet|_{X'} \longrightarrow \mathcal{Q}^\bullet|_{X'}.$$

Here we should think of $j_t^*[-1](\gamma)$ for $t \in \mathbb{C}$ as a family of perverse sheaf morphisms $\mathcal{P}^\bullet \to \mathcal{Q}^\bullet$, defined near x in X locally in the étale topology. But morphisms of perverse sheaves are discrete (to see this, note that we can take $A = \mathbb{Z}$), so as $j_t^*[-1](\gamma)$ depends continuously on t, it should be locally constant in t near x, in a suitable sense. The conclusion $\alpha|_{X'} = \beta|_{X'}$ essentially says that $j_0^*[-1](\gamma) = j_1^*[-1](\gamma)$ near x.

If $P \to X$ is a principal $\mathbb{Z}/2\mathbb{Z}$-bundle on a $\mathbb{C}$-scheme X, and $\mathcal{Q}^\bullet \in \mathrm{Perv}(X)$, we will define a perverse sheaf $\mathcal{Q}^\bullet \otimes_{\mathbb{Z}/2\mathbb{Z}} P$, which will be important in §5–§6.

Definition 2.9. Let X be a $\mathbb{C}$-scheme. A principal $\mathbb{Z}/2\mathbb{Z}$-bundle $P \to X$ is a proper, surjective, étale morphism of $\mathbb{C}$-schemes $\pi : P \to X$ together with a free involution $\sigma : P \to P$, such that the orbits of $\mathbb{Z}/2\mathbb{Z} = \{1, \sigma\}$ are the fibres of π. We will use the ideas of *isomorphism* of principal

bundles $\iota : P \to P'$, *section* $s : X \to P$, *tensor product* $P \otimes_{\mathbb{Z}/2\mathbb{Z}} P'$, and *pullback* $f^*(P) \to W$ under a $\mathbb{C}$-scheme morphism $f : W \to X$, all of which are defined in the obvious ways.

Let $P \to X$ be a principal $\mathbb{Z}/2\mathbb{Z}$-bundle. Write $\mathcal{L}_P \in D^b_c(X)$ for the rank one A-local system on X induced from P by the nontrivial representation of $\mathbb{Z}/2\mathbb{Z} \cong \{\pm 1\}$ on A. It is characterized by $\pi_*(A_P) \cong A_X \oplus \mathcal{L}_P$. For each $\mathcal{Q}^\bullet \in D^b_c(X)$, write $\mathcal{Q}^\bullet \otimes_{\mathbb{Z}/2\mathbb{Z}} P \in D^b_c(X)$ for $\mathcal{Q}^\bullet \overset{L}{\otimes} \mathcal{L}_P$, and call it $\mathcal{Q}^\bullet$ *twisted by* P. If $\mathcal{Q}^\bullet$ is perverse then $\mathcal{Q}^\bullet \otimes_{\mathbb{Z}/2\mathbb{Z}} P$ is perverse.

Perverse sheaves and complexes twisted by principal $\mathbb{Z}/2\mathbb{Z}$-bundles have the obvious functorial behaviour. For example, if $P \to X$, $P' \to X$ are principal $\mathbb{Z}/2\mathbb{Z}$-bundles and $\mathcal{Q}^\bullet \in D^b_c(X)$ there is a canonical isomorphism $(\mathcal{Q}^\bullet \otimes_{\mathbb{Z}/2\mathbb{Z}} P) \otimes_{\mathbb{Z}/2\mathbb{Z}} P' \cong \mathcal{Q}^\bullet \otimes_{\mathbb{Z}/2\mathbb{Z}} (P \otimes_{\mathbb{Z}/2\mathbb{Z}} P')$, and if $f : W \to X$ is a $\mathbb{C}$-scheme morphism there is a canonical isomorphism

$$f^*(\mathcal{Q}^\bullet \otimes_{\mathbb{Z}/2\mathbb{Z}} P) \cong f^*(\mathcal{Q}^\bullet) \otimes_{\mathbb{Z}/2\mathbb{Z}} f^*(P).$$

2.3. **Nearby cycles and vanishing cycles on $\mathbb{C}$-schemes.** We explain nearby cycles and vanishing cycles, as in Dimca [14, §4.2]. The definition is complex analytic, $\widetilde{X^{\mathrm{an}}_*}, \widetilde{\mathbb{C}^*}$ in (2.2) do not come from $\mathbb{C}$-schemes.

Definition 2.10. Let X be a $\mathbb{C}$-scheme, and let $f : X \to \mathbb{C}$ be a regular function. Define $X_0 = f^{-1}(0)$, as a $\mathbb{C}$-subscheme of X, and $X_* = X \setminus X_0$. Consider the commutative diagram of complex analytic spaces:

$$
\begin{array}{ccccccc}
X^{\mathrm{an}}_0 & \xrightarrow{\;\;i\;\;} & X^{\mathrm{an}} & \xleftarrow{\;\;j\;\;} & X^{\mathrm{an}}_* & \xleftarrow{\;\;p\;\;} & \widetilde{X^{\mathrm{an}}_*} \\
\downarrow{f} & & \downarrow{f} & & \downarrow{f} & & \downarrow{\tilde{f}} \\
\{0\} & \longrightarrow & \mathbb{C} & \longleftarrow & \mathbb{C}^* & \xleftarrow{\;\;\rho\;\;} & \widetilde{\mathbb{C}^*}.
\end{array}
\tag{2.2}
$$

with an arc labelled π from $\widetilde{X^{\mathrm{an}}_*}$ over X^{an}_* to X^{an}.

Here $X^{\mathrm{an}}, X^{\mathrm{an}}_0, X^{\mathrm{an}}_*$ are the complex analytic spaces associated to the $\mathbb{C}$-schemes X_0, X, X_*, and $i : X^{\mathrm{an}}_0 \hookrightarrow X^{\mathrm{an}}$, $j : X^{\mathrm{an}}_* \hookrightarrow X^{\mathrm{an}}$ are the inclusions, $\rho : \widetilde{\mathbb{C}^*} \to \mathbb{C}^*$ is the universal cover of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and $\widetilde{X^{\mathrm{an}}_*} = X^{\mathrm{an}}_* \times_{f,\mathbb{C}^*,\rho} \widetilde{\mathbb{C}^*}$ the corresponding cover of X^{an}_*, with covering map $p : \widetilde{X^{\mathrm{an}}_*} \to X^{\mathrm{an}}_*$, and $\pi = j \circ p$.

As in §2.6, the triangulated categories $D(X), D^b_c(X)$ and six operations f^*, $f^!$, Rf_*, $Rf_!$, $\mathcal{R}Hom, \overset{L}{\otimes}$ also make sense for complex analytic spaces. So we can define the *nearby cycle functor* $\psi_f : D^b_c(X) \to D^b_c(X_0)$ to be $\psi_f = i^* \circ R\pi_* \circ \pi^*$. Since this definition goes via $\widetilde{X^{\mathrm{an}}_*}$ which is not a $\mathbb{C}$-scheme, it is not obvious that ψ_f maps to (algebraically) constructible complexes $D^b_c(X_0)$ rather than just to $D(X_0)$, but it does [14, p. 103], [27, p. 352].

There is a natural transformation $\Xi : i^* \Rightarrow \psi_f$ between the functors

$$i^*, \psi_f : D^b_c(X) \longrightarrow D^b_c(X_0).$$

The *vanishing cycle functor* $\phi_f : D^b_c(X) \to D^b_c(X_0)$ is a functor such that for every $\mathcal{C}^\bullet$ in $D^b_c(X)$ we have a distinguished triangle

$$i^*(\mathcal{C}^\bullet) \xrightarrow{\;\Xi(\mathcal{C}^\bullet)\;} \psi_f(\mathcal{C}^\bullet) \longrightarrow \phi_f(\mathcal{C}^\bullet) \xrightarrow{\;[+1]\;} i^*(\mathcal{C}^\bullet)$$

in $D^b_c(X_0)$. Following Dimca [14, p. 108], we write ψ^p_f, ϕ^p_f for the shifted functors $\psi_f[-1]$, $\phi_f[-1] : D^b_c(X) \to D^b_c(X_0)$.

The generator of $\mathbb{Z} = \pi_1(\mathbb{C}^*)$ on $\widetilde{\mathbb{C}^*}$ induces a deck transformation $\delta_{\mathbb{C}^*} : \widetilde{\mathbb{C}^*} \to \widetilde{\mathbb{C}^*}$ which lifts to a deck transformation $\delta_{X^*} : \widetilde{X^*} \to \widetilde{X^*}$ with $p \circ \delta_{X^*} = p$ and $\tilde{f} \circ \delta_{X^*} = \delta_{\mathbb{C}^*} \circ \tilde{f}$. As in [14, p. 103,

p. 105], we can use δ_{X^*} to define natural transformations $M_{X,f} : \psi_f^p \Rightarrow \psi_f^p$ and $M_{X,f} : \phi_f^p \Rightarrow \phi_f^p$, called *monodromy*.

Alternative definitions of ψ_f, ϕ_f in terms of specialization and microlocalization functors are given by Kashiwara and Schapira [27, Prop. 8.6.3]. Here are some properties of nearby and vanishing cycles. Parts (i),(ii) can be found in Dimca [14, Th. 5.2.21 & Prop. 4.2.11]. Part (iv) is proved by Massey [37]; compare also Proposition A.1 in the Appendix.

Theorem 2.11. (i) *If X is a $\mathbb{C}$-scheme and $f : X \to \mathbb{C}$ is regular, then the functors $\psi_f^p, \phi_f^p :$ $D_c^b(X) \to D_c^b(X_0)$ both map $\mathrm{Perv}(X)$ to $\mathrm{Perv}(X_0)$.*

(ii) *Let $\Phi : X \to Y$ be a proper morphism of $\mathbb{C}$-schemes, and $g : Y \to \mathbb{C}$ be regular. Write $f = g \circ \Phi : X \to \mathbb{C}$, $X_0 = f^{-1}(0) \subseteq X$, $Y_0 = g^{-1}(0) \subseteq Y$, and $\Phi_0 = \Phi|_{X_0} : X_0 \to Y_0$. Then we have natural isomorphisms*

$$R(\Phi_0)_* \circ \psi_f^p \cong \psi_g^p \circ R\Phi_* \quad and \quad R(\Phi_0)_* \circ \phi_f^p \cong \phi_g^p \circ R\Phi_*. \tag{2.3}$$

Note too that $R\Phi_* \cong R\Phi_!$ and $R(\Phi_0)_* \cong R(\Phi_0)_!$, as Φ, Φ_0 are proper.

(iii) *Let $\Phi : X \to Y$ be an étale morphism of $\mathbb{C}$-schemes, and $g : Y \to \mathbb{C}$ be regular. Write $f = g \circ \Phi : X \to \mathbb{C}$, $X_0 = f^{-1}(0) \subseteq X$, $Y_0 = g^{-1}(0) \subseteq Y$, and $\Phi_0 = \Phi|_{X_0} : X_0 \to Y_0$. Then we have natural isomorphisms*

$$\Phi_0^* \circ \psi_f^p \cong \psi_g^p \circ \Phi^* \quad and \quad \Phi_0^* \circ \phi_f^p \cong \phi_g^p \circ \Phi^*. \tag{2.4}$$

Note too that $\Phi^* \cong \Phi^!$ and $\Phi_0^* \cong \Phi_0^!$, as Φ, Φ_0 are étale.

More generally, if $\Phi : X \to Y$ is smooth of relative (complex) dimension d and g, f, X_0, Y_0, Φ_0 are as above, then we have natural isomorphisms

$$\Phi_0^*[d] \circ \psi_f^p \cong \psi_g^p \circ \Phi^*[d] \quad and \quad \Phi_0^*[d] \circ \phi_f^p \cong \phi_g^p \circ \Phi^*[d]. \tag{2.5}$$

Note too that $\Phi^*[d] \cong \Phi^![-d]$ and $\Phi_0^*[d] \cong \Phi_0^![-d]$.

(iv) *If X is a $\mathbb{C}$-scheme and $f : X \to \mathbb{C}$ is regular, then there are natural isomorphisms $\psi_f^p \circ \mathbb{D}_X \cong \mathbb{D}_{X_0} \circ \psi_f^p$ and $\phi_f^p \circ \mathbb{D}_X \cong \mathbb{D}_{X_0} \circ \phi_f^p$.*

2.4. **Perverse sheaves of vanishing cycles on $\mathbb{C}$-schemes.** We can now define the main subject of this paper, the perverse sheaf of vanishing cycles $\mathcal{PV}_{U,f}^\bullet$ for a regular function $f : U \to \mathbb{C}$.

Definition 2.12. Let U be a smooth $\mathbb{C}$-scheme, and $f : U \to \mathbb{C}$ a regular function. Write $X = \mathrm{Crit}(f)$, as a closed $\mathbb{C}$-subscheme of U.

Then as a map of topological spaces, $f|_X : X \to \mathbb{C}$ is locally constant, with finite image $f(X)$, so we have a decomposition $X = \coprod_{c \in f(X)} X_c$, for $X_c \subseteq X$ the open and closed $\mathbb{C}$-subscheme with $f(x) = c$ for each $\mathbb{C}$-point $x \in X_c$.

(Note that if X is non-reduced, then $f|_X : X \to \mathbb{C}$ need not be locally constant as a morphism of $\mathbb{C}$-schemes, but $f|_{X^{\mathrm{red}}} : X^{\mathrm{red}} \to \mathbb{C}$ is locally constant, where X^{red} is the reduced $\mathbb{C}$-subscheme of X. Since X, X^{red} have the same topological space, $f|_X : X \to \mathbb{C}$ is locally constant on topological spaces.)

For each $c \in \mathbb{C}$, write $U_c = f^{-1}(c) \subseteq U$. Then as in §2.3, we have a vanishing cycle functor $\phi_{f-c}^p : \mathrm{Perv}(U) \to \mathrm{Perv}(U_c)$. So we may form $\phi_{f-c}^p(A_U[\dim U])$ in $\mathrm{Perv}(U_c)$, since $A_U[\dim U] \in \mathrm{Perv}(U)$ by Theorem 2.6(f). One can show $\phi_{f-c}^p(A_U[\dim U])$ is supported on the closed subset $X_c = \mathrm{Crit}(f) \cap U_c$ in U_c, where $X_c = \emptyset$ unless $c \in f(X)$. That is, $\phi_{f-c}^p(A_U[\dim U])$ lies in $\mathrm{Perv}(U_c)_{X_c}$.

But Theorem 2.6(c) says $\mathrm{Perv}(U_c)_{X_c}$ and $\mathrm{Perv}(X_c)$ are equivalent categories, so we may regard $\phi^p_{f-c}(A_U[\dim U])$ as a perverse sheaf on X_c. That is, we can consider

$$\phi^p_{f-c}(A_U[\dim U])|_{X_c} = i^*_{X_c,U_c}\big(\phi^p_{f-c}(A_U[\dim U])\big)$$

in $\mathrm{Perv}(X_c)$, where $i_{X_c,U_c} : X_c \to U_c$ is the inclusion morphism.

As $X = \coprod_{c \in f(X)} X_c$ with each X_c open and closed in X, we have

$$\mathrm{Perv}(X) = \bigoplus_{c \in f(X)} \mathrm{Perv}(X_c).$$

Define the *perverse sheaf of vanishing cycles* $\mathcal{PV}^\bullet_{U,f}$ of U, f in $\mathrm{Perv}(X)$ to be

$$\mathcal{PV}^\bullet_{U,f} = \bigoplus_{c \in f(X)} \phi^p_{f-c}(A_U[\dim U])|_{X_c}.$$

That is, $\mathcal{PV}^\bullet_{U,f}$ is the unique perverse sheaf on $X = \mathrm{Crit}(f)$ with

$$\mathcal{PV}^\bullet_{U,f}|_{X_c} = \phi^p_{f-c}(A_U[\dim U])|_{X_c}$$

for all $c \in f(X)$.

Under Verdier duality, we have $A_U[\dim U] \cong \mathbb{D}_U(A_U[\dim U])$ by Theorem 2.6(f), so

$$\phi^p_{f-c}(A_U[\dim U]) \cong \mathbb{D}_{U_c}\big(\phi^p_{f-c}(A_U[\dim U])\big)$$

by Theorem 2.11(iv). Applying $i^*_{X_c,U_c}$ and using $\mathbb{D}_{X_c} \circ i^*_{X_c,U_c} \cong i^!_{X_c,U_c} \circ \mathbb{D}_{U_c}$ by Theorem 2.4(iv) and $i^!_{X_c,U_c} \cong i^*_{X_c,U_c}$ on $\mathrm{Perv}(U_c)_{X_c}$ by Theorem 2.6(c) also gives

$$\phi^p_{f-c}(A_U[\dim U])|_{X_c} \cong \mathbb{D}_{X_c}\big(\phi^p_{f-c}(A_U[\dim U])|_{X_c}\big).$$

Summing over all $c \in f(X)$ yields a canonical isomorphism

$$\sigma_{U,f} : \mathcal{PV}^\bullet_{U,f} \overset{\cong}{\longrightarrow} \mathbb{D}_X(\mathcal{PV}^\bullet_{U,f}). \tag{2.6}$$

For $c \in f(X)$, we have a monodromy operator

$$M_{U,f-c} : \phi^p_{f-c}(A_U[\dim U]) \longrightarrow \phi^p_{f-c}(A_U[\dim U]),$$

which restricts to $\phi^p_{f-c}(A_U[\dim U])|_{X_c}$. Define the *twisted monodromy operator*

$$\tau_{U,f} : \mathcal{PV}^\bullet_{U,f} \longrightarrow \mathcal{PV}^\bullet_{U,f}$$

by

$$\tau_{U,f}|_{X_c} = (-1)^{\dim U} M_{U,f-c}|_{X_c} : \phi^p_{f-c}(A_U[\dim U])|_{X_c} \longrightarrow \phi^p_{f-c}(A_U[\dim U])|_{X_c}, \tag{2.7}$$

for each $c \in f(X)$.

Here 'twisted' refers to the sign $(-1)^{\dim U}$ in (2.7). We include this sign change as it makes monodromy act naturally under transformations which change dimension — without it, equation (5.15) below would only commute up to a sign $(-1)^{\dim V - \dim U}$, not commute — and it normalizes the monodromy of any nondegenerate quadratic form to be the identity, as in (2.13). The sign $(-1)^{\dim U}$ also corresponds to the twist '$(\frac{1}{2} \dim U)$' in the definition (2.24) of the mixed Hodge module of vanishing cycles $\mathcal{HV}^\bullet_{U,f}$ in §2.10.

The (compactly-supported) hypercohomology $\mathbb{H}^*(\mathcal{PV}^\bullet_{U,f}), \mathbb{H}^*_c(\mathcal{PV}^\bullet_{U,f})$ from (2.1) is an important invariant of U, f. If A is a field then the isomorphism $\sigma_{U,f}$ in (2.6) implies that $\mathbb{H}^k(\mathcal{PV}^\bullet_{U,f}) \cong \mathbb{H}^{-k}_c(\mathcal{PV}^\bullet_{U,f})^*$, a form of Poincaré duality.

We defined $\mathcal{PV}_{U,f}^\bullet$ in perverse sheaves over a base ring A. Writing $\mathcal{PV}_{U,f}^\bullet(A)$ to denote the base ring, one can show that

$$\mathcal{PV}_{U,f}^\bullet(A) \cong \mathcal{PV}_{U,f}^\bullet(\mathbb{Z}) \overset{L}{\otimes}_{\mathbb{Z}} A.$$

Thus, we may as well take $A = \mathbb{Z}$, or $A = \mathbb{Q}$ if we want A to be a field, since the case of general A contains no more information.

There is a Thom–Sebastiani Theorem for perverse sheaves, due to Massey [35] and Schürmann [50, Cor. 1.3.4]. Applied to $\mathcal{PV}_{U,f}^\bullet$, it yields:

Theorem 2.13. *Let U, V be smooth $\mathbb{C}$-schemes and $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ be regular, so that $f \boxplus g : U \times V \to \mathbb{C}$ is regular with $(f \boxplus g)(u, v) := f(u) + g(v)$. Set $X = \mathrm{Crit}(f)$ and $Y = \mathrm{Crit}(g)$ as $\mathbb{C}$-subschemes of U, V, so that $\mathrm{Crit}(f \boxplus g) = X \times Y$. Then there is a natural isomorphism*

$$\mathcal{TS}_{U,f,V,g} : \mathcal{PV}_{U \times V, f \boxplus g}^\bullet \longrightarrow \mathcal{PV}_{U,f}^\bullet \overset{L}{\boxtimes} \mathcal{PV}_{V,g}^\bullet \tag{2.8}$$

in $\mathrm{Perv}(X \times Y)$, such that the following diagrams commute:

$$\tag{2.9}$$

$$\tag{2.10}$$

The next example will be important later.

Example 2.14. Define $f : \mathbb{C}^n \to \mathbb{C}$ by

$$f(z_1, \ldots, z_n) = z_1^2 + \cdots + z_n^2$$

for $n > 1$. Then $\mathrm{Crit}(f) = \{0\}$, so $\mathcal{PV}_{\mathbb{C}^n, z_1^2 + \cdots + z_n^2}^\bullet = \phi_f^p(A_{\mathbb{C}^n}[n])|_{\{0\}}$ is a perverse sheaf on the point $\{0\}$. Following Dimca [14, Prop. 4.2.2, Ex. 4.2.3 & Ex. 4.2.6], we find that there is a canonical isomorphism

$$\mathcal{PV}_{\mathbb{C}^n, z_1^2 + \cdots + z_n^2}^\bullet \cong H^{n-1}\big(MF_f(0); A\big) \otimes_A A_{\{0\}}, \tag{2.11}$$

where $MF_f(0)$ is the *Milnor fibre* of f at 0, as in [14, p. 103]. Since $f(z) = z_1^2 + \cdots + z_n^2$ is homogeneous, we see that

$$MF_f(0) \cong \big\{(z_1, \ldots, z_n) \in \mathbb{C}^n : f(z_1, \ldots, z_n) = 1\big\} \cong T^*\mathcal{S}^{n-1},$$

so that $H^{n-1}\big(MF_f(0); A\big) \cong H^{n-1}\big(\mathcal{S}^{n-1}; A\big) \cong A$. Therefore we have

$$\mathcal{PV}_{\mathbb{C}^n, z_1^2 + \cdots + z_n^2}^\bullet \cong A_{\{0\}}. \tag{2.12}$$

This isomorphism (2.12) is *natural up to sign* (unless the base ring A has characteristic 2, in which case (2.12) is natural), as it depends on the choice of isomorphism $H^{n-1}(\mathcal{S}^{n-1}, A) \cong A$, which corresponds to an orientation for $\mathcal{S}^{n-1}$. This uncertainty of signs will be important in §5–§6.

We can also use Milnor fibres to compute the monodromy operator on $\mathcal{PV}_{\mathbb{C}^n, z_1^2 + \cdots + z_n^2}^\bullet$. There is a monodromy map $\mu_f : MF_f(0) \to MF_f(0)$, natural up to isotopy, which is the monodromy

in the Milnor fibration of f at 0. Under the identification $MF_f(0) \cong T^*\mathcal{S}^{n-1}$ we may take μ_f to be the map $\mathrm{d}(-1) : T^*\mathcal{S}^{n-1} \to T^*\mathcal{S}^{n-1}$ induced by $-1 : \mathcal{S}^{n-1} \to \mathcal{S}^{n-1}$ mapping

$$-1 : (x_1, \ldots, x_n) \mapsto (-x_1, \ldots, -x_n).$$

This multiplies orientations on $\mathcal{S}^{n-1}$ by $(-1)^n$. Thus, $\mu_{f*} : H^{n-1}(\mathcal{S}^{n-1}, A) \to H^{n-1}(\mathcal{S}^{n-1}, A)$ multiplies by $(-1)^n$.

By [14, Prop. 4.2.2], equation (2.11) identifies the action of the monodromy operator $M_{\mathbb{C}^n, f}|_{\{0\}}$ on $\mathcal{PV}_{\mathbb{C}^n, z_1^2 + \cdots + z_n^2}^\bullet$ with the action of μ_{f*} on $H^{n-1}(\mathcal{S}^{n-1}, A)$. So $M_{\mathbb{C}^n, f}|_{\{0\}}$ is multiplication by $(-1)^n$. Combining this with the sign change $(-1)^{\dim U}$ in (2.7) for $U = \mathbb{C}^n$ shows that the twisted monodromy is

$$\tau_{\mathbb{C}^n, z_1^2 + \cdots + z_n^2} = \mathrm{id} : \mathcal{PV}_{\mathbb{C}^n, z_1^2 + \cdots + z_n^2}^\bullet \longrightarrow \mathcal{PV}_{\mathbb{C}^n, z_1^2 + \cdots + z_n^2}^\bullet. \tag{2.13}$$

Equations (2.12)–(2.13) also hold for $n = 0, 1$, though (2.11) does not.

Note also that these results are compatible with the Thom–Sebastiani Theorem 2.13, and can be deduced from it and the case $n = 1$.

We introduce some notation for pullbacks of $\mathcal{PV}_{V,g}^\bullet$ by étale morphisms.

Definition 2.15. Let U, V be smooth $\mathbb{C}$-schemes, $\Phi : U \to V$ an étale morphism, and $g : V \to \mathbb{C}$ a regular function. Write $f = g \circ \Phi : U \to \mathbb{C}$, and $X = \mathrm{Crit}(f)$, $Y = \mathrm{Crit}(g)$ as $\mathbb{C}$-subschemes of U, V. Then $\Phi|_X : X \to Y$ is étale. Define an isomorphism

$$\mathcal{PV}_\Phi : \mathcal{PV}_{U,f}^\bullet \longrightarrow \Phi|_X^* (\mathcal{PV}_{V,g}^\bullet) \quad \text{in } \mathrm{Perv}(X) \tag{2.14}$$

by the commutative diagram for each $c \in f(X) \subseteq g(Y)$:

$$\begin{array}{ccc} \mathcal{PV}_{U,f}^\bullet|_{X_c} = \phi_{f-c}^p(A_U[\dim U])|_{X_c} & \xrightarrow{\ \alpha\ } & \phi_{f-c}^p \circ \Phi^*(A_V[\dim V]))|_{X_c} \\ \Big\downarrow{\scriptstyle \mathcal{PV}_\Phi|_{X_c}} & & \beta\Big\downarrow \\ \Phi|_{X_c}^* (\mathcal{PV}_{V,g}^\bullet) \ =\!=\!=\!=\!=\!=\!= & & \Phi_0^* \circ \phi_{g-c}^p \circ (A_V[\dim V]))|_{X_c}. \end{array} \tag{2.15}$$

Here α is ϕ_{f-c}^p applied to the canonical isomorphism $A_U \to \Phi^*(A_V)$, noting that, as Φ is étale, $\dim U = \dim V$ and β is induced by (2.4).

By naturality of the isomorphisms α, β in (2.15) we find the following commute, where $\sigma_{U,f}, \tau_{U,f}$ are as in (2.6)–(2.7):

$$\begin{array}{ccc} \mathcal{PV}_{U,f}^\bullet & \xrightarrow{\ \ \ \ \ \sigma_{U,f}\ \ \ \ \ } & \mathbb{D}_X(\mathcal{PV}_{U,f}^\bullet) \\ \Big\downarrow{\scriptstyle \mathcal{PV}_\Phi} & & {\scriptstyle \mathbb{D}_X(\mathcal{PV}_\Phi)}\Big\uparrow \\ \Phi|_X^*(\mathcal{PV}_{V,g}^\bullet) \xrightarrow{\scriptstyle \Phi|_X^*(\sigma_{V,g})} \Phi|_X^*(\mathbb{D}_Y(\mathcal{PV}_{V,g}^\bullet)) \xrightarrow{\ \cong\ } & & \mathbb{D}_X(\Phi|_X^*(\mathcal{PV}_{V,g}^\bullet)), \end{array} \tag{2.16}$$

$$\begin{array}{ccc} \mathcal{PV}_{U,f}^\bullet & \xrightarrow{\ \ \ \ \ \tau_{U,f}\ \ \ \ \ } & \mathcal{PV}_{U,f}^\bullet \\ \Big\downarrow{\scriptstyle \mathcal{PV}_\Phi} & & {\scriptstyle \mathcal{PV}_\Phi}\Big\downarrow \\ \Phi|_X^*(\mathcal{PV}_{V,g}^\bullet) & \xrightarrow{\ \ \Phi|_X^*(\tau_{V,g})\ \ } & \Phi|_X^*(\mathcal{PV}_{V,g}^\bullet). \end{array} \tag{2.17}$$

If $U = V$, $f = g$ and $\Phi = \mathrm{id}_U$ then $\mathcal{PV}_{\mathrm{id}_U} = \mathrm{id}_{\mathcal{PV}_{U,f}^\bullet}$.

If W is another smooth $\mathbb{C}$-scheme, $\Psi : V \to W$ is étale, and $h : W \to \mathbb{C}$ is regular with $g = h \circ \Psi : V \to \mathbb{C}$, then composing (2.15) for Φ with $\Phi|_{X_c}^*$ of (2.15) for Ψ shows that

$$\mathcal{PV}_{\Psi \circ \Phi} = \Phi|_X^*(\mathcal{PV}_\Psi) \circ \mathcal{PV}_\Phi : \mathcal{PV}_{U,f}^\bullet \longrightarrow (\Psi \circ \Phi)|_X^* (\mathcal{PV}_{W,h}^\bullet). \tag{2.18}$$

That is, the isomorphisms $\mathcal{PV}_\Phi$ are functorial.

Example 2.16. In Definition 2.15, set $U = V = \mathbb{C}^n$ and

$$f(z_1, \ldots, z_n) = g(z_1, \ldots, z_n) = z_1^2 + \cdots + z_n^2,$$

so that $Y = Z = \{0\} \subset \mathbb{C}^n$. Let $M \in \mathrm{O}(n, \mathbb{C})$ be an orthogonal matrix, so that $M : \mathbb{C}^n \to \mathbb{C}^n$ is an isomorphism with $f = g \circ M$ and $M|_{\{0\}} = \mathrm{id}_{\{0\}}$. As $M|_Y = \mathrm{id}_Y$, Definition 2.15 defines an isomorphism

$$\mathcal{PV}_M : \mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2 + \cdots + z_n^2} \longrightarrow \mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2 + \cdots + z_n^2}. \tag{2.19}$$

Equation (2.11) describes $\mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2 + \cdots + z_n^2}$ in terms of $MF_f(0) \cong T^*\mathcal{S}^{n-1}$. Now

$$M|_{MF_f(0)} : MF_f(0) \longrightarrow MF_f(0)$$

multiplies orientations on $\mathcal{S}^{n-1}$ by $\det M$, so

$$(M|_{MF_f(0)})_* : H^{n-1}\big(MF_f(0); A\big) \longrightarrow H^{n-1}\big(MF_f(0); A\big)$$

is multiplication by $\det M$. Thus (2.11) implies that $\mathcal{PV}_M$ in (2.19) is multiplication by $\det M = \pm 1$.

2.5. **Summary of the properties we use in this paper.** Since parts of §2.1–§2.4 do not work for the other kinds of perverse sheaves, $\mathscr{D}$-modules and mixed Hodge modules in §2.6–§2.10, we list what we will need for §3–§6, to make it easy to check they are also valid in the settings of §2.6–§2.10.

(i) There should be an A-linear abelian category $\mathcal{P}(X)$ of $\mathcal{P}$-objects defined for each scheme or complex analytic space X, over a fixed, well-behaved base ring A. We do not require A to be a field.

(ii) There should be a Verdier duality functor $\mathbb{D}_X$ with $\mathbb{D}_X \circ \mathbb{D}_X \cong \mathrm{id}$, defined on a suitable subcategory of $\mathcal{P}$-objects on X which includes the objects we are interested in. We do not need $\mathbb{D}_X$ to be defined on all objects in $\mathcal{P}(X)$.

(iii) If U is a smooth scheme or complex manifold, then there should be a canonical object $A_U[\dim U] \in \mathcal{P}(U)$, with a canonical isomorphism

$$\mathbb{D}_U(A_U[\dim U]) \cong A_U[\dim U].$$

(iv) Let $f : X \hookrightarrow Y$ be a closed embedding of schemes or complex analytic spaces; this implies f is proper. Then $f_*, f_! : \mathcal{P}(X) \to \mathcal{P}(Y)$ should exist, inducing an equivalence of categories $\mathcal{P}(X) \xrightarrow{\sim} \mathcal{P}_X(Y)$ as in Theorem 2.6(c), where $\mathcal{P}_X(Y)$ is the full subcategory of objects in $\mathcal{P}(Y)$ supported on X.

(v) Let $f : X \to Y$ be an étale morphism. Then the pullbacks $f^*, f^! : \mathcal{P}(Y) \to \mathcal{P}(X)$ should exist. More generally, if $f : X \to Y$ is smooth of relative dimension d, then there should be pullbacks $f^*[d], f^![-d]$ mapping $\mathcal{P}(Y) \to \mathcal{P}(X)$. If X, Y are smooth, there should be a canonical isomorphism $f^*[d](A_Y[\dim Y]) \cong A_X[\dim X]$. We do not need pullbacks to exist for general morphisms $f : X \to Y$, though see (xi) below.

(vi) An external tensor product $\overset{L}{\boxtimes} : \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y)$ should exist for all X, Y.

(vii) If X is a scheme or complex analytic space, $P \to X$ a principal $\mathbb{Z}/2\mathbb{Z}$-bundle, and $\mathcal{Q}^\bullet \in \mathcal{P}(X)$, the twisted perverse sheaf $\mathcal{Q}^\bullet \otimes_{\mathbb{Z}/2\mathbb{Z}} P \in \mathcal{P}(X)$ should make sense as in Definition 2.9, and have the obvious functorial properties.

(viii) A vanishing cycle functor $\phi_f^p : \mathcal{P}(U) \to \mathcal{P}(U_0)$ and a monodromy transformation

$$M_{U,f} : \phi_f^p \Rightarrow \phi_f^p$$

in §2.4 should exist for all smooth U and regular/holomorphic $f : U \to \mathbb{A}^1$.

(ix) The functors $\mathbb{D}_X, f^*, f^!, f_*, f_!, \phi_f^p$ should satisfy the natural isomorphisms in Theorems 2.4 and 2.11, provided they exist. They should have the obvious compatibilities with $\overset{L}{\boxtimes}$, and restriction to (Zariski) open sets.

(x) There should be suitable subcategories of $\mathcal{P}$-objects which form a stack in the étale or complex analytic topologies, as in Theorem 2.7. In the algebraic case we only need Theorem 2.7(ii) to hold for Zariski open covers, not étale open covers.

(xi) Proposition 2.8 must hold. This involves pullbacks j_t^* by a morphism $j_t : W_t \hookrightarrow W$ which is not étale or smooth, as in (v) above. But on objects we only consider

$$j_t^*\big(\pi_X^*(\mathcal{P}^\bullet)\big) = \pi_X|_{W_t}^*(\mathcal{P}^\bullet)$$

which exists in $\mathcal{P}(W_t)$ by (v) as $\pi_X|_{W_t}$ is étale, so j_t^* is defined on the objects we need.

(xii) There should be a Thom–Sebastiani Theorem for $\mathcal{P}$-objects, so that the analogue of Theorem 2.13 holds.

Remark 2.17. The existence of a (bounded) derived category of $\mathcal{P}$-objects will not be assumed, or used, in this paper. On the other hand, in all the cases we consider, there will be a realization functor from the category of $\mathcal{P}$-objects to an appropriate category of constructible complexes, and the notation used above reflects this. So in (iii),(v) above, [1] does not stand for a shift in any derived category; the notation means a $\mathcal{P}$-object or morphism whose realization is the appropriate constructible object or morphism. See Remark 2.20 below.

2.6. **Perverse sheaves on complex analytic spaces.** Next we discuss perverse sheaves on complex analytic spaces, as in Dimca [14]. The theory follows §2.1–§2.4, replacing (smooth) $\mathbb{C}$-schemes by complex analytic spaces (complex manifolds), and regular functions by holomorphic functions.

Let X be a complex analytic space, always assumed locally of finite type (that is, locally embeddable in $\mathbb{C}^n$). In the analogue of Definition 2.2, we fix a well-behaved commutative ring A, and consider sheaves of A-modules $\mathcal{S}$ on X in the complex analytic topology. A sheaf $\mathcal{S}$ is called (*analytically*) *constructible* if all the stalks $\mathcal{S}_x$ for $x \in X$ are finite type A-modules, and there is a locally finite stratification $X = \coprod_{j \in J} X_j$ of X, where now $X_j \subseteq X$ for $j \in J$ are complex analytic subspaces of X, such that $\mathcal{S}|_{X_j}$ is an A-local system for all $j \in J$.

Write $D(X)$ for the derived category of complexes $\mathcal{C}^\bullet$ of sheaves of A-modules on X, exactly as in §2.1, and $D_c^b(X)$ for the full subcategory of bounded complexes $\mathcal{C}^\bullet$ in $D(X)$ whose cohomology sheaves $\mathcal{H}^m(\mathcal{C}^\bullet)$ are analytically constructible for all $m \in \mathbb{Z}$. Then $D(X), D_c^b(X)$ are triangulated categories.

When we wish to distinguish the complex algebraic and complex analytic theories, we will write $D_c^b(X)^{\mathrm{alg}}, \mathrm{Perv}(X)^{\mathrm{alg}}$ for the algebraic versions in §2.1–§2.2 with X a $\mathbb{C}$-scheme, and $D_c^b(X)^{\mathrm{an}}, \mathrm{Perv}(X)^{\mathrm{an}}$ for the analytic versions.

Here are the main differences between the material of §2.1–§2.4 for perverse sheaves on $\mathbb{C}$-schemes and on complex analytic spaces:

(a) If $f : X \to Y$ is an arbitrary morphism of $\mathbb{C}$-schemes, then as in §2.1 the pushforwards $Rf_*, Rf_! : D(X) \to D(Y)$ also map $D_c^b(X)^{\mathrm{alg}} \to D_c^b(Y)^{\mathrm{alg}}$.

However, if $f : X \to Y$ is a morphism of complex analytic spaces, then

$$Rf_*, Rf_! : D(X) \longrightarrow D(Y)$$

need not map $D_c^b(X)^{\mathrm{an}} \to D_c^b(Y)^{\mathrm{an}}$ without extra assumptions on f, for example, if $f : X \to Y$ is proper.

(b) The analogue of Theorem 2.7 says that perverse sheaves on a complex analytic space X form a stack in the complex analytic topology. This is proved in the subanalytic context

in [27, Th. 10.2.9]; the analytic case follows upon noting that a sheaf is complex analytically constructible if and only if is locally at all points, as proved in [14, Prop. 4.1.13]. See also [21, Prop. 8.1.26].

The analogues of (i)–(xii) in §2.5 work for complex analytic perverse sheaves, and so our main results hold in this context.

If X is a $\mathbb{C}$-scheme, and X^{an} the corresponding complex analytic space, then $D(X)$ in §2.1 for X a $\mathbb{C}$-scheme coincides with $D(X^{\mathrm{an}})$ for X^{an} a complex analytic space, and

$$D_c^b(X)^{\mathrm{alg}} \subset D_c^b(X^{\mathrm{an}})^{\mathrm{an}}, \quad \mathrm{Perv}(X)^{\mathrm{alg}} \subset \mathrm{Perv}(X^{\mathrm{an}})^{\mathrm{an}}$$

are full subcategories, and the six functors f^*, $f^!$, Rf_*, $Rf_!$, $\mathcal{R}Hom$, $\overset{L}{\otimes}$ for $\mathbb{C}$-scheme morphisms $f : X \to Y$ agree in the algebraic and analytic cases.

2.7. $\mathscr{D}$**-modules on $\mathbb{C}$-schemes and complex analytic spaces.** $\mathscr{D}$-modules on a smooth $\mathbb{C}$-scheme or smooth complex analytic space X are sheaves of modules over a certain sheaf of rings of differential operators $\mathscr{D}_X$ on X. Some books on them are Borel et al. [8], Coutinho [12], and Hotta, Takeuchi and Tanisaki [21] in the $\mathbb{C}$-scheme case, and Björk [7] and Kashiwara [26] in the complex analytic case. For a singular complex $\mathbb{C}$-scheme or complex analytic space X, the definition of a well-behaved category of $\mathscr{D}$-modules is given by Saito [48], via locally embedding X into a smooth scheme or space.

The analogue of perverse sheaves on X are called *regular holonomic $\mathscr{D}$-modules*, which form an abelian category $\mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X)$, the heart in the derived category $D_{\mathrm{rh}}^b(\mathrm{Mod}(\mathscr{D}_X))$ of bounded complexes of $\mathscr{D}_X$-modules with regular holonomic cohomology modules. The whole package of §2.1–§2.4 works for $\mathscr{D}$-modules. Our next theorem is known as the *Riemann–Hilbert correspondence* [7, §V.5], [21, Th. 7.2.1], see Borel [8, §14.4] for $\mathbb{C}$-schemes, Kashiwara [25] for complex manifolds, and Saito [48, §6] for complex analytic spaces, and also Maisonobe and Mekhbout [34].

Theorem 2.18. *Let X be a $\mathbb{C}$-scheme or complex analytic space. Then there is a* **de Rham functor** $\mathrm{DR} : D_{\mathrm{rh}}^b(\mathrm{Mod}(\mathscr{D}_X)) \overset{\sim}{\longrightarrow} D_c^b(X, \mathbb{C})$, *which is an equivalence of categories, restricts to an equivalence* $\mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X) \overset{\sim}{\longrightarrow} \mathrm{Perv}(X, \mathbb{C})$, *and commutes with* $f^*, f^!, Rf_*, Rf_!, \mathcal{R}Hom, \overset{L}{\otimes}$, *and also with* ψ_f^p, ϕ_f^p *for X smooth. Here $D_c^b(X, \mathbb{C}), \mathrm{Perv}(X, \mathbb{C})$ are constructible complexes and perverse sheaves over the base ring $A = \mathbb{C}$.*

Because of the Riemann–Hilbert correspondence, all our results on perverse sheaves of vanishing cycles on $\mathbb{C}$-schemes and complex analytic spaces in §3–§6 over a well-behaved base ring A, translate immediately when $A = \mathbb{C}$ to the corresponding results for $\mathscr{D}$-modules of vanishing cycles, with no extra work.

2.8. **Mixed Hodge modules: basics.** We write this section in the minimal generality needed for our applications. The statements made work equally well in the category of (algebraic) $\mathbb{C}$-schemes and the category of complex analytic spaces. By space, we will mean an object in either of these categories. The theory of mixed Hodge modules works with reduced spaces; should a space X be non-reduced, the following constructions are taken by definition on its reduction.

For a space X, let $\mathrm{HM}(X)$ denote Saito's category [45] of polarizable pure Hodge modules, (locally) a direct sum of subcategories $\mathrm{HM}(X)^w$ of pure Hodge modules of fixed weight w. On a smooth X, a pure Hodge module $M^\bullet$ consists of a triple of data: a filtered holonomic $\mathscr{D}$-module (M, F), a $\mathbb{Q}$-perverse sheaf, and a comparison map identifying the former with the complexification of the latter under the Riemann–Hilbert correspondence; see [45, §5.1.1, p. 952] and [47, §4]. This triple has to satisfy many other properties; in particular, the underlying holonomic $\mathscr{D}$-module is automatically regular, and algebraic Hodge modules are asked to be extendable to an algebraic compactification. Thus there is a forgetful functor $\mathrm{HM}(X) \to \mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X)$ from Hodge

modules to regular holonomic (algebraic) $\mathscr{D}$-modules. Hodge modules on singular spaces are defined, similarly to $\mathscr{D}$-modules, via embeddings into smooth varieties; see Saito [47] and also Maxim, Saito and Schürmann [39, §1.8].

There is a duality functor $\mathbb{D}_X^H : \mathrm{HM}(X) \to \mathrm{HM}(X)$. Pure Hodge modules also admit a Tate twist functor $M^\bullet \mapsto M^\bullet(1)$, see [45, §5.1.3, p. 952]. This functor shifts the filtration and rotates the rational structure on the underlying perverse sheaf: the $\mathscr{D}$-module filtration (M, F) is shifted to $(M, F[n])$ with $(F[n])_i = F_{i-n}$; the underlying perverse sheaf is tensored by $\mathbb{Z}(n) = (2\pi i)^n \mathbb{Z} \subset \mathbb{C}$, as in [45, (2.0.2), p. 876].

A polarization of weight w on a pure Hodge module $M^\bullet \in \mathrm{HM}(X)^w$ is a morphism of pure Hodge modules
$$\sigma : M^\bullet \longrightarrow \mathbb{D}_X^H(M^\bullet)(-w),$$
satisfying the extra conditions using vanishing cycles described on [45, (5.1.6.2) on p. 956 and (5.2.10.2) on p. 968], as well as the condition that on points it should correspond to the classical notion of a polarization of a pure Hodge structure (including positive definiteness).

Next, let $\mathrm{MHM}(X)$ denote the category of graded polarizable mixed Hodge modules [45, 47]. A graded polarizable mixed Hodge module carries a functorial weight filtration W, with graded pieces being polarizable pure Hodge modules, see [45, §5.2.10, p. 967-8]. The forgetful functor $\mathbf{rat} : \mathrm{MHM}(X) \to \mathrm{Perv}(X)$ to the appropriate category of perverse $\mathbb{Q}$-sheaves on X is faithful and exact; faithfulness in particular means that a morphism in $\mathrm{MHM}(X)$ is uniquely determined by the underlying morphism of perverse sheaves. The Tate twist functor extends to $\mathrm{MHM}(X)$; under this functor, the weight filtration W of the mixed Hodge module is changed to $W[2n]$ with $W[2n]_i = W_{i+2n}$ as on [45, p. 855]. The duality functor $\mathbb{D}_X^H$ also extends to $\mathrm{MHM}(X)$ and is compatible with Verdier duality on the perverse realization. There is also a forgetful functor $\mathrm{MHM}(X) \to \mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X)$ to regular holonomic $\mathscr{D}$-modules, even for singular spaces.

Theorem 2.19. *The categories of graded polarizable mixed Hodge modules have the following properties:*

- **(i)** *By* [47, Th. 3.9, p. 288], *the category of mixed Hodge modules for X a point is canonically equivalent to Deligne's category of graded polarizable mixed Hodge structures.*
- **(ii)** *For a smooth space U, we have a canonical object of weight $\dim U$*
$$\mathbb{Q}_U^H[\dim U] \in \mathrm{HM}(U) \subset \mathrm{MHM}(U),$$
which by [45, Prop. 5.2.16, p. 971] *possesses a canonical polarization*
$$\sigma : \mathbb{Q}_U^H[\dim U] \longrightarrow \mathbb{D}_U^H \mathbb{Q}_U^H[\dim U](-\dim U).$$
- **(iii)** *For an open inclusion $f : Y \hookrightarrow X$ of spaces, there is a pullback functor*
$$f^* = f^! : \mathrm{MHM}(X) \longrightarrow \mathrm{MHM}(Y).$$
  More generally, by [47, Prop. 2.19, p. 258], *for an arbitrary morphism $f : Y \to X$, there exist cohomological pullback functors $L^j f^*, L^j f^! : \mathrm{MHM}(X) \to \mathrm{MHM}(Y)$ compatible with (perverse) cohomological pullback on the perverse sheaf level.*
- **(iv)** *For a closed embedding $i : X \hookrightarrow Y$, there is a pushforward functor*
$$i_* = i_! : \mathrm{MHM}(X) \longrightarrow \mathrm{MHM}(Y),$$
  whose essential image is the full subcategory $\mathrm{MHM}_X(Y)$ of objects in $\mathrm{MHM}(Y)$ supported on X. Its inverse is $i^* = i^! : \mathrm{MHM}_X(Y) \to \mathrm{MHM}(X)$. More generally, by [45, Th. 5.3.1, p. 977] *and* [47, Th. 2.14, p. 252], *for a **projective** map $f : X \to Y$ there are cohomological pushforward functors*
$$R^j f_* : \mathrm{MHM}(X) \longrightarrow \mathrm{MHM}(Y).$$

(v) *There is an external tensor product functor*

$$\overset{L}{\boxtimes} : \mathrm{MHM}(X) \times \mathrm{MHM}(Y) \longrightarrow \mathrm{MHM}(X \times Y),$$

which is compatible with duality in the sense that for all $M^\bullet \in \mathrm{MHM}(X)$ *and* $N^\bullet \in \mathrm{MHM}(Y)$, *there is a natural isomorphism*

$$\mathbb{D}_X^H M^\bullet \overset{L}{\boxtimes} \mathbb{D}_Y^H N^\bullet \cong \mathbb{D}_{X \times Y}^H (M^\bullet \overset{L}{\boxtimes} N^\bullet).$$

Remark 2.20. We will not need to use any derived category $D^? \mathrm{MHM}(X)$ of mixed Hodge modules in this paper, which is just as well since on singular analytic X, the appropriate boundedness conditions do not appear to be well understood, and the general pullback and pushforward functors of Theorem 2.19(iii),(iv) are not known to exist as derived functors outside of the algebraic context of [47, §4]. Hence, in part (ii) above, [1] does not stand for a shift in the derived category; $\mathbb{Q}_U^H[\dim U]$ just denotes a mixed Hodge module whose realization is the perverse sheaf $\mathbb{Q}_U[\dim U]$ on U. Compare Remark 2.17 above.

Using the functors above, we can now define the twist of a mixed Hodge module by a principal $\mathbb{Z}/2\mathbb{Z}$-bundle. In the setup of Definition 2.9, given a $\mathbb{Z}/2\mathbb{Z}$-bundle $\pi : P \to X$, and an object $M^\bullet \in \mathrm{MHM}(X)$ on a space X, we have a natural map $M^\bullet \to \pi_* \pi^* M^\bullet$, which is an injection by faithfulness of the realization functor and the fact that it is an injection on the perverse sheaf level. The quotient object will be denoted, by abuse of notation, by $M^\bullet \otimes_{\mathbb{Z}/2\mathbb{Z}} P$ in $\mathrm{MHM}(X)$.

2.9. **Monodromic mixed Hodge modules.** To discuss nearby and vanishing cycle functors in a way consistent with monodromy, we need an extension of the category of mixed Hodge modules. For a space X, following Saito [49, §4.2] denote by $\mathrm{MHM}(X; T_s, N)$ the category of mixed Hodge modules $M^\bullet$ on X with commuting actions of a finite order operator $T_s : M^\bullet \to M^\bullet$ and a locally nilpotent operator $N : M^\bullet \to M^\bullet(-1)$. There is an embedding of categories $\mathrm{MHM}(X) \to \mathrm{MHM}(X; T_s, N)$ defined by setting $T = \mathrm{id}$ and $N = 0$. As proved by [49, (4.6.2)], the category $\mathrm{MHM}(X; T_s, N)$ is equivalent to the category $\mathrm{MHM}(X \times \mathbb{C})_{\mathrm{mon},!}$ of monodromic mixed Hodge modules on $X \times \mathbb{C}^*$ extended by zero to $X \times \mathbb{C}$; compare also [33, §4.2].

Every object $M^\bullet \in \mathrm{MHM}(X; T_s, N)$ decomposes into a direct sum $M^\bullet = M_1^\bullet \oplus M_{\neq 1}^\bullet$ of the T_s-invariant part and its T_s-equivariant complement. The Tate twist, and appropriate cohomological pullback and pushforward functors continue to exist. There is a duality functor

$$\mathbb{D}_X^T : \mathrm{MHM}(X; T_s, N) \longrightarrow \mathrm{MHM}(X; T_s, N)$$

defined by

$$\mathbb{D}_X^T(M^\bullet) = \mathbb{D}_X^H(M_1^\bullet) \oplus \mathbb{D}_T^H(M_{\neq 1}^\bullet)(1),$$

equipped with the finite-order operator $\mathbb{D}_X(T_s)^{-1}$ and the nilpotent operator $-\mathbb{D}_X(N)$. This duality functor still satisfies $\mathbb{D}_X^T \circ \mathbb{D}_X^T = \mathrm{id}$.

Saito [49, §5.1] also defines an external tensor product

$$\overset{T}{\boxtimes} : \mathrm{MHM}(X_1; T_s, N) \times \mathrm{MHM}(X_2; T_s, N) \longrightarrow \mathrm{MHM}(X_1 \times X_2; T_s, N).$$

defined on the monodromic category as follows. The addition map on fibres

$$\pi : (X_1 \times \mathbb{C}) \times (X_2 \times \mathbb{C}) \longrightarrow (X_1 \times X_2) \times \mathbb{C}$$

induces the additive convolution

$$\pi_*(- \boxtimes -) : \mathrm{MHM}(X_1 \times \mathbb{C})_{\mathrm{mon},!} \times \mathrm{MHM}(X_2 \times \mathbb{C})_{\mathrm{mon},!} \longrightarrow \mathrm{MHM}(X_1 \times x_2 \times \mathbb{C})_{\mathrm{mon},!}.$$

One can translate this external tensor product $\overset{T}{\boxtimes}$ to the $\mathrm{MHM}(X; T_s, N)$ defined by concrete data $(M^\bullet, T_s, N)$. On the underlying $\mathscr{D}$-modules and perverse sheaves, it is just the usual product

$\boxtimes$. The operators are defined by $T_s = T_s \boxtimes T_s$ and $N = N \boxtimes \mathrm{id} + \mathrm{id} \boxtimes N$. However, the Hodge and weight filtrations on the underlying $\mathscr{D}$-modules and perverse sheaves are shifted using the finite order endomorphisms T_s; for details, see [49, (5.1.1)–(5.1.2)]. Note that as a consequence of these definitions, the forgetful functors

$$\mathrm{MHM}(-; T_s, N) \longrightarrow \mathrm{MHM}(-)$$

do not map $\overset{T}{\boxtimes}$ to $\overset{L}{\boxtimes}$. Twisted duality and the twisted tensor product commute in the sense that given $M^\bullet \in \mathrm{MHM}(X; T_s, N)$ and $N^\bullet \in \mathrm{MHM}(Y; T_s, N)$, we have a natural isomorphism in $\mathrm{MHM}(X \times Y; T_s, N)$:

$$\mathbb{D}_X^T(M^\bullet) \overset{T}{\boxtimes} \mathbb{D}_Y^T(N^\bullet) \cong \mathbb{D}_{X \times Y}^T(M^\bullet \overset{T}{\boxtimes} N^\bullet). \tag{2.20}$$

For an object $M^\bullet \in \mathrm{MHM}(X; T_s, N)$ whose weight filtration is a (suitable shifted) monodromy filtration of the nilpotent morphism N, there is a stronger notion of polarization which will be useful for us. A *strong polarization of weight w* of such an object $M^\bullet$ is a morphism

$$\sigma : M^\bullet \longrightarrow \mathbb{D}_X^T(M^\bullet)(-w)$$

in $\mathrm{MHM}(X)$, compatible with T_s and N, such that σ defines polarizations on the N-primitive parts of $M^\bullet$, compatible with Hodge filtrations; for precise conditions, see [45, p. 855]. A polarization on a pure Hodge module is a strong polarization (with $N = 0$); a strongly polarized mixed Hodge module is graded polarizable. The partial twist in the definition of $\mathbb{D}_X^T$ implies that $M^\bullet$ is of weight w if and only if $M_1^\bullet$, respectively $M_{\neq 1}^\bullet$ are of weights $w, w - 1$ in the sense of [45, p. 855].

Given strongly polarized mixed Hodge modules $M_i^\bullet \in \mathrm{MHM}(X_i; T_s, N)$ of weight w_i for $i = 1, 2$, polarized by $\sigma_i : M_i^\bullet \to \mathbb{D}_{X_i}^T(M_i^\bullet)(-w_i)$, there is an induced morphism σ in a commutative diagram

$$
\begin{array}{ccc}
M_1^\bullet \overset{T}{\boxtimes} M_2^\bullet & \longrightarrow & \mathbb{D}_{X_1}^T(M_1^\bullet)(-w_1) \overset{T}{\boxtimes} \mathbb{D}_{X_2}^T(M_2^\bullet)(-w_2) \\
& \underset{\sigma}{\searrow} & \downarrow \\
& & \mathbb{D}_{X_1 \times X_2}^T(M_1^\bullet \overset{T}{\boxtimes} M_2^\bullet)(-w_1 - w_2),
\end{array}
$$

where the top map is $\sigma_1 \overset{T}{\boxtimes} \sigma_2$ and the right is the isomorphism (2.20). In general, it is *not clear* whether this morphism is a strong polarization of the tensor product $M_1^\bullet \overset{T}{\boxtimes} M_2^\bullet$; this result is not available in the literature. However, in this paper we only use this construction in cases where one of the monodromic mixed Hodge modules is essentially trivial, living on $X_1 = \mathrm{pt}$ with $N = 0$, in which case it is easy to check that the resulting σ is a strong polarization.

Note also that if $M^\bullet$ is strongly polarized by $\sigma : M^\bullet \to \mathbb{D}_X^T(M^\bullet)(-w)$, then its Tate twist is also strongly polarized by the composition

$$M^\bullet(1) \xrightarrow{\sigma(1)} \mathbb{D}_X^T(M^\bullet)(-w + 1) \xrightarrow{\sim} \mathbb{D}_X^T(M^\bullet(1))(-w + 2). \tag{2.21}$$

The notion of strong polarization leads to gluing, in the following way.

Theorem 2.21. *Let $X = \bigcup_i U_i$ be an open cover of a space X, in any of the Zariski, étale or complex analytic topologies. Then:*

   (i) *Suppose we are given mixed Hodge modules $M^\bullet, N^\bullet \in \mathrm{MHM}(X)$, with morphisms $f_i : M^\bullet|_{U_i} \to N^\bullet|_{U_i}$ in $\mathrm{MHM}(U_i)$ which agree on overlaps U_{ij}. Then there is a unique $f \in \mathrm{Hom}_{\mathrm{MHM}(X)}(M^\bullet, N^\bullet)$ with $f|_{U_i} = f_i$.*

(ii) *Suppose we are given mixed Hodge modules $M_i^\bullet \in \mathrm{MHM}(U_i; T_s, N)$, each equipped with a strong polarization σ_i. Suppose also that we are given isomorphisms*

$$\alpha_{ij} : M_i^\bullet|_{U_{ij}} \longrightarrow M_j^\bullet|_{U_{ij}}$$

on intersections, commuting with the restrictions of the maps T_{si}, N_i and σ_i, with

$$\alpha_{jk}|_{U_{ijk}} \circ \alpha_{ij}|_{U_{ijk}} = \alpha_{ik}|_{U_{ijk}}$$

on triple intersections. Then there is a strongly polarized mixed Hodge module

$$M^\bullet \in \mathrm{MHM}(X; T_s, N),$$

restricting to $M_i^\bullet$ on U_i.

Proof. To prove (i), it is enough to note that the f_i glue on the perverse sheaf and $\mathscr{D}$-module levels, respecting filtrations.

To prove (ii), we begin with the case of pure Hodge modules of fixed weight w. The data of a pure Hodge module consists of a pair of a filtered holonomic $\mathscr{D}$-module and a $\mathbb{Q}$-perverse sheaf, with an identification of the former with the complexification of the latter under the Riemann–Hilbert correspondence. Since both filtered holonomic $\mathscr{D}$-modules and $\mathbb{Q}$-perverse sheaves form stacks (both in the algebraic and the analytic case), this data glues over X. As for the (strong) polarization, $T_{si} = \mathrm{id}$ and $N_i = 0$ glue to $T_s = \mathrm{id}$ and $N = 0$, whereas the map σ on the perverse sheaf level glues from the maps σ_i once again from the stack property (now for morphisms) of perverse sheaves.

The conditions [45, §5.1.6, p. 955] which make such a pair a pure Hodge module come from local conditions as well as conditions on vanishing cycles; the latter glue by induction on the dimension. So strongly polarized pure Hodge modules form a stack. The case of mixed Hodge modules is similar: we need to glue filtrations and polarizations, as well as the maps T_{si}, N_i and σ_i, first on the level of perverse sheaves, and then checking the axioms, which are local or follow by induction. $\square$

Remark 2.22. Given a projective $\mathbb{C}$-scheme X, and a polarizable mixed Hodge module $M^\bullet$ in $\mathrm{MHM}(X)$ on it, the second part of Theorem 2.19(iv) applied to $f : X \to \mathrm{pt}$ shows that the hypercohomology $\mathbb{H}^*(X, M^\bullet)$ carries a mixed Hodge structure. In particular, it carries a weight filtration and therefore has a weight polynomial, which will be useful in refinements of Donaldson–Thomas theory, see the discussion in Remark 6.14 below. So we need to glue polarizable objects from local data. On the other hand, graded polarizable mixed Hodge modules may fail to form a stack in the analytic category unless the polarizations glue. This is the reason for using the stronger form of polarization, which allows for gluing as shown above.

2.10. **Mixed Hodge modules of vanishing cycles.** By Saito's work [45–47], for a regular function $f : U \to \mathbb{C}$ on a smooth space U, the perverse nearby and vanishing cycle functors ψ_f^p, ϕ_f^p defined on perverse sheaves in §2.3 lift to functors

$$\psi_f^H, \phi_f^H : \mathrm{MHM}(U) \longrightarrow \mathrm{MHM}(U_0; T_s, N),$$

where $U_0 = f^{-1}(0)$. The actions of the finite order and nilpotent operators T_s, N are given by the semisimple part of the monodromy operator, and the logarithm of its unipotent part. The analogue

$$\phi_f^H \circ \mathbb{D}_U^T \cong \mathbb{D}_{U_0}^T \circ \phi_f^H$$

of Theorem 2.11(iv) is proved in [46]; to make this isomorphism work is the reason for the the twist in the definition of $\mathbb{D}_U^T$. Note also that [46, Th. 1.6] fits with the convention that T_s and N are defined on dual objects as $\mathbb{D}_U(T_s)^{-1}$ and $\mathbb{D}_U(N)$, respectively.

By [45, §5.2], if $M^\bullet \in \mathrm{HM}(U)$ is a pure Hodge module, then a polarization of $M^\bullet$ induces a strong polarization on the (mixed) Hodge module of vanishing cycles $\phi_f^H(M^\bullet)$, of the same weight. In particular, if $M^\bullet = \mathbb{Q}_U^H[\dim U]$ is the canonical object with its canonical polarization from Theorem 2.19(ii), then $\phi_f^H(\mathbb{Q}_U^H[\dim U]) \in \mathrm{MHM}(\mathrm{Crit}(f); T_s, N)$ is a strongly polarized mixed Hodge module on the critical locus of f, with polarization

$$\sigma : \phi_f^H\big(\mathbb{Q}_U^H[\dim U]\big) \longrightarrow \mathbb{D}_{\mathrm{Crit}(f)}^T \circ \phi_f^H\big(\mathbb{Q}_U^H[\dim U]\big)(-\dim U). \tag{2.22}$$

Example 2.23. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^2$. Then $\mathrm{Crit}(f) = \{0\}$, and we obtain an object $\phi_f^H\big(\mathbb{Q}_\mathbb{C}^H[1]\big)$ in $\mathrm{MHM}(\mathrm{pt}; T_s, N)$, a one-dimensional polarized mixed Hodge structure with monodromy acting by $T_s = -\mathrm{id}$ and $N = 0$. For $g : \mathbb{C}^2 \to \mathbb{C}$ given by $g(z_1, z_2) = z_1^2 + z_2^2$, it is well known that

$$\phi_{z_1^2+z_2^2}^H\big(\mathbb{Q}_{\mathbb{C}^2}^H[2]\big) \cong \mathbb{Q}(-1),$$

with trivial monodromy action. Applying the Thom–Sebastiani formula for mixed Hodge modules [49, Th. 5.4], we see that

$$\phi_{z^2}^H\big(\mathbb{Q}_\mathbb{C}^H[1]\big) \overset{T}{\boxtimes} \phi_{z^2}^H\big(\mathbb{Q}_\mathbb{C}^H[1]\big) \cong \mathbb{Q}(-1)$$

in the category $\mathrm{MHM}(\mathrm{pt}; T_s, N)$. The objects $\mathbb{Q}(1)$ and $\mathbb{Q}(-1)$ thus admit square roots under $\overset{T}{\boxtimes}$ in this category, which we will denote by $\mathbb{Q}(\frac{1}{2})$ and $\mathbb{Q}(-\frac{1}{2})$, where

$$\phi_{z^2}^H\big(\mathbb{Q}_\mathbb{C}^H[1]\big) = \mathbb{Q}(-\tfrac{1}{2}). \tag{2.23}$$

More explicitly, we have

$$\mathbb{Q}(-\tfrac{1}{2}) = (\mathbb{Q}(0), -\mathrm{id}, 0)$$

and

$$\mathbb{Q}(\tfrac{1}{2}) = (\mathbb{Q}(1), -\mathrm{id}, 0).$$

Define an object $\mathbb{Q}(\frac{n}{2}) \in \mathrm{MHM}(\mathrm{pt}; T_s, N)$ for each $n \in \mathbb{Z}$ by $\mathbb{Q}(\frac{n}{2}) = \mathbb{Q}(\frac{1}{2})^{\overset{T}{\boxtimes}n}$ for $n \geqslant 0$, and $\mathbb{Q}(\frac{n}{2}) = \mathbb{Q}(-\frac{1}{2})^{\overset{T}{\boxtimes}-n}$ for $n < 0$. For any space X with structure morphism $\pi : X \to \mathrm{pt}$, and any $M^\bullet \in D^b\,\mathrm{MHM}(X; T_s, N)$, we define the $\frac{n}{2}$ *twist of* $M^\bullet$ to be $M^\bullet(\frac{n}{2}) = M^\bullet \overset{T}{\boxtimes}\big(\mathbb{Q}(\frac{n}{2})\big)$. If $M^\bullet$ is strongly polarized, then this tensor product is also strongly polarized by the tensor polarization by our comments above.

Let U be a smooth space, $f : U \to \mathbb{C}$ a regular function, and $X = \mathrm{Crit}(f)$ its critical locus, as a subspace of U. The perverse sheaf of vanishing cycles $\mathcal{PV}_{U,f}^\bullet \in \mathrm{Perv}(X)$ from §2.4 has a lift to a mixed Hodge module $\mathcal{HV}_{U,f}^\bullet$ in $\mathrm{MHM}(X; T_s, N)$, defined for each $c \in f(X)$ by

$$\mathcal{HV}_{U,f}^\bullet|_{X_c} = \phi_{f-c}^H\big(\mathbb{Q}_U^H[\dim U]\big)\big|_{X_c}\big(\tfrac{1}{2}\dim U\big) \in \mathrm{MHM}(X_c; T_s, N). \tag{2.24}$$

This mixed Hodge module inherits a strong polarization of weight 0 (compare (2.21) and (2.22))

$$\sigma_{U,f}^H : \mathcal{HV}_{U,f}^\bullet \longrightarrow \mathbb{D}_X^T\big(\mathcal{HV}_{U,f}^\bullet\big). \tag{2.25}$$

The twist $(\frac{1}{2}\dim U)$ in (2.24), using the notation of Example 2.23, is included for the same reason as the $(-1)^{\dim U}$ in the definition (2.7) of $\tau_{U,f}$. It makes $\mathcal{HV}_{U,f}^\bullet$ act naturally under transformations which change dimension — without it, the mixed Hodge module version of (5.15) below would have to include a twist $(\frac{1}{2}n)$ for $n = \dim V - \dim U$. Then

$$\mathcal{HV}_{U,f}^\bullet, \quad T_s : \mathcal{HV}_{U,f}^\bullet \to \mathcal{HV}_{U,f}^\bullet, \quad N : \mathcal{HV}_{U,f}^\bullet \to \mathcal{HV}_{U,f}^\bullet(-1), \quad \text{and} \quad \sigma_{U,f}^H : \mathcal{HV}_{U,f}^\bullet \to \mathbb{D}_X^T\big(\mathcal{HV}_{U,f}^\bullet\big)$$

are related to

$$\mathcal{PV}_{U,f}^\bullet, \quad \tau_{U,f} : \mathcal{PV}_{U,f}^\bullet \to \mathcal{PV}_{U,f}^\bullet, \quad \text{and} \quad \sigma_{U,f} : \mathcal{PV}_{U,f}^\bullet \overset{\cong}{\longrightarrow} \mathbb{D}_X(\mathcal{PV}_{U,f}^\bullet)$$

in §2.4 by

$$\mathcal{PV}^\bullet_{U,f} = \mathbf{rat}\big(\mathcal{HV}^\bullet_{U,f}\big), \ \ \tau_{U,f} = \mathbf{rat}(T_s) \circ \exp\big(2\pi i\,\mathbf{rat}(N)\big), \ \ \sigma_{U,f} = \mathbf{rat}(\sigma^H_{U,f});$$

for the last statement, see Proposition A.1 in the Appendix.

The following Thom–Sebastiani type result is the analogue of Theorem 2.13.

Theorem 2.24. *Let U,V be smooth spaces and $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ be regular functions, so that $f \boxplus g : U \times V \to \mathbb{C}$ is given by $(f \boxplus g)(u,v) := f(u) + g(v)$. Set $X = \mathrm{Crit}(f)$ and $Y = \mathrm{Crit}(g)$ as subspaces of U,V, so that $\mathrm{Crit}(f \boxplus g) = X \times Y$. Then there is a natural isomorphism*

$$\mathcal{TS}^H_{U,f,V,g} : \mathcal{HV}^\bullet_{U \times V, f \boxplus g} \overset{\cong}{\longrightarrow} \mathcal{HV}^\bullet_{U,f} \overset{T}{\boxtimes} \mathcal{HV}^\bullet_{V,g} \ \ in \ \mathrm{MHM}(X \times Y; T_s, N), \tag{2.26}$$

so that the following diagram commutes:

$$\tag{2.27}$$

$$
\begin{array}{ccc}
\mathcal{HV}^\bullet_{U \times V, f \boxplus g} & \xrightarrow{\quad\quad\quad\quad\sigma_{U \times V, f \boxplus g}\quad\quad\quad\quad} & \mathbb{D}^T_{X \times Y}(\mathcal{HV}^\bullet_{U \times V, f \boxplus g}) \\
\Big\downarrow{\scriptstyle\mathcal{TS}^H_{U,f,V,g}} & & \Big\uparrow{\scriptstyle\mathbb{D}^T_{X \times Y}(\mathcal{TS}^H_{U,f,V,g})} \\
\mathcal{HV}^\bullet_{U,f} \overset{T}{\boxtimes} \mathcal{HV}^\bullet_{V,g} & \xrightarrow[\sigma_{U,f} \overset{T}{\boxtimes} \sigma_{V,g}]{} \ \mathbb{D}^T_X(\mathcal{HV}^\bullet_{U,f}) \overset{T}{\boxtimes} \mathbb{D}^T_Y(\mathcal{HV}^\bullet_{V,g}) \xrightarrow[\quad\cong\quad]{} & \mathbb{D}^T_{X \times Y}\big(\mathcal{HV}^\bullet_{U,f} \overset{T}{\boxtimes} \mathcal{HV}^\bullet_{V,g}\big).
\end{array}
$$

Proof. The existence of the isomorphism (2.26) follows from the Thom–Sebastiani Theorem for mixed Hodge modules due to Saito [49, Th. 5.4], applied to $\mathcal{HV}^\bullet_{U,f}$. The diagram (2.27) exists by (2.20); its commutativity can be checked on the level of the underlying perverse sheaves which is (2.9), in light of Propositions A.1–A.2 in the Appendix. Note that (2.26) also includes the analogue of (2.10) in Theorem 2.13, according to which we have a matching of the monodromy actions

$$\tau_{U \times V, f \boxplus g} \cong \tau_{U,f} \overset{L}{\boxtimes} \tau_{V,g},$$

as (2.26) holds in $\mathrm{MHM}(X \times Y; T_s, N)$ rather than just $\mathrm{MHM}(X \times Y)$. $\square$

In this paper we will only ever apply Theorem 2.24 when $V = \mathbb{C}^n$, $g = z_1^2 + \cdots + z_n^2$ and $Y = \{0\}$. Combining (2.23) and (2.24) shows that

$$\mathcal{HV}^\bullet_{\mathbb{C}, z^2} = \big(\mathbb{Q}(-\tfrac{1}{2})\big)(\tfrac{1}{2}) \cong \mathbb{Q}(0) \cong \mathbb{Q}^H_{\{0\}}.$$

Thus, by Theorem 2.24, $\mathbb{Q}^H_{\{0\}} \overset{T}{\boxtimes} \mathbb{Q}^H_{\{0\}} \cong \mathbb{Q}^H_{\{0\}}$, and induction on n, we see that

$$\mathcal{HV}^\bullet_{\mathbb{C}^n, z_1^2 + \cdots + z_n^2} \cong \mathbb{Q}^H_{\{0\}}.$$

As for (2.12), this isomorphism is natural up to sign, depending on a choice of orientation for the complex Euclidean space $(\mathbb{C}^n, \mathrm{d}z_1^2 + \cdots + \mathrm{d}z_n^2)$.

3. Action of symmetries on vanishing cycles

Here is our first main result.

Theorem 3.1. *Let U,V be smooth $\mathbb{C}$-schemes, $\Phi, \Psi : U \to V$ étale morphisms, and $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ regular functions with $g \circ \Phi = f = g \circ \Psi$. Write $X = \mathrm{Crit}(f)$ and $Y = \mathrm{Crit}(g)$ as $\mathbb{C}$-subschemes of U,V, so that $\Phi|_X, \Psi|_X : X \to Y$ are étale morphisms. Suppose $\Phi|_X = \Psi|_X$. Then:*

(a) *As Φ, Ψ are étale, $\mathrm{d}\Phi : TU \to \Phi^*(TV)$, $\mathrm{d}\Psi : TU \to \Psi^*(TV)$ are isomorphisms of vector bundles. Restricting to the reduced $\mathbb{C}$-subscheme X^{red} of X, and using $\Phi|_{X^{\mathrm{red}}} = \Psi|_{X^{\mathrm{red}}}$ as $\Phi|_X = \Psi|_X$, gives isomorphisms*

$$\mathrm{d}\Phi|_{X^{\mathrm{red}}}, \mathrm{d}\Psi|_{X^{\mathrm{red}}} : TU|_{X^{\mathrm{red}}} \longrightarrow \Phi|_{X^{\mathrm{red}}}^*(TV),$$

$$\text{and thus} \quad \mathrm{d}\Psi|_{X^{\mathrm{red}}}^{-1} \circ \mathrm{d}\Phi|_{X^{\mathrm{red}}} : TU|_{X^{\mathrm{red}}} \longrightarrow TU|_{X^{\mathrm{red}}}.$$

Hence

$$\det\!\big(\mathrm{d}\Psi|_{X^{\mathrm{red}}}^{-1} \circ \mathrm{d}\Phi|_{X^{\mathrm{red}}}\big) : X^{\mathrm{red}} \longrightarrow \mathbb{C} \setminus \{0\}$$

is a regular function. Then $\det\!\big(\mathrm{d}\Psi|_{X^{\mathrm{red}}}^{-1} \circ \mathrm{d}\Phi|_{X^{\mathrm{red}}}\big)$ is a locally constant map

$$X^{\mathrm{red}} \longrightarrow \{\pm 1\} \subset \mathbb{C} \setminus \{0\}.$$

(b) *Definition 2.15 defines isomorphisms $\mathcal{PV}_\Phi, \mathcal{PV}_\Psi : \mathcal{PV}_{U,f}^\bullet \to \Phi|_X^*\big(\mathcal{PV}_{V,g}^\bullet\big)$ in $\mathrm{Perv}(X)$. These are related by*

$$\mathcal{PV}_\Phi = \det\!\big(\mathrm{d}\Psi|_{X^{\mathrm{red}}}^{-1} \circ \mathrm{d}\Phi|_{X^{\mathrm{red}}}\big) \cdot \mathcal{PV}_\Psi, \tag{3.1}$$

regarding $\det\!\big(\mathrm{d}\Psi|_{X^{\mathrm{red}}}^{-1} \circ \mathrm{d}\Phi|_{X^{\mathrm{red}}}\big) : X \to \{\pm 1\}$ as a locally constant map of topological spaces, where X, X^{red} have the same topological space.

The analogues of these results also hold for $\mathscr{D}$-modules and mixed Hodge modules on $\mathbb{C}$-schemes, and (with Φ, Ψ local biholomorphisms and f, g analytic functions) for perverse sheaves, $\mathscr{D}$-modules and mixed Hodge modules on complex analytic spaces, as in §2.6–§2.10.

By taking $U = V$, $f = g$, Φ an isomorphism and $\Psi = \mathrm{id}_U$, we deduce a result on the action of symmetries on perverse sheaves of vanishing cycles:

Corollary 3.2. *Let U be a smooth $\mathbb{C}$-scheme, $\Phi : U \to U$ an isomorphism, and $f : U \to \mathbb{C}$ be regular with $f \circ \Phi = f$. Write $X = \mathrm{Crit}(f)$ as a $\mathbb{C}$-subscheme of U and X^{red} for its reduced $\mathbb{C}$-subscheme, and suppose $\Phi|_X = \mathrm{id}_X$. Then $\det\!\big(\mathrm{d}\Phi|_{X^{\mathrm{red}}} : TU|_{X^{\mathrm{red}}} \to TU|_{X^{\mathrm{red}}}\big)$ is a locally constant map $X^{\mathrm{red}} \to \{\pm 1\}$, and $\mathcal{PV}_\Phi : \mathcal{PV}_{U,f}^\bullet \overset{\cong}{\longrightarrow} \mathcal{PV}_{U,f}^\bullet$ in $\mathrm{Perv}(X)$ from Definition 2.15 is multiplication by $\det\!\big(\mathrm{d}\Phi|_{X^{\mathrm{red}}}\big) = \pm 1$. The analogues hold in the settings of §2.6–§2.10.*

Example 3.3. Let $U = V = \mathbb{C}^n$ and $f(z_1, \ldots, z_n) = g(z_1, \ldots, z_n) = z_1^2 + \cdots + z_n^2$, so that $X = Y = \{0\} \subset \mathbb{C}^n$. Let $\Phi, \Psi \in \mathrm{O}(n, \mathbb{C})$ be orthogonal matrices, so that $\det \Phi, \det \Psi \in \{\pm 1\}$ and $\Phi, \Psi : \mathbb{C}^n \to \mathbb{C}^n$ are isomorphisms with $f = g \circ \Phi = g \circ \Phi$ and $\Phi|_{\{0\}} = \Psi|_{\{0\}} = \mathrm{id}_{\{0\}}$. In Theorem 3.1(a) we have

$$\mathrm{d}\Psi|_{X^{\mathrm{red}}}^{-1} \circ \mathrm{d}\Phi|_{X^{\mathrm{red}}} = \Psi^{-1} \circ \Phi : \mathbb{C}^n \longrightarrow \mathbb{C}^n,$$

so that $\det\!\big(\mathrm{d}\Psi|_{X^{\mathrm{red}}}^{-1} \circ \mathrm{d}\Phi|_{X^{\mathrm{red}}}\big) = \det \Psi^{-1} \det \Phi = \pm 1$.

For Theorem 3.1(b), Example 2.16 shows that $\mathcal{PV}_\Phi, \mathcal{PV}_\Psi : A_{\{0\}} \to A_{\{0\}}$ are multiplication by $\det \Phi, \det \Psi$, so $\mathcal{PV}_\Phi = (\det \Psi^{-1} \det \Phi) \cdot \mathcal{PV}_\Psi$, as in (3.1).

The proof of Theorem 3.1(b) uses the following proposition. To interpret it, pretend for simplicity that the étale morphisms $\pi_U|_{W_t} : W_t \to U$ in (b) are invertible. Then

$$\Theta_t := \pi_V|_{W_t} \circ \pi_U|_{W_t}^{-1}$$

for $t \in \mathbb{C}$ are a 1-parameter family of morphisms $U \to V$, which satisfy $f = g \circ \Theta_t$ and $\Theta_t|_X = \Phi|_X = \Psi|_X$ for $t \in \mathbb{C}$, with $\Theta_0 = \Phi$ and $\Theta_1 = \Psi$. Thus, modulo taking étale covers of U, the family $\{\Theta_t : t \in \mathbb{C}\}$ interpolates between Φ and Ψ.

Proposition 3.4. *Let U, V be smooth $\mathbb{C}$-schemes, let $\Phi, \Psi : U \to V$ be étale morphisms, and let $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ be regular functions with $g \circ \Phi = f = g \circ \Psi$. Write $X = \mathrm{Crit}(f)$ and $Y = \mathrm{Crit}(g)$ as $\mathbb{C}$-subschemes of U, V, so that $\Phi|_X, \Psi|_X : X \to Y$ are étale. Suppose $\Phi|_X = \Psi|_X$, and $x \in X$ such that $\mathrm{d}\Psi|_x^{-1} \circ \mathrm{d}\Phi|_x : T_x U \to T_x U$ satisfies*

$$\left(\mathrm{d}\Psi|_x^{-1} \circ \mathrm{d}\Phi|_x - \mathrm{id}_{T_x U} \right)^2 = 0.$$

Then there exist a smooth $\mathbb{C}$-scheme W and morphisms $\pi_{\mathbb{C}} : W \to \mathbb{C}$, $\pi_U : W \to U$, $\pi_V : W \to V$ and $\iota : \mathbb{C} \to W$ such that:

(a) *$\pi_{\mathbb{C}} \circ \iota(t) = t$, $\pi_U \circ \iota(t) = x$ and $\pi_V \circ \iota(t) = \Phi(x)$ for all $t \in \mathbb{C}$;*

(b) *$\pi_{\mathbb{C}} \times \pi_U : W \to \mathbb{C} \times U$ and $\pi_{\mathbb{C}} \times \pi_V : W \to \mathbb{C} \times V$ are étale. Thus, $W_t := \pi_{\mathbb{C}}^{-1}(t)$ is a smooth $\mathbb{C}$-scheme for each $t \in \mathbb{C}$, and $\pi_U|_{W_t} : W_t \to U$, $\pi_V|_{W_t} : W_t \to V$ are étale, and $\iota(t) \in W_t$ with $\pi_U : \iota(t) \mapsto x$, $\pi_V : \iota(t) \mapsto \Phi(x)$;*

(c) *$h := f \circ \pi_U = g \circ \pi_V : W \to \mathbb{C}$. Thus,*

$$(\pi_{\mathbb{C}} \times \pi_U)|_Z : Z \longrightarrow \mathbb{C} \times X \qquad \text{and} \qquad (\pi_{\mathbb{C}} \times \pi_V)|_Z : Z \longrightarrow \mathbb{C} \times Y$$

are étale, where $Z := \mathrm{Crit}(h)$;

(d) *$\Phi|_X \circ \pi_U|_Z = \Psi|_X \circ \pi_U|_Z = \pi_V|_Z : Z \to Y \subseteq V$; and*

(e) *$\Phi \circ \pi_U|_{W_0} = \pi_V|_{W_0}$ and $\Psi \circ \pi_U|_{W_1} = \pi_V|_{W_1}$, for W_0, W_1 as in (b).*

We will prove Proposition 3.4 in §3.1, and Theorem 3.1 in §3.2–§3.4.

3.1. **Proof of Proposition 3.4.** Let $U, V, \Phi, \Psi, f, g, X, Y, x$ be as in Proposition 3.4. Choose a Zariski open neighbourhood V' of $\Phi(x) = \Psi(x)$ in V and étale coordinates $(z_1, \ldots, z_n) : V' \to \mathbb{C}^n$ on V', with

$$z_1 = \cdots = z_n = 0$$

at $\Phi(x)$. Let m be the rank of the symmetric matrix $\left(\frac{\partial^2 g}{\partial z_i \partial z_j} |_{\Phi(x)} \right)_{i,j=1}^n$, so that $m \in \{0, \ldots, n\}$. By applying an element of $\mathrm{GL}(n, \mathbb{C})$ to the coordinates $(z_1, \ldots, z_n)$ we can suppose that

$$\frac{\partial^2 g}{\partial z_i \partial z_j} \Big|_{\Phi(x)} = \begin{cases} 1, & i = j \in \{1, \ldots, m\}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

Then $\frac{1}{2} \frac{\partial g}{\partial z_i}$ agrees with z_i to first order at $\Phi(x)$ for $i = 1, \ldots, m$, so replacing z_i by $\frac{1}{2} \frac{\partial g}{\partial z_i}$ for $i = 1, \ldots, m$ and making V' smaller, we can suppose (3.2) holds and $z_1, \ldots, z_m$ lie in the ideal $\left(\frac{\partial g}{\partial z_i}, i = 1, \ldots, n \right)$ in $\mathcal{O}_{V'}$. Thus we may write

$$z_i = \sum_{j=1}^n A_{ij} \cdot \frac{\partial g}{\partial z_j}, \qquad i = 1, \ldots, m, \tag{3.3}$$

where $A_{ij} : V' \to \mathbb{A}^1$ are regular functions for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Taking $\frac{\partial}{\partial z_j}$ of (3.3) for $j = 1, \ldots, m$ and using (3.2) gives

$$A_{ij}|_{\Phi(x)} = \begin{cases} 1, & i = j, \ i, j \in \{1, \ldots, m\}, \\ 0, & i \neq j, \ i, j \in \{1, \ldots, m\}. \end{cases} \tag{3.4}$$

Set $U' = \Phi^{-1}(V') \cap \Psi^{-1}(V')$, so that U' is a Zariski open neighbourhood of x in U. Define étale coordinates $(x_1, \ldots, x_n) : U' \to \mathbb{C}^n$ and $(y_1, \ldots, y_n) : U' \to \mathbb{C}^n$ by $x_i = z_i \circ \Phi$ and $y_i = z_i \circ \Psi$, so that $x_1 = \cdots = x_n = y_1 = \cdots = y_n = 0$ at x. Since $f = g \circ \Phi = g \circ \Psi$ we have $\frac{\partial f}{\partial x_j} = \frac{\partial g}{\partial z_j} \circ \Phi$

and $\frac{\partial f}{\partial y_j} = \frac{\partial g}{\partial z_j} \circ \Psi$. Thus (3.2) and (3.3) imply that

$$\frac{\partial^2 f}{\partial x_i \partial x_j}\Big|_x = \frac{\partial^2 f}{\partial y_i \partial y_j}\Big|_x = \begin{cases} 1, & i = j \in \{1, \ldots, m\}, \\ 0, & \text{otherwise}, \end{cases} \tag{3.5}$$

$$x_i = \sum_{j=1}^{n} (A_{ij} \circ \Phi) \cdot \frac{\partial f}{\partial x_j}, \quad y_i = \sum_{j=1}^{n} (A_{ij} \circ \Psi) \cdot \frac{\partial f}{\partial y_j}, \quad i = 1, \ldots, m. \tag{3.6}$$

Now $d\Phi|_x : T_x U \to T_{\Phi(x)} V$ maps $\frac{\partial}{\partial x_j} \mapsto \frac{\partial}{\partial z_j}$, as $x_j = z_j \circ \Phi$, and $d\Psi|_x : T_x U \to T_{\Phi(x)} V$ maps $\frac{\partial}{\partial y_j} \mapsto \frac{\partial}{\partial z_j}$. Hence $d\Psi|_x^{-1} \circ d\Phi|_x : T_x U \to T_x U$ maps $\frac{\partial}{\partial x_j} \mapsto \frac{\partial}{\partial y_j} = \sum_{i=1}^{n} \frac{\partial x_i}{\partial y_j} \cdot \frac{\partial}{\partial x_i}$. Define $B_{ij} \in \mathbb{C}$ for $i, j = 1, \ldots, n$ by

$$\delta_{ij} + B_{ij} = \frac{\partial x_i}{\partial y_j}\Big|_x. \tag{3.7}$$

Then $(\delta_{ij} + B_{ij})_{i,j=1}^{n}$ is the matrix of $d\Psi|_x^{-1} \circ d\Phi|_x$ w.r.t. the basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$, and $(B_{ij})_{i,j=1}^{n}$ is the matrix of $d\Psi|_x^{-1} \circ d\Phi|_x - \mathrm{id}_{T_x U}$, so by assumption $(B_{ij})^2 = 0$. Therefore the inverse matrix of $(\delta_{ij} + B_{ij})$ is $(\delta_{ij} - B_{ij})$, so (3.7) gives

$$\delta_{ij} - B_{ij} = \frac{\partial y_i}{\partial x_j}\Big|_x. \tag{3.8}$$

More generally, $(\delta_{ij} + t B_{ij})$ is invertible for $t \in \mathbb{C}$, with inverse $(\delta_{ij} - t B_{ij})$.

Now $\Phi|_X = \Psi|_X$ implies that $d\Psi|_x^{-1} \circ d\Phi|_x$ is the identity on $T_x X \subseteq T_x U$, and $T_x X = \mathrm{Ker}(\mathrm{Hess}_x f) = \langle \frac{\partial}{\partial x_{m+1}}, \ldots, \frac{\partial}{\partial x_n} \rangle$ by (3.5), so

$$B_{ij} = 0 \quad \text{for all } i = 1, \ldots, n \text{ and } j = m+1, \ldots, n. \tag{3.9}$$

We have $\frac{\partial^2 f}{\partial y_i \partial y_l}\Big|_x = \sum_{j,k} \frac{\partial x_j}{\partial y_i} \frac{\partial^2 f}{\partial x_j \partial x_k} \frac{\partial x_k}{\partial y_l}\Big|_x$, $\frac{\partial^2 f}{\partial x_i \partial x_l}\Big|_x = \sum_{j,k} \frac{\partial y_j}{\partial x_i} \frac{\partial^2 f}{\partial y_j \partial y_k} \frac{\partial y_k}{\partial x_l}\Big|_x$, which by (3.5) and (3.7)–(3.9) give equations equivalent to

$$B_{ij} + B_{ji} = \sum_{k=1}^{n} B_{ki} B_{kj} = 0 \quad \text{for all } i, j = 1, \ldots, m. \tag{3.10}$$

Define regular $t', x_1', \ldots, x_n', y_1', \ldots, y_n', z_1', \ldots, z_n' : \mathbb{C} \times U' \times V' \to \mathbb{C}$ by

$$t'(t, u, v) = t, \qquad\qquad x_i'(t, u, v) = x_i(u) = z_i \circ \Phi(u),$$
$$y_i'(t, u, v) = y_i(u) = z_i \circ \Psi(u), \qquad z_i'(t, u, v) = z_i(v).$$

Then $(t', y_1', \ldots, y_n', z_1', \ldots, z_n')$ are étale coordinates on $\mathbb{C} \times U' \times V'$.

Let S be an affine Zariski open neighbourhood of $\mathbb{C} \times (x, \Phi(x))$ in $\mathbb{C} \times U' \times V'$, satisfying a series of smallness conditions we will give during the proof. Regard $(t', y_1', \ldots, y_n', z_1', \ldots, z_n')$ as étale coordinates on S, and write $\pi_{\mathbb{C}} : S \to \mathbb{C}$, $\pi_U : S \to U$, $\pi_V : S \to V$ for the projections. We will work with (sheaves of) ideals in $\mathcal{O}_S$, using notation $(x_i' - z_i', \ i = 1, \ldots, n)$ to denote the ideal generated by the functions $x_1' - z_1', \ldots, x_n' - z_n'$, and

$$f \circ \pi_U - g \circ \pi_V \in (x_i' - z_i', \ i = 1, \ldots, n)$$

to mean that $f \circ \pi_U - g \circ \pi_V \in H^0(\mathcal{O}_S)$ is a section of the ideal $(x_i' - z_i', \ i = 1, \ldots, n)$. Write $I_X \subset \mathcal{O}_U$, $I_Y \subset \mathcal{O}_V$ for the ideals of functions on U, V vanishing on X, Y, and

$$\pi_U^{-1}(I_X), \pi_V^{-1}(I_Y) \subset \mathcal{O}_S$$

for the preimage ideals.

Since $x_i = z_i \circ \Phi$, the functions $x_i' - z_i'$ for $i = 1, \ldots, n$ vanish on the smooth, closed $\mathbb{C}$-subscheme $(\mathbb{C} \times (\mathrm{id} \times \Phi)(U)) \cap S$ in S, and locally these functions cut out this $\mathbb{C}$-subscheme. So making S smaller we can suppose $(\mathbb{C} \times (\mathrm{id} \times \Phi)(U)) \cap S$ is the $\mathbb{C}$-subscheme

$$x_1' - z_1' = \cdots = x_n' - z_n' = 0$$

in S. As $f = g \circ \Phi$, the function $f \circ \pi_U - g \circ \pi_V$ is zero on $\big(\mathbb{C} \times (\mathrm{id} \times \Phi)(U) \big) \cap S$. Hence

$$f \circ \pi_U - g \circ \pi_V \in \big(x_i' - z_i', \ i = 1, \ldots, n \big) \subset \mathcal{O}_S. \tag{3.11}$$

Lifting (3.11) from $\big(x_i' - z_i', \ i = 1, \ldots, n \big)$ to $\big(x_i' - z_i', \ i = 1, \ldots, n \big)^2$, making S smaller if necessary, we may choose regular $C_i : S \to \mathbb{C}$ for $i = 1, \ldots, n$ with

$$f \circ \pi_U - g \circ \pi_V - \sum_{i=1}^n C_i \cdot (x_i' - z_i') \in \big(x_i' - z_i', \ i = 1, \ldots, n \big)^2. \tag{3.12}$$

Apply $\frac{\partial}{\partial z_i'}$ to (3.12), using the étale coordinates $(t', y_1', \ldots, y_n', z_1', \ldots, z_n')$ on S. Since

$$\frac{\partial}{\partial z_i'} \big(g \circ \pi_V \big) = \frac{\partial g}{\partial z_i} \circ \pi_V \qquad \text{and} \qquad \frac{\partial}{\partial z_i'} (f \circ \pi_U) = 0 = \frac{\partial x_j'}{\partial z_i'},$$

this gives

$$C_i - \tfrac{\partial g}{\partial z_i} \circ \pi_V \in \big(x_i' - z_i', \ i = 1, \ldots, n \big).$$

Combining this with (3.12) yields

$$f \circ \pi_U - g \circ \pi_V - \sum_{i=1}^n \big(\tfrac{\partial g}{\partial z_i} \circ \pi_V \big) \cdot (x_i' - z_i') \in \big(x_i' - z_i', \ i = 1, \ldots, n \big)^2.$$

So making S smaller we can choose regular $D_{ij} : S \to \mathbb{C}$ for $i, j = 1, \ldots, n$ with $D_{ij} = D_{ji}$ and

$$f \circ \pi_U - g \circ \pi_V = \sum_{i=1}^n \big(\tfrac{\partial g}{\partial z_i} \circ \pi_V \big) \cdot (x_i' - z_i') + \sum_{i,j=1}^n D_{ij} \cdot (x_i' - z_i')(x_j' - z_j'). \tag{3.13}$$

Similarly, starting from $y_i = z_i \circ \Psi$ and $f = g \circ \Psi$ we may choose regular $E_{ij} : S \to \mathbb{C}$ for $i, j = 1, \ldots, n$ with $E_{ij} = E_{ji}$ and

$$f \circ \pi_U - g \circ \pi_V = \sum_{i=1}^n \big(\tfrac{\partial g}{\partial z_i} \circ \pi_V \big) \cdot (y_i' - z_i') + \sum_{i,j=1}^n E_{ij} \cdot (y_i' - z_i')(y_j' - z_j'). \tag{3.14}$$

Applying $\frac{\partial^2}{\partial z_i' \partial z_j'}$ to (3.13) and (3.14), restricting to $(t, x, \Phi(x))$ for $t \in \mathbb{C}$, noting that $x_i' = y_i' = z_i' = 0$ at $(t, x, \Phi(x))$, and using (3.2), we deduce that

$$D_{ij}(t, x, \Phi(x)) = E_{ij}(t, x, \Phi(x)) = \tfrac{1}{2} \tfrac{\partial^2 g}{\partial z_i \partial z_j} \Big|_{\Phi(x)}$$

$$= \begin{cases} \frac{1}{2}, & i = j \in \{1, \ldots, m\}, \\ 0, & \text{otherwise}. \end{cases} \tag{3.15}$$

Summing $1 - t'$ times (3.13) with t' times (3.14) and rearranging yields

$$f \circ \pi_U - g \circ \pi_V = \sum_{i=1}^n \Big[\tfrac{\partial g}{\partial z_i} \circ \pi_V + 2t'(1-t') \sum_{j=1}^n (D_{ij} - E_{ij})(x_j' - y_j') \Big]$$
$$\cdot \big((1-t') x_i' + t' y_i' - z_i' \big)$$
$$+ \sum_{i,j=1}^n \big[(1-t') D_{ij} + t' E_{ij} \big] \cdot \big((1-t') x_i' + t' y_i' - z_i' \big) \big((1-t') x_j' + t' y_j' - z_j' \big)$$
$$+ \sum_{i,j=1}^n t'(1-t') \big[t' D_{ij} + (1-t') E_{ij} \big] \cdot (x_i' - y_i')(x_j' - y_j'). \tag{3.16}$$

Since $x_i' - y_i' = (x_i - y_i) \circ \pi_U$, and $(x_i - y_i)|_X = z_i \circ \Phi|_X - z_i \circ \Psi|_X = 0$ as $\Phi|_X = \Psi|_X$, we see that $x_i' - y_i' \in \pi_U^{-1}(I_X)$. Thus making S smaller if necessary, we may choose regular $F_{ij} : S \to \mathbb{C}$ such that

$$x_i' - y_i' = \textstyle\sum_{j=1}^n F_{ij} \cdot \big(\tfrac{\partial f}{\partial y_j} \circ \pi_U \big) \quad \text{for } i = 1, \ldots, n. \tag{3.17}$$

Furthermore, by (3.6) when $i = 1, \ldots, m$ we may take

$$F_{ij} = \Big(\sum_{k=1}^{n} (A_{ik} \circ \Phi) \cdot \tfrac{\partial y_j}{\partial x_k} - A_{ij} \circ \Psi \Big) \circ \pi_U.$$

Restricting to $(t, x, \Phi(x))$ and using (3.4), (3.8), (3.9) and $\Phi(x) = \Psi(x)$ gives

$$F_{ij}(t, x, \Phi(x)) = -B_{ji} \quad \text{for } i = 1, \ldots, m \text{ and } j = 1, \ldots, n. \tag{3.18}$$

Applying $\frac{\partial}{\partial x_i'}$ to equation (3.13) shows that

$$\tfrac{\partial g}{\partial z_i} \circ \pi_V - \tfrac{\partial f}{\partial x_i} \circ \pi_U \in \big(x_j' - z_j', \ j = 1, \ldots, n \big).$$

Thus we may write

$$\tfrac{\partial g}{\partial z_i} \circ \pi_V = \sum_{j=1}^{n} \big(\tfrac{\partial y_j}{\partial x_i} \circ \pi_U \big) \cdot \big(\tfrac{\partial f}{\partial y_j} \circ \pi_U \big) + \sum_{j=1}^{n} G_{ij} \cdot (x_j' - z_j'), \tag{3.19}$$

where $G_{ij} : S \to \mathbb{C}$ are regular. Applying $\frac{\partial}{\partial z_j'}$ to (3.19), restricting to $(t, x, \Phi(x))$ and using (3.2) yields

$$G_{ij}(t, x, \Phi(x)) = \begin{cases} -1, & i = j \in \{1, \ldots, m\}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.20}$$

From (3.17) and (3.19) we see that

$$\tfrac{\partial g}{\partial z_i} \circ \pi_V = \sum_{j=1}^{n} H_{ij} \cdot \big(\tfrac{\partial f}{\partial y_j} \circ \pi_U \big) + \sum_{j=1}^{n} G_{ij} \cdot \big((1 - t') x_j' + t' y_j' - z_j' \big), \tag{3.21}$$

where $H_{ij} = \frac{\partial y_j}{\partial x_i} \circ \pi_U + t' \sum_{k=1}^{n} G_{ik} F_{kj}$, so that from equations (3.8), (3.9), (3.18) and (3.20) we deduce that

$$H_{ij}(t, x, \Phi(x)) = \delta_{ij} - (1 - t) B_{ji}. \tag{3.22}$$

Combining (3.16), (3.17) and (3.21) gives

$$f \circ \pi_U - g \circ \pi_V - \sum_{i=1}^{n} I_i \cdot \big((1 - t') x_i' + t' y_i' - z_i' \big)$$

$$- \sum_{i,j,k,l=1}^{n} t'(1 - t') \big[t' D_{ij} + (1 - t') E_{ij} \big] F_{ik} F_{jl} \cdot \big(\tfrac{\partial f}{\partial y_k} \circ \pi_U \big) \big(\tfrac{\partial f}{\partial y_l} \circ \pi_U \big)$$

$$\in \big((1 - t') x_i' + t' y_i' - z_i', \ i = 1, \ldots, n \big)^2, \tag{3.23}$$

$$\text{where} \quad I_i = \sum_{j=1}^{n} \Big[H_{ij} + 2t'(1 - t') \sum_{k=1}^{n} (D_{ik} - E_{ik}) F_{kj} \Big] \cdot \big(\tfrac{\partial f}{\partial y_j} \circ \pi_U \big). \tag{3.24}$$

Consider the matrix of functions $[H_{ij} + \cdots]_{i,j=1}^{n}$ appearing in (3.24). Equations (3.15) and (3.22) imply that at $(t, x, \Phi(x))$ this reduces to $(\delta_{ij} - (1 - t) B_{ji})$, which is invertible from above. Thus, making S smaller, we can suppose that $[H_{ij} + \cdots]_{i,j=1}^{n}$ in (3.24) is an invertible matrix on S. Write $[J_{ij}]_{i,j=1}^{n}$ for the inverse matrix. Then we have

$$\sum_{i,j,k,l=1}^{n} t'(1 - t') \big[t' D_{ij} + (1 - t') E_{ij} \big] F_{ik} F_{jl} \cdot \big(\tfrac{\partial f}{\partial y_k} \circ \pi_U \big) \big(\tfrac{\partial f}{\partial y_l} \circ \pi_U \big)$$

$$= t'(1 - t') \sum_{i,j=1}^{n} K_{ij} \cdot I_i I_j, \quad \text{where} \tag{3.25}$$

$$K_{ij} = \sum_{k,l,p,q=1}^{n} t'(1 - t') \big[t' D_{kl} + (1 - t') E_{kl} \big] F_{kp} F_{lq} J_{pi} J_{qj}.$$

Using (3.9), (3.10), (3.15) and (3.18) we find that

$$K_{ij}(t, x, \Phi(x)) = 0 \quad \text{for all } t \in \mathbb{C}. \tag{3.26}$$

Combining (3.23) and (3.25), making S smaller if necessary we may write

$$f \circ \pi_U - g \circ \pi_V = \sum_{i=1}^{n} I_i \cdot \left((1-t')x_i' + t'y_i' - z_i'\right) + t'(1-t') \sum_{i,j=1}^{n} K_{ij} \cdot I_i I_j$$

$$+ \sum_{i,j=1}^{n} L_{ij} \cdot \left((1-t')x_i' + t'y_i' - z_i'\right)\left((1-t')x_j' + t'y_j' - z_j'\right), \qquad (3.27)$$

for regular $L_{ij} : S \to \mathbb{C}$ for $i, j = 1, \dots, n$.

Write $(r_{ij})_{i,j=1}^{n}$ for the coordinates on $\mathbb{C}^{n^2}$. Let T be a Zariski open neighbourhood of

$$\mathbb{C} \times \left(x, \Phi(x), (0)_{i,j=1}^{n}\right)$$

in $S \times \mathbb{C}^{n^2}$ to be chosen shortly, and let W be the closed $\mathbb{C}$-subscheme of T defined by

$$W = \Big\{ \left(t, u, v, (r_{ij})_{i,j=1}^{n}\right) \in T \subseteq S \times \mathbb{C}^{n^2} \subseteq \mathbb{C} \times U \times V \times \mathbb{C}^{n^2} :$$

$$\left((1-t')x_i' + t'y_i' - z_i'\right)(t, u, v) = \sum_{j=1}^{n} r_{ij} \cdot I_j(t, u, v), \;\; i = 1, \dots, n, \qquad (3.28)$$

$$r_{ij} + t(1-t)K_{ij}(t, u, v) + \sum_{k,l=1}^{n} L_{kl}(t, u, v) \cdot r_{ki}r_{lj} = 0, \;\; i, j = 1, \dots, n \Big\}.$$

Define $\mathbb{C}$-scheme morphisms $\pi_{\mathbb{C}} : W \to \mathbb{C}$, $\pi_U : W \to U$, and $\pi_V : W \to V$ to map $\left(t, u, v, (r_{ij})_{i,j=1}^{n}\right)$ to t, u, v, respectively.

At $(t, x, \Phi(x)) \in S$ for $t \in \mathbb{C}$ we have $x_i' = y_i' = z_i' = 0$, and $I_i = 0$ by (3.24) as $\frac{\partial f}{\partial y_j}\big|_x = 0$, and $K_{ij} = 0$ by (3.26). Hence $\left(t, x, \Phi(x), (0)_{i,j=1}^{n}\right)$ satisfies the equations of (3.28), and lies in W. Define $\iota : \mathbb{C} \to W$ by

$$\iota(t) = \left(t, x, \Phi(x), (0)_{i,j=1}^{n}\right). \qquad (3.29)$$

Now $T \subseteq \mathbb{C} \times U \times V \times \mathbb{C}^{n^2}$ is smooth of dimension $1 + n + n + n^2$, and in (3.28) we impose $n + n^2$ equations, so the expected dimension of W is $(1 + 2n + n^2) - (n + n^2) = n + 1$. The linearizations of the $n + n^2$ equations in (3.28) at

$$\left(t, u, v, (r_{ij})_{i,j=1}^{n}\right) = \left(t, x, \Phi(x), (0)_{i,j=1}^{n}\right) = \iota(t)$$

are

$$dy_i|_x(\delta u) - dz_i|_{\Phi(x)}(\delta v) = 0, \quad i = 1, \dots, n,$$

$$\delta r_{ij} + dK_{ij}|_{(t,x,\Phi(x))}(\delta t \oplus \delta u \oplus \delta v) = 0, \quad i, j = 1, \dots, n, \qquad (3.30)$$

for $\delta t \in T_t\mathbb{C}$, $\delta u \in T_xU$, $\delta v \in T_{\Phi(x)}V$, and $(\delta r_{ij})_{i,j=1}^{n} \in T_{(0)_{i,j=1}^{n}}\mathbb{C}^{n^2}$, where we have used $dx_i' = dy_i'$ and $I_j = K_{ij} = 0$ at $(t, x, \Phi(x))$. As $dy_1|_x, \dots, dy_n|_x$ are a basis for T_x^*U, equations (3.30) are transverse, so W is smooth of dimension $n+1$ near $\iota(t)$. Hence, taking T small enough, we can suppose W is smooth.

It remains to prove Proposition 3.4(a)–(e). Part (a) is immediate from (3.29). For (b), the vector space of solutions $\left(\delta t, \delta u, \delta v, (\delta r_{ij})_{i,j=1}^{n}\right)$ to (3.30) is $T_{\iota(t)}W$, where

$$d(\pi_{\mathbb{C}} \times \pi_U)|_{\iota(t)} : T_{\iota(t)}W \to T_{(t,x)}(\mathbb{C} \times U)$$

and $d(\pi_{\mathbb{C}} \times \pi_V)|_{\iota(t)} : T_{\iota(t)}W \to T_{(t,\Phi(x))}(\mathbb{C} \times V)$ map $\left(\delta t, \delta u, \delta v, (\delta r_{ij})_{i,j=1}^{n}\right)$ to $(\delta t, \delta u)$ and $(\delta t, \delta v)$. By (3.30), these are isomorphisms, so $\pi_{\mathbb{C}} \times \pi_U$ and $\pi_{\mathbb{C}} \times \pi_V$ are étale near $\iota(\mathbb{C})$. Making T, W smaller, we can suppose $\pi_{\mathbb{C}} \times \pi_U$ and $\pi_{\mathbb{C}} \times \pi_V$ are étale.

For (c), we have

$$
\begin{aligned}
\big(f \circ \pi_W - g \circ \pi_U\big)\big(t, u, v, (r_{ij})_{i,j=1}^n\big) &= f(u) - g(v) = (f \circ \pi_U - g \circ \pi_V)(t, u, v) \\
&= \sum_{i=1}^n I_i \cdot \big((1-t')x_i' + t'y_i' - z_i'\big) + t'(1-t') \sum_{i,j=1}^n K_{ij} I_i I_j \\
&\quad + \sum_{i,j=1}^n L_{ij} \cdot \big((1-t')x_i' + t'y_i' - z_i'\big)\big((1-t')x_j' + t'y_j' - z_j'\big) \\
&= \sum_{i=1}^n I_i \cdot \Big(\sum_{j=1}^n r_{ij} \cdot I_j\Big) + t'(1-t') \sum_{i,j=1}^n K_{ij} I_i I_j \\
&\quad + \sum_{i,j=1}^n L_{ij} \cdot \Big(\sum_{k=1}^n r_{ik} \cdot I_k\Big)\Big(\sum_{l=1}^n r_{jl} \cdot I_l\Big) \\
&= \sum_{i,j=1}^n I_i I_j \cdot \Big[r_{ij} + t'(1-t')K_{ij} + \sum_{k,l=1}^n L_{kl} \cdot r_{ki} r_{lj}\Big] = 0,
\end{aligned}
$$

using (3.27) in the third step, the first equation of (3.28) in the fourth, rearranging and exchanging labels i, k and j, l in the fifth, and the second equation of (3.28) in the sixth. Hence $f \circ \pi_U - g \circ \pi_V = 0 : W \to \mathbb{C}$, proving (c).

For (d), from (3.28) we can show that $\big(\mathbb{C} \times (\mathrm{id} \times \Phi)(X) \times \mathbb{C}^{n^2}\big) \cap W$ is open and closed in $Z = \mathrm{Crit}(h)$, and contains $\iota(\mathbb{C})$. So making T, W smaller we can take

$$
Z = \big(\mathbb{C} \times (\mathrm{id} \times \Phi)(X) \times \mathbb{C}^{n^2}\big) \cap W,
$$

and then (d) follows as $\Phi|_X = \Psi|_X$.

For (e), observe that when $t = 0$ in (3.28), the second equation reduces to $r_{ij} = 0$ near $\iota(\mathbb{C})$ as $t(1-t)K_{ij}(t, u, v) = 0$, so making T, W smaller gives

$$
\begin{aligned}
W_0 &= \big\{\big(0, u, v, (0)_{i,j=1}^n\big) \in T : \big(x_i' - z_i'\big)(0, u, v) = 0,\ i = 1, \dots, n\big\} \\
&= \big\{\big(0, u, v, (0)_{i,j=1}^n\big) \in T : v = \Phi(u)\big\}.
\end{aligned}
$$

Hence $\Phi \circ \pi_W|_{W_0} = \pi_U|_{W_0}$. Similarly, when $t = 1$ we have

$$
\begin{aligned}
W_1 &= \big\{\big(1, u, v, (0)_{i,j=1}^n\big) \in T : \big(y_i' - z_i'\big)(0, u, v) = 0,\ i = 1, \dots, n\big\} \\
&= \big\{\big(0, u, v, (0)_{i,j=1}^n\big) \in T : v = \Psi(u)\big\},
\end{aligned}
$$

so that $\Psi \circ \pi_W|_{W_1} = \pi_U|_{W_1}$. This proves (e), and Proposition 3.4.

3.2. **Part (a):** $\det(\mathrm{d}\Psi|_{X^{\mathrm{red}}}^{-1} \circ \mathrm{d}\Phi|_{X^{\mathrm{red}}}) = \pm 1$. We work in the situation of Theorem 3.1. For each $x \in X \subseteq U$, consider the diagram of linear maps of vector spaces:

$$
\begin{CD}
0 @>>> T_x X @>>> T_x U @>\mathrm{Hess}_x f>> T_x^* U @>>> T_x^* X @>>> 0 \\
@. @V\mathrm{d}(\Phi|_X)|_x VV @V\mathrm{d}\Phi|_x VV @V(\mathrm{d}\Phi|_x^{-1})^* VV @V(\mathrm{d}(\Phi|_X)|_x^{-1})^* VV @. \\
0 @>>> T_{\Phi(x)} Y @>>> T_{\Phi(x)} V @>\mathrm{Hess}_{\Phi(x)} g>> T_{\Phi(x)}^* V @>>> T_{\Phi(x)}^* Y @>>> 0,
\end{CD}
\tag{3.31}
$$

where $T_x X$ is the Zariski tangent space of X, and $\mathrm{Hess}_x f = (\partial^2 f)|_x$ the Hessian of f at x. The rows of (3.31) are exact, and the columns isomorphisms. The outer squares of (3.31) clearly commute. We can show the central square commutes by taking second derivatives of $f = g \circ \Phi$ to get $\partial^2 f|_x = \partial^2 g|_{\Phi(x)} \circ (\mathrm{d}\Phi|_x \otimes \mathrm{d}\Phi|_x)$, and composing with $\mathrm{id} \otimes \mathrm{d}\Phi|_x^{-1}$. Thus (3.31) is commutative.

There is also an analogue of (3.31) for Ψ. Since $\Psi(x) = \Phi(x)$, we may compose the columns of (3.31) for Φ with the inverses of the columns of (3.31) for Ψ to get a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & T_x X & \longrightarrow & T_x U & \xrightarrow{\text{Hess}_x f} & T_x^* U & \longrightarrow & T_x^* X & \longrightarrow 0 \\
& & \downarrow{\substack{\text{d}(\Psi|_X)|_x^{-1}\circ \\ \text{d}(\Phi|_X)|_x \\ =\text{id}_{T_x X}}} & & \downarrow{\substack{\text{d}\Psi|_x^{-1}\circ \\ \text{d}\Phi|_x}} & & \downarrow{\substack{\text{d}\Psi|_x^*\circ \\ (\text{d}\Phi|_x^{-1})^*}} & & \downarrow{\substack{(\text{d}(\Psi|_X)|_x)^*\circ \\ (\text{d}(\Phi|_X)|_x^{-1})^* \\ =\text{id}_{T_x^* X}}} \\
0 & \longrightarrow & T_x X & \longrightarrow & T_x U & \xrightarrow{\text{Hess}_x f} & T_x^* U & \longrightarrow & T_x^* X & \longrightarrow 0,
\end{array}
\tag{3.32}
$$

where the outer morphisms are identities as $\Phi|_X = \Psi|_X$.

Choose a complementary vector subspace N_x to $T_x X$ in $T_x U$, which we think of as the normal to X in U at x, so that $T_x U = T_x X \oplus N_x$. Write $\text{Hess}'_x f$ for the restriction of $\text{Hess}_x f$ to a symmetric bilinear form on N_x. Since $T_x X = \text{Ker}(\text{Hess}_x f)$, we see that $\text{Hess}'_x f$ is a nondegenerate symmetric bilinear form on N_x. We may write equation (3.32) as

$$
\begin{array}{ccccccccc}
0 \to & T_x X & \longrightarrow & T_x X \oplus N_x & \longrightarrow & T_x^* X \oplus N_x^* & \longrightarrow & T_x^* X & \to 0 \\
& \downarrow{\text{id}} & {\tiny\begin{pmatrix}\text{id}\\0\end{pmatrix}} & \downarrow & {\tiny\begin{pmatrix}0 & 0\\0 & \text{Hess}'_x f\end{pmatrix}} & \downarrow & {\tiny\begin{pmatrix}\text{id}&0\end{pmatrix}} & \downarrow{\text{id}} & \\
0 \to & T_x X & \longrightarrow & T_x X \oplus N_x & \longrightarrow & T_x^* X \oplus N_x^* & \longrightarrow & T_x^* X & \to 0,
\end{array}
\tag{3.33}
$$

with $\text{d}\Psi|_x^{-1}\circ\text{d}\Phi|_x = \begin{pmatrix}\text{id} & A\\0 & B\end{pmatrix}$ and $\text{d}\Psi|_x^*\circ(\text{d}\Phi|_x^{-1})^* = \begin{pmatrix}\text{id} & -AB^{-1}\\0 & B^{-1}\end{pmatrix}$,

for some linear $A : N_x \to T_x X$ and $B : N_x \to N_x$. Then (3.33) commuting implies that B preserves the nondegenerate symmetric bilinear form $\text{Hess}'_x f$ on N_x, and $\det B = \pm 1$. So $\det\left(\text{d}\Psi|_x^{-1} \circ \text{d}\Phi|_x\right) = \det\left(\begin{smallmatrix}\text{id} & A\\0 & B\end{smallmatrix}\right) = \det B = \pm 1$ for $x \in X$.

Thus, as a map of topological spaces, $\det\left(\text{d}\Psi|_{X^{\text{red}}}^{-1} \circ \text{d}\Phi|_{X^{\text{red}}}\right) : X^{\text{red}} \to \mathbb{C} \setminus \{0\}$ actually maps $X^{\text{red}} \to \{\pm 1\}$. Since it is continuous, it is locally constant. Now if $f, g : Y \to Z$ are morphisms of $\mathbb{C}$-schemes with Y reduced, then $f = g$ if and only if $f(y) = g(y)$ for each point $y \in Y$. Applying this to compare $\det\left(\text{d}\Psi|_{X^{\text{red}}}^{-1} \circ \text{d}\Phi|_{X^{\text{red}}}\right) : X^{\text{red}} \to \mathbb{C} \setminus \{0\}$ locally with the constant maps 1 or -1 on X^{red} shows that $\det\left(\text{d}\Psi|_{X^{\text{red}}}^{-1} \circ \text{d}\Phi|_{X^{\text{red}}}\right)$ is a locally constant map $X^{\text{red}} \to \{\pm 1\} \subset \mathbb{C} \setminus \{0\}$ as a $\mathbb{C}$-scheme morphism. This proves Theorem 3.1(a).

3.3. **Part (b):** $\mathcal{PV}_\Phi = \det\left(\text{d}\Psi|_{X^{\text{red}}}^{-1} \circ \text{d}\Phi|_{X^{\text{red}}}\right) \cdot \mathcal{PV}_\Psi$. For Theorem 3.1(b), we begin with the following proposition.

Proposition 3.5. *Let* $U, V, \Phi, \Psi, f, g, X, Y$ *be as in Theorem* 3.1, *and suppose* $x \in X$ *with* $\left(\text{d}\Psi|_x^{-1} \circ \text{d}\Phi|_x - \text{id}_{T_x U}\right)^2 = 0$. *Then there exists a Zariski open neighbourhood* X' *of* x *in* X *such that* $\mathcal{PV}_\Phi|_{X'} = \mathcal{PV}_\Psi|_{X'}$.

Proof. Apply Proposition 3.4 to get $W, \pi_{\mathbb{C}}, \pi_U, \pi_V, \iota, h, Z$. Then apply Proposition 2.8 with $Z, X, x, \pi_{\mathbb{C}}|_Z, \pi_U|_Z, \iota, \mathcal{PV}_{U,f}^\bullet, \Phi|_X^*\left(\mathcal{PV}_{V,g}^\bullet\right), \mathcal{PV}_\Phi, \mathcal{PV}_\Psi$ in place of $W, X, x, \pi_{\mathbb{C}}, \pi_X, \iota, \mathcal{P}^\bullet, \mathcal{Q}^\bullet, \alpha,$

β, respectively, and with γ defined by the commuting diagram of isomorphisms:

$$
\begin{array}{ccc}
\mathcal{PV}^{\bullet}_{W,h} & \xrightarrow{\;\;\mathcal{PV}_{\pi_{\mathbb{C}}\times\pi_V}\;\;} & (\pi_{\mathbb{C}}\times\pi_V)|^*_Z\big(\mathcal{PV}^{\bullet}_{\mathbb{C}\times V,0\boxplus g}\big) \\
\Big\downarrow{\scriptstyle\mathcal{PV}_{\pi_{\mathbb{C}}\times\pi_U}} & & \Big\downarrow{\scriptstyle(\pi_{\mathbb{C}}\times\pi_V)|^*_Z(\mathcal{TS}_{\mathbb{C},0,V,g})} \\
(\pi_{\mathbb{C}}\times\pi_U)|^*_Z\big(\mathcal{PV}^{\bullet}_{\mathbb{C}\times U,0\boxplus f}\big) & & (\pi_{\mathbb{C}}\times\pi_V)|^*_Z\big(\mathcal{PV}^{\bullet}_{\mathbb{C},0}\overset{L}{\boxtimes}\mathcal{PV}^{\bullet}_{V,g}\big) \\
\Big\downarrow{\scriptstyle(\pi_{\mathbb{C}}\times\pi_V)|^*_Z(\mathcal{TS}_{\mathbb{C},0,U,f})} & & \Big\downarrow{\scriptstyle\delta'} \\
(\pi_{\mathbb{C}}\times\pi_U)|^*_Z\big(\mathcal{PV}^{\bullet}_{\mathbb{C},0}\overset{L}{\boxtimes}\mathcal{PV}^{\bullet}_{U,f}\big) & \big[\pi_{\mathbb{C}}|^*_Z\big(A_{\mathbb{C}}[1]\big)\big]\overset{L}{\otimes}\big[\pi_V|^*_Z\big(\mathcal{PV}^{\bullet}_{V,g}\big)\big] \\
\Big\downarrow{\scriptstyle\delta} & & \Big\downarrow{\scriptstyle\epsilon'} \\
\big[\pi_{\mathbb{C}}|^*_Z\big(A_{\mathbb{C}}[1]\big)\big]\overset{L}{\otimes}\big[\pi_U|^*_Z\big(\mathcal{PV}^{\bullet}_{U,f}\big)\big] & & \pi_V|^*_Z[1]\big(\mathcal{PV}^{\bullet}_{V,g}\big) \\
\Big\downarrow{\scriptstyle\epsilon} & & \Big\| \\
\pi_U|^*_Z[1]\big(\mathcal{PV}^{\bullet}_{U,f}\big) & \xrightarrow{\;\;\gamma\;\;} & \pi_U|^*_Z[1]\big(\Phi|^*_X(\mathcal{PV}^{\bullet}_{V,g})\big).
\end{array}
\tag{3.34}
$$

Here $\mathcal{TS}_{\mathbb{C},0,U,f},\mathcal{TS}_{\mathbb{C},0,V,g}$ are as in (2.8), δ,δ' come from $\mathcal{PV}_{\mathbb{C},0}\cong A_{\mathbb{C}}[1]$, and ϵ,ϵ' come from $\pi_{\mathbb{C}}|^*_Z(A_{\mathbb{C}})\cong A_Z$ and $A_Z\otimes\mathcal{P}^{\bullet}\cong\mathcal{P}^{\bullet}$ for $\mathcal{P}^{\bullet}\in\mathrm{Perv}(Z)$.

Then the hypothesis $\pi_X|^*_{W_0}(\alpha)=j^*_0[-1](\gamma)$ in Proposition 2.8 follows from comparing $j^*_0[-1]$ applied to (3.34) with the commuting diagram

$$
\begin{array}{ccc}
j^*_0[-1]\big(\mathcal{PV}^{\bullet}_{W,h}\big) & \xrightarrow{\;j^*_0[-1](\mathcal{PV}_{\pi_{\mathbb{C}}\times\pi_V})\;} & j^*_0[-1]\circ(\pi_{\mathbb{C}}\times\pi_V)|^*_Z\big(\mathcal{PV}^{\bullet}_{\mathbb{C}\times V,0\boxplus g}\big) \\
\Big\downarrow{\scriptstyle\cong} & & \Big\downarrow{\scriptstyle\cong} \\
\mathcal{PV}^{\bullet}_{W_0,h|_{W_0}} & \xrightarrow{\;\;\mathcal{PV}_{\pi_V|_{W_0}}\;\;} & \pi_V|^*_{Z\cap W_0}\big(\mathcal{PV}^{\bullet}_{V,g}\big) \\
\Big\downarrow{\scriptstyle\mathcal{PV}_{\pi_U|_{W_0}}} & & \Big\| \\
\pi_V|^*_{Z\cap W_0}\big(\mathcal{PV}^{\bullet}_{U,f}\big) & \xrightarrow{\;\pi_U|^*_{Z\cap W_0}(\mathcal{PV}_{\Phi})\;} & \pi_U|^*_{Z\cap W_0}\circ\Phi|^*_X\big(\mathcal{PV}^{\bullet}_{V,g}\big),
\end{array}
$$

where $j_0:Z\cap W_0\hookrightarrow Z$ is the inclusion, and the bottom square commutes by Proposition 3.4(e) and (2.18). Similarly $\pi_X|^*_{W_1}(\beta)=j^*_1[-1](\gamma)$. Hence Proposition 2.8 gives Zariski open $x\in X'\subseteq X$ with $\mathcal{PV}_{\Phi}|_{X'}=\mathcal{PV}_{\Psi}|_{X'}$. $\qquad\square$

Now to prove Theorem 3.1(b), let $x\in X$ be arbitrary. As in §3.2, we can choose a splitting $T_xU=T_xX\oplus N_x$ such that

$$
\mathrm{d}\Psi|^{-1}_x\circ\mathrm{d}\Phi|_x=\begin{pmatrix}\mathrm{id} & A\\ 0 & B\end{pmatrix}:\begin{matrix}T_xX\oplus\\ N_x\end{matrix}\longrightarrow\begin{matrix}T_xX\oplus\\ N_x\end{matrix}
\tag{3.35}
$$

for linear $A:N_x\to T_xX$ and $B:N_x\to N_x$, where B preserves the nondegenerate symmetric bilinear form $\mathrm{Hess}'_x f$ on N_x.

Choose a Zariski open neighbourhood U' of x in U and a splitting $TU'=E\oplus F$ for algebraic vector subbundles $E,F\subseteq TU$ with $E|_x=T_xX$ and $F|_x=N_x$. Then $\mathrm{d}f|_{U'}=\alpha\oplus\beta$ for unique $\alpha\in H^0(E)$ and $\beta\in H^0(F)$, and $X\cap U'$ is defined by $\alpha=\beta=0$.

Since $\mathrm{Hess}_x f=\partial(\mathrm{d}f)|_x$ is nondegenerate on N_x, we see that $\nabla\beta|_x:T_xU\to F|_x$ induces an isomorphism $N_x\to F|_x$, so $\nabla\beta|_x$ is surjective. Therefore $S:=\beta^{-1}(0)$ is a smooth $\mathbb{C}$-subscheme of U' near x, and making U' smaller, we can suppose S is smooth.

Set $e=f|_S:S\to\mathbb{C}$. Then the isomorphism $T^*S\cong E|_S$ identifies $\mathrm{d}e\in H^0(T^*S)$ with $\alpha|_S\in H^0(E|_S)$. Hence $\mathrm{Crit}(e:S\to\mathbb{C})=\mathrm{Crit}(f|_{U'}:U'\to\mathbb{C})=X\cap U'$, as $\mathbb{C}$-subschemes of U.

By [23, Prop. 2.23] quoted in Theorem 5.1(i) below, there exist a smooth $\mathbb{C}$-scheme R, morphisms $\gamma:R\to U'$, $\delta:R\to S$, $\epsilon:R\to\mathbb{C}^n$ where $n=\dim U'-\dim S$, and $r\in R$, such that

$\gamma(r) = x$, $\gamma|_Q = \delta|_Q$, $f \circ \gamma = e \circ \delta + (z_1^2 + \cdots + z_n^2) \circ \epsilon : R \to \mathbb{C}$, and the following commutes with horizontal morphisms étale:

$$
\begin{array}{ccccc}
S & \xleftarrow{\quad \gamma|_Q \quad} & Q := \gamma^{-1}(S) & \xrightarrow{\quad \gamma|_Q \quad} & S \\
\downarrow{\scriptstyle\subset} & & \downarrow{\scriptstyle\subset} & & \downarrow{\scriptstyle \mathrm{id}_S \times 0} \\
U' & \xleftarrow{\quad \gamma \quad} & R & \xrightarrow{\quad \delta \times \epsilon \quad} & S \times \mathbb{C}^n.
\end{array}
\tag{3.36}
$$

Taking derivatives at $r \in Q \subseteq R$ in (3.36) gives a commutative diagram

$$
\begin{array}{ccccc}
T_x X = T_x S & \xleftarrow[\mathrm{d}(\gamma|_Q)|_r]{\cong} & T_r Q & \xrightarrow[\mathrm{d}(\gamma|_Q)|_r]{\cong} & T_x X = T_x S \\
\downarrow{\scriptstyle\subset} & & \downarrow{\scriptstyle\subset} & & \downarrow{\scriptstyle \mathrm{id} \times 0} \\
T_x X \oplus N_x = T_x U & \xleftarrow[\cong]{\mathrm{d}\gamma|_r} & T_r R & \xrightarrow[\cong]{\mathrm{d}(\delta \times \epsilon)|_r} & T_x X \oplus T_0 \mathbb{C}^n.
\end{array}
$$

Therefore $\mathrm{d}(\delta \times \epsilon)|_r \circ \mathrm{d}\gamma|_r^{-1} : T_x X \oplus N_x \to T_x X \oplus T_0 \mathbb{C}^n$ is the identity on $T_x X$, and induces an isomorphism $N_x \to T_0 \mathbb{C}^n$, which as $f \circ \gamma = e \circ \delta + (z_1^2 + \cdots + z_n^2) \circ \epsilon$ identifies $\mathrm{Hess}'_x f$ on N_x with $\mathrm{Hess}_0(z_1^2 + \cdots + z_n^2) = \mathrm{d}z_1 \otimes \mathrm{d}z_1 + \cdots + \mathrm{d}z_n \otimes \mathrm{d}z_n$ on $T_0 \mathbb{C}^n$. Thus, the linear isomorphism $B : N_x \to N_x$ above preserving $\mathrm{Hess}'_x f$ is identified with a linear isomorphism $M : \mathbb{C}^n \to \mathbb{C}^n$ preserving $\mathrm{d}z_1 \otimes \mathrm{d}z_1 + \cdots + \mathrm{d}z_n \otimes \mathrm{d}z_n$, that is, $M \in \mathrm{O}(n, \mathbb{C})$ satisfies

$$
\begin{pmatrix} \mathrm{id} & 0 \\ 0 & B \end{pmatrix} \circ \mathrm{d}\gamma|_r \circ \mathrm{d}(\delta \times \epsilon)|_r^{-1} = \mathrm{d}\gamma|_r \circ \mathrm{d}(\delta \times \epsilon)|_r^{-1} \circ \begin{pmatrix} \mathrm{id} & 0 \\ 0 & M \end{pmatrix}.
\tag{3.37}
$$

Define P to be the $\mathbb{C}$-scheme fibre product $P = R \times_{\delta \times (M \circ \epsilon), S \times \mathbb{C}^n, \delta \times \epsilon} R$, with projections $\pi_1, \pi_2 : P \to R$. Then P is smooth and π_1, π_2 are étale, as $R, S \times \mathbb{C}^n$ are smooth and

$$
\delta \times (M \circ \epsilon), \delta \times \epsilon : R \to S \times \mathbb{C}^n
$$

are étale. As $r \in R$ with $(\delta \times (M \circ \epsilon))(r) = (x, 0) = (\delta \times \epsilon)(r)$, we have a point $p \in P$ with $\pi_1(p) = \pi_2(p) = r$. Define $d = f \circ \gamma \circ \pi_1 : P \to \mathbb{C}$ and $Z = \mathrm{Crit}(d)$. Then

$$
\begin{aligned}
d = f \circ \gamma \circ \pi_1 &= (e \boxplus z_1^2 + \cdots + z_n^2) \circ (\delta \times \epsilon) \circ \pi_1 \\
&= (e \boxplus z_1^2 + \cdots + z_n^2) \circ (\delta \times (M \circ \epsilon)) \circ \pi_1 \\
&= (e \boxplus z_1^2 + \cdots + z_n^2) \circ (\delta \times \epsilon) \circ \pi_2 = f \circ \gamma \circ \pi_2.
\end{aligned}
\tag{3.38}
$$

Consider the étale morphisms $\Phi \circ \gamma \circ \pi_1, \Psi \circ \gamma \circ \pi_2 : P \to V$. Both map $p \mapsto \Phi(x)$, and satisfy $g \circ (\Phi \circ \gamma \circ \pi_1) = d = g \circ (\Psi \circ \gamma \circ \pi_2)$ by (3.38) and $g \circ \Phi = f = g \circ \Psi$. Taking derivatives at p to get linear maps $T_p P \to T_{\Phi(x)} V$, we find that

$$
\begin{aligned}
\mathrm{d}(\Psi \circ \gamma \circ \pi_2)|_p &= \mathrm{d}\Psi|_x \circ \mathrm{d}\gamma|_r \circ \mathrm{d}(\delta \times \epsilon)|_r^{-1} \circ \mathrm{d}((\delta \times \epsilon) \circ \pi_2)|_p \\
&= \mathrm{d}\Psi|_x \circ \mathrm{d}\gamma|_r \circ \mathrm{d}(\delta \times \epsilon)|_r^{-1} \circ \mathrm{d}((\delta \times (M \circ \epsilon)) \circ \pi_1)|_p \\
&= \mathrm{d}\Psi|_x \circ \mathrm{d}\gamma|_r \circ \mathrm{d}(\delta \times \epsilon)|_r^{-1} \circ \begin{pmatrix} \mathrm{id} & 0 \\ 0 & M \end{pmatrix} \circ \mathrm{d}(\delta \times \epsilon)|_r \circ \mathrm{d}\pi_1|_p \\
&= \mathrm{d}\Psi|_x \circ \begin{pmatrix} \mathrm{id} & 0 \\ 0 & B \end{pmatrix} \circ \mathrm{d}\gamma|_r \circ \mathrm{d}(\delta \times \epsilon)|_r^{-1} \circ \mathrm{d}(\delta \times \epsilon)|_r \circ \mathrm{d}\pi_1|_p \\
&= \mathrm{d}\Psi|_x \circ \begin{pmatrix} \mathrm{id} & -AB^{-1} \\ 0 & \mathrm{id} \end{pmatrix} \begin{pmatrix} \mathrm{id} & A \\ 0 & B \end{pmatrix} \circ \mathrm{d}\gamma|_r \circ \mathrm{d}\pi_1|_p \\
&= \mathrm{d}\Psi|_x \circ \begin{pmatrix} \mathrm{id} & -AB^{-1} \\ 0 & \mathrm{id} \end{pmatrix} \circ \mathrm{d}\Psi|_x^{-1} \circ \mathrm{d}(\Phi \circ \gamma \circ \pi_1)|_p,
\end{aligned}
\tag{3.39}
$$

using $(\delta \times (M \circ \epsilon)) \circ \pi_1 = (\delta \times \epsilon) \circ \pi_2$ in the second step, (3.37) in the fourth, and (3.35) in the sixth. Since $\left[\left(\begin{smallmatrix} \mathrm{id} & -AB^{-1} \\ 0 & \mathrm{id} \end{smallmatrix}\right) - \mathrm{id}\right]^2 = 0$, equation (3.39) implies that

$$\left(\mathrm{d}(\Psi \circ \gamma \circ \pi_2)|_p^{-1} \circ \mathrm{d}(\Phi \circ \gamma \circ \pi_1)|_p - \mathrm{id}_{T_p P}\right)^2 = 0,$$

and thus Proposition 3.5 gives a Zariski open neighbourhood P' of p in P such that

$$\mathcal{PV}_{\Phi \circ \gamma \circ \pi_1}|_{P'} = \mathcal{PV}_{\Psi \circ \gamma \circ \pi_2}|_{P'} : \mathcal{PV}_{P,d}^\bullet|_{P'} \longrightarrow (\Phi \circ \gamma \circ \pi_1)|_Z^* \left(\mathcal{PV}_{V,g}^\bullet\right)|_{P'}. \tag{3.40}$$

Since $(\delta \times (M \circ \epsilon)) \circ \pi_1 = (\delta \times \epsilon) \circ \pi_2 : P \to S \times \mathbb{C}^n$ are étale with

$$(e \boxplus z_1^2 + \cdots + z_n^2) \circ (\delta \times (M \circ \epsilon)) \circ \pi_1 = d = (e \boxplus z_1^2 + \cdots + z_n^2) \circ (\delta \times \epsilon) \circ \pi_2,$$

we see using (2.8) and (2.18) that

$$
\begin{aligned}
\pi_1|_Z^* \left[\mathcal{PV}_\delta \overset{L}{\boxtimes} \left(M|_{\{0\}}^*(\mathcal{PV}_\epsilon) \circ \mathcal{PV}_M\right)\right] \circ \mathcal{PV}_{\pi_1} &= \pi_1|_Z^* \left[\mathcal{PV}_\delta \overset{L}{\boxtimes} \mathcal{PV}_{M \circ \epsilon}\right] \circ \mathcal{PV}_{\pi_1} \\
&\cong \pi_1|_Z^*(\mathcal{PV}_{\delta \times (M \circ \epsilon)}) \circ \mathcal{PV}_{\pi_1} = \mathcal{PV}_{(\delta \times (M \circ \epsilon)) \circ \pi_1} = \mathcal{PV}_{(\delta \times \epsilon) \circ \pi_2} \\
&= \pi_2|_Z^*(\mathcal{PV}_{\delta \times \epsilon}) \circ \mathcal{PV}_{\pi_2} \cong \pi_2|_Z^* \left[\mathcal{PV}_\delta \overset{L}{\boxtimes} \mathcal{PV}_\epsilon\right] \circ \mathcal{PV}_{\pi_2},
\end{aligned}
\tag{3.41}
$$

where '$\cong$' are equalities after identifying both sides of (2.8). Since $\pi_1|_Z = \pi_2|_Z$, and $M|_{\{0\}} = \mathrm{id}_{\{0\}}$, and Example 2.16 shows that $\mathcal{PV}_M$ in (2.19) is multiplication by $\det M$, equation (3.41) implies that

$$\det M \cdot \pi_1|_Z^* \left[\mathcal{PV}_\delta \overset{L}{\boxtimes} \mathcal{PV}_\epsilon\right] \circ \mathcal{PV}_{\pi_1} = \pi_1|_Z^* \left[\mathcal{PV}_\delta \overset{L}{\boxtimes} \mathcal{PV}_\epsilon\right] \circ \mathcal{PV}_{\pi_2}.$$

As $\pi_1|_Z^* \left[\mathcal{PV}_\delta \overset{L}{\boxtimes} \mathcal{PV}_\epsilon\right]$ is an isomorphism, this gives

$$\det M \cdot \mathcal{PV}_{\pi_1} = \mathcal{PV}_{\pi_2} : \mathcal{PV}_{P,d}^\bullet \longrightarrow \pi_1|_Z^* \left(\mathcal{PV}_{R,e}^\bullet\right). \tag{3.42}$$

Writing $Z' = Z \cap P'$, we now have

$$
\begin{aligned}
(\gamma \circ \pi_1)|_{Z'}^*(\mathcal{PV}_\Phi) \circ \mathcal{PV}_{\gamma \circ \pi_1}|_{P'} &= \mathcal{PV}_{\Phi \circ \gamma \circ \pi_1}|_{P'} = \mathcal{PV}_{\Psi \circ \gamma \circ \pi_2}|_{P'} \\
&= \pi_2|_{Z'}^*(\mathcal{PV}_{\Psi \circ \gamma}) \circ \mathcal{PV}_{\pi_2}|_{P'} \\
&= \det M \cdot \pi_1|_{Z'}^*(\mathcal{PV}_{\Psi \circ \gamma}) \circ \mathcal{PV}_{\pi_1}|_{P'} = \det M \cdot \mathcal{PV}_{\Psi \circ \gamma \circ \pi_1}|_{P'} \\
&= \det M \cdot (\gamma \circ \pi_1)|_{Z'}^*(\mathcal{PV}_\Psi) \circ \mathcal{PV}_{\gamma \circ \pi_1}|_{P'},
\end{aligned}
\tag{3.43}
$$

using (3.40) in the second step, (3.42) and $\pi_1|_{Z'} = \pi_2|_{Z'}$ in the fourth, and (2.18) in the rest. As $\mathcal{PV}_{\gamma \circ \pi_1}|_{P'}$ is an isomorphism, (3.43) implies that

$$(\gamma \circ \pi_1)|_{Z'}^*(\mathcal{PV}_\Phi) = \det M \cdot (\gamma \circ \pi_1)|_{Z'}^*(\mathcal{PV}_\Psi),$$

and by Theorem 2.7(i) this implies that

$$\mathcal{PV}_\Phi|_{X'} = \det M \cdot \mathcal{PV}_\Phi|_{X'}, \tag{3.44}$$

where $X' = (\gamma \circ \pi_1)(Z')$ is a Zariski open neighbourhood of x in X, since $(\gamma \circ \pi_1)|_{Z'} : Z' \to X$ is étale with $\gamma \circ \pi_1(p) = x$. Now (3.35) and (3.37) give

$$\det\left(\mathrm{d}\Psi|_x^{-1} \circ \mathrm{d}\Phi|_x\right) = \det\left(\begin{smallmatrix} \mathrm{id} & A \\ 0 & B \end{smallmatrix}\right) = \det B = \det M.$$

So (3.44) proves that (3.1) holds near x in X. As this is true for all $x \in X$, Theorem 3.1(b) follows.

3.4. $\mathscr{D}$-modules and mixed Hodge modules.

The proof of Proposition 3.4 applies verbatim also in the analytic context. Theorem 3.1(a),(b) then follow from Proposition 3.4 and the argument given above, using §2.5, including the Sheaf Property (x) for morphisms. Hence all these results carry over to our other contexts §2.6–§2.10.

4. Dependence of $\mathcal{PV}^\bullet_{U,f}$ on f

We will use the following notation:

Definition 4.1. Let U be a smooth $\mathbb{C}$-scheme, let $f : U \to \mathbb{C}$ be a regular function, and let X equal $\mathrm{Crit}(f)$ as a closed $\mathbb{C}$-subscheme of U. Write $I_X \subseteq \mathcal{O}_U$ for the sheaf of ideals of regular functions $U \to \mathbb{C}$ vanishing on X, so that $I_X = I_{\mathrm{d}f}$. For each $k = 1, 2, \ldots$, write $X^{(k)}$ for the kth *order thickening of X in U*, that is, $X^{(k)}$ is the closed $\mathbb{C}$-subscheme of U defined by the sheaf of ideals I_X^k in $\mathcal{O}_U$. Also write X^{red} for the reduced $\mathbb{C}$-subscheme of U.

Then we have a chain of inclusions of closed $\mathbb{C}$-subschemes

$$X^{\mathrm{red}} \subseteq X = X^{(1)} \subseteq X^{(2)} \subseteq X^{(3)} \subseteq \cdots \subseteq U. \tag{4.1}$$

Write $f^{(k)} := f|_{X^{(k)}} : X^{(k)} \to \mathbb{C}$, and $f^{\mathrm{red}} := f|_{X^{\mathrm{red}}} : X^{\mathrm{red}} \to \mathbb{C}$, so that $f^{(k)}, f^{\mathrm{red}}$ are regular functions on the $\mathbb{C}$-schemes $X^{(k)}, X^{\mathrm{red}}$. Note that $f^{\mathrm{red}} : X^{\mathrm{red}} \to \mathbb{C}$ is locally constant, since $X = \mathrm{Crit}(f)$.

We also use the same notation for complex analytic spaces.

In §2.4 we defined the perverse sheaf of vanishing cycles $\mathcal{PV}^\bullet_{U,f}$ in $\mathrm{Perv}(X)$. So we can ask: how much of the sequence (4.1) does $\mathcal{PV}^\bullet_{U,f}$ depend on? That is, is $\mathcal{PV}^\bullet_{U,f}$ (canonically?) determined by $(X^{\mathrm{red}}, f^{\mathrm{red}})$, or by $(X^{(k)}, f^{(k)})$ for some $k \geq 1$, as well as by (U, f)? Our next theorem shows that $\mathcal{PV}^\bullet_{U,f}$ is determined up to canonical isomorphism by $(X^{(3)}, f^{(3)})$, and hence a fortiori also by $(X^{(k)}, f^{(k)})$ for $k > 3$:

Theorem 4.2. *Let U, V be smooth $\mathbb{C}$-schemes, $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ be regular functions, and $X = \mathrm{Crit}(f)$, $Y = \mathrm{Crit}(g)$ as closed $\mathbb{C}$-subschemes of U, V, so that §2.4 defines perverse sheaves $\mathcal{PV}^\bullet_{U,f}, \mathcal{PV}^\bullet_{V,g}$ on X, Y. Define $X^{(3)}, f^{(3)}$ and $Y^{(3)}, g^{(3)}$ as in Definition 4.1, and suppose $\Phi : X^{(3)} \to Y^{(3)}$ is an isomorphism with $g^{(3)} \circ \Phi = f^{(3)}$, so that $\Phi|_X : X \to Y \subseteq Y^{(3)}$ is an isomorphism.*

Then there is a canonical isomorphism in $\mathrm{Perv}(X)$

$$\Omega_\Phi : \mathcal{PV}^\bullet_{U,f} \longrightarrow \Phi|_X^*(\mathcal{PV}^\bullet_{V,g}), \tag{4.2}$$

which is characterized by the property that if T is a smooth $\mathbb{C}$-scheme and $\pi_U : T \to U$, $\pi_V : T \to V$ are étale morphisms with $e := f \circ \pi_U = g \circ \pi_V : T \to \mathbb{C}$, so that $\pi_U|_Q : Q \to X$, $\pi_V|_Q : Q \to Y$ are étale for $Q := \mathrm{Crit}(e)$, and $\Phi \circ \pi_U|_{Q^{(2)}} = \pi_V|_{Q^{(2)}} : Q^{(2)} \to Y^{(2)}$, then

$$\pi_U|_Q^*(\Omega_\Phi) \circ \mathcal{PV}_{\pi_U} = \mathcal{PV}_{\pi_V} : \mathcal{PV}^\bullet_{T,e} \longrightarrow \pi_V|_Q^*(\mathcal{PV}^\bullet_{U,f}). \tag{4.3}$$

Also the following commute, where $\sigma_{U,f}, \sigma_{V,g}, \tau_{U,f}, \tau_{V,g}$ are as in (2.6)–(2.7):

$$
\begin{array}{ccc}
\mathcal{PV}^\bullet_{U,f} & \xrightarrow{\quad\quad\quad\quad \sigma_{U,f} \quad\quad\quad\quad} & \mathbb{D}_X(\mathcal{PV}^\bullet_{U,f}) \\
\Big\downarrow{\scriptstyle \Omega_\Phi} & & {\scriptstyle \mathbb{D}_X(\Omega_\Phi)} \Big\uparrow \\
\Phi|_X^*(\mathcal{PV}^\bullet_{V,g}) \xrightarrow{\;\Phi|_X^*(\sigma_{V,g})\;} \Phi|_X^*\big(\mathbb{D}_Y(\mathcal{PV}^\bullet_{V,g})\big) & \xrightarrow{\;\cong\;} & \mathbb{D}_X\big(\Phi|_X^*(\mathcal{PV}^\bullet_{V,g})\big),
\end{array} \tag{4.4}
$$

$$
\begin{array}{ccc}
\mathcal{PV}^\bullet_{U,f} & \xrightarrow{\quad\quad\quad\quad \tau_{U,f} \quad\quad\quad\quad} & \mathcal{PV}^\bullet_{U,f} \\
\Big\downarrow{\scriptstyle \Omega_\Phi} & & {\scriptstyle \Omega_\Phi} \Big\downarrow \\
\Phi|_X^*(\mathcal{PV}^\bullet_{V,g}) & \xrightarrow{\quad\quad \Phi|_X^*(\tau_{V,g}) \quad\quad} & \Phi|_X^*(\mathcal{PV}^\bullet_{V,g}).
\end{array} \tag{4.5}
$$

If there exists an étale morphism $\Xi : U \to V$ with

$$g \circ \Xi = f : U \to \mathbb{C} \quad\quad \text{and} \quad\quad \Xi|_{X^{(3)}} = \Phi : X^{(3)} \to Y^{(3)},$$

then $\Omega_\Phi = \mathcal{PV}_\Xi$, for $\mathcal{PV}_\Xi$ as in (2.14).

If W is another smooth $\mathbb{C}$-scheme, $h : W \to \mathbb{C}$ is a regular function, $Z = \mathrm{Crit}(h)$, and $\Psi : Y^{(3)} \to Z^{(3)}$ is an isomorphism with $h^{(3)} \circ \Psi = g^{(3)}$, then

$$\Omega_{\Psi \circ \Phi} = \Phi|_X^*(\Omega_\Psi) \circ \Omega_\Phi : \mathcal{PV}_{U,f}^\bullet \longrightarrow (\Psi \circ \Phi)|_X^*(\mathcal{PV}_{W,h}^\bullet). \tag{4.6}$$

If $U = V$, $f = g$, $X = Y$ and $\Phi = \mathrm{id}_{X^{(3)}}$ then $\Omega_{\mathrm{id}_{X^{(3)}}} = \mathrm{id}_{\mathcal{PV}_{U,f}^\bullet}$.

The analogues of all the above also hold with appropriate modifications for $\mathscr{D}$-modules on $\mathbb{C}$-schemes, for perverse sheaves and $\mathscr{D}$-modules on complex analytic spaces, and for mixed Hodge modules on $\mathbb{C}$-schemes and complex analytic spaces, as in §2.6–§2.10.

We will prove Theorem 4.2 in §4.2–§4.3. The proof for $\mathbb{C}$-schemes depends on the case $k = 2$ of the following proposition, proved in §4.1:

Proposition 4.3. *Let U, V be smooth $\mathbb{C}$-schemes, $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ be regular functions, and $X = \mathrm{Crit}(f) \subseteq U$, $Y = \mathrm{Crit}(g) \subseteq V$. Using the notation of Definition 4.1, suppose $\Phi : X^{(k+1)} \to Y^{(k+1)}$ is an isomorphism with $g^{(k+1)} \circ \Phi = f^{(k+1)}$ for some $k \geqslant 2$. Then for each $x \in X$ we can choose a smooth $\mathbb{C}$-scheme T and étale morphisms $\pi_U : T \to U$, $\pi_V : T \to V$ such that*

(a) $e := f \circ \pi_U = g \circ \pi_V : T \to \mathbb{C}$;
(b) *setting $Q = \mathrm{Crit}(e)$, then $\pi_U|_{Q^{(k)}} : Q^{(k)} \to X^{(k)} \subseteq U$ is an isomorphism with a Zariski open neighbourhood $\tilde{X}^{(k)}$ of x in $X^{(k)}$; and*
(c) $\Phi \circ \pi_U|_{Q^{(k)}} = \pi_V|_{Q^{(k)}} : Q^{(k)} \to Y^{(k)}$.

The proof of Proposition 4.3 is similar to that of Proposition 3.5 in §3.1. One can also prove an analogue of Proposition 4.3 when $k = 1$, but in part (b) $\pi_U|_{Q^{(1)}} : Q^{(1)} \to X^{(1)}$ must be étale rather than a Zariski open inclusion.

In Proposition 4.3, we start with $\Phi : X^{(k+1)} \xrightarrow{\cong} Y^{(k+1)}$, but we construct T, π_U, π_V with $\Phi \circ \pi_U|_{Q^{(k)}} = \pi_V|_{Q^{(k)}}$. One might expect to find T, π_U, π_V with $\Phi \circ \pi_U|_{Q^{(k+1)}} = \pi_V|_{Q^{(k+1)}}$, but the next example shows this is not possible.

Example 4.4. Let U, V be open neighbourhoods of 0 in $\mathbb{C}$, and $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ be regular functions given as power series by $f(x) = x^{m+1}$ and $g(y) = y^{m+1} + Ay^{(k+1)m} + \cdots$, for $k, m \geqslant 2$ and $0 \neq A \in \mathbb{C}$, where 0 is the only critical point of g.

Then $X := \mathrm{Crit}(f) = \mathrm{Spec}\big(\mathbb{C}[x]/(x^m)\big)$ and $Y := \mathrm{Crit}(g) = \mathrm{Spec}\big(\mathbb{C}[y]/(y^m)\big)$, so $X^{(k+1)} = \mathrm{Spec}\big(\mathbb{C}[x]/(x^{(k+1)m})\big)$, $f^{(k+1)} = x^{m+1} + (x^{(k+1)m})$, $Y^{(k+1)} = \mathrm{Spec}\big(\mathbb{C}[y]/(y^{(k+1)m})\big)$, and $g^{(k+1)} = y^{m+1} + (y^{(k+1)m})$.

Thus $\Phi : X^{(k+1)} \to Y^{(k+1)}$ acting on functions by $y + (y^{(k+1)m}) \mapsto x + (x^{(k+1)m})$ is an isomorphism with $f^{(k+1)} = g^{(k+1)} \circ \Phi$.

Suppose T, π_U, π_V are as in Proposition 4.3, and use $w = x \circ \pi_U$ as a coordinate on T. Then $e(w) = w^{m+1}$, and $Q = \mathrm{Crit}(e) = \mathrm{Spec}\big(\mathbb{C}[w]/(w^m)\big)$. We have $\pi_U(w) = w$, so $\Phi \circ \pi_U(w) = w$, but $w^{m+1} = \pi_V(w)^{m+1} + A\pi_V(w)^{(k+1)m} + \cdots$, so that $\pi_V(w) = w - \frac{1}{m+1}Aw^{km} + \cdots$. Thus, $\Phi \circ \pi_U : T \to V$ and $\pi_V : T \to V$ differ by $-\frac{1}{m+1}Aw^{km} + \cdots$, which is zero on $Q^{(k)}$ but not on $Q^{(k+1)}$. Hence in this example there do not exist T, π_U, π_V with $\Phi \circ \pi_U|_{Q^{(k+1)}} = \pi_V|_{Q^{(k+1)}}$.

Remark 4.5. We can also ask: can we improve $(X^{(3)}, f^{(3)})$ in Theorem 4.2 to $(X^{(2)}, f^{(2)})$ or $(X^{(1)}, f^{(1)})$ or $(X^{\mathrm{red}}, f^{\mathrm{red}})$? Here are some thoughts on this.

(a) The analogue of Proposition 4.3 for $k = 1$ mentioned above implies that étale or complex analytically locally on X, (U, f) and hence $\mathcal{PV}_{U,f}^\bullet$ are determined up to *non-canonical* isomorphism by $(X^{(2)}, f^{(2)})$. Using the ideas of §5–§6, one can show that these non-canonical isomorphisms of $\mathcal{PV}_{U,f}^\bullet$ are unique *up to sign*.

(b) Consider the following example: let $U = (\mathbb{C} \setminus \{0\}) \times \mathbb{C} = V$, and define $f : U \to \mathbb{C}$ and $g : V \to \mathbb{C}$ by $f(x, y) = y^2$ and $g(x, y) = xy^2$. Then $X := \mathrm{Crit}(f) = \{y = 0\} = \mathrm{Crit}(g) =: Y$, and $f^{(2)} = g^{(2)} = 0$, so that $(X^{(2)}, f^{(2)}) = (Y^{(2)}, g^{(2)})$. However, as in Example 5.5 below, $\mathcal{PV}_{U,f}^{\bullet} \not\cong \mathcal{PV}_{V,g}^{\bullet}$. Thus, globally, $\mathcal{PV}_{U,f}^{\bullet}$ is not determined up to isomorphism by $(X^{(2)}, f^{(2)})$.

(c) Suppose U is a complex manifold and $f : U \to \mathbb{C}$ is holomorphic, with $\mathrm{Crit}(f)$ a single (not necessarily reduced) point x. The *Mather–Yau Theorem* [38] shows that the germ of (U, f) at x is determined up to non-canonical isomorphism by the complex analytic subspace $f^{(1)} = 0$ in $X^{(1)}$, and hence by the pair $(X^{(1)}, f^{(1)})$. Therefore, for *isolated* singularities, $\mathcal{PV}_{U,f}^{\bullet}$ is determined up to non-canonical isomorphism by $(X^{(1)}, f^{(1)})$.

(d) Define $f : U \to \mathbb{C}$ by $U = \mathbb{C}$ and $f(z) = cz^n$ for $0 \neq c \in \mathbb{C}$ and $n > 2$. This has an isolated singularity at 0, and $(X^{(1)}, f^{(1)})$ is independent of c. By moving c in a circle round zero, we see that in this example $\mathcal{PV}_{U,f}^{\bullet}$ is determined up to a $\mathbb{Z}/n\mathbb{Z}$ group of automorphisms. So the non-canonical isomorphisms of $\mathcal{PV}_{U,f}^{\bullet}$ are not unique up to sign, in contrast to **(a)**.

(e) Parts **(a)**–**(d)** leave open the question of whether $\mathcal{PV}_{U,f}^{\bullet}$ is determined locally up to non-canonical isomorphism by $(X^{(1)}, f^{(1)})$ for non-isolated singularities. We do not have a counterexample to this.

However, Gaffney and Hauser [19, §4] give examples of complex manifolds U and holomorphic $f : U \to \mathbb{C}$ with $X = \mathrm{Crit}(f)$ non-isolated, such that the germ of (U, f) at $x \in X$ is not determined up to non-canonical isomorphism by the germ of $(X^{(1)}, f^{(1)})$ at x, in contrast to the Mather–Yau Theorem, and continuous families of distinct germs $[U, f, x]$ can have the same germ $[X^{(1)}, f^{(1)}, x]$. It seems likely that in examples of this kind, the mixed Hodge module $\mathcal{HV}_{U,f}^{\bullet}$ (which contains continuous Hodge-theoretic information) is not locally determined up to non-canonical isomorphism by $(X^{(1)}, f^{(1)})$.

(f) For the example in **(d)**, $\mathcal{PV}_{U,f}^{\bullet}$ depends on $n = 3, 4, \ldots$, but $(X^{\mathrm{red}}, f^{\mathrm{red}}) = (\{0\}, 0)$ is independent of n. So $\mathcal{PV}_{U,f}^{\bullet}$ is not determined even locally up to non-canonical isomorphism by $(X^{\mathrm{red}}, f^{\mathrm{red}})$.

4.1. **Proof of Proposition 4.3.** The $\mathbb{C}$-subscheme $X^{(k+1)}$ in U is the zeroes of the ideal $I_X^{k+1} \subset \mathcal{O}_U$, which vanishes to order $k+1 \geqslant 2$ at $x \in X \subseteq X^{(k+1)} \subseteq U$. Hence $T_x X^{(k+1)} = T_x U$. As $\Phi : X^{(k+1)} \to Y^{(k+1)}$ is an isomorphism, it follows that

$$T_x U = T_x X^{(k+1)} \cong T_{\Phi(x)} Y^{(k+1)} = T_{\Phi(x)} V. \tag{4.7}$$

Therefore $n := \dim U = \dim V$.

Choose a Zariski open neighbourhood V' of $\Phi(x)$ in V and étale coordinates

$$(y_1, \ldots, y_n) : V' \to \mathbb{C}^n$$

on V'. Write $g' = g|_{V'}$ and $Y' = \mathrm{Crit}(g') = Y \cap V'$, so that $Y'^{(k+1)} = Y^{(k+1)} \cap V'$. Then $y_a \circ \Phi$ are regular functions on the open neighbourhood $\Phi^{-1}(V') \subseteq X^{(k+1)}$ of x in $X^{(k+1)}$, so they extend Zariski locally from $X^{(k+1)}$ to U. Thus we can choose a Zariski open neighbourhood U' of x in U with $\Phi(X^{(k+1)} \cap U') \subseteq Y^{(k+1)} \cap V'$, and regular functions $x_i : U' \to \mathbb{C}$ with $x_i|_{X^{(k+1)} \cap U'} = y_i \circ \Phi|_{X^{(k+1)} \cap U'}$ for $i = 1, \ldots, n$.

Write $f' = f|_{U'}$ and $X' = \mathrm{Crit}(f') = X \cap U'$, so that $X'^{(k+1)} = X^{(k+1)} \cap X'$. Since $(y_1, \ldots, y_n)$ are étale coordinates,

$$\mathrm{d}y_1|_{\Phi(x)}, \ldots, \mathrm{d}y_n|_{\Phi(x)}$$

are a basis for $T_{\Phi(x)}^* V$, so $\mathrm{d}x_1|_x, \ldots, \mathrm{d}x_n|_x$ are a basis for $T_x^* X$ by (4.7). Hence by making U' smaller, we can suppose $(x_1, \ldots, x_n)$ are étale coordinates on U'.

Consider the $\mathbb{C}$-scheme $U' \times V'$, with projections

$$\pi_{U'} : U' \times V' \longrightarrow U' \qquad \text{and} \qquad \pi_{V'} : U' \times V' \longrightarrow V',$$

and write
$$x'_i = x_i \circ \pi_{U'} : U' \times V' \to \mathbb{C}, \qquad y'_i = y_i \circ \pi_{V'} : U' \times V' \to \mathbb{C},$$
so that $(x'_1, \ldots, x'_n, y'_1, \ldots, y'_n)$ are étale coordinates on $U' \times V'$. We have a morphism
$$\mathrm{id} \times \Phi|_{X'^{(k+1)}} : X'^{(k+1)} \longrightarrow U' \times V'$$
which embeds $X'^{(k+1)}$ as a closed $\mathbb{C}$-subscheme of $U' \times V'$. The image $\big(\mathrm{id} \times \Phi|_{X'^{(k+1)}}\big)\big(X'^{(k+1)}\big)$ is locally the zeroes of the sheaf of ideals
$$\big(x'_i - y'_i, \ i = 1, \ldots, n\big) + \pi_{U'}^{-1}\big(I_X^{k+1}\big) \subset \mathcal{O}_{U' \times V'},$$
where $\big(x'_i - y'_i, i = 1, \ldots, n\big)$ denotes the ideal generated by $x'_i - y'_i : U' \times V' \to \mathbb{C}$ for $i = 1, \ldots, n$, and $\pi_{U'}^{-1}\big(I_X^{k+1}\big) \subset \mathcal{O}_{U' \times V'}$ the preimage ideal of $I_X^{k+1}|_{U'} \subset \mathcal{O}_{U'}$.

Now $\big(f \circ \pi_{U'} - g \circ \pi_{V'}\big)|_{(\mathrm{id} \times \Phi)(X'^{(k+1)})} = 0$ as $f^{(k+1)} = g^{(k+1)} \circ \Phi$. Hence
$$f \circ \pi_{U'} - g \circ \pi_{V'} \in \big(x'_i - y'_i, \ i = 1, \ldots, n\big) + \pi_{U'}^{-1}\big(I_X^{k+1}\big). \tag{4.8}$$

Lifting (4.8) from $\big(x'_i - y'_i, \ i = 1, \ldots, n\big)$ to $\big(x'_i - y'_i, \ i = 1, \ldots, n\big)^2$, after making U', V' smaller if necessary, we can choose regular functions $A_i : U' \times V' \to \mathbb{C}$ for $i = 1, \ldots, n$ such that
$$f \circ \pi_{U'} - g \circ \pi_{V'} - \sum_{i=1}^n A_i \cdot (x'_i - y'_i) \in \big(x'_i - y'_i, \ i = 1, \ldots, n\big)^2 + \pi_{U'}^{-1}\big(I_X^{k+1}\big). \tag{4.9}$$
Apply $\frac{\partial}{\partial x'_i}$ to (4.9), using the étale coordinates $(x'_1, \ldots, x'_n, y'_1, \ldots, y'_n)$ on $U' \times V'$. Since
$$\frac{\partial}{\partial x'_i}\big(f \circ \pi_{U'}\big) = \frac{\partial f}{\partial x_i} \circ \pi_{U'}$$
and $\frac{\partial}{\partial x'_i}\big(g \circ \pi_{V'}\big) = 0$, this gives
$$\frac{\partial f}{\partial x_i} \circ \pi_{U'} - A_i \in \big(x'_i - y'_i, \ i = 1, \ldots, n\big) + \pi_{U'}^{-1}\big(I_X^k\big). \tag{4.10}$$

Changing A_i by an element of $\big(x'_i - y'_i, \ i = 1, \ldots, n\big)$ can be absorbed in the ideal
$$\big(x'_i - y'_i, \ i = 1, \ldots, n\big)^2$$
in (4.9), so we can suppose $\frac{\partial f}{\partial x_i} \circ \pi_{U'} - A_i \in \pi_{U'}^{-1}\big(I_X^k\big)$. As
$$I_X = \big(\frac{\partial f}{\partial x_j}, \ j = 1, \ldots, n\big),$$
after making U', V' smaller we may write
$$A_i = \frac{\partial f}{\partial x_i} \circ \pi_{U'} + \sum_{j=1}^n B_{ij} \cdot \frac{\partial f}{\partial x_j} \circ \pi_{U'}, \tag{4.11}$$
with $B_{ij} \in \pi_{U'}^{-1}\big(I_X^{k-1}\big)$ for $i, j = 1, \ldots, n$. Consider the matrix of functions $\big(\delta_{ij} + B_{ij}\big)_{i,j=1}^n$ on $U' \times V'$. At the point $(x, \Phi(x))$ in $U' \times V'$ this matrix is the identity, since $B_{ij}(x, \Phi(x)) = 0$ as $B_{ij} \in \pi_{U'}^{-1}(I_X^{k-1})$ with $k \geqslant 2$, so $\big(\delta_{ij} + B_{ij}\big)_{i,j=1}^n$ is invertible near $(x, \Phi(x))$, and making U', V' smaller we can suppose $\big(\delta_{ij} + B_{ij}\big)_{i,j=1}^n$ is invertible on $U' \times V'$. But in matrix notation we have
$$\big(A_i\big)_{i=1}^n = \big(\delta_{ij} + B_{ij}\big)_{i,j=1}^n \big(\frac{\partial f}{\partial x_j} \circ \pi_{U'}\big)_{j=1}^n.$$
Hence in ideals in $\mathcal{O}_{U' \times V'}$ we have
$$\big(A_i, \ i = 1, \ldots, n\big) = \big(\frac{\partial f}{\partial x_j} \circ \pi_{U'}, \ j = 1, \ldots, n\big) = \pi_{U'}^{-1}\big(I_X\big) \subset \mathcal{O}_{U' \times V'}. \tag{4.12}$$

Now by (4.9), after making U', V' smaller if necessary, we may write

$$
\begin{aligned}
f \circ \pi_{U'} - g \circ \pi_{V'} = {} & \sum_{i=1}^{n} A_i \cdot \left(x_i' - y_i' \right) \\
& + \sum_{i,j=1}^{n} C_{ij} \cdot \left(x_i' - y_i' \right)\left(x_j' - y_j' \right) + \sum_{i,j=1}^{n} D_{ij} \cdot A_i A_j,
\end{aligned}
\tag{4.13}
$$

for regular functions $C_{ij}, D_{ij} : U' \times V' \to \mathbb{C}$ with $D_{ij} \in H^0\big(\pi_{U'}^{-1}(I_X^{k-1})\big)$ for $i,j = 1, \ldots, n$, where in the last term we have used (4.12) to write two factors of $\pi_{U'}^{-1}(I_X)$ in terms of $A_1, \ldots, A_n$.

Write $(z_{ij})_{i,j=1}^{n}$ for the coordinates on $\mathbb{C}^{n^2}$. Let W be a Zariski open neighbourhood of $\big(x, \Phi(x), (0)_{i,j=1}^{n}\big)$ in $U' \times V' \times \mathbb{C}^{n^2}$ to be chosen shortly, and let T be the $\mathbb{C}$-subscheme of W defined by

$$
\begin{aligned}
T = \Big\{ & \big(u, v, (z_{ij})_{i,j=1}^{n}\big) \in W \subseteq U' \times V' \times \mathbb{C}^{n^2} : \\
& x_i(u) - y_i(v) = \sum_{j=1}^{n} z_{ij} \cdot A_j(u,v), \quad i = 1, \ldots, n, \\
& z_{ij} + \sum_{l,m=1}^{n} C_{lm}(u,v) \cdot z_{li} z_{mj} + D_{ij}(u,v) = 0, \quad i,j = 1, \ldots, n \Big\}.
\end{aligned}
\tag{4.14}
$$

Define $\mathbb{C}$-scheme morphisms $\pi_U : T \to U$ by $\pi_U : \big(u, v, (z_{ij})_{i,j=1}^{n}\big) \mapsto u$ and $\pi_V : T \to V$ by $\pi_V : \big(u, v, (z_{ij})_{i,j=1}^{n}\big) \mapsto v$.

Now $W \subseteq U' \times V' \times \mathbb{C}^{n^2}$ is smooth of dimension $n + n + n^2$, and in (4.14) we impose $n + n^2$ equations, so the expected dimension of T is $(2n + n^2) - (n + n^2) = n$. The linearizations of the $n + n^2$ equations in (4.14) at $\big(u, v, (z_{ij})_{i,j=1}^{n}\big) = \big(x, \Phi(x), (0)_{i,j=1}^{n}\big)$ are

$$
\begin{aligned}
\mathrm{d}x_i|_x(\delta u) - \mathrm{d}y_i|_{\Phi(x)}(\delta v) = 0, \quad i = 1, \ldots, n, \\
\delta z_{ij} + \mathrm{d}D_{ij}|_{(x,\Phi(x))}(\delta u \oplus \delta v) = 0, \quad i,j = 1, \ldots, n,
\end{aligned}
\tag{4.15}
$$

for $\delta u \in T_x U'$, $\delta v \in T_{\Phi(x)} V'$, and $(\delta z_{ij})_{i,j=1}^{n} \in T_{(0)_{i,j=1}^{n}} \mathbb{C}^{n^2}$. As $\mathrm{d}x_1|_x, \ldots, \mathrm{d}x_n|_x$ are a basis for $T_x^* U'$, the equations (4.15) are transverse, so that T is smooth of dimension n near $\big(x, \Phi(x), (0)_{i,j=1}^{n}\big)$.

The vector space of solutions $\big(\delta u, \delta v, (\delta z_{ij})_{i,j=1}^{n}\big)$ to (4.15) is $T_{(x,\Phi(x),(0))} T$, where

$$
\mathrm{d}\pi_U|_{(x,\Phi(x),(0))} : T_{(x,\Phi(x),(0))} T \longrightarrow T_x U
$$

maps $\big(\delta u, \delta v, (\delta z_{ij})_{i,j=1}^{n}\big) \mapsto \delta u$, and

$$
\mathrm{d}\pi_V|_{(x,\Phi(x),(0))} : T_{(x,\Phi(x),(0))} T \longrightarrow T_{\Phi(x)} V
$$

maps $\big(\delta u, \delta v, (\delta z_{ij})_{i,j=1}^{n}\big) \mapsto \delta v$. Clearly, $\mathrm{d}\pi_U|_{(x,\Phi(x),(0))}, \mathrm{d}\pi_V|_{(x,\Phi(x),(0))}$ are isomorphisms, so as T is smooth near $\big(x, \Phi(x), (0)_{i,j=1}^{n}\big)$ and U, V are smooth, we see that π_U, π_V are étale near $\big(x, \Phi(x), (0)\big)$. Thus, by choosing the open neighbourhood $\big(x, \Phi(x), (0)\big) \in W \subseteq U' \times V' \times \mathbb{C}^{n^2}$ sufficiently small, we can suppose that T is smooth of dimension n and $\pi_U : T \to U$ and $\pi_V : T \to V$ are étale.

It remains to prove Proposition 4.3(a)–(c). For (a), we have

$$\big(f \circ \pi_U - g \circ \pi_V\big)\big(u, v, (z_{ij})_{i,j=1}^n\big) = f(u) \ - g(v) = (f \circ \pi_{U'} \ - \ g \circ \pi_{V'})(u, v)$$

$$= \sum_{i=1}^n A_i(u, v) \cdot \big(x_i(u) - y_i(v)\big) + \sum_{i,j=1}^n C_{ij}(u, v) \cdot \big(x_i(u) - y_i(v)\big)\big(x_j(u) - y_j(v)\big)$$

$$+ \sum_{i,j=1}^n D_{ij}(u, v) \cdot A_i(u, v) A_j(u, v)$$

$$= \sum_{i=1}^n A_i(u, v) \cdot \Big(\sum_{j=1}^n z_{ij} \cdot A_j(u, v) \Big) + \sum_{i,j=1}^n C_{ij}(u, v) \cdot \Big(\sum_{l=1}^n z_{il} \cdot A_l(u, v) \Big) \Big(\sum_{m=1}^n z_{jm} \cdot A_m(u, v) \Big)$$

$$+ \sum_{i,j=1}^n D_{ij}(u, v) \cdot A_i(u, v) A_j(u, v)$$

$$= \sum_{i,j=1}^n A_i(u, v) A_j(u, v) \Big[z_{ij} + \sum_{l,m=1}^n C_{lm}(u, v) \cdot z_{li} z_{mj} + D_{ij}(u, v) \Big] = 0,$$

using (4.13) in the third step, the first equation of (4.14) in the fourth, rearranging and exchanging labels i, l and j, m in the fifth, and the second equation of (4.14) in the sixth. Hence $f \circ \pi_U - g \circ \pi_V = 0 : T \to \mathbb{C}$, proving (a).

For (b), using the morphism $\mathrm{id} \times \Phi|_X \times (0) : X \to U \times V \times \mathbb{C}^{n^2} \supseteq W$, define

$$\tilde{X} = \big(\mathrm{id} \times \Phi|_X \times (0)\big)^{-1}(W),$$

so that $\tilde{X}$ is a Zariski open neighbourhood of x in X. Then $\big(\mathrm{id} \times \Phi|_X \times (0)\big)(\tilde{X})$ is a closed $\mathbb{C}$-subscheme of W. We claim that:

(i) $\big(\mathrm{id} \times \Phi|_X \times (0)\big)(\tilde{X})$ is a closed $\mathbb{C}$-subscheme of $T \subseteq W$; and

(ii) $\big(\mathrm{id} \times \Phi|_X \times (0)\big)(\tilde{X})$ is open and closed in $Q := \mathrm{Crit}(e) \subseteq T$,

where $e := f \circ \pi_U = g \circ \pi_V : T \to \mathbb{C}$. To prove (i), we have to show that the equations of (4.14) hold on $\big(\mathrm{id} \times \Phi|_X \times (0)\big)(\tilde{X})$, which is true as $x_i|_{\tilde{X}} = y_i \circ \Phi|_{\tilde{X}}$, and $z_{ij} \circ (0) = 0$, and $D_{ij} \circ (\mathrm{id} \times \Phi|_{\tilde{X}}) = 0$ as $D_{ij} \in H^0\big(\pi_{U'}^{-1}(I_X^{k-1})\big)$ for $k \geqslant 2$.

For (ii), as $\pi_U : T \to U$ is étale with $e = f \circ \pi_U$, we see that $\pi_U|_Q : Q \to X$ is étale. But $\pi_U|_Q \circ \big(\mathrm{id} \times \Phi|_X \times (0)\big)|_{\tilde{X}} = \mathrm{id}_{\tilde{X}}$. Hence $\big(\mathrm{id} \times \Phi|_X \times (0)\big)(\tilde{X})$ is open in Q, and is also closed in Q as it is closed in T. Thus, by making W, T smaller to delete other components of Q, we can suppose that $Q = \big(\mathrm{id} \times \Phi|_X \times (0)\big)(\tilde{X})$. Then $\pi_U|_Q : Q \to \tilde{X}$ is an isomorphism with the Zariski open neighbourhood $\tilde{X}$ of x in X. Since $\pi_U : T \to U$ is étale with $e = f \circ \pi_U$, this extends to the kth order thickenings, so $\pi_U|_{Q^{(k)}} : Q^{(k)} \to \tilde{X}^{(k)}$ is an isomorphism, proving (b).

For (c), first note that $Q = \big(\mathrm{id} \times \Phi|_X \times (0)\big)(\tilde{X})$, so $\Phi \circ \pi_U|_Q = \pi_V|_Q$ is immediate. We have to extend this to the thickening $Q^{(k)}$. Write $I_Q \subset \mathcal{O}_T$ for the ideal of functions vanishing on Q. Then $I_Q = \pi_U^{-1}(I_X)$ as π_U identifies Q with $\tilde{X} \subseteq X$. We have

$$A_i \circ \pi_{U' \times V'} \in I_Q \quad \text{and} \quad D_{ij} \circ \pi_{U' \times V'} \in I_Q^{k-1},$$

as $A_i \in H^0\big(\pi_{U'}^{-1}(I_X)\big)$, $D_{ij} \in H^0\big(\pi_{U'}^{-1}(I_X^{k-1})\big)$. The second equation of (4.14) then shows that

$$z_{ij} \circ \pi_{\mathbb{C}^{n^2}} \in I_Q^{k-1},$$

since $Q = \big(\mathrm{id} \times \Phi|_X \times (0)\big)(\tilde{X})$ implies that $z_{ij} \circ \pi_{\mathbb{C}^{n^2}} = 0$ on Q, so we can neglect the terms $\sum_{l,m=1}^n C_{lm}(u, v) \cdot z_{li} z_{mj}$. Hence the first equation of (4.14) gives

$$x_i \circ \pi_U - y_i \circ \pi_V \in I_Q^k.$$

As I_Q^k vanishes on $Q^{(k)}$, and $x_i|_{X'} = y_i \circ \Phi|_{X'}$, this gives

$$y_i \circ \left(\Phi \circ \pi_U|_{Q^{(k)}} \right) = x_i \circ \pi_U|_{Q^{(k)}} = y_i \circ \left(\pi_V|_{Q^{(k)}} \right).$$

Thus $\Phi \circ \pi_U|_{Q^{(k)}} = \pi_V|_{Q^{(k)}}$ follows, as $(y_1, \ldots, y_n)$ are étale coordinates on V near $\pi_V(Q)$ and $\Phi \circ \pi_U|_Q = \pi_V|_Q$. This proves (c), and Proposition 4.3.

4.2. **Proof of Theorem 4.2 for $\mathbb{C}$-schemes.** Let U, V, f, g, X, Y and $\Phi : X^{(3)} \to Y^{(3)}$ be as in Theorem 4.2. Pick $x \in X$, and apply Proposition 4.3 with $k = 2$. This gives a smooth $\mathbb{C}$-scheme T and étale morphisms $\pi_U : T \to U$, $\pi_V : T \to V$ with $e := f \circ \pi_U = g \circ \pi_V : T \to \mathbb{C}$ and $Q := \mathrm{Crit}(e)$, such that $\pi_U|_{Q^{(2)}} : Q^{(2)} \to X^{(2)}$ is an étale open neighbourhood of x in $X^{(2)}$, and

$$\Phi \circ \pi_U|_{Q^{(2)}} = \pi_V|_{Q^{(2)}} : Q^{(2)} \longrightarrow Y^{(2)}.$$

Actually Proposition 4.3 proves more, that $\pi_U|_{Q^{(2)}} : Q^{(2)} \to X^{(2)}$ is an isomorphism with a Zariski open set $x \in \tilde{X}^{(2)} \subseteq X^{(2)}$, but we will not use this.

Thus, we can choose $\left\{ (T^a, \pi_U^a, \pi_V^a, e^a, Q^a) : a \in A \right\}$, where A is an indexing set, such that $T^a, \pi_U^a, \pi_V^a, e^a, Q^a$ satisfy the conditions above for each $a \in A$, and $\left\{ \pi_U^a|_{Q^a} : Q^a \to X \right\}_{a \in A}$ is an étale open cover of X. Then for each $a \in A$, by Definition 2.15 we have isomorphisms

$$\mathcal{PV}_{\pi_U^a} : \mathcal{PV}_{T^a, e^a}^\bullet \longrightarrow \pi_U^a|_{Q^a}^* \left(\mathcal{PV}_{U,f}^\bullet \right), \quad \mathcal{PV}_{\pi_V^a} : \mathcal{PV}_{T^a, e^a}^\bullet \longrightarrow \pi_V^a|_{Q^a}^* \left(\mathcal{PV}_{V,f}^\bullet \right).$$

Noting that $\pi_V^a|_{Q^a} = \Phi|_X \circ \pi_U^a|_{Q^a}$, we may define an isomorphism

$$\Omega^a = \mathcal{PV}_{\pi_V^a} \circ \mathcal{PV}_{\pi_U^a}^{-1} : \pi_U^a|_{Q^a}^* \left(\mathcal{PV}_{U,f}^\bullet \right) \longrightarrow \pi_U^a|_{Q^a}^* \left(\Phi|_X^* \left(\mathcal{PV}_{V,f}^\bullet \right) \right). \tag{4.16}$$

For $a, b \in A$, define $T^{ab} = T^a \times_{\pi_U^a, U, \pi_U^b} T^b$ to be the $\mathbb{C}$-scheme fibre product, so that T^{ab} is a smooth $\mathbb{C}$-scheme and the projections $\Pi_{T^a} : T^{ab} \to T^a$, $\Pi_{T^b} : T^{ab} \to T^b$ are étale. Define $e^{ab} = e^a \circ \Pi_{T^a} : T^a \to \mathbb{C}$. Then

$$\begin{aligned} e^{ab} = e^a \circ \Pi_{T^a} &= g \circ \pi_V^a \circ \Pi_{T^a} = f \circ \pi_U^a \circ \Pi_{T^a} \\ &= f \circ \pi_U^b \circ \Pi_{T^b} = g \circ \pi_V^b \circ \Pi_{T^b} = e^b \circ \Pi_{T^b}. \end{aligned} \tag{4.17}$$

Write $Q^{ab} = \mathrm{Crit}(e^{ab})$. Then $\Pi_{T^a}|_{Q^{ab}} : Q^{ab} \to Q^a$ and $\Pi_{T^b}|_{Q^{ab}} : Q^{ab} \to Q^b$ are étale. Now $\pi_U^a \circ \Pi_{T^a} = \pi_U^b \circ \Pi_{T^b}$ and $\Phi \circ \pi_U^a|_{Q^a(2)} = \pi_V^a|_{Q^a(2)}$ imply that

$$\begin{aligned} \left(\pi_V^a \circ \Pi_{T^a} \right)|_{Q^{ab}(2)} &= \pi_V^a|_{Q^a(2)} \circ \Pi_{T^a}|_{Q^{ab}(2)} \\ &= \Phi|_{X^{(2)}} \circ \pi_U^a|_{Q^a(2)} \circ \Pi_{T^a}|_{Q^{ab}(2)} = \Phi|_{X^{(2)}} \circ \left(\pi_U^a \circ \Pi_{T^a} \right)|_{Q^{ab}(2)} \\ &= \Phi|_{X^{(2)}} \circ \left(\pi_U^b \circ \Pi_{T^b} \right)|_{Q^{ab}(2)} = \Phi|_{X^{(2)}} \circ \pi_U^a|_{Q^b(2)} \circ \Pi_{T^b}|_{Q^{ab}(2)} \\ &= \pi_V^a|_{Q^b(2)} \circ \Pi_{T^b}|_{Q^{ab}(2)} = \left(\pi_V^b \circ \Pi_{T^b} \right)|_{Q^{ab}(2)}. \end{aligned} \tag{4.18}$$

Hence $(\pi_V^a \circ \Pi_{T^a})|_{Q^{ab}} = (\pi_V^b \circ \Pi_{T^b})|_{Q^{ab}}$. Moreover, as $TQ^{ab(2)}|_{Q^{ab}} = T(T^{ab})|_{Q^{ab}}$, we see that $\mathrm{d}(\pi_V^a \circ \Pi_{T^a})|_{Q^{ab}} = \mathrm{d}(\pi_V^b \circ \Pi_{T^b})|_{Q^{ab}}$, so that

$$\mathrm{d}(\pi_V^b \circ \Pi_{T^b})|_{Q^{ab}}^{-1} \circ \mathrm{d}(\pi_V^a \circ \Pi_{T^a})|_{Q^{ab}} = \mathrm{id} : T(T^{ab})|_{Q^{ab}} \longrightarrow T(T^{ab})|_{Q^{ab}}.$$

So $\det \left(\mathrm{d}(\pi_V^b \circ \Pi_{T^b})|_{Q^{ab}}^{-1} \circ \mathrm{d}(\pi_V^a \circ \Pi_{T^a})|_{Q^{ab}} \right) = 1$. Thus, applying Theorem 3.1 with T^{ab}, V, Q^{ab}, $\pi_V^a \circ \Pi_{T^a}, \pi_V^b \circ \Pi_{T^b}, e^{ab}, f$ in place of $V, W, X, \Phi, \Psi, f, g$ gives

$$\mathcal{PV}_{\pi_V^a \circ \Pi_{T^a}} = \mathcal{PV}_{\pi_V^b \circ \Pi_{T^b}} : \mathcal{PV}_{T^{ab}, e^{ab}}^\bullet \longrightarrow (\pi_V^a \circ \Pi_{T^a})|_{Q^{ab}}^* \left(\mathcal{PV}_{V,g}^\bullet \right). \tag{4.19}$$

Now

$$
\begin{aligned}
\Pi_{T^a}|_{Q^{ab}}^*(\Omega^a) &= \Pi_{T^a}|_{Q^{ab}}^*(\mathcal{PV}_{\pi_V^a}) \circ \Pi_{T^a}|_{Q^{ab}}^*(\mathcal{PV}_{\pi_U^a})^{-1} \\
&= \left[\Pi_{T^a}|_{Q^{ab}}^*(\mathcal{PV}_{\pi_V^a}) \circ \mathcal{PV}_{\Pi_{T^a}}\right] \circ \left[\Pi_{T^a}|_{Q^{ab}}^*(\mathcal{PV}_{\pi_U^a}) \circ \mathcal{PV}_{\Pi_{T^a}}\right]^{-1} \\
&= \mathcal{PV}_{\pi_V^a \circ \Pi_{T^a}} \circ \mathcal{PV}_{\pi_U^a \circ \Pi_{T^a}}^{-1} = \mathcal{PV}_{\pi_V^b \circ \Pi_{T^b}} \circ \mathcal{PV}_{\pi_U^b \circ \Pi_{T^b}}^{-1} \\
&= \left[\Pi_{T^b}|_{Q^{ab}}^*(\mathcal{PV}_{\pi_V^a}) \circ \mathcal{PV}_{\Pi_{T^b}}\right] \circ \left[\Pi_{T^b}|_{Q^{ab}}^*(\mathcal{PV}_{\pi_U^b}) \circ \mathcal{PV}_{\Pi_{T^b}}\right]^{-1} \\
&= \Pi_{T^b}|_{Q^{ab}}^*(\mathcal{PV}_{\pi_V^b}) \circ \Pi_{T^b}|_{Q^{ab}}^*(\mathcal{PV}_{\pi_U^b})^{-1} = \Pi_{T^b}|_{Q^{ab}}^*(\Omega^b),
\end{aligned} \tag{4.20}
$$

using (4.16) in the first and seventh steps, (2.18) in the third and fifth, and (4.19) and

$$
\pi_U^a \circ \Pi_{T^a} = \pi_U^b \circ \Pi_{T^b}
$$

in the fourth. Therefore Theorem 2.7(i) applied to the étale open cover $\left\{\pi_U^a|_{Q^a} : Q^a \to X\right\}_{a \in A}$ of X shows that there is a unique isomorphism Ω_Φ in (4.2) with $\pi_U^a|_{Q^a}^*(\Omega_\Phi) = \Omega^a$ for all $a \in A$.

Suppose $\left\{(T^a, \dots, Q^a) : a \in A\right\}$ and $\left\{(T'^a, \dots, Q'^a) : a \in A'\right\}$ are alternative choices above, yielding morphisms Ω_Φ and Ω'_Φ in (4.2). By running the same construction using the family $\left\{(T^a, \dots, Q^a) : a \in A\right\} \amalg \left\{(T'^a, \dots, Q'^a) : a \in A'\right\}$, we get a third morphism Ω''_Φ in (4.2), such that $\pi_U^a|_{Q^a}^*(\Omega_\Phi) = \Omega^a = \pi_U^a|_{Q^a}^*(\Omega''_\Phi)$ for $a \in A$, giving $\Omega_\Phi = \Omega''_\Phi$, and

$$
\pi'^a_U|_{Q'^a}^*(\Omega'_\Phi) = \Omega'^a = \pi'^a_U|_{Q'^a}^*(\Omega''_\Phi)
$$

for $a \in A'$, which forces $\Omega'_\Phi = \Omega''_\Phi$. Thus $\Omega_\Phi = \Omega'_\Phi$, so Ω_Φ is independent of the choice of $\left\{(T^a, \dots, Q^a) : a \in A\right\}$ above.

Let T, π_U, π_V, e, Q be as in Theorem 4.2. Applying the argument above using the family $\left\{(T^a, \dots, Q^a) : a \in A\right\} \amalg \left\{(T, \pi_U, \pi_V, e, Q)\right\}$ shows that Ω_Φ satisfies

$$
\pi_U|_Q^*(\Omega_\Phi) = \mathcal{PV}_{\pi_V} \circ \mathcal{PV}_{\pi_U}^{-1},
$$

by (4.16). Thus (4.3) holds.

To show that (4.4)–(4.5) commute, we can combine equations (2.16)–(2.17), (4.16) and $\pi_U^a|_{Q^a}^*(\Omega_\Phi) = \Omega^a$ to show that $\pi_U^a|_{Q^a}^*$ applied to (4.4)–(4.5) commute in $\mathrm{Perv}(Q^a)$ for each $a \in A$, so (4.4)–(4.5) commute by Theorem 2.7(i).

Suppose there exists an étale morphism $\Xi : U \to V$ with $f = g \circ \Xi : U \to \mathbb{C}$ and

$$
\Xi|_{X^{(3)}} = \Phi : X^{(3)} \to Y^{(3)}.
$$

Then as we have to prove, we have

$$
\mathcal{PV}_\Xi = \mathrm{id}_X^*(\Omega_\Phi) \circ \mathcal{PV}_{\mathrm{id}_U} = \Omega_\Phi \circ \mathrm{id}_{\mathcal{PV}_{U,f}^\bullet} = \Omega_\Phi,
$$

where in the first step we use (4.3) with $T = U$, $\pi_U = \mathrm{id}_U$, $\pi_V = \Xi$, $e = f$, and $Q = X$, and in the second we use $\mathcal{PV}_{\mathrm{id}_U} = \mathrm{id}_{\mathcal{PV}_{U,f}^\bullet}$ from Definition 2.15.

Suppose W is another smooth $\mathbb{C}$-scheme, $h : W \to \mathbb{C}$ is regular, $Z = \mathrm{Crit}(h)$, and $\Psi : Y^{(3)} \to Z^{(3)}$ is an isomorphism with $h^{(3)} \circ \Psi = g^{(3)}$. Let $x \in X$, and set $y = \Phi(x) \in Y$. Proposition 4.3 for x, Φ gives a smooth T and étale $\pi_U : T \to U$, $\pi_V : T \to V$ with $e := f \circ \pi_U = g \circ \pi_V$ and $Q := \mathrm{Crit}(e)$, such that $\pi_U|_{Q^{(2)}} : Q^{(2)} \to X^{(2)}$ is an étale open neighbourhood of x, and $\Phi \circ \pi_U|_{Q^{(2)}} = \pi_V|_{Q^{(2)}}$. Proposition 4.3 for y, Ψ gives smooth $\tilde{T}$ and étale $\tilde{\pi}_V : \tilde{T} \to V$, $\tilde{\pi}_W : \tilde{T} \to W$ with $\tilde{e} := g \circ \tilde{\pi}_V = h \circ \tilde{\pi}_W$ and $\tilde{Q} := \mathrm{Crit}(\tilde{e})$, such that $\tilde{\pi}_V|_{\tilde{Q}^{(2)}} : \tilde{Q}^{(2)} \to Y^{(2)}$ is an étale open neighbourhood of y, and $\Psi \circ \tilde{\pi}_V|_{\tilde{Q}^{(2)}} = \tilde{\pi}_W|_{\tilde{Q}^{(2)}}$.

Define $\hat{T} = T \times_{\pi_V, V, \tilde{\pi}_V} \tilde{T}$ with projections $\Pi_T : \hat{T} \to T$, $\Pi_{\tilde{T}} : \hat{T} \to \tilde{T}$. Then $\hat{T}$ is smooth and $\Pi_T, \Pi_{\tilde{T}}$ are étale, as $T, \tilde{T}, V$ are smooth and $\pi_V, \tilde{\pi}_V$ étale. Define $\hat{\pi}_U = \pi_U \circ \Pi_T : \hat{T} \to U$

and $\hat{\pi}_W = \tilde{\pi}_W \circ \Pi_{\tilde{T}} : \hat{T} \to W$. Then $\hat{\pi}_U, \hat{\pi}_W$ are étale. Set $\hat{e} = f \circ \hat{\pi}_U : \hat{T} \to \mathbb{C}$, and write $\hat{Q} = \mathrm{Crit}(\hat{e})$. Then

$$\hat{e} = f \circ \hat{\pi}_U = f \circ \pi_U \circ \Pi_T = g \circ \pi_V \circ \Pi_T = g \circ \tilde{\pi}_V \circ \Pi_{\tilde{T}} = h \circ \tilde{\pi}_W \circ \Pi_{\tilde{T}} = h \circ \hat{\pi}_W.$$

Also $\hat{\pi}_U|_{\hat{Q}^{(2)}} : \hat{Q}^{(2)} \to X^{(2)}$ is an étale open neighbourhood of x, and

$$(\Psi \circ \Phi) \circ \hat{\pi}_U|_{\hat{Q}^{(2)}} = \Psi \circ \Phi \circ \pi_U|_{Q^{(2)}} \circ \Pi_T|_{\hat{Q}^{(2)}} = \Psi \circ \pi_V|_{Q^{(2)}} \circ \Pi_T|_{\hat{Q}^{(2)}}$$
$$= \Psi \circ \tilde{\pi}_V|_{\tilde{Q}^{(2)}} \circ \Pi_{\tilde{T}}|_{\hat{Q}^{(2)}} = \tilde{\pi}_W|_{\tilde{Q}^{(2)}} \circ \Pi_{\tilde{T}}|_{\hat{Q}^{(2)}} = \hat{\pi}_W|_{\hat{Q}^{(2)}}.$$

Thus we may apply (4.3) for Ω_Φ with $T, \pi_U, \pi_V, \ldots, Q$, and for Ω_Ψ with $\tilde{T}, \tilde{\pi}_V, \tilde{\pi}_W, \ldots, \tilde{Q}$, and for $\Omega_{\Psi \circ \Phi}$ with $\hat{T}, \hat{\pi}_U, \hat{\pi}_W, \ldots, \hat{Q}$. This yields

$$\pi_U|_Q^*(\Omega_\Phi) = \mathcal{PV}_{\pi_V} \circ \mathcal{PV}_{\pi_U}^{-1}, \quad \tilde{\pi}_V|_{\tilde{Q}}^*(\Omega_\Psi) = \mathcal{PV}_{\tilde{\pi}_W} \circ \mathcal{PV}_{\tilde{\pi}_V}^{-1}, \tag{4.21}$$
$$\hat{\pi}_U|_{\hat{Q}}^*(\Omega_{\Psi \circ \Phi}) = \mathcal{PV}_{\hat{\pi}_W} \circ \mathcal{PV}_{\hat{\pi}_U}^{-1}.$$

Now

$$\hat{\pi}_U|_{\hat{Q}}^*(\Omega_{\Psi \circ \Phi}) = \mathcal{PV}_{\hat{\pi}_W} \circ \mathcal{PV}_{\hat{\pi}_U}^{-1} = \mathcal{PV}_{\tilde{\pi}_W \circ \Pi_{\tilde{T}}} \circ \mathcal{PV}_{\pi_U \circ \Pi_T}^{-1}$$
$$= \left[\Pi_{\tilde{T}}|_{\hat{Q}}^*(\mathcal{PV}_{\tilde{\pi}_W}) \circ \mathcal{PV}_{\Pi_{\tilde{T}}}\right] \circ \left[\Pi_T|_{\hat{Q}}^*(\mathcal{PV}_{\pi_U}) \circ \mathcal{PV}_{\Pi_T}\right]^{-1}$$
$$= \Pi_{\tilde{T}}|_{\hat{Q}}^*\left(\mathcal{PV}_{\tilde{\pi}_W} \circ \mathcal{PV}_{\tilde{\pi}_V}^{-1}\right) \circ \Pi_{\tilde{T}}|_{\hat{Q}}^*(\mathcal{PV}_{\tilde{\pi}_V}) \circ \mathcal{PV}_{\Pi_{\tilde{T}}} \circ \mathcal{PV}_{\Pi_T}^{-1} \circ \Pi_T|_{\hat{Q}}^*(\mathcal{PV}_{\pi_U}^{-1})$$
$$= \Pi_{\tilde{T}}|_{\hat{Q}}^*\left(\tilde{\pi}_V|_{\tilde{Q}}^*(\Omega_\Psi)\right) \circ \mathcal{PV}_{\tilde{\pi}_V \circ \Pi_{\tilde{T}}} \circ \mathcal{PV}_{\Pi_T}^{-1} \circ \Pi_T|_{\hat{Q}}^*(\mathcal{PV}_{\pi_U}^{-1})$$
$$= \left[\tilde{\pi}_V|_{\tilde{Q}} \circ \Pi_{\tilde{T}}|_{\hat{Q}}\right]^*(\Omega_\Psi) \circ \mathcal{PV}_{\pi_V \circ \Pi_T} \circ \mathcal{PV}_{\Pi_T}^{-1} \circ \Pi_T|_{\hat{Q}}^*(\mathcal{PV}_{\pi_U}^{-1})$$
$$= \left[\pi_V|_Q \circ \Pi_T|_{\hat{Q}}\right]^*(\Omega_\Psi) \circ \Pi_T|_{\hat{Q}}^*(\mathcal{PV}_{\pi_V}) \circ \mathcal{PV}_{\Pi_T} \circ \mathcal{PV}_{\Pi_T}^{-1} \circ \Pi_T|_{\hat{Q}}^*(\mathcal{PV}_{\pi_U}^{-1})$$
$$= \left[\Phi|_X \circ \pi_U|_Q \circ \Pi_T|_{\hat{Q}}\right]^*(\Omega_\Psi) \circ \Pi_T|_{\hat{Q}}^*(\mathcal{PV}_{\pi_V} \circ \mathcal{PV}_{\pi_U}^{-1})$$
$$= (\pi_U \circ \Pi_T)|_{\hat{Q}}^*(\Phi|_X^*(\Omega_\Psi)) \circ \Pi_T|_{\hat{Q}}^*(\pi_U|_Q^*(\Omega_\Phi)) = \hat{\pi}_U|_{\hat{Q}}^*\left(\Phi|_X^*(\Omega_\Psi) \circ \Omega_\Phi\right),$$

using (4.21) in the first, fifth and ninth steps, (2.18) in the third, fifth and seventh steps, $\pi_V \circ \Pi_T = \tilde{\pi}_V \circ \Pi_{\tilde{T}}$ in the sixth and seventh, and $\Phi \circ \pi_U|_Q = \pi_V|_Q$ in the eighth. Thus, for each $x \in X$, we have constructed an étale open neighbourhood $\hat{\pi}_U|_{\hat{Q}} : \hat{Q} \to X$ such that $\hat{\pi}_U|_{\hat{Q}}^*$ applied to (4.6) holds. Equation (4.6) follows by Theorem 2.7(i). Finally, if $U = V$, $f = g$, $X = Y$ and $\Phi = \mathrm{id}_{X^{(3)}}$ then $\Omega_{\mathrm{id}_{X^{(3)}}} = \mathrm{id}_{\mathcal{PV}_{U,f}^\bullet}$ follows by taking $\Xi = \mathrm{id}_U$ in the fourth paragraph of the theorem. This proves Theorem 4.2 for perverse sheaves on $\mathbb{C}$-schemes.

4.3. $\mathscr{D}$-modules and mixed Hodge modules.

Once again, the proof of Proposition 4.3 is completely algebraic, so applies in the other contexts of §2.6–§2.10. Theorem 4.2 then follows for our other contexts from that and the general framework of §2.5.

5. Stabilizing vanishing cycles

To set up notation for our main result, which is Theorem 5.4 below, we need the following theorem, which is proved in Joyce [23, Prop.s 2.22, 2.23 & 2.25].

Theorem 5.1 (Joyce [23]). *Let U, V be smooth $\mathbb{C}$-schemes, $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ be regular, and $X = \mathrm{Crit}(f)$, $Y = \mathrm{Crit}(g)$ as $\mathbb{C}$-subschemes of U, V. Let $\Phi : U \hookrightarrow V$ be a closed embedding of $\mathbb{C}$-schemes with $f = g \circ \Phi : U \to \mathbb{C}$, and suppose $\Phi|_X : X \to V \supseteq Y$ is an isomorphism $\Phi|_X : X \to Y$. Then:*

(i) For each $x \in X \subseteq U$ there exist smooth $\mathbb{C}$-schemes U', V', a point $x' \in U'$ and morphisms $\iota : U' \to U$, $\jmath : V' \to V$, $\Phi' : U' \to V'$, $\alpha : V' \to U$ and $\beta : V' \to \mathbb{C}^n$, where $n = \dim V - \dim U$,

such that $\iota(x') = x$, *and* $\iota, \jmath$ *and* $\alpha \times \beta : V \to U \times \mathbb{C}^n$ *are étale, and the following diagram commutes*

$$\begin{array}{ccccc}
U & \xleftarrow{\;\iota\;} & U' & \xrightarrow{\;\iota\;} & U \\
\Big\downarrow{\scriptstyle\Phi} & & \Big\downarrow{\scriptstyle\Phi'} & & \Big\downarrow{\scriptstyle \mathrm{id}_U \times 0} \\
V & \xleftarrow{\;\jmath\;} & V' & \xrightarrow{\;\alpha\times\beta\;} & U \times \mathbb{C}^n,
\end{array} \tag{5.1}$$

and $g \circ \jmath = f \circ \alpha + (z_1^2 + \cdots + z_n^2) \circ \beta : V' \to \mathbb{C}$. *Thus, setting* $f' := f \circ \iota : U' \to \mathbb{C}$, $g' := g \circ \jmath : V' \to \mathbb{C}$, $X' := \mathrm{Crit}(f') \subseteq U'$, *and* $Y' := \mathrm{Crit}(g') \subseteq V'$, *then* $f' = g' \circ \Phi' : U' \to \mathbb{C}$, *and* $\Phi'|_{X'} : X' \to Y'$, $\iota|_{X'} : X' \to X$, $\jmath|_{Y'} : Y' \to Y$, $\alpha|_{Y'} : Y' \to X$ *are étale. We also require that* $\Phi \circ \alpha|_{Y'} = \jmath|_{Y'} : Y' \to Y$.

(ii) *Write* N_{UV} *for the normal bundle of* $\Phi(U)$ *in* V, *regarded as an algebraic vector bundle on* U *in the exact sequence of vector bundles on* U:

$$0 \longrightarrow TU \xrightarrow{\;\mathrm{d}\Phi\;} \Phi^*(TU) \xrightarrow{\;\Pi_{UV}\;} N_{UV} \longrightarrow 0. \tag{5.2}$$

Then there exists a unique $q_{UV} \in H^0(S^2 N_{UV}^*|_X)$ *which is a nondegenerate quadratic form on* $N_{UV}|_X$, *such that whenever* $U', V', \iota, \jmath, \Phi', \beta, n, X'$ *are as in* **(i)**, *writing* $\langle \mathrm{d}z_1, \ldots, \mathrm{d}z_n\rangle_{U'}$ *for the trivial vector bundle on* U' *with basis* $\mathrm{d}z_1, \ldots, \mathrm{d}z_n$, *there is a natural isomorphism* $\hat{\beta} : \langle \mathrm{d}z_1, \ldots, \mathrm{d}z_n\rangle_{U'} \to \iota^*(N_{UV}^*)$ *making the following diagram commute:*
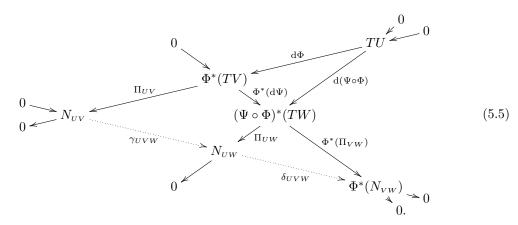
$$\begin{array}{ccc}
\iota^*(N_{UV}^*) & \xrightarrow{\quad\iota^*(\Pi_{UV}^*)\quad} & \iota^* \circ \Phi^*(T^*V) = \Phi'^* \circ \jmath^*(T^*V) \\
{\scriptstyle\hat{\beta}}\Big\uparrow & & \Big\downarrow{\scriptstyle\Phi'^*(\mathrm{d}\jmath^*)} \\
\langle \mathrm{d}z_1, \ldots, \mathrm{d}z_n\rangle_{U'} = \Phi'^* \circ \beta^*(T_0^*\mathbb{C}^n) & \xrightarrow{\quad\Phi'^*(\mathrm{d}\beta^*)\quad} & \Phi'^*(T^*V'),
\end{array} \tag{5.3}$$

$$and \qquad \iota|_{X'}^*(q_{UV}) = (S^2\hat{\beta})|_{X'}(\mathrm{d}z_1 \otimes \mathrm{d}z_1 + \cdots + \mathrm{d}z_n \otimes \mathrm{d}z_n). \tag{5.4}$$

(iii) *Now suppose* W *is another smooth* $\mathbb{C}$-*scheme,* $h : W \to \mathbb{C}$ *is regular,* $Z = \mathrm{Crit}(h)$ *as a* $\mathbb{C}$-*subscheme of* W, *and* $\Psi : V \hookrightarrow W$ *is a closed embedding of* $\mathbb{C}$-*schemes with* $g = h \circ \Psi : V \to \mathbb{C}$ *and* $\Psi|_Y : Y \to Z$ *an isomorphism. Define* N_{VW}, q_{VW} *and* N_{UW}, q_{UW} *using* $\Psi : V \hookrightarrow W$ *and* $\Psi \circ \Phi : U \hookrightarrow W$ *as in* **(ii)** *above. Then there are unique morphisms* $\gamma_{UVW}, \delta_{UVW}$ *which make the following diagram of vector bundles on* U *commute, with straight lines exact:*

$$ \tag{5.5}$$

Restricting to X *gives an exact sequence of vector bundles:*

$$0 \longrightarrow N_{UV}|_X \xrightarrow{\;\gamma_{UVW}|_X\;} N_{UW}|_X \xrightarrow{\;\delta_{UVW}|_X\;} \Phi|_X^*(N_{VW}) \longrightarrow 0. \tag{5.6}$$

Then there is a natural isomorphism of vector bundles on X

$$N_{UW}|_X \cong N_{UV}|_X \oplus \Phi|_X^*(N_{VW}), \tag{5.7}$$

compatible with the exact sequence (5.6), *which identifies*

$$q_{UW} \cong q_{UV} \oplus \Phi|_X^*(q_{VW}) \oplus 0 \qquad \text{under the splitting}$$
$$S^2 N_{UW}^*|_X \cong S^2 N_{UV}^*|_X \oplus \Phi|_X^* \big(S^2 N_{VW}^*|_Y \big) \oplus \big(N_{UV}^*|_X \otimes \Phi|_X^*(N_{VW}^*) \big). \tag{5.8}$$

(iv) *Analogues of* **(i)**–**(iii)** *hold for complex analytic spaces, replacing the smooth $\mathbb{C}$-schemes U, V, W by complex manifolds, the regular functions f, g, h by holomorphic functions, the $\mathbb{C}$-schemes X, Y, Z by complex analytic spaces, the étale open sets $\iota : U' \to U$, $\jmath : V' \to V$ by complex analytic open sets $U' \subseteq U$, $V' \subseteq V$, and with $\alpha \times \beta : V' \to U \times \mathbb{C}^n$ a biholomorphism with a complex analytic open neighbourhood of $(x, 0)$ in $U \times \mathbb{C}^n$.*

Following [23, Def.s 2.26 & 2.34], we define:

Definition 5.2. Let U, V be smooth $\mathbb{C}$-schemes, $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ be regular, and $X = \mathrm{Crit}(f)$, $Y = \mathrm{Crit}(g)$ as $\mathbb{C}$-subschemes of U, V. Suppose $\Phi : U \hookrightarrow V$ is a closed embedding of $\mathbb{C}$-schemes with $f = g \circ \Phi : U \to \mathbb{C}$ and $\Phi|_X : X \to Y$ an isomorphism. Then Theorem 5.1(ii) defines the normal bundle N_{UV} of U in V, a vector bundle on U of rank $n = \dim V - \dim U$, and a nondegenerate quadratic form $q_{UV} \in H^0(S^2 N_{UV}^*|_X)$. Taking top exterior powers in the dual of (5.2) gives an isomorphism of line bundles on U

$$\rho_{UV} : K_U \otimes \Lambda^n N_{UV}^* \xrightarrow{\cong} \Phi^*(K_V),$$

where K_U, K_V are the canonical bundles of U, V.

Write X^{red} for the reduced $\mathbb{C}$-subscheme of X. As q_{UV} is a nondegenerate quadratic form on $N_{VW}|_X$, its determinant $\det(q_{VW})$ is a nonzero section of $(\Lambda^n N_{VW}^*)|_X^{\otimes^2}$. Define an isomorphism of line bundles on X^{red}:

$$J_\Phi = \rho_{UV}^{\otimes^2} \circ \big(\mathrm{id}_{K_U^2|_{X^{\mathrm{red}}}} \otimes \det(q_{UV})|_{X^{\mathrm{red}}} \big) : K_U^{\otimes^2} \big|_{X^{\mathrm{red}}} \xrightarrow{\cong} \Phi|_{X^{\mathrm{red}}}^* \big(K_V^{\otimes^2} \big). \tag{5.9}$$

Since principal $\mathbb{Z}/2\mathbb{Z}$-bundles $\pi : P \to X$ in the sense of Definition 2.9 are an (étale or complex analytic) topological notion, and X^{red} and X have the same topological space (even in the étale or complex analytic topology), principal $\mathbb{Z}/2\mathbb{Z}$-bundles on X^{red} and on X are equivalent. Define $\pi_\Phi : P_\Phi \to X$ to be the principal $\mathbb{Z}/2\mathbb{Z}$-bundle which parametrizes square roots of J_Φ on X^{red}. That is, (étale or complex analytic) local sections $s_\alpha : X \to P_\Phi$ of P_Φ correspond to local isomorphisms $\alpha : K_U|_{X^{\mathrm{red}}} \to \Phi|_{X^{\mathrm{red}}}^*(K_V)$ on X^{red} with $\alpha \otimes \alpha = J_\Phi$.

Now suppose W is another smooth $\mathbb{C}$-scheme, $h : W \to \mathbb{C}$ is regular, $Z = \mathrm{Crit}(h)$ as a $\mathbb{C}$-subscheme of W, and $\Psi : V \hookrightarrow W$ is a closed embedding of $\mathbb{C}$-schemes with $g = h \circ \Psi : V \to \mathbb{C}$ and $\Psi|_Y : Y \to Z$ an isomorphism. Then Theorem 5.1(iii) applies, and from (5.7)–(5.8) we can deduce that

$$J_{\Psi \circ \Phi} = \Phi|_{X^{\mathrm{red}}}^*(J_\Psi) \circ J_\Phi : K_U^{\otimes^2} \big|_{X^{\mathrm{red}}} \xrightarrow{\cong} (\Psi \circ \Phi)|_{X^{\mathrm{red}}}^* \big(K_W^{\otimes^2} \big)$$
$$= \Phi|_{X^{\mathrm{red}}}^* \big[\Psi|_{Y^{\mathrm{red}}}^* \big(K_W^{\otimes^2} \big) \big]. \tag{5.10}$$

For the principal $\mathbb{Z}/2\mathbb{Z}$-bundles $\pi_\Phi : P_\Phi \to X$, $\pi_\Psi : P_\Psi \to Y$, $\pi_{\Psi \circ \Phi} : P_{\Psi \circ \Phi} \to X$, equation (5.10) implies that there is a canonical isomorphism

$$\Xi_{\Psi, \Phi} : P_{\Psi \circ \Phi} \xrightarrow{\cong} \Phi|_X^*(P_\Psi) \otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Phi. \tag{5.11}$$

It is also easy to see that these $\Xi_{\Psi, \Phi}$ have an associativity property under triple compositions, that is, given another smooth $\mathbb{C}$-scheme T, regular $e : T \to \mathbb{C}$ with $Q := \mathrm{Crit}(e)$, and $\Upsilon : T \hookrightarrow U$

a closed embedding with $e = f \circ \Upsilon : T \to \mathbb{C}$ and $\Upsilon|_Q : Q \to X$ an isomorphism, then

$$\big(\mathrm{id}_{(\Phi \circ \Upsilon)|_Q^*(P_\Psi)} \otimes \Xi_{\Phi,\Upsilon}\big) \circ \Xi_{\Psi,\Phi \circ \Upsilon} = \big(\Upsilon|_Q^*(\Xi_{\Psi,\Phi}) \otimes \mathrm{id}_{P_\Upsilon}\big) \circ \Xi_{\Psi \circ \Phi,\Upsilon} :$$
$$P_{\Psi \circ \Phi \circ \Upsilon} \longrightarrow (\Phi \circ \Upsilon)|_Q^*(P_\Psi) \otimes_{\mathbb{Z}/2\mathbb{Z}} \Upsilon|_Q^*(P_\Phi) \otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Upsilon. \tag{5.12}$$

Analogues of all the above also work for complex manifolds and complex analytic spaces, as in Theorem 5.1(v).

The reason for restricting to X^{red} above is the following [23, Prop. 2.27], whose proof uses the fact that X^{red} is reduced in an essential way.
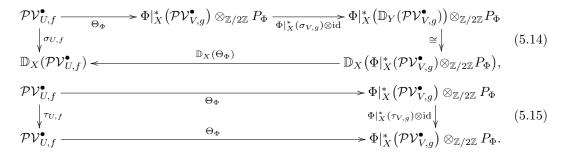
Lemma 5.3. *In Definition 5.2, the isomorphism J_Φ in (5.9) and the principal $\mathbb{Z}/2\mathbb{Z}$-bundle $\pi_\Phi : P_\Phi \to X$ depend only on U, V, X, Y, f, g and $\Phi|_X : X \to Y$. That is, they do not depend on $\Phi : U \to V$ apart from $\Phi|_X : X \to Y$.*

Using the notation of Definition 5.2, we can state our main result:

Theorem 5.4. (a) *Let U, V be smooth $\mathbb{C}$-schemes, $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ be regular, and $X = \mathrm{Crit}(f)$, $Y = \mathrm{Crit}(g)$ as $\mathbb{C}$-subschemes of U, V. Let $\Phi : U \hookrightarrow V$ be a closed embedding of $\mathbb{C}$-schemes with $f = g \circ \Phi : U \to \mathbb{C}$, and suppose $\Phi|_X : X \to V \supseteq Y$ is an isomorphism $\Phi|_X : X \to Y$. Then there is a natural isomorphism of perverse sheaves on X:*

$$\Theta_\Phi : \mathcal{PV}_{U,f}^\bullet \longrightarrow \Phi|_X^*\big(\mathcal{PV}_{V,g}^\bullet\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Phi, \tag{5.13}$$

where $\mathcal{PV}_{U,f}^\bullet, \mathcal{PV}_{V,g}^\bullet$ are the perverse sheaves of vanishing cycles from §2.4, and P_Φ the principal $\mathbb{Z}/2\mathbb{Z}$-bundle from Definition 5.2, and if $\mathcal{Q}^\bullet$ is a perverse sheaf on X then $\mathcal{Q}^\bullet \otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Phi$ is as in Definition 2.9. Also the following diagrams commute, where $\sigma_{U,f}, \sigma_{V,g}, \tau_{U,f}, \tau_{V,g}$ are as in (2.6)–(2.7):

$$\begin{array}{ccccc}
\mathcal{PV}_{U,f}^\bullet & \xrightarrow{\;\;\Theta_\Phi\;\;} & \Phi|_X^*\big(\mathcal{PV}_{V,g}^\bullet\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Phi & \xrightarrow{\Phi|_X^*(\sigma_{V,g}) \otimes \mathrm{id}} & \Phi|_X^*\big(\mathbb{D}_Y(\mathcal{PV}_{V,g}^\bullet)\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Phi \\
\Big\downarrow{\scriptstyle\sigma_{U,f}} & & & & \cong\Big\downarrow \\
\mathbb{D}_X(\mathcal{PV}_{U,f}^\bullet) & \xleftarrow{\;\;\mathbb{D}_X(\Theta_\Phi)\;\;} & & & \mathbb{D}_X\big(\Phi|_X^*(\mathcal{PV}_{V,g}^\bullet) \otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Phi\big),
\end{array} \tag{5.14}$$

$$\begin{array}{ccc}
\mathcal{PV}_{U,f}^\bullet & \xrightarrow{\;\;\;\;\;\;\;\;\;\;\Theta_\Phi\;\;\;\;\;\;\;\;\;\;} & \Phi|_X^*\big(\mathcal{PV}_{V,g}^\bullet\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Phi \\
\Big\downarrow{\scriptstyle\tau_{U,f}} & & {\scriptstyle\Phi|_X^*(\tau_{V,g}) \otimes \mathrm{id}}\Big\downarrow \\
\mathcal{PV}_{U,f}^\bullet & \xrightarrow{\;\;\;\;\;\;\;\;\;\;\Theta_\Phi\;\;\;\;\;\;\;\;\;\;} & \Phi|_X^*\big(\mathcal{PV}_{V,g}^\bullet\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Phi.
\end{array} \tag{5.15}$$

If $U = V$, $f = g$, $\Phi = \mathrm{id}_U$ then $\pi_\Phi : P_\Phi \to X$ is trivial, and Θ_Φ corresponds to $\mathrm{id}_{\mathcal{PV}_{U,f}^\bullet}$ under the natural isomorphism $\mathrm{id}_X^*(\mathcal{PV}_{U,f}^\bullet) \otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Phi \cong \mathcal{PV}_{U,f}^\bullet$.

(b) *The isomorphism Θ_Φ in (5.13) depends only on U, V, X, Y, f, g and $\Phi|_X : X \to Y$. That is, if $\tilde{\Phi} : U \to V$ is an alternative choice for Φ with $\Phi|_X = \tilde{\Phi}|_X : X \to Y$, then $\Theta_\Phi = \Theta_{\tilde{\Phi}}$, noting that $P_\Phi = P_{\tilde{\Phi}}$ by Lemma 5.3.*

(c) *Now suppose W is another smooth $\mathbb{C}$-scheme, $h : W \to \mathbb{C}$ is a regular function, $Z = \mathrm{Crit}(h)$, and $\Psi : V \hookrightarrow W$ is a closed embedding with $g = h \circ \Psi : V \to \mathbb{C}$ and $\Psi|_Y : Y \to Z$ an isomorphism. Then Definition 5.2 defines principal $\mathbb{Z}/2\mathbb{Z}$-bundles*

$$\pi_\Phi : P_\Phi \longrightarrow X, \quad \pi_\Psi : P_\Psi \longrightarrow Y, \quad \pi_{\Psi \circ \Phi} : P_{\Psi \circ \Phi} \longrightarrow X$$

and an isomorphism $\Xi_{\Psi,\Phi}$ *in* (5.11), *and part* **(a)** *defines isomorphisms of perverse sheaves* $\Theta_\Phi, \Theta_{\Psi\circ\Phi}$ *on* X *and* Θ_Ψ *on* Y. *Then the following commutes in* $\mathrm{Perv}(X)$:

$$
\begin{array}{ccc}
\mathcal{PV}^\bullet_{U,f} & \xrightarrow{\quad\quad\Theta_{\Psi\circ\Phi}\quad\quad} & (\Psi\circ\Phi)|_X^*\big(\mathcal{PV}^\bullet_{W,h}\big)\otimes_{\mathbb{Z}/2\mathbb{Z}}P_{\Psi\circ\Phi} \\
\Big\downarrow{\scriptstyle\Theta_\Phi} & & \Big\downarrow{\scriptstyle\mathrm{id}\otimes\Xi_{\Psi,\Phi}} \\
\Phi|_X^*\big(\mathcal{PV}^\bullet_{V,g}\big)\otimes_{\mathbb{Z}/2\mathbb{Z}}P_\Phi & \xrightarrow{\Phi|_X^*(\Theta_\Psi)\otimes\mathrm{id}} & \Phi|_X^*\circ\Psi|_Y^*\big(\mathcal{PV}^\bullet_{W,h}\big)\otimes_{\mathbb{Z}/2\mathbb{Z}}\Phi|_X^*(P_\Psi)\otimes_{\mathbb{Z}/2\mathbb{Z}}P_\Phi.
\end{array}
\tag{5.16}
$$

(d) *The analogues of* **(a)**–**(c)** *also hold for* $\mathscr{D}$-*modules on* $\mathbb{C}$-*schemes, for perverse sheaves and* $\mathscr{D}$-*modules on complex analytic spaces, and for mixed Hodge modules on* $\mathbb{C}$-*schemes and complex analytic spaces, as in* §2.6–§2.10.

Example 5.5. Let $U = \mathbb{C}\setminus\{0\}$ and $V = (\mathbb{C}\setminus\{0\})\times\mathbb{C}$ as smooth $\mathbb{C}$-schemes, define regular $f : U \to \mathbb{C}$ and $g : V \to \mathbb{C}$ by $f(x) = 0$ and $g(x,y) = x^k y^2$ for fixed $k \in \mathbb{Z}$, and define $\Phi : U \to V$ by $\Phi : x \mapsto (x,0)$, so that $f = g\circ\Phi : U \to \mathbb{C}$. Then $X := \mathrm{Crit}(f) = U$, and $Y := \mathrm{Crit}(g) = \big\{(x,y)\in V : kx^{k-1}y^2 = 2x^k y = 0\big\} = \big\{(x,y)\in V : y = 0\big\}$, as $x \neq 0$. Thus $\Phi|_X : X \to Y$ is an isomorphism.

In Theorem 5.1(ii), N^*_{UV} is the trivial line bundle on U with basis $\mathrm{d}y$, and $q_{UV} = x^k\mathrm{d}y\otimes\mathrm{d}y$. In Definition 5.2, $K_U|_X$ and $\Phi|_X^*(K_V)$ are the trivial line bundles on $X = X^{\mathrm{red}} = U$ with bases $\mathrm{d}x$ and $\mathrm{d}x\wedge\mathrm{d}y$, and J_Φ in (5.9) maps

$$
J_\Phi : \mathrm{d}x\otimes\mathrm{d}x \longmapsto x^k\,(\mathrm{d}x\wedge\mathrm{d}y)\otimes(\mathrm{d}x\wedge\mathrm{d}y).
$$

The principal $\mathbb{Z}/2\mathbb{Z}$-bundle $\pi_\Phi : P_\Phi \to X$ in Definition 5.2 parametrizes $\alpha : K_U|_X \to \Phi|_X^*(K_V)$ with $\alpha\otimes\alpha = J_\Phi$. Writing $\alpha : \mathrm{d}x\mapsto p\,\mathrm{d}x\wedge\mathrm{d}y$ for p a local function on $X = \mathbb{C}\setminus\{0\}$, $\alpha\otimes\alpha = J_\Phi$ reduces to $p^2 = x^k$. Thus, P_Φ parametrizes (étale local) square roots p of $x^k : \mathbb{C}\setminus\{0\} \to \mathbb{C}\setminus\{0\}$.

If k is even then x^k has a global square root $p = x^{k/2}$, so the principal $\mathbb{Z}/2\mathbb{Z}$-bundle P_Φ has a global section, and is trivial. If k is odd then x^k has no global square root on $X = \mathbb{C}\setminus\{0\}$, so P_Φ has no global section, and is nontrivial.

Thus, Theorem 5.4 implies that if k is even then $\mathcal{PV}^\bullet_{V,g} \cong A_Y[1]$ is the constant perverse sheaf on Y, but if k is odd then $\mathcal{PV}^\bullet_{V,g}$ is the twist of $A_Y[1]$ by the unique nontrivial principal $\mathbb{Z}/2\mathbb{Z}$-bundle on $Y \cong \mathbb{C}\setminus\{0\}$.

5.1. **Theorem 5.4(a): the isomorphism** Θ_Φ. Let U, V, f, g, X, Y, Φ be as in Theorem 5.4(a), and use the notation N_{UV}, q_{UV} from Theorem 5.1(ii) and J_Φ, P_Φ from Definition 5.2. We will show that there exists a unique perverse sheaf morphism Θ_Φ in (5.13) which is characterized by the property that whenever $U', V', \iota, \jmath, \Phi', \alpha, \beta, X', Y', f', g'$ are as in Theorem 5.1(i) then the following diagram of isomorphisms in $\mathrm{Perv}(X')$ commutes:

$$
\begin{array}{ccc}
\iota|_{X'}^*\big(\mathcal{PV}^\bullet_{U,f}\big) & \xrightarrow{\quad\iota|_{X'}^*(\gamma)\quad} & \iota|_{X'}^*\circ(\mathrm{id}_X\times 0)^*\big(\mathcal{PV}^\bullet_{U,f}\overset{L}{\boxtimes}\mathcal{PV}^\bullet_{\mathbb{C}^n,z_1^2+\cdots+z_n^2}\big) \\[2pt]
\Big\downarrow{\scriptstyle\iota|_{X'}^*(\Theta_\Phi)} & & \Big\downarrow{\scriptstyle\iota|_{X'}^*\circ(\mathrm{id}_X\times 0)^*(\mathcal{TS}^{-1}_{U,f,\mathbb{C}^n,z_1^2+\cdots+z_n^2})} \\[2pt]
\iota|_{X'}^*\circ\Phi|_X^*\big(\mathcal{PV}^\bullet_{V,g}\big)\otimes_{\mathbb{Z}/2\mathbb{Z}}\iota|_{X'}^*(P_\Phi) & & \begin{array}{c}\iota|_{X'}^*\circ(\mathrm{id}_X\times 0)^*\big(\mathcal{PV}^\bullet_{U\times\mathbb{C}^n,f\boxplus z_1^2+\cdots+z_n^2}\big) = \\ \Phi'|_{X'}^*\circ(\alpha\times\beta)|_{Y'}^*\big(\mathcal{PV}^\bullet_{U\times\mathbb{C}^n,f\boxplus z_1^2+\cdots+z_n^2}\big)\end{array} \\[2pt]
\Big\downarrow{\scriptstyle\delta} & & \Big\downarrow{\scriptstyle\Phi'|_{X'}^*(\mathcal{PV}^{-1}_{\alpha\times\beta})} \\[2pt]
\begin{array}{c}\iota|_{X'}^*\circ\Phi|_X^*\big(\mathcal{PV}^\bullet_{V,g}\big) = \\ \Phi'|_{X'}^*\circ\jmath|_{Y'}^*\big(\mathcal{PV}^\bullet_{V,g}\big)\end{array} & \xleftarrow{\quad\Phi'|_{X'}^*(\mathcal{PV}_\jmath)\quad} & \Phi'|_{X'}^*\big(\mathcal{PV}^\bullet_{V',g'}\big),
\end{array}
\tag{5.17}
$$

where $\mathcal{TS}_{U,f,\mathbb{C}^n,z_1^2+\cdots+z_n^2}$ is as in (2.8), and γ, δ are defined as follows:

(A) $\gamma : \mathcal{PV}^\bullet_{U,f} \rightarrow (\mathrm{id}_X \times 0)^* \big(\mathcal{PV}^\bullet_{U,f} \overset{L}{\boxtimes} \mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2 + \cdots + z_n^2} \big)$ in $\mathrm{Perv}(X)$ comes from the isomorphism $\mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2 + \cdots + z_n^2} \cong A_{\{0\}}$ in (2.12).

(B) The principal $\mathbb{Z}/2\mathbb{Z}$-bundle $P_\Phi \rightarrow X$ comes from $(N_{UV}|_{X^{\mathrm{red}}}, q_{UV}|_{X^{\mathrm{red}}})$, as the bundle of square roots of $\det(q_{UV}|_{X^{\mathrm{red}}})$. Thus, the pullback $\iota|^*_{X'}(P_\Phi) \rightarrow X'$ comes from $\big(\iota|^*_{X'^{\mathrm{red}}}(N_{UV}), \iota|^*_{X'^{\mathrm{red}}}(q_{UV}) \big)$. Now Theorem 5.1(ii) defines

$$\hat{\beta}|_{X'^{\mathrm{red}}} : \langle \mathrm{d}z_1, \ldots, \mathrm{d}z_n \rangle_{X'^{\mathrm{red}}} \overset{\cong}{\longrightarrow} \iota|^*_{X'^{\mathrm{red}}}(N^*_{UV})$$

identifying $\sum_{j=1}^n \mathrm{d}z_j^2$ with $\iota|^*_{X'^{\mathrm{red}}}(q_{UV})$. Thus, $\hat{\beta}|_{X'^{\mathrm{red}}}$ induces a trivialization of

$$\iota|^*_{X'}(P_\Phi) \longrightarrow X'.$$

Then $\delta : \iota|^*_{X'} \circ \Phi|^*_X \big(\mathcal{PV}^\bullet_{V,g} \big) \otimes_{\mathbb{Z}/2\mathbb{Z}} \iota|^*_{X'}(P_\Phi) \rightarrow \iota|^*_{X'} \circ \Phi|^*_X \big(\mathcal{PV}^\bullet_{V,g} \big)$ in $\mathrm{Perv}(X')$ comes from this trivialization of the principal $\mathbb{Z}/2\mathbb{Z}$-bundle $\iota|^*_{X'}(P_\Phi) \rightarrow X'$.

Since Theorem 5.1(i) holds for each $x \in X$, we may choose a family

$$\big\{ (U'_a, V'_a, \iota_a, \jmath_a, \Phi'_a, \alpha_a, \beta_a, f'_a, g'_a, X'_a, Y'_a) : a \in A \big\}$$

such that $U'_a, V'_a, \ldots, Y'_a$ satisfy Theorem 5.1(i) for each $a \in A$, and $\big\{ \iota'_a|_{X'_a} : X'_a \rightarrow X \big\}_{a \in A}$ is an étale open cover of X. For each $a \in A$, define an isomorphism

$$\Theta_a : \iota_a|^*_{X'_a} \big(\mathcal{PV}^\bullet_{U,f} \big) \longrightarrow \iota_a|^*_{X'_a} \circ \Phi|^*_X \big(\mathcal{PV}^\bullet_{V,g} \big)$$

to make the following diagram of isomorphisms commute:

$$
\begin{array}{ccc}
\iota_a|^*_{X'_a}\big(\mathcal{PV}^\bullet_{U,f}\big) & \xrightarrow{\quad \iota_a|^*_{X'_a}(\gamma) \quad} & \iota_a|^*_{X'_a} \circ (\mathrm{id}_X \times 0)^* \big(\mathcal{PV}^\bullet_{U,f} \overset{L}{\boxtimes} \mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2 + \cdots + z_n^2} \big) \\[2mm]
\Big\downarrow {\scriptstyle \Theta_a} & & \Big\downarrow {\scriptstyle \iota_a|^*_{X'_a} \circ (\mathrm{id}_X \times 0)^* (\mathcal{TS}^{-1}_{U,f,\mathbb{C}^n, z_1^2 + \cdots + z_n^2})} \\[2mm]
\begin{array}{c} \iota_a|^*_{X'_a} \circ \Phi|^*_X \big(\mathcal{PV}^\bullet_{V,g} \big) \\ \otimes_{\mathbb{Z}/2\mathbb{Z}} \iota_a|^*_{X'_a}(P_\Phi) \end{array} & & \begin{array}{c} \iota_a|^*_{X'_a} \circ (\mathrm{id}_X \times 0)^* \big(\mathcal{PV}^\bullet_{U \times \mathbb{C}^n, f \boxplus z_1^2 + \cdots + z_n^2} \big) = \\ \Phi'_a|^*_{X'_a} \circ (\alpha_a \times \beta_a)|^*_{Y'_a} \big(\mathcal{PV}^\bullet_{U \times \mathbb{C}^n, f \boxplus z_1^2 + \cdots + z_n^2} \big) \end{array} \\[3mm]
\Big\downarrow {\scriptstyle \delta_a} & & \Big\downarrow {\scriptstyle \Phi'_a|^*_{X'_a}(\mathcal{PV}^{-1}_{\alpha_a \times \beta_a})} \\[2mm]
\begin{array}{c} \iota_a|^*_{X'_a} \circ \Phi|^*_X \big(\mathcal{PV}^\bullet_{V,g} \big) = \\ \Phi'_a|^*_{X'_a} \circ \jmath_a|^*_{Y'_a} \big(\mathcal{PV}^\bullet_{V,g} \big) \end{array} & \xleftarrow{\quad \Phi'_a|^*_{X'_a}(\mathcal{PV}_{\jmath_a}) \quad} & \Phi'_a|^*_{X'_a} \big(\mathcal{PV}^\bullet_{V'_a, g'_a} \big),
\end{array}
$$
(5.18)

where γ is as in (A), and δ_a defined as in (B) above.

For $a, b \in A$, define

$$U'_{ab} = U'_a \times_{\iota_a, U, \iota_b} U'_b \quad \text{and} \quad V'_{ab} = V'_a \times_{\jmath_a, V, \jmath_b} V'_b,$$

with projections $\Pi_{U'_a} : U'_{ab} \rightarrow U'_a$, $\Pi_{U'_b} : U'_{ab} \rightarrow U'_b$, $\Pi_{V'_a} : V'_{ab} \rightarrow V'_a$, $\Pi_{V'_b} : V'_{ab} \rightarrow V'_b$. Then U'_{ab}, V'_{ab} are smooth and $\Pi_{U'_a}, \Pi_{U'_b}, \Pi_{V'_a}, \Pi_{V'_b}$ étale. The universal property of $V'_a \times_{\jmath_a, V, \jmath_b} V'_b$ gives a unique morphism $\Phi'_{ab} : U'_{ab} \rightarrow V'_{ab}$ with

$$\Pi_{V'_a} \circ \Phi'_{ab} = \Phi'_a \circ \Pi_{U'_a} \quad \text{and} \quad \Pi_{V'_b} \circ \Phi'_{ab} = \Phi'_b \circ \Pi_{U'_b}. \tag{5.19}$$

Set $f'_{ab} = f'_a \circ \Pi_{U'_a} : U'_{ab} \to \mathbb{C}$, $g'_{ab} = g'_a \circ \Pi_{V'_a} : V'_{ab} \to \mathbb{C}$ and $X'_{ab} = \mathrm{Crit}(f'_{ab}) \subseteq U'_{ab}$, $Y'_{ab} = \mathrm{Crit}(g'_{ab}) \subseteq V'_{ab}$. As for (4.17) we have

$$f'_{ab} = f'_a \circ \Pi_{U'_a} = f \circ \iota_a \circ \Pi_{U'_a} = f \circ \iota_b \circ \Pi_{U'_b} = f'_b \circ \Pi_{U'_b},$$
$$g'_{ab} = g'_a \circ \Pi_{V'_a} = g \circ \jmath_a \circ \Pi_{V'_a} = g \circ \jmath_b \circ \Pi_{V'_b} = g'_b \circ \Pi_{V'_b}$$
$$= (f \boxplus z_1^2 + \cdots + z_n^2) \circ (\alpha_a \times \beta_a) \circ \Pi_{V'_a}$$
$$= (f \boxplus z_1^2 + \cdots + z_n^2) \circ (\alpha_b \times \beta_b) \circ \Pi_{V'_b}.$$

Apply Theorem 3.1 with $V'_{ab}, U \times \mathbb{C}^n, (\alpha_a \times \beta_a) \circ \Pi_{V'_a}, (\alpha_b \times \beta_b) \circ \Pi_{V'_b}, g'_{ab}$, and $f \boxplus z_1^2 + \cdots + z_n^2$ in place of V, W, Φ, Ψ, f, g. The analogue of $\Phi|_X = \Psi|_X$ is

$$\begin{aligned}
(\alpha_a \times \beta_a) \circ \Pi_{V'_a}|_{Y'_{ab}} &= ((\Phi|_X^{-1} \circ \Phi|_X \circ \alpha_a)|_{Y'_a} \times 0) \circ \Pi_{V'_a}|_{Y'_{ab}} \\
&= (\Phi|_X^{-1} \circ \jmath_a \circ \Pi_{V'_a}|_{Y'_{ab}}) \times 0 = (\Phi|_X^{-1} \circ \jmath_b \circ \Pi_{V'_b}|_{Y'_{ab}}) \times 0 \qquad (5.20)\\
&= ((\Phi|_X^{-1} \circ \Phi|_X \circ \alpha_b)|_{Y'_b} \times 0) \circ \Pi_{V'_b}|_{Y'_{ab}} = (\alpha_b \times \beta_b) \circ \Pi_{V'_b}|_{Y'_{ab}},
\end{aligned}$$

using $\Phi|_X : X \to Y$ an isomorphism and $\beta_a|_{Y'_a} = 0$ in the first step, $\jmath_a|_{Y'_a} = \Phi|_X \circ \alpha_a|_{Y'_a}$ in the second, $\jmath_a \circ \Pi_{V'_a} = \jmath_b \circ \Pi_{V'_b}$ in the third, $\jmath_b|_{Y'_b} = \Phi|_X \circ \alpha_b|_{Y'_b}$ in the fourth, and $\beta_b|_{Y'_b} = 0$ in the fifth. Thus Theorem 3.1 gives

$$\begin{aligned}
&\mathcal{PV}_{(\alpha_a \times \beta_a) \circ \Pi_{V'_a}} = \det\left[\mathrm{d}\big((\alpha_b \times \beta_b) \circ \Pi_{V'_b}\big)|_{Y'^{\mathrm{red}}_{ab}}^{-1} \circ \mathrm{d}\big((\alpha_a \times \beta_a) \circ \Pi_{V'_a}\big)|_{Y'^{\mathrm{red}}_{ab}}\right] \cdot \\
&\mathcal{PV}_{(\alpha_b \times \beta_b) \circ \Pi_{V'_b}} : \mathcal{PV}^\bullet_{V'_{ab}, g'_{ab}} \longrightarrow (\alpha_a \times \beta_a) \circ \Pi_{V'_a}|_{Y'_{ab}}^* \big(\mathcal{PV}^\bullet_{U \times \mathbb{C}^n, f \boxplus z_1^2 + \cdots + z_n^2}\big)
\end{aligned} \qquad (5.21)$$

in $\mathrm{Perv}(Y'_{ab})$, where $\det[\cdots]$ maps $Y'^{\mathrm{red}}_{ab} \to \{\pm 1\}$.

Consider the morphisms

$$\begin{aligned}
\Pi_{U'_a}|_{X'_{ab}}^*(\delta_a), \Pi_{U'_b}|_{X'_{ab}}^*(\delta_b) : (\Phi \circ \iota_a \circ \Pi_{U'_a})|_{X'_{ab}}^* \big(\mathcal{PV}^\bullet_{V,g}\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} (\iota_a \circ \Pi_{U'_a})|_{X'_{ab}}^*(P_\Phi) \\
\longrightarrow (\Phi \circ \iota_a \circ \Pi_{U'_a})|_{X'_{ab}}^* \big(\mathcal{PV}^\bullet_{V,g}\big).
\end{aligned} \qquad (5.22)$$

As in (B) above, these are defined using two different trivializations of the principal $\mathbb{Z}/2\mathbb{Z}$-bundle $(\iota_a \circ \Pi_{U'_a})|_{X'_{ab}}^*(P_\Phi) \to X'_{ab}$, defined using

$$\Pi_{U'_a}|_{X'^{\mathrm{red}}_{ab}}^*(\hat{\beta}_a), \Pi_{U'_b}|_{X'^{\mathrm{red}}_{ab}}^*(\hat{\beta}_b) : \langle \mathrm{d}z_1, \ldots, \mathrm{d}z_n \rangle_{X'^{\mathrm{red}}_{ab}} \longrightarrow (\iota_a \circ \Pi_{U'_a})|_{X'^{\mathrm{red}}_{ab}}^*(N^*_{UV}),$$

which are isomorphisms of vector bundles on X'^{red}_{ab} identifying the nondegenerate quadratic forms $\sum_{j=1}^n \mathrm{d}z_j^2$ on $\langle \mathrm{d}z_1, \ldots, \mathrm{d}z_n \rangle_{X'^{\mathrm{red}}_{ab}}$ and $(\iota_a \circ \Pi_{U'_a})|_{X'^{\mathrm{red}}_{ab}}^*(q_{UV})$ on $(\iota_a \circ \Pi_{U'_a})|_{X'^{\mathrm{red}}_{ab}}^*(N^*_{UV})$, for $\hat{\beta}_a, \hat{\beta}_b$ as in (5.3). Thus we see that

$$\Pi_{U'_a}|_{X'_{ab}}^*(\delta_a) = \det\left[\Pi_{U'_a}|_{X'^{\mathrm{red}}_{ab}}^*(\hat{\beta}_a) \circ \Pi_{U'_b}|_{X'^{\mathrm{red}}_{ab}}^*(\hat{\beta}_b)^{-1}\right] \cdot \Pi_{U'_b}|_{X'_{ab}}^*(\delta_b), \qquad (5.23)$$

where $\det[\cdots]$ maps $X'^{\mathrm{red}}_{ab} \to \{\pm 1\}$ since both isomorphisms in (5.22) identify the same nondegenerate quadratic forms.

We have an exact sequence of vector bundles on X'^{red}_{ab}:

$$0 \longrightarrow TU'_{ab}|_{X'^{\mathrm{red}}_{ab}} \longrightarrow \Phi'_{ab}|_{X'^{\mathrm{red}}_{ab}}^*(TV'_{ab}) \longrightarrow (\iota_a \circ \Pi_{U'_a})|_{X'^{\mathrm{red}}_{ab}}^*(N_{UV}) \longrightarrow 0.$$

Choosing a local splitting of this sequence, we may identify

$$\begin{aligned}
&\Phi'_{ab}|_{X'^{\mathrm{red}}_{ab}}^* \left[\mathrm{d}\big((\alpha_b \times \beta_b) \circ \Pi_{V'_b}\big)|_{Y'^{\mathrm{red}}_{ab}}^{-1} \circ \mathrm{d}\big((\alpha_a \times \beta_a) \circ \Pi_{V'_a}\big)|_{Y'^{\mathrm{red}}_{ab}}\right] \\
&\qquad \cong \begin{pmatrix} \mathrm{id}_{TU'_{ab}|_{X'^{\mathrm{red}}_{ab}}} & * \\ 0 & \big(\Pi_{U'_a}|_{X'^{\mathrm{red}}_{ab}}^*(\hat{\beta}_a) \circ \Pi_{U'_b}|_{X'^{\mathrm{red}}_{ab}}^*(\hat{\beta}_b)^{-1}\big)^* \end{pmatrix}.
\end{aligned}$$

Therefore

$$
\begin{aligned}
&\Phi'_{ab}|^*_{X'^{\mathrm{red}}_{ab}}\left(\det\left[\mathrm{d}\left((\alpha_b\times\beta_b)\circ\Pi_{V'_b}\right)|^{-1}_{Y'^{\mathrm{red}}_{ab}}\circ\mathrm{d}\left((\alpha_a\times\beta_a)\circ\Pi_{V'_a}\right)|_{Y'^{\mathrm{red}}_{ab}}\right]\right)\\
&\quad=\det\left[\Pi_{U'_a}|^*_{X'^{\mathrm{red}}_{ab}}(\hat{\beta}_a)\circ\Pi_{U'_b}|^*_{X'^{\mathrm{red}}_{ab}}(\hat{\beta}_b)^{-1}\right]:X'^{\mathrm{red}}_{ab}\longrightarrow\{\pm1\}.
\end{aligned}
\tag{5.24}
$$

Now

$$
\begin{aligned}
&\Pi_{U'_a}|^*_{X'_{ab}}(\Theta_a)=\Pi_{U'_a}|^*_{X'_{ab}}(\delta_a^{-1})\circ(\Phi'_a\circ\Pi_{U'_a})|^*_{X'_{ab}}(\mathcal{PV}_{\jmath_a})\circ(\Phi'_a\circ\Pi_{U'_a})|^*_{X'_{ab}}(\mathcal{PV}^{-1}_{\alpha_a\times\beta_a})\\
&\circ((\mathrm{id}_X\times0)\circ\iota_a\circ\Pi_{U'_a})|^*_{X'_{ab}}(\mathcal{TS}^{-1}_{U,f,\mathbb{C}^n,\Sigma_j z_j^2})\circ(\iota_a\circ\Pi_{U'_a})|^*_{X'_{ab}}(\gamma)\\
&=\Pi_{U'_a}|^*_{X'_{ab}}(\delta_a^{-1})\circ(\Pi_{V'_a}\circ\Phi'_{ab})|^*_{X'_{ab}}(\mathcal{PV}_{\jmath_a})\circ\Phi'_{ab}|^*_{X'_{ab}}(\mathcal{PV}_{\Pi_{V'_a}})\circ\Phi'_{ab}|^*_{X'_{ab}}(\mathcal{PV}^{-1}_{\Pi_{V'_a}})\\
&\circ(\Pi_{V'_a}\circ\Phi'_{ab})|^*_{X'_{ab}}(\mathcal{PV}^{-1}_{\alpha_a\times\beta_a})\circ((\mathrm{id}_X\times0)\circ\iota_a\circ\Pi_{U'_a})|^*_{X'_{ab}}(\mathcal{TS}^{-1}_{U,f,\mathbb{C}^n,\Sigma_j z_j^2})\circ(\iota_a\circ\Pi_{U'_a})|^*_{X'_{ab}}(\gamma)\\
&=\Pi_{U'_a}|^*_{X'_{ab}}(\delta_a^{-1})\circ\Phi'_{ab}|^*_{X'_{ab}}(\mathcal{PV}_{\jmath_a\circ\Pi_{V'_a}})\circ\Phi'_{ab}|^*_{X'_{ab}}(\mathcal{PV}^{-1}_{(\alpha_a\times\beta_a)\circ\Pi_{V'_a}})\\
&\circ((\mathrm{id}_X\times0)\circ\iota_a\circ\Pi_{U'_a})|^*_{X'_{ab}}(\mathcal{TS}^{-1}_{U,f,\mathbb{C}^n,\Sigma_j z_j^2})\circ(\iota_a\circ\Pi_{U'_a})|^*_{X'_{ab}}(\gamma)\\
&=\det\left[\Pi_{U'_a}|^*_{X'^{\mathrm{red}}_{ab}}(\hat{\beta}_a)\circ\Pi_{U'_b}|^*_{X'^{\mathrm{red}}_{ab}}(\hat{\beta}_b)^{-1}\right]^{-1}\cdot\\
&\Phi'_{ab}|^*_{X'^{\mathrm{red}}_{ab}}\left(\det\left[\mathrm{d}\left((\alpha_b\times\beta_b)\circ\Pi_{V'_b}\right)|^{-1}_{Y'^{\mathrm{red}}_{ab}}\circ\mathrm{d}\left((\alpha_a\times\beta_a)\circ\Pi_{V'_a}\right)|_{Y'^{\mathrm{red}}_{ab}}\right]\right)\cdot\\
&\Pi_{U'_b}|^*_{X'_{ab}}(\delta_b^{-1})\circ\Phi'_{ab}|^*_{X'_{ab}}(\mathcal{PV}_{\jmath_b\circ\Pi_{V'_b}})\circ\Phi'_{ab}|^*_{X'_{ab}}(\mathcal{PV}^{-1}_{(\alpha_b\times\beta_b)\circ\Pi_{V'_b}})\\
&\circ((\mathrm{id}_X\times0)\circ\iota_b\circ\Pi_{U'_b})|^*_{X'_{ab}}(\mathcal{TS}^{-1}_{U,f,\mathbb{C}^n,\Sigma_j z_j^2})\circ(\iota_b\circ\Pi_{U'_b})|^*_{X'_{ab}}(\gamma)\\
&=\Pi_{U'_b}|^*_{X'_{ab}}(\delta_b^{-1})\circ(\Pi_{V'_b}\circ\Phi'_{ab})|^*_{X'_{ab}}(\mathcal{PV}_{\jmath_b})\circ\Phi'_{ab}|^*_{X'_{ab}}(\mathcal{PV}_{\Pi_{V'_b}})\circ\Phi'_{ab}|^*_{X'_{ab}}(\mathcal{PV}^{-1}_{\Pi_{V'_b}})\\
&\circ(\Pi_{V'_b}\circ\Phi'_{ab})|^*_{X'_{ab}}(\mathcal{PV}^{-1}_{\alpha_b\times\beta_b})\circ((\mathrm{id}_X\times0)\circ\iota_b\circ\Pi_{U'_b})|^*_{X'_{ab}}(\mathcal{TS}^{-1}_{U,f,\mathbb{C}^n,\Sigma_j z_j^2})\circ(\iota_b\circ\Pi_{U'_b})|^*_{X'_{ab}}(\gamma)\\
&=\Pi_{U'_b}|^*_{X'_{ab}}(\delta_b^{-1})\circ(\Phi'_b\circ\Pi_{U'_b})|^*_{X'_{ab}}(\mathcal{PV}_{\jmath_b})\circ(\Phi'_b\circ\Pi_{U'_b})|^*_{X'_{ab}}(\mathcal{PV}^{-1}_{\alpha_b\times\beta_b})\\
&\circ((\mathrm{id}_X\times0)\circ\iota_b\circ\Pi_{U'_b})|^*_{X'_{ab}}(\mathcal{TS}^{-1}_{U,f,\mathbb{C}^n,\Sigma_j z_j^2})\circ(\iota_b\circ\Pi_{U'_b})|^*_{X'_{ab}}(\gamma)=\Pi_{U'_b}|^*_{X'_{ab}}(\Theta_b),
\end{aligned}
\tag{5.25}
$$

using (5.18) in the first and seventh steps, (5.19) in the second and sixth, (2.18) in the third, (5.21), (5.23), $\iota_a\circ\Pi_{U'_a}=\iota_b\circ\Pi_{U'_b}$ and $\jmath_a\circ\Pi_{V'_a}=\jmath_b\circ\Pi_{V'_b}$ in the fourth, and (2.18) and (5.24) in the fifth. Therefore Theorem 2.7(i) applied to the étale open cover $\left\{\iota_a|_{X'_a}:X'_a\to X\right\}_{a\in A}$ of X shows that there is a unique isomorphism Θ_Φ in (5.13) with $\iota_a|^*_{X'_a}(\Theta_\Phi)=\Theta_a$ for all $a\in A$.

Suppose $\left\{(U'_a,\ldots,Y'_a):a\in A\right\}$ and $\left\{(\tilde{U}'_a,\ldots,\tilde{Y}'_a):a\in\tilde{A}\right\}$ are alternative choices above, yielding morphisms Θ_Φ and $\tilde{\Theta}_\Phi$ in (5.13). By running the same construction using the family $\left\{(U'_a,\ldots,Y'_a):a\in A\right\}\amalg\left\{(\tilde{U}'_a,\ldots,\tilde{Y}'_a):a\in\tilde{A}\right\}$, we can show that $\Theta_\Phi=\tilde{\Theta}_\Phi$, so Θ_Φ is independent of the choice of $\left\{(U'_a,\ldots,Y'_a):a\in A\right\}$ above. Let $U',V',\iota,\jmath,\Phi',\alpha,\beta,X',Y',f',g'$ be as in Theorem 5.1(i). Constructing Θ_Φ using $\left\{(U'_a,\ldots,Y'_a):a\in A\right\}\amalg\left\{(U',\ldots,Y')\right\}$, we see from (5.18) that (5.17) commutes. This completes the construction of Θ_Φ.

To see that (5.14)–(5.15) commute, in the situation of (5.17) we show that Verdier duality and monodromy operators commute with each morphism in (5.17). Going clockwise from the top left corner, $\iota|^*_{X'}(\gamma)$ is compatible with Verdier duality and monodromy because of the commutative diagrams

$$
\begin{array}{ccc}
A_{\{0\}} & \xrightarrow{\ \cong\ } & \mathbb{D}_{\{0\}}\left(A_{\{0\}}\right)\\
\downarrow{\scriptstyle\Gamma} & & \uparrow{\scriptstyle\mathbb{D}(\Gamma)}\\
\mathcal{PV}^\bullet_{\mathbb{C}^n,\Sigma_j z_j^2} & \xrightarrow[\ \sigma_{\mathbb{C}^n,\Sigma_j z_j^2}\ ]{} & \mathbb{D}_{\{0\}}\left(\mathcal{PV}^\bullet_{\mathbb{C}^n,\Sigma_j z_j^2}\right),
\end{array}
\qquad
\begin{array}{ccc}
A_{\{0\}} & \xrightarrow{\ \mathrm{id}\ } & A_{\{0\}}\\
\downarrow{\scriptstyle\Gamma} & & \downarrow{\scriptstyle\Gamma}\\
\mathcal{PV}^\bullet_{\mathbb{C}^n,\Sigma_j z_j^2} & \xrightarrow[\ \tau_{\mathbb{C}^n,\Sigma_j z_j^2}\ ]{} & \mathcal{PV}^\bullet_{\mathbb{C}^n,\Sigma_j z_j^2},
\end{array}
$$

where $\Gamma : A_{\{0\}} \to \mathcal{PV}^\bullet_{\mathbb{C}^n, \Sigma_j z_j^2}$ is the isomorphism used to define γ in (A) above. Equations (2.9)–(2.10) imply that $\iota|^*_{X'} \circ (\mathrm{id}_X \times 0)^*(\mathcal{TS}^{-1}_{U,f,\mathbb{C}^n, \Sigma_j z_j^2})$ is compatible with Verdier duality and monodromy, and (2.16)–(2.17) imply that $\Phi'|^*_{X'}(\mathcal{PV}_j), \Phi'|^*_{X'}(\mathcal{PV}^{-1}_{\alpha \times \beta})$ are. Also δ is compatible with Verdier duality and monodromy, since these do not affect the trivialization of $\iota|^*_{X'}(P_\Phi) \to X'$ used to define δ in (B) above.

Thus by (5.17) we see that $\iota|^*_{X'}(\Theta_\Phi)$ is compatible with Verdier duality and monodromy, that is, $\iota|^*_{X'}$ applied to (5.14)–(5.15) commute. Since we can form an étale open cover of X by such $\iota|_{X'} : X' \to X$, Theorem 2.7(i) implies that (5.14)–(5.15) commute.

Finally, if $U = V$, $f = g$ and $\Phi = \mathrm{id}_U$ then $J_\Phi = \mathrm{id} : K^2_U|_{X^{\mathrm{red}}} \to K^2_U|_{X^{\mathrm{red}}}$ in (5.9), which has a natural square root $\alpha = \mathrm{id} : K_U|_{X^{\mathrm{red}}} \to K_U|_{X^{\mathrm{red}}}$, so $\pi_\Phi : P_\Phi \to X$ is trivial in Definition 5.2. In (5.17) we may put $U' = V' = U$, $\iota = \jmath = \alpha = \mathrm{id}_U$, $n = 0$, $\beta = 0$, $X' = Y' = X$, $f' = g' = f$, and then each morphism in (5.17) is essentially the identity on $\mathcal{PV}^\bullet_{U,f}$, so $\Theta_\Phi = \mathrm{id}^*_X(\Theta_\Phi) = \mathrm{id}_{\mathcal{PV}^\bullet_{U,f}}$. This proves Theorem 5.4(a).

5.2. Theorem 5.4(b): Θ_Φ depends only on $\Phi|_X : X \to Y$.

Suppose $\Phi, \tilde{\Phi} : U \to V$ are alternative choices in Theorem 5.4(a) with

$$\Phi|_X = \tilde{\Phi}|_X : X \longrightarrow Y,$$

so that $P_\Phi = P_{\tilde{\Phi}}$ by Lemma 5.3. Fix $x \in X$, let $a \neq b$ be labels, and let U'_a, V'_a, ι_a, $\jmath_a$, Φ'_a, α_a, β_a, X'_a, Y'_a, f'_a, g'_a be as in Theorem 5.1(i) for x, Φ and $U'_b, V'_b, \ldots, g'_b$ as in Theorem 5.1(i) for $x, \tilde{\Phi}$. As in §5.1, define Θ_a, Θ_b and $U'_{ab}, V'_{ab}, \Pi_{U'_a}, \Pi_{U'_b}, \Pi_{V'_a}, \Pi_{V'_b}, \Phi'_{ab}, f'_{ab}, g'_{ab}, X'_{ab}, Y'_{ab}$, and follow the proof in §5.1 from (5.19) as far as (5.25).

This proof does not actually need $U'_a, \ldots, g_a, \Theta_a$ and $U'_b, \ldots, g_b, \Theta_b$ to be defined using the same $\Phi : U \to V$, it only uses in (5.20)–(5.22) that $\Phi|_X : X \to Y$ is the same for $U'_a, \ldots, \Theta_a$ and $U'_b, \ldots, \Theta_b$. Thus we can apply it with $U'_a, \ldots, \Theta_a$ defined using Φ, and $U'_b, \ldots, \Theta_b$ defined using $\tilde{\Phi}$. Hence

$$\begin{aligned}
(\iota_a \circ \Pi_{U'_a})|^*_{X'_{ab}}(\Theta_\Phi) &= \Pi_{U'_a}|^*_{X'_{ab}}(\Theta_a) = \Pi_{U'_b}|^*_{X'_{ab}}(\Theta_b) \\
&= (\iota_b \circ \Pi_{U'_b})|^*_{X'_{ab}}(\Theta_{\tilde{\Phi}}) = (\iota_a \circ \Pi_{U'_a})|^*_{X'_{ab}}(\Theta_{\tilde{\Phi}}),
\end{aligned}$$

using $\iota_a|^*_{X'_a}(\Theta_\Phi) = \Theta_a$ in the first step, (5.25) in the second, $\iota_b|^*_{X'_b}(\Theta_{\tilde{\Phi}}) = \Theta_b$ in the third, and $\iota_a \circ \Pi_{U'_a} = \iota_b \circ \Pi_{U'_b}$ in the fourth. As such $\iota_a \circ \Pi_{U'_a}|_{X'_{ab}} : X'_{ab} \to X$ form an étale open cover of X, this implies that $\Theta_\Phi = \Theta_{\tilde{\Phi}}$ by Theorem 2.7(i).

5.3. Theorem 5.4(c): composition of the Θ_Φ.

Let $U, V, W, f, g, h, X, Y, Z, \Phi, \Psi$ be as in Theorem 5.4(c). Let $x \in X$, and set $y = \Phi(x) \in Y$. Apply Theorem 5.1(i) to $U, V, f, g, X, Y, \Phi, x$ to get $\mathbb{C}$-schemes U', V', a point $x' \in U'$, morphisms $\iota : U' \to U$, $\jmath : V' \to V$, $\Phi' : U' \to V'$, $\alpha : V' \to U$ and $\beta : V' \to \mathbb{C}^m$ where $m = \dim V - \dim U$, and $f' := f \circ \iota : U' \to \mathbb{C}$, $g' := g \circ \jmath : V' \to \mathbb{C}$, $X' := \mathrm{Crit}(f') \subseteq U'$, $Y' := \mathrm{Crit}(g') \subseteq V'$, satisfying conditions including $\iota, \jmath, \alpha \times \beta$ étale, (5.1) commutes, and $\iota(x') = x$.

Similarly, apply Theorem 5.1(i) to $V, W, g, h, Y, Z, \Psi, y$ to get $\mathbb{C}$-schemes $\tilde{V}, \tilde{W}$, a point $\tilde{y} \in \tilde{V}$, morphisms $\tilde{\iota} : \tilde{V} \to V$, $\tilde{\jmath} : \tilde{W} \to W$, $\tilde{\Psi} : \tilde{V} \to \tilde{W}$, $\tilde{\alpha} : \tilde{W} \to V$ and $\tilde{\beta} : \tilde{W} \to \mathbb{C}^n$ where $n = \dim W - \dim V$, and $\tilde{g} := g \circ \tilde{\iota} : \tilde{V} \to \mathbb{C}$, $\tilde{h} := h \circ \tilde{\jmath} : \tilde{W} \to \mathbb{C}$, $\tilde{Y} := \mathrm{Crit}(\tilde{g}) \subseteq \tilde{V}$, $\tilde{Z} := \mathrm{Crit}(\tilde{h}) \subseteq \tilde{W}$, satisfying conditions.

Define $\hat{U} = U' \times_{\Phi \circ \iota, V, \tilde{\iota}} \tilde{V}$ and $\hat{W} = V' \times_{\jmath, V, \tilde{\alpha}} \tilde{W}$, with projections $\Pi_{U'} : \hat{U} \to U'$, $\Pi_{\tilde{V}} : \hat{U} \to \tilde{V}$, $\Pi_{V'} : \hat{W} \to V'$, $\Pi_{\tilde{W}} : \hat{W} \to \tilde{W}$. As $x' \in U'$ and $\tilde{y} \in \tilde{V}$ with $\Phi \circ \iota(x') = y = \tilde{\iota}(\tilde{y})$, there exists $\hat{x} \in \hat{U}$ with $\Pi_{U'}(\hat{x}) = x'$ and $\Pi_{\tilde{V}}(\hat{x}) = \tilde{y}$. Set $\hat{f} := f' \circ \Pi_{U'} : \hat{U} \to \mathbb{C}$ and $\hat{h} := \tilde{h} \circ \Pi_{\tilde{W}} : \hat{W} \to \mathbb{C}$,

and $\hat{X} := \mathrm{Crit}(\hat{f}) \subseteq \hat{U}$, $\hat{Z} := \mathrm{Crit}(\hat{h}) \subseteq \hat{W}$. The morphisms $\Phi' \circ \Pi_{U'} : \hat{U} \to V'$, $\tilde{\Psi} \circ \Pi_{\tilde{V}} : \hat{U} \to \tilde{W}$ satisfy

$$\jmath \circ (\Phi' \circ \Pi_{U'}) = \Phi \circ \iota \circ \Pi_{U'} = \tilde{\iota} \circ \Pi_{\tilde{V}} = \tilde{\alpha} \circ (\tilde{\Psi} \circ \Pi_{\tilde{V}}).$$

Hence there exists a unique morphism $\widehat{\Psi \circ \Phi} : \hat{U} \to \hat{W}$ such that $\Pi_{V'} \circ \widehat{\Psi \circ \Phi} = \Phi' \circ \Pi_{U'}$ and $\Pi_{\tilde{W}} \circ \widehat{\Psi \circ \Phi} = \tilde{\Psi} \circ \Pi_{\tilde{V}}$. Then the following diagram

is the analogue of (5.1) for $U, W, f, h, X, Z, \Psi \circ \Phi, x$, and the conclusions of Theorem 5.1(i) hold. Thus (5.17) holds for Θ_Φ using $U', V', X', Y'\iota, \jmath, \Phi', \alpha, \beta, m$, and for Θ_Ψ using $\tilde{V}, \tilde{W}, \tilde{Y}, \tilde{Z}, \tilde{\iota}, \tilde{\jmath}, \tilde{\Psi}, \tilde{\alpha}, \tilde{\beta}, n$, and for $\Theta_{\Psi \circ \Phi}$ using $\hat{U}, \hat{W}, \hat{X}, \hat{Z}, \hat{\iota}, \hat{\jmath}, \widehat{\Psi \circ \Phi}, \hat{\alpha}, \hat{\beta}, m + n$.

We have a commutative diagram in $\mathrm{Perv}(\hat{X})$:

(5.26)

where the top right quadrilateral commutes because of associativity in the Thom–Sebastiani Theorem for $\mathcal{PV}_{V,f}^\bullet$, Theorem 2.13.

Also we have

$$(5.27)$$

$$
\begin{array}{ccc}
(\Psi\circ\Phi\circ\hat{\imath})|^*_{\hat{X}}\big(\mathcal{PV}^\bullet_{W,h}\big) & \xleftarrow{\quad\hat{\delta}\quad} & (\Psi\circ\Phi\circ\hat{\imath})|^*_{\hat{X}}\big(\mathcal{PV}^\bullet_{W,h}\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} \hat{\imath}|^*_{\hat{X}}(P_{\Psi\circ\Phi}) \\
\Big\uparrow{\scriptstyle\Pi_{\tilde{V}}|^*_{\hat{X}}(\tilde{\delta})} & & \Big\downarrow{\scriptstyle\mathrm{id}\,\otimes\,\hat{\imath}|^*_{\hat{X}}(\Xi_{\Psi,\Phi})} \\
\begin{array}{c}(\Psi\circ\Phi\circ\hat{\imath})|^*_{\hat{X}}\big(\mathcal{PV}^\bullet_{W,h}\big)\\ \otimes_{\mathbb{Z}/2\mathbb{Z}}(\tilde{\imath}\circ\tilde{\Pi}_{\tilde{V}})|^*_{\hat{X}}(P_\Psi)\end{array} & & \begin{array}{c}(\Psi\circ\Phi\circ\hat{\imath})|^*_{\hat{X}}\big(\mathcal{PV}^\bullet_{W,h}\big)\\ \otimes_{\mathbb{Z}/2\mathbb{Z}}(\tilde{\imath}\circ\tilde{\Pi}_{\tilde{V}})|^*_{\hat{X}}(P_\Psi) \otimes_{\mathbb{Z}/2\mathbb{Z}} \hat{\imath}|^*_{\hat{X}}(P_\Phi)\end{array} \\
\Big\uparrow{\scriptstyle(\Phi\circ\hat{\imath})|^*_{\hat{X}}(\Theta_\Psi)} & & \Big\downarrow{\scriptstyle(\Phi\circ\hat{\imath})|^*_{\hat{X}}(\Theta_\Psi^{-1})\otimes\mathrm{id}} \\
(\Phi\circ\hat{\imath})|^*_{\hat{X}}\big(\mathcal{PV}^\bullet_{V,g}\big) & \xleftarrow{\quad\Pi_{U'}|^*_{\hat{X}}(\delta)\quad} & (\Phi\circ\hat{\imath})|^*_{\hat{X}}\big(\mathcal{PV}^\bullet_{V,g}\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} \hat{\imath}|^*_{\hat{X}}(P_\Phi),
\end{array}
$$

which commutes because the trivializations of $\iota|_{X'}(P_\Phi), \tilde{\iota}|_{\tilde{X}}(P_\Psi), \hat{\imath}|_{\hat{X}}(P_{\Psi\circ\Phi})$ used to define $\delta, \tilde{\delta}, \hat{\delta}$ are compatible with $\Xi_{\Psi,\Phi}$.

Combining (5.26) and (5.27) with $\Pi_{U'}|^*_{\hat{X}}$ applied to (5.17) for Θ_Φ, and $\Pi_{\tilde{V}}|^*_{\hat{X}}$ applied to (5.17) for Θ_Ψ, and (5.17) for $\Theta_{\Psi\circ\Phi}$, we can show that the following diagram commutes in $\mathrm{Perv}(\hat{X})$:

$$
\begin{array}{ccc}
\hat{\imath}|^*_{\hat{X}}\big(\mathcal{PV}^\bullet_{U,f}\big) & \xrightarrow{\quad\hat{\imath}|^*_{\hat{X}}(\Theta_{\Psi\circ\Phi})\quad} & (\Psi\circ\Phi\circ\hat{\imath})|^*_{\hat{X}}\big(\mathcal{PV}^\bullet_{W,h}\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} \hat{\imath}|^*_{\hat{X}}(P_{\Psi\circ\Phi}) \\
\Big\downarrow{\scriptstyle\hat{\imath}|^*_{\hat{X}}(\Theta_\Phi)} & & \Big\downarrow{\scriptstyle\mathrm{id}\,\otimes\,\hat{\imath}|^*_{\hat{X}}(\Xi_{\Psi,\Phi})} \\
\begin{array}{c}(\Phi\circ\hat{\imath})|^*_{\hat{X}}\big(\mathcal{PV}^\bullet_{V,g}\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} \hat{\imath}|^*_{\hat{X}}(P_\Phi)\end{array} & \xrightarrow{\quad(\Phi\circ\hat{\imath})|^*_{\hat{X}}(\Theta_\Psi)\otimes\mathrm{id}\quad} & \begin{array}{c}(\Psi\circ\Phi\circ\hat{\imath})|^*_{\hat{X}}\big(\mathcal{PV}^\bullet_{W,h}\big)\\ \otimes_{\mathbb{Z}/2\mathbb{Z}}(\tilde{\imath}\circ\tilde{\Pi}_{\tilde{V}})|^*_{\hat{X}}(P_\Psi) \otimes_{\mathbb{Z}/2\mathbb{Z}} \hat{\imath}|^*_{\hat{X}}(P_\Phi),\end{array}
\end{array}
$$

which is $\hat{\imath}|^*_{\hat{X}}$ applied to (5.16). Since such $\hat{\imath}|_{\hat{X}} : \hat{X} \to X$ form an étale cover of X, equation (5.16) commutes by Theorem 2.7(i). This proves Theorem 5.4(c).

5.4. **$\mathscr{D}$-modules and mixed Hodge modules.** By Theorem 5.1(iv),(v), the earlier parts of that result hold for our other contexts in §2.6–§2.10. Once again, the proofs of Theorem 5.4(a)–(c) then carry over to the other contexts using the general framework of §2.5, now also making use of property (vii).

6. Perverse sheaves on oriented d-critical loci

6.1. **Background material on d-critical loci.** Here are some of the main definitions and results on d-critical loci, from Joyce [23, Th.s 2.1, 2.20, 2.28 & Def.s 2.5, 2.18, 2.31]. For the algebraic case we work with $\mathbb{C}$-schemes.

Theorem 6.1. *Let X be a $\mathbb{C}$-scheme. Then there exists a sheaf $\mathcal{S}_X$ of $\mathbb{C}$-vector spaces on X, unique up to canonical isomorphism, which is uniquely characterized by the following two properties:*

(i) *Suppose $R \subseteq X$ is Zariski open, U is a smooth $\mathbb{C}$-scheme, and $i : R \hookrightarrow U$ is a closed embedding. Then we have an exact sequence of sheaves of $\mathbb{C}$-vector spaces on R:*

$$0 \longrightarrow I_{R,U} \longrightarrow i^{-1}(\mathcal{O}_U) \xrightarrow{\;i^\sharp\;} \mathcal{O}_X|_R \longrightarrow 0,$$

where $\mathcal{O}_X, \mathcal{O}_U$ are the sheaves of regular functions on X, U, and $i^\sharp$ is the morphism of sheaves of $\mathbb{C}$-algebras on R induced by i.

There is an exact sequence of sheaves of $\mathbb{C}$-vector spaces on R:

$$0 \longrightarrow \mathcal{S}_X|_R \xrightarrow{\;\iota_{R,U}\;} \frac{i^{-1}(\mathcal{O}_U)}{I^2_{R,U}} \xrightarrow{\;\mathrm{d}\;} \frac{i^{-1}(T^*U)}{I_{R,U}\cdot i^{-1}(T^*U)},$$

where d *maps* $f + I_{R,U}^2 \mapsto \mathrm{d}f + I_{R,U} \cdot i^{-1}(T^*U)$.

(ii) *Let $R \subseteq S \subseteq X$ be Zariski open, U, V be smooth $\mathbb{C}$-schemes, $i : R \hookrightarrow U$, $j : S \hookrightarrow V$ closed embeddings, and $\Phi : U \to V$ a morphism with $\Phi \circ i = j|_R : R \to V$. Then the following diagram of sheaves on R commutes:*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{S}_X|_R & \xrightarrow{\iota_{S,V}|_R} & \dfrac{j^{-1}(\mathcal{O}_V)}{I_{S,V}^2}\Big|_R & \xrightarrow{\ \mathrm{d}\ } & \dfrac{j^{-1}(T^*V)}{I_{S,V} \cdot j^{-1}(T^*V)}\Big|_R & & \\[1em]
& & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle i^{-1}(\Phi^\sharp)} & & \Big\downarrow{\scriptstyle i^{-1}(\mathrm{d}\Phi)} & & \\[1em]
0 & \longrightarrow & \mathcal{S}_X|_R & \xrightarrow{\iota_{R,U}} & \dfrac{i^{-1}(\mathcal{O}_U)}{I_{R,U}^2} & \xrightarrow{\ \mathrm{d}\ } & \dfrac{i^{-1}(T^*U)}{I_{R,U} \cdot i^{-1}(T^*U)} & . &
\end{array}
\tag{6.1}
$$

Here $\Phi : U \to V$ induces $\Phi^\sharp : \Phi^{-1}(\mathcal{O}_V) \to \mathcal{O}_U$ on U, so we have

$$
i^{-1}(\Phi^\sharp) : j^{-1}(\mathcal{O}_V)|_R = i^{-1} \circ \Phi^{-1}(\mathcal{O}_V) \longrightarrow i^{-1}(\mathcal{O}_U),
\tag{6.2}
$$

a morphism of sheaves of $\mathbb{C}$-algebras on R.

As $\Phi \circ i = j|_R$, equation (6.2) maps $I_{S,V}|_R \to I_{R,U}$, and so maps $I_{S,V}^2|_R \to I_{R,U}^2$. Thus (6.2) induces the morphism in the second column of (6.1).

Similarly, $\mathrm{d}\Phi : \Phi^{-1}(T^*V) \to T^*U$ induces the third column of (6.1).

There is a natural decomposition $\mathcal{S}_X = \mathcal{S}_X^0 \oplus \mathbb{C}_X$, where $\mathbb{C}_X$ is the constant sheaf on X with fibre $\mathbb{C}$, and $\mathcal{S}_X^0 \subset \mathcal{S}_X$ is the kernel of the composition

$$
\mathcal{S}_X \xrightarrow{\ \beta_X\ } \mathcal{O}_X \xrightarrow{\ i_X^\sharp\ } \mathcal{O}_{X^{\mathrm{red}}},
$$

with X^{red} the reduced $\mathbb{C}$-subscheme of X, and $i_X : X^{\mathrm{red}} \hookrightarrow X$ the inclusion.

The analogue of all the above also holds for complex analytic spaces.

Definition 6.2. An *algebraic d-critical locus* over $\mathbb{C}$ is a pair (X, s), where X is a $\mathbb{C}$-scheme, and $s \in H^0(\mathcal{S}_X^0)$ for $\mathcal{S}_X^0$ as in Theorem 6.1, satisfying the condition that for each $x \in X$, there exists a Zariski open neighbourhood R of x in X, a smooth $\mathbb{C}$-scheme U, a regular function $f : U \to \mathbb{A}^1 = \mathbb{C}$, and a closed embedding $i : R \hookrightarrow U$, such that $i(R) = \mathrm{Crit}(f)$ as $\mathbb{C}$-subschemes of U, and $\iota_{R,U}(s|_R) = i^{-1}(f) + I_{R,U}^2$.

Similarly, a *complex analytic d-critical locus* is a pair (X, s), where X is a complex analytic space, and $s \in H^0(\mathcal{S}_X^0)$ for $\mathcal{S}_X$ as in Theorem 6.1, such that each $x \in X$ has an open neighbourhood $R \subset X$ with a closed embedding $i : R \hookrightarrow U$ into a complex manifold U and a holomorphic function $f : U \to \mathbb{C}$, such that $i(R) = \mathrm{Crit}(f)$, and $\iota_{R,U}(s|_R) = i^{-1}(f) + I_{R,U}^2$.

In both cases we call the quadruple (R, U, f, i) a *critical chart* on (X, s).

Let (X, s) be a d-critical locus (either algebraic or complex analytic), and (R, U, f, i) be a critical chart on (X, s). Let $U' \subseteq U$ be (Zariski) open, and set

$$
R' = i^{-1}(U') \subseteq R, \qquad i' = i|_{R'} : R' \hookrightarrow U', \qquad \text{and} \qquad f' = f|_{U'}.
$$

Then (R', U', f', i') is also a critical chart on (X, s), and we call it a *subchart* of (R, U, f, i). As a shorthand we write $(R', U', f', i') \subseteq (R, U, f, i)$.

Let $(R, U, f, i), (S, V, g, j)$ be critical charts on (X, s), with $R \subseteq S \subseteq X$. An *embedding* of (R, U, f, i) in (S, V, g, j) is a locally closed embedding $\Phi : U \hookrightarrow V$ such that $\Phi \circ i = j|_R$ and $f = g \circ \Phi$. As a shorthand we write $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$.

If $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$ and $\Psi : (S, V, g, j) \hookrightarrow (T, W, h, k)$ are embeddings, then $\Psi \circ \Phi : (R, U, f, i) \hookrightarrow (T, W, h, k)$ is also an embedding.

Theorem 6.3. *Let* (X, s) *be a d-critical locus (either algebraic or complex analytic), and let* $(R, U, f, i), (S, V, g, j)$ *be critical charts on* (X, s).

Then, for each $x \in R \cap S \subseteq X$, *there exist subcharts* $(R', U', f', i') \subseteq (R, U, f, i)$, *and* $(S', V', g', j') \subseteq (S, V, g, j)$ *with* $x \in R' \cap S' \subseteq X$, *a critical chart* (T, W, h, k) *on* (X, s), *and embeddings* $\Phi : (R', U', f', i') \hookrightarrow (T, W, h, k)$, *and* $\Psi : (S', V', g', j') \hookrightarrow (T, W, h, k)$.

Theorem 6.4. *Let* (X, s) *be a d-critical locus (either algebraic or complex analytic), and* $X^{\mathrm{red}} \subseteq X$ *the associated reduced* $\mathbb{C}$*-scheme or reduced complex analytic space. Then there exists an (algebraic or holomorphic) line bundle* $K_{X,s}$ *on* X^{red} *which we call the **canonical bundle** of* (X, s), *which is natural up to canonical isomorphism, and is characterized by the following properties:*

   (i) *If* (R, U, f, i) *is a critical chart on* (X, s), *there is a natural isomorphism*

$$\iota_{R,U,f,i} : K_{X,s}|_{R^{\mathrm{red}}} \longrightarrow i^* \big(K_U^{\otimes^2} \big)|_{R^{\mathrm{red}}}, \tag{6.3}$$

   where $K_U = \Lambda^{\dim U} T^* U$ *is the canonical bundle of* U *in the usual sense.*
   (ii) *Let* $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$ *be an embedding of critical charts on* (X, s). *Then* (5.9) *defines an isomorphism of line bundles on* $\mathrm{Crit}(f)^{\mathrm{red}}$:

$$J_\Phi : K_U^{\otimes^2}|_{\mathrm{Crit}(f)^{\mathrm{red}}} \overset{\cong}{\longrightarrow} \Phi|_{\mathrm{Crit}(f)^{\mathrm{red}}}^* \big(K_V^{\otimes^2} \big).$$

   Since $i : R \to \mathrm{Crit}(f)$ *is an isomorphism with* $\Phi \circ i = j|_R$, *this gives*

$$i|_{R^{\mathrm{red}}}^* (J_\Phi) : i|_{R^{\mathrm{red}}}^* \big(K_U^{\otimes^2} \big) \overset{\cong}{\longrightarrow} j|_{R^{\mathrm{red}}}^* \big(K_V^{\otimes^2} \big),$$

   and we must have

$$\iota_{S,V,g,j}|_{R^{\mathrm{red}}} = i|_{R^{\mathrm{red}}}^* (J_\Phi) \circ \iota_{R,U,f,i} : K_{X,s}|_{R^{\mathrm{red}}} \longrightarrow j^* \big(K_V^{\otimes^2} \big)|_{R^{\mathrm{red}}}. \tag{6.4}$$

Definition 6.5. Let (X, s) be a d-critical locus (either algebraic or complex analytic), and $K_{X,s}$ its canonical bundle from Theorem 6.4. An *orientation* on (X, s) is a choice of square root line bundle $K_{X,s}^{1/2}$ for $K_{X,s}$ on X^{red}. That is, an orientation is an (algebraic or holomorphic) line bundle L on X^{red}, together with an isomorphism $L^{\otimes^2} = L \otimes L \cong K_{X,s}$. A d-critical locus with an orientation will be called an *oriented d-critical locus.*

In [9, Th. 6.6] we show that algebraic d-critical loci are classical truncations of objects in derived algebraic geometry known as -1-*shifted symplectic derived schemes*, introduced by Pantev, Toën, Vaquié and Vezzosi [42].

Theorem 6.6 (Bussi, Brav and Joyce [9]). *Suppose* $(\boldsymbol{X}, \omega)$ *is a* -1-*shifted symplectic derived scheme in the sense of Pantev et al.* [42] *over* $\mathbb{C}$, *and let* $X = t_0(\boldsymbol{X})$ *be the associated classical* $\mathbb{C}$-*scheme of* $\boldsymbol{X}$. *Then* X *extends naturally to an algebraic d-critical locus* (X, s). *The canonical bundle* $K_{X,s}$ *from Theorem* 6.4 *is naturally isomorphic to the determinant line bundle* $\det(\mathbb{L}_{\boldsymbol{X}})|_{X^{\mathrm{red}}}$ *of the cotangent complex* $\mathbb{L}_{\boldsymbol{X}}$ *of* $\boldsymbol{X}$.

Now Pantev et al. [42] show that derived moduli schemes of coherent sheaves, or complexes of coherent sheaves, on a Calabi–Yau 3-fold Y have -1-shifted symplectic structures. Using this, in [9, Cor. 6.7] we deduce:

Corollary 6.7. *Suppose* Y *is a Calabi–Yau* 3-*fold over* $\mathbb{C}$, *and* $\mathcal{M}$ *is a classical moduli* $\mathbb{C}$-*scheme of simple coherent sheaves in* $\mathrm{coh}(Y)$, *or simple complexes of coherent sheaves in* $D^b \mathrm{coh}(Y)$, *with (symmetric) obstruction theory* $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$ *as in Behrend* [2], *Thomas* [52], *or Huybrechts and Thomas* [22]. *Then* $\mathcal{M}$ *extends naturally to an algebraic d-critical locus* $(\mathcal{M}, s)$. *The canonical bundle* $K_{\mathcal{M},s}$ *from Theorem* 6.4 *is naturally isomorphic to* $\det(\mathcal{E}^\bullet)|_{\mathcal{M}^{\mathrm{red}}}$.

Here we call $F \in \mathrm{coh}(Y)$ *simple* if $\mathrm{Hom}(F,F) = \mathbb{C}$, and we call $F^{\bullet} \in D^b \mathrm{coh}(Y)$ *simple* if $\mathrm{Hom}(F^{\bullet}, F^{\bullet}) = \mathbb{C}$ and $\mathrm{Ext}^{<0}(F^{\bullet}, F^{\bullet}) = 0$. Thus, d-critical loci will have applications in Donaldson–Thomas theory for Calabi–Yau 3-folds [24, 32, 33, 52]. Orientations on $(\mathcal{M}, s)$ are closely related to *orientation data* in the work of Kontsevich and Soibelman [32, 33].

Pantev et al. [42] also show that derived intersections $L \cap M$ of algebraic Lagrangians L, M in an algebraic symplectic manifold (S, ω) have -1-shifted symplectic structures, so that Theorem 6.6 gives them the structure of algebraic d-critical loci. Bussi [10, §3] will prove a complex analytic version of this:

Theorem 6.8 (Bussi [10]). *Suppose (S, ω) is a complex symplectic manifold, and L, M are complex Lagrangian submanifolds in S. Then the intersection $X = L \cap M$, as a complex analytic subspace of S, extends naturally to a complex analytic d-critical locus (X, s). The canonical bundle $K_{X,s}$ from Theorem 6.4 is naturally isomorphic to $K_L|_{X^{\mathrm{red}}} \otimes K_M|_{X^{\mathrm{red}}}$.*

6.2. **The main result, and applications.** Here is our main result, which will be proved in §6.3–§6.4.

Theorem 6.9. *Let (X, s) be an oriented algebraic d-critical locus over $\mathbb{C}$, with orientation $K_{X,s}^{1/2}$. Then for any well-behaved base ring A, such as $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{C}$, there exists a perverse sheaf $P_{X,s}^{\bullet}$ in $\mathrm{Perv}(X)$ over A, which is natural up to canonical isomorphism, and Verdier duality and monodromy isomorphisms*

$$\Sigma_{X,s} : P_{X,s}^{\bullet} \longrightarrow \mathbb{D}_X\big(P_{X,s}^{\bullet}\big), \qquad \mathrm{T}_{X,s} : P_{X,s}^{\bullet} \longrightarrow P_{X,s}^{\bullet}, \tag{6.5}$$

which are characterized by the following properties:

(i) *If (R, U, f, i) is a critical chart on (X, s), there is a natural isomorphism*

$$\omega_{R,U,f,i} : P_{X,s}^{\bullet}|_R \longrightarrow i^*\big(\mathcal{PV}_{U,f}^{\bullet}\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R,U,f,i}, \tag{6.6}$$

where $\pi_{R,U,f,i} : Q_{R,U,f,i} \to R$ is the principal $\mathbb{Z}/2\mathbb{Z}$-bundle parametrizing local isomorphisms $\alpha : K_{X,s}^{1/2} \to i^*(K_U)|_{R^{\mathrm{red}}}$ with $\alpha \otimes \alpha = \iota_{R,U,f,i}$, for $\iota_{R,U,f,i}$ as in (6.3). Furthermore the following commute in $\mathrm{Perv}(R)$:

$$
\begin{array}{ccc}
P_{X,s}^{\bullet}|_R & \xrightarrow{\quad\quad\quad\quad \omega_{R,U,f,i} \quad\quad\quad\quad} & i^*\big(\mathcal{PV}_{U,f}^{\bullet}\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R,U,f,i} \\[4pt]
{\scriptstyle \Sigma_{X,s}|_R} \Big\downarrow & & \Big\downarrow {\scriptstyle i^*(\sigma_{U,f}) \otimes \mathrm{id}_{Q_{R,U,f,i}}} \\[4pt]
& & i^*\big(\mathbb{D}_{\mathrm{Crit}(f)}(\mathcal{PV}_{U,f}^{\bullet})\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R,U,f,i} \\[2pt]
\mathbb{D}_R\big(P_{X,s}^{\bullet}|_R\big) & \xleftarrow{\quad \mathbb{D}_R(\omega_{R,U,f,i}) \quad} & \cong \mathbb{D}_R\big(i^*(\mathcal{PV}_{U,f}^{\bullet}) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R,U,f,i}\big),
\end{array}
\tag{6.7}
$$

$$
\begin{array}{ccc}
P_{X,s}^{\bullet}|_R & \xrightarrow{\quad\quad\quad \omega_{R,U,f,i} \quad\quad\quad} & i^*\big(\mathcal{PV}_{U,f}^{\bullet}\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R,U,f,i} \\[4pt]
{\scriptstyle \mathrm{T}_{X,s}|_R} \Big\downarrow & & \Big\downarrow {\scriptstyle i^*(\tau_{U,f}) \otimes \mathrm{id}_{Q_{R,U,f,i}}} \\[4pt]
P_{X,s}^{\bullet}|_R & \xrightarrow{\quad\quad\quad \omega_{R,U,f,i} \quad\quad\quad} & i^*\big(\mathcal{PV}_{U,f}^{\bullet}\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R,U,f,i}.
\end{array}
\tag{6.8}
$$

(ii) *Let $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$ be an embedding of critical charts on (X, s). Then there is a natural isomorphism of principal $\mathbb{Z}/2\mathbb{Z}$-bundles*

$$\Lambda_{\Phi} : Q_{S,V,g,j}|_R \xrightarrow{\ \cong\ } i^*(P_{\Phi}) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R,U,f,i} \tag{6.9}$$

on R, for P_{Φ} as in Definition 5.2, defined as follows: local isomorphisms

$$\alpha : K_{X,s}^{1/2}|_{R^{\mathrm{red}}} \longrightarrow i^*(K_U)|_{R^{\mathrm{red}}}, \quad \beta : K_{X,s}^{1/2}|_{R^{\mathrm{red}}} \longrightarrow j^*(K_V)|_{R^{\mathrm{red}}},$$

$$and \qquad \gamma : i^*(K_U)|_{R^{\mathrm{red}}} \longrightarrow j^*(K_V)|_{R^{\mathrm{red}}}$$

with $\alpha \otimes \alpha = \iota_{R,U,f,i}$, $\beta \otimes \beta = \iota_{S,V,g,j}|_{R^{\mathrm{red}}}$, $\gamma \otimes \gamma = i|_{R^{\mathrm{red}}}^*(J_\Phi)$ *correspond to local sections* $s_\alpha : R \to Q_{R,U,f,i}$, $s_\beta : R \to Q_{S,V,g,j}|_R$, $s_\gamma : R \to i^*(P_\Phi)$. *Equation* (6.4) *shows that* $\beta = \gamma \circ \alpha$ *is a possible solution for* β, *and we define* Λ_Φ *in* (6.9) *such that* $\Lambda_\Phi(s_\beta) = s_\gamma \otimes_{\mathbb{Z}/2\mathbb{Z}} s_\alpha$ *if and only if* $\beta = \gamma \circ \alpha$.

Then the following diagram commutes in $\mathrm{Perv}(R)$, *for* Θ_Φ *as in* (5.13):

$$
\begin{array}{ccc}
P_{X,s}^\bullet|_R & \xrightarrow{\;\;\;\;\omega_{R,U,f,i}\;\;\;\;} & i^*\big(\mathcal{PV}_{U,f}^\bullet\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R,U,f,i} \\[2pt]
{\scriptstyle\omega_{S,V,g,j}|_R}\big\downarrow & & \big\downarrow{\scriptstyle i^*(\Theta_\Phi)\otimes\mathrm{id}_{Q_{R,U,f,i}}} \\[2pt]
\begin{array}{c} j^*\big(\mathcal{PV}_{V,g}^\bullet\big)|_R \\ \otimes_{\mathbb{Z}/2\mathbb{Z}}Q_{S,V,g,j}|_R \end{array} & \xrightarrow{\;\;\mathrm{id}_{j^*(\mathcal{PV}_{V,g}^\bullet)}\otimes\Lambda_\Phi\;\;} & \begin{array}{c} i^*\big(\Phi^*(\mathcal{PV}_{V,g}^\bullet) \otimes_{\mathbb{Z}/2\mathbb{Z}} P_\Phi\big) \\ \otimes_{\mathbb{Z}/2\mathbb{Z}}Q_{R,U,f,i}. \end{array}
\end{array}
\tag{6.10}
$$

The analogues of all the above also hold for $\mathscr{D}$*-modules on oriented algebraic d-critical loci over* $\mathbb{C}$, *for perverse sheaves and* $\mathscr{D}$*-modules on oriented complex analytic d-critical loci, and for mixed Hodge modules on oriented algebraic d-critical loci over* $\mathbb{C}$ *and oriented complex analytic d-critical loci, as in* §2.6–§2.10.

Remark 6.10. This sheaf-theoretic result is compatible with the motivic result of Bussi, Joyce and Meinhardt in [11]. Given (X, s) an oriented algebraic d-critical locus over $\mathbb{C}$, [11] proves the existence of a natural motivic element $MF_{X,s} \in \overline{\mathcal{M}}_X^{\hat{\mu}}$ in a version of the relative Grothendieck ring of varieties over X, equivariant with respect to suitable actions of the group $\hat{\mu}$ of all roots of unity (for detailed definitions, see [11]). Since the mixed Hodge module realization factorizes over the additional relation one has to impose in [11] on the Grothendieck group, the ring $\mathcal{M}_X^{\hat{\mu}}$ has a map to $K_0(\mathrm{MHM}(X; T_s))$, the K-group of algebraic mixed Hodge modules on X with a finite order automorphism (note that the Grothendieck group only sees the semisimple part T_s of the monodromy and not the nilpotent part N). By a Čech-type argument using the corresponding comparison result of [20, Prop. 3.17], the image of $MF_{X,s}$ in $K_0(\mathrm{MHM}(X; T_s))$ agrees with the image of the mixed Hodge module realization of $P_{X,s}^\bullet$, since both sides are Zariski locally modelled by the same vanishing cycles. Thus, for example, they give the same weight polynomial for global cohomology with compact support.

From Theorem 6.6, Corollary 6.7 and Theorem 6.8 we deduce:

Corollary 6.11. *Let* $(\boldsymbol{X}, \omega)$ *be a* -1*-shifted symplectic derived scheme over* $\mathbb{C}$ *in the sense of Pantev et al.* [42], *and* $X = t_0(\boldsymbol{X})$ *the associated classical* $\mathbb{C}$*-scheme. Suppose we are given a square root* $\det(\mathbb{L}_{\boldsymbol{X}})|_X^{1/2}$ *for* $\det(\mathbb{L}_{\boldsymbol{X}})|_X$. *Then we may define* $P_{\boldsymbol{X},\omega}^\bullet \in \mathrm{Perv}(X)$, *uniquely up to canonical isomorphism, and isomorphisms* $\Sigma_{\boldsymbol{X},\omega} : P_{\boldsymbol{X},\omega}^\bullet \to \mathbb{D}_X(P_{\boldsymbol{X},\omega}^\bullet)$, $T_{\boldsymbol{X},\omega} : P_{\boldsymbol{X},\omega}^\bullet \to P_{\boldsymbol{X},\omega}^\bullet$.

The same applies for $\mathscr{D}$*-modules and mixed Hodge modules on* X.

Corollary 6.12. *Let* Y *be a Calabi–Yau* 3*-fold over* $\mathbb{C}$, *and* $\mathcal{M}$ *a classical moduli* $\mathbb{C}$*-scheme of simple coherent sheaves in* $\mathrm{coh}(Y)$, *or simple complexes of coherent sheaves in* $D^b\mathrm{coh}(Y)$, *with natural (symmetric) obstruction theory* $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$ *as in Behrend* [2], *Thomas* [52], *or Huybrechts and Thomas* [22]. *Suppose we are given a square root* $\det(\mathcal{E}^\bullet)^{1/2}$ *for* $\det(\mathcal{E}^\bullet)$. *Then we may define* $P_{\mathcal{M}}^\bullet \in \mathrm{Perv}(\mathcal{M})$, *uniquely up to canonical isomorphism, and isomorphisms* $\Sigma_{\mathcal{M}} : P_{\mathcal{M}}^\bullet \to \mathbb{D}_{\mathcal{M}}(P_{\mathcal{M}}^\bullet)$, $T_{\mathcal{M}} : P_{\mathcal{M}}^\bullet \to P_{\mathcal{M}}^\bullet$.

The same applies for $\mathscr{D}$*-modules and mixed Hodge modules on* $\mathcal{M}$.

Corollary 6.13. *Let* (S, ω) *be a complex symplectic manifold and* L, M *complex Lagrangian submanifolds in* S, *and write* $X = L \cap M$, *as a complex analytic subspace of* S. *Suppose we are given square roots* $K_L^{1/2}, K_M^{1/2}$ *for* K_L, K_M. *Then we may define* $P_{L,M}^\bullet \in \mathrm{Perv}(X)$,

uniquely up to canonical isomorphism, and isomorphisms $\Sigma_{L,M} : P^\bullet_{L,M} \to \mathbb{D}_X(P^\bullet_{L,M})$, and $\mathrm{T}_{L,M} : P^\bullet_{L,M} \to P^\bullet_{L,M}$.

The same applies for $\mathscr{D}$-modules and mixed Hodge modules on X.

The next two remarks discuss applications of Corollaries 6.12 and 6.13 to Donaldson–Thomas theory, and to Lagrangian Floer cohomology.

Remark 6.14. If Y is a Calabi–Yau 3-fold over $\mathbb{C}$ and τ a suitable stability condition on coherent sheaves on Y, the *Donaldson–Thomas invariants* $DT^\alpha(\tau)$ are integers which 'count' the moduli schemes $\mathcal{M}^\alpha_{\mathrm{st}}(\tau)$ of τ-stable coherent sheaves on Y with Chern character $\alpha \in H^{\mathrm{even}}(Y;\mathbb{Q})$, provided there are no strictly τ-semistable sheaves in class α on Y. They were defined by Thomas [52], who showed they are unchanged under deformations of Y, following a suggestion of Donaldson and Thomas [16].

Behrend [2] showed that $DT^\alpha(\tau)$ may be written as a weighted Euler characteristic $\chi(\mathcal{M}^\alpha_{\mathrm{st}}(\tau), \nu)$, where $\nu : \mathcal{M}^\alpha_{\mathrm{st}}(\tau) \to \mathbb{Z}$ is a certain constructible function called the *Behrend function*. Joyce and Song [24] extended the definition of $DT^\alpha(\tau)$ to classes α including τ-semistable sheaves (with $DT^\alpha(\tau) \in \mathbb{Q}$), and proved a wall-crossing formula for $DT^\alpha(\tau)$ under change of stability condition τ. Kontsevich and Soibelman [32] gave a (partly conjectural) motivic generalization of Donaldson–Thomas invariants, also with a wall-crossing formula.

Corollary 6.12 is relevant to the *categorification* of Donaldson–Thomas theory. As in [2, §1.2], the perverse sheaf $P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}$ has pointwise Euler characteristic $\chi\big(P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}\big) = \nu$. This implies that when A is a field, say $A = \mathbb{Q}$, the (compactly-supported) hypercohomologies $\mathbb{H}^*\big(P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}\big), \mathbb{H}^*_{\mathrm{c}}\big(P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}\big)$ from (2.1) satisfy

$$\sum_{k\in\mathbb{Z}}(-1)^k \dim \mathbb{H}^k\big(P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}\big) = \sum_{k\in\mathbb{Z}}(-1)^k \dim \mathbb{H}^k_{\mathrm{c}}\big(P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}\big) = \chi\big(\mathcal{M}^\alpha_{\mathrm{st}}(\tau),\nu\big) = DT^\alpha(\tau),$$

where $\mathbb{H}^k\big(P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}\big) \cong \mathbb{H}^{-k}_{\mathrm{c}}\big(P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}\big)^*$ by Verdier duality. That is, we have produced a natural graded $\mathbb{Q}$-vector space $\mathbb{H}^*\big(P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}\big)$, thought of as some kind of generalized cohomology of $\mathcal{M}^\alpha_{\mathrm{st}}(\tau)$, whose graded dimension is $DT^\alpha(\tau)$. This gives a new interpretation of the Donaldson–Thomas invariant $DT^\alpha(\tau)$.

In fact, as discussed at length in [51, §3], the first natural "refinement" or "quantization" direction of a Donaldson–Thomas invariant $DT^\alpha(\tau) \in \mathbb{Z}$ is not the Poincaré polynomial of this cohomology, but its weight polynomial

$$w\big(\mathbb{H}^*(P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}), t\big) \in \mathbb{Z}\big[t^{\pm\frac{1}{2}}\big],$$

defined using the mixed Hodge structure on the cohomology of the mixed Hodge module version of $P^\bullet_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}$ (which exists assuming that $\mathcal{M}^\alpha_{\mathrm{st}}(\tau)$ is projective, for example, see Remark 2.22).

The material above is related to work by other authors. The idea of categorifying Donaldson–Thomas invariants using perverse sheaves or $\mathscr{D}$-modules is probably first due to Behrend [2], and for Hilbert schemes $\mathrm{Hilb}^n(Y)$ of a Calabi–Yau 3-fold Y is discussed by Dimca and Szendrői [15] and Behrend, Bryan and Szendrői [3, §3.4], using mixed Hodge modules. Corollary 6.12 answers a question of Joyce and Song [24, Question 5.7(a)].

As in [24,32] representations of *quivers with superpotentials* (Q, W) give 3-Calabi–Yau triangulated categories, and one can define Donaldson–Thomas type invariants $DT^\alpha_{Q,W}(\tau)$ 'counting' such representations, which are simple algebraic 'toy models' for Donaldson–Thomas invariants of Calabi–Yau 3-folds. Kontsevich and Soibelman [33] explain how to categorify these quiver invariants $DT^\alpha_{Q,W}(\tau)$, and define an associative multiplication on the categorification to make a *Cohomological Hall Algebra*. This paper was motivated by the aim of extending [33] to define Cohomological Hall Algebras for Calabi–Yau 3-folds.

The square root $\det(\mathcal{E}^\bullet)^{1/2}$ required in Corollary 6.12 corresponds roughly to *orientation data* in the work of Kontsevich and Soibelman [32, §5], [33].

In a paper written independently of our programme [9,11,23], Kiem and Li [31] have recently proved an analogue of Corollary 6.12 by complex analytic methods, beginning from Joyce and Song's result [24, Th. 5.4], proved using gauge theory, that $\mathcal{M}_{\mathrm{st}}^\alpha(\tau)$ is locally isomorphic to $\mathrm{Crit}(f)$ as a complex analytic space, for V a complex manifold and $f : V \to \mathbb{C}$ holomorphic.

Remark 6.15. In the situation of Corollary 6.13, with $\dim_\mathbb{C} S = 2n$, we claim that there ought morally to be some kind of approximate comparison

$$\mathbb{H}^k(P_{L,M}^\bullet) \approx HF^{k+n}(L, M), \tag{6.11}$$

where $HF^*(L, M)$ is the *Lagrangian Floer cohomology* of Fukaya, Oh, Ohta and Ono [18]. We can compare and contrast the two sides of (6.11) as follows:

(a) $\mathbb{H}^*(P_{L,M}^\bullet)$ is defined over any well-behaved base ring A, e.g. $A = \mathbb{Z}$ or $\mathbb{Q}$, but $HF^*(L, M)$ is defined over a Novikov ring of power series Λ_{nov}.

(b) $\mathbb{H}^*(P_{L,M}^\bullet)$ has extra structure not visible in $HF^*(L, M)$, from Verdier duality and monodromy operators $\Sigma_{L,M}, \mathrm{T}_{L,M}$, plus the mixed Hodge module version has a mixed Hodge structure.

(c) $\mathbb{H}^*(P_{L,M}^\bullet)$ is defined for arbitrary complex Lagrangians L, M, not necessarily compact or closed in S, but $HF^*(L, M)$ is only defined for L, M compact, or at least for L, M closed and well-behaved at infinity.

(d) To define $HF^*(L, M)$ one generally assumes L, M intersect transversely, or at least cleanly. But $\mathbb{H}^*(P_{L,M}^\bullet)$ is defined when $L \cap M$ is arbitrarily singular, and the construction is only really interesting for singular $L \cap M$.

(e) To define $HF^*(L, M)$ we need L, M to be oriented and spin, to orient moduli spaces of J-holomorphic curves. When L, M are complex Lagrangians they are automatically oriented, and spin structures on L, M correspond to choices of square roots $K_L^{1/2}, K_M^{1/2}$, as used in Corollary 6.13.

Some of the authors are working on defining a 'Fukaya category' of complex Lagrangians in a complex symplectic manifold, using $\mathbb{H}^*(P_{L,M}^\bullet)$ as morphisms.

We now discuss related work. Nadler and Zaslow [40, 41] show that if X is a real analytic manifold (for instance, a complex manifold), then the derived category $D_c^b(X)$ of constructible sheaves on X is equivalent to a certain derived Fukaya category $D^b\mathcal{F}(T^*X)$ of exact Lagrangians in T^*X.

Let L, M be complex Lagrangians in a complex symplectic manifold (S, ω). Regarding $\mathcal{O}_L, \mathcal{O}_M$ as coherent sheaves on S, Behrend and Fantechi [4, Th.s 4.3 & 5.2] claim to construct canonical $\mathbb{C}$-linear (not $\mathcal{O}_S$-linear) differentials

$$\mathrm{d} : \mathcal{E}xt_{\mathcal{O}_S}^i(\mathcal{O}_L, \mathcal{O}_M) \longrightarrow \mathcal{E}xt_{\mathcal{O}_S}^{i+1}(\mathcal{O}_L, \mathcal{O}_M)$$

with $\mathrm{d}^2 = 0$, such that $\big(\mathcal{E}xt_{\mathcal{O}_S}^*(\mathcal{O}_L, \mathcal{O}_M), \mathrm{d}\big)$ is a constructible complex. There is a mistake in the proof of [4, Th. 4.3]. To fix this one should instead work with $\mathcal{E}xt_{\mathcal{O}_S}^*(K_L^{1/2}, K_M^{1/2})$ for square roots $K_L^{1/2}, K_M^{1/2}$ as in Corollary 6.13. Also the proof of the constructibility of $\big(\mathcal{E}xt_{\mathcal{O}_S}^*(K_L^{1/2}, K_M^{1/2}), \mathrm{d}\big)$ in [4, Th. 5.2] depended on a result of Kapranov, which later turned out to be false.

Our $P_{L,M}^\bullet$ over $A = \mathbb{C}$ should be the natural perverse sheaf on $L \cap M$ conjectured by Behrend and Fantechi [4, Conj. 5.16], who also suggest there should be a spectral sequence from $\big(\mathcal{E}xt_{\mathcal{O}_S}^*(K_L^{1/2}, K_M^{1/2}), \mathrm{d}\big)[n]$ to $P_{L,M}^\bullet$. (See Sabbah [44, Th. 1.1] for a related result.) In [4, §5.3], Behrend and Fantechi discuss how to define a 'Fukaya category' using their ideas.

Kashiwara and Schapira [29] develop a theory of *deformation quantization modules*, or *DQ-modules*, on a complex symplectic manifold (S, ω), which roughly may be regarded as symplectic versions of $\mathscr{D}$-modules. *Holonomic* DQ-modules $\mathcal{D}^\bullet$ are supported on (possibly singular) complex Lagrangians L in S. If L is a smooth, closed, complex Lagrangian in S and $K_L^{1/2}$ a square root of K_L, D'Agnolo and Schapira [13] show that there exists a simple holonomic DQ-module $\mathcal{D}^\bullet$ supported on L.

If $\mathcal{D}^\bullet, \mathcal{E}^\bullet$ are simple holonomic DQ-modules on S supported on smooth Lagrangians L, M, then Kashiwara and Schapira [28] show that $R\mathscr{H}om(\mathcal{D}^\bullet, \mathcal{E}^\bullet)[n]$ is a perverse sheaf on S over the field $\mathbb{C}((\hbar))$, supported on $X = L \cap M$. Pierre Schapira explained to the authors how to prove that $R\mathscr{H}om(\mathcal{D}^\bullet, \mathcal{E}^\bullet)[n] \cong P_{L,M}^\bullet$, when $P_{L,M}^\bullet$ is defined over the base ring $A = \mathbb{C}((\hbar))$.

Now let L, M, N be Lagrangians in S, with square roots $K_L^{1/2}, K_M^{1/2}, K_N^{1/2}$. We have a product $HF^k(L, M) \times HF^l(M, N) \to HF^{k+l}(L, N)$ from composition of morphisms in $D^b\mathcal{F}(S)$. So (6.11) suggests there should be a product

$$\mathbb{H}^k(P_{L,M}^\bullet) \times \mathbb{H}^l(P_{M,N}^\bullet) \longrightarrow \mathbb{H}^{k+l+n}(P_{L,N}^\bullet), \qquad (6.12)$$

which would naturally be induced by a morphism in $D_c^b(S)$

$$\mu_{L,M,N} : P_{L,M}^\bullet \overset{L}{\otimes} P_{M,N}^\bullet \longrightarrow P_{L,N}^\bullet[n]. \qquad (6.13)$$

Observe that the work of Behrend–Fantechi and Kashiwara–Schapira cited above supports the existence of (6.12)–(6.13): there are natural products

$$\mathcal{E}xt_{\mathcal{O}_S}^k(K_L^{1/2}, K_M^{1/2}) \otimes_{\mathcal{O}_S} \mathcal{E}xt_{\mathcal{O}_S}^l(K_M^{1/2}, K_N^{1/2}) \longrightarrow \mathcal{E}xt_{\mathcal{O}_S}^{k+l}(K_L^{1/2}, K_N^{1/2}),$$

$$R\mathscr{H}om(\mathcal{D}^\bullet, \mathcal{E}^\bullet) \overset{L}{\otimes} R\mathscr{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \longrightarrow R\mathscr{H}om(\mathcal{D}^\bullet, \mathcal{F}^\bullet).$$

But since (6.13) is a morphism of complexes, not of perverse sheaves, Theorem 2.7(i) does not apply, so we cannot construct $\mu_{L,M,N}$ by naïvely gluing data on an open cover, as we have been doing in §3–§6.

6.3. **Proof of Theorem 6.9 for $\mathbb{C}$-schemes.** Let (X, s) be an oriented algebraic d-critical locus over $\mathbb{C}$, with orientation $K_{X,s}^{1/2}$. By Definition 6.2 we may choose a family

$$\{(R_a, U_a, f_a, i_a) : a \in A\}$$

of critical charts (R_a, U_a, f_a, i_a) on (X, s) such that $\{R_a : a \in A\}$ is a Zariski open cover of the $\mathbb{C}$-scheme X. Then for each $a \in A$ we have a perverse sheaf

$$i_a^*\big(\mathcal{PV}_{U_a,f_a}^\bullet\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R_a,U_a,f_a,i_a} \in \mathrm{Perv}(R_a), \qquad (6.14)$$

for Q_{R_a,U_a,f_a,i_a} as in Theorem 6.9(i). The idea of the proof is to use Theorem 2.7(ii) to glue the perverse sheaves (6.14) on the Zariski open cover $\{R_a : a \in A\}$ to get a global perverse sheaf $P_{X,s}^\bullet$ on X. Note that Theorem 2.7(ii) is written for étale open covers, but this immediately implies the simpler Zariski version.

To do this, for all $a, b \in A$ we have to construct isomorphisms

$$\begin{aligned}
\alpha_{ab} : \big[i_a^*\big(\mathcal{PV}_{U_a,f_a}^\bullet\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R_a,U_a,f_a,i_a}\big]\big|_{R_a \cap R_b} &\longrightarrow \\
\big[i_b^*\big(\mathcal{PV}_{U_b,f_b}^\bullet\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R_b,U_b,f_b,i_b}\big]\big|_{R_a \cap R_b} &\in \mathrm{Perv}(R_a \cap R_b),
\end{aligned} \qquad (6.15)$$

satisfying $\alpha_{aa} = \mathrm{id}$ for all $a \in A$ and

$$\alpha_{bc}|_{R_a \cap R_b \cap R_c} \circ \alpha_{ab}|_{R_a \cap R_b \cap R_c} = \alpha_{ac}|_{R_a \cap R_b \cap R_c} \quad \text{for all } a, b, c \in A. \qquad (6.16)$$

Fix $a, b \in A$. By applying Theorem 6.3 to the critical charts (R_a, U_a, f_a, i_a), (R_b, U_b, f_b, i_b) at each $x \in R_a \cap R_b$, we can choose an indexing set D_{ab} and for each $d \in D_{ab}$ subcharts

$(R_a'^d, U_a'^d, f_a'^d, i_a'^d) \subseteq (R_a, U_a, f_a, i_a)$ and $(R_b'^d, U_b'^d, f_b'^d, i_b'^d) \subseteq (R_b, U_b, f_b, i_b)$, a critical chart (S^d, V^d, g^d, j^d) on (X, s), and embeddings

$$\Phi^d : (R_a'^d, U_a'^d, f_a'^d, i_a'^d) \hookrightarrow (S^d, V^d, g^d, j^d) \quad \text{and} \quad \Psi^d : (R_b'^d, U_b'^d, f_b'^d, i_b'^d) \hookrightarrow (S^d, V^d, g^d, j^d)$$

such that $\{R_a'^d \cap R_b'^d : d \in D_{ab}\}$ is a Zariski open cover of $R_a \cap R_b$.

For each $d \in D_{ab}$, define an isomorphism

$$\alpha_{ab}^d : \left[i_a^*\big(\mathcal{PV}_{U_a, f_a}^\bullet\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R_a, U_a, f_a, i_a}\right]\big|_{R_a'^d \cap R_b'^d} \longrightarrow \left[i_b^*\big(\mathcal{PV}_{U_b, f_b}^\bullet\big) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R_b, U_b, f_b, i_b}\right]\big|_{R_a'^d \cap R_b'^d}$$

by the commutative diagram

where $\Theta_{\Phi^d}, \Theta_{\Psi^d}$ are as in Theorem 5.4, and $\Lambda_{\Phi^d}, \Lambda_{\Psi^d}$ as in (6.9).

We claim that for all $d, e \in D_{ab}$ we have

$$\alpha_{ab}^d\big|_{R_a'^d \cap R_b'^d \cap R_a'^e \cap R_b'^e} = \alpha_{ab}^e\big|_{R_a'^d \cap R_b'^d \cap R_a'^e \cap R_b'^e}. \tag{6.18}$$

To see this, let $x \in R_a'^d \cap R_b'^d \cap R_a'^e \cap R_b'^e$, and apply Theorem 6.3 to the critical charts (S^d, V^d, g^d, j^d), (S^e, V^e, g^e, j^e) and point $x \in S^d \cap S^e$. This gives subcharts

$$(S'^d, V'^d, g'^d, j'^d) \subseteq (S^d, V^d, g^d, j^d) \qquad \text{and} \qquad (S'^e, V'^e, g'^e, j'^e) \subseteq (S^e, V^e, g^e, j^e)$$

with $x \in S'^d \cap S'^e$, a critical chart (T, W, h, k) on (X, s), and embeddings

$$\Omega : (S'^d, V'^d, g'^d, j'^d) \hookrightarrow (T, W, h, k) \qquad \text{and} \qquad \Upsilon : (S'^e, V'^e, g'^e, j'^e) \hookrightarrow (T, W, h, k).$$

Set $R^{de} = R_a'^d \cap R_b'^d \cap R_a'^e \cap R_b'^e \cap S'^d \cap S'^e$, and consider the diagram:

$$
\begin{array}{ccc}
\left[i_a^*\big(\mathcal{PV}_{U_a,f_a}^\bullet\big)\otimes_{\mathbb{Z}/2\mathbb{Z}} \atop Q_{R_a,U_a,f_a,i_a}\right]\big|_{R^{de}} & \xrightarrow{\;(\mathrm{id}\otimes\Lambda_{\Phi^d}^{-1})\circ(i_a^*(\Theta_{\Phi^d})\otimes\mathrm{id})|_{R^{de}}\;} & \left[(j^d)^*\big(\mathcal{PV}_{V^d,g^d}^\bullet\big)\otimes_{\mathbb{Z}/2\mathbb{Z}} \atop Q_{S^d,V^d,g^d,j^d}\right]\big|_{R^{de}}
\end{array}
$$

(6.19)

Here we have given two expressions for the top left diagonal morphism in (6.19). To see these are equal, set $R_a'^{de} = R_a'^d \cap R_a'^e \cap S'^d \cap S'^e$, $U_a'^{de} = (\Phi^d)^{-1}(V'^d) \cap (\Phi^e)^{-1}(V'^e)$, $f_a'^{de} = f_a|_{U_a'^{de}}$, and $i_a'^{de} = i_a|_{R_a'^{de}}$. Then $(R_a'^{de}, U_a'^{de}, f_a'^{de}, i_a'^{de}) \subseteq (R_a, U_a, f_a, i_a)$ is a subchart and

$$\Omega \circ \Phi^d|_{U_a'^{de}}, \Upsilon \circ \Phi^e|_{U_a'^{de}} : (R_a'^{de}, U_a'^{de}, f_a'^{de}, i_a'^{de}) \longrightarrow (T, W, h, k)$$

are embeddings.

As $\Omega \circ \Phi^d \circ i_a'^{de} = k|_{R_a'^{de}} = \Upsilon \circ \Phi^e \circ i_a'^{de}$, Theorem 5.4(b) gives $\Theta_{\Omega\circ\Phi^d}|_{i_a(R_a'^{de})} = \Theta_{\Upsilon\circ\Phi^e}|_{i_a(R_a'^{de})}$, so that $i_a|_{R^{de}}^*(\Theta_{\Omega\circ\Phi^d}) = i_a|_{R^{de}}^*(\Theta_{\Upsilon\circ\Phi^e})$ as $R^{de} \subseteq R_a'^{de}$. Also $\Lambda_{\Omega\circ\Phi^d} = \Lambda_{\Upsilon\circ\Phi^e}$ as these are defined in Theorem 6.9(ii) using $J_{\Omega\circ\Phi^d}, J_{\Upsilon\circ\Phi^e}$, which are equal by Lemma 5.3. So the two expressions are equal, and similarly for the bottom right diagonal morphism.

The upper triangle in (6.19) commutes because (5.16) gives

$$(\mathrm{id}\otimes\Xi_{\Omega,\Phi^d}) \circ \Theta_{\Omega\circ\Phi^d}|_{i_a(R^{de})} = (\Phi^d|_{i_a(R^{de})}^*(\Theta_\Omega) \otimes \mathrm{id}) \circ \Theta_{\Phi^d}|_{i_a(R^{de})},$$

and the definitions of Ξ_{Ω,Φ^d} in (5.11) and $\Lambda_\Omega, \Lambda_{\Phi^d}, \Lambda_{\Omega\circ\Phi^d}$ in (6.9) imply that

$$(i_a|_{R^{de}}^*(\Xi_{\Omega,\Phi^d}) \otimes \mathrm{id}) \circ \Lambda_{\Omega\circ\Phi^d}|_{R^{de}} = (\mathrm{id}\otimes\Lambda_{\Phi^d}) \circ \Lambda_\Omega|_{R^{de}} :$$
$$Q_{T,W,h,k}|_{R^{de}} \longrightarrow j^d|_{R^{de}}^*(P_\Omega) \otimes_{\mathbb{Z}/2\mathbb{Z}} i_a|_{R^{de}}^*(P_{\Phi^d}) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R_a,U_a,f_a,i_a}|_{R^{de}}.$$

Similarly, the other three triangles in (6.19) commute, so (6.19) commutes.

By (6.17), the two routes round the outside of (6.19) are $\alpha_{ab}^d|_{R^{de}}$ and $\alpha_{ab}^e|_{R^{de}}$, which are equal as (6.19) commutes. As we can cover $R_a'^d \cap R_b'^d \cap R_a'^e \cap R_b'^e$ by such Zariski open R^{de}, equation (6.18) follows. Therefore by the Zariski open cover version of Theorem 2.7(i), there is a unique isomorphism α_{ab} in (6.15) such that $\alpha_{ab}|_{R_a'^d \cap R_b'^d} = \alpha_{ab}^d$ for all $d \in D_{ab}$.

If $D_{ab}, R_a'^d, \ldots, \Phi^d, \Psi^d$ are used to define α_{ab} and $\tilde{D}_{ab}, \tilde{R}_a'^d, \ldots, \tilde{\Phi}^d, \tilde{\Psi}^d$ are alternative choices yielding $\tilde{\alpha}_{ab}$, then by our usual argument using $D_{ab} \amalg \tilde{D}_{ab}$ and both sets of data we see that $\alpha_{ab} = \tilde{\alpha}_{ab}$, so α_{ab} is independent of choices.

Because the $\Theta_{\Phi^d}, \Theta_{\Psi^d}$ used to define α_{ab} are compatible with Verdier duality and monodromy by (5.14)–(5.15), and the $\Lambda_{\Phi^d}, \Lambda_{\Psi^d}$ affect only the principal $\mathbb{Z}/2\mathbb{Z}$-bundles rather than the perverse sheaves, we can show α_{ab} is compatible with Verdier duality and monodromy, in that the

following commute:

$$
\begin{array}{ccc}
\begin{aligned}&\big[i_a^*(\mathcal{PV}_{U_a,f_a}^\bullet)\otimes_{\mathbb{Z}/2\mathbb{Z}}\\ &Q_{R_a,U_a,f_a,i_a}\big]\big|_{R_a\cap R_b}\end{aligned} & \xrightarrow{\quad\quad\quad\quad\quad\alpha_{ab}\quad\quad\quad\quad\quad} & \begin{aligned}&\big[i_b^*(\mathcal{PV}_{U_b,f_b}^\bullet)\otimes_{\mathbb{Z}/2\mathbb{Z}}\\ &Q_{R_b,U_b,f_b,i_b}\big]\big|_{R_a\cap R_b}\end{aligned}
\end{array}
$$

$$\downarrow{\scriptstyle i_a^*(\sigma_{U_a,f_a})\otimes\mathrm{id}_{Q_{R_a,U_a,f_a,i_a}}\,|_{R_a\cap R_b}}\qquad\qquad {\scriptstyle i_b^*(\sigma_{U_b,f_b})\otimes\mathrm{id}_{Q_{R_b,U_b,f_b,i_b}}\,|_{R_a\cap R_b}}\downarrow$$

$$
\begin{array}{ccc}
\begin{aligned}&\big[i_a^*(\mathbb{D}_{\mathrm{Crit}(f_a)}(\mathcal{PV}_{U_a,f_a}^\bullet))\\ &\otimes_{\mathbb{Z}/2\mathbb{Z}}Q_{R_a,U_a,f_a,i_a}\big]\big|_{R_a\cap R_b})\\ &\cong\mathbb{D}_{R_a\cap R_b}\big(\big[i_a^*(\mathcal{PV}_{U_a,f_a}^\bullet)\\ &\otimes_{\mathbb{Z}/2\mathbb{Z}}Q_{R_a,U_a,f_a,i_a}\big]|_{R_a\cap R_b}\big)\end{aligned} & \xleftarrow{\quad\mathbb{D}_{R_a\cap R_b}(\alpha_{ab})\quad} & \begin{aligned}&\big[i_b^*(\mathbb{D}_{\mathrm{Crit}(f_b)}(\mathcal{PV}_{U_b,f_b}^\bullet))\\ &\otimes_{\mathbb{Z}/2\mathbb{Z}}Q_{R_b,U_b,f_b,i_b}\big]\big|_{R_a\cap R_b})\\ &\cong\mathbb{D}_{R_a\cap R_b}\big(\big[i_b^*(\mathcal{PV}_{U_b,f_b}^\bullet)\\ &\otimes_{\mathbb{Z}/2\mathbb{Z}}Q_{R_b,U_b,f_b,i_b}\big]|_{R_a\cap R_b}\big),\end{aligned}
\end{array}
$$

(6.20)

$$
\begin{array}{ccc}
\begin{aligned}&\big[i_a^*(\mathcal{PV}_{U_a,f_a}^\bullet)\otimes_{\mathbb{Z}/2\mathbb{Z}}\\ &Q_{R_a,U_a,f_a,i_a}\big]\big|_{R_a\cap R_b}\end{aligned} & \xrightarrow{\quad\quad\quad\quad\quad\alpha_{ab}\quad\quad\quad\quad\quad} & \begin{aligned}&\big[i_b^*(\mathcal{PV}_{U_b,f_b}^\bullet)\otimes_{\mathbb{Z}/2\mathbb{Z}}\\ &Q_{R_b,U_b,f_b,i_b}\big]\big|_{R_a\cap R_b}\end{aligned}
\end{array}
$$

$$\downarrow{\scriptstyle i_a^*(\tau_{U_a,f_a})\otimes\mathrm{id}_{Q_{R_a,U_a,f_a,i_a}}\,|_{R_a\cap R_b}}\qquad\qquad {\scriptstyle i_b^*(\tau_{U_b,f_b})\otimes\mathrm{id}_{Q_{R_b,U_b,f_b,i_b}}\,|_{R_a\cap R_b}}\downarrow$$

(6.21)

$$
\begin{array}{ccc}
\begin{aligned}&\big[i_a^*(\mathcal{PV}_{U_a,f_a}^\bullet)\otimes_{\mathbb{Z}/2\mathbb{Z}}\\ &Q_{R_a,U_a,f_a,i_a}\big]\big|_{R_a\cap R_b}\end{aligned} & \xrightarrow{\quad\quad\quad\quad\quad\alpha_{ab}\quad\quad\quad\quad\quad} & \begin{aligned}&\big[i_b^*(\mathcal{PV}_{U_b,f_b}^\bullet)\otimes_{\mathbb{Z}/2\mathbb{Z}}\\ &Q_{R_b,U_b,f_b,i_b}\big]\big|_{R_a\cap R_b}.\end{aligned}
\end{array}
$$

When $a=b$ we can take $\Psi^d=\Phi^d$, so (6.17) gives $\alpha_{aa}^d=\mathrm{id}$, and $\alpha_{aa}=\mathrm{id}$.

To prove (6.16), let $a,b,c\in A$, and $x\in R_a\cap R_b\cap R_c$. Applying Theorem 6.3 twice and composing the embeddings, we can construct subcharts $(R_a',U_a',f_a',i_a')\subseteq(R_a,U_a,f_a,i_a)$, $(R_b',U_b',f_b',i_b')\subseteq(R_b,U_b,f_b,i_b)$, and $(R_c',U_c',f_c',i_c')\subseteq(R_c,U_c,f_c,i_c)$ with $x\in R_a'\cap R_b'\cap R_c'$, a critical chart (S,V,g,j) on (X,s), and embeddings $\Phi:(R_a',U_a',f_a',i_a')\hookrightarrow(S,V,g,j)$, $\Psi:(R_b',U_b',f_b',i_b')\hookrightarrow(S,V,g,j)$, $\Upsilon:(R_c',U_c',f_c',i_c')\hookrightarrow(S,V,g,j)$. Then the construction of α_{ab} above yields

$$
\begin{aligned}
\alpha_{ab}|_{R_a'\cap R_b'\cap R_c'} &= \big((i_b^*(\Theta_\Psi^{-1})\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Lambda_\Psi)\circ(\mathrm{id}\otimes\Lambda_\Phi^{-1})\circ(i_a^*(\Theta_\Phi)\otimes\mathrm{id})\big)|_{R_a'\cap R_b'\cap R_c'},\\
\alpha_{bc}|_{R_a'\cap R_b'\cap R_c'} &= \big((i_c^*(\Theta_\Upsilon^{-1})\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Lambda_\Upsilon)\circ(\mathrm{id}\otimes\Lambda_\Psi^{-1})\circ(i_b^*(\Theta_\Psi)\otimes\mathrm{id})\big)|_{R_a'\cap R_b'\cap R_c'},\\
\alpha_{ac}|_{R_a'\cap R_b'\cap R_c'} &= \big((i_c^*(\Theta_\Upsilon^{-1})\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Lambda_\Upsilon)\circ(\mathrm{id}\otimes\Lambda_\Phi^{-1})\circ(i_a^*(\Theta_\Phi)\otimes\mathrm{id})\big)|_{R_a'\cap R_b'\cap R_c'},
\end{aligned}
$$

so that $\alpha_{bc}|_{R_a'\cap R_b'\cap R_c'}\circ\alpha_{ab}|_{R_a'\cap R_b'\cap R_c'}=\alpha_{ac}|_{R_a'\cap R_b'\cap R_c'}$. As we can cover $R_a\cap R_b\cap R_c$ by such Zariski open $R_a'\cap R_b'\cap R_c'$, equation (6.16) follows by Theorem 2.7(i).

The Zariski open cover version of Theorem 2.7(ii) now implies that there exists $P_{X,s}^\bullet$ in $\mathrm{Perv}(X)$, unique up to canonical isomorphism, with isomorphisms

$$
\omega_{R_a,U_a,f_a,i_a}:P_{X,s}^\bullet|_{R_a}\longrightarrow i_a^*(\mathcal{PV}_{U_a,f_a}^\bullet)\otimes_{\mathbb{Z}/2\mathbb{Z}}Q_{R_a,U_a,f_a,i_a}
$$

as in (6.6) for each $a\in A$, with $\alpha_{ab}\circ\omega_{R_a,U_a,f_a,i_a}|_{R_a\cap R_b}=\omega_{R_b,U_b,f_b,i_b}|_{R_a\cap R_b}$ for all $a,b\in A$. Also, (6.7)–(6.8) with (R_a,U_a,f_a,i_a) in place of (R,U,f,i) define isomorphisms $\Sigma_{X,s}|_{R_a}$, $\mathrm{T}_{X,s}|_{R_a}$ for each $a\in A$. Equations (6.20)–(6.21) imply that the prescribed values for $\Sigma_{X,s}|_{R_a}$, $\mathrm{T}_{X,s}|_{R_a}$ and $\Sigma_{X,s}|_{R_b}$, $\mathrm{T}_{X,s}|_{R_b}$ agree when restricted to $R_a\cap R_b$ for all $a,b\in A$. Hence, Theorem 2.7(i) gives unique isomorphisms $\Sigma_{X,s}$, $\mathrm{T}_{X,s}$ in (6.5) such that (6.7)–(6.8) commute with (R_a,U_a,f_a,i_a) in place of (R,U,f,i) for all $a\in A$.

Suppose $\big\{(R_a,U_a,f_a,i_a):a\in A\big\}$ and $\big\{(\tilde R_a,\tilde U_a,\tilde f_a,\tilde i_a):a\in\tilde A\big\}$ are alternative choices above, yielding $P_{X,s}^\bullet,\Sigma_{X,s},\mathrm{T}_{X,s}$ and $\tilde P_{X,s}^\bullet,\tilde\Sigma_{X,s},\tilde{\mathrm{T}}_{X,s}$. Then applying the same construction to the family $\big\{(R_a,U_a,f_a,i_a):a\in A\big\}\amalg\big\{(\tilde R_a,\tilde U_a,\tilde f_a,\tilde i_a):a\in\tilde A\big\}$ to get $\hat P_{X,s}^\bullet$, we have

canonical isomorphisms $P^\bullet_{X,s} \cong \hat{P}^\bullet_{X,s} \cong \tilde{P}^\bullet_{X,s}$, which identify $\Sigma_{X,s}, T_{X,s}$ with $\tilde{\Sigma}_{X,s}, \tilde{T}_{X,s}$. Thus $P^\bullet_{X,s}, \Sigma_{X,s}, T_{X,s}$ are independent of choices up to canonical isomorphism.

Now fix $\{(R_a, U_a, f_a, i_a) : a \in A\}$, $P^\bullet_{X,s}, \Sigma_{X,s}, T_{X,s}$ and ω_{R_a,U_a,f_a,i_a} for $a \in A$ above for the rest of the proof. Suppose (R, U, f, i) is a critical chart on (X, s). Running the construction above with the family $\{(R_a, U_a, f_a, i_a) : a \in A\} \amalg \{(R, U, f, i)\}$, we can suppose it yields the same (not just isomorphic) $P^\bullet_{X,s}, \Sigma_{X,s}, T_{X,s}$ and ω_{R_a,U_a,f_a,i_a}, but it also yields a unique $\omega_{R,U,f,i}$ in (6.6) which makes (6.7)–(6.8) commute. This proves Theorem 6.9(i).

Let $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$ be an embedding of critical charts on (X, s). The definition of Λ_Φ in Theorem 6.9(ii) is immediate. Run the construction above using the family $\{(R_a, U_a, f_a, i_a) : a \in A\} \amalg \{(R, U, f, i), (S, V, g, j)\}$, and follow the definition of α_{ab} with $(R, U, f, i), (S, V, g, j)$ in place of $(R_a, U_a, f_a, i_a), (R_b, U_b, f_b, i_b)$. We can take

$$D_{ab} = \{d\}, \quad (R'^d_a, U'^d_a, f'^d_a, i'^d_a) = (R, U, f, i), \quad (R'^d_b, U'^d_b, f'^d_b, i'^d_b) = (S^d, V^d, g^d, j^d) = (S, V, g, j),$$

$\Phi^d = \Phi$ and $\Psi^d = \mathrm{id}_V$. Then (6.17) gives $\alpha_{ab} = \alpha^d_{ab} = (\mathrm{id} \otimes \Lambda^{-1}_\Phi) \circ (i^*(\Theta_\Phi) \otimes \mathrm{id})$. Thus,

$$\alpha_{ab} \circ \omega_{R_a,U_a,f_a,i_a}|_{R_a \cap R_b} = \omega_{R_b,U_b,f_b,i_b}|_{R_a \cap R_b}$$

implies that (6.10) commutes, proving Theorem 6.9(ii).

6.4. $\mathscr{D}$-modules and mixed Hodge modules.

Once again, the proof of Theorem 6.9 carries over to our other contexts in §2.6–§2.10 using the general framework of §2.5, now also making use of the Stack Property (x) for objects. For the case of mixed Hodge modules, we use Theorem 2.21(ii) to glue the $i^*_a(\mathcal{H}\mathcal{V}^\bullet_{U_a,f_a}) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{R_a,U_a,f_a,i_a}$ on $R_a \subseteq X$ for $a \in A$ *with their natural strong polarizations* (2.25), which are preserved by the isomorphisms α_{ab} in §6.3 on overlaps $R_a \cap R_b$.

APPENDIX A. COMPATIBILITY RESULTS, BY JÖRG SCHÜRMANN

In the main body of the paper, when comparing results for mixed Hodge modules to those involving perverse sheaves, we rely on the compatibility between duality and Thom–Sebastiani type isomorphisms of perverse sheaves and mixed Hodge modules. These compatibility statements cannot easily be read off from the existing literature, so we provide proofs here.

Proposition A.1. *If X is a $\mathbb{C}$-scheme and $f : X \to \mathbb{C}$ is regular, then Massey's natural isomorphisms from [37] quoted as Theorem 2.11(iv) coincide with the image under the realization functor of Saito's analogous isomorphisms [45] between functors on mixed Hodge modules. There is also an analogous compatibility result for X a complex analytic space equipped with an analytic function f.*

Proof. Massey's construction in [37] of the duality isomorphisms uses the definition of the vanishing cycle functor in terms of the local cohomology of suitable real half-spaces, compare also [50]. Using their notation, the compatibility comes down to compatibility of the diagram

$$
\begin{array}{ccc}
\phi^p_f \circ \mathbb{D}_X & \xrightarrow{\quad\sim\quad} & \mathbb{D}_{X_0} \circ \phi^p_f \\
\cong \downarrow & & \cong \uparrow \\
(R\Gamma_{\{\mathrm{Re}(f) \geq 0\}}(-)|_{X_0}) \circ \mathbb{D}_X & \xrightarrow{\quad\sim\quad} & \mathbb{D}_{X_0} \circ (R\Gamma_{\{\mathrm{Re}(f) \leq 0\}}(-)|_{X_0}).
\end{array}
\tag{A.1}
$$

Here the upper, respectively lower horizontal isomorphisms are the ones of Saito, respectively Massey, and the vertical isomorphisms follow for example from [50, Lem. 1.3.2, p. 69]. Saito deduces his duality isomorphism in [45, Lem. 5.2.4, p. 965] from a pairing on nearby cycles induced by a pairing $F \otimes G \to a^!_X A$ for $F, G \in D^b_c(X)$, with $a_X : X \to \mathrm{pt}$ the constant map and

$A \subset \mathbb{C}$ a coefficient field. But Massey's duality isomorphism can be also be induced from such a pairing fitting into a commutative diagram, with

$$L_0 = \{\mathrm{Re}(f) = 0\} \quad \text{and} \quad j : \{\mathrm{Re}(f) = 0, f \neq 0\} \to L_0$$

the open inclusion:

$$
\begin{array}{ccc}
(R\Gamma_{\{\mathrm{Re}(f)<0\}}(F)|_{X_0}) \otimes (R\Gamma_{\{\mathrm{Re}(f)>0\}}(G)|_{X_0}) & \xleftarrow{\ \sim\ } & \psi_f(F) \otimes \psi_f(G) \\
\downarrow & & \downarrow \\
(Rj_*j^*(a_{L_0}^! A|_{L_0}))|_{X_0} & \longleftarrow & \psi_f(a_X^! A) \\
\downarrow{\scriptstyle\cong} & & \downarrow \\
(Rj_*j^*(a_{L_0}^! A))|_{X_0}[1] & \longrightarrow & a_{X_0}^! A[2].
\end{array}
\tag{A.2}
$$

Here the isomorphism $j^*(a_X^! A|_{L_0}) \cong j^*(a_{L_0}^! A)[1]$ comes from the fact that $\mathrm{Re}(f)$ has no critical points (in a stratified sense) in $X \setminus X_0$, locally near X_0. But then the commutativity of (A.2) implies by [45, Lem. 5.2.4, p. 965] the commutativity of (A.1), concluding the proof. $\qquad\square$

A similar compatibility question arises for the Thom–Sebastiani isomorphism. Here the precise statement is the following.

Proposition A.2. *Let $f_i : Y_i \to \mathbb{C}$ be regular functions on smooth $\mathbb{C}$-schemes, for $i = 1, 2$. Let $f = f_1 \boxplus f_2 : Y_1 \times Y_2 \to \mathbb{C}$ be as in Theorem 2.13. Then the isomorphism (2.8) of Massey [35,50] coincides with the image under the realization functor of Saito's analogous isomorphism (2.26) of [49] for mixed Hodge modules.*

Proof. The Thom–Sebastiani isomorphism (2.26) is constructed by Saito [49, Th. 2.6] based on the Verdier specialization [53]. First, let $f : Y \to \mathbb{C}$ be a regular function, with $X = f^{-1}(0)$ of codimension one, so that the normal cone $C_X Y = X \times \mathbb{C}$ becomes a trivial line bundle with $f' : C_X Y \to \mathbb{C}$ given by the projection. Let $p : D_X Y \to \mathbb{C}$ be the deformation to the normal cone with $C_X Y = p^{-1}(0) \subset D_X Y$, with $q : D_X Y \to Y$ the natural map. Then f' extends to a function $g : D_X Y \to \mathbb{C}$, with $g = f/s$ on $p \neq 0 = Y \times \mathbb{C}$ for s the usual coordinate on $\mathbb{C}$. For $F \in D_c^b(Y)$, we get a commutative diagram

$$
\begin{array}{ccc}
\phi_{f'}^p(sp_X F) & \xleftarrow{\qquad \sim \qquad} & \phi_f^p(F) \\
\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow \\
R\Gamma_{\{\mathrm{Re}(f')\geq 0\}}(sp_X F)|_X & \longleftarrow & R\Gamma_{\{\mathrm{Re}(f)\geq 0\}}(F)|_X.
\end{array}
$$

Here the monodromical sheaf complex $sp_X F \in D_c^b(C_X Y)_{\mathrm{mon}}$ is the Verdier specialization of F as in [49,53]. The upper horizontal isomorphism is the one of [49, Lem. 2.2], whereas the vertical isomorphisms are those of [50, Lem. 1.3.2, p. 69]. The lower horizontal map is defined by the natural base change morphism

$$R\Gamma_{\{\mathrm{Re}(f')\geq 0\}}(sp_X F)|_X \longleftarrow \psi_p(R\Gamma_{\{\mathrm{Re}(g)\geq 0\}}(q^* F))|_X \cong R\Gamma_{\{\mathrm{Re}(f)\geq 0\}}(F)|_X,$$

where the last isomorphism follows as in [50, Lem. 1.3.3, p. 70-71].

Consider now the situation in the proposition, with

$$f = f_1 \boxplus f_2 : Y = Y_1 \times Y_2 \to \mathbb{C},$$

also $X_i = f_i^{-1}(0)$ and $X = f^{-1}(0)$; finally let $\mu_{f_i} = R\Gamma_{\{\mathrm{Re}(f_i)\geq 0\}}$ to shorten the notation. Let

$$\pi : C_{X_1} Y_1 \times C_{X_2}(Y_2) = (X_1 \times X_2) \times \mathbb{C}^2 \to (X_1 \times X_2) \times \mathbb{C} \subset C_X(Y)$$

be the map induced by addition in the fibres. Then, for $F_i \in D_c^b(Y_i)$, one gets a commutative diagram

$$
\begin{array}{ccc}
\phi_f^p(F_1 \boxtimes F_2)|_{X_1 \times X_2} & \xleftarrow{\quad\sim\quad} & \phi_{f_i}^p(F_1) \otimes \phi_{f_2}^p(F_2) \\
\downarrow{\cong} & & \cong\downarrow \\
\mu_{f'}(\pi_*(sp_{X_1}F_1 \boxtimes sp_{X_2}F_2))|_{X_1 \times X_2} & \xleftarrow{\quad} & \mu_{f_1'}(sp_{X_1}F_1)|_{X_1} \otimes \mu_{f_2'}(sp_{X_2}F_2)|_{X_2} \\
\uparrow & & \cong\uparrow \\
\mu_f(F_1 \boxtimes F_2)|_{X_1 \times X_2} & \xleftarrow{\quad\sim\quad} & \mu_{f_1}(F_1)|_{X_1} \otimes \mu_{f_2}(F_2)|_{X_2}.
\end{array}
$$

The upper horizontal and left vertical isomorphisms form the Thom–Sebastiani isomorphism (2.26) of [49], whereas the lower horizontal isomorphism is the Thom–Sebastiani isomorphism (2.8) of [35, 50]. This concludes the proof. $\qquad\square$

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