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SPECIAL GENERIC MAPS ON OPEN 4-MANIFOLDS

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ABSTRACT. We characterize those smooth 1-connected open 4-manifolds with certain finite type properties which admit proper special generic maps into 3-manifolds. As a corollary, we show that a smooth 4-manifold homeomorphic to $\mathbf{R}^4$ admits a proper special generic map into $\mathbf{R}^n$ for some $n = 1, 2$ or 3 if and only if it is diffeomorphic to $\mathbf{R}^4$. We also characterize those smooth 4-manifolds homeomorphic to $L \times \mathbf{R}$ for some closed orientable 3-manifold L which admit proper special generic maps into $\mathbf{R}^3$.

1. INTRODUCTION

A *special generic map* $f : M \to N$ between smooth manifolds is a smooth map with at most *definite fold singularities*, which have the normal form

$$(x_1, x_2, \dots, x_m) \mapsto (x_1, x_2, \dots, x_{n-1}, x_n^2 + x_{n+1}^2 + \cdots + x_m^2), \tag{1.1}$$

where $m = \dim M \geq \dim N = n$. In particular, submersions are considered special generic maps.

In [24, 25], the author has shown that a smooth connected closed m-dimensional manifold M admits a special generic map into $\mathbf{R}^n$ for every n with $1 \leq n \leq m$ if and only if M is diffeomorphic to the standard m-sphere S^m. Furthermore, certain cobordism groups of special generic maps into $\mathbf{R}$ are naturally isomorphic to the h-cobordism groups of homotopy spheres in higher dimensions [26]. In [27, 28] Sakuma and the author found some pairs of homeomorphic smooth closed 4-manifolds such that one of them admits a special generic map into $\mathbf{R}^3$, while the other does not. These show that special generic maps are sensitive to detecting distinct differentiable structures on a given topological manifold.

On the other hand, it has been known that a smooth m-dimensional manifold is homeomorphic to $\mathbf{R}^m$ if and only if it is diffeomorphic to the standard $\mathbf{R}^m$, provided $m \neq 4$ (see [18, 31]), while for $m = 4$, there exist uncountably many distinct differentiable structures on $\mathbf{R}^4$ (for example, see [4, 8, 10, 32]). In fact, it is known that most open 4-manifolds admit infinitely (and very often, uncountably) many distinct differentiable structures [1, 3, 7, 9].

In this paper, we characterize those smooth 1-connected open 4-manifolds of "finite type" which admit proper special generic maps into 3-manifolds, using the solution to the Poincaré Conjecture in dimension three (see [19, 20, 21] or [17], for example). Here, an open 4-manifold is of finite type if its homology is finitely generated and it has only finitely many ends, whose associated fundamental groups

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are stable and finitely presentable. As a corollary, we show that a smooth 4-manifold homeomorphic to $\mathbf{R}^4$ is diffeomorphic to the standard $\mathbf{R}^4$ if and only if it admits a proper special generic map into $\mathbf{R}^n$ for some $n = 1, 2$ or 3. We also prove similar results for certain standard 1-connected open 4-manifolds.

Furthermore, in §4 we show that if a smooth 4-manifold M is homeomorphic to $L \times \mathbf{R}$ for some connected closed orientable 3-manifold L and if M admits a proper special generic map into $\mathbf{R}^3$, then M is diffeomorphic to $L \times \mathbf{R}$ and the 3-manifold L admits a special generic map into $\mathbf{R}^2$.

All these results claim that among the (uncountably or infinitely) many distinct differentiable structures on a certain open topological 4-manifold, there is at most one smooth structure that allows the existence of a proper special generic map into a lower dimensional manifold.

Throughout the paper, manifolds and maps between them are differentiable of class C^∞ unless otherwise indicated. The (co)homology groups are with integer coefficients unless otherwise specified. The symbol "$\cong$" denotes a diffeomorphism between smooth manifolds or an appropriate isomorphism between algebraic objects. For a topological space X, the symbol "id_X" denotes the identity map of X.

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2. Preliminaries

Let us first recall the following notion of a Stein factorization, which will play an important role in this paper.

Definition 2.1. Let $f : M \to N$ be a smooth map between smooth manifolds. For two points $x, x' \in M$, we define $x \sim_f x'$ if $f(x) = f(x')(= y)$, and the points x and x' belong to the same connected component of $f^{-1}(y)$. We define $W_f = M/\sim_f$ to be the quotient space with respect to this equivalence relation, and denote by $q_f : M \to W_f$ the quotient map. Then, we see easily that there exists a unique continuous map $\bar{f} : W_f \to N$ that makes the diagram

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "q_f"'] & & N \\ & W_f \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

commutative. The above diagram is called the *Stein factorization* of f (see [15]).

The Stein factorization is a very useful tool for studying topological properties of special generic maps. In fact, we can prove the following, which is folklore (for example, see [24]). (In the following, a continuous map is *proper* if the inverse image of a compact set is always compact.)

Proposition 2.2. *Let $f : M \to N$ be a proper special generic map between smooth manifolds with $m = \dim M > \dim N = n$. Then, we have the following.*

1. *The set of singular points $S(f)$ of f is a regular submanifold of M of dimension $n - 1$, which is closed as a subset of M.*
2. *The quotient space W_f has the structure of a smooth n-dimensional manifold possibly with boundary such that $\bar{f} : W_f \to N$ is an immersion.*

(3) *The quotient map* $q_f : M \to W_f$ *restricted to* $S(f)$ *is a diffeomorphism onto* ∂W_f.
(4) *The quotient map* q_f *restricted to* $M \setminus S(f)$ *is a smooth fiber bundle over* $\operatorname{Int} W_f$ *with fiber the standard* $(m-n)$*-sphere* S^{m-n}.

In the following, we recall several notions concerning ends of manifolds. For details, the reader is referred to Siebenmann's thesis [30].

Definition 2.3. Let X be a Hausdorff space. Consider a collection ε of subsets of X with the following properties.

(i) Each $G \in \varepsilon$ is a connected open non-empty set with compact frontier $\overline{G} - G$,
(ii) If $G, G' \in \varepsilon$, then there exists $G'' \in \varepsilon$ with $G'' \subset G \cap G'$,
(iii) $\bigcap_{G \in \varepsilon} \overline{G} = \emptyset$.

Adding to ε every connected open non-empty set $H \subset X$ with compact frontier such that $G \subset H$ for some $G \in \varepsilon$, we produce a collection ε' satisfying (i), (ii) and (iii), which we call the *end* of X determined by ε.

An *end* of a Hausdorff space X is a collection ε of subsets of X which is maximal with respect to the properties (i), (ii) and (iii) above.

A *neighborhood* of an end ε is any set $N \subset X$ that contains some member of ε.

Definition 2.4. Let ε be an end of a topological manifold X. The fundamental group π_1 is *stable* at ε if there exists a sequence of path connected neighborhoods of ε, $X_1 \supset X_2 \supset \cdots$, with $\bigcap \overline{X}_i = \emptyset$ such that (with base points and base paths chosen) the sequence

$$\pi_1(X_1) \xleftarrow{f_1} \pi_1(X_2) \xleftarrow{f_2} \cdots$$

induced by the inclusions induces isomorphisms

$$\operatorname{Im}(f_1) \xleftarrow{\cong} \operatorname{Im}(f_2) \xleftarrow{\cong} \cdots .$$

The following lemma is proved in [30].

Lemma 2.5. *If* π_1 *is stable at* ε *and* $Y_1 \supset Y_2 \supset \cdots$ *is any path connected sequence of neighborhoods of* ε *such that* $\bigcap \overline{Y}_i = \emptyset$, *then for any choice of base points and base paths, the inverse sequence*

$$\mathcal{G} : \qquad \pi_1(Y_1) \xleftarrow{g_1} \pi_1(Y_2) \xleftarrow{g_2} \cdots$$

induced by the inclusions is stable, i.e. there exists a subsequence

$$\pi_1(Y_{i_1}) \xleftarrow{h_1} \pi_1(Y_{i_2}) \xleftarrow{h_2} \cdots$$

inducing isomorphisms

$$\operatorname{Im}(h_1) \xleftarrow{\cong} \operatorname{Im}(h_2) \xleftarrow{\cong} \cdots ,$$

where each h_j *is a suitable composition of* g_i*'s.*

Definition 2.6. When π_1 is stable at an end ε, we define $\pi_1(\varepsilon)$ to be the projective limit $\varprojlim \mathcal{G}$ for some fixed system $\mathcal{G}$ as above. According to [30], $\pi_1(\varepsilon)$ is well defined up to isomorphism.

3. Open 4-manifolds that admit special generic maps

In the following, a manifold is *open* if it has no boundary and each of its component is non-compact, while a manifold is *closed* if it has no boundary and is compact.

Let us begin by the following.

Lemma 3.1. *Let M be a smooth connected open 4-manifold with finitely many ends such that $H_2(M; \mathbf{Z}_2)$ is finitely generated. We assume that for each end ε, π_1 is stable and $\pi_1(\varepsilon)$ is finitely presentable. If $f : M \to N$ is a proper special generic map into a smooth orientable 3-manifold N, then there exists a smooth compact 3-manifold $\widetilde{W}$ possibly with boundary and a smooth embedding $h : W_f \to \widetilde{W}$ such that $h(\operatorname{Int} W_f) = \operatorname{Int} \widetilde{W}$.*

Proof. Suppose that $S(f) \cong \partial W_f$ has infinitely many components. Let S_i, $i = 0, 1, 2, \ldots$, be an infinite family of distinct components of ∂W_f. Since M is connected and q_f is surjective, W_f is connected. Thus, there exists an infinite family of disjointly embedded arcs α_i, $i \geq 1$, connecting S_0 and S_i in the 3-manifold W_f such that each α_i intersects ∂W_f transversely at its end points and $\operatorname{Int} \alpha_i \subset \operatorname{Int} W_f$. Then, $\{q_f^{-1}(\alpha_i)\}_{i \geq 1}$ is an infinite family of disjointly embedded 2-spheres in M. Furthermore, for each $i \geq 1$, $q_f^{-1}(S_i)$ is a submanifold of M which is closed as a subset of M, intersects $q_f^{-1}(\alpha_i)$ transversely at one point, and does not intersect $q_f^{-1}(\alpha_j)$ for $j \neq i$. This implies that the homology classes in $H_2(M; \mathbf{Z}_2)$ represented by $q_f^{-1}(\alpha_i)$, $i \geq 1$, are linearly independent. This contradicts our assumption that $H_2(M; \mathbf{Z}_2)$ is finitely generated. Therefore, ∂W_f has at most finitely many components.

Let the number of ends of M be denoted by r. Let K be an arbitrary compact subset of W_f. Since f is proper, so is q_f, and hence $K' = q_f^{-1}(K)$ is a compact subset of M. Therefore, $M \setminus K'$ has at most r unbounded components[1] (see [30, Lemma 1.8]). Thus, $q_f(M \setminus K') = W_f \setminus K$ has at most r unbounded components, since q_f is proper. Hence, W_f has finitely many ends.

Let ε be an end of W_f and $U_1 \supset U_2 \supset \cdots$ be any path connected sequence of neighborhoods of ε such that $\bigcap \overline{U}_i = \emptyset$. Then, for $V_i = q_f^{-1}(U_i)$, $V_1 \supset V_2 \supset \cdots$ is a path connected sequence of neighborhoods of the corresponding end of M with $\bigcap \overline{V}_i = \emptyset$. By Lemma 2.5 together with our assumption, there exists a subsequence $V_{i_1} \supset V_{i_2} \supset \cdots$ such that the sequence

$$\pi_1(V_{i_1}) \xleftarrow{f_1} \pi_1(V_{i_2}) \xleftarrow{f_2} \cdots$$

induced by the inclusions induces isomorphisms

$$\operatorname{Im}(f_1) \xleftarrow{\cong} \operatorname{Im}(f_2) \xleftarrow{\cong} \cdots .$$

Since U_{i_j} is open in W_f, every V_{i_j} contains an S^1-fiber of q_f. Thus, each f_j induces an isomorphism between the cyclic subgroups generated by the S^1-fibers. Since $(q_f)_* : \pi_1(V_{i_j}) \to \pi_1(U_{i_j})$ is an epimorphism whose kernel coincides with the cyclic subgroup generated by the S^1-fibers, we see that the sequence

$$\pi_1(U_{i_1}) \xleftarrow{g_1} \pi_1(U_{i_2}) \xleftarrow{g_2} \cdots$$

[1] A subset of a topological space is *bounded* if its closure is compact; otherwise, it is *unbounded*.

induced by the inclusions induces isomorphisms

$$\mathrm{Im}(g_1) \xleftarrow{\cong} \mathrm{Im}(g_2) \xleftarrow{\cong} \cdots .$$

Therefore, for each end of W_f, π_1 is stable. Furthermore, by our assumption, π_1 is finitely presentable.

On the other hand, since $\bar{f} : W_f \to N$ is an immersion and N is orientable, W_f is also orientable. Therefore, by [13] (see also [14]), we have the desired conclusion. (In fact, what we need here is [13, Theorem 3] with the condition $\pi_1(\varepsilon_i) \not\cong \mathbf{Z}_2$ for each i being replaced by the orientability of the 3-manifold M. This version of the theorem holds by the same reason as explained in the proof of [13, Corollary 2.1]: when the manifold is orientable, no projective plane appears in the boundary, and the argument works.) □

Remark 3.2. By [5], the compact 3-manifold $\widetilde{W}$ as in Lemma 3.1 is unique up to diffeomorphism.

Using Lemma 3.1, we prove the following.

Theorem 3.3. *Let M be a smooth connected open orientable 4-manifold with finitely many ends such that $H_*(M)$ is finitely generated. We assume that for each end ε, π_1 is stable and $\pi_1(\varepsilon)$ is finitely presentable. If $f : M \to N$ is a proper special generic map into a smooth orientable 3-manifold N, then there exists a smooth connected closed 4-manifold $\widetilde{M}$ and a compact orientable surface F possibly with boundary smoothly embedded in $\widetilde{M}$ such that M is diffeomorphic to $\widetilde{M} \setminus F$.*

Proof. By [24], there exists an orientable linear D^2-bundle $\pi : E_f \to W_f$ such that M is diffeomorphic to ∂E_f, where an ℓ-dimensional disk bundle is *linear* if its structure group can be reduced to a subgroup of the orthogonal group $O(\ell)$. Moreover, if C denotes a small closed collar neighborhood of ∂W_f in W_f, then $N_S = q_f^{-1}(C)$ is a tubular neighborhood of $S(f)$ in M and π restricted to $(\partial E_f) \cap \pi^{-1}(W_f \setminus C)$ can be identified with the smooth S^1-bundle $q_f|_{M \setminus N_S} : M \setminus N_S \to W_f \setminus C$.

Now, let us consider the cohomology exact sequence for the pair $(E_f, M \setminus N_S) \simeq (E_f, M \setminus S(f))$:

$$\widetilde{H}^k(E_f) \to \widetilde{H}^k(M \setminus S(f)) \to \widetilde{H}^{k+1}(E_f, M \setminus S(f)).$$

We have $\widetilde{H}^k(E_f) \cong \widetilde{H}^k(W_f)$, since $E_f \to W_f$ is a D^2-bundle. Furthermore, by the Thom isomorphism theorem (for example, see [16]), we have $\widetilde{H}^{k+1}(E_f, M \setminus S(f)) \cong \widetilde{H}^{k-1}(W_f)$. Therefore, putting $k = 2$, we have the exact sequence

$$H^2(W_f) \to H^2(M \setminus S(f)) \to H^1(W_f).$$

Since $H^*(W_f) \cong H^*(\widetilde{W})$ is finitely generated, so is $H^2(M \setminus S(f))$, where $\widetilde{W}$ is the compact orientable 3-manifold as in Lemma 3.1.

By excision, we have $\widetilde{H}^{k+1}(M, M \setminus S(f)) \cong \widetilde{H}^{k+1}(N_S, \partial N_S)$. Since M and $S(f)$ are orientable, N_S is an orientable D^2-bundle over $S(f)$. Therefore, $\widetilde{H}^{k+1}(N_S, \partial N_S)$ is isomorphic to $\widetilde{H}^{k-1}(S(f))$ by the Thom isomorphism theorem. Thus, we have $\widetilde{H}^{k+1}(M, M \setminus S(f)) \cong \widetilde{H}^{k-1}(S(f))$.

Let us consider the cohomology exact sequence for the pair $(M, M \setminus S(f))$:

$$H^2(M \setminus S(f)) \to H^3(M, M \setminus S(f)) \to H^3(M).$$

Since $H^2(M \setminus S(f))$ and $H^3(M)$ are finitely generated, so is $H^3(M, M \setminus S(f)) \cong H^1(S(f))$. This implies that $H_*(S(f))$ is finitely generated, since $S(f)$ has finitely many components by the proof of Lemma 3.1. Then, we see that $S(f) \cong \partial W_f$ is diffeomorphic to $\partial \widetilde{W} \setminus F_1$, where $F_1(\subset \partial \widetilde{W})$ is a compact orientable surface possibly with boundary (see [13, Proposition 2]). In fact, we can prove that W_f is diffeomorphic to $\widetilde{W} \setminus F_1$.

Let $\widetilde{\pi} : \widetilde{E} \to \widetilde{W}$ be the linear D^2-bundle which naturally extends $\pi : E_f \to W_f$. Then, by the above arguments, we see that $M \cong \partial E_f$ is diffeomorphic to $\partial \widetilde{E} \setminus \widetilde{\pi}^{-1}(F_1)$. Set $\widetilde{M} = \partial \widetilde{E}$ and let F be the compact surface in $\widetilde{M}$ which corresponds to the zero section of $\widetilde{\pi}$ over F_1. Then the desired conclusion follows. □

Remark 3.4. As the above proof shows, the closed 4-manifold $\widetilde{M}$ in Theorem 3.3 is the boundary of an orientable linear D^2-bundle over the compact orientable 3-manifold $\widetilde{W}$ as in Lemma 3.1. In particular, it admits a special generic map $\widetilde{f} : \widetilde{M} \to \mathbf{R}^3$ whose quotient space can be identified with $\widetilde{W}$ (see [24]). Furthermore, the surface F in Theorem 3.3 is a codimension zero submanifold of $S(\widetilde{f})$ and the quotient map $q_f : M \to W_f$ can be identified with $q_{\widetilde{f}}|_{\widetilde{M} \setminus F}$.

Remark 3.5. Theorem 3.3 holds true even if N is non-orientable, provided that W_f is orientable. If for each end ε, $\pi_1(\varepsilon)$ contains no cyclic subgroup of index two, then even the orientability of W_f is not necessary (but, in this case, the surface F may possibly be non-orientable).

As a corollary, we have the following characterization of smooth 1-connected open 4-manifolds of "finite type" which admit proper special generic maps into 3-manifolds.

Corollary 3.6. *Let M be a smooth 1-connected open 4-manifold with finitely many ends such that $H_*(M)$ is finitely generated. We assume that for each end ε, π_1 is stable and $\pi_1(\varepsilon)$ is finitely presentable. Then there exists a proper special generic map $f : M \to N$ into a smooth 3-manifold N with $S(f) \neq \emptyset$ if and only if M is diffeomorphic to the connected sum of a finite number of copies of the following 4-manifolds:*

(1) $\mathbf{R}^4$,
(2) *the interior of the boundary connected sum of a finite number of copies of $S^2 \times D^2$,*
(3) *the total space of a 2-plane bundle over S^2,*
(4) *the total space of an S^2-bundle over S^2,*

where at least one manifold of the form (1), (2) or (3) should appear in the connected sum. In particular, each end of such an open 4-manifold has a neighborhood diffeomorphic to $L \times \mathbf{R}$, where L is the 3-sphere S^3, a lens space, or a connected sum of a finite number of copies of $S^1 \times S^2$.

Proof. Suppose that there exists a proper special generic map $f : M \to N$ into a 3-manifold N with $S(f) \neq \emptyset$. Since $(q_f)_* : \pi_1(M) \to \pi_1(W_f)$ is an isomorphism (see [24]), W_f is also 1-connected and hence is orientable. Let $\widetilde{W}$ be the compact 3-manifold as in Lemma 3.1 (see also Remark 3.5). Note that $\widetilde{W}$ is 1-connected. Then by the solution to the 3-dimensional Poincaré Conjecture (see [19, 20, 21] or [17], for example), $\widetilde{W}$ is diffeomorphic either to the 3-disk or to the boundary connected sum of a finite number of copies of $S^2 \times I$, where $I = [0,1]$. By the proof of Theorem 3.3, there exists a compact surface F possibly with boundary in $\partial\widetilde{W}$ such that W_f is diffeomorphic to $\widetilde{W} \setminus F$. Note that $\partial W_f \cong \partial\widetilde{W} \setminus F \neq \emptyset$, since $S(f) \neq \emptyset$.

We can decompose $\widetilde{W}$ as the boundary connected sum of a finite number of compact 3-manifolds W_i such that

(i) each W_i contains at most one component of F, say F_i,
(ii) if W_i contains no component of F, then we put $F_i = \emptyset$ and $W_i \cong S^2 \times I$,
(iii) if $F_i \neq \emptyset$ has no boundary, then $F_i \cong S^2$ is a component of ∂W_i and $W_i \cong S^2 \times I$,
(iv) if F_i has non-empty boundary, then $W_i \cong D^3$.

The 3-manifold W_f can also be decomposed as the boundary connected sum of the manifolds $W_i' = W_i \setminus F_i$. Then, M is decomposed into the connected sum of the 4-manifolds M_i, which is obtained by attaching 4-disks to $q_f^{-1}(W_i')$ along the boundary 3-spheres (for details, see [24]).

If W_i contains no component of F, then M_i admits a special generic map whose quotient space in the Stein factorization is diffeomorphic to $S^2 \times I$. Therefore, M_i is diffeomorphic to an S^2-bundle over S^2 (see [24]).

If $F_i \neq \emptyset$ has no boundary, then M_i admits a special generic map whose quotient space in the Stein factorization is diffeomorphic to $S^2 \times [0,1)$. Then, M_i is diffeomorphic to a 2-plane bundle over S^2.

If F_i has non-empty boundary, then by Theorem 3.3 M_i is diffeomorphic to $\partial\widetilde{E}_i \setminus F_i$, where $\widetilde{E}_i$ is a D^2-bundle over $W_i \cong D^3$ and F_i is identified with the zero section over F_i. Therefore, M_i is diffeomorphic to $S^4 \setminus \Sigma$, where Σ is a connected non-empty surface with non-empty boundary embedded in S^4. If Σ is a disk, then M_i is diffeomorphic to $\mathbf{R}^4$. Otherwise, Σ is homotopy equivalent to a bouquet of a finite number of circles. Then, $S^4 \setminus \Sigma$ is diffeomorphic to the interior of the boundary connected sum of a finite number of copies of $S^2 \times D^2$.

Thus, we have proved that M is diffeomorphic to a manifold of a desired form.

Conversely, each 4-manifold in the list admits a proper special generic map into a 3-manifold with non-empty set of singularities. By the connected sum construction with respect to the quotient space (for details, see [24]), we see that their connected sums also admit proper special generic maps into 3-manifolds.

This completes the proof. □

Remark 3.7. The 4-manifold $S^2 \times \mathbf{R}^2$ admits at least two types of proper special generic maps into $\mathbf{R}^3$ as follows. Let $g : S^2 \to \mathbf{R}$ be the standard height function with exactly two critical points, which are non-degenerate. Then, $g \times \mathrm{id}_{\mathbf{R}^2} : S^2 \times \mathbf{R}^2 \to \mathbf{R} \times \mathbf{R}^2$ is a proper special generic map whose quotient space is diffeomorphic to $[-1,1] \times \mathbf{R}^2$. On the other hand, let $h : \mathbf{R}^2 \to [0,\infty)$ be the proper smooth function defined by $h(x,y) = x^2 + y^2$. Then, $\mathrm{id}_{S^2} \times h : S^2 \times \mathbf{R}^2 \to S^2 \times [0,\infty)$ composed with a proper embedding $S^2 \times [0,\infty) \to \mathbf{R}^3$ is a proper special generic map whose quotient space is diffeomorphic to $S^2 \times [0,\infty)$.

The above observation corresponds to the fact that $S^2 \times \mathbf{R}^2$ appears twice in Corollary 3.6: it is the interior of $S^2 \times D^2$, and at the same time it is the total space of the trivial 2-plane bundle over S^2.

The 4-manifold $(\mathbf{C}P^2 \sharp \overline{\mathbf{C}P^2}) \setminus \{\text{two points}\}$ is another such example. It is the connected sum of a non-trivial S^2-bundle over S^2 and two copies of $\mathbf{R}^4$, and at the same time it is the connected sum of two 2-plane bundles over S^2 with Euler numbers $+1$ and -1.

Remark 3.8. For 4-manifolds as in Corollary 3.6, two are homeomorphic if and only if they are diffeomorphic. Note that every 4-manifold in Corollary 3.6 admits infinitely many distinct differentiable structures by [1]. In fact, most of them admit uncountably many distinct differentiable structures (see [3, 7, 9]).

Remark 3.9. In Corollary 3.6 we assumed that $S(f) \neq \emptyset$. If f is a proper submersion, then W_f is still 1-connected and is diffeomorphic to the interior of the connected sum of a finite number of copies of $S^2 \times [0, 1]$. Furthermore, M is diffeomorphic to the total space of an orientable S^1-bundle over W_f. Since M is 1-connected, the Euler class of the S^1-bundle should be primitive.

As a corollary, we have the following.

Corollary 3.10. *Let M be a smooth 4-manifold homeomorphic to $\mathbf{R}^4$. Then there exists a proper special generic map $f : M \to \mathbf{R}^n$ for some $n = 1, 2$ or 3 if and only if M is diffeomorphic to the standard $\mathbf{R}^4$.*

Proof. First note that the standard $\mathbf{R}^4$ admits a special generic map into $\mathbf{R}^n$ for $n = 1, 2$ and 3: just consider the map defined by (1.1) for $m = 4$ globally. Therefore, if $M \cong \mathbf{R}^4$, then M also admits proper special generic maps into $\mathbf{R}^n$ for $n = 1, 2$ and 3.

Suppose that there exists a proper special generic map $f : M \to \mathbf{R}^3$. If f is a submersion, then M must be diffeomorphic to $\mathbf{R}^3 \times S^1$, which is a contradiction. Then by Corollary 3.6, M must be diffeomorphic to $\mathbf{R}^4$.

Suppose now that there exists a proper special generic map $f : M \to \mathbf{R}^2$. Then, the quotient space W_f is a 1-connected non-compact surface with non-empty boundary.

Lemma 3.11. *The boundary ∂W_f is connected and non-compact.*

Proof. Suppose that $S(f) \cong \partial W_f$ is not connected. Let S_1 and S_2 be distinct connected components of ∂W_f. Note that W_f is connected, since so is M. Therefore, there exists an arc α in W_f which intersects S_1 and S_2 at its end points transversely such that $\operatorname{Int} \alpha \subset \operatorname{Int} W_f$. Then, $q_f^{-1}(\alpha)$ is a smooth submanifold of M diffeomorphic to S^3. Furthermore, it intersects the component $q_f^{-1}(S_1)$ of $S(f)$ transversely at one point. Note that $q_f^{-1}(S_1)$ is a 1-dimensional submanifold of M, which is a closed subset of M. This is a contradiction, since M is contractible and $H_3(M) = 0$. Therefore, $S(f)$ must be connected.

Suppose that $S(f)$ is compact. Since M is non-compact and q_f is proper, W_f is non-compact. Therefore, there exists a proper embedding $\gamma : [0, \infty) \to W_f$ which intersects with $\partial W_f (\cong S(f))$ transversely at its end point. Then, $q_f^{-1}(\gamma([0, \infty)))$ is a properly embedded open 3-disk in M which intersects $S(f)$ transversely at one point. This implies that $S(f)$ represents a nontrivial homology class in $H_1(M)$, which is a contradiction, since $H_1(M) = 0$. Therefore, $S(f)$ must be non-compact. □

Therefore, W_f is diffeomorphic to $\mathbf{R} \times [0, \infty)$ (for example, see [13, Proposition 2] or [23]). Then, we see that $M \cong \partial(W_f \times D^3)$ is diffeomorphic to the standard $\mathbf{R}^4$.

Finally, suppose that M admits a proper special generic map into $\mathbf{R}$. Then, W_f is diffeomorphic to $[0, \infty)$ and M is diffeomorphic to the boundary of $[0, \infty) \times D^4$, which is diffeomorphic to the standard $\mathbf{R}^4$. □

Remark 3.12. It has been known that a smooth m-dimensional manifold is homeomorphic to $\mathbf{R}^m$ if and only if it is diffeomorphic to the standard $\mathbf{R}^m$, provided that $m \neq 4$ (see [18, 31]), while for $m = 4$, there exist uncountably many distinct differentiable structures on $\mathbf{R}^4$ (for example, see [4, 8, 10, 32]). This shows that among the uncountably many differentiable structures on $\mathbf{R}^4$, the standard one is the unique structure that allows the existence of a proper special generic map into $\mathbf{R}^n$ for $n \leq 3$.

Remark 3.13. If a smooth 4-manifold M is homeomorphic to $\mathbf{R}^4$, then there always exists a proper special generic map $M \to \mathbf{R}^4$. See [6] and [11, The Folding Theorem (p. 27)] for details.

Remark 3.14. If we omit the properness, then every smooth 4-manifold homeomorphic to $\mathbf{R}^4$ admits a submersion into $\mathbf{R}^n$ for all n with $1 \leq n \leq 4$ (see [22]).

In fact, by virtue of [22], an open 4-manifold admits a submersion into $\mathbf{R}^n$ if and only if it has n everywhere linearly independent vector fields. Therefore, we have the following.

Proposition 3.15. *Let M be a smooth connected open orientable 4-manifold. Then we have the following.*

1. *There always exists a submersion $M \to \mathbf{R}$.*
2. *There exists a submersion $M \to \mathbf{R}^2$ if and only if $W_3(M) = \beta w_2(M) = 0$, where W_3 (or w_2) denotes the 3rd Whitney (resp. 2nd Stiefel–Whitney) class.*
3. *There exists a submersion $M \to \mathbf{R}^3$ if and only if $w_2(M) = 0$.*
4. *There exists a submersion $M \to \mathbf{R}^4$ if and only if $w_2(M) = 0$.*

Remark 3.16. Let $f : \mathbf{R}^4 \to \mathbf{R}^3$ be a proper special generic map. Then, we can show that the quotient map $q_f : \mathbf{R}^4 \to W_f$ is C^∞ right-left equivalent to the standard map $g : \mathbf{R}^4 \to \mathbf{R}^2 \times [0, \infty)$ defined by (1.1) with $(n, m) = (4, 3)$.

Note that the map $\bar{f} : W_f \to \mathbf{R}^3$ is a proper immersion. Since there are plenty of proper immersions $\mathbf{R}^2 \times [0, \infty) \to \mathbf{R}^3$, the C^∞ right-left equivalence class of a proper special generic map $f : \mathbf{R}^4 \to \mathbf{R}^3$ is far from being unique. In fact, we can show that two proper special generic maps $f_i : \mathbf{R}^4 \to \mathbf{R}^3$, $i = 0, 1$, are C^∞ right-left equivalent if and only if the proper immersions $\bar{f}_i : W_{f_i} \to \mathbf{R}^3$ are C^∞ right-left equivalent.

Remark 3.17. By [24] together with the solution to the 3-dimensional Poincaré Conjecture, we have the following: a smooth 4-manifold M homeomorphic to S^4 admits a special generic map into $\mathbf{R}^n$ for some $n = 1, 2$ or 3 if and only if M is diffeomorphic to the standard S^4. Furthermore, when $n = 3$, the singular set of a special generic map $M \to \mathbf{R}^3$ is always isotopic to the standardly embedded 2-sphere in S^4. (For details, see [29].)

Similarly, we have the following.[2]

[2]Corollaries 3.18 and 3.19, and Theorem 4.1 in §4 were first conjectured by Kazuhiro Sakuma to whom the author would like to express his sincere gratitude.

Corollary 3.18. *Let M be a smooth 4-manifold homeomorphic to $S^3 \times \mathbf{R}$. Then there exists a proper special generic map $f : M \to \mathbf{R}^n$ for some $n = 1, 2$ or 3 if and only if M is diffeomorphic to the standard $S^3 \times \mathbf{R}$.*

Note that $S^3 \times \mathbf{R} \cong \mathbf{R}^4 \sharp \mathbf{R}^4$.

Corollary 3.19. *Let M be a smooth 4-manifold homeomorphic to $S^2 \times \mathbf{R}^2$. Then there exists a proper special generic map $f : M \to \mathbf{R}^n$ for some $n = 2$ or 3 if and only if M is diffeomorphic to the standard $S^2 \times \mathbf{R}^2$.*

4. Manifolds homeomorphic to $L^3 \times \mathbf{R}$

In this section, we prove the following.

Theorem 4.1. *Let L be a smooth connected closed orientable 3-manifold. A smooth 4-manifold M homeomorphic to $L \times \mathbf{R}$ admits a proper special generic map into $\mathbf{R}^3$ if and only if M is diffeomorphic to $L \times \mathbf{R}$ and L is a smooth closed 3-manifold that admits a special generic map into $\mathbf{R}^2$.*

Proof. First suppose that M admits a proper special generic map $f : M \to \mathbf{R}^3$. Note that $S(f) \neq \emptyset$, since otherwise M is diffeomorphic to $S^1 \times \mathbf{R}^3$, which leads to a contradiction.

By the proof of Theorem 3.3, there exist a compact orientable 3-manifold $\widetilde{W}$ and a compact surface F possibly with boundary embedded in $\partial\widetilde{W}$ such that W_f is diffeomorphic to $\widetilde{W} \setminus F$. In particular, for each end of W_f, there exists a neighborhood C_i diffeomorphic to $F_i \times [0, \infty)$ for some compact connected orientable surface F_i possibly with boundary. Then, each $\widetilde{C}_i = q_f^{-1}(C_i)$ is a neighborhood of an end of M. Set $\widetilde{F}_i = q_f^{-1}(F_i \times \{1\})$, which is a connected closed orientable 3-manifold. Since M has exactly two ends and each of them has a neighborhood homeomorphic to $L \times [0, \infty)$, we see that W_f also has exactly two ends and the inclusions $\widetilde{F}_i \to M$ induce homotopy equivalences.

Let us consider the following commutative diagram:

$$\begin{tikzcd}
\pi_1(\widetilde{F}_i) \arrow[r, "(\widetilde{\iota}_i)_*"] \arrow[d, "(q_f)_*"'] & \pi_1(M) \arrow[d, "(q_f)_*"] \\
\pi_1(F_i) \arrow[r, "(\iota_i)_*"] & \pi_1(W_f),
\end{tikzcd}$$

where $\widetilde{\iota}_i : \widetilde{F}_i \to M$ and $\iota_i : F_i \to W_f$ are the inclusions. Since $(q_f)_* \circ (\widetilde{\iota}_i)_*$ is an isomorphism, $(q_f)_* : \pi_1(\widetilde{F}_i) \to \pi_1(F_i)$ is a monomorphism. Since it is an epimorphism, it must be an isomorphism. Therefore, $(\iota_i)_*$ is also an isomorphism and W_f has a surface fundamental group.

Then by [12, Theorem 10.6] together with the solution to the 3-dimensional Poincaré Conjecture, we see that $\widetilde{W}$ is diffeomorphic to $(F_i \times [0, 1]) \sharp \left(\sharp^k B^3\right)$ for some $k \geq 0$, and hence W_f is diffeomorphic to $(F_i \times \mathbf{R}) \sharp \left(\sharp^k B^3\right)$, where B^3 denotes the 3-dimensional ball. Then, by an argument similar to that in [27], we can show that M is diffeomorphic to the connected sum of $\widetilde{F}_i \times \mathbf{R}$ and S^2-bundles over S^2. Since M is homeomorphic to $L \times \mathbf{R}$ for a closed orientable 3-manifold L, we see that M is diffeomorphic to $\widetilde{F}_i \times \mathbf{R}$.

If F_i has no boundary, then $\widetilde{F}_i$ is an S^1-bundle over F_i. Since $(q_f)_* : \pi_1(\widetilde{F}_i) \to \pi_1(F_i)$ is an isomorphism, we see that $\widetilde{F}_i$ is diffeomorphic to S^3 and the S^1-bundle is the Hopf fibration. If F_i has non-empty boundary, then for any immersion $\eta : F_i \to \mathbf{R}^2$, the composition $\eta \circ q_f : \widetilde{F}_i \to \mathbf{R}^2$ is a special generic map. In either case, $\widetilde{F}_i$ admits a special generic map into $\mathbf{R}^2$.

Note that $\widetilde{F}_i$ has a free fundamental group. Since the inclusion $\widetilde{F}_i \to M$ induces a homotopy equivalence, L also has a free fundamental group. Therefore, L is diffeomorphic to S^3 or the connected sum of some copies of $S^1 \times S^2$ by virtue of [12, Chapter 5] and the solution to the 3-dimensional Poincaré Conjecture. In particular, L admits a special generic map into $\mathbf{R}^2$ (see [2]).

Conversely, if M is diffeomorphic to $L \times \mathbf{R}$ and L admits a special generic map $g : L \to \mathbf{R}^2$, then the map

$$M \cong L \times \mathbf{R} \xrightarrow{g \times \mathrm{id}_{\mathbf{R}}} \mathbf{R}^2 \times \mathbf{R} = \mathbf{R}^3$$

is a proper special generic map. This completes the proof. □

Remark 4.2. As has been seen in the above proof, the 3-manifold L in Theorem 4.1 is diffeomorphic to S^3 or the connected sum of a finite number of copies of $S^1 \times S^2$. For details, see [2].

Remark 4.3. We can also show that if M is homeomorphic to $L \times \mathbf{R}$ for some connected closed orientable 3-manifold L and M admits a proper special generic map into $\mathbf{R}^2$, then L is diffeomorphic to S^3 and M is diffeomorphic to $S^3 \times \mathbf{R}$.

The following conjecture seems plausible.

Conjecture 4.4. For a topological 4-manifold M, there exists at most one differentiable structure on M that allows the existence of a proper special generic map into $\mathbf{R}^3$.

Remark 4.5. In the above conjecture, the properness of the special generic map is essential. Suppose that $f : M \to N$ is a special generic map of a smooth open 4-manifold M into a smooth manifold N with $\dim N < 4$. Let us consider a homeomorphism $h : M' \to M$, where M' is another smooth open 4-manifold. Then, by using h, we can construct a "formal solution" over M' on the jet level for the open differential relation corresponding to special generic maps (see [11]). Then, by virtue of the Gromov h-principle for open manifolds, we see that M' also admits a special generic map into N. Note that even if the original special generic map f is proper, the resulting special generic map on M' may not be proper.

Compare this with the situation in Remark 3.13, where the target has dimension four. In the equidimensional case, the C^0 dense h-principle holds for special generic maps and the properness can be preserved (see [11]).

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Théorèmes d'annulation et groupes de Picard

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Résumé

Dans cet article nous donnons des théorèmes du type de Lefschetz pour les groupes de Picard des variétés quasi-projectives. En particulier pour leur démonstration nous démontrons une généralisation du théorème d'annulation de Kodaira que nous interprétons comme un théorème de Lefschetz pour le faisceau structural.

Abstract

In this paper we give Lefschetz type theorems for Picard groups of quasi-projective varieties. In particular we prove a generalization of the Kodaira vanishing theorem that we understand as a Lefschetz theorem for the structural sheaf.

Introduction

Dans SGA2 ([9]) A. Grothendieck étudie le théorème de Lefschetz sur les sections hyperplanes pour la cohomologie des faisceaux cohérents sur les schémas projectifs. Il considère également en topologie le groupe fondamental et en géométrie algébrique le groupe de Picard. En particulier il démontre l'isomorphisme entre le groupe de Picard d'un schéma algébrique projectif et celui d'une section hyperplane sous une hypothèse *ad hoc* d'annulation de cohomologie analogue à celle du théorème de Kodaira.

Dans cet article nous étudions systématiquement le comportement du groupe de Picard d'une variété algébrique complexe quasi-projective par section hyperplane. Nous obtenons des énoncés pour des variétés algébriques complexes quasi-projectives mais nous faisons des hypothèses topologiques et analytiques. Une étape importante est la démonstration d'un théorème d'annulation qui généralise à notre situation le théorème d'annulation de Kodaira et qui est équivalent à un théorème du type de Lefschetz pour le faisceau structural.

Nous donnons trois méthodes pour établir des théorèmes du type de Lefschetz pour le groupe de Picard. Outre le calcul de Grothendieck, les méthodes utilisées sont les suivantes. L'une d'elles consiste à se ramener au cas où la variété est quasi-projective et non singulière. L'autre identifie sous certaines hypothèses le groupe de Picard algébrique et le groupe de Picard analytique que l'on étudie à l'aide de la suite exacte exponentielle, puis, comme dans [16], où l' on étudie le cas projectif, on utilise des théorèmes du type de Lefschetz pour la cohomologie à coefficients entiers et pour la cohomologie du faisceau structural. Nous utilisons essentiellement des méthodes transcendantes, mais les résultats de P. Deligne et L. Illusie dans [4] permettent d'espérer que les résultats basés sur la méthode de Grothendieck s'étendent aux corps de caractéristique zéro.

1 Théorème de Lefschetz singulier pour le groupe de Picard

Soit Y un espace analytique complexe réduit (resp. une variété algébrique complexe). Soit $\mathcal{O}_Y$ son faisceau structural. Soit m un entier. On note (cf [2] Part II §2):

$$S_m(\mathcal{O}_Y) = \{x \in Y \,|\, \mathrm{prof}\, \mathcal{O}_{Y,x} \leq m\},$$

où prof $\mathcal{O}_{Y,x}$ désigne la profondeur de l'anneau local $\mathcal{O}_{Y,x}$ (dans le cas algébrique pour définir $S_m(\mathcal{O}_Y)$ on ne considère que les points fermés x de Y). D'après un théorème de G. Scheja ([27]), $S_m(\mathcal{O}_Y)$ est un sous-espace analytique fermé de Y.

Pour un sous-espace analytique fermé A, on peut prendre pour définition:

$$\mathrm{prof}_A \mathcal{O}_Y \geq n :\Leftrightarrow \dim(A \cap S_{\ell+n}(\mathcal{O}_Y)) \leq \ell, \forall \ell,$$

en convenant $\dim(\emptyset) = -\infty$. On a le théorème suivant (voir [2] Part II Theorem 3.6):

1.1 Théorème. *Soit Y un espace analytique, A un sous-espace analytique fermé de Y. Pour tout entier $n \geq 1$, les conditions suivantes sont équivalentes:*

1. $\mathrm{prof}_A \mathcal{O}_Y \geq n$;
2. *pour tout ouvert U de Y, on a:*

$$H^i_{A\cap U}(U, \mathcal{O}_Y) = 0$$

pour $i < n$.

Ce théorème montre clairement comment une condition sur la profondeur se traduit par l'annulation de cohomologies. Remarquons que la condition 2 du théorème ci-dessus peut s'écrire:

2 bis. pour tout ouvert U de Y, le morphisme naturel:

$$H^i(U, \mathcal{O}_Y) \to H^i(U \setminus A, \mathcal{O}_Y)$$

est bijectif pour $i \leq n-2$ et injectif pour $i = n-1$.

Rappelons que le groupe de Picard d'un espace annelé est le groupe des classes d'isomorphismes des faisceaux inversibles sur l'espace annelé. Donc, le groupe de Picard d'une variété algébrique est le groupe des classes d'isomorphismes des faisceaux inversibles sur la variété. Le groupe de Picard analytique est le groupe des classes d'isomorphismes des faisceaux analytiques inversibles sur l'espace analytique considéré.

De façon analogue au cas du théorème de Lefschetz sur les sections hyperplanes (voir [20]), nous avons un théorème du type de Lefschetz pour le groupe de Picard dans lequel nous avons des hypothèses de profondeur, i.e. des hypothèses d'annulation sur certaines cohomologies:

1.2 Théorème. *Soit X une variété projective complexe dans $\mathbb{P}_m$, Z un sous-espace fermé. Fixons une stratification de Whitney de X qui soit compatible avec Z et $\operatorname{Sing} X$. Soit H un hyperplan de $\mathbb{P}_m$ qui coupe X transversalement au sens stratifié. On fait l'hypothèse (H) suivante:*

$$\dim(X \setminus Z) \geq 4, \ \operatorname{prof}_{(\operatorname{Sing} X^{an} \setminus Z^{an})} \mathcal{O}_{(X^{an} \setminus Z^{an})} \geq 3$$

et on suppose de plus que $H^3(X^{an}, X^{an} \setminus \{x\}; \mathbb{Z}) = 0$ pour tout $x \in X^{an} \setminus Z^{an}$. Alors

$$\operatorname{Pic}(X \setminus Z) \simeq \operatorname{Pic}(X \cap H \setminus Z).$$

Démonstration: Comme X est quasi-projectif, d'après [10] Prop. 21.3.3 et Cor. 2.3.5, le groupe de Picard de $X \setminus Z$ est isomorphe au groupe des classes de diviseurs de Cartier $\operatorname{CaCl}(X \setminus Z)$:

$$\operatorname{Pic}(X \setminus Z) \simeq \operatorname{CaCl}(X \setminus Z).$$

La même chose vaut pour $X \cap H \setminus Z$. En plus, l'application canonique

$$\operatorname{CaCl}(X \setminus Z) \longrightarrow \operatorname{Cl}(X \setminus Z)$$

dans le groupe des classes de diviseurs de Weil est injective, car avec les hypothèses du théorème, $X \setminus Z$ est normal (voir la Proposition 21.3.4b) et le Corollaire 21.6.10 de [10] ou aussi Lemma 2.2 de [16]).

D'après la définition de S_m avec $m = \dim(X^{an} \setminus Z^{an})$, on a évidemment:

$$S_{\dim (X^{an} \setminus Z^{an})}(\mathcal{O}_{(X^{an} \setminus Z^{an})}) = X^{an} \setminus Z^{an}$$

Pour $\ell = \dim\, X^{an} \setminus Z^{an} - 3$, l'hypothèse (H) implique que

$$\dim(\operatorname{Sing}(X \setminus Z)) \leq \dim(X^{an} \setminus Z^{an}) - 3,$$

c'est à dire

$$\operatorname{codim}_{(X \setminus Z)} \operatorname{Sing}(X \setminus Z) \geq 3 \text{ (donc} \geq 2).$$

Par [17] Theorem 1.5, comme $\operatorname{codim}_{(X \setminus Z)} \operatorname{Sing}(X \setminus Z) \geq 2$ nous savons que les groupes de classes de diviseurs de Weil $\operatorname{Cl}(X \setminus Z)$ et $\operatorname{Cl}(X \cap H \setminus Z)$ sont isomorphes:

$$\operatorname{Cl}(X \setminus Z) \simeq \operatorname{Cl}(X \cap H \setminus Z).$$

Ceci implique, en considérant le diagramme commutatif suivant:

$$\begin{array}{ccccccc}
\operatorname{Pic}(X \setminus Z) & \simeq & \operatorname{CaCl}(X \setminus Z) & \rightarrow & \operatorname{CaCl}(X \cap H \setminus Z) & \simeq & \operatorname{Pic}(X \cap H \setminus Z) \\
 & & \downarrow & & \downarrow & & \\
 & & \operatorname{Cl}(X \setminus Z) & \simeq & \operatorname{Cl}(X \cap H \setminus Z) & &
\end{array}$$

que l'application canonique $\mathrm{CaCl}(X \setminus Z) \longrightarrow \mathrm{CaCl}(X \cap H \setminus Z)$ est injective puisqu'on vient de voir que $\mathrm{CaCl}(X \setminus Z) \longrightarrow \mathrm{Cl}(X \setminus Z)$ est injective. Il reste donc à démontrer la surjectivité de

$$h : \mathrm{Pic}(X \setminus Z) \longrightarrow \mathrm{Pic}(X \cap H \setminus Z)$$

Soit $[D_0] \in \mathrm{CaCl}(X \cap H \setminus Z) \simeq \mathrm{Pic}(X \cap H \setminus Z)$. Comme on vient de voir que

$$\mathrm{Cl}(X \setminus Z) \simeq \mathrm{Cl}(X \cap H \setminus Z),$$

la classe $[D_0]$ a une image inverse $[D]$ dans $\mathrm{Cl}(X \setminus Z)$. Il faut démontrer que $[D]$ est dans l'image de $\mathrm{CaCl}(X \setminus Z)$. Comme $X \setminus Z$ est normal, il suffit de démontrer que le faisceau $\mathcal{O}_{(X \setminus Z)}(D)$ des germes de sections méromorphes de $X \setminus Z$ dont la valuation le long d'une composante D_i de D est $\geq -n_i$, où n_i est la multiplicité de D_i dans D (voir [23] Definition p. 126), est inversible, ou bien que les fibres de $\mathcal{O}_{(X \setminus Z)}(D)$ sont isomorphes à celles de $\mathcal{O}_{(X \setminus Z)}$. En effet, comme le faisceau $\mathcal{O}_{(X \setminus Z)}(D)$ est engendré sur la partie non singulière de $X \setminus Z$ par des équations locales de D, le diviseur de Cartier ainsi associé au faisceau inversible $\mathcal{O}_{(X \setminus Z)}(D)$ donne un diviseur de Weil qui coïncide avec le diviseur D sur la partie non-singulière de $X \setminus Z$, ce qui établit notre assertion car $X \setminus Z$ est normal.

Pour démontrer que le faisceau $\mathcal{O}_{(X \setminus Z)}(D)$ est inversible on considère l'espace analytique X^{an} sous-jacent à la variété X. Par fidèle platitude, nous sommes emmenés à démontrer que les fibres de $\mathcal{O}_{(X^{an} \setminus Z^{an})}(D^{an})$ sont libres de rang 1, c'est-à-dire que $\mathcal{O}_{(X^{an} \setminus Z^{an})}(D^{an})$ est inversible.

Soit Σ le lieu où $\mathcal{O}_{(X^{an} \setminus Z^{an})}(D^{an})$ n'est pas libre de rang 1. L'ensemble Σ est un sous-espace algébrique fermé de $X^{an} \setminus Z^{an}$ car il est défini par un idéal de Fitting du $\mathcal{O}_{(X^{an} \setminus Z^{an})}$-module $\mathcal{O}_{(X^{an} \setminus Z^{an})}(D^{an})$.

Comme H est transverse à une stratification de Whitney de X (adaptée à Z et $\mathrm{Sing}\, X$), l'inclusion de $X \cap H$ dans X est normalement non-singulière au sens de §1.1 de [12]. On peut donc choisir un voisinage tubulaire ouvert V de $X^{an} \cap H^{an}$ dans X^{an} au sens stratifié. Soit $\pi : V \longrightarrow X^{an} \cap H^{an}$ la rétraction correspondante.

Nous allons montrer que $\Sigma \cap V = \emptyset$.

Soit $z \in X^{an} \cap H^{an}$ et W un voisinage de z dans $X^{an} \cap H^{an}$ tel que $\pi^{-1}(W)$ soit homéomorphe à $W \times N$, avec $N := \pi^{-1}(z)$. Remarquons que N est homéomorphe à un disque. Pour $z' \in W \setminus Z^{an}$, comme par hypothèse $\mathcal{O}_{((X^{an} \setminus Z^{an}) \cap H^{an})}(D_0^{an})$ est localement libre, il existe un voisinage W' de z' dans $W \setminus Z^{an}$ tel que $\mathcal{O}_{((X^{an} \setminus Z^{an}) \cap H^{an})}(D_0^{an})_{|W'}$ soit trivial. Par ailleurs, comme D^{an} est un diviseur de Weil, sur la partie non-singulière $X^{an} \setminus (\mathrm{Sing}\, X^{an} \cup Z^{an})$ le faisceau

$$\mathcal{O}_{(X^{an} \setminus Z^{an})}(D^{an})_{|X^{an} \setminus (\mathrm{Sing}\, X^{an} \cup Z^{an})}$$

est localement trivial. Soit x un point de $\pi^{-1}(W')$, il existe un voisinage de Stein V' de x dans $\pi^{-1}(W')$. Donc, $H^1(V', \mathcal{O}_{V'}) = 0$. Ceci implique que:

1.3 Lemme. *Le faisceau $\mathcal{O}_{(X^{an} \setminus Z^{an})}(D^{an})_{|V' \setminus \mathrm{Sing}\, X^{an}}$ est trivial.*

Preuve : Le faisceau $\mathcal{O}_{(X^{an} \setminus Z^{an})}(D^{an})_{|V' \setminus \mathrm{Sing}\, X^{an}}$ est inversible. Sa classe ξ dans le groupe de Picard analytique $\mathrm{Pic}_{(an)}(V' \setminus \mathrm{Sing}\, X^{an})$ est en fait nulle. Ceci provient de l'étude que nous allons faire de la suite exacte suivante :

$$H^1(V' \setminus \mathrm{Sing}\, X^{an}, \mathcal{O}_{V'}) \to \mathrm{Pic}_{(an)}(V' \setminus \mathrm{Sing}\, X^{an}) \to H^2(V' \setminus \mathrm{Sing}\, X^{an}, \mathbb{Z}).$$

Comme le faisceau $\mathcal{O}_{((X^{an}\setminus Z^{an})\cap H^{an}))}(D_0^{an})_{|W'}$ est trivial, la classe de Chern

$$c_1(\mathcal{O}_{((X^{an}\setminus Z^{an})\cap H^{an})}(D_0^{an})_{|W'}) = 0.$$

Par conséquent, on a aussi par restriction :

$$c_1(\mathcal{O}_{((X^{an}\setminus Z^{an})\cap H^{an})}(D_0^{an})_{|W'\setminus \operatorname{Sing} X^{an}}) = 0.$$

Comme H est un hyperplan assez général, le premier lemme d'isotopie de Thom (voir [25]) implique que l'espace $\pi^{-1}(W') \setminus \operatorname{Sing} X^{an}$ est homéomorphe au produit $(W' \setminus \operatorname{Sing} X^{an}) \times N$. La cohomologie $H^2(W' \setminus \operatorname{Sing} X^{an}, \mathbb{Z})$ est isomorphe par π^* à la cohomologie $H^2(\pi^{-1}(W') \setminus \operatorname{Sing} X^{an}, \mathbb{Z})$. Par $\pi*$ la classe de Chern

$$c_1(\mathcal{O}_{(X^{an}\setminus Z^{an})}(D^{an})_{|\pi^{-1}(W')\setminus \operatorname{Sing} X^{an}})$$

a pour image la classe de la restriction de $\mathcal{O}_{(X^{an}\setminus Z^{an})}(D^{an})_{|\pi^{-1}(W')\setminus \operatorname{Sing} X^{an}}$ à H^{an}. On vient de voir qu'elle est nulle. Comme π^* est un isomorphisme, on a:

$$c_1(\mathcal{O}_{(X^{an}\setminus Z^{an})}(D^{an})_{|\pi^{-1}(W')\setminus \operatorname{Sing} X^{an}}) = 0$$

ce qui donne par restriction :

$$c_1(\mathcal{O}_{(X^{an}\setminus Z^{an})}(D^{an})_{|V'\setminus \operatorname{Sing} X^{an}}) = 0.$$

La suite exacte ci-dessus montre alors que la classe ξ est dans l'image du groupe:

$$H^1(V' \setminus \operatorname{Sing} X^{an}, \mathcal{O}_{V'}).$$

D'autre part d'après le Théorème 3.6 du Chap.2 de [2] (voir 1.1), l'hypothèse (H) sur la profondeur

$$\dim(\operatorname{Sing}(X \setminus Z) \cap S_{\ell+3}(\mathcal{O}_{X^{an}\setminus Z^{an}})) \leq \ell$$

donne que le groupe de cohomologie $H^1(V' \setminus \operatorname{Sing} X^{an}, \mathcal{O}_{V'})$ est isomorphe au groupe $H^1(V', \mathcal{O}_{V'})$ qui est nul. Ceci implique que ξ est nul et que le faisceau inversible restriction de $(\mathcal{O}_{X^{an}\setminus Z^{an}})(D^{an})$ à $V' \setminus \operatorname{Sing} X^{an}$ est trivial.

Suite de la démonstration de Théorème 1.2: Grâce au Lemme 1.3, il y a une fonction méromorphe f sur $V' \setminus \operatorname{Sing} X^{an}$ dont le diviseur coïncide avec $D^{an} \cap V' \setminus \operatorname{Sing} X^{an}$. Soit U un voisinage ouvert de x dans V'. D'après le théorème 1.1 l'hypothèse (H) implique que

$$H^1(U, U \setminus \operatorname{Sing} X^{an}; \mathcal{O}_{X^{an}}) = 0,$$

ce qui implique que V' est localement irréductible en x. Cette irréducibilité locale est aussi conséquence de la normalité de $X \setminus Z$ donnée par les hypothèses du Théorème. Quitte à rétrécir V' nous pouvons donc supposer que V' est irréductible. Or, comme ci-dessus, l'espace $V' \cap \operatorname{Sing} X^{an}$ est de codimension $\geq 3 > 2$ à cause de l'hypothèse (H) . Il y a donc une extension méromorphe $\hat{f}$ de f sur V', voir [24] §53A.9. Le diviseur de $\hat{f}$ doit coïncider avec $D^{an} \cap V'$. Le faisceau $(\mathcal{O}_{X^{an} \setminus Z^{an}})(D^{an})_{|V'}$ est donc trivial.

Ceci démontre que la restriction $(\mathcal{O}_{X^{an} \setminus Z^{an}})(D^{an})_{|V \setminus Z^{an}}$ est inversible.

Par conséquent on a bien obtenu $\Sigma \cap V = \emptyset$.

Ceci implique que Σ est de dimension 0, donc fini. D'autre part on a évidemment:

$$\Sigma \subset \operatorname{Sing}(X^{an} \setminus Z^{an}).$$

Supposons $x \in \Sigma$. Soit U un voisinage de Stein convenable du point $x \in \Sigma$ dans l'espace $X^{an} \setminus Z^{an}$ pour lequel $U \cap \Sigma = \{x\}$. Alors D^{an} définit un fibré en droites sur $U \setminus \{x\}$. On sait que $c_1(\mathcal{O}_{(X^{an} \setminus Z^{an})}(D^{an})_{|U \setminus \{x\}}) = 0$, parce que:

$$H^2(U \setminus \{x\}; \mathbb{Z}) = H^3(X^{an}, X^{an} \setminus \{x\}; \mathbb{Z}) = 0.$$

Par un raisonnement analogue à celui fait précédemment, on montre que $\mathcal{O}_{(X^{an} \setminus Z^{an})}(D^{an})$ est inversible et trivial sur $U \setminus \{x\}$. En effet, on montre que:

$$c_1(\mathcal{O}_{(X^{an} \setminus Z^{an})}(D^{an}) | U \setminus \{x\}) = 0$$

donc la classe dans $\operatorname{Pic}_{(an)}(U \setminus \{x\})$ du faisceau inversible $\mathcal{O}_{(X^{an} \setminus Z^{an})}(D^{an}) | U \setminus \{x\}$ provient du groupe de cohomologie $H^1(U \setminus \{x\}, \mathcal{O}_U)$.
D'après le Théorème 1.1 et l'hypothèse (H) qui donne prof $\mathcal{O}_{X^{an},x} \geq 3$, la cohomologie locale $H^i_{\{x\}}(U, \mathcal{O}_U) = 0$ pour $i < 3$ donc, en particulier, que le groupe de cohomologie $H^1(U, \mathcal{O}_U)$ est isomorphe à $H^1(U \setminus \{x\}, \mathcal{O}_U)$. Or l'ouvert U étant de Stein, $H^1(U, \mathcal{O}_U)$ est nul, ce qui montre que le faisceau $(\mathcal{O}_{X^{an} \setminus Z^{an}})(D^{an}) | U \setminus \{x\}$ est trivial.
Comme précédemment, comme $\operatorname{codim}_U(\{x\}) \geq 2$ ce faisceau se prolonge à U en un faisceau inversible trivial, car la fonction méromorphe qui définit son diviseur se prolonge à U et définit une extension de ce faisceau.

Ceci contredit $x \in \Sigma$. Donc $\Sigma = \emptyset$, ce qu'il fallait démontrer. □

Observation: En utilisant la Proposition 2.6 de [15], on peut montrer:

$$\operatorname{prof}_{(\operatorname{Sing} X^{an} \setminus Z^{an})} \mathcal{O}_{(X^{an} \setminus Z^{an})} \simeq \operatorname{prof}_{(\operatorname{Sing} X \setminus Z)} \mathcal{O}_{(X \setminus Z)}.$$

Remarque: Dans la démonstration précédente nous avons en fait établi (voir Lemma 2.6 de [16]) :

1.4 Proposition. *Soit X une variété algébrique complexe. Soit $x \in X$ un point fermé de X. On suppose que* $\operatorname{prof}(\mathcal{O}_{X^{an},x}) \geq 3$ *et que* $H^3(X^{an}, X^{an} \setminus \{x\}, \mathbb{Z}) = 0$, *alors X est parafactoriel en x.*

Preuve: La notion de parafactorialité a été introduite par A. Grothendieck (cf. [9] XI, §3, Définition 3.1). Il suffit de démontrer que tout faisceau inversible sur $X \setminus \{0\}$ se prolonge uniquement à isomorphisme près à X. C'est précisément ce que nous avons fait.

1.5 Corollaire. *Soient X une variété projective complexe dans $\mathbb{P}_m$ et Z un sous-espace fermé de X. On suppose que $X^{an} \setminus Z^{an}$ est localement une intersection complète de dimension ≥ 4 et* $\operatorname{Sing}(X^{an} \setminus Z^{an})$ *de codimension ≥ 3. Fixons une* *stratification de Whitney de X qui soit compatible avec Z et* $\operatorname{Sing} X$. *Soit H un hyperplan de $\mathbb{P}_m$ qui coupe X transversalement au sens stratifié. Alors*

$$\operatorname{Pic}(X \setminus Z) \simeq \operatorname{Pic}(X \cap H \setminus Z).$$

Preuve: Ce corollaire est une conséquence immédiate du théorème précédent, parce que l'hypothèse que $X^{an} \setminus Z^{an}$ est localement une intersection complète de dimension ≥ 4 implique les hypothèses du théorème 1.2, car

$$\operatorname{prof}(\mathcal{O}_{X^{an},x}) = \dim X$$

car une intersection complète est localement Cohen-Macaulay et topologiquement:

$$H^3(X^{an}, X^{an} \setminus \{x\}; \mathbb{Z}) = 0$$

si $\dim X \geq 3$.

2 Cas affine

Comme nous utilisons des méthodes analytiques, nous rappelons les points suivants.

Pour un espace analytique général il y a une différence entre le groupe des classes diviseurs de Cartier et le groupe de Picard.

Soit Y un espace analytique complexe normal. Soit $\operatorname{Cl}(Y)$ le groupe des classes de diviseurs de Weil sur Y. On remarque que $\operatorname{CaCl}(Y)$ est le sous-groupe de $\operatorname{Cl}(Y)$ formé par les éléments $[D]$ tels que le faisceau $\mathcal{O}_Y(D)$ soit inversible. On a une injection canonique $\operatorname{CaCl}(Y) \longrightarrow \operatorname{Pic}(Y)$.

On peut définir $\operatorname{CaCl}(Y)$ d'une autre façon: Soit $\mathcal{M}_Y^*$ le faisceau de germes de fonctions méromorphes non-triviales. Alors $H^0(Y, \mathcal{M}_Y^*/\mathcal{O}_Y^*)$ est l'espace des distributions multiplicatives de Cousin. On a une flèche canonique injective $H^0(Y, \mathcal{M}_Y^*/\mathcal{O}_Y^*) \longrightarrow \operatorname{Div} Y$ où $\operatorname{Div} Y$ désigne le groupe des diviseurs de Weil sur Y. La suite exacte

$$0 \longrightarrow \mathcal{O}_Y^* \longrightarrow \mathcal{M}_Y^* \longrightarrow \mathcal{M}_Y^*/\mathcal{O}_Y^* \longrightarrow 0$$

donne une suite exacte de cohomologie

$$H^0(Y, \mathcal{M}_Y^*) \xrightarrow{i} H^0(Y, \mathcal{M}_Y^*/\mathcal{O}_Y^*) \longrightarrow H^1(Y, \mathcal{O}_Y^*) \longrightarrow H^1(Y, \mathcal{M}_Y^*)$$

On a donc une injection de $Coker\, i$ dans $\operatorname{Pic}(Y) = H^1(Y, \mathcal{O}_Y^*)$ qui est bijective si $H^1(Y, \mathcal{M}_Y^*) = 0$.
De plus, on obtient une flèche injective $Coker\, i \longrightarrow \operatorname{Cl}(Y)$. Évidemment, on peut identifier $Coker\, i$ avec $\operatorname{CaCl}(Y)$.

Il y a un cas où il y a une coincidence de $\operatorname{CaCl}(Y)$ et de $\operatorname{Pic}(Y)$:

2.1 Lemme. *Si Y est un espace analytique de Stein on a:*

$$\operatorname{CaCl}(Y) \simeq \operatorname{Pic}(Y) \simeq H^2(Y; \mathbb{Z}).$$

Démonstration: D'après [24] Cor. 54.8, $\mathrm{CaCl}(Y) \simeq H^2(Y;\mathbb{Z})$. Comme Y est de Stein, $H^1(Y, \mathcal{O}_Y) = 0$ et la suite exacte exponentielle donne $\mathrm{Pic}(Y) = H^1(Y, \mathcal{O}_Y^*) \simeq H^2(Y;\mathbb{Z})$.

Rappelons la définition de la profondeur cohomologique rectifiée $\mathrm{pcr}(Z)$ d'un espace analytique complexe Z (voir Definition 1.2 of [16]). Soit:

$$\mathcal{S}_m(Z) := \{x \in Z \mid H^m(Z, Z \setminus \{x\}, \mathbb{Z}) \neq 0\}.$$

On définit

$$\mathrm{pcr}(Z) \geq n :\Leftrightarrow \dim \mathcal{S}_{n+m}(Z) \leq m, \forall m.$$

Remarquons qu'avec la définition 1.2.1 de [20] nous avons $\mathrm{pcr}(Z) \geq n$ si et seulement si le faisceau constant $\mathbb{Z}_Z$ est dans ${}^{1/2}D^{\geq n}(Z, \mathbb{Z})$ ce qui s'exprime aussi par l'annulation de cohomologies. On obtient alors:

2.2 Théorème. *Soit X une variété projective complexe dans $\mathbb{P}_m$, Z un sous-espace fermé. Supposons que $X \setminus Z$ est affine. Fixons une stratification de Whitney de X qui soit compatible avec Z et $\mathrm{Sing}\, X$. Soit H un hyperplan de $\mathbb{P}_m$ qui coupe X transversalement au sens stratifié.*

a) Supposons $\mathrm{pcr}(X^{an} \setminus Z^{an}) \geq 3$. Alors la flèche canonique

$$\mathrm{Pic}_{(an)}(X^{an} \setminus Z^{an}) \longrightarrow \mathrm{Pic}_{(an)}(X^{an} \cap H^{an} \setminus Z^{an})$$

est injective.

b) Soit $\mathrm{pcr}(X^{an} \setminus Z^{an}) \geq 4$. Alors

$$\mathrm{Pic}_{(an)}(X^{an} \setminus Z^{an}) \simeq \mathrm{Pic}_{(an)}(X^{an} \cap H^{an} \setminus Z^{an}).$$

Démonstration. Comme la variété affine $X \setminus Z$ est un espace analytique de Stein, le groupe de Picard analytique de $X \setminus Z$ est isomorphe à $H^2(X^{an} \setminus Z^{an}; \mathbb{Z})$. De même pour $X \cap H \setminus Z$. L'assertion a) revient à démontrer que l'homomorphisme

$$H^2(X^{an} \setminus Z^{an}; \mathbb{Z}) \to H^2(X^{an} \cap H^{an} \setminus Z^{an}; \mathbb{Z})$$

est injectif. Comme $\mathrm{pcr}(X^{an} \setminus Z^{an}) \geq 3$, cette assertion est un théorème du type de Lefschetz comme dans [19].

L'assertion b) revient à montrer que l'homorphisme précédent est surjectif, ce qui est impliqué par $\mathrm{pcr}(X^{an} \setminus Z^{an}) \geq 4$, d'après les théorèmes du type de Lefschetz de [19].

3 Comparaison entre les cas algébriques et analytiques

Dans le cas où Z est de codimension ≥ 2 (dans ce cas $Z \cap H$ est aussi de codimension ≥ 2 dans $X \cap H$):

3.1 Lemme. *Soient X une variété projective complexe, Z un sous-espace fermé de codimension ≥ 2.*
a) $\mathrm{Cl}(X^{an} \setminus Z^{an}) \simeq \mathrm{Cl}(X \setminus Z)$ *et* $\mathrm{CaCl}(X^{an} \setminus Z^{an}) \simeq \mathrm{Pic}(X \setminus Z)$.
b) S'il existe un voisinage ouvert $U(Z)$ de Z dans X pour lequel

$$\dim U(Z) \cap S_{k+2}(\mathcal{O}_{X^{an}\setminus Z^{an}}) \leq k$$

pour tout $k \leq \dim Z$, on a

$$\mathrm{Pic}_{(an)}(X^{an} \setminus Z^{an}) \simeq \mathrm{Pic}(X \setminus Z).$$

Démonstration: a) La flèche $\mathrm{Cl}(X \setminus Z) \longrightarrow \mathrm{Cl}(X^{an} \setminus Z^{an})$ est bijective.
En effet, par le théorème de Remmert-Stein ([13] p.169), chaque diviseur de $X^{an} \setminus Z^{an}$ peut être étendu à X^{an}. Par le théorème de Chow l'extension est algébrique. Il en résulte que les groupes des diviseurs de Weil $\mathrm{Div}(X^{an} \setminus Z^{an}) \simeq \mathrm{Div}(X) \simeq \mathrm{Div}(X \setminus Z)$ sont isomorphes. Comme chaque fonction méromorphe sur $X^{an} \setminus Z^{an}$ peut être étendue à X^{an} par [24] 53.A.9 et que l'extension est algébrique par le théorème de Hurwitz [7] 4.7 on a $\mathrm{Cl}(X \setminus Z) \simeq \mathrm{Cl}(X^{an} \setminus Z^{an})$.

Comme $\mathrm{CaCl}(X \setminus Z)$ correspond au sous-groupe de $\mathrm{Cl}(X \setminus Z)$ formé par les éléments $[D]$ tels que $\mathcal{O}_{X\setminus Z}(D)$ soit inversible, on obtient que

$$\mathrm{Pic}(X \setminus Z) \simeq \mathrm{CaCl}(X \setminus Z) \simeq \mathrm{CaCl}(X^{an} \setminus Z^{an}).$$

b) La flèche $\mathrm{Pic}(X \setminus Z) \longrightarrow \mathrm{Pic}_{(an)}(X^{an} \setminus Z^{an})$ est surjective. En effet soit $\mathcal{L}$ un faisceau inversible sur $X^{an} \setminus Z^{an}$ et $j : X^{an} \setminus Z^{an} \longrightarrow X^{an}$ l'inclusion. Alors $j_*\mathcal{L}$ est cohérent à cause de l'hypothèse sur la profondeur, voir [8] Cor. VII.4 ou [31] Theorem 2 (voir la remarque à la fin de l'article); par GAGA ce faisceau provient d'un faisceau algébrique cohérent $\mathcal{F}$ sur X. Alors la restriction $\mathcal{F}_{|X\setminus Z}$ est inversible et représente l'image inverse cherchée.
La composition

$$\mathrm{Pic}(X \setminus Z) \longrightarrow \mathrm{CaCl}(X \setminus Z) \longrightarrow \mathrm{CaCl}(X^{an} \setminus Z^{an}) \longrightarrow \mathrm{Pic}_{(an)}(X^{an} \setminus Z^{an})$$

montre que $\mathrm{Pic}(X \setminus Z) \longrightarrow \mathrm{Pic}_{(an)}(X^{an} \setminus Z^{an})$ est injective parce que les premières flèches sont bijectives et la troisième injective (§2), comme on l'a vu ci-dessus.

Dans la démonstration du lemme précédent on ne suppose pas qu'une variété algébrique est irréductible.

3.2 Corollaire. *Sous les hypothèses du Théorème 1.2, si* $\mathrm{codim}_X Z \geq 3$, *on a:*

$$\mathrm{Pic}_{(an)}(X^{an} \setminus Z^{an}) \simeq \mathrm{Pic}(X \setminus Z) \simeq Pic(X \cap H \setminus Z) \simeq \mathrm{Pic}_{(an)}(X^{an} \cap H^{an} \setminus Z^{an}).$$

Démonstration: Si $k + 2 < \dim(X^{an} \setminus Z^{an})$, on a évidemment

$$S_{k+2}(\mathcal{O}_{X^{an}\setminus Z^{an}}) \subset \mathrm{Sing}(X^{an} \setminus Z^{an}),$$

donc $\dim S_{k+2}(\mathcal{O}_{X^{an}\setminus Z^{an}}) \leq k - 1$, à cause de l'hypothèse (H) du théorème 1.2.
Pour $k + 2 \geq \dim(X^{an} \setminus Z^{an})$ nous avons $k > \dim Z$ car on a supposé $\mathrm{codim}_X Z \geq 3$.

Nous pouvons donc appliquer le lemme 3.1 et ceci nous donne le corollaire. $\square$

Ce qui se passe dans le cas où Z est de codimension ≤ 2 sans que $X \setminus Z$ soit affine n'est pas clair.

4 Théorème de Lefschetz pour le faisceau structural

Du paragraphe qui précède nous pouvons déduire un théorème du type de Lefschetz pour le faisceau structural:

4.1 Théorème. *Sous les hypothèses de Théorème 1.2, supposons en plus que la profondeur cohomologique rectifiée vérifie l'inégalité* $\mathrm{pcr}_{\mathbb{Z}}(X^{an} \setminus Z^{an}) \geq 3$ *et* $\mathrm{codim}_X Z \geq 3$. *Alors*

$$H^1(X^{an} \setminus Z^{an}, \mathcal{O}_{X^{an}}) \simeq H^1(X^{an} \cap H^{an} \setminus Z^{an}, \mathcal{O}_{X^{an} \cap Z^{an}}).$$

Démonstration: On applique le lemme des cinq à la suite exacte exponentielle. D'abord, on a

$$H^1(X^{an} \setminus Z^{an}, \mathcal{O}^*_{X^{an}}) \simeq H^1(X^{an} \cap H^{an} \setminus Z^{an}, \mathcal{O}^*_{X^{an} \cap H^{an}})$$

à cause du Corollaire 3.2. Le théorème de Lefschetz géométrique (cf. [18]) donne que l'homomorphisme

$$H^k(X^{an} \setminus Z^{an}; \mathbb{Z}) \longrightarrow H^k(X^{an} \cap H^{an} \setminus Z^{an}; \mathbb{Z})$$

est bijectif pour $k = 1$ et injectif pour $k = 2$. Il reste à démontrer la surjectivité de

$$H^0(X^{an} \setminus Z^{an}, \mathcal{O}^*_{X^{an}}) \to H^0(X^{an} \cap H^{an} \setminus Z^{an}, \mathcal{O}^*_{X^{an} \cap H^{an}}).$$

Soit f une section de $H^0(X^{an} \cap H^{an} \setminus Z^{an}, \mathcal{O}^*_{X^{an} \cap H^{an}})$. L'hypothèse (H) du théorème 1.2 implique (par normalité) que f s'étend à $X^{an} \cap H^{an}$. Comme $X^{an} \cap H^{an}$ est compact, f est localement constante et la surjectivité provient de l'isomorphisme de Lefschetz géométrique ci-dessus quand $k = 0$. $\square$

Nous allons maintenant donner un théorème d'annulation qui nous donnera une généralisation du théorème de Kodaira et nous permettra d'étudier le groupe de Picard.

La notion de la profondeur par rapport à un sous-espace introduite dans le §1 peut se généraliser aux faisceaux cohérents de la façon suivante:

Soit X une variété algébrique complexe, Y un sous-espace fermé. Soit $\mathcal{F}$ un faisceau algébrique cohérent sur X. On définit:

$$\mathrm{prof}_Y \mathcal{F} \geq n \Longleftrightarrow \dim\{x \in Y,\ x \text{ point fermé de } Y \mid \mathrm{prof}\, \mathcal{F}_x \leq n+k\} \leq k \ \text{ pour tout } k.$$

Ici $\mathrm{prof}\, \mathcal{F}_x$ est la profondeur du $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$ (cf. [28]).

D'abord on a le théorème suivant qui est bien connu dans le cas $Z = \emptyset$ (cf. [9] Exposé XII, corollaire 1.4):

4.2 Théorème. *Soient X une variété projective complexe, Z un sous-espace algébrique fermé, $\mathcal{S}$ un faisceau algébrique cohérent sur X,* $\mathrm{prof}\, \mathcal{S}_{|X \setminus Z} \geq n$. *Soit $\mathcal{L}$ un faisceau ample sur X. Alors*

$$H^q(X \setminus Z, \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{L}^{-l}) = 0$$

pour $q < n - \max(\dim Z, -1) - 1$, $l \gg 0$.

Démonstration: Soit $X \subset \mathbb{P}_m$ et $i : X \longrightarrow \mathbb{P}_m$ l'inclusion. En remplaçant $\mathcal{S}$ par $i_*\mathcal{S}$ on peut supposer que $X = \mathbb{P}_m$. Soit $d := \dim Z$.
Notons que $n \leq m$. On procède par récurrence sur $m - n$.
Si $n = m$ on a que la restriction $\mathcal{S}_{|\mathbb{P}_m \setminus Z}$ est localement libre, car $\mathbb{P}_m \setminus Z$ est lisse. En effet, soit x un point fermé de $\mathbb{P}_m \setminus Z$, alors $\operatorname{prof} \mathcal{S}_x + \operatorname{dh} \mathcal{S}_x = m$, où dh désigne la dimension homologique (voir [28] IV D Prop. 21); comme $\operatorname{prof} \mathcal{S}_x = m$, $\mathcal{S}_x$ est projectif, donc libre (voir [28] IV Prop. 20).

On remarque que sur $\mathbb{P}_m \setminus Z$ on a

$$\mathcal{S}_{|\mathbb{P}_m \setminus Z} \simeq Hom(Hom(\mathcal{S}, \mathcal{O}_{\mathbb{P}_m}), \mathcal{O}_{\mathbb{P}_m})_{|\mathbb{P}_m \setminus Z}$$

Soit $\mathcal{S}' = Hom(\mathcal{S}, \mathcal{O}_{\mathbb{P}_m})$. Choisissons une résolution localement libre à gauche $\mathcal{F}'_*$ de $\mathcal{S}'$ (cf. [21] Corollary II 5.18). Comme $\mathcal{S}'_{|\mathbb{P}_m \setminus Z}$ est localement libre, le faisceau $Ext^q(\mathcal{S}', \mathcal{O}_{\mathbb{P}_m})$ est concentré sur Z pour $q > 0$, donc

$$0 \to Hom(\mathcal{S}', \mathcal{O}_{\mathbb{P}_m})_{|\mathbb{P}_m \setminus Z} \to \mathcal{F}^0_{|\mathbb{P}_m \setminus Z} \to \mathcal{F}^1_{|\mathbb{P}_m \setminus Z} \to \dots$$

avec $\mathcal{F}^q := Hom(\mathcal{F}'_q, \mathcal{O}_{\mathbb{P}_m})$ est exacte. Or on a

$$H^q(\mathbb{P}_m \setminus Z, \mathcal{S} \otimes \mathcal{L}^{-l}) = H^q(\mathbb{P}_m \setminus Z, Hom(\mathcal{S}', \mathcal{O}_{\mathbb{P}_m}) \otimes \mathcal{L}^{-l})$$

qui est isomorphe à l'hypercohomologie $\mathbb{H}^q(\mathbb{P}_m \setminus Z, \mathcal{F}^* \otimes \mathcal{L}^{-l})$.

D'autre part, pour $k \geq 0$, dans [9] III Lemme 3.1, comme $\operatorname{prof}_Z \mathcal{F}^k \otimes \mathcal{L}^{-l} \geq m - \dim Z$ un résultat de A. Grothendieck donne que $H^q(\mathbb{P}_m \setminus Z, \mathcal{F}^k \otimes \mathcal{L}^{-l}) \simeq H^q(\mathbb{P}_m, \mathcal{F}^k \otimes \mathcal{L}^{-l})$, pour $q < m - \dim Z - 1$. Un autre résultat de Grothendieck (voir [9] XII Corollaire 1.4) donne

$$H^q(\mathbb{P}_m, \mathcal{F}^k \otimes \mathcal{L}^{-l}) = 0$$

pour $q \leq m$ et $l \gg 0$. Une suite spectrale donne l'annulation de l'hypercohomologie :

$$\mathbb{H}^q(\mathbb{P}_m \setminus Z, \mathcal{F}^* \otimes \mathcal{L}^{-l}) = 0$$

pour $q < m - \dim Z - 1$ et $l \gg 0$.

Par conséquent

$$H^q(\mathbb{P}_m \setminus Z, \mathcal{S} \otimes \mathcal{L}^{-l}) = 0$$

pour $q < m - \dim Z - 1$ et $l \gg 0$.

Soit $n < m$. Il y a une suite exacte $0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{F} \longrightarrow \mathcal{S} \longrightarrow 0$ avec $\mathcal{F}$ localement libre (voir e.g. [21] Cor. 5.18 Chap II). On a $\operatorname{prof} \mathcal{T}_{|\mathbb{P}_m \setminus Z} \geq n + 1$ (voir [2] I Cor. 1.13) et $\operatorname{prof} \mathcal{F} = m$. La suite exacte de cohomologie et l'hypothèse de récurrence donnent le résultat cherché.

Le théorème suivant a déjà été énoncé dans le cas $Z = \emptyset$ avec une démonstration par Anapura et Jaffe [1] (Proposition 1.1), par une démonstration différente :

4.3 Théorème. *Soient X une variété projective complexe dans $\mathbb{P}_m$, Z un sous-espace fermé, H un hyperplan dans $\mathbb{P}_m$ qui ne contient pas X. Soit $n \in \mathbb{N}$:*

$$\operatorname{codim}_X Z \geq n + 1 \ \textit{ et } \operatorname{prof}_{\operatorname{Sing}(X \setminus Z)} \mathcal{O}_{X \setminus Z} \geq n,$$

$\dim X \geq n$. *Alors*

$$H^q(X \setminus Z, \mathcal{O}_{X\setminus Z}) \longrightarrow H^q(X \cap H \setminus Z, \mathcal{O}_{X\cap H\setminus Z})$$

est bijectif pour $q < n-1$ *et injectif pour* $q = n-1$.

Démonstration: Soit $\mathcal{I} = \mathcal{O}_X(-H)$ l'idéal de H dans $\mathcal{O}_X$. Il faut démontrer que

$$H^q(X \setminus Z, \mathcal{I}) = 0,\ q < n.$$

On peut remplacer H par un hyperplan générique L car

$$H^q(X \setminus Z, \mathcal{O}_X(-H)) \simeq H^q(X \setminus Z, \mathcal{O}_X(-L)).$$

On va procéder par récurrence sur $\dim X$.

Dans le cas $\dim X = n$, nécessairement $Z = \emptyset$. La condition sur la profondeur donne $\mathrm{prof}(\mathcal{O}_X) = n$ et signifie que X est de Cohen-Macaulay. De plus les singularités de X sont isolées. Si X est lisse,

$$H^k(X, \mathcal{O}_X(-L)) = 0,$$

pour $k < n$, par le théorème de Kodaira, voir [21] p. 248, car, pour L assez général, $\mathcal{O}_X(L))$ est un faisceau très ample.

Supposons que X ait des singularités isolées. Comme L est assez général, on a que

$$H^k(X^{an} \setminus \mathrm{Sing}\, X^{an}; \mathbb{Z}) \longrightarrow H^k((X^{an} \setminus \mathrm{Sing}\, X^{an}) \cap L^{an}; \mathbb{Z})$$

est bijectif pour $k < n-1$, et injectif pour $k = n-1$, d'après le Théorème de Lefschetz de [18]. Comme les singularités sont isolées, on a

$$(X^{an} \setminus \mathrm{Sing}\, X^{an}) \cap L^{an} = X^{an} \cap L^{an},$$

c'est à dire

$$H^k(X^{an} \setminus \mathrm{Sing}\, X^{an}; \mathbb{Z}) \longrightarrow H^k(X^{an} \cap L^{an}; \mathbb{Z})$$

est bijectif pour $k < n-1$, et injectif pour $k = n-1$.

Soit $\pi : \tilde{X} \longrightarrow X$ une désingularisation de X telle que $D := \pi^{-1}(\mathrm{Sing}\, X)$ soit un diviseur à croisements normaux. Alors l'homorphisme

$$H^k(X^{an} \setminus \mathrm{Sing}\, X^{an}; \mathbb{C}) \longrightarrow H^k(X^{an} \cap L^{an}; \mathbb{C})$$

s'identifie avec l'homorphisme d'hypercohomologie

$$\mathbb{H}^k(\tilde{X}, {\Omega^*}_{\tilde{X}}(log\, D)) \longrightarrow \mathbb{H}^k(X \cap H, {\Omega^*}_{X\cap L}),$$

où $X \cap L$ est identifié à son image inverse par π puisque π est un isomorphisme en dehors des singularités isolées. Or, il s'agit d'une application de structures de Hodge mixtes; en considérant le terme Gr_F^0 en degré 0, on obtient que $H^k(\tilde{X}, \mathcal{O}_{\tilde{X}}) \longrightarrow H^k(X \cap H, \mathcal{O}_{X\cap L})$ est bijectif pour $k < n-1$ et injectif pour $k = n-1$ (Voir [3]).

Comme $H^k(\tilde{X}, \mathcal{O}_{\tilde{X}}(-L)) = H^k(\tilde{X}^{an}, \mathcal{O}_{\tilde{X}^{an}}(-L))$ d'après GAGA, ce qui précède implique que $H^k(\tilde{X}, \mathcal{O}_{\tilde{X}}(-L)) = H^k(\tilde{X}^{an}, \mathcal{O}_{\tilde{X}^{an}}(-L)) = 0$, $k < n$, où $\mathcal{O}_{\tilde{X}^{an}}(-L)$ désigne l'idéal qui définit L dans $\tilde{X}$.

Soit U un voisinage fermé convenable de $X^{an} \cap L^{an}$ dans X^{an} tel que U ne contienne pas de singularités et que $X^{an} \setminus U$ soit un ouvert de Stein. Alors (voir [2] I Theorem 3.6)

$$H^k_c(X^{an} \setminus U, \mathcal{O}_{X^{an}}(-L)) = 0, \; k < n,$$

car on a vu que $\mathrm{prof}\mathcal{O}_X = n$.

La composition des flèches

$$H^k(X^{an}, \mathcal{O}_{X^{an}}(-L)) \longrightarrow H^k(\tilde{X}^{an}, \mathcal{O}_{\tilde{X}^{an}}(-L)) \longrightarrow H^k(U, \mathcal{O}_{X^{an}}(-L))$$

est donc injective pour $k < n$; comme $H^k(\tilde{X}^{an}, \mathcal{O}_{X^{an}}(-L)) = 0$, $k < n$, par GAGA on obtient que $H^k(X, \mathcal{O}_X(-L)) \simeq H^k(X^{an}, \mathcal{O}_{X^{an}}(-L)) = 0$, $k < n$. Cette annulation est aussi obtenue grâce au théorème (7.80) de [29] et ceci est l'annulation cherchée.

Soit $\dim X > n$. Soit Y le lieu des points qui sont ou bien dans Z ou bien en lesquels la profondeur $\mathrm{prof}\, \mathcal{O}_{X,x} \leq n + d$, avec $d = \dim Z$. Alors $\dim Y = d$. En effet, comme on a supposé $\mathrm{prof}_{\mathrm{Sing}(X \setminus Z)} \mathcal{O}_{X \setminus Z} \geq n$, on a $\dim(Y \cap \mathrm{Sing}(X \setminus Z)) \leq d$. Par ailleurs $Y \cap (X \setminus Z) \setminus \mathrm{Sing}(X \setminus Z) = \emptyset$, car, pour $x \in (X \setminus Z) \setminus \mathrm{Sing}(X \setminus Z)$, la profondeur $\mathrm{prof}_x(\mathcal{O}_X) \geq \dim X \geq n + 1 + \dim Z$, car on a supposé $\mathrm{codim}_X Z \geq n + 1$. On peut se réduire au cas $Z = Y$. En effet, l'hypothèse garantit que $H^k(X \setminus Z, \mathcal{O}_X(-L)) \to H^k(X \setminus Y, \mathcal{O}_X(-L))$ est bijectif pour $k < n$ et injectif pour $k = n$ (voir [2] II Theorem 3.6.).

Supposons donc que $Z = Y$, c.-à d. que $\mathrm{prof}\, \mathcal{O}_{X \setminus Z} \geq n + d + 1$. On a choisi L générique, donc en particulier transverse aux strates d' une stratification de Whitney de (X, Z). On a $\dim X \geq n + 1$. Pour $l \gg 0$ on a $H^q(X \setminus Z, \mathcal{O}_X(-L)^l) = 0$, $q < n$, parce que l'on a $\mathrm{prof}\, \mathcal{O}_{X \setminus Z, x} \geq n + d + 1$ pour tout $x \in X \setminus Z$, d'après le théorème 2.1 précédent.

Il reste à démontrer que $H^q(X \setminus Z, \mathcal{O}_X(-L)^l / \mathcal{O}_X(-L)^{l+1}) = 0$, $q < n$, $l \geq 0$.

Mais $\mathcal{O}_X(-L)^l / \mathcal{O}_X(-L)^{l+1}$ un faisceau très ample sur L. On peut donc procéder par récurrence en montrant

$$H^q((X \setminus Z) \cap L, \mathcal{O}_X(-L)^l / \mathcal{O}_X(-L)^{l+1}) = 0$$

pour $q < n$, $l \geq 0$.

Notons que

$$\mathrm{prof}_{\mathrm{Sing}\, X \cap L \setminus Z} \mathcal{O}_{X \cap L \setminus Z} \geq \mathrm{prof}_{\mathrm{Sing}\, X \setminus Z} \mathcal{O}_{X \setminus Z} \geq n$$

et $\dim\, X \cap L \geq n$.

Un corollaire intéressant du Théorème précédent est la généralisation suivante du Théorème de Kodaira (voir e.g. Theorem 1.2 §1 de [5]) ainsi considéré comme le théorème d'annulation de cohomologie associé à notre théorème du type de Lefschetz pour le faisceau structural:

4.4 Corollaire. *Soient X une variété projective complexe, Z un sous-espace fermé, $\mathcal{L}$ un faisceau ample sur X. Soit $n \in \mathbb{N}$ et supposons que* $\operatorname{codim}_X Z \geq n+1$, $\operatorname{prof}_{\operatorname{Sing}(X \setminus Z)} \mathcal{O}_{X \setminus Z} \geq n$, $\dim X \geq n$. *Alors*

$$H^q(X \setminus Z, \mathcal{L}^{-1}) = 0$$

pour $q < n$.

Démonstration: Si on suppose que $\mathcal{L}$ est très ample, on peut plonger la variété projective X dans $\mathbb{P}_m$ de telle sorte que $\mathcal{L}$ soit la restriction à X du faisceau $\mathcal{O}_{\mathbb{P}_m}(1)$. L'annulation de $H^q(X \setminus Z, \mathcal{L}^{-1})$ équivaut donc à l'annulation de $H^q(X \setminus Z, \mathcal{I})$, où $\mathcal{I}$ est l'déal dans X d'une section hyperplane de X dans $\mathbb{P}_m$. Or cette annulation pour $q < n$ est une conséquence immédiate du Théorème 4.3.

Dans le cas général, on procède de façon analogue à [26] Lemma 1, dans le cas non-singulier. Soit $l \gg 0$ tel que $\mathcal{L}^l$ soit très ample. Fixons un tel entier l. On a dans $H^0(X, \mathcal{L}^l) - \{0\}$ une section σ de diviseur $D := [\sigma]$. Dans le fibré $\pi : L \to X$ défini par $\mathcal{L}$ sur X, la sous-variété X' des points $x \in L$ où $x^l = \sigma(\pi(x)) \in L^l$ est un revêtement cyclique

$$f : X' \to X$$

sur X ramifié le long de D. Localement L est trivial sur X donc

$$\operatorname{prof}_{\operatorname{Sing}(L \setminus \pi^{-1}(Z))} \mathcal{O}_{L \setminus \pi^{-1}(Z)} \geq n+1.$$

Comme X' est donné localement par $s - t^l = 0$ dans L, on a

$$\operatorname{prof}_{\operatorname{Sing}(X' \setminus f^{-1}(Z))} \mathcal{O}_{X' \setminus f^{-1}(Z)} \geq n.$$

On peut donc appliquer le résultat précédent à X' et $f^*(\mathcal{L}^{-1})$ qui est très ample:

$$H^q(X' \setminus f^{-1}(Z), f^*(\mathcal{L}^{-1})) = 0$$

pour $q < n$. Comme f est fini,

$$H^q(X' \setminus f^{-1}(Z), f^*(\mathcal{L}^{-1})) = H^q(X \setminus Z, f_*(f^*(\mathcal{L}^{-1}))).$$

D'autre part f est fini et galoisien et ramifié le long de D, donc la partie invariante de $f_*(f^*(\mathcal{L}^{-1}))$ sous l'action du groupe de Galois est $\mathcal{L}^{-1}$ (comparer avec [26]). Comme la partie invariante de la cohomologie est la cohomologie de la partie invariante du faisceau, on obtient le résultat cherché

$$H^q(X \setminus Z, \mathcal{L}^{-1}) = 0$$

pour $q < n$.

Remarque. Le théorème 4.3 est une conséquence immédiate du corollaire 4.4. Ils sont donc équivalents.

5 Cas des faisceaux cohérents analytiques

On a les analogues analytiques suivants des Théorèmes 4.2 et 4.3:

5.1 Théorème. *Soient X une variété projective, Z un sous-espace algébrique fermé, $\mathcal{S}$ un faisceau analytique cohérent sur X et supposons que* $\operatorname{prof} \mathcal{S}|X \setminus Z \geq n$. *Soit $\mathcal{L}$ un faisceau ample sur X. Alors*

$$H^q(X^{an} \setminus Z^{an}, \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{L}^{-l}) = 0$$

pour $q < n - \dim Z - 1$ et $l \gg 0$.

Démonstration. On procède comme dans le cas algébrique. Au lieu de [9] III Lemme 3.1 et [9] XII Cor. 1.4 on utilise [27], [33], [32], [2] II Theorem 3.6 et [2] IV Cor. 3.3, respectivement.

5.2 Théorème. *Soient X une variété projective complexe dans $\mathbb{P}_m$, Z un sous-espace fermé, H un hyperplan dans $\mathbb{P}_m$ qui ne contient aucune composante irréductible de X. Soit $n \in \mathbb{N}$ et supposons que* $\operatorname{codim}_X Z \geq n + 1$,

$$\operatorname{prof}_{\operatorname{Sing}(X^{an} \setminus Z^{an})} \mathcal{O}_{X^{an} \setminus Z^{an}} \geq n,$$

$\dim X \geq n$. *Alors*

$$H^q(X^{an} \setminus Z^{an}, \mathcal{O}_{X^{an} \setminus Z^{an}}) \longrightarrow H^q(X^{an} \cap H^{an} \setminus Z^{an}, \mathcal{O}_{X^{an} \cap H^{an} \setminus Z^{an}})$$

est bijectif pour $q < n - 1$ et injectif pour $q = n - 1$.

Démonstration: Au lieu de [9] on utilise [2] IV Theorem 3.1, de plus au lieu du théorème 4.2 on utilise le théorème 5.1 précédent.

On obtient le corollaire suivant qui est aussi obtenu par Y.T. Siu dans [30] (Theorems A and B p. 348) :

5.3 Corollaire. *Soient X une variété projective complexe dans $\mathbb{P}_m$, Z un sous-espace fermé. Soit $n \in \mathbb{N}$ et supposons que* $\operatorname{codim}_X Z \geq n + 1$, $\operatorname{prof}_{\operatorname{Sing}(X \setminus Z)} \mathcal{O}_{X \setminus Z} \geq n$, $\dim X \geq n$. *Alors*

$$H^q(X \setminus Z, \mathcal{O}_{X \setminus Z}) \simeq H^q(X^{an} \setminus Z^{an}, \mathcal{O}^{an}_{X \setminus Z})$$

pour $q < n - 1$.

Démonstration: On a un morphisme naturel

$$H^q(X \setminus Z, \mathcal{O}_{X \setminus Z}) \to H^q(X^{an} \setminus Z^{an}, \mathcal{O}_{X \setminus Z}).$$

On procède par récurrence sur $\dim X$. On a $\dim X \geq n$. Pour $\dim X = n$, on a $Z = \emptyset$, l'assertion est donc vraie dans ce cas à cause de GAGA.
Soit donc $\dim X > n$, H un hyperplan générique dans $\mathbb{P}_m$. Alors les hypothèses restent valables pour $X \cap H$ et $Z \cap H$ au lieu de X et Z, respectivement, on peut donc leur appliquer l'hypothèse de récurrence. Avec le Théorème 4.3 (resp. 5.2) on obtient le résultat cherché.
Remarque: On a aussi l'analogue du corollaire 4.4 :

5.4 Corollaire. *Soit X une variété projective complexe dans $\mathbb{P}_m$, Z un sous-espace fermé. Soit $\mathcal{L}$ un faisceau ample sur X. Soit $n \in \mathbb{N}$, $\operatorname{codim}_X Z \geq n+1$, $\operatorname{prof}_{\operatorname{Sing}(X^{an} \setminus Z^{an})} \mathcal{O}_{X^{an} \setminus Z^{an}} \geq n$, $\dim X \geq n$. Alors,*

$$H^q(X^{an} \setminus Z^{an}, \mathcal{L}^{-1}) = 0$$

pour $q < n$.

6 Applications au groupe de Picard

On peut appliquer le Théorème 5.2 au groupe de Picard analytique, en utilisant la suite exacte exponentielle:

6.1 Théorème. *Soient X une variété projective complexe dans $\mathbb{P}_m$, Z un sous-espace fermé. Fixons une stratification de Whitney de (X, Z). Soit H un hyperplan dans $\mathbb{P}_m$ qui est transverse aux strates de Z.*
a) Supposons

$$\operatorname{pcr}(X^{an} \setminus (Z \cup H)^{an}) \geq 3,\ \operatorname{codim}_X Z \geq 3,\ \operatorname{prof}_{\operatorname{Sing} X^{an} \setminus Z^{an}} \mathcal{O}_{X^{an} \setminus Z^{an}} \geq 2.$$

Alors la flèche $\operatorname{Pic}_{(an)}(X^{an} \setminus Z^{an}) \longrightarrow \operatorname{Pic}_{(an)}(X^{an} \cap H^{an} \setminus Z^{an})$ est injective.
b) Supposons

$$\operatorname{pcr}_Z(X^{an} \setminus (Z \cup H)^{an}) \geq 4, \operatorname{codim}_X Z \geq 4, \operatorname{prof}_{\operatorname{Sing} X^{an} \setminus Z^{an}} \mathcal{O}_{X^{an} \setminus Z^{an}} \geq 3.$$

Alors $\operatorname{Pic}_{(an)}(X^{an} \setminus Z^{an}) \simeq \operatorname{Pic}_{(an)}(X^{an} \cap H^{an} \setminus Z^{an})$.

Démonstration: Si $\operatorname{pcr}_Z(X^{an} \setminus (Z \cup H)^{an}) \geq n$, le théorème de type de Lefschetz sur les sections hyperplanes (voir par exemple [20]) nous dit que

$$H^k(X^{an} \setminus Z^{an}, \mathbb{Z}) \longrightarrow H^k(X^{an} \cap H^{an} \setminus Z^{an}, \mathbb{Z})$$

est bijectif pour $k \leq n-2$ et injectif pour $k = n-1$, car on a supposé que H est transverse aux strates de Z. D'autre part on dispose du Théorème 5.2.

La suite exacte de l'exponentielle

$$0 \to \mathbb{Z}_{X^{an} \setminus Z^{an}} \to \mathcal{O}_{X^{an} \setminus Z^{an}} \to \mathcal{O}^*_{X^{an} \setminus Z^{an}} \to 0$$

et la suite analogue pour l'espace $X^{an} \cap H^{an} \setminus Z^{an}$ conduisent à des suites exactes longues de cohomologie que l'on compare. On conclut par le lemme des cinq que

$$H^1(X^{an} \setminus Z^{an}, \mathcal{O}^*_{X^{an} \setminus Z^{an}}) \to H^1(X^{an} \cap H^{an} \setminus Z^{an}, \mathcal{O}^*_{X^{an} \cap H^{an} \setminus Z^{an}})$$

est injectif dans le cas a) (resp. bijectif dans le cas b)).
Remarque: Sous les hypothèses qu'on a faites on a $\operatorname{Pic}(X \setminus Z) \simeq \operatorname{CaCl}(X^{an} \setminus Z^{an}) \simeq \operatorname{Pic}_{(an)}(X^{an} \setminus Z^{an})$ et l'énoncé analogue pour $X \cap H \setminus Z$ d'après le Lemme 3.1. On peut comparer ce résultat avec le Théorème 1.2 où il y a une hypothèse de transversalité plus forte.

6.2 Corollaire. *Soient X une variété projective complexe dans $\mathbb{P}_m$, Z un sous-espace fermé, H un hyperplan dans $\mathbb{P}_m$.*
a) Soit $X^{an} \setminus Z^{an}$ localement intersection complète de dimension $n \geq 3$ et supposons que $\operatorname{codim}_{X\cap H} Z \cap H \geq 3$, $\operatorname{codim}_X \operatorname{Sing} X \geq 2$. *Alors la flèche*

$$\operatorname{Pic}_{(an)}(X^{an} \setminus Z^{an}) \longrightarrow \operatorname{Pic}_{(an)}(X^{an} \cap H^{an} \setminus Z^{an})$$

est injective.
b) Soit $X^{an} \setminus Z^{an}$ localement intersection complète de dimension $n \geq 4$, $\operatorname{codim}_{X\cap H} Z \cap H \geq 4$, $\operatorname{codim}_X \operatorname{Sing} X \geq 3$. *Alors*

$$\operatorname{Pic}_{(an)}(X^{an} \setminus Z^{an}) \simeq \operatorname{Pic}_{(an)}(X^{an} \cap H^{an} \setminus Z^{an}).$$

Démonstration: On peut supposer que $X^{an} \setminus Z^{an}$ est connexe. L'énoncé est trivial si $X^{an} \setminus Z^{an}$ est contenu dans H. Nous pouvons donc supposer que:

$$\dim(X^{an} \setminus Z^{an}) \cap H = \dim(X^{an} \setminus Z^{an}) - 1.$$

Choisissons un sous-espace projectif générique L de H tel que $\dim L = m - n + 2$ (resp. $\dim L = m - n + 3$). Alors $L \cap Z = \emptyset$. Le Théorème 6.1 précédent nous donne dans le cas a) (resp. b))

$$\operatorname{Pic}_{(an)}(X^{an} \setminus Z^{an}) \to \operatorname{Pic}_{(an)}(X^{an} \cap L \setminus Z^{an})$$

et

$$\operatorname{Pic}_{(an)}(X^{an} \cap H \setminus Z^{an}) \to \operatorname{Pic}_{(an)} X^{an} \cap L \setminus Z^{an})$$

sont injectifs (resp. bijectifs), d'où le résultat.

Rappelons qu'il n'y a pas de différence entre le groupe de Picard algébrique et analytique dans ce cas. Ce corollaire se compare donc avec le Corollaire 1.5.

En fait, on obtient à partir du Corollaire 6.2 :

6.3 Corollaire. *Soient X une variété projective complexe dans $\mathbb{P}_m$, H un hyperplan dans $\mathbb{P}_m$. Soit X^{an} localement une intersection complète de dimension ≥ 4 et supposons que* $\operatorname{codim}_X \operatorname{Sing} X \geq 3$. *Alors*

$$\operatorname{Pic} X \simeq \operatorname{Pic}(X \cap H).$$

En comparaison avec Corollaire 1.5 avec $Z = \emptyset$ il n'y a plus besoin de condition de transversalité!

Il y a une variante du Théorème 6.1 sans hypothèse de transversalité:

6.4 Théorème. *Soient X une variété projective complexe dans $\mathbb{P}_m$, Z un sous-espace fermé, H l'hyperplan défini par $z_0 = 0$, $U := \{(z_0 : \ldots : z_m) \in X^{an} \mid |z_1|^2 + \ldots + |z_m|^2 \leq R|z_0|^2\}$ avec $R > 0$; il s'agit d'un voisinage de $X^{an} \cap H^{an}$ dans X^{an}.*
a) Supposons $\operatorname{pcr}(X^{an} \setminus Z^{an}) \geq 3$, $\operatorname{prof}_{Z^{an}} \mathcal{O}_{X^{an}} \geq 3$, $\operatorname{prof} \mathcal{O}_{X^{an} \setminus H^{an}} \geq 2$. *Alors la flèche*

$$\operatorname{Pic}_{(an)}(X^{an} \setminus Z^{an}) \longrightarrow \operatorname{Pic}_{(an)}(U \setminus Z^{an})$$

est injective.
b) Supposons que $\operatorname{pcr} Z(X^{an} \setminus Z^{an}) \geq 4$, $\operatorname{prof}_{Z^{an}} \mathcal{O}_{X^{an}} \geq 4$, $\operatorname{prof} \mathcal{O}_{X^{an} \setminus H^{an}} \geq 3$. *Alors*

$$\operatorname{Pic}_{(an)}(X^{an} \setminus Z^{an}) \simeq \operatorname{Pic}_{(an)}(U \setminus Z^{an}).$$

Démonstration: b) $H_c^k(X^{an} \setminus U, \mathcal{O}_{X^{an}}) = 0$ pour $k \leq 2$ parce que $X^{an} \setminus U$ est de Stein et $X^{an} \setminus H^{an}$ est de profondeur ≥ 3, voir [2] I Theorem 3.6. La flèche $H^k(X^{an}, \mathcal{O}_{X^{an}}) \longrightarrow H^k(U, \mathcal{O}_{X^{an}})$ est donc bijective pour $k = 1$ et injective pour $k = 2$. À cause de l'hypothèse sur la profondeur par rapport à Z^{an} on a $H^k(X^{an}, \mathcal{O}_{X^{an}}) \simeq H^k(X^{an} \setminus Z^{an}, \mathcal{O}_{X^{an}})$, $k = 1, 2$, et un énoncé pareil pour U au lieu de X^{an}. On obtient le reste à cause de la suite exacte exponentielle en utilisant un théorème de Zariski-Lefschetz qui nous donne

$$H^k(X^{an} \setminus Z^{an}, \mathbb{Z}) \simeq H^k(U \setminus Z^{an}, \mathbb{Z})$$

pour $k = 0, 1, 2$ (voir e.g. Cor. 4.3.8 de [19], avec la définition de pcr donnée ci-dessus et le fait que U est un bon voisinage de $X^{an} \cap H^{an}$).

a) La démonstration de a) est analogue à celle de b).

Remarque. En utilisant [14] on peut donner des hypothèses plus faibles que celles du théorème précédent en ne supposant que prof $\mathcal{O}_{X^{an} \setminus (H^{an} \cup Z^{an})} \geq 2$ (resp. prof $\mathcal{O}_{X^{an} \setminus (H^{an} \cup Z^{an})} \geq 3$).

7 Approche algébrique

Dans cette section nous reprenons le point de vue de A. Grothendieck développé dans [9]. On considèrera la complétion $\hat{X}$ du schéma projectif X le long d'une section hyperplane $X \cap H$ (voir [21] (Chap. II §9)

Le point de vue de A. Grothendieck consiste à établir les isomorphismes :

$$\lim_{\rightarrow} \mathrm{Pic}(U) \simeq \mathrm{Pic}(\hat{X} \setminus \hat{Z}) \simeq \mathrm{Pic}(X \cap H \setminus Z)$$

pour des sous-variétés projectives X de $\mathbb{P}_N(k)$, où Z une partie fermé de X, H un hyperplan de $\mathbb{P}_N(k)$ satisfaisant certaines hypothèses et U parcourt le système inductif des voisinages de Zariski ouverts de $X \cap H \setminus Z$ dans $X \setminus Z$.

La première comparaison est donnée par le théorème suivant (comparer au corollaire 3.6 de l'exp. XII de [9]):

7.1 Théorème. *Soient $X \subset \mathbb{P}_N(k)$ un sous-schéma projectif sur le corps k, Z une partie fermée de X, H un hyperplan de $\mathbb{P}_N(k)$ dont l'équation donne un élément non-diviseur de zéro sur X. Soit $\hat{X}$ la complétion de X le long de $X \cap H$. Supposons que:*

$$\dim S_\ell(\mathcal{O}_{X \cap H \setminus Z}) \leq \ell - 2, \text{ pour tout } \ell \leq 2 + \dim Z \cap H.$$

Alors

$$\lim_{\rightarrow} \mathrm{Pic}\,(U) \simeq \mathrm{Pic}\,(\hat{X} \setminus \hat{Z}),$$

où U parcourt le système inductif des voisinages ouverts de $X \cap H \setminus Z$ dans $X \setminus Z$.

En fait ce théorème est une conséquence immédiate du théorème suivant inspiré par le Corollaire 3.4 de [9] Exposé XII, (voir aussi dans le cas du groupe de Picard le point 2 de la démonstration du Theorem 3.1 de [22] Chapter 4 §3):

7.2 Théorème. *Soient $X \subset \mathbb{P}_N(k)$ un sous-schéma projectif sur le corps k, Z une partie fermée de X, H un hyperplan de $\mathbb{P}_N(k)$ dont l'équation donne un élément non-diviseur de zéro sur X. Soit $\hat{X}$ la complétion de X le long de $X \cap H$. Supposons que:*

$$\dim S_\ell(\mathcal{O}_{X\cap H\setminus Z}) \leq \ell - 2, \text{ pour tout } \ell \leq 2 + \dim Z \cap H.$$

Alors

$$\lim_{\to} Vect\,(U) \simeq Vect\,(\hat{X} \setminus \hat{Z}) \simeq \lim_{\leftarrow} Vect\,(X_n \setminus Z_n),$$

où U parcourt le système inductif des voisinages ouverts de Zariski de $X \cap H \setminus Z$ dans $X \setminus Z$, $Vect\,(U)$ est le semi-anneau des classes d'isomorphismes des fibrés k-vectoriels algébriques sur U et X_n le voisinage infinitésimal d'ordre n de $X \cap H$ dans X.

Démonstration. Rappelons que:

$$S_\ell(\mathcal{O}_{X\cap H\setminus Z}) := \{x \text{ point fermé de } X \cap H \setminus Z \,|\, \operatorname{prof} \mathcal{O}_{X\cap H\setminus Z,x} \leq \ell\}.$$

Par ailleurs on a

$$\dim S_\ell(\mathcal{O}_{X\cap H\setminus Z}) \leq \ell - 2, \text{ pour tout } \ell \leq 2 + \dim Z \cap H.$$

Donc sur un voisinage ouvert U' de $X \cap H \setminus Z$ dans $X \setminus Z$ on a

$$\dim S_{\ell+1}(\mathcal{O}_{U'}) \leq \ell - 2, \text{ pour tout } \ell \leq 2 + \dim Z \cap H.$$

On peut donc supposer que sur un voisinage ouvert U' de $X \cap H \setminus Z$ dans $X \setminus Z$, on a:

$$\dim S_\ell(\mathcal{O}_{U'}) \leq \ell - 3, \text{ pour tout } \ell \leq 3 + \dim Z \cap H.$$

Donc avec $\ell = 1$ on a $\operatorname{prof} \mathcal{O}_{U'} \geq 2$ et, avec $Z' = X \setminus U'$ et $Z'' := Z' \cup S_m(\mathcal{O}_{X\setminus Z'})$, où

$$m := \dim(Z \cap H) + 3,$$

par définition (cf. §1), on a:

$$\operatorname{prof}_{Z''\setminus Z'} \mathcal{O}_U \geq 2.$$

Donc la Proposition 2.6 de [15] montre que

$$\mathcal{H}^i_{Z''\setminus Z'}(\mathcal{O}_U) = 0$$

pour $i \leq 1$.

On obtient alors l'injectivité de $Vect\,(U) \to Vect\,(X \setminus Z'')$. En effet, soient $\mathcal{E}$ et $\mathcal{E}'$ deux fibrés vectoriels sur U dont les restrictions à $X \setminus Z''$ soient isomorphes. Le fibré $Hom(\mathcal{E}, \mathcal{E}')$ des morphismes de fibrés k-vectoriels a donc une section sur $X \setminus Z''$ qui se prolonge uniquement à U. Pour cela, on considère la suite exacte:

$$H^0_{Z''\setminus Z'}(X \setminus Z',\, Hom\,(\mathcal{E}.\mathcal{E}')) \to \Gamma(X \setminus Z',\, Hom\,(\mathcal{E}.\mathcal{E}')) \to$$

$$\to \Gamma(X \setminus Z'',\, Hom\,(\mathcal{E}.\mathcal{E}')) \to H^1_{Z''\setminus Z'}(X \setminus Z',\, Hom\,(\mathcal{E}.\mathcal{E}')) \to \ldots$$

Il suffit d''etablir que $H^i_{Z''\setminus Z'}(X \setminus Z',\, Hom\,(\mathcal{E}.\mathcal{E}')) = 0$, pour $i = 0, 1$.

Comme $Hom\,(\mathcal{E}.\mathcal{E}')$ est localement libre, $\mathcal{H}^i_{Z''\setminus Z'}(\mathcal{O}_U) = 0$, pour $i \leq 1$, implique:

$$\mathcal{H}^i_{Z''\setminus Z'}(\,Hom\,(\mathcal{E}.\mathcal{E}')) = 0,$$

pour $i \leq 1$.

Comme on a une suite spectrale (cf. [9], exposé 1, théorème 2.6):

$$H^p(X \setminus Z', \mathcal{H}^q_{Z''\setminus Z'}(\,Hom\,(\mathcal{E}.\mathcal{E}')) \Rightarrow H^{p+q}_{Z''\setminus Z'}(X \setminus Z',\, Hom\,(\mathcal{E}.\mathcal{E}')),$$

on obtient le résultat cherché avec $p, q \leq 1$.

On a donc un isomorphisme:

$$\Gamma(X \setminus Z',\, Hom\,(\mathcal{E}.\mathcal{E}')) \to \Gamma(X \setminus Z'',\, Hom\,(\mathcal{E}.\mathcal{E}')).$$

Donc tout homomorphisme sur $X \setminus Z$" s'étend uniquement en un homorphisme sur $X \setminus Z'$. En choisissant l'isomorphisme inverse de $\mathcal{E}'$ sur $\mathcal{E}$ sur $X \setminus Z$", l'isomorphisme sur $X \setminus Z$" entre $\mathcal{E}$ et $\mathcal{E}'$ s'étend sur $X \setminus Z'$ uniquement en un isomorphisme. Ceci donne l'injectivité $Vect\,(U) \to Vect\,(X \setminus Z'')$.

Montrons maintenant l'injectivité de $Vect\,(X\setminus Z'')$ dans $\lim\limits_{\leftarrow} Vect\,(X_n \setminus Z''_n)$. Soient $\mathcal{E}$ et $\mathcal{E}'$ deux fibrés vectoriels sur $X \setminus Z''$ tels que $\mathcal{E}|X_n \setminus Z''_n$ et $\mathcal{E}'|X_n \setminus Z''_n$ soient isomorphes pour tout $n \geq 0$. Par définition de Z'', on a prof$(Hom(\mathcal{E}, \mathcal{E}')|X \setminus Z'') \geq \dim\,(Z \cap H) + 4 \geq \dim\, Z + 3$.

D'après le théorème 4.2, on a:

$$H^i(X \setminus Z'', Hom(\mathcal{E}, \mathcal{E}') \otimes_{\mathcal{O}_X \setminus Z''} \mathcal{L}^{-n}|X \setminus Z'') = 0,$$

pour $n \gg 0$ et $i = 0, 1$ avec un faisceau ample $\mathcal{L}$ sur X.

Comme H est défini localement par une fonction qui n'est pas localement diviseur de zéro sur X, on peut remplacer $\mathcal{L}^{-1}$ par l'idéal $\mathcal{I}$ qui définit H. On a donc:

$$H^i(X \setminus Z'', \mathcal{I}^n Hom(\mathcal{E}, \mathcal{E}')) = 0,$$

pour $n \gg 0$ et $i = 0, 1$. Ceci donne:

$$H^0(X \setminus Z'', Hom(\mathcal{E}, \mathcal{E}')) \simeq H^0(X_n \setminus Z''_n, Hom(\mathcal{E}, \mathcal{E}')),$$

en considérant la suite exacte de faisceaux:

$$0 \to \mathcal{I}^n Hom(\mathcal{E}, \mathcal{E}') \to Hom(\mathcal{E}, \mathcal{E}') \to Hom(\mathcal{E}, \mathcal{E}')/\mathcal{I}^n Hom(\mathcal{E}, \mathcal{E}') \to 0.$$

On a donc une application injective $Vect\,(U) \to \lim\limits_{\leftarrow} Vect\,(X_n \setminus Z''_n)$. Comme ceci factorise par $Vect\,(U) \to \lim\limits_{\leftarrow} Vect\,(X_n \setminus Z_n)$, cette dernière application est aussi injective. Donc:

$$\lim_{\rightarrow} Vect\,(U) \to \lim_{\leftarrow} Vect\,(X_n \setminus Z_n)$$

est injective. Ceci donne immédiatement l'injection $\lim\limits_{\rightarrow} Vect\,(U) \to Vect\,(\hat{X} \setminus \hat{Z})$.

Démontrons maintenant la surjectivité de $\lim\limits_{\rightarrow} Vect\,(U) \to Vect\,(\hat{X} \setminus \hat{Z})$. Pour cela nous allons démontrer qu'un fibré vectoriel $\mathcal{E}$ sur $\hat{X} \setminus \hat{Z}$ définit un fibré vectoriel sur un voisinage de $X \cap H \setminus Z$. Pour cela il est pratique de passer au cône affine $\tilde{X}$ de X et au complété $\hat{\tilde{X}}$ le long de $\tilde{H} \cap \tilde{X}$ de $\tilde{X}$.

Le fibré vectoriel $\mathcal{E}$ induit un faisceau $\tilde{\mathcal{E}}$ sur $\hat{\tilde{X}} \setminus \hat{\tilde{Z}}$. Soit $\hat{\tilde{j}}$ l'inclusion de $\hat{\tilde{X}} \setminus \hat{\tilde{Z}}$ dans $\hat{\tilde{X}}$. D'après le théorème 2.2 de [9], le faisceau $\mathcal{T} := \hat{\tilde{j}}_* \mathcal{E}$ est cohérent. D'après le Théorème 10.10.2 de [11], les sections globales $T := \Gamma(\hat{\tilde{X}}, \mathcal{T})$ forment donc un module de type fini sur l'anneau $A := \Gamma(\hat{\tilde{X}}, \mathcal{O}_{\hat{\tilde{X}}})$.

Comme $\hat{\tilde{X}}$ est le complété d'un cône, on a une action de $k[t, t^{-1}]$ sur T induite par l'action de $k[t, t^{-1}]$ sur $\Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}/\mathcal{I}^n \mathcal{O}_{\tilde{X}})$. Le sous-groupe de T où la multiplication par t est donnée par $t \mapsto t^n$ est noté T^n. On a aussi une section à l'inclusion $T^n \subset T$. En effet, si $\tilde{\mathcal{I}}$ définit la section hyperplane sur $\tilde{X}$, $\mathcal{T}/\hat{\tilde{\mathcal{I}}}^k \mathcal{T}$ est un $\mathcal{O}_{\hat{\tilde{X}}}$-module cohérent, donc un $\mathcal{O}_{\tilde{X}}$-module cohérent, puisque la complétion est faite le long de $\tilde{H}$. On obtient donc:

$$T/\hat{\tilde{I}}^k T \simeq \oplus_{n \in \mathbb{Z}} (T/\hat{\tilde{I}}^k T)^n.$$

On obtient donc une section naturelle de l'inclusion $(T/\hat{\tilde{I}}^k T)^n$ dans $T/\hat{\tilde{I}}^k T$. Par passage à la limite projective, on a donc une flèche de T dans T^n qui est la section cherchée.

Soient $g_1, \ldots, g_m$ les générateurs du A-module T. Pour tout $k \geq 1$, comme $T/\hat{\tilde{I}}^k T \simeq \oplus_{n \in \mathbb{Z}} (T/\hat{\tilde{I}}^k T)^n$, il existe un ensemble fini M tel que, pour tout $n \in \mathbb{Z} \setminus M$ l'image de g_i soit 0 dans $(T/\hat{\tilde{I}}^k T)^n$ pour $1 \leq i \leq m$. Soient $\tilde{g}_1, \ldots, \tilde{g}_m$ les images de $g_1, \ldots, g_m$ dans l'application de T dans $\oplus_{n \in M} T^n$. Remarquons que $g_1, \ldots, g_m$ et $\tilde{g}_1, \ldots, \tilde{g}_m$ ont les mêmes images respectives dans $T/\hat{\tilde{I}}^k T$.

Soit $x \in \tilde{X} \cap \tilde{H}$. Les éléments $(\tilde{g}_1)_x, \ldots, (\tilde{g}_m)_x$ représentent des générateurs du $\mathcal{O}_{\tilde{X} \cap \tilde{H}, x}$-module $(\mathcal{T}/\hat{\tilde{\mathcal{I}}} \mathcal{T})_x$, en considérant $k = 1$. Le lemme de Nakayama montre que ces générateurs de $(\mathcal{T}/\hat{\tilde{\mathcal{I}}} \mathcal{T})_x$ donnent des générateurs du $\mathcal{O}_{\hat{\tilde{X}}, x}$-module $\mathcal{T}_x$. En effet $\mathcal{T}_x/(< \tilde{g_1})_x, \ldots, \tilde{g_m})_x > + \hat{\tilde{\mathcal{I}}}_x \mathcal{T}_x) = 0$, et $\hat{\tilde{\mathcal{I}}}_x$ est contenu dans l'idéal maximal de $\mathcal{O}_{\hat{\tilde{X}}, x}$.

On a donc un épimorphisme de faisceaux:

$$(\mathcal{O}_{\hat{\tilde{X}}})^m \to \mathcal{T}$$

Le théorème de [21] Chapter II 9.7 permet d'établir que $(\tilde{g}_1)_x, \ldots, (\tilde{g}_m)_x$ donnent des générateurs du A-module T.

On peut supposer que les $(\tilde{g}_1)_x, \ldots, (\tilde{g}_m)_x$ sont homogènes. Comme $B := \Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}})$ est gradué, les $(\tilde{g}_1)_x, \ldots, (\tilde{g}_m)_x$ engendrent un sous B-module gradué T' de $\oplus_{n \in \mathbb{Z}} T^n$. D'après [21] Chapter II 5.11, le faisceau $\mathcal{T}'$ correspondant définit une extension cohérente $\mathcal{F}$ de $\mathcal{E}$ à $X = Proj B$.

Dans un voisinage U de $\hat{X} \setminus \hat{Z}$ dans $X \setminus Z$, le faisceau $\mathcal{F}$ est localement libre (voir [11] Chapitre I 10.8.15).

Ce raisonnement donne la surjectivité de $\lim\limits_{\rightarrow} Vect\,(U) \to Vect\,(\hat{X} \setminus \hat{Z})$.

La surjectivité de $Vect\,(\hat{X} \setminus \hat{Z})$ sur $\lim\limits_{\leftarrow} Vect\,(X_n \setminus Z_n)$ est obtenue e.g. avec [11] Chapitre I 10.11.10.

On a cité ci-dessus des résultats énoncés pour $k = \mathbb{C}$, mais qui sont vrais pour k arbitraire.

En particulier on a en plus :

7.3 Corollaire. *Sous les mêmes hypothèses que le théorème 7.1, on a :*

$$\lim_{\rightarrow} \mathrm{Pic}\,(U) \simeq Pic\,(\hat{X} \setminus \hat{Z}) \simeq \lim_{\leftarrow} \mathrm{Pic}\,(X_n \setminus Z_n).$$

Remarque: En utilisant un résultat de G. Faltings (voir [6] Corollary 3), on peut établir que $\mathrm{CaCl}(\hat{X} \setminus \hat{Z}) \simeq \mathrm{Pic}(\hat{X} \setminus \hat{Z})$ dans le cas où X est géométriquement intègre. Pour cela on utilise les suites exactes de cohomologie associées aux suites exactes de faisceaux :

$$0 \to \mathcal{O}^*_{\hat{X}\setminus\hat{Z}} \to \mathcal{M}^*_{\hat{X}\setminus\hat{Z}} \to \mathcal{M}^*_{\hat{X}\setminus\hat{Z}}/\mathcal{O}^*_{\hat{X}\setminus\hat{Z}} \to 0$$

$$0 \to \mathcal{O}^*_U \to \mathcal{M}^*_U \to \mathcal{M}^*_U/\mathcal{O}^*_U \to 0$$

ce qui nous donne

$$\begin{array}{ccccccc}
\lim\limits_{\rightarrow} H^0(U, \mathcal{O}^*_U) & \to & \lim\limits_{\rightarrow} H^0(U, \mathcal{M}^*_U) & \to & \lim\limits_{\rightarrow} H^0(U, \mathcal{M}^*_U/\mathcal{O}^*_U) \to \\
\downarrow & & \downarrow & & \downarrow \\
H^0(\hat{X} \setminus \hat{Z}, \mathcal{O}^*_{\hat{X}\setminus\hat{Z}}) & \to & H^0(\hat{X} \setminus \hat{Z}, \mathcal{M}^*_{\hat{X}\setminus\hat{Z}}) & \to & H^0(\hat{X} \setminus \hat{Z}, \mathcal{M}^*_{\hat{X}\setminus\hat{Z}}/\mathcal{O}^*_{\hat{X}\setminus\hat{Z}}) \to \\
 & & & & \\
 & \to & \lim\limits_{\rightarrow} H^1(U, \mathcal{O}^*_U) & \to & 0 \\
 & & \downarrow & & \downarrow \\
 & \to & H^1(\hat{X} \setminus \hat{Z}, \mathcal{O}^*_{\hat{X}\setminus\hat{Z}}) & \to & H^1(\hat{X} \setminus \hat{Z}, \mathcal{M}^*_{\hat{X}\setminus\hat{Z}})
\end{array}$$

La deuxième flèche verticale est un isomorphisme pour le faisceau $\mathcal{M}$ d'après [6]. On en déduit l'isomorphisme pour le sous-faisceau $\mathcal{M}^*$. On conclut en utilisant le lemme des cinq.

8 Application

En utilisant des hypothèses et des techniques transcendantes, nos résultats conduisent au théorème suivant (voir [16] Théorème 5.1 dans le cas projectif):

8.1 Théorème. *Soient X une sous-variété projective complexe de $\mathbb{P}_m$, Z un sous-ensemble algébrique fermé de X, H un hyperplan de $\mathbb{P}_m$. Supposons que H ne contienne aucune composante irréductible de X et soit transverse à une stratification de Whitney donnée $\mathcal{S}$ de Z dans $\mathbb{P}_m$,* $\mathrm{codim}_{X\cap H} Z \cap H \geq 4$ *et*

$$\mathrm{pcr}(X^{an} \setminus Z^{an}) \geq 4,\ \mathrm{prof}_{\,\mathrm{Sing}(X\cap H\setminus Z)}\mathcal{O}_{X\cap H\setminus Z} \geq 3,\ \mathrm{prof}_{\,\mathrm{Sing}(X\setminus Z)}\mathcal{O}_{X\setminus Z} \geq 3.$$

Alors:

$$\mathrm{Pic}\,(X \setminus Z) \simeq \mathrm{Pic}\,(X \cap H \setminus Z).$$

Démonstration. Reprenons les notations de 7.1 et 7.2. Nous avons

$$\lim_{\to} \mathrm{Pic}\,(U) \simeq \lim_{\leftarrow} \mathrm{Pic}\,(X_n \setminus Z_n).$$

Par conséquent, comme les voisinages ouverts de $X \cap H \setminus Z$ dans $X \setminus Z$ sont de la forme $X \setminus Z'$ où Z' est un sous-ensemble algébrique fermé de X tel que $Z \subset Z' \subset X$ et $Z \cap H = Z' \cap H$, il nous suffira de démontrer que

1. pour tous les fermés Z' de X tels que $Z \subset Z' \subset X$ et $Z \cap H = Z' \cap H$, $\mathrm{Pic}\,(X \setminus Z) \simeq \mathrm{Pic}\,(X \setminus Z')$;
2. pour tout $n \geq 1$, $\mathrm{Pic}\,(X_n \setminus Z_n) \simeq \mathrm{Pic}\,(X_{n+1} \setminus Z_{n+1})$.

Preuve de 1. Remarquons que X est normal (voir par exemple [16] Lemma 2.2) donc, pour tous les fermés Z' de X tels que $Z \subset Z' \subset X$ et $Z \cap H = Z' \cap H$, l'homomorphisme $\mathrm{Pic}\,(X \setminus Z) \to \mathrm{Pic}\,(X \setminus Z')$ est injectif par un raisonnement analogue à celui donné pour les fibrés vectoriels dans la démonstration du Théorème 7.2.

Maintenant considérons un élément $[D']$ de $\mathrm{Pic}\,(X \setminus Z')$ et un diviseur de Cartier D' qui le représente. Soit $\overline{D}$ la fermeture dans $X \setminus Z$ du diviseur de Weil D associé à D'. Nous devons démontrer que $\overline{D}$ est le diviseur de Weil associé à un diviseur de Cartier de $X \setminus Z$. Comme dans la section §1, il nous faut établir que le faisceau $(\mathcal{O}_{X \setminus Z})(\overline{D})$ est localement libre sur $X \setminus Z$. Par platitude fidèle cela revient à montrer que $(\mathcal{O}_{X^{an} \setminus Z^{an}})(\overline{D}^{an})$ est localement libre.

Pour cela considérons S le sous-espace des points de $X \cap H$ où H n'intersecte pas stratification de Whitney $\mathcal{S}$ de X transversalement. Ce sous-espace est un fermé de Zariski dans X et par hypothèse $S \cap Z = \emptyset$.

Nous allons recouvrir $X^{an} \cap H^{an}$ par deux ouverts U_1 et U_2 de X^{an}, tels que les restrictions de $(\mathcal{O}_{X^{an} \setminus Z^{an}})(\overline{D}^{an})$ à U_1 et U_2 soient localement libres.

En fait U_1 est un voisinage tubulaire au sens stratifié de $X^{an} \cap H^{an} \setminus S$ dans X^{an}. L'ouvert U_2 est $X^{an} \setminus Z'^{an}$. Clairement l'ouvert $U_1 \cup U_2$ contient $X^{an} \cap H^{an}$. Par hypothèse la restriction de $(\mathcal{O}_{X^{an} \setminus Z^{an}})(\overline{D}^{an})$ à U_2 est inversible.

Comme dans la démonstration du Théorème 1.2, appelons Σ le sous-espace de $X \setminus Z$ des points au voisinage desquels le faisceau $(\mathcal{O}_{X^{an} \setminus Z^{an}})(\overline{D}^{an})$ n'est pas localement libre. Le même raisonnement conduit à $U_1 \cap \Sigma = \emptyset$. Comme $U_1 \cup U_2$ est un voisinage ouvert de $X^{an} \cap H^{an}$ dans X^{an}, on en déduit que Σ est un ensemble fini. En terminant comme dans la démonstration du Théorème 1.2 on obtient que le faisceau $(\mathcal{O}_{X^{an} \setminus Z^{an}})(\overline{D}^{an})$ est inversible.

Preuve de 2. Soit $\mathcal{I}$ le faisceau d'idéaux de $\mathcal{O}_{\mathbb{P}_m}$ qui définit H, alors la restriction de $\mathcal{I}$ à X engendre un faisceau I sur $\mathcal{O}_X$, tel que I^{-1} soit ample sur X, et $(I \otimes_{\mathcal{O}_X} (\mathcal{O}_{X \cap H}))^{-1}$ est un faisceau ample $\mathcal{L}$ sur $X \cap H$.

Si l'on suppose que H ne contienne aucune composante irréductible de X, comme X est réduit, on a:

$$I^n / I^{n+1} \simeq I^n \otimes_{\mathcal{O}_X} (\mathcal{O}_{X \cap H})$$

qui est donc $\mathcal{L}^{-n}$.

Comme dans [9] (exposé XI §1(1,1)), avec l'isomorphisme précédent, on a la suite exacte

$$0 \to I^n \otimes \mathcal{O}_{X \cap H} \to \mathcal{O}^*_{X_{n+1}} \to \mathcal{O}^*_{X_n} \to 0.$$

Pour démontrer 2, il suffit de démontrer que, pour tout $n \geq 1$, on a :

$$H^k(X \cap H, \mathcal{L}^{-n}) = 0, \text{ pour } k = 1, 2,$$

qui est conséquence du Corollaire 4.4 car $\dim X \geq 4$.

Ceci termine la démonstration de 8.1.

Remarque. Dans [16] (Theorem 5.1) nous donnons un théorème analogue, mais avec l'hypothèse $Z = \emptyset$. Ce dernier résultat est en fait conséquence du Corollaire 3.2 de l'exp. XII de [9]. La différence essentielle de nos résultats avec celui de A. Grothendieck est que l'hypothèse de parafactorialité de Grothendieck est remplacée dans ce cas sur le corps des complexes par des hypothèses topologiques et analytiques qui l'impliquent.

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MOTIVIC MILNOR CLASSES

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ABSTRACT. The Milnor class is a generalization of the Milnor number, defined as the difference (up to sign) of Chern–Schwartz–MacPherson's class and Fulton–Johnson's canonical Chern class of a local complete intersection variety in a smooth variety. In this paper we introduce a "motivic" Grothendieck group $K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S)$ and natural transformations from this Grothendieck group to the homology theory. We capture the Milnor class, more generally Milnor–Hirzebruch class, as a special value of a distinguished element under these natural transformations. We also show a Verdier-type Riemann–Roch formula for our motivic Milnor–Hirzebruch class. We use Fulton–MacPherson's bivariant theory and the motivic Hirzebruch class.

1. INTRODUCTION

The Milnor class is defined for a local complete intersection variety X in a non-singular variety M as follows. The local complete intersection variety X defines a normal bundle N_X in M, from which we can define the virtual tangent bundle T_X of X by

$$T_X := TM|_X - N_X M$$

which is a well-defined element of the Grothendieck group $K^0(X)$. Then Fulton-Johnson's or Fulton's canonical (Chern) class of X (see [FJ] and [Fu]) is defined by

$$c^{FJ}_*(X) := c(T_X) \cap [X].$$

Here $c(T_X)$ is the total Chern class of the virtual bundle T_X.

In general, Fulton-Johnson's and Fulton's canonical (Chern) classes are defined for any scheme X embedded as a closed subscheme of a non-singular variety M(see [Fu, Example 4.2.6]): Fulton–Johnson's canonical class $c^{FJ}_*(X)$ ([Fu, Example 4.2.6 (c)]) is defined by

$$c(TM|_X) \cap s(\mathcal{N}_X M),$$

where TM is the tangent bundle of M and $s(\mathcal{N}_X M)$ is the Segre class of the conormal sheaf $\mathcal{N}_X M$ of X in M [Fu, §4.2]. Fulton's canonical class $c^F_*(X)$ ([Fu, Example 4.2.6 (a)]) is defined by

$$c(TM|_X) \cap s(X, M),$$

where $s(X, M)$ is the relative Segre class [Fu, §4.2]. As shown in [Fu, Example 4.2.6], for a local complete intersection variety X in a non-singular variety M these two classes are both equal to $c(T_X) \cap [X]$.

On the other hand there is another well-known notion of Chern class for possibly singular varieties. That is Chern–Schwartz–MacPherson's class $c_*(X)$ [Mac1, Schw1, Schw2,

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Schw3, BrSc]. Then the Milnor class of the local complete intersection variety X, denoted by $\mathcal{M}(X)$, is defined by, up to sign, the difference of Fulton–Johnson's class and Chern–Schwartz–MacPherson's class $c_*(X)$; more precisely

$$\mathcal{M}(X) := (-1)^{\dim X}\left(c_*^{FJ}(X) - c_*(X)\right).$$

Since Chern–Schwartz–MacPherson's class $c_*(X)$ and Fulton–Johnson's class $c_*^{FJ}(X)$ are identical for a nonsingular variety, the Milnor class is certainly supported on the singular locus of the given variety, thus is an invariant of singularities. Prototypes of the Milnor class were studied by P. Aluffi [Alu1, Alu2], A. Parusiński [Pa1, Pa2], A. Parusiński and P. Pragacz [PP2] and T. Suwa [Su3]. Many people have been investigating on the Milnor class from their own viewpoints or interests, and many papers are now available [Alu2, Alu3, Br, BLSS1, BLSS2, Max, Pa3, PP1, PP3, Sea1, SeSu, Su2, Yo2, Yo3]. A category-functorial aspect of the Milnor class is its connection to the so-called Verdier–Riemann–Roch theorem for MacPherson's Chern class [Yo4, Sch1].

Some functoriality of the Milnor class was investigated in [Yo4], but so far it has never been captured as a natural transformation from a certain covariant functor to the homology theory. In this paper we try to capture the Milnor class from a motivic viewpoint and we show that in fact we can capture it as a natural transformation from a pre-motivic covariant functor to the homology theory. For this we need to use the motivic Hirzebruch class [BSY1, BSY2] and a key idea comes from the construction of a universal bivariant theory given in [Yo5].

In §2 we make a quick review of the motivic Hirzebruch class, following [BSY1] (also see [Yo6] and [Sch4]). In §3 we construct the motivic Grothendieck group $K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S)$, motivated by the construction of an oriented bivariant theory [Yo5]. The main results are given in §4 and §5. In §4 we construct a motivic Milnor–Hirzebruch class as a natural transformation from the above motivic Grothendieck group to Fulton–MacPherson's bivariant homology theory, a special case of which captures the Milnor class as a natural transformation from the motivic Grothendieck group to the Borel–Moore homology theory. In §5 we show a Verdier-type Riemann–Roch theorem for the motivic Milnor–Hirzebruch class.

In [CMSS] (also see [CLMS1, CLMS2, CMS1, CMS2, CS2, CS3]) Sylvain Cappell et al. independently consider the motivic Hirzebruch–Milnor class and they describe it in terms of other invariants of singularities, thus dealing more with singularities. Our present work is more category-functorial, compared with [CMSS]. A more general work is done in [Yo8].

2. Motivic Hirzebruch classes

In the following sections we use the motivic Hirzebruch class [BSY1, BSY2], thus we very quickly recall some ingredients which are needed later.

Let $\mathcal{V}$ denote the category of complex algebraic varieties. The relative Grothendieck group $K_0(\mathcal{V}/X)$ of a variety X is the quotient of the free abelian group $\mathrm{Iso}^{\mathcal{P}rop}(\mathcal{V}/X)$ of isomorphism classes $[V \xrightarrow{h} X]$ of proper morphisms to X , modulo the following *additivity* relation:

$$[V \xrightarrow{h} X] = [Z \hookrightarrow V \xrightarrow{h} X] + [V \setminus Z \hookrightarrow Y \xrightarrow{h} X]$$

for $Z \subset Y$ a closed subvariety of Y. We set the quotient homomorphism by

$$\Theta : \mathrm{Iso}^{\mathcal{P}rop}(\mathcal{V}/X) \to K_0(\mathcal{V}/X).$$

From now on the equivalence class $\Theta([V \xrightarrow{h} X])$ of the isomorphism class $[V \xrightarrow{h} X]$ is denoted by the same symbol $[V \xrightarrow{h} X]$ unless some possible confusion occurs.

Remark 2.1. Furthermore it follows from Hironaka's resolution of singularities that the restriction $\Theta^{sm} := \Theta|_{\mathrm{Iso}^{\mathcal{P}rop}(Sm/X)}$ of Θ to the subgroup $\mathrm{Iso}^{\mathcal{P}rop}(Sm/X)$ of isomorphism classes $[V \xrightarrow{h} X]$ of proper morphisms from smooth varieties V to X is surjective:

$$\Theta^{sm} : \mathrm{Iso}^{\mathcal{P}rop}(Sm/X) \to K_0(\mathcal{V}/X).$$

Here we just remark that F. Bittner [Bit] identified the kernel of the above map $\Theta^{sm} : \mathrm{Iso}^{\mathcal{P}rop}(Sm/X) \to K_0(\mathcal{V}/X)$ by some "blow-up relation", for the details of which see [Bit]. This "blow-up relation" plays an important role for constructing a bivariant analogue of the motivic Hirzebruch classes. Since we do not deal with this bivariant analogue, we do not go further into details of this "blow-up relation".

If we use the above "pre-motivic" group $\mathrm{Iso}^{\mathcal{P}rop}(Sm/X)$ we can get the following *"pre-motivic" characteristic classes of singular varieties for an arbitrary characteristic class $c\ell$ of complex vector bundles*.

For a proper morphism $f : X \to Y$ we have the obvious pushforward

$$f_* : \mathrm{Iso}^{\mathcal{P}rop}(Sm/X) \to \mathrm{Iso}^{\mathcal{P}rop}(Sm/Y)$$

defined by $f_*([V \xrightarrow{h} X]) := [V \xrightarrow{f \circ h} Y]$. Let $c\ell$ be any characteristic class of complex vector bundles with values in the cohomology theory $H^*(\quad) \otimes R$, where R is a coefficient ring. Then we define

$$\gamma_{c\ell} : \mathrm{Iso}^{\mathcal{P}rop}(Sm/X) \to H_*^{BM}(X) \otimes R$$

by

$$\gamma_{c\ell}([V \xrightarrow{h} X]) := h_*(c\ell(TV) \cap [V]).$$

Then it is clear that

$$\gamma_{c\ell} : \mathrm{Iso}^{\mathcal{P}rop}(Sm/\quad) \to H_*^{BM}(\quad) \otimes R$$

is a unique natural transformation satisfying the normalization condition that for a smooth variety X the homomorphism $\gamma_{c\ell} : \mathrm{Iso}^{\mathcal{P}rop}(Sm/X) \to H_*^{BM}(X) \otimes R$ satisfies that

$$\gamma_{c\ell}([X \xrightarrow{\mathrm{id}_X} X]) := c\ell(TX) \cap [X].$$

A naïve question is whether $\gamma_{c\ell}$ can be pushed down to the relative Grothendieck group $K_0(\mathcal{V}/X)$, i.e., for some natural transformation $? : K_0(\mathcal{V}/X) \to H_*^{BM}(\quad) \otimes R$ so that the following diagram commutes:

$$\begin{tikzcd}
& \mathrm{Iso}^{\mathcal{P}rop}(Sm/X) \arrow[dl, "\Theta^{sm}"'] \arrow[dr, "\gamma_{c\ell}"] & \\
K_0(\mathcal{V}/X) \arrow[rr, "?"'] & & H_*^{BM}(X) \otimes R
\end{tikzcd}$$

If we require that $c\ell$ is a multiplicative characteristic class, the above normalization condition and another extra condition that the degree of the 0-dimensional component of the class $\gamma_{c\ell}(\mathbb{CP}^n)$ equals $1 - y + y^2 + \cdots (-y)^n$, then the characteristic class $c\ell$ can be identified as the Hirzebruch class. Namely, let α_i's be the Chern roots of a complex vector bundle E over X. Then

$$td(E) = \prod_{i=1}^{\operatorname{rank} E} \frac{\alpha_i}{1 - e^{-\alpha_i}} \in H^{2*}(X; \mathbb{Q})$$

is the *Todd class* of E, and its modified version of it

$$td_{(y)}(V) := \prod_{i=1}^{\operatorname{rank} E} \left(\frac{\alpha_i(1+y)}{1 - e^{-\alpha_i(1+y)}} - \alpha_i y \right) \in H^*(X) \otimes \mathbb{Q}[y]$$

is called the *Hirzebruch class* (see [Hir] and [HBJ]. In fact, the Hirzebruch class unifies Chern class, Todd class and Thom–Hirzebruch L-class:

(1) $\underline{y = -1}$: $td_{(-1)}(E) = c(E)$ Chern class,
(2) $\underline{y = 0}$: $td_{(0)}(E) = td(E)$ Todd class,
(3) $\underline{y = 1}$: $td_{(1)}(E) = L(E)$ Thom–Hirzebruch L-class.

Our previous paper [BSY1] (also see [BSY2] and [SY]) showed the following theorem (originally using Saito's theory of mixed Hodge modules [Sai]):

Theorem 2.2. *(Motivic Hirzebruch class of singular varieties) There exists a unique natural transformation*

$$T_{y_*} : K_0(\mathcal{V}/\quad) \to H_*^{BM}(\quad) \otimes \mathbb{Q}[y]$$

satisfying the normalization condition that for a smooth variety X

$$T_{y_*}([X \xrightarrow{\mathrm{id}_X} X]) = td_{(y)}(TX) \cap [X].$$

This motivic Hirzebruch class $T_{y_*} : K_0(\mathcal{V}/\quad) \to H_*^{BM}(\quad) \otimes \mathbb{Q}[y]$ in a sense "unifies" the following three well-known characteristic classes of singular varieties:

Theorem 2.3. *(A "unification" of three characteristic classes)*

(1) *$\underline{c = \text{Chern class}}$: There exists a unique natural transformation*

$$\gamma_F : K_0(\mathcal{V}/\quad) \to F(\quad)$$

such that for X nonsingular $\gamma_F([X \xrightarrow{\mathrm{id}} X]) = 1\!\!1_X$. And the following diagram commutes

$$\begin{tikzcd} & K_0(\mathcal{V}/X) \arrow[dl, "\gamma_F"'] \arrow[dr, "T_{-1_*}"] & \\ F(X) \arrow[rr, "c_* \otimes \mathbb{Q}"'] & & H_*^{BM}(X) \otimes \mathbb{Q}. \end{tikzcd}$$

Here $c_* : F(X) \to H_*^{BM}(X)$ is MacPherson's Chern class transformation [Mac1] *defined on the group $F(X)$ of complex algebraically constructible functions.*

(2) *td = Todd class: There exists a unique natural transformation*

$$\gamma_{G_0} : K_0(\mathcal{V}/\quad) \to G_0(\quad)$$

such that for X nonsingular $\gamma([X \xrightarrow{\mathrm{id}} X]) = [\mathcal{O}_X]$. And the following diagram commutes

$$\begin{tikzcd} & K_0(\mathcal{V}/X) \arrow[dl, "\gamma_{G_0}"'] \arrow[dr, "T_{0*}"] & \\ G_0(X) \arrow[rr, "td_*"'] & & H_*^{BM}(X) \otimes \mathbb{Q}. \end{tikzcd}$$

Here $td_* : G_0(X) \to H_*^{BM}(X) \otimes \mathbb{Q}$ is Baum–Fulton–MacPherson's Todd class (or Riemann–Roch) transformation [BFM1] *defined on the Grothendieck group $G_0(X)$ of coherent algebraic $\mathcal{O}_X$-sheaves.*

(3) *L = Thom-Hirzebruch L-class:There exists a unique natural transformation*

$$\gamma_\Omega : K_0(\mathcal{V}/\quad) \to \Omega(\quad)$$

such that for X nonsingular $\gamma_\Omega([X \xrightarrow{\mathrm{id}} X]) = [\mathbb{Q}_X[\dim X]]$. And the following diagram commutes

$$\begin{tikzcd} & K_0(\mathcal{V}/X) \arrow[dl, "\gamma_{\Omega}"'] \arrow[dr, "T_{1*}"] & \\ \Omega(X) \arrow[rr, "L_*"'] & & H_*^{BM}(X) \otimes \mathbb{Q}. \end{tikzcd}$$

Here $\Omega(X)$ is the Cappell–Shaneson–Youssin's cobordism group of self-dual constructible sheaves (see [CS1] *and* [You]*) and $L_* : \Omega(X) \to H_*^{BM}(X) \otimes \mathbb{Q}$ is Cappell–Shaneson's homology L-class transformation* [CS1] *(also see* [GM]*).*

We also have the following

Corollary 2.4. *The following diagram commutes:*

$$\begin{tikzcd} & \mathrm{Iso}^{\mathcal{P}rop}(Sm/X) \arrow[dl, "\Theta^{sm}"'] \arrow[dr, "\gamma_{td_{(y)}}"] & \\ K_0(\mathcal{V}/X) \arrow[rr, "T_{y*}"'] & & H_*^{BM}(X) \otimes R \end{tikzcd}$$

Definition 2.5. For a complex algebraic variety X

$$T_{y*}(X) := T_{y*}([X \xrightarrow{\mathrm{id}} X]) \in H_*^{BM}(X) \otimes \mathbb{Q}[y]$$

is called *the motivic Hirzebruch class of X*.

Remark 2.6. As to the homomorphism $\gamma_F : K_0(\mathcal{V}/X) \to F(X)$ we have that for *any* variety X

$$\gamma_F([X \xrightarrow{\mathrm{id}} X]) = 1\!\!1_X, \text{therefore} \quad T_{-1*}(X) = c_*(X) \otimes \mathbb{Q},$$

whether X is singular or non-singular. However, as to the other two homomorphisms $\gamma_{G_0} : K_0(\mathcal{V}/X) \to G_0(X)$ and $\gamma_\Omega : K_0(\mathcal{V}/X) \to \Omega(X)$, if X is singular, in general we have that

$$\gamma_{G_0}([X \xrightarrow{\mathrm{id}} X]) \neq [\mathcal{O}_X], \quad \gamma_\Omega([X \xrightarrow{\mathrm{id}} X]) \neq [\mathcal{IC}_X],$$

where $\mathcal{IC}_X$ is the middle intersection homology complex of Goresky–MacPherson [GM]. Hence, if X is singular, in general we have that

$$T_{0*}(X) \neq td_*(X), \quad T_{1*}(X) \neq L_*(X).$$

If X is a Du Bois variety, i.e., a variety with Du Bois singularities, then we have that

$$\gamma_{G_0}([X \xrightarrow{\mathrm{id}} X]) = [\mathcal{O}_X], \text{therefore} \quad T_{0*}(X) = td_*(X).$$

If X is a rational homology manifold, then conjecturally

$$\gamma_\Omega([X \xrightarrow{\mathrm{id}} X]) = [\mathcal{IC}_X], \text{therefore} \quad T_{1*}(X) = L_*(X).$$

For more details, see [BSY1] and also [CMSS, Theorem 4.3], where the conjecture is proved in some special cases.

3. The Grothendieck group $K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S)$

Let S be a complex algebraic variety and fixed. Let $\mathcal{V}_S$ be the category of S-varieties, i.e., an object is a morphism $h : X \to S$ and a morphism from $h : X \to S$ to $k : Y \to S$ is a morphism $f : X \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "k"] \\ & S & \end{tikzcd} .$$

A morphism $f : X \to Y$ is called *a local complete intersection ($\ell.c.i.$) morphism* if f admits a factorization into a closed regular embedding followed by a smooth morphism (e.g., see [Fu] or [FM]). In particular, regular embeddings and smooth morphisms are $\ell.c.i.$ morphisms. The composite of $\ell.c.i.$ morphisms are again an $\ell.c.i.$ morphism.

Definition 3.1. Let $M^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S)$ be the monoid consisting of isomorphism classes $[V \xrightarrow{p} X]$ of proper morphisms $p : V \to X$ such that the composite $h \circ p : V \to S$ is an $\ell.c.i.$ morphism, with the addition $(+)$ and zero (0) defined by

- $[V \xrightarrow{h} X] + [V' \xrightarrow{h'} X] := [V \sqcup V' \xrightarrow{h+h'} X]$,
- $0 := [\phi \to X]$.

Then we define

$$K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S)$$

to be the Grothendieck group of the monoid $M^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S)$.

Remark 3.2. In other words, $K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S)$ is the free abelian group generated by the set of all isomorphism classes of $[V \xrightarrow{p} X]$ of proper morphisms $p : V \to X$ such that the composite $h \circ p : V \to S$ is an $\ell.c.i.$ morphism, modulo the subgroup generated by the elements of the following form

$$[V \xrightarrow{h} X] + [V' \xrightarrow{h'} X] - [V \sqcup V' \xrightarrow{h+h'} X].$$

Lemma 3.3. (1) *The Grothendieck group $K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S)$ is a covariant functor with pushforwards for proper morphisms, i.e., for a proper morhism $f : X \to Y \in \mathcal{V}_S$*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "k"] \\ & S & \end{tikzcd}$$

the pushforward

$$f_* : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S) \to K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/Y \xrightarrow{k} S)$$

defined by

$$f_*([V \xrightarrow{p} X]) := [V \xrightarrow{f \circ p} Y]$$

is covariantly functorial.

(2) *The Grothendieck group $K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S)$ is a contravariant functor with pullbacks for smooth morphisms, i.e., for a smooth morhism $f : X \to Y \in \mathcal{V}_S$ the pullback*

$$f^* : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/Y \xrightarrow{k} S) \to K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S)$$

defined by

$$f^*([W \xrightarrow{p} Y]) := [W' \xrightarrow{p'} X]$$

is contravariantly functorial. Here we consider the following commutative diagrams whose top square is a fiber square:

$$\begin{tikzcd} W' \arrow[rr, "f'"] \arrow[d, "p'"'] & & W \arrow[d, "p"] \\ X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "k"] \\ & S. & \end{tikzcd}$$

Proof. (1) The well-definedness of the pushforward homomorphism f_* is clear.

(2) In the diagram of Lemma 3.3 (2) , by the definition $k \circ p : W \to S$ is an $\ell.c.i.$ morphism, and $f' : W' \to W$ is smooth since it is a base change of a smooth morphism $f : X \to Y$. The composite $h \circ p' : W' \to S$ is equal to the composite $k \circ p \circ f'$, thus it is an $\ell.c.i.$ morphism because it is the composite of two $\ell.c.i.$ morphisms. Thus the pullback homomorphism f^* is well-defined. □

Remark 3.4. (1) As to the contravariance of the Grothendieck group $K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S)$, one might be tempted to consider the pullback for a local complete intersection morphism $f : X \to Y$ instead of a smooth morphism. But a crucial problem for this is that the pullback of a local complete intersection morphism is not necessarily a local complete intersection morphism, thus in the diagram of Lemma 3.3 (2) $f' : W' \to W$ is not necessarily a local complete intersection morphism and hence we do not know whether or not the composite $k \circ p \circ f' = h \circ p'$ is a local complete intersection morphism.

(2) If we consider the finer class $\mathcal{S}m$ of smooth morphisms instead of the class $\mathcal{L}.c.i$ of local complete intersection morphisms, we do have a bivariant theory, from which we can construct a motivic bivariant characteristic class [Yo7].

4. MOTIVIC MILNOR–HIRZEBRUCH CLASSES

For a morphism $f : X \to Y$, $\mathbb{H}(X \to Y)$ is the Fulton–MacPherson bivariant homology theory [FM]. Since the main theme of the present paper is not a bivariant theoretic, we do not recall a general bivariant theory, thus see [FM] for details. In the paper $\bullet$ denotes the bivariant product, i.e., for morphisms $f : X \to Y$, $g : Y \to Z$ the bivariant product $\bullet$ is

$$\bullet : \mathbb{H}(X \xrightarrow{f} Y) \times \mathbb{H}(Y \xrightarrow{g} Z) \to \mathbb{H}(X \xrightarrow{g\circ f} Z).$$

Then $\mathbb{H}(X \xrightarrow{\mathrm{id}_X} X)$ is the usual cohomology theory $H^*(X)$ and $\mathbb{H}(X \to pt)$ (for a mapping to a point) is the Borel–Moore homology theory $H_*^{BM}(X)$.

Proposition 4.1. *Let $c\ell : K^0 \to H^*(\quad) \otimes R$ be a characteristic class of complex vector bundles with a suitable coefficients R. Then on the category $\mathcal{V}_S$ we have that*

(1) *There exists a unique natural transformation (not a Grotendieck transformation)*

$$\widetilde{\gamma_{c\ell *}} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S) \to \mathbb{H}(X \xrightarrow{h} S) \otimes R$$

such that for a local complete intersection morphism $h : X \to S$

$$\widetilde{\gamma_{c\ell *}}([X \xrightarrow{\mathrm{id}_X} X]) = c\ell(T_h) \bullet U_h.$$

Here T_h is the (virtual) relative tangent bundle of h and $U_h \in \mathbb{H}(X \xrightarrow{h} S)$ is the canonical orientation.

(2) *There exists a unique natural transformation*

$$\gamma_{c\ell *} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S) \to H_*^{BM}(X) \otimes R$$

such that for a local complete intersection morphism $h : X \to S$

$$\gamma_{c\ell *}([X \xrightarrow{\mathrm{id}_X} X]) = c\ell(T_h) \cap [X].$$

Proof. (1) We define $\widetilde{\gamma_{c\ell *}} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S) \to \mathbb{H}(X \xrightarrow{h} S) \otimes R$ by

$$\widetilde{\gamma_{c\ell *}}([V \xrightarrow{p} X]) := p_*(c\ell(T_{h\circ p}) \bullet U_{h\circ p}).$$

First we observe that $\widetilde{\gamma_{c\ell *}}$ is well-defined. Let $p' : V' \to X$ be another representative of $[V \xrightarrow{p} X]$, i.e., the composite $h \circ p'$ is an $\ell.c.i.$ morphism and there is an isomorphism $g : V' \cong V$ such that the following diagram commutes:

$$\begin{tikzcd} V' \arrow[rr, "g", "\cong"'] \arrow[dr, "p'"'] & & V \arrow[dl, "p"] \\ & X. & \end{tikzcd}$$

Then we have

$$
\begin{aligned}
\widetilde{\gamma_{c\ell *}}([V' \xrightarrow{p'} X]) &= p'_*(c\ell(T_{h\circ p'}) \bullet U_{h\circ p'}) \\
&= p_* g_*(c\ell(g^* T_{h\circ p}) \bullet U_{h\circ p'}) \\
&= p_* g_*(c\ell(g^* T_{h\circ p}) \bullet U_{h\circ p'}) \\
&= p_* g_*(g^* c\ell(T_{h\circ p}) \bullet U_{h\circ p'}) \\
&= p_*(c\ell(T_{h\circ p}) \bullet g_* U_{h\circ p'}) \quad \text{(projection formula)} \\
&= p_*(c\ell(T_{h\circ p}) \bullet U_{h\circ p}) \quad \text{(since } g \text{ is an isomorphism)} \\
&= \widetilde{\gamma_{c\ell *}}([V \xrightarrow{p} X]).
\end{aligned}
$$

The equality $g_* U_{h\circ p'} = U_{h\circ p}$ is due to the following observation. By the definition or the construction of Fulton–MacPherson's bivariant homology theory $\mathbb{H}$ (see [FM]), for the isomorphism $g : V' \xrightarrow{\cong} V$ we have

- $\mathbb{H}^i(V' \xrightarrow{g} V) = H^i(V)$
- $g_* : \mathbb{H}^i(V' \xrightarrow{g} V) \to \mathbb{H}^i(V \xrightarrow{\mathrm{id}_V} V)$ is the identity map,
- $U_g = 1_V \in H^0(V)$.

Since $h \circ p' = (h \circ p) \circ g$ and g is also an $\ell.c.i.$ morphism, it follows from [FM, Part II, §1.3] that we have

$$U_{h\circ p'} = U_{(h\circ p)\circ g} = U_g \bullet U_{h\circ p}.$$

Then we have

$$
\begin{aligned}
g_* U_{h\circ p'} &= g_*\big(U_g \bullet U_{h\circ p}\big) \\
&= g_* U_g \bullet U_{h\circ p} \quad \text{([FM, } A_{12}\text{, p.20])} \\
&= U_{h\circ p} \quad \text{(since } g_* U_g = 1_V)
\end{aligned}
$$

Thus $\widetilde{\gamma_{c\ell *}}$ is well-defined.

Now, for a morphism $f : X \to Y$, i.e., for the following commutative diagram

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "k"] \\
& S &
\end{tikzcd}
$$

the following diagram commutes:

$$
\begin{tikzcd}
K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S) \arrow[r, "\widetilde{\gamma_{c\ell *}}"] \arrow[d, "f_*"'] & \mathbb{H}(X \xrightarrow{h} S) \otimes R \arrow[d, "f_*"] \\
K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/Y \xrightarrow{k} S) \arrow[r, "\widetilde{\gamma_{c\ell *}}"'] & \mathbb{H}(Y \xrightarrow{k} S) \otimes R,
\end{tikzcd}
$$

Indeed, for $[V \xrightarrow{p} X] \in K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S)$ we have that

$$\begin{aligned} f_*\left(\widetilde{\gamma_{c\ell *}}([V \xrightarrow{p} X])\right) &= f_*\left(p_*(c\ell(T_{h\circ p}) \bullet U_{h\circ p})\right) \\ &= (f\circ p)_*\left(c\ell(T_{h\circ p}) \bullet U_{h\circ p}\right) \\ &= (f\circ p)_*\left(c\ell(T_{k\circ f\circ p}) \bullet U_{k\circ f\circ p}\right) \\ &= (f\circ p)_*\left(c\ell(T_{k\circ f\circ p}) \bullet U_{k\circ f\circ p}\right) \\ &= \gamma_{c\ell *}([V \xrightarrow{f\circ p} Y]) \\ &= \widetilde{\gamma_{c\ell *}}\left(f_*([V \xrightarrow{p} X])\right). \end{aligned}$$

Since, for a local complete intersection morphism $h : X \to S$, by definition of $\gamma_{c\ell *}$ we have $\gamma_{c\ell *}([X \xrightarrow{\mathrm{id}_X} X]) = c\ell(T_h) \bullet U_h$, the uniqueness of $\gamma_{c\ell *}$ follows.

(2) We define $\gamma_{c\ell *} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S) \to H^{BM}_*(X) \otimes R$ by

$$\gamma_{c\ell *}([V \xrightarrow{p} X]) := p_*(c\ell(T_{h\circ p}) \cap [V]).$$

The well-definedness of $\gamma_{c\ell *}$ is similar to the above, but more straightforward. Indeed, we have

$$\begin{aligned} \gamma_{c\ell *}([V' \xrightarrow{p} X]) &= p'_*(c\ell(T_{h\circ p'}) \cap [V']) \\ &= p_*g_*(c\ell(g^*T_{h\circ p}) \cap [V']) \\ &= p_*g_*(c\ell(g^*T_{h\circ p}) \cap [V']) \\ &= p_*g_*(g^*c\ell(T_{h\circ p}) \cap [V']) \\ &= p_*(c\ell(T_{h\circ p}) \cap g_*[V']) \\ &= p_*(c\ell(T_{h\circ p}) \cap [V]) \\ &= \gamma_{c\ell *}([V \xrightarrow{p} X]). \end{aligned}$$

Then the following diagram commutes:

$$\begin{array}{ccc} K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S) & \xrightarrow{\gamma_{c\ell *}} & H^{BM}_*(X) \otimes R \\ {\scriptstyle f_*}\big\downarrow & & \big\downarrow{\scriptstyle f_*} \\ K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/Y \xrightarrow{k} S) & \xrightarrow[\gamma_{c\ell *}]{} & H^{BM}_*(Y) \otimes R, \end{array}$$

which follows from replacing $\bullet U_{h\circ p}$ and $\bullet U_{k\circ f\circ p}$ by $\cap[V]$ in the proof of (1). $\square$

Remark 4.2. For a local complete intersection morphism $f : X \to S$, we have

$$\bullet U_h \bullet [S] = \cap [X].$$

Here $[W]$ is the fundamental class of W and $[W] \in \mathbb{H}(W \to pt) = H^{BM}_*(W)$. Thus the relation between the above two natural transformations $\widetilde{\gamma_{c\ell *}}$ and $\gamma_{c\ell *}$ is that

$$\gamma_{c\ell *} = \widetilde{\gamma_{c\ell *}} \bullet [S].$$

Remark 4.3. When the fixed variety S is a point, the above two natural transformations $\widetilde{\gamma_{c\ell *}}$ and $\gamma_{c\ell *}$ are the same: $\gamma_{c\ell *} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X) \to H^{BM}_*(X) \otimes R$.

If S is a point and $c\ell = c$ the Chern class, then for a local complete intersection variety X in a smooth manifold, we have that

$$\gamma_{c*}([X \xrightarrow{\mathrm{id}_X} X]) = c(T_X) \cap [X]$$

which is Fulton–Johnson's class $c_*^{FJ}(X)$. Thus the above natural transformations

$$\widetilde{\gamma_{c\ell *}} : K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/X \xrightarrow{h} S) \to \mathbb{H}(X \xrightarrow{h} S) \otimes R$$

$$\gamma_{c\ell *} : K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/X \xrightarrow{h} S) \to H_*^{BM}(X) \otimes R$$

are both generalizations of Fulton–Johnson's class as natural transformations. They are respecively called *a motivic "bivariant" FJ-$c\ell$ class*, denoted by $\widetilde{c\ell_*^{FJ}}$, and *a motivic FJ-$c\ell$ class*, denoted by $c\ell_*^{FJ}$, since it is modelled after Fulton–Johnson's class c_*^{FJ}.

From here on we consider the Hirzebruch class $td_{(y)}$, instead of an arbitrary characteristic class $c\ell$, because we use the motivic Hirzebruch class $T_{y_*} : K_0(\mathcal{V}/X) \to H_*^{BM}(X) \otimes \mathbb{Q}[y]$ below. We use the above natural transformations

$$\widetilde{\gamma_{td_{(y)}*}} : K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/X \xrightarrow{h} S) \to \mathbb{H}(X \xrightarrow{h} S) \otimes \mathbb{Q}[y],$$

$$\gamma_{td_{(y)}*} : K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/X \xrightarrow{h} S) \to H_*^{BM}(X) \otimes \mathbb{Q}[y],$$

which are respectively called *the motivic "bivariant" FJ-Hirzebruch class* and *the motivic FJ-Hirzebruch class* and denoted by $\widetilde{T_{y_*}^{FJ}}$ and $T_{y_*}^{FJ}$.

We define the twisted pushforward for homology as follows: for a morphism $f : X \to Y$, the relative dimension of f and the co-relative dimension of f are respectively defined by

$$\dim(f) := \dim X - \dim Y \quad \operatorname{codim}(f) := \dim Y - \dim X.$$

For the Borel–Moore homology theory H_*, the twisted pushforward for a proper morphism $f : X \to Y$ is define by

$$f_{**} := (-1)^{\operatorname{codim}(f)} f_* : H_*^{BM}(X) \to H_*^{BM}(Y).$$

With this twisted pushforward the Borel–Moore homology theory is still a covariant functor. To avoid a possible confusion we denote $H_{**}^{BM}(X)$ for the Borel–Moore homology theory with the twisted pushforward.

Corollary 4.4. *On the category $\mathcal{V}_S$ there exists a unique natural transformation*

$$\mathcal{M}T_{y_*} : K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/X \xrightarrow{h} S) \to H_{**}^{BM}(X) \otimes \mathbb{Q}[y]$$

such that for a local complete intersection morphism $h : X \to S$ the homomorphism $\mathcal{M}T_{y_*} : K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/X \xrightarrow{h} S) \to H_{**}^{BM}(X) \otimes \mathbb{Q}[y]$ satisfies that

$$\mathcal{M}T_{y_*}([X \xrightarrow{\mathrm{id}_X} X]) = (-1)^{\dim X} \left(T_{y_*}^{FJ} - T_{y_*} \circ \Theta\right)([X \xrightarrow{\mathrm{id}_X} X]).$$

Proof. We define $\mathcal{M}T_{y_*} : K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/X \xrightarrow{h} S) \to H_{**}^{BM}(X) \otimes \mathbb{Q}[y]$ by

$$\mathcal{M}T_{y_*}([V \xrightarrow{p} X]) := (-1)^{\dim V} \left(T_{y_*}^{FJ} - T_{y_*} \circ \Theta\right)([V \xrightarrow{p} X]).$$

This is equal to

$$(-1)^{\dim X} p_* \left(td_{(y)}(T_{p\circ h}) \cap [V] - T_{y_*}(V)\right).$$

□

From here on we denote $T_{y_*} \circ \Theta$ simply by T_{y_*}. When S is a point, the above motivic natural transformation

$$\mathcal{M}T_{y_*} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X) \to H^{BM}_{**}(X) \otimes \mathbb{Q}[y]$$

shall be called *a motivic Milnor–Hirzebruch class*, even though $K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X)$ is not (a subgroup of) the motivic group $K_0(\mathcal{V}/X)$, but because it is defined by using the motivic Hirzebruch class $T_{y_*} : K_0(\mathcal{V}/X) \to H^{BM}_*(X) \otimes \mathbb{Q}[y]$ and because, if we specialize $\mathcal{M}T_{y_*}$ to the case when $y = -1$ and X is a local complete intersection variety in a smooth manifold, we have

$$\begin{aligned}
&\mathcal{M}T_{-1_*}([X \xrightarrow{\mathrm{id}} X]) \\
&= (-1)^{\dim X} \left\{ td_{(-1)}(T_X) \cap [X] - T_{-1_*}\left(\Theta([X \xrightarrow{\mathrm{id}} X])\right)\right\} \\
&= (-1)^{\dim X} \left(c^{FJ}_*(X) - c_*(X) \right),
\end{aligned}$$

which is the Milnor class $\mathcal{M}(X)$ of X. Thus $\mathcal{M}T_{-1_*} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X) \to H^{BM}_{**}(X) \otimes \mathbb{Q}[y]$ is called *the motivic Milnor class (or Milnor–Chern class).* The more general one

$$\mathcal{M}T_{y_*} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S) \to H^{BM}_{**}(X) \otimes \mathbb{Q}[y]$$

is called *a generalized motivic Milnor–Hirzebruch class*.

In fact, if the base variety S is a $\mathbb{Q}$-homology manifold or a rational homology manifold, the fundamental class $[S] \in \mathbb{H}(S \to pt) = H^{BM}_*(S)$ is a strong orientation (see [FM, Part I, §2.6]), namely we have the following isomorphism (see [BSY3])

$$\bullet[S] : \mathbb{H}(X \xrightarrow{h} S) \otimes \mathbb{Q} \xrightarrow{\cong} \mathbb{H}(X \to pt) \otimes \mathbb{Q} = H^{BM}_*(X) \otimes \mathbb{Q}.$$

Which is a generalized Poincaré duality isomorphism, hence denoted by $\mathcal{PD}_h$. Indeed, when X is a rational homology compact manifold, for the identity $id_X : X \to X$, the above isomorphism is nothing but the classical Poincaré duality isomorphism

$$\cap[X] : H^*(X) \otimes \mathbb{Q} \to H_*(X) \otimes \mathbb{Q}.$$

Examples of a $\mathbb{Q}$-homology manifold (e.g., see [BM, §1.4 Rational homology manifolds]) are surfaces with Kleinian singularities, the moduli space of curves of a given genus, Satake's V-manifolds or orbifolds, in particular, the quotient of a nonsingular variety by a finite group action on.

Thus we can get the following corollary:

Corollary 4.5. *Let the base variety S be a $\mathbb{Q}$-homology manifold. On the category $\mathcal{V}_S$ there exists a unique natural transformation*

$$\widetilde{\mathcal{M}T_{y_*}} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S) \to \mathbb{H}_{**}(X \xrightarrow{h} S) \otimes \mathbb{Q}[y]$$

such that for a local complete intersection morphism $h : X \to S$ the homomorphism $\widetilde{\mathcal{M}T_{y_*}} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S) \to \mathbb{H}(X \xrightarrow{h} S) \otimes \mathbb{Q}[y]$ satisfies that

$$\widetilde{\mathcal{M}T_{y_*}}([X \xrightarrow{\mathrm{id}_X} X]) = (-1)^{\dim X} \left(\widetilde{T^{FJ}_{y_*}} - \mathcal{PD}_h^{-1} \circ T_{y_*} \right) ([X \xrightarrow{\mathrm{id}_X} X]).$$

Here $\mathbb{H}_{**}(X \xrightarrow{h} S)$ is the twisted bivariant homology theory with the twisted pushforward $f_{**} := (-1)^{\mathrm{codim}(f)} f_*$.

Remark 4.6. (1) $\widetilde{\mathcal{M}T_{y_*}} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S) \to \mathbb{H}_{**}(X \xrightarrow{h} S) \otimes \mathbb{Q}[y]$ shall be called *a motivic "bivariant" Milnor–Hirzebruch class*, even thought the source group $K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S)$ is not a bivariant theory, but the target group $\mathbb{H}_{**}(X \xrightarrow{h} S) \otimes \mathbb{Q}[y]$ is a bivariant theory.

(2) Note that when the base variety S is a point, $\widetilde{\mathcal{M}T_{y_*}} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S) \to \mathbb{H}_{**}(X \xrightarrow{h} S) \otimes \mathbb{Q}[y]$ is the same as $\mathcal{M}T_{y_*} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X) \to H^{BM}_{**}(X) \otimes \mathbb{Q}[y]$.

Proposition 4.7. *In the case when $y = 0$, the Milnor–Todd class $\mathcal{M}T_{0*} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X) \to H^{BM}_*(X) \otimes \mathbb{Q}$ vanishes on the subgroup generated by $[V \xrightarrow{p} X]$ with V being Du Bois varieties:*

$$\mathcal{M}T_{0*}([V \xrightarrow{p} X]) = 0 \quad \text{if } V \text{ is a Du Bois variety.}$$

Proof. For a local complete intersection variety V in a smooth variety M, we have that

$$\begin{aligned} &\mathcal{M}T_{0*}([V \xrightarrow{p} X]) \\ &= p_{**}\mathcal{M}T_{0*}([V \xrightarrow{\mathrm{id}} V]) \\ &= (-1)^{\dim X} p_* \Big(td(T_V) \cap [V] - T_{0*}([V \xrightarrow{\mathrm{id}} V]) \Big) \\ &= (-1)^{\dim X} p_* \big(td(T_V) \cap [V] - T_{0*}(V) \big). \end{aligned}$$

If V is a Du Bois variety, it follows from Remark 2.6 that $T_{0*}(V) = td_*(\mathcal{O}_V)$. On the other hand we observe that it follows from the properties of the Baum–Fulton–MacPherson's Riemann–Roch $td_* : G_0(X) \to H^{BM}_*(X) \otimes \mathbb{Q}$ (see [Fu, Corollary 18.3.1 (b)], or more generally [FM, PART II, §0.2 Summary of results]) that for a local complete intersection variety V in a smooth variety M we have

$$td_*(\mathcal{O}_V) = td(T_V) \cap [V],$$

for T_V the virtual tangent bundle of V in M. Therefore, if V is a local complete intersection variety V in a smooth variety M and V is also a Du Bois variety, then we have

$$\mathcal{M}T_{0*}([V \xrightarrow{p} X]) = 0.$$

□

Corollary 4.8. *If the base variety S is a $\mathbb{Q}$-homology manifold, then the motivic bivariant Milnor–Todd class $\widetilde{\mathcal{M}T_{0*}} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S) \to \mathbb{H}_{**}(X \xrightarrow{h} S) \otimes \mathbb{Q}$ vanishes on the subgroup generated by $[V \xrightarrow{p} X]$ with V being Du Bois varieties.*

Proof. This follows from the fact that for an element $[V \xrightarrow{p} X]$ with V a Du Bois variety $\widetilde{\mathcal{M}T_{0*}}([V \xrightarrow{p} X]) \bullet [S] = \mathcal{M}T_{0*}([V \xrightarrow{p} X]) = 0$ and $\bullet[S] : \mathbb{H}(X \xrightarrow{h} S) \otimes \mathbb{Q} \xrightarrow{\cong} \mathbb{H}(X \to pt) \otimes \mathbb{Q} = H^{BM}_*(X) \otimes \mathbb{Q}$ is an isomorphism when S is a $\mathbb{Q}$-homology manifold. □

Remark 4.9. Let us compare with the results in Theorem 2.3. Neither of the following three diagrams commutes in general:

- $\underline{y = -1}$:

$$\begin{tikzcd} & K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X) \arrow[dl, "\gamma_F"'] \arrow[dr, "{T_{-1}}^{FJ}_*"] & \\ F(X) \arrow[rr, "c_*"'] & & H^{BM}_*(X) \otimes \mathbb{Q}. \end{tikzcd}$$

- $\underline{y=0}$:

$$\begin{tikzcd} & K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X) \arrow[dl, "\gamma_{G_0}"'] \arrow[dr, "{T_0}_*^{FJ}"] & \\ G_0(X) \arrow[rr, "td_*"'] & & H_*^{BM}(X) \otimes \mathbb{Q}. \end{tikzcd}$$

- $\underline{y=1}$:

$$\begin{tikzcd} & K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X) \arrow[dl, "\gamma_{\Omega}"'] \arrow[dr, "{T_1}_*^{FJ}"] & \\ \Omega(X) \arrow[rr, "L_*"'] & & H_*^{BM}(X) \otimes \mathbb{Q}. \end{tikzcd}$$

Hence it is natural or reasonable to consider the following commutative diagrams with the corresponding Milnor classes and the corresponding looked-for natural transformations

$$\mathcal{M}T_{-1_*} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X) \to H_{**}^{BM}(X) \otimes \mathbb{Q}, \quad \mathcal{M}c_* : F(X) \to H_{**}^{BM}(X) \otimes \mathbb{Q}$$
$$\mathcal{M}T_{0*} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X) \to H_{**}^{BM}(X) \otimes \mathbb{Q}, \quad \mathcal{M}td_* : G_0(X) \to H_{**}^{BM}(X) \otimes \mathbb{Q},$$
$$\mathcal{M}T_{1*} : K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X) \to H_{**}^{BM}(X) \otimes \mathbb{Q}, \quad \mathcal{M}L_* : \Omega(X) \to H_{**}^{BM}(X) \otimes \mathbb{Q} :$$

- $\underline{y=-1}$:

$$\begin{tikzcd} & K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X) \arrow[dl, "\gamma_{F}"'] \arrow[dr, "\mathcal{M}T_{-1_*}"] & \\ F(X) \arrow[rr, "\mathcal{M}c_*"'] & & H_{**}^{BM}(X) \otimes \mathbb{Q}. \end{tikzcd}$$

- $\underline{y=0}$:

$$\begin{tikzcd} & K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X) \arrow[dl, "\gamma_{G_0}"'] \arrow[dr, "\mathcal{M}T_{0*}"] & \\ G_0(X) \arrow[rr, "\mathcal{M}td_*"'] & & H_{**}^{BM}(X) \otimes \mathbb{Q}. \end{tikzcd}$$

- $\underline{y=1}$:

$$\begin{tikzcd} & K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X) \arrow[dl, "\gamma_{\Omega}"'] \arrow[dr, "\mathcal{M}T_{1*}"] & \\ \Omega(X) \arrow[rr, "\mathcal{M}L_*"'] & & H_{**}^{BM}(X) \otimes \mathbb{Q}. \end{tikzcd}$$

5. Verdier-type Riemann–Roch formulas

In this section we show *Verdier-type Riemann–Roch formulas*.

First we show *a Verdier-type Riemann–Roch formula for the motivic canonical $c\ell$ class for a smooth morphism*. Here we emphasize that we need a smooth morphism instead of a local complete intersection morphism:

Proposition 5.1. *Let $f : X \to Y$ be a smooth morphism in the category $\mathcal{V}_S$:*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "k"] \\ & S. & \end{tikzcd}$$

Then the following diagram commutes:

$$\begin{tikzcd} K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/Y \xrightarrow{k} S) \arrow[r, "c\ell^{FJ}_*"] \arrow[d, "f^*"'] & H^{BM}_*(Y) \otimes R \arrow[d, "c\ell(T_f)\cap f^*"] \\ K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/X \xrightarrow{h} S) \arrow[r, "c\ell^{FJ}_*"'] & H^{BM}_*(X) \otimes R, \end{tikzcd}$$

Here $f^* : H^{BM}_*(Y) \to H^{BM}_*(X)$ is the Gysin pullback homomorphism.

Proof. Let $[W \xrightarrow{p} Y] \in K^{\mathcal{P}rop}_{\ell.c.i}(\mathcal{V}/Y \xrightarrow{k} S)$ and consider the following diagram whose top square is a fiber square:

(5.2)
$$\begin{tikzcd} W' \arrow[rr, "f'"] \arrow[d, "p'"'] & & W \arrow[d, "p"] \\ X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "k"] \\ & S. & \end{tikzcd}$$

We want to show that

$$c\ell^{FJ}_* f^*([W \xrightarrow{p} Y]) = c\ell(T_f) \cap f^* \left(c\ell^{FJ}_*([W \xrightarrow{p} Y]) \right).$$

$$\begin{aligned} c\ell^{FJ}_* f^*([W \xrightarrow{p} Y]) &= c\ell^{FJ}_*([W' \xrightarrow{p'} X]) \\ &= p'_*(c\ell(T_{h\circ p'}) \cap [W']) \quad \text{(by definition of } c\ell^{FJ}_*) \end{aligned}$$

$$c\ell(T_f) \cap f^* \left(c\ell^{FJ}_*([W \xrightarrow{p} Y]) \right) = c\ell(T_f) \cap f^* \left(p_*(c\ell(T_{k\circ p}) \cap [W]) \right).$$

Since $p : W \to Y$ is proper and $f : X \to Y$ is smooth, hence flat, it follows from [Fu, Proposition 1.7] that we have the *base change formula*:$f^* p_* = p'_* {f'}^*$. The above equality continues as follows:

$$\begin{aligned} &= c\ell(T_f) \cap p'_* {f'}^* (c\ell(T_{k\circ p}) \cap [W]) \\ &= p'_* \left({p'}^* c\ell(T_f) \cap {f'}^* (c\ell(T_{k\circ p}) \cap [W]) \right) \ \text{(projection formula)} \\ &= p'_* \left(c\ell({p'}^* T_f) \cap (c\ell({f'}^* T_{k\circ p}) \cap {f'}^*[W]) \right) \ \text{(by [Fu, Theorem 3.2])} \\ &= p'_* \left((c\ell(T_{f'}) \cup c\ell({f'}^* T_{k\circ p})) \cap [{f'}^{-1}(W)] \right) \ \text{(by [Fu, Lemma1.7.1])} \\ &= p'_* \left(c\ell(T_{f'} + {f'}^* T_{k\circ p}) \cap [W'] \right) \\ &= p'_* \left(c\ell(T_{k\circ p\circ f'}) \cap [W'] \right) \quad (T_{k\circ p\circ f'} = T_{f'} + {f'}^* T_{k\circ p} \in K^0(W')) \\ &= p'_* \left(c\ell(T_{h\circ p'}) \cap [W'] \right). \end{aligned}$$

Therefore we get that $c\ell_*^{FJ} f^*([W \xrightarrow{p} Y]) = c\ell(T_f) \cap f^* \left(c\ell_*^{FJ}([W \xrightarrow{p} Y]) \right).$ $\square$

By the definition $K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/X \xrightarrow{k} S)$ is the Grothendieck group of the monoid consisting of some elements of $\mathrm{Iso}^{Prop}(\mathcal{V}/X)$, hence a homomorphism

$$\Psi : K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/X \xrightarrow{k} S) \to H_*^{BM}(X) \otimes \mathbb{Q}[y]$$

satisfying

$$\Psi([V \xrightarrow{p} X]) = T_{y*}([V \xrightarrow{p} X]) \, (= T_{y*} \circ \Theta([V \xrightarrow{p} X]))$$

is uniquely determined. So we denote Ψ by the same symbol T_{y*}:

$$T_{y*} : K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/X \xrightarrow{k} S) \to H_*^{BM}(X) \otimes \mathbb{Q}[y],$$

which is also called a motivic Hirzebruch class in the present set-up.

Secondly we show *a Verdier-type Riemann–Roch formula for the motivic Hirzebruch class for a smooth morphism*:

Proposition 5.3. *Let $f : X \to Y$ be a smooth morphism in the category $\mathcal{V}_S$ as in Proposition 5.1. Then the following diagram commutes:*

$$\begin{array}{ccc}
K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/Y \xrightarrow{k} S) & \xrightarrow{T_{y*}} & H_*^{BM}(Y) \otimes \mathbb{Q}[y] \\
{\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle td_{(y)}(T_f)\cap f^*} \\
K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/X \xrightarrow{h} S) & \xrightarrow[T_{y*}]{} & H_*^{BM}(X) \otimes \mathbb{Q}[y].
\end{array}$$

Proof. For the above diagram (5.2) we want to show that

$$T_{y*} f^*([W \xrightarrow{p} Y]) = td_{(y)}(T_f) \cap f^* \left(T_{y*}([W \xrightarrow{p} Y]) \right).$$

Since it follows from Hironaka's resolution of singularities that any $[W \xrightarrow{p} Y]$ can be expressed as a linear combination

$$\sum_V a_V [V \xrightarrow{p_V} Y]$$

where $a_V \in \mathbb{Z}$, V is a smooth variety , and $p_V : V \to Y$ is proper, it suffices to show that

$$T_{y*} f^*([V \xrightarrow{p_V} Y]) = td_{(y)}(T_f) \cap f^* \left(T_{y*}([V \xrightarrow{p_V} Y]) \right).$$

Hence, from the beginning we can assume that in the above diagram 5.2 W is smooth and $p : W \to Y$ is proper, but here note that we DO NOT need the requirement that the composite $k \circ p : W \to S$ is a local complete intersection morphism. Here it should be noted that since W is smooth and $f' : W' \to W$ is smooth (because f' is the pullback of the smooth morphism $f : X \to Y$), W' is also smooth, which is crucial below.

$$\begin{aligned}
T_{y*} f^*([W \xrightarrow{p} Y]) &= T_{y*}([W' \xrightarrow{p'} X]) \\
&= T_{y*}(p'_*[W' \xrightarrow{\mathrm{id}_{W'}} W']) \\
&= p'_* T_{y*}([W' \xrightarrow{\mathrm{id}_{W'}} W']) \\
&= p'_*(td_{(y)}(TW') \cap [W']) \quad \text{(since } W' \text{ is smooth).}
\end{aligned}$$

On the other hand we have

$$
\begin{aligned}
& td_{(y)}(T_f) \cap f^* T_{y*}([W \xrightarrow{p} Y]) \\
&= td_{(y)}(T_f) \cap f^* T_{y*}(p_*[W \xrightarrow{\mathrm{id}_W} W]) \\
&= td_{(y)}(T_f) \cap f^* p_*(T_{y*}([W \xrightarrow{\mathrm{id}_W} W])) \\
&= td_{(y)}(T_f) \cap f^* p_*(td_{(y)}(TW) \cap [W])) \quad \text{(since W is smooth)} \\
&= td_{(y)}(T_f) \cap p'_* {f'}^*(td_{(y)}(TW) \cap [W])) \\
&= p'_* \left({p'}^* td_{(y)}(T_f) \cap {f'}^*(td_{(y)}(TW) \cap [W])\right) \\
&= p'_* \left(td_{(y)}({p'}^* T_f) \cap \left({f'}^* td_{(y)}(TW) \cap {f'}^*[W]\right)\right) \\
&= p'_* \Big(\left(td_{(y)}(T_{f'}) \cup td_{(y)}({f'}^* TW)\right) \cap [{f'}^{-1} W]\Big) \Big) \\
&= p'_* \left(td_{(y)}(T_{f'} + {f'}^* TW) \cap [W']\right)\big) \\
&= p'_* \left(td_{(y)}(TW') \cap [W']\right)\big) \quad \text{(since $T_{f'} = TW' - {f'}^* TW$).}
\end{aligned}
$$

Therefore we get that $T_{y*} f^*([W \xrightarrow{p} Y]) = td_{(y)}(T_f) \cap f^* \Big(T_{y*}([W \xrightarrow{p} Y])\Big)$. $\square$

Remark 5.4. The above proof of course implies that the following Verdier-type Riemann–Roch formula holds for the motivic Hirzebruch class $T_{y_*} : K_0(\mathcal{V}/X) \to H_*^{BM}(X) \otimes \mathbb{Q}[y]$: for a smooth morphism $f : X \to Y$ in the category $\mathcal{V}$ the following diagram commutes:

$$
\begin{array}{ccc}
K_0(\mathcal{V}/Y) & \xrightarrow{T_{y_*}} & H_*^{BM}(Y) \otimes \mathbb{Q}[y] \\
{\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle td_{(y)}(T_f) \cap f^*} \\
K_0(\mathcal{V}/X) & \xrightarrow[T_{y_*}]{} & H_*^{BM}(X) \otimes \mathbb{Q}[y].
\end{array}
$$

Definition 5.5. For a smooth morphism $f : X \to Y$, the twisted Gysin pullback homomophism $f^{**} : H_*^{BM}(Y) \to H_*^{BM}(X)$ is defined by

$$
f^{**} = (-)^{\dim(f)} f^* = (-1)^{\dim X - \dim Y} f^*.
$$

(In other words, $(-)^{\mathrm{codim}(f)} f^{**} = (-1)^{\dim Y - \dim X} f^{**} = f^*$.) The contravariant Borel–Moore homology theory with this twisted pullback homomotphism for smoth morphisms is denoted by H_{**}^{BM}.

In [Yo4, Theorem 2.2] we obtained a Verdier-type Riemann–Roch formula of the Milnor class in a special case. The following Verdier-type Riemann–Roch formula of the motivc Milnor–Hirzebruch class is a generalization of this result:

Theorem 5.6. *For a smooth morphism $f : X \to Y$ in the category $\mathcal{V}_S$ as in Proposition 5.1, the following diagram commutes:*

$$
\begin{array}{ccc}
K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/Y \xrightarrow{k} S) & \xrightarrow{\mathcal{M}T_{y_*}} & H_{**}^{BM}(Y) \otimes \mathbb{Q}[y] \\
{\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle td_{(y)}(T_f) \cap f^{**}} \\
K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/X \xrightarrow{h} S) & \xrightarrow[\mathcal{M}T_{y_*}]{} & H_{**}^{BM}(X) \otimes \mathbb{Q}[y].
\end{array}
$$

Proof. Let $[W \xrightarrow{p} Y] \in K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/Y \xrightarrow{k} S)$. Then we have that

$$
\begin{aligned}
&\mathcal{M}T_{y_*}f^*([W \xrightarrow{p} Y])\\
&= \mathcal{M}T_{y_*}[W' \xrightarrow{p'} X])\\
&= (-1)^{\dim W'}\left(T_{y_*}^{FJ} - T_{y_*}\right)([W' \xrightarrow{p'} X])\\
&= (-1)^{\dim W'}\left(T_{y_*}^{FJ} - T_{y_*}\right)(f^*[W \xrightarrow{p} Y])\\
&= (-1)^{\dim W'}\left(T_{y_*}^{FJ}f^* - T_{y_*}f^*\right)([W \xrightarrow{p} Y])\\
&= (-1)^{\dim W'}\left(td_{(y)}(T_f) \cap f^*T_{y_*}^{FJ} - td_{(y)}(T_f) \cap f^*T_{y_*}\right)([W \xrightarrow{p} Y])\\
&= (-1)^{\dim W'}td_{(y)}(T_f) \cap f^*\left(T_{y_*}^{FJ} - T_{y_*}\right)([W \xrightarrow{p} Y])\\
&= (-1)^{\dim W'}(-)^{\mathrm{codim}(f)}td_{(y)}(T_f) \cap f^{**}\left(T_{y_*}^{FJ} - T_{y_*}\right)([W \xrightarrow{p} Y])\\
&= (-1)^{\dim W' + \dim Y - \dim X}td_{(y)}(T_f) \cap f^{**}\left(T_{y_*}^{FJ} - T_{y_*}\right)([W \xrightarrow{p} Y])\\
&= (-1)^{\dim W}td_{(y)}(T_f) \cap f^{**}\left(T_{y_*}^{FJ} - T_{y_*}\right)([W \xrightarrow{p} Y])\\
&= td_{(y)}(T_f) \cap f^{**}\left((-1)^{\dim W}\left(T_{y_*}^{FJ} - T_{y_*}\right)([W \xrightarrow{p} Y])\right)\\
&= td_{(y)}(T_f) \cap f^{**}\left(\mathcal{M}T_{y_*}([W \xrightarrow{p} Y])\right).
\end{aligned}
$$

□

Finally we give a "bivariant version" of Theorem 5.6:

Corollary 5.7. *For a smooth morphism $f : X \to Y$ in the category $\mathcal{V}_S$ as in Proposition 5.1, the following diagram commutes:*

$$
\begin{array}{ccc}
K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/Y \xrightarrow{k} S) & \xrightarrow{\widetilde{\mathcal{M}T_{y_*}}} & \mathbb{H}(Y \xrightarrow{h} S) \otimes \mathbb{Q}[y]\\
{\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle (-1)^{\dim(f)}td_{(y)}(T_f)\bullet U_f \bullet}\\
K_{\ell.c.i}^{\mathcal{P}rop}(\mathcal{V}/X \xrightarrow{h} S) & \xrightarrow[\widetilde{\mathcal{M}T_{y_*}}]{} & \mathbb{H}(X \xrightarrow{h} S) \otimes \mathbb{Q}[y],
\end{array}
$$

Proof. The commutativity of the above diagram follows from Theorem 5.6, the following commutative diagram

$$
\begin{array}{ccc}
\mathbb{H}(Y \xrightarrow{k} S) \otimes \mathbb{Q}[y] & \xrightarrow{\bullet[S]} & \mathbb{H}_*^{BM}(Y) \otimes \mathbb{Q}[y]\\
{\scriptstyle (-1)^{\dim(f)}td_{(y)}(T_f)\bullet U_f \bullet}\downarrow & & \downarrow{\scriptstyle td_{(y)}(T_f)\cap f^{**}}\\
\mathbb{H}(X \xrightarrow{h} S) \otimes \mathbb{Q}[y] & \xrightarrow[\bullet[S]]{} & \mathbb{H}_*^{BM}(X) \otimes \mathbb{Q}[y],
\end{array}
$$

and the fact (see [FM]) that for any $\beta \in \mathbb{H}(Y \to pt) = H_*^{BM}(Y)$

$$U_f \bullet \beta = f^*\beta$$

and also using the fact that $\bullet[S] : \mathbb{H}(X \xrightarrow{h} S) \xrightarrow{\cong} H_*^{BM}(X)$ is an isomorphism.

□

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Multi-variable Poincaré series associated with Newton diagrams

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Abstract

We define a multi-index filtration on the ring of germs of functions on a hypersurface singularity associated with its Newton diagram and compute the multivariable Poincaré series of this filtration in some cases.

Introduction

Poincaré series of filtrations (including multi-index ones) on the ring of germs of functions on a complex analytic variety are of interest for some problems (see, e.g., [2], [4], [7], ...). In a number of cases they look like the A'Campo formula for the monodromy zeta function (being products/ratios of binomials of the form $(1 - \underline{t}^{\underline{m}})$). Moreover, in some cases they are connected with monodromy zeta functions corresponding to the singularity (see, e.g., [6, 5]).

For quasi-homogeneous singularities one has the classical Poincaré series in one variable. A Poincaré series of one variable also corresponds to the semi-group of values of an irreducible curve singularity. The initial motivation to consider multi-variable Poincaré series stems from the study of reducible curve singularities [2]. Recently they were found to be connected with the study of Seiberg-Witten invariants for surface singularities [7].

We define a multi-index filtration on the ring of germs of functions on a hypersurface singularity associated with its Newton diagram. One can say that this filtration is a multi-index generalization of the quasi-homogeneous one. We compute the Poincaré series of this filtration for curve singularities and for some singularities of more variables. In the computed cases, they turn out to be of A'Campo type.

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1 Multi-index filtrations and their Poincaré series

A function v on the ring $\mathcal{O}_{X,0}$ of germs of functions on the germ $(X, 0)$ of an analytic space with values in $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ is called a *valuation* if

(1) $v(g_1 + g_2) \geq \min\{v(g_1), v(g_2)\}$,

(2) $v(g_1 g_2) = v(g_1) + v(g_2)$,

(3) $v(c) = 0$ for a non-zero constant $c \in \mathcal{O}_{X,0}$.

If a function $v : \mathcal{O}_{X,0} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ possesses the properties (1) and (3) but in general not the property (2), it is called an *order function*.

A family $\{v_1, \ldots, v_s\}$ of order functions on the ring $\mathcal{O}_{X,0}$ defines a multi-index filtration of the ring $\mathcal{O}_{X,0}$. For $g \in \mathcal{O}_{X,0}$, let

$$\underline{v}(g) := (v_1(g), \ldots, v_s(g)) \in (\mathbb{Z}_{\geq 0} \cup \{\infty\})^s.$$

For $\underline{v} = (v_1, \ldots, v_s) \in \mathbb{Z}^s$ the corresponding subspace is defined by

$$J(\underline{v}) = \{g \in \mathcal{O}_{X,0} \, : \, \underline{v}(g) \geq \underline{v}\}.$$

(Here $\underline{v}(g) \geq \underline{v}$ means that $v_i(g) \geq v_i$ for all $i = 1, \ldots, s$.)

The notion of the Poincaré series of the multi-index filtration $\{J(\underline{v})\}$ defined by a family $\{v_i\}$ of order functions was given in [3]. For $\underline{v} \in \mathbb{Z}^s$, let $d(\underline{v}) = \dim J(\underline{v})/J(\underline{v} + \underline{1})$ where $\underline{1} = (1, 1, \ldots, 1)$. Let

$$L(\underline{t}) = \sum_{\underline{v} \in \mathbb{Z}^s} d(\underline{v})\, \underline{t}^{\,\underline{v}}\,,$$

where $\underline{t} = (t_1, \ldots, t_s)$, $\underline{t}^{\,\underline{v}} = t_1^{v_1} \cdot \ldots \cdot t_s^{v_s}$. (Pay attention that the sum is over all $\underline{v}$ from $\mathbb{Z}^s$, not from $\mathbb{Z}^s_{\geq 0}$. For $s > 1$, the series $L(\underline{t})$ contains monomials with negative exponents.) The *Poincaré series* of the multi-index filtration $\{J(\underline{v})\}$ is the power series in $\underline{t} = (t_1, \ldots, t_s)$ defined by

$$P_{\{v_i\}}(\underline{t}) = \frac{L(\underline{t}) \cdot \prod_{i=1}^{s} (t_i - 1)}{t_1 \cdot \ldots \cdot t_s - 1}\,. \tag{1}$$

(This makes sense if all the dimensions $d(\underline{v})$ are finite.)

Equation (1) implies that the coefficient at $\underline{t}^{\,\underline{v}}$ in the Poincaré series $P_{\{v_i\}}(\underline{t})$ is equal to

$$\sum_{I \subset I_0} (-1)^{\#I} \dim J(\underline{v} + \underline{1}_I)/J(\underline{v} + \underline{1})\,, \tag{2}$$

where $I_0 = \{1, 2, \ldots, s\}$, $\underline{1}_I$ is the s-tuple in which the i-th component is equal to 0 for $i \notin I$ and is equal to 1 otherwise.

Remark. One can easily see that, if all the subspaces $J(\underline{v}+\underline{1}_{\{i\}})$ (and therefore all the subspaces $J(\underline{v}+\underline{1}_I)$ for $I \neq \emptyset$) are contained in one of them, say, in $J(\underline{v}+\underline{1}_{\{1\}})$, then the coefficient in equation (2) is equal to $\dim J(\underline{v})/J(\underline{v}+\underline{1}_{\{1\}})$.

Equation (2) (together with the inclusion-exclusion formula) implies that the coefficient at $\underline{t}^{\underline{v}}$ in the Poincaré series is equal to the Euler characteristic $\chi(\mathbb{P}F_{\underline{v}})$ of the projectivisation $\mathbb{P}F_{\underline{v}} = F_{\underline{v}}/\mathbb{C}^*$ of the space

$$F_{\underline{v}} = (J(\underline{v})/J(\underline{v}+\underline{1})) \setminus \bigcup_{i=1}^{s} \Big(J(\underline{v}+\underline{1}_{\{i\}})/J(\underline{v}+\underline{1}) \Big)$$

(see [2]).

To compute the Euler characteristic $\chi(\mathbb{P}F_{\underline{v}})$, it can be convenient to define a $\mathbb{C}^*$-action on the space $\mathbb{P}F_{\underline{v}}$ (or on elements of a constructible partitioning of it). In this case the Euler characteristic of the total space coincides with the Euler characteristic of the set of fixed points. In particular, if the $\mathbb{C}^*$-action is free, the Euler characteristic $\chi(\mathbb{P}F_{\underline{v}})$ is equal to zero.

One says that a multi-index filtration $\{J(\underline{v})\}$ on the ring $\mathcal{O}_{X,0}$ is induced by a (multi-)grading if there exist subspaces $A_{\underline{v}} \subset \mathcal{O}_{X,0}$, $\underline{v} \in \mathbb{Z}^s_{\geq 0}$, such that the ring $\mathcal{O}_{X,0}$ is a completion of the graded algebra $\bigoplus\limits_{\underline{v}\in\mathbb{Z}^s_{\geq 0}} A_{\underline{v}}$ and $J(\underline{v})$ is the corresponding completion of $\bigoplus\limits_{\underline{v}'\geq\underline{v}} A_{\underline{v}'}$. One can easily see that, if a filtration $\{J(\underline{v})\}$ is induced by a grading $\{A_{\underline{v}}\}$ with finite-dimensional subspaces $A_{\underline{v}}$, the Poincaré series of the filtration $\{J(\underline{v})\}$ is given by the equation

$$P(\underline{t}) = \sum_{\underline{v}\in\mathbb{Z}^s_{\geq 0}} \dim A_{\underline{v}} \cdot \underline{t}^{\underline{v}}.$$

This is not the case in general. Coefficients of the Poincaré series are not, generally speaking, dimensions of some spaces. They may be negative (see, e.g., Example 1 at the end of the paper).

2 Multi-index filtration corresponding to a Newton diagram

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function with an isolated critical point at the origin, non-degenerate with respect to its Newton diagram $\Gamma = \Gamma(f)$ [1]. Let $V = \{f = 0\}$ be the corresponding hypersurface singularity. Here we shall define a multi-index filtration on the ring $\mathcal{O}_{V,0}$ of germs of functions on the hypersurface $(V, 0)$. This filtration is a generalization of the quasi-homogeneous filtrations defined by the equations of the $(n-1)$-dimensional faces of the Newton diagram.

Suppose that the Newton diagram Γ has s faces $\gamma_1, \ldots, \gamma_s$ of dimension $n-1$ (facets), and let γ_i lie in the hyperplane given by the equation

$$\ell_i(k_1, \ldots, k_n) = a_1^{(i)} k_1 + \cdots + a_n^{(i)} k_n = d^{(i)}$$

where $a_1^{(i)}, \ldots, a_n^{(i)}$ and $d^{(i)}$ are positive integers with greatest common divisor equal to 1.

For a monomial $\underline{x}^{\underline{k}} = x_1^{k_1} \cdot \ldots \cdot x_n^{k_n}$, let

$$u_i(\underline{x}^{\underline{k}}) := \ell_i(k_1, \ldots, k_n) = \sum_{j=1}^{n} a_j^{(i)} k_j \, .$$

For a germ $g(x_1, \cdots, x_n) = \sum\limits_{\underline{k} \in \mathbb{Z}^n_{\geq 0}} c_{\underline{k}} \, \underline{x}^{\underline{k}} \in \mathcal{O}_{\mathbb{C}^n,0}$, let $u_i(g) := \min\limits_{\underline{k}: c_{\underline{k}} \neq 0} u_i(\underline{x}^{\underline{k}})$. The function u_i is a valuation on the ring $\mathcal{O}_{\mathbb{C}^n,0}$.

Proposition 1 *The Poincaré series $P_{\{u_i\}}(\underline{t})$ of the family $\{u_i\}$ of valuations is given by the equation*

$$P_{\{u_i\}}(\underline{t}) = \prod_{j=1}^{n} (1 - \underline{t}^{\,\underline{u}(x_j)})^{-1}$$

(x_j is the j-th coordinate function on the space $\mathbb{C}^n$).

The proof easily follows from the fact that, in this case, the filtration $\{J(\underline{u})\}$ is induced by a grading. The corresponding subspace $A_{\underline{u}}$, $\underline{u} = (u_1, \ldots, u_s) \in \mathbb{Z}^s_{\geq 0}$, is generated by the monomials $\underline{x}^{\underline{k}}$ with $\ell_i(\underline{k}) = u_i$, $i = 1, \ldots, s$.

For a function $g \in \mathcal{O}_{V,0} = \mathcal{O}_{\mathbb{C}^n,0}/(f)$, let

$$v_i(g) := \max_{g': g' \equiv g \mod f} u_i(g') \, .$$

The function v_i on the ring $\mathcal{O}_{V,0}$ is not, generally speaking, a valuation. For example, for $f(x,y) = x^5 + x^2y^2 + y^5$ and for the face of the Newton diagram given by the equation $\ell(k_x, k_y) = 2k_x + 3k_y = 10$, one has $u(x^2) = 4$, $u(x^3+y^2) = 6$, but $u(x^5 + x^2y^2) = 15$. However, it is an order function. In this way one gets a family $\{v_1, \ldots, v_s\}$ of order functions and the corresponding s-index filtration on the ring $\mathcal{O}_{V,0}$. We shall call it the *Newton filtration*.

3 Poincaré series of the Newton filtration for curve singularities

Let f be the germ of a holomorphic function of two variables with an isolated critical point at the origin, non-degenerate with respect to its Newton diagram Γ. Let $v_1, \ldots, v_s$ be the order functions on the ring $\mathcal{O}_{V,0}$ ($V = \{f = 0\}$) corresponding to the one-dimensional faces $\gamma_1, \ldots, \gamma_s$ of the diagram Γ. These order functions are induced by the valuations $u_1, \ldots, u_s$ on the ring $\mathcal{O}_{\mathbb{C}^2,0}$.

Theorem 1 *One has*

$$P_{\{v_i\}}(\underline{t}) = \begin{cases} (1 - \underline{t}^{\,\underline{u}(f)}) \cdot P_{\{u_i\}}(\underline{t}) & \text{for } s = 2, \\ P_{\{u_i\}}(\underline{t}) & \text{for } s > 2. \end{cases}$$

Proof. Let the Newton diagram Γ consist of two faces (i.e., $s = 2$), and let $\underline{m} = (m_1, m_2)$ be the intersection point of them. (The coordinates m_1 and m_2 are integers.) To compute the coefficient at $\underline{t}^{\,\underline{v}}$, $\underline{v} = (v_1, v_2) \in \mathbb{Z}^2_{\geq 0}$, consider the lines $L_i = \{\ell_i(\underline{k}) = v_i\}$, $i = 1, 2$, and let $\underline{k} = (k_1, k_2)$ be their intersection point.

Suppose that the intersection point of the lines L_1 and L_2 is either non-integral, or one of its coordinates is negative, or it satisfies the condition $\underline{k} \geq \underline{m}$ (i.e., $k_i \geq m_i$ for $i = 1, 2$). In $\mathcal{O}_{\mathbb{C}^2,0}$ the space $J(\underline{v})/J(\underline{v}+\underline{1})$ is freely generated by the monomials $\underline{x}^{\,\underline{k}}$ whose exponents $\underline{k}$ are the integer points on the boundary of the domain $\{\ell_i(\underline{k}) \geq v_i$ for $i = 1, 2\}$ (and thus lie on the lines L_1 and L_2). Using the relation $f = 0$ in $\mathcal{O}_{V,0}$, one eliminates some monomials (if any) on the lines L_1 and L_2 starting from the intersection point of these lines. (If the intersection point is integral, it is eliminated.) Let p_i be the number of remaining points (monomials) on the line L_i, $i = 1, 2$. Then, in $\mathcal{O}_{V,0}$, one has

$$\begin{aligned}
&\dim J(\underline{v})/J(\underline{v}+\underline{1}) = p_1 + p_2,\\
&\dim J(\underline{v}+\underline{1}_{\{1\}})/J(\underline{v}+\underline{1}) = p_2, \quad \dim J(\underline{v}+\underline{1}_{\{2\}})/J(\underline{v}+\underline{1}) = p_1,
\end{aligned}$$

and the equation (2) implies that the coefficient at $\underline{t}^{\,\underline{v}}$ is equal to zero.

Suppose that the intersection point of the lines L_1 and L_2 is integral, non-negative (i.e., $k_i \geq 0$ for $i = 1, 2$) and satisfies the condition $k_1 < m_1$. (The case $k_2 < m_2$ is treated in the same way.) In this case the relation $f = 0$ permits one to eliminate points (if any) only on the line L_2. In particular, the intersection point of the lines L_1 and L_2 is not eliminated. As above, let p_i be the number of remaining points on the line L_i, $i = 1, 2$. (The intersection point is counted on both of them.) Then, in $\mathcal{O}_{V,0}$, one has

$$\begin{aligned}
&\dim J(\underline{v})/J(\underline{v}+\underline{1}) = p_1 + p_2 - 1,\\
&\dim J(\underline{v}+\underline{1}_{\{1\}})/J(\underline{v}+\underline{1}) = p_2 - 1, \quad \dim J(\underline{v}+\underline{1}_{\{2\}})/J(\underline{v}+\underline{1}) = p_1 - 1,
\end{aligned}$$

and the equation (2) implies that the coefficient at $\underline{t}^{\,\underline{v}}$ is equal to 1.

Therefore

$$P_{\{v_i\}}(\underline{t}) = \sum_{\underline{k}\in\mathbb{Z}^2_{\geq 0}:\underline{k}\not\geq\underline{m}} \underline{t}^{\,\underline{u}(\underline{x}^{\,\underline{k}})} = (1-\underline{t}^{\,\underline{u}(\underline{x}^{\,\underline{m}})})\cdot\prod_{i=1}^{2}(1-\underline{t}^{\,\underline{u}(x_i)})^{-1} = (1-\underline{t}^{\,\underline{u}(f)})\cdot P_{\{u_i\}}(\underline{t})\,.$$

Let the Newton diagram consist of more than 2 faces (i.e., $s > 2$). For $\underline{v} = (v_1, \ldots, v_s) \in \mathbb{Z}^s_{\geq 0}$, let L_i be the line $\{\ell_i(\underline{k}) = v_i\}$, $i = 1, \ldots, s$. Suppose first that all the lines L_i have (one) common integral point $\underline{k} \geq \underline{0}$. The relation $f = 0$ permits one to eliminate some points (if any) on two of the lines L_i (the extreme ones), but not the intersection point $\underline{k}$ itself. In the space $J(\underline{v})/J(\underline{v}+\underline{1})$ each subspace $J(\underline{v}+\underline{1}_{\{i\}})/J(\underline{v}+\underline{1})$ is contained in the subspace $\{c_{\underline{k}} = 0\}$ ($c_{\underline{k}}$ is the coefficient at $\underline{x}^{\,\underline{k}}$ in the power series decomposition of a function) and some of them (in fact all but at most two) coincide with this subspace. This implies that the coefficient at $\underline{t}^{\,\underline{v}}$ ($\underline{v} = (\ell_1(\underline{k}), \ldots, \ell_s(\underline{k}))$) in the Poincaré series is equal to

$$\dim J(\underline{v})/J(\underline{v}+\underline{1}) - \dim\{c_{\underline{k}} = 0\} = 1 \tag{3}$$

(see the Remark in Section 1).

Now suppose that the lines L_i do not have a common point in the non-negative orthant. We shall show that in this case the coefficient at $\underline{t}^{\underline{v}}$ in the Poincaré series is equal to zero. We may suppose that each line L_i intersects the boundary of the domain $B = \{\ell_i(\underline{k}) \geq v_i \text{ for } i = 1, \ldots, s\}$. Otherwise $J(\underline{v} + \underline{1}_{\{i\}})/J(\underline{v} + \underline{1}) = J(\underline{v})/J(\underline{v} + \underline{1})$ and the coefficient at $\underline{t}^{\underline{v}}$ is equal to zero according to the Remark in Section 1.

As written above, in $\mathcal{O}_{\mathbb{C}^2,0}$, the factorspace $J(\underline{v})/J(\underline{v}+\underline{1})$ is freely generated by the monomials $\underline{x}^{\underline{k}}$ with $\underline{k}$ from the boundary of the domain B. The relations between these generators in $\mathcal{O}_{V,0} = \mathcal{O}_{\mathbb{C}^2,0}/(f)$ correspond to non-negative integer translations of the Newton diagram Γ such that the translate of the diagram is contained in the domain B and intersects the boundary of the domain. Suppose that there is a face $\beta_i = L_i \cap B$ of the domain B (possibly of length zero) which cannot intersect a translate (in the described way) of the Newton diagram. In this case the monomials corresponding to integer points on the face β_i do not participate in any relation. Multiplication of the coefficients of all these monomials in the power series decomposition of a function by $\lambda \in \mathbb{C}^*$ defines a free $\mathbb{C}^*$-action on the part of the space $\mathbb{P}F_{\underline{v}}$ where at least one coefficient of a monomial corresponding to a point on the boundary of the domain B outside of the face β_i is different from zero. Taking into account the condition that all the lines L_i do not have a common non-negative integer point, one can see that the complement to this part may be non-empty only if the length of the face β_i is finite, but not zero, both ends $\underline{k}_1$ and $\underline{k}_2$ of this face are integral and the boundary of the domain B consists of β_i and two rays (corresponding to the extreme two lines among L_j). (All the other lines (if any) go through the points $\underline{k}_1$ or $\underline{k}_2$.) Moreover, both coefficients at $\underline{x}^{\underline{k}_1}$ and at $\underline{x}^{\underline{k}_2}$ in the power series decomposition of a function from the complement under consideration are different from zero. Therefore a free $\mathbb{C}^*$-action on this part can be defined by multiplying the coefficient at $\underline{x}^{\underline{k}_1}$ by $\lambda \in \mathbb{C}^*$.

If all faces of the domain B intersect translates of the Newton diagram, a vertex $\underline{k}$ of it (in fact any one) lies on a translate of Γ. This means that the coefficient at $\underline{x}^{\underline{k}}$ can be eliminated with the help of the corresponding relation. There are no other relations which include points of the boundary ∂B from both connected components of $\partial B \setminus \{\underline{k}\}$. In this case a free $\mathbb{C}^*$-action on the space $\mathbb{P}F_{\underline{v}}$ can be defined by multiplying coefficients at all the monomials from one of the connected components of $\partial B \setminus \{\underline{k}\}$ by $\lambda \in \mathbb{C}^*$. Combining with equation (3), this implies the statement of Theorem 1 for $s > 2$. $\square$

4 Poincaré series of some singularities of more than 2 variables

It looks somewhat complicated to get the Poincaré series corresponding to an arbitrary Newton diagram. However, a special property of some Newton diagrams permits one to compute the Poincaré series for them in a uniform way.

Definition: We say that a Newton diagram is *stellar* if all its facets (faces of maximal dimension) have a common vertex.

Example. Singularities with stellar Newton diagrams include surface singularities of type $T_{p,q,r}$, suspensions of singularities, and singularities with the Newton diagram consisting of 2 facets, in particular, bimodal singularities.

Theorem 2 *Let the Newton diagram* Γ *of a germ* $f \in \mathcal{O}_{\mathbb{C}^n,0}$ *be stellar. Then*

$$P_{\{v_i\}}(\underline{t}) = (1 - \underline{t}^{\,\underline{u}(f)}) \cdot P_{\{u_i\}}(\underline{t})\,.$$

Proof. Let $\underline{m}$ be a vertex of the Newton diagram Γ on the intersection of all facets of Γ. One can easily see that $\underline{v}(f) = \underline{v}(\underline{x}^{\,\underline{m}})$. Therefore, for all points $\underline{k}$ in Γ, one has $\ell_i(\underline{k}) \geq \ell_i(\underline{m})$ for $i = 1, \ldots, s$. For $\underline{v} = (v_1, \ldots, v_s) \in \mathbb{Z}^s_{\geq 0}$, let $L_i = \{\ell_i(\underline{k}) = v_i\}$, $i = 1, \ldots, s$, be the corresponding affine hyperplanes in $\mathbb{R}^s$. In $\mathcal{O}_{\mathbb{C}^n,0}$, the factorspace $J(\underline{v})/J(\underline{v} + \underline{1})$ is freely generated by the monomials $\underline{x}^{\,\underline{k}}$ with $\underline{k}$ from the boundary of the domain $B = \{\ell_i(\underline{k}) \geq v_i \text{ for } i = 1, \ldots, s\}$. The relations between these generators in $\mathcal{O}_{V,0} = \mathcal{O}_{\mathbb{C}^2,0}/(f)$ correspond to non-negative integer translations of the Newton diagram Γ such that the translate of Γ is contained in B and intersects the boundary ∂B. For each such translation, the translate $\underline{m}'$ of the vertex $\underline{m}$ lies on ∂B as well. (If the values of all linear functions ℓ_i at the point $\underline{m}'$ are greater than v_i, this holds for the translates of other points of Γ as well.) Let $\lambda(\underline{k})$ be a generic linear function such that it has different values at different integer points (i.e., it is "irrational") and its value at the vertex $\underline{m}$ is greater than at all other points of the Newton diagram Γ. Using translates of Γ in order of decreasing values of the function λ on the translation vectors, one eliminates all the translates of the vertex $\underline{m}$. In this way one eliminates all the integer points $\underline{k}$ on ∂B with $\underline{k} \geq \underline{m}$. The monomials corresponding to the remaining integral non-negative points on ∂B form a basis of the factor space $J(\underline{v})/J(\underline{v}+\underline{1})$ in $\mathcal{O}_{V,0}$. Moreover the space $J(\underline{v}+\underline{1}_{\{i\}})/J(\underline{v}+\underline{1})$ is freely generated by the monomials corresponding to those points which do not lie on L_i. The equation (2) and the inclusion-exclusion formula imply that the coefficient at $\underline{t}^{\,\underline{v}}$ in the Poincaré series is equal to the number of those non-negative integer points $\underline{k} \in \partial B$ with $\underline{k} \not\geq \underline{m}$ which belong to all the hyperplanes L_i. Therefore

$$P_{\{v_i\}}(\underline{t}) = \sum_{\underline{k} \in \mathbb{Z}^s_{\geq 0} : \underline{k} \not\geq \underline{m}} \underline{t}^{\,\underline{u}(\underline{x}^{\,\underline{k}})} = (1 - \underline{t}^{\,\underline{u}(\underline{x}^{\,\underline{m}})}) \cdot \prod_{i=1}^{s} (1 - \underline{t}^{\,\underline{u}(x_i)})^{-1} = (1 - \underline{t}^{\,\underline{u}(f)}) \cdot P_{\{u_i\}}(\underline{t})\,.$$

□

Up to now, we have had only two types of equations, the types in Theorem 1, for the Poincaré series of Newton filtrations. Moreover all coefficients in these series were non-negative. This could produce a hope that Newton filtrations are induced by gradings of the coordinate ring. The following examples show that, in general, neither all the Poincaré series of Newton filtrations are of these types nor all the coefficients in them are non-negative. In these examples we compute the coefficient of the Poincaré series at $\underline{t}^{\,\underline{v}(f)}$.

Examples. 1. For $f(x,y,z) = x^5 + y^5 + z^5 + x^2yz + xy^2z + xyz^2$ the Newton diagram consists of 4 facets γ_0, γ_1, γ_2, and γ_3 lying on the hyperplanes with the equations $k_x + k_y + k_z = 4$, $2k_x + k_y + k_z = 5$, $k_x + 2k_y + k_z = 5$, and $k_x + k_y + 2k_z = 5$ respectively. Besides the vertices $(2,1,1), \ldots, (0,0,5)$ of the Newton diagram, there are 12 integral points on the diagram: 4 on each of the facets γ_1, γ_2, and γ_3. One integer point, say $(0,0,5)$ can be eliminated using the relation $f = 0$. The space $J(\underline{v})/J(\underline{v}+\underline{1})$ ($\underline{v} = \underline{v}(f)$) is freely generated by the 17 remaining monomials. The subspace $F_{\underline{v}}$ in it is the complement of the union of the 4 subspaces given by the equations ($c_{\underline{k}}$ is the coefficient at $\underline{x}^{\underline{k}}$ in the power series decomposition of a function):

$$\begin{aligned}
& J(\underline{v}+\underline{1}_{\{0\}})/J(\underline{v}+\underline{1}) \\
& \quad = \quad \{c_{211} = c_{121} = c_{112}\}, \\
& J(\underline{v}+\underline{1}_{\{1\}})/J(\underline{v}+\underline{1}) \\
& \quad = \quad \{c_{050} = c_{041} = c_{032} = c_{023} = c_{014} = c_{121} = c_{112} = 0\}, \\
& J(\underline{v}+\underline{1}_{\{2\}})/J(\underline{v}+\underline{1}) \\
& \quad = \quad \{c_{500} = c_{401} = c_{302} = c_{203} = c_{104} = c_{211} = c_{112} = 0\}, \\
& J(\underline{v}+\underline{1}_{\{3\}})/J(\underline{v}+\underline{1}) \\
& \quad = \quad \{c_{500} = c_{410} = c_{320} = c_{230} = c_{140} = c_{050} = c_{211} = c_{121}\}.
\end{aligned}$$

From these data one can easily compute that the coefficient at $\underline{t}^{\underline{v}(f)} = t_0^4 t_1^5 t_2^5 t_3^5$ in the Poincaré series is equal to -1. Therefore, in this case, the Newton filtration is not induced by a grading.

2. For $f(x,y,z) = x^{20} + y^{20} + z^{16} + x^8y^8 + x^6y^6z^2 + x^2y^2z^{10} + x^3y^8z^3 + x^8y^3z^3$ the Newton diagram consists of 5 facets γ_0, γ_1, γ_2, γ_3, and γ_4 lying on the hyperplanes with the equations $k_x + k_y + k_z = 14$, $2k_x + 3k_y + 5k_z = 40$, $3k_x + 2k_y + 5k_z = 40$, $11k_x + 4k_y + 5k_z = 80$, and $4k_x + 11k_y + 5k_z = 80$ respectively. Computations like in Example 1 yield the coefficient at $\underline{t}^{\underline{v}(f)} = t_0^{14} t_1^{40} t_2^{40} t_3^{80} t_4^{80}$ to be equal to 1. Since $\underline{v}(f) = (14, 40, 40, 80, 80)$ is not a linear combination of $\underline{v}(x) = (1,2,3,11,4)$, $\underline{v}(y) = (1,3,2,4,11)$, and $\underline{v}(z) = (1,5,5,5,5)$, the Poincaré series is not of one of the types of Theorem 1.

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SINGULARITIES OF PIECEWISE LINEAR SADDLE SPHERES ON $\mathbb{S}^3$

GAIANE PANINA

Abstract. Segre's theorem asserts the following: *let a smooth closed simple curve $c \subset S^2$ have a non-empty intersection with any closed hemisphere. Then c has at least 4 inflection points.*

In the paper, we prove two Segre-type theorems. The first one is a version of Segre's theorem for piecewise linear closed curves on S^2. Here we have *inflection edges* instead of inflection points.

Next, we go one dimension higher: we replace S^2 by S^3. Instead of simple curves, we treat immersed saddle surfaces which are homeomorphic to S^2 ("saddle spheres"). We prove that a piecewise linear saddle sphere $\Gamma \subset S^3$ necessarily has *inflection* or *reflex faces*. The latter replace inflection points and should be considered as singular phenomena.

As an application, we prove that a piecewise linear saddle surface cannot be altered in a neighborhood of its vertex maintaining its saddle property.

1. Introduction

Let us start with the following classical theorems.

Theorem 1.1. *Segre's theorem, see* [12], [17].

Let a smooth closed simple (i.e., embedded) curve $c \subset S^2$ have a non-empty intersection with any closed hemisphere. Then c has at least four inflection points. □

Here are its two famous corollaries:

Theorem 1.2. *V. Arnold's tennis ball theorem, see* [3], [12].

Any smooth closed simple curve $c \subset S^2$ bisecting the area of the sphere has at least four inflection points. □

Theorem 1.3. *Möbius theorem, see* [12].

A smooth closed simple non-contractible curve $c \subset \mathbb{R}P^2$ has at least three inflection points. □

Segre's theorem has various applications, generalizations and refinements. In the paper, we present one more Segre-type phenomenon. However, unlike the already existent ones, it deals with closed saddle surfaces on S^3 rather than closed curves. This object is not chosen just by chance: the study of closed saddle surfaces was originally motivated by A.D. Alexandrov's problem (see "Motivations" below).

Key words and phrases. Saddle surface, piecewise linear surface, inflection point, Segre's theorem.

Definitions and the main result. By $S^3 \subset \mathbb{R}^4$ we denote the unit sphere centered at the origin O. A *plane on the sphere* S^3 is a plane in the sense of spherical geometry, i.e., the intersection of S^3 with a Euclidean hyperplane passing through O.

Definition 1.4. A closed surface Γ immersed in S^3 is called *saddle* if no (spherical) plane intersects Γ locally at just one point.

Definition 1.5. A (spherical) *polygon* on the two-dimensional sphere S^2 is a part of S^2 bounded by a piecewise geodesic closed simple curve.

An angle of a polygon is called *convex* (respectively, *reflex*) if it is smaller (respectively, greater) than π.

A vertex of a polygon is called *convex* (respectively, *reflex*) if it is incident to a convex (respectively, *reflex*) angle.

Definition 1.6. A *piecewise linear saddle sphere* (a *PLS-sphere*, for short) on S^3 is an immersed piecewise linear saddle surface which is homeomorphic to S^2.

To avoid degeneracies and non-interesting exceptions, we assume in addition that all edges of a PLS-sphere are shorter than π, and that its vertex-edge graph is 3-connected.

Besides, we assume that the dihedral angle at each edge does not equal π, so the vertex-edge graph has no redundant edges.

Given an oriented PLS-sphere, we can speak of its *convex and concave* edges. In the sequel, we paint all the convex (respectively, concave) edges red (respectively, blue).

Definition 1.7. A PLS-sphere is called *elementary Barner* if there is a point $p \in S^3$ such that each great semicircle with endpoints at p and at its antipode $-p$ hits the surface exactly once.

Equivalently, an elementary Barner PLS-sphere admits a bijective projection π onto some equator $S^2 \subset S^3$, see Fig. 2.

Elementary Barner saddle spheres are of a particular interest because of a relationship to A.D. Alexandrov's problem (see "Motivations" below).

The interplay between PLS-spheres and smooth saddle spheres is not well understood yet. On the one hand, it seems plausible that a piecewise linear saddle sphere can be approximated by a smooth saddle sphere and vice versa. On the other hand, there is just one proven result (see [13]). It asserts that an elementary Barner PLS-sphere with a trivalent vertex-edge graph has a C^∞-smooth saddle approximation.

By topological reasons, a smooth saddle sphere necessarily has flattening points. In some sense, the below defined inflection and reflex faces play the role of flattening phenomena of a piecewise linear saddle sphere.

Definition 1.8. • A face f of a PLS-sphere Γ is *an inflection face* if

(1) f is bounded by two convex broken lines (say, by L_1 and L_2) such that the convexity directions look like in Fig. 1.
(NB. A polygon with such convexity properties does not exist in Euclidean plane.)

(2) All the edges of L_1 are convex, whereas all the edges of L_2 are concave.

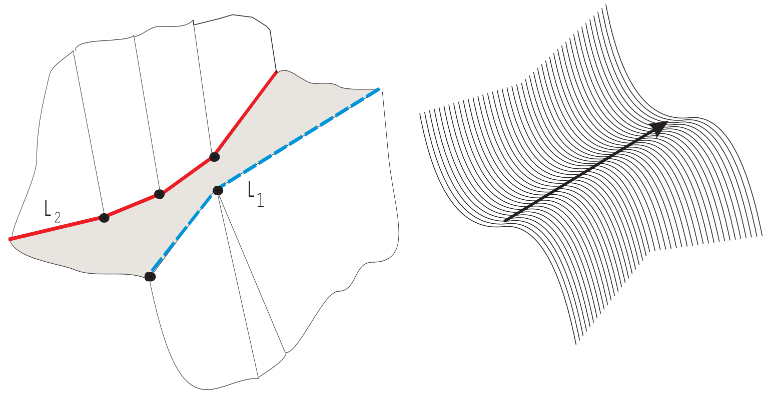

FIGURE 1. A fragment of an inflection face. An inflection arch of a smooth saddle surface

- A face f of a PLS-sphere Γ is called *a reflex face* if it contains a (two-dimensional) hemisphere.

Inflection faces as well as reflex faces represent a kind of singularity of the surface Γ: none of them fits in a hemisphere (see Lemma 5.1).

The main result of the paper describes singularities of a saddle sphere:

Theorem 1.9. (1) *Each saddle sphere $\Gamma \subset S^3$ belongs to one of the following disjoint classes:*
   (a) *Γ has at least two reflex faces.*
   (b) *Γ has exactly one reflex face and at least two inflection faces.*
   (c) *Γ has no reflex faces and at least 4 inflection faces.*

(2) *There are saddle spheres with*
   (a) *exactly two reflex faces.*
   (b) *exactly one reflex face and exactly two inflection faces.*
   (c) *no reflex faces and any number of inflection faces greater than 4.*

(3) *There are no embedded PLS-spheres on S^3 of type (1a).*

(4) *There are no embedded PLS-spheres on $\mathbb{R}P^3$.*

(5) *There exist immersed PLS-spheres on $\mathbb{R}P^3$.*

(6) *There are no elementary Barner PLS-spheres of types (1a) and (1b).*

(7) *There exist elementary Barner PLS-spheres of type (1c) with any number of inflection faces greater than 4. Moreover, the set of elementary Barner PLS-spheres with a fixed number of inflection faces is disconnected.* □

Outline of the proof. Combinatorially, a PLS-sphere is a planar graph with additional equipment: its edges are colored and some of the angles (the reflex ones) are marked. This equipment necessarily has some properties which follow from the discrete Segre's theorem proven in Section 2.

This leads to a combinatorial notion of a saddle graph. Reflex and inflection faces are easily encoded in the combinatorial language, and we prove their existence using just combinatorics. Some similar phenomena are already discussed in [5] and [6]; our approach combines in a sense these ideas.

Motivations.

- The proof of the Theorem 1.9 is based on and generalizes the Segre's theorem. Here is one more link to the Segre's theorem: a surface Γ is saddle if and only if its intersection with a small sphere centered at any of its vertices satisfies the condition of the discrete Segre's theorem.
- There exist embedded saddle tori on $\mathbb{R}P^3$. V. Arnold [2] formulated some conjectures about them (and about their higher dimensional versions). Some of the conjectures proved to be wrong [11], in partial cases some of them are true [7, 8], but two of them still stand open for $\mathbb{R}P^3$. In particular, Arnold conjectured that the set of all smooth saddle tori embedded in $\mathbb{R}P^3$ is connected (compare with Theorem 1.9, (7)). This paper sheds no light to Arnold's conjecture, but it treats some similar objects.
- Smooth elementary Barner saddle spheres arose originally in a relationship (see [10, 13]) to the following uniqueness conjecture proven for analytic surfaces by A. D. Alexandrov in [1]:

  Let $K \subset \mathbb{R}^3$ be a smooth convex body. If for a constant C, at every point of ∂K, we have $R_1 \leq C \leq R_2$, then K is a ball. (R_1 and R_2 stand for the principal curvature radii of ∂K).

  Here is the link: let K be a counterexample to the conjecture. Denote by h_K its support function and denote by h_C the support function of the ball of radius C. The graph γ of the difference $h_K - h_C$ is a conical surface in $\mathbb{R}^4$ with the apex at the origin O. Its intersection with S^3 is an elementary Barner saddle sphere (see Fig. 2).

  Vice versa, a smooth elementary Barner saddle sphere yields a cone in $\mathbb{R}^4$ which can be interpreted as the graph of some positively homogeneous function h. For a sufficiently large C, the sum $h + h_C$ is a convex function. Then it is a support function of some convex body K which is a counterexample to the conjecture.

  To summarize, each smooth elementary Barner saddle sphere yields a counterexample to the conjecture. An observation was made that all saddle spheres constructed in [10] and [13] have *inflection arches*. Later, the existence of at least four inflection arches for elementary Barner saddle spheres was proven in [14]. The above defined *inflection faces* represent a piecewise linear counterpart of inflection arches.
- We were also motivated by the following toy problem:

  Given a piecewise linear saddle surface in $\mathbb{R}^3$, is it possible to alter it locally (i.e., in a neighborhood of a vertex), maintaining its saddle property?

  In Section 5 we show that it is never possible.

A convention about figures. Fix a hyperplane $H \subset \mathbb{R}^4$ not passing through the origin O. The projection from the origin $pr : S^3 \to H$ maps bijectively some open hemisphere onto H. Spherical planes and lines are mapped to Euclidean planes and lines. Therefore, pr preserves convexity and saddle property. By this reason, we will sometimes depict spherical objects as their images under pr and refer to the convexity type of the image, as in Fig. 1, 3, 12.

Alternatively, if a spherical drawing does not fit in a hemisphere, it makes sense to depict it schematically, as in Fig. 6, 14.

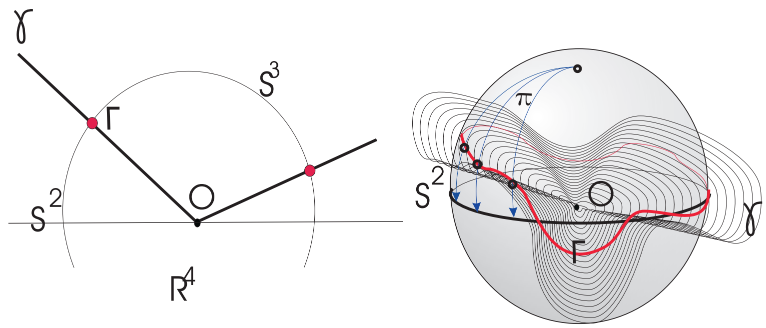

Figure 2. Elementary Barner sphere

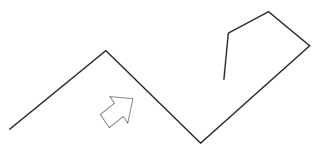

Figure 3. An inflection edge

2. Discrete Segre's theorem

We consider piecewise linear simple closed curves c on the unit sphere S^2. If an edge of such a curve is shorter than π, it is called *short*. Otherwise, we call it *long*.

Definition 2.1. A closed simple (i.e., embedded) curve $c \subset S^2$ is *spanning* if it intersects each closed hemisphere.

A closed simple curve c is *strongly spanning* if it intersects each open hemisphere.

Definition 2.2. Let $c \subset S^2$ be a piecewise linear simple closed curve. It splits S^2 into two (spherical) polygons. After fixing one of them, it makes sense to speak of convex and reflex angles of c.

An edge is called an *inflection edge* of c (see Fig. 3) if it is incident to both convex and reflex angles.

Theorem 2.3. *(Discrete Segre's Theorem)*

(1) *A strongly spanning piecewise linear closed simple curve has at least 4 inflection edges.*
(2) *Let $c \subset S^2$ be a spanning piecewise linear closed simple curve. We assume that c has more than 2 vertices. Then one of the two (non-disjoint) assertions hold:*
  (a) *c has at least 4 inflection edges,*
  (b) *c has a long edge (say, e) and at least 2 inflection edges among the edges excluding e.*
(3) *Let $c = (P_1, ..., P_n) \subset S^2$ be a spanning piecewise linear closed simple curve with vertices $\{P_1, ..., P_n\}$. Assume that c has more than 2 vertices and at least two long edges. Then for any two long edges P_iP_{i+1} and P_jP_{j+1}, there is at least one inflection edge among the edges lying between them (that is, among the edges $P_{i+1}P_{i+2}, P_{i+2}P_{i+3}, ..., P_{j-1}P_j$).*

Proof. The idea of the proof is to approximate c by an appropriate smooth curve c' and to apply then Segre's theorem. However, this needs some accuracy: if the curve c is not strongly spanning, its smooth approximation c' can be non-spanning.

(1) Suppose c is strongly spanning. Then it can be approximated by a smooth curve c' such that c' has only isolated inflection points which are in a natural bijection with inflection edges of c. It remains to observe that a sufficiently close c' is spanning, and to apply Segre's theorem to the curve c'.

(2) Suppose c is spanning, but not strongly spanning. Then there exists a closed hemisphere S^+ containing c. Denote by b its boundary (b is a great circle). We may assume that $b \cap c$ is a union of some geodesic segments $e_{i_1}, ..., e_{i_m}$ of non-zero length (see Fig. 4). Two cases should be treated separately:

   (a) Suppose all the edges $e_{i_1}, ..., e_{i_m}$ are short.
   Note first that each semicircle $b^+ \subset b$ intersects the curve c. Take a smooth approximation c' of the curve c such that c' has only isolated inflection points which are in a natural bijection with inflection edges of c plus the following additional property: the curve c' tangents each of the segments $e_{i_1}, ..., e_{i_m}$, and each semicircle b^+ contains at least one tangent point (see Fig. 4). This is always possible by Caratheodory theorem. This guarantees that c' is spanning. It remains to apply Segre's theorem to the curve c'.

   (b) Suppose one of the edges (say, e) is long. We may assume that $|e| > \pi$. We approximate c by a smooth curve c' such that c' has only isolated inflection points which are in a natural bijection with inflection edges of c except for two extra inflection points lying on e (see Fig. 5). It remains to apply Segre's theorem to the curve c'.
   We explore here the following phenomenon: suppose a (geodesic) segment in the plane is approximated by a smooth curve which tangents the segment at the endpoints. Then by Möbius Theorem, the curve has at least 2 inflection points (except for the endpoints). For a long segment on the sphere, such a curve can have no inflection points.

(3) The curve c is strongly spanning and has therefore at least 4 inflection edges.

   Assume the contrary, i.e., that the chain $P_{i+1}, P_{i+2}, ..., P_j$ contains no inflection edges. The (non-closed) curve $P_{i+1}, P_{i+2}, ..., P_j$ is contained in the lune bounded by P_iP_{i+1} and the extension of $P_{i+1}P_{i+2}$ (see Fig. 6).

   Indeed, if not, i.e., if $P_{i+1}, P_{i+2}, ..., P_j$ hits the extension of $P_{i+1}P_{i+2}$ (the dotted line) at a point A, then the curve c'' depicted in Fig. 6, 2 has at least two inflection edges. A contradiction.

   Now prove the theorem. We replace the curve c by another curve c' as is depicted in Fig. 6, 1. By the above proven, c' is simple. Since the curve c' is strongly spanning, it has at least 4 inflection edges. A contradiction. $\square$

3. Saddle graphs

By a *graph* we mean a tuple $G = (V, E)$ where V is a (finite) set of vertices and E is the set of edges (unordered pairs of different vertices).

For $v \in V$, denote by $E(v)$ the set of edges incident to the vertex v.

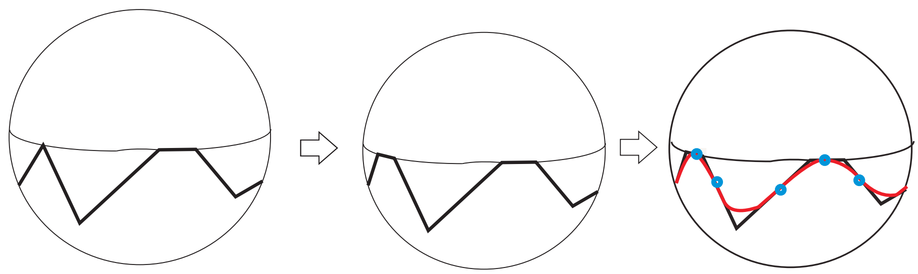

FIGURE 4. Smoothing of a curve without long edges

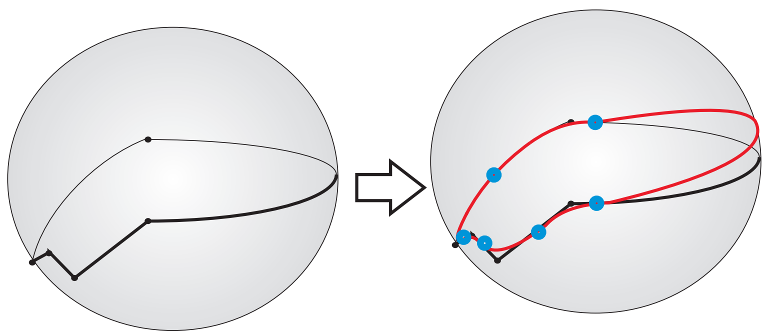

FIGURE 5. Smoothing of a non-strongly spanning curve with a long edge

Let $G = (V, E)$ be a 3-connected planar graph. All its embeddings in the sphere S^2 are known to have one and the same facial structure. Therefore, we have a natural notion of a *face* of the graph and a cyclic ordering on the set $E(V)$. Besides, we have a well-defined notion of angles:

Definition 3.1. An unordered pair of edges (e_1, e_2) is called an *angle* of G if the edges e_1 and e_2 are consecutive edges of a face of the graph G. The set of all angles we denote by $A(G)$. The set of all angles incident to a vertex v we denote by $A(v)$.

The next idea is to add the so called saddle structure to a graph G. Namely, we paint convex edges red and we paint concave edges blue. Besides, we mark all the reflex angles.

Till now, a graph G is just a combinatorial object, so in the below definition, the combinatorial convexity and concavity have no geometrical meaning. The saddle structure is defined axiomatically.

However, later we shall see that if a graph G together with a coloring on its edges arise from some saddle sphere, then it satisfies the axioms from the below definition.

Definition 3.2. Let $G = (V, E)$ be a 3-connected planar graph.

Let $Col : E \to \{red, blue\}$ and $Refl : A(G) \to \{0, 1\}$ be some mappings.

Angles with $Refl(a) = 1$ we call (combinatorially) *reflex angles*.

We say that a triple $(G, Col, Refl)$ is a *graph equipped with a saddle structure* (*a saddle graph*, for short) if for any vertex v, we have the following (see Fig. 7):

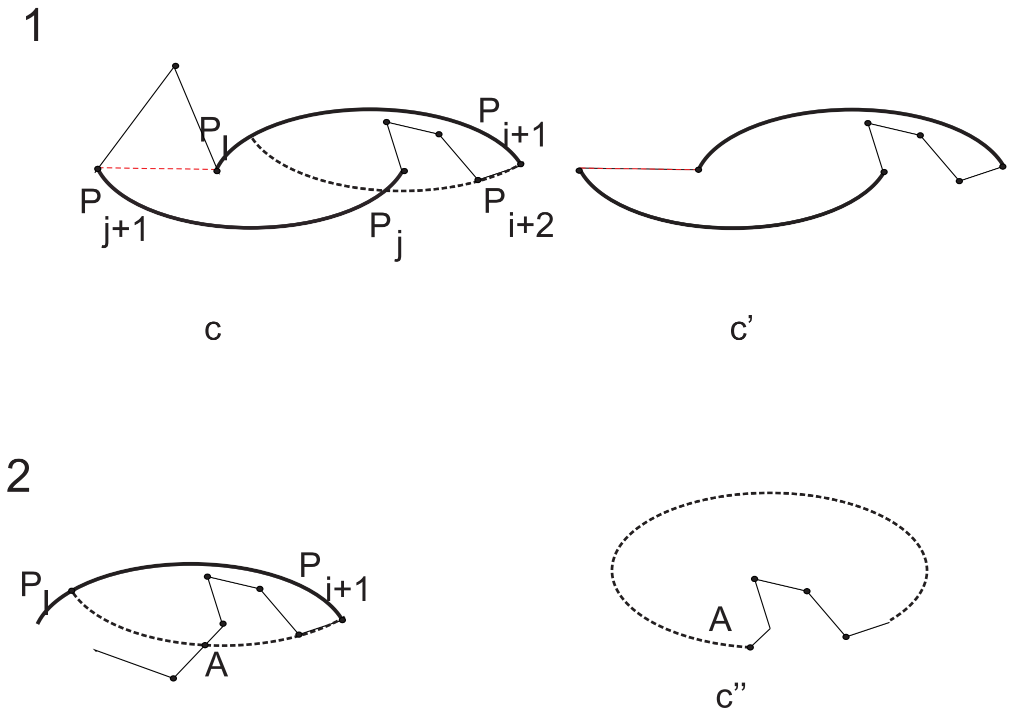

FIGURE 6.

(1) **"No reflex angles condition"** If $Refl$ is identically 0 on $A(v)$ (i.e., there are no reflex angles incident to v), then the number of changes of the function Col when going around the vertex v is greater or equal than 4.
(2) **"Exactly one reflex angle condition"** If there is exactly one reflex angle at v, (say, $Refl(e_i, e_j) = 1$), then the function Col changes at least twice when going around the vertex v from e_i to e_j.
(3) **"More than one reflex angle condition"** If there are more than one reflex angle at v, then we claim two things: (1) that the total number of color changes when going around the vertex v is greater or equal than 4 and (2) that the color changes at least once when going from one edge of a reflex angle to the edge of the next reflex angle.

Definition 3.3. For a face f of a saddle graph, we algorithmically define its *index* $i(f)$, see Fig. 8:

(1) At the beginning, put $i(f) := 0$. Start going along the boundary of the face f.
(2) Once we pass by a vertex at which the color changes, put
$$i(f) := i(f) + 1.$$
(3) Once we pass by a vertex, if the color does not change and the angle we are passing by is reflex, we keep $i(f)$ unchanged.
(4) Once we pass by a vertex, if the color does not change and the angle we are passing by is not reflex, put $i(f) := i(f) + 2$.

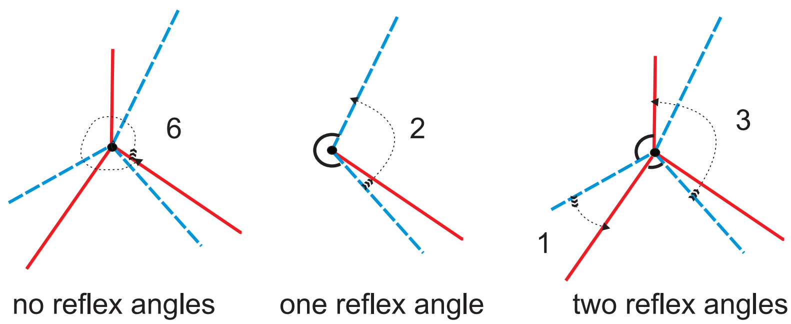

FIGURE 7. Counting color changes

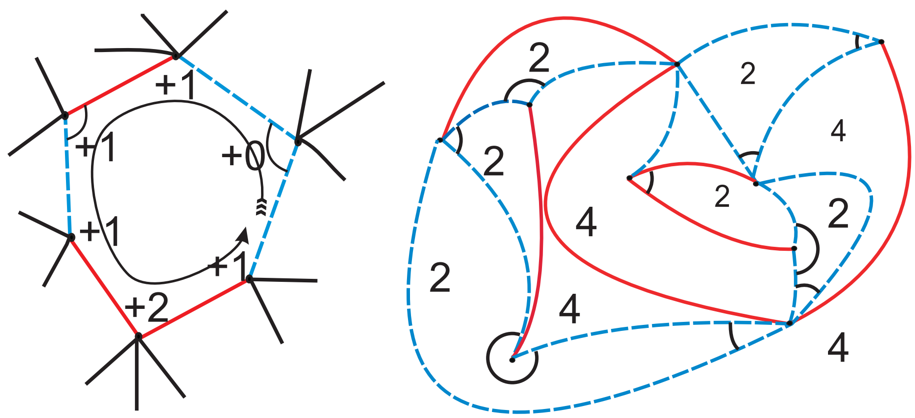

FIGURE 8. A saddle graph and the values of $i(f)$

Definition 3.4. Let v be a vertex of a saddle graph. An edge e incident to v is called *superfluous with respect to the vertex* v if its deletion maintains the properties (1)–(3) of the Definition 3.2 at the vertex v.

We describe below some local graph transformations, the *elementary splittings* of three types.

Definition 3.5. (1) For two neighbor edges of different colors, one of which is superfluous, the local graph transformation depicted in Fig. 9 is called the *first elementary splitting*.

Here are the formalities: if a blue edge av is superfluous with respect to v and a red edge bv is neighbor to av at the vertex v, then the first elementary splitting looks as follows:

(a) Remove from the graph the edges av and bv
(b) Add a new vertex d, red edges bd and dv, and a blue edge ad
(c) Mark the angle bdv as reflex.

(2) Suppose that a vertex v has no adjacent reflex angles and exactly 4 incident edges. The local graph transformation depicted in Fig. 10, 1 is called the *second elementary splitting*.

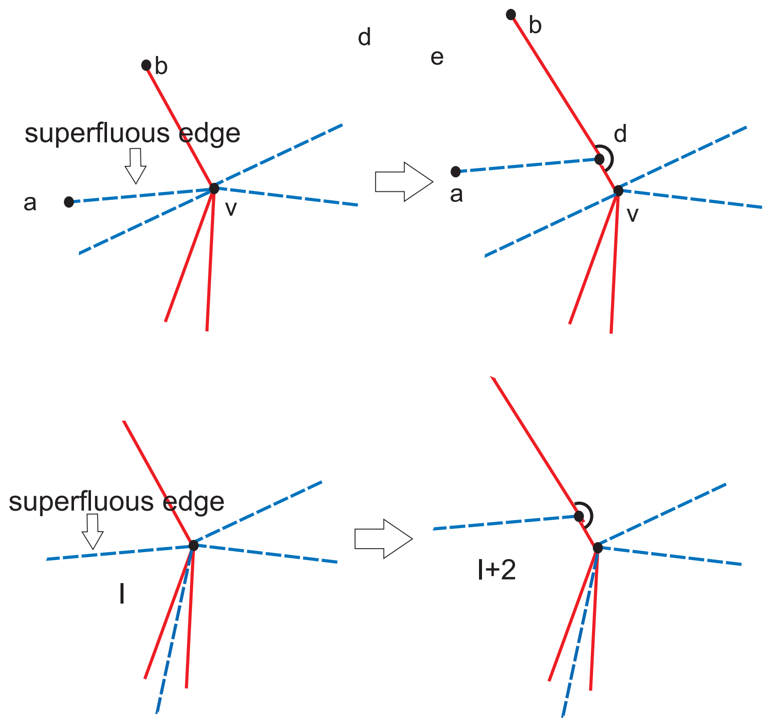

FIGURE 9. Two elementary splittings of the first type. For the first example, the index i is maintained. For one face of the second example, it increases on 2.

More precisely, let a, b, c, and d be vertices adjacent to v. Assume that the edges av and cv are red. We do the following:
(a) Remove from the graph the edges cv and bv
(b) Add a new vertex e, red edges ve and ce, and a blue edge be
(c) Mark the angles ave and vec as reflex.

Definition 3.6. Suppose a vertex v is incident to more than one reflex angles. The following procedure describes the splitting which takes reflex angles apart.

(1) Choose two edges e and e' of one and the same color (say, red) incident to the vertex v such that the edges e and e' are separated by reflex angles, see Fig. 11.
(2) Split the vertex v into two vertices, split also the two edges e and e' and add one more edge of the other color (here it is blue) as is shown in Fig. 11. This local graph transformation is called the *third elementary splitting*.

More precisely, let av and bv be the edges e and e'. Assume that they are red. We do the following:
(a) The set of all the edges incident to the vertex v (except for the edges e and e') is divided by the broken line avb into two parts E_1 and E_2.
(b) Add a new vertex v', red edges av' and bv', and a blue edge vv'
(c) Each edge xv from E_2 replace by the new edge xv'.

An easy check proves that:

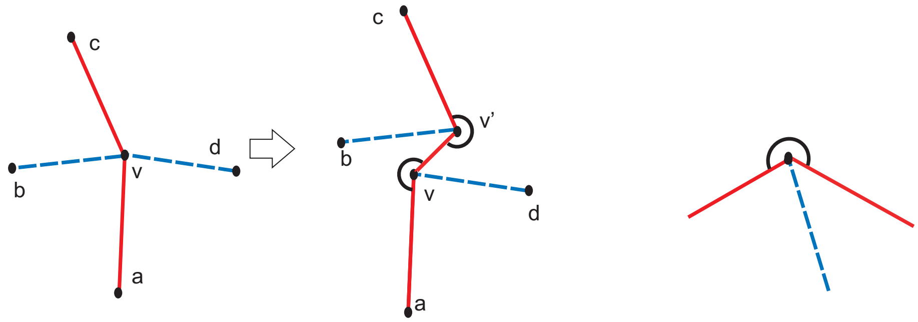

FIGURE 10. Second elementary splitting. All the indices are maintained.

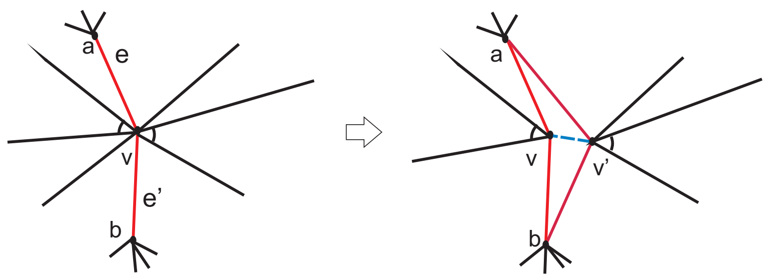

FIGURE 11. Third elementary splitting adds two faces with $i =$ 4. All the other indices are maintained.

Lemma 3.7. (1) *An elementary splitting of a saddle graph yields a saddle graph.*
(2) *For any first or second elementary splitting, the faces of the new graph are in a natural bijection with the faces of the original graph.*
(3) *A third elementary splitting adds two faces with $i = 4$.*
(4) *The index i of a face does not decrease after any elementary splitting.* □

Lemma 3.8. *Each saddle graph is reducible to a trivalent saddle graph via a chain of elementary splittings.*

Proof. Third elementary splittings enable us to get a graph with at most one reflex angle at each vertex. Next, we treat all the vertices one by one. After fixing a vertex v, we first get rid of all superfluous edges incident to v. We arrive at one of the two possible cases depicted in Fig. 10. In the second case, we are done. In the first case it remains to apply the second splitting. □

The following theorem is a combinatorial version of Theorem 1.9, (1).

Theorem 3.9. *For each saddle graph, one of the following statements is valid:*
(1) *The graph has at least two faces with $i(f) = 0$.*
(2) *The graph has one face with $i(f) = 0$ and at least 2 faces with $i(f) = 2$.*
(3) *The graph has no faces with $i(f) = 0$ and at least 4 faces with $i(f) = 2$.*

Proof. Due to Lemma 3.8 and Lemma 3.7, we may assume that the graph is trivalent. At each of its vertex it looks like the graph in Fig. 10, 2 (up to color reverting). Count the total sum Σ of indices $i(f)$ for all the faces f. The contribution of each vertex equals 2, therefore we have $\Sigma = 2 \mid V \mid$. The Euler formula for trivalent graphs $2 \mid F \mid = \mid V \mid +4$ implies that $\Sigma = 4 \mid F \mid -8$. Since the index $i(f)$ is always positive and even, we are done. (Here $\mid F \mid$ and $\mid V \mid$ denote the number of faces and vertices respectively.) □

Proof of Theorem 1.9, (1)

We associate a saddle graph $SG(\Gamma)$ to a saddle sphere Γ:

(1) Set the graph G equal the vertex-edge graph of the surface Γ.
(2) Fix an orientation of Γ. Now it makes sense to speak of convex and concave edges. For an edge e, set

$$Col(e) = \begin{cases} \text{red}, & \text{if } e \text{ is convex;} \\ \text{blue}, & \text{otherwise.} \end{cases}$$

(3) For an angle a, set

$$Refl(a) = \begin{cases} 1, & a \text{ is a reflex angle on } \Gamma; \\ 0, & \text{otherwise.} \end{cases}$$

For a saddle sphere, $SG(\Gamma)$ is a saddle graph. Indeed, the properties from Definition 3.2 follow from the discrete Segre's theorem. Given a vertex v of the surface Γ, take its Euclidean image and a small sphere S_v centered at the point v. The intersection $\Gamma \cap S_v$ is a piecewise linear simple spanning curve with more than two vertices.

Next, we apply Theorem 3.9 to the saddle graph $SG(\Gamma)$. To conclude the proof, it remains to understand the geometrical meaning of the index $i(f)$.

Lemma 3.10. *For a a face f of a saddle sphere Γ, we have:*

(1) $i(f) = 0$ *implies that f is a reflex face.*
(2) $i(f) = 2$ *implies that f is either a reflex face or an inflection face.*

Proof. (1). $i(f) = 0$ implies that the complement of f is a (spherical) polygon with convex angles. Such polygons are known to lie in an open hemisphere.

(2). If $i(f) = 2$, three cases are possible:

(1) The face f has no convex angles. Then its complement lies in an open hemisphere.
(2) The face f has exactly one convex angle. This means that the boundary of f has exactly 2 inflection edges (the ones adjacent to the only convex vertex). By Segre's Theorem, the boundary of f is not a strongly spanning curve, and therefore, fits in an closed hemisphere.
(3) The face f has two convex angles. This implies that the boundary of f has both blue and red edges and the color changes at the convex vertices. This means by definition that f is an inflection face.

□

4. Proof of Theorem 1.9, (2-7)

(2,a). Here is the construction of a saddle sphere with two reflex faces (see Fig. 12): take two (spherical) planes an join them by a polytopal tube.

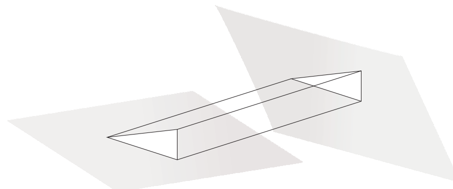

Figure 12. A saddle sphere with two reflex faces

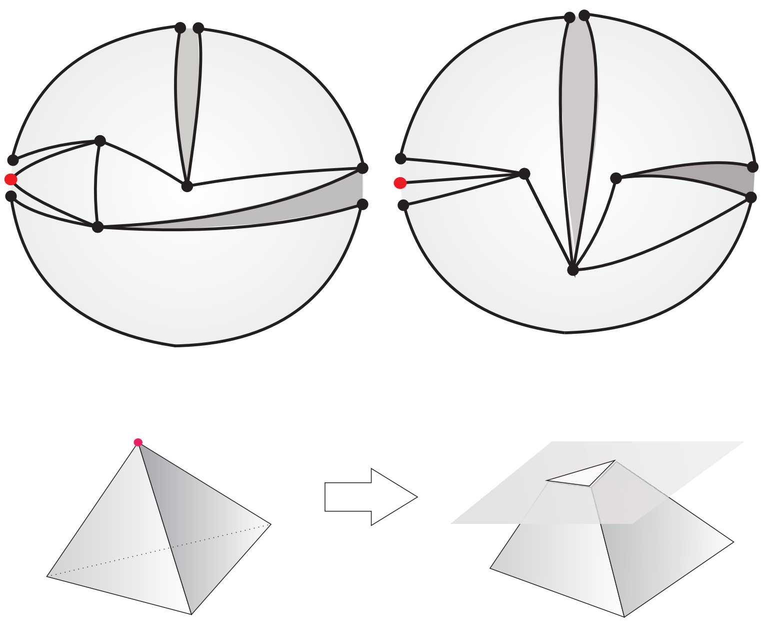

Figure 13. Saddle sphere with one reflex face and two inflection faces. The shadowed tiles correspond to inflection faces

(2,b). The construction of a saddle sphere with just one reflex face and two inflection faces is based on Maxwell-Cremona theorem and Laman theory for planar graphs embedded in S^2 (see details in [4] and [16]).

Figure 13 depicts a tiling of the sphere S^2 generated by an embedded graph. The graph is a *rigidity circuit*, therefore it has a 3D *lifting*, that is, there exists a piecewise linear surface Γ embedded in S^3 whose bijective projection π (see Fig. 2) onto S^2 yields this tiling. All the vertices of Γ (except for a single one) have an incident reflex angle. Therefore, the surface Γ is saddle everywhere except for just one vertex (marked red in Fig. 13). Next, we truncate Γ at the convex vertex and patch a reflex face. The result is the desired surface.

(2,c). An example of an elementary Barner sphere with any number of inflection faces greater than 4 was constructed in [9].

(3). A saddle sphere with two reflex faces is never embedded since the reflex faces necessarily have an intersection.

(4). Suppose the contrary: there exists an embedded PLS-sphere $\Gamma \subset \mathbb{R}P^3$. Consider the standard covering $\varphi : S^3 \to \mathbb{R}P^3$. The preimage of Γ is a union of two embedded saddle spheres on S^3. Each of them has either an inflection face or a reflex face f. But φ is not injective on f.

(5). The mapping φ maps an immersed saddle sphere to an immersed saddle sphere.

(6). Projections of two reflex faces (or a reflex face and an inflection face) on any (spherical) plane necessarily have an intersection. This is because each such projection necessarily contains a lune, see Lemma 5.1.

(7). The existence of an elementary Barner saddle sphere with any number of inflection faces greater than 3 was proven in [9]. The set of all elementary Barner saddle spheres with exactly 4 inflection faces is disconnected. This was proven in [14].

Furthermore, paper [15] gives a combinatorial classification of elementary Barner saddle sphere with any number of inflection faces greater than 3. Each elementary Barner saddle sphere $\Gamma \subset S^3$ generates an arrangement of (at least four) non-crossing oriented great semicircles on S^2. Namely, take the bijective projection of Γ onto some equator S^2 (it exists by definition). The projection of each inflection face (see Lemma 5.1) contains a great semicircle which carries an orientation generated by red-blue sides of the projection. If we take one oriented great semicircle for each inflection face, we get an arrangement of non-crossing oriented great semicircles on S^2. In the paper [15] the converse is proven: each spanning arrangement of non-crossing oriented great semicircles is generated by an elementary Barner saddle sphere. Since there exist non-isotopic arrangements with one and the same number of great semicircles, the theorem is proven.

In particular, this means the diversity of saddle spheres on S^3. □

5. An application to saddle surfaces in Euclidean space

Lemma 5.1. (1) *Two inflection faces of an elementary Barner saddle sphere cannot have a common convex vertex.*

(2) *For an inflection face f, let s_1 and s_2 be linear segments lying on L_1 and L_2 respectively (we use notation of Definition 1.8). Then the lune bounded by extended s_1 and s_2 lies in f.*

(3) *An inflection face contains a geodesic arc (a great semicircle) joining two antipodal points of S^3.*

Proof. (1). Indeed, in this case, projections of the faces to any spherical plane have a nonempty intersection. (2) follows from convexity properties of L_1 and L_2 and implies (3). □

Consider a piecewise linear saddle surface M in $\mathbb{R}^3$ with the only vertex O (i.e., M is a conical surface, as in Fig. 15). Assume in addition that M can be bijectively projected onto some plane E. A natural question which arose in attempts to develop a saddle approximation technique was the following:

Can we alter M locally, maintaining its saddle properties?

The answer is "No":

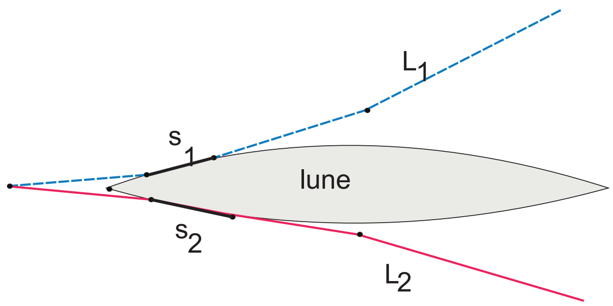

FIGURE 14. An inflection face (and its projection) contains a lune

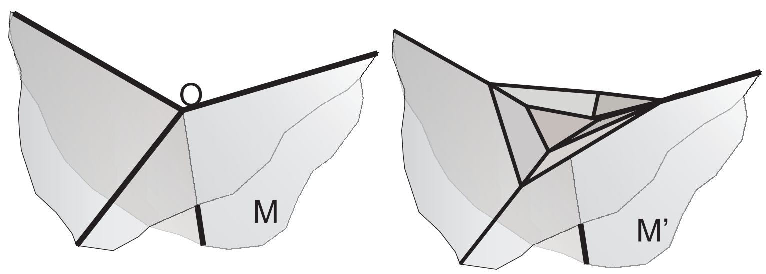

FIGURE 15.

Proposition 5.2. *In the above notation, suppose that for a piecewise linear saddle surface $M' \in R^3$ the following is true:*

- *M' coincides with M outside a ball centered at O;*
- *M' can be bijectively projected onto the plane E.*

Then $M = M'$.

Proof. Assume that $M' \neq M$. We raise the surface M' to the sphere S^3. Namely, we take the preimage $pr^{-1}(M')$ under the central projection $pr : S^3 \to \mathbb{R}^3$. The closure of the preimage is some elementary Barner saddle sphere M'_{sph}. By Theorem 1.9, surface M'_{sph} necessarily has 4 inflection faces. The only candidates are those coming from unbounded faces of M'. But each two of them have an intersection, which is impossible. □

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FAST LOOPS ON SEMI-WEIGHTED HOMOGENEOUS HYPERSURFACE SINGULARITIES

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Abstract. We show the existence of essential fast loops on semi-weighted homogeneous hypersurface singularities with weights $w_1 \geq w_2 > w_3$. In particular we show that semi-weighted homogeneous hypersurface singularities are metrically conical only if their two lowest weights are equal.

1. Introduction

Let $X \subset \mathbb{R}^n$ be a subanalytic set with a singularity at x. It is well-known for small real numbers $\epsilon > 0$ that there exists a homeomorphism from the Euclidean ball $B(x, \epsilon)$ to itself which maps $X \cap B(x, \epsilon)$ onto the straight cone over $X \cap S(x, \epsilon)$ with vertex at x. The homeomorphism h is called a *topologically conical structure* of X at x and, since John Milnor proved the existence of topologically conical structure for algebraic complex hypersurfaces with an isolated singularity [9], some authors say ϵ is a *Milnor radius* of X at x. Some developments of the Lipschitz geometry of complex algebraic singularities come from the following question: given an algebraic subset $X \subset \mathbb{C}^n$ with an isolated singularity at x, is there $\epsilon > 0$ such that $X \cap B(x, \epsilon)$ is bi-Lipschitz homeomorphic to the cone over $X \cap S(x, \epsilon)$ with vertex at x? When we have a positive answer for this question we say that (X, x) *is metrically conical*. Some motivations for this question were given in [3], [6] and, in the same papers, the above question was answered negatively. The strategy used in [6] to show that some examples of complex weighted homogeneous surface singularities (X, x) are not metrically conical was to exhibit nontrivial loops on $X \cap S(x, \epsilon)$ which the diameter goes to 0 faster than linearly as $\epsilon \to 0$. In this paper we analyze semi-weighted homogeneous hypersurface singularities under the same

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point of view above and, in particular, we show that semi-weighted homogeneous hypersurface singularities are metrically conical only if its two lowest weights are equal.

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2. Preliminaries

2.1. **Inner metric.** Given an arc $\gamma\colon [0,1] \to \mathbb{R}^n$, we remember that the *length of* γ is defined by

$$l(\gamma) = \inf\{\sum_{i=1}^{m} |\gamma(t_i) - \gamma(t_{i-1})| \ : \ 0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = 1\}.$$

Let $X \subset \mathbb{R}^n$ be a subanalytic connected subset. It is well-know that the function

$$d_X\colon X \times X \to [0, +\infty)$$

defined by

$$d_X(x,y) = \inf\{l(\gamma) \ : \ \gamma\colon [0,1] \to X;\ \gamma(0) = x,\ \gamma(1) = y\}$$

is a metric on X, so-called *inner metric* on X.

Theorem 2.1 (Pancake Decomposition [8])**.** *Let $X \subset \mathbb{R}^n$ be a subanalytic connected subset. Then, there exist $\lambda > 0$ and $X_1, \ldots, X_m$ subanalytic subsets such that:*

a. $X = \bigcup_{i=1}^m X_i$,
b. $d_X(x,y) \leq \lambda|x-y|$ *for any* $x, y \in X_i$, $i = 1, \ldots, m$.

2.2. **Horn exponents.** Let $\beta \geq 1$ be a rational number. The germ of

$$H_\beta = \{(x,y,z) \in \mathbb{R}^3 \ : \ x^2 + y^2 = z^\beta,\ z \geq 0\}$$

at $0 \in \mathbb{R}^3$ is called a *β-horn*.

By results of [1], we know that a β_1-horn is bi-Lipschitz equivalent, with respect to the inner metric, to a β_2-horn if, and only if $\beta_1 = \beta_2$. Let $\Omega \subset \mathbb{R}^n$ be a 2-dimensional subanalytic set. Let $x_0 \in \Omega$ be a point such that Ω is a topological 2-dimensional manifold without boundary near x_0.

Theorem 2.2. [1] *There exists a unique rational number $\beta \geq 1$ such that the germ of Ω at x_0 is bi-Lipschitz equivalent, with respect to the inner metric, to a β-horn.*

The number β is called *the horn exponent of* Ω *at* x_0. We use the notation $\beta(\Omega, x_0)$. By Theorem 2.2, $\beta(\Omega, x_0)$ is a complete intrinsic bi-Lipschitz invariant of germs of subanalytic sets which are topological 2-dimensional manifold without boundary. In the following, we show a way to compute horn exponents.

According to [2], $\beta(\Omega, x_0) + 1$ is the volume growth number of Ω at x_0, i. e.

$$\beta(\Omega, x_0) + 1 = \lim_{r \to 0+} \frac{\log \mathcal{H}^2[\Omega \cap B(x_0, r)]}{\log r}$$

where $\mathcal{H}^2$ denotes the 2-dimensional Hausdorff measure with respect to Euclidean metric on $\mathbb{R}^n$.

2.3. **Order of contact of arcs.** Let $\gamma_1 \colon [0, \epsilon) \to \Omega$ and $\gamma_2 \colon [0, \epsilon) \to \Omega$ be two continuous semianalytic arcs with $\gamma_1(0) = \gamma_2(0) = x_0$ and not identically equal to x_0. We suppose that the arcs are parameterized in the following way:

$$\|\gamma_i(t) - x_0\| = t, \ i = 1, 2.$$

Let $\rho(t)$ be a function defined as follows: $\rho(t) = \|\gamma_1(t) - \gamma_2(t)\|$. Since ρ is a subanalytic function there exist numbers $\lambda \in \mathbb{Q}$ and $a \in \mathbb{R}$, $a \neq 0$, such that

$$\rho(t) = at^\lambda + o(t^\lambda).$$

The number λ is called *an order of contact of* γ_1 *and* γ_2. We use the notation $\lambda(\gamma_1, \gamma_2)$ (see [4]).

Let K be the field of germs of subanalytic functions $f \colon (0, \epsilon) \to \mathbb{R}$. Let $\nu \colon K \to \mathbb{R}$ be a canonical valuation on K. Namely, if $f(t) = \alpha t^\beta + o(t^\beta)$ with $\alpha \neq 0$ we put $ord_t(f(t)) = \beta$.

Lemma 2.3. *Let γ_1, γ_2 be a pair of semianalytic continuous arcs such that $\gamma_1(0) = \gamma_2(0) = x_0$ and $\gamma_i \neq x_0$ $(i = 1, 2)$. Let $\tilde{\gamma_1}(\tau)$ and $\tilde{\gamma_2}(\tau)$ be semianalytic parameterizations of γ_1 and γ_2 such that $\|\tilde{\gamma_i}(\tau) - x_0\| = \tau + o_i(\tau)$, $i = 1, 2$. Then $ord_\tau \|\tilde{\gamma_1}(\tau) - \tilde{\gamma_2}(\tau)\| \leq \lambda(\gamma_1, \gamma_2)$.*

The following result is an alternative way to compute horn exponents of germs of subanalytic sets which are topological 2-dimensional manifold without boundary.

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^n$ be a 2-dimensional subanalytic set. Let $x_0 \in \Omega$ be a point such that Ω is a topological 2-dimensional manifold without boundary near x_0. Then $\beta(\Omega, x_0) = \min\{\lambda(\gamma_1, \gamma_2) \ : \ \gamma_1, \gamma_2$ are semianalytic arcs on Ω with $\gamma_1(0) = \gamma_2(0) = x_0\}$.*

Lemma 2.3 and Theorem 2.4 were proved in [5].

3. Fast loops

Let $X \subset \mathbb{R}^n$ be a subanalytic set with a singularity at x. Let $\epsilon > 0$ be a Milnor radius of X at x and let us denote by X^* the set $X \cap B(x, \epsilon) \setminus \{x\}$. Given a positive real number α, a continuous map $\gamma\colon S^1 \to X^*$ is called an α-fast loop if there exists a homotopy $H\colon S^1 \times [0,1] \to X \cap B(x, \epsilon)$ such that

(1) $H(\theta, 0) = x$ and $H(\theta, 1) = \gamma(\theta)$, $\forall\ \theta \in S^1$,
(2) $\displaystyle\lim_{r \to 0^+} \frac{1}{r^a} \mathcal{H}^2(Im(H) \cap B(x, r)) = 0$ for each $0 < a < \alpha$,

where $Im(H)$ denotes the image of H.

Given a subanalytic set X and a singular point $x \in X$, according to [2], there exists a positive number c such that any α-fast loop $\gamma\colon S^1 \to X^*$ with $\alpha > c$ is necessarily homotopically trivial. Such a number c is called *distinguished for* (X, x). We define the υ invariant in the following way:

$$\upsilon(X, x) = \inf\{c \ : \ c \text{ is distinguished for } (X, x)\}.$$

The number $\upsilon(X, x)$ defined above is inspired by the first characteristic exponent for the local metric homology presented in [2].

Example 3.1. Let $K \subset \mathbb{R}^n$ be a straight cone over a Nash submanifold $N \subset \mathbb{R}^n$, with vertex at p. Then every loop $\gamma\colon S^1 \to K^*$ is a 2-fast loop. Moreover, if $\alpha > 2$, then each α-fast loop $\gamma\colon S^1 \to K^*$ is homotopically trivial. We can sum up it saying $\upsilon(K, p) = 2$.

Proposition 3.2. *Let (X, x) and (Y, y) be subanalytic germs. If there exists a germ of a bi-Lipschitz homeomorphism, with respect to inner metric, between (X, x) and (Y, y), then $\upsilon(X, x) = \upsilon(Y, y)$.*

Proof. Let $f\colon (X,x) \to (Y,y)$ be a bi-Lipschitz homeomorphism, with respect to the inner metric. Given $A \subset X$, let us denote $\tilde{A} = f(A)$. In this case, $A = f^{-1}(\tilde{A})$, where f^{-1} denotes the inverse map of $f\colon (X,x) \to (Y,y)$.

`Claim.` There are positive constants $k_1, k_2, \lambda_1, \lambda_2$ such that

$$\frac{1}{k_1}\mathcal{H}^2(\tilde{A} \cap B(y, \frac{r}{\lambda_2})) \leq \mathcal{H}^2(A \cap B(x,r)) \leq k_2\mathcal{H}^2(\tilde{A} \cap B(y, \lambda_1 r)).$$

In fact, using Pancake Decomposition Theorem (see Subsection 2.1) and using that f and f^{-1} are Lipschitz maps, we obtain positive constants λ_1, λ_2 such that

$$f(A \cap B(x,r)) \subset (\tilde{A} \cap B(y\lambda_1 r)) \text{ and } f(\tilde{A} \cap B(y,r)) \subset (A \cap B(x\lambda_2 r))$$

and we also obtain positive constants k_1, k_2 such that

$$\mathcal{H}^2(f(A \cap B(x,r))) \leq k_1\mathcal{H}^2(A \cap B(x,r)) \text{ and } \mathcal{H}^2(f^{-1}(\tilde{A} \cap B(y,r))) \leq k_2\mathcal{H}^2(\tilde{A} \cap B(y,r)).$$

Our claim follows from these two inequalities and the two inclusions above.

Now, we use this claim to show that given $\alpha > 0$, a loop $\gamma\colon S^1 \to X \setminus \{x\}$ is an α-fast loop if, and only if, $f \circ \gamma\colon S^1 \to Y \setminus \{y\}$ is an α-fast loop. In fact, let $\gamma\colon S^1 \to X \setminus \{x\}$ be a loop and $H\colon S^1 \times [0,1] \to X$ a homotopy such that $H(\theta, 0) = x$ and $H(\theta, 1) = \gamma(\theta)$, $\forall\ \theta \in S^1$. Thus, $f \circ \gamma\colon S^1 \to Y \setminus \{y\}$ is a loop and $f \circ H\colon S^1 \times [0,1] \to Y$ is a homotopy such that $f \circ H(\theta, 0) = x$ and $f \circ H(\theta, 1) = f \circ \gamma(\theta)$, $\forall\ \theta \in S^1$. Let us denote $A = Im(H)$ and $\tilde{A} = Im(f \circ H)$, i. e., $\tilde{A} = f(A)$. Given $0 < a < \alpha$, by the above claim, we have that

$$\lim_{r \to 0^+} \frac{1}{r^a}\mathcal{H}^2(A \cap B(x,r)) = 0$$

if, and only if,

$$\lim_{r \to 0^+} \frac{1}{r^a}\mathcal{H}^2(\tilde{A} \cap B(y,r)) = 0.$$

In other words, it was shown that $\gamma\colon S^1 \to X \setminus \{x\}$ is an α-fast loop if, and only if, $f \circ \gamma\colon S^1 \to Y \setminus \{y\}$ is an α-fast loop, hence $\upsilon(X,x) = \upsilon(Y,y)$. $\square$

Corollary 3.3. *Let $X \subset \mathbb{R}^n$ be a subanalytic set and $x \in X$ an isolated singular point. If $\upsilon(X,x) > 2$, then (X,x) is not metrically conical.*

Proof. Let N be the intersection $X \cap S(x,\epsilon)$ where $\epsilon > 0$ is chosen sufficiently small. Since x is an isolated singular point of X, we have $N \subset \mathbb{R}^n$ is a Nash submanifold. If (X,x) is metrically conical, $X \cap B(x,\epsilon)$ must be bi-Lipschitz homeomorphic (with respect to the inner metric) to the straight cone over N with vertex at x. Thus, it follows from Proposition 3.2 that $\upsilon(X,x)=2$. □

4. SEMI-WEIGHTED HOMOGENEOUS HYPERSURFACE SINGULARITIES

Remind that a polynomial function $f\colon \mathbb{C}^3 \to \mathbb{C}$ is called *semi-weighted homogeneous* of degree $d \in \mathbb{N}$ with respect to the weights $w_1, w_2, w_3 \in \mathbb{N}$ if f can be presented in the following form: $f = h + \theta$ where h is a weighted homogeneous polynomial of degree d with respect to the weights w_1, w_2, w_3, the origin is an isolated singularity of h and θ contains only monomials $x_1^{m_1}x_2^{m_2}x_3^{m_3}$ such that $w_1m_1 + w_2m_2 + w_3m_3 > d$.

An algebraic surface $S \subset \mathbb{C}^3$ is called *semi-weighted homogeneous* if there exists a semi-weighted homogeneous polynomial $f = h + \theta$ such that

$$S = \{(x_1,x_2,x_3) \in \mathbb{C}^3 \ : \ f(x_1,x_2,x_3) = 0\}.$$

The set

$$S_0 = \{(x_1,x_2,x_3) \in \mathbb{C}^3 \ : \ f(x_1,x_2,x_3) = 0\}$$

is called a *weighted approximation* of S.

Theorem 4.1. *Let $S \subset \mathbb{C}^3$ be a semi-weighted homogeneous algebraic surface with an isolated singularity at origin $0 \in \mathbb{C}^3$. If the weights of S satisfy $w_1 \geq w_2 > w_3$, then $\upsilon(S,0) > 2$. In particular, $(S,0)$ is not metrically conical.*

Proof. Let $S \subset \mathbb{C}^3$ be defined by the semi-weighted polynomial $f = h + \theta$ of degree d and let S_0 be the following weighted homogeneous approximation of S:

$$S_0 = \{(x_1,x_2,x_3) \in \mathbb{C}^3 \ : \ h(x_1,x_2,x_3) = 0\}.$$

Let us consider a family of functions defined as follows:

$$F(X,u) = h(X) + u\theta(X),$$

where $u \in [0,1]$, $X = (x_1,x_2,x_3)$. Let $V(X,u)$ be the vector field defined by:

$$V(X,u) = -\sum_{i=1}^{3} \frac{Q_i(X,u)}{P(X,u)}\frac{\partial}{\partial x_i} + \frac{\partial}{\partial u}$$

where

$$P(X,u) = \sum_{i=1}^{3} |\frac{\partial F}{\partial x_i}(X,u)|^{2\alpha_i} \text{ and } Q_i(X,u) = \theta(X)|\frac{\partial F}{\partial x_i}(X,u)|^{2\alpha_i - 2}\overline{\frac{\partial F}{\partial x_i}}(X,u)$$

and $\alpha_i = \frac{(d-w_1)(d-w_2)(d-w_3)}{d-w_i}$, $i = 1, 2, 3$.

It was shown, by L. Fukui and L. Paunescu (see [7] p. 445), that the flow of this vector field gives a modified analytic trivialization [7] of the family $F^{-1}(0)$. In particular, we obtain a homeomorphism $\Phi\colon (S_0, 0) \to (S, 0)$ which defines a correspondence of subanalytic continuous arcs. Moreover, Φ satisfies the following equation

$$\Phi(X) = X + \int_0^1 W(\Phi(X), u)du \tag{4.1}$$

where $W(X, u) = V(X, u) - \frac{\partial}{\partial u}$.

Proposition 4.2. *Let $\gamma(t) = (t^{w_1}x_1(t), t^{w_2}x_2(t), t^{w_3}x_3(t))$ be such that $x_1(t), x_2(t)$ and $x_3(t)$ are subanalytic continuous functions, $0 \le t < \epsilon$, with $(x_1(0), x_2(0), x_3(0)) \neq (0, 0, 0)$. If*

$$\eta(t) = \int_0^1 W(\gamma(t), u)du$$

with $\eta(t) = (\eta_1(t), \eta_2(t), \eta_3(t))$, then $ord_t|\eta_i(t)| > w_i$ for all $i = 1, 2, 3$.

Proof of the proposition. Let $m = (d - w_1)(d - w_2)(d - w_3)$. Since h has isolated singularity at $0 \in \mathbb{C}^3$, $\exists\ \lambda_1 > 0$ such that

$$P(\gamma(t), u) \ge \lambda_1 \sum_{i=1}^{3} |\frac{\partial h}{\partial x_i}(\gamma(t))|^{2\alpha_i}.$$

Moreover, since each $\dfrac{\partial h}{\partial x_i}$ is weighted homogeneous of degree $d - w_i$, $\exists\ \lambda_2 > 0$ such that

$$|\frac{\partial h}{\partial x_i}(\gamma(t))|^{2\alpha_i} \ge \lambda_2 t^{2m}.$$

Hence, $ord_t P(\gamma(t), u) \le 2m$. By hypothesis,

$$ord_t|\theta(\gamma(t))| > d \quad \text{and} \quad ord_t|\frac{\partial F}{\partial x_i}(\gamma(t), u)|^{2\alpha_i - 1} \ge (2\alpha_i - 1)(d - w_i).$$

Now, we can conclude that ord_t of $\dfrac{\theta}{P}|\dfrac{\partial F}{\partial x_i}|^{2\alpha_i - 1}$ on $\gamma(t)$ is bigger than

$$d + (2\alpha_i - 1)(d - w_i) - 2m = w_i.$$

Finally, since

$$\eta_i(t) = \int_0^1 \frac{\theta(\gamma(t))}{P(\gamma(t), u)} |\frac{\partial F}{\partial x_i}(\gamma(t), u)|^{2\alpha_i - 2} \overline{\frac{\partial F}{\partial x_i}}(\gamma(t), u)du,$$

we have proved the proposition. $\square$

According to Lemma 1 of [6], we can take an essential loop Γ from S^1 to the link of the weighted homogeneous approximation S_0 of S of the form:

$$\Gamma(\theta) = (x_1(\theta), x_2(\theta), 1).$$

Let $H_0 \colon [0,1] \times S^1 \to S_0$ be defined by

$$H_0(r,\theta) = (r^{\frac{w_1}{w_3}} x_1(\theta), r^{\frac{w_2}{w_3}} x_2(\theta), r).$$

Then, $H \colon [0,1] \times S^1 \to S$ defined by

$$H(r,\theta) = \Phi \circ H_0(r,\theta)$$

is a subanalytic homotopy satisfying: $H(0,\theta) = x$ and $H(1,\theta) = \Phi \circ \Gamma(\theta)$. We are going to show the image of H $(Im(H) = \Omega)$ has volume growth number at origin bigger than 2. Actually, since the volume growth number of Ω at 0 is $1 + \beta(\Omega, 0)$, we are going to show that $\beta(\Omega, 0)$ is bigger than 1. So, let us consider two arcs γ_1 and γ_2 on $(\Omega, 0)$.

`Claim.` Each γ_i can be parameterized in the following form:

$$\gamma_i(s) = (s^{\frac{w_1}{w_3}} x_{i1}(s), s^{\frac{w_2}{w_3}} x_{i2}(s), s x_{i3}(s))$$

where $x_{i1}(s), x_{i2}(s)$ and $x_{i3}(s)$ are subanalytic continuous functions and $x_{i3}(0) = 1$, $(i = 1, 2)$.

In fact, first of all, let us fix i and denote $\gamma = \gamma_i$. For each $s > 0$, let $\gamma(s)$ be the point on the arc γ such that $\rho(\gamma(s)) = s^{\frac{1}{w_3}}$, where

$$\rho(x_1, x_2, x_3) := [|x_1|^{w_2 w_3} + |x_2|^{w_1 w_3} + |x_3|^{w_1 w_2}]^{\frac{1}{w_1 w_2 w_3}}.$$

In particular, $\gamma(s) = (s^{\frac{w_1}{w_3}} x_1(s), s^{\frac{w_2}{w_3}} x_2(s), s x_3(s))$ where $x_1(s), x_2(s)$ and $x_3(s)$ are subanalytic continuous functions, with $(x_1(0), x_2(0), x_3(0)) \neq (0,0,0)$. For each $s > 0$, let $\xi(s)$ be the point on the image $Im(H_0) \subset S_0$ such that $\Phi(\xi(s)) = \gamma(s)$. It follows from eq. (4.1) that

$$\gamma(s) = \xi(s) + \eta(s)$$

where $\eta(s) = \int_0^1 W(\gamma(s), u)du$. By Proposition 4.2 (taking $s = t^{w_3}$), it follows that

$$\xi(s) = (s^{\frac{w_1}{w_3}} z_1(s), s^{\frac{w_2}{w_3}} z_2(s), s z_3(s)) \quad \text{with} \quad (z_1(0), z_2(0), z_3(0)) = (x_1(0), x_2(0), x_3(0)).$$

Since the image $Im(H_0)$ is invariant by the $\mathbb{R}_+$-action

$$s \cdot (x_1, x_2, x_3) = (s^{\frac{w_1}{w_3}} x_1, s^{\frac{w_2}{w_3}} x_2, s x_3)$$

and $\xi(s) \in Im(H_0)$ for all $s > 0$, we have that $(z_1(0), z_2(0), z_3(0)) \in Im(H_0)$, hence $z_3(0) = x_3(0) = a$ is a positive real number. Finally, via the simple change $s \mapsto a^{-1}s$, we show what was claimed above.

In order to finalize the proof of Theorem 4.1, by Lemma 2.3, we have

$$\lambda(\gamma_1, \gamma_2) \geq ord_s(|\gamma_1(s) - \gamma_2(s)|)$$

and, since

$$ord_s(|\gamma_1(s) - \gamma_2(s)|) > 1,$$

$\lambda(\gamma_1, \gamma_2) > 1$. Therefore, we can use Theorem 2.4 to get $\beta(\Omega, 0) > 1$. □

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NOTES ON BEILINSON'S "HOW TO GLUE PERVERSE SHEAVES"

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Abstract. The titular, foundational work of Beilinson not only gives a technique for gluing perverse sheaves but also implicitly contains constructions of the nearby and vanishing cycles functors of perverse sheaves. These constructions are completely elementary and show that these functors preserve perversity and respect Verdier duality on perverse sheaves. The work also defines a new, "maximal extension" functor, which is left mysterious aside from its role in the gluing theorem. In these notes, we present the complete details of all of these constructions and theorems.

In this paper we discuss Alexander Beilinson's "How to glue perverse sheaves" [1] with three goals. The first arose from a suggestion of Dennis Gaitsgory that the author study the construction of the unipotent nearby cycles functor $R\psi^{\mathrm{un}}$ which, as Beilinson observes in his concluding remarks, is implicit in the proof of his Key Lemma 2.1. Here, we make this construction explicit, since it is invaluable in many contexts not necessarily involving gluing. The second goal is to restructure the presentation around this new perspective; in particular, we have chosen to eliminate the two-sided limit formalism in favor of the straightforward setup indicated briefly in [3, §4.2] for D-modules. We also emphasize this construction as a simple demonstration that $R\psi^{\mathrm{un}}[-1]$ and Verdier duality $\mathbb{D}$ commute, and de-emphasize its role in the gluing theorem. Finally, we provide complete proofs; with the exception of the Key Lemma, [1] provides a complete program of proof which is not carried out in detail, making a technical understanding of its contents more difficult given the density of ideas. This paper originated as a learning exercise for the author, so we hope that in its final form it will be helpful as a learning aid for others. We do not intend it to supplant, but merely to supplement, the original, and we are grateful to Beilinson for his generosity in permitting this.

The author would like to offer three additional thanks: to Gaitsgory, who explained how this beautiful construction can be understood concretely, thus providing the basis for the perspective taken here; to Sophie Morel, for confirming the author's understanding of nearby and vanishing cycles as presented below; and to Mark de Cataldo, for his generous contribution of time and effort to the improvement of these notes.

In order to maintain readability, we will work with sheaves of vector spaces in the classical topology on complex algebraic varieties, except in the second part of Section 4, where we will require the field of coefficients to be algebraically closed. For the necessary modifications to étale sheaves, one should consult Beilinson's paper: aside from the shift in definitions the only change is some Tate twists. For the D-modules case, one should read Sam Lichtenstein's undergraduate thesis, [11], in which the two-sided limit construction is also given in detail.

1. Theoretical preliminaries

The topic at hand is perverse sheaves and nearby cycles; for greater accessibility of these notes, we give a summary of the definitions and necessary properties here.

Diagram chases. Occasionally, we indicate diagram chases in a proof. For ease of reading we have tried not to make this an essential point, but in case the reader should find such a chase to be a convincing informal argument, we indicate here why it is also a convincing formal one.

Every object in an abelian category $\mathbf{A}$ can be considered, via Yoneda's lemma, to be a sheaf, namely its functor of points, on the *canonical topology* of $\mathbf{A}$. This is, by definition, the largest Grothendieck topology in which all representable functors $\mathrm{Hom}_{\mathbf{A}}(\bullet, x)$ are sheaves, and its open covers are precisely the *universal strict epimorphisms*. Such a map is, in a more general category, a map $f\colon u \to x$ such that the fibered product $u' = u \times_x u$ exists, the coequalizer $x' = \mathrm{coker}(u' \rightrightarrows u)$ exists, the natural map $x' \to x$ is an isomorphism, and that all of this is *also* true when we make any base change along a map $g\colon y \to x$, for the induced map $f \times_x \mathrm{id}\colon u \times_x y \to y$. In an abelian category, however, this is all equivalent merely to the statement that f is a surjection.

Recall the definitions of the various constructions on sheaves:

(1) Kernels of maps are taken sectionwise; i.e. for a map $f\colon \mathcal{F} \to \mathcal{G}$, $\ker(f)(U) = \ker(f(U)\colon \mathcal{F}(U) \to \mathcal{G}(U))$. Likewise, products and limits are taken sectionwise.
(2) Cokernels are *locally* taken sectionwise: any section $s \in \mathrm{coker}(f)(U)$ is, on some open cover V of U, of the form $\bar{t}$ for $t \in \mathcal{G}(V)$. Likewise, images, coproducts, and colimits are taken locally.

In an abelian category, where all of these constructions exist by assumption, these descriptions are even prescriptive: if one forms the sheaves thus described, they are representable by the objects claimed. Therefore, the following common arguments in diagram chasing are valid:

(1) A map $f\colon x \to y$ is surjective if and only if for every $s \in y$, there is some $t \in x$ such that $s = f(t)$. This is code for: for every "open set" U and every $s \in y(U)$, there is a surjection $V \to U$ and a section $t \in y(V)$ such that $s|_V = f(t)$.
(2) If $s \in y$, then $\overline{s} = 0 \in \mathrm{coker}(f)$ if and only if $s \in \mathrm{im}(f)$. This is code for: if $s \in y(U)$ and $\overline{s} = 0 \in \mathrm{coker}(f)(U)$, then there is some surjection $V \to U$ and $t \in x(V)$ with $s|_V = f(t)$.
(3) For $s, t \in x$, $s = t$ if and only if $s - t = 0$. Here, the sum of maps $s\colon U \to x$ and $t\colon V \to x$ is obtained by forming the fibered product $W = U \times_x V$ which covers both U and V, and then taking the sum of the maps $s|_W, t|_W \in \mathrm{Hom}(W, x)$; the condition for equality is just the statement that a section of a sheaf vanishes if only it vanishes on an open cover.

Any other arguments involving elements and some concept related to exactness can also be phrased in this way. Thus, a naïve diagram-chasing argument can be converted into a rigorous one simply by replacing statements like $s \in x$ with correct ones $s \in x(U)$ for some open set U, and passing to surjective covers of U when necessary.

Derived category and functors. All the action takes place in the derived category; specifically, let X be an algebraic variety and denote by $\mathbf{D}(X)$ its derived category of bounded complexes of sheaves of vector spaces with constructible cohomology. By definition, a map of complexes $f\colon A^\bullet \to B^\bullet$ defines an isomorphism in $\mathbf{D}(X)$ if and only if its associated map on cohomology sheaves $H^i(f)\colon H^i(A^\bullet) \to H^i(B^\bullet)$ is an isomorphism for all i. We have a notation for the index-shift: $A^{i+1} = (A[1])^i$ (technically, the differential maps also change sign, but we will never need to think about this). The derived category $\mathbf{D}(X)$ is a "triangulated category", which means merely that in it are a class of triples, called "distinguished triangles", of complexes and maps

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

in which two consecutive arrows compose to zero, satisfying the axioms given in, for example, [9] (but see also Section 4), and with the property that the associated long sequence of cohomology sheaves

$$\dots H^{-1}(C^\bullet) \to H^0(A^\bullet) \to H^0(B^\bullet) \to H^0(C^\bullet) \to H^1(A^\bullet) \to \dots$$

is exact (note that $H^0(A^\bullet[1]) = H^1(A^\bullet)$); we say that the H^i are "cohomological". If $f\colon A^\bullet \to B^\bullet$ is given, there always exists a triangle whose third term $C^\bullet = \mathrm{Cone}(f)$ is the "cone" of f; this cone is unique up to nonunique isomorphism and any commutative diagram of maps f induces a map on cones, but this is not functorial. It follows that the induced triangle itself is unique up to a nonunique isomorphism whose component morphisms on $A^\bullet$ and $B^\bullet$ are the identity maps. A functor between two triangulated categories is "triangulated" if it sends triangles in one to triangles in the other.

In $\mathbf{D}(X)$ we also have some standard constructions of sheaf theory. For any two complexes there is the "total tensor product" $A^\bullet \otimes B^\bullet$ obtained by taking in degree n the direct sum of all products $A^i \otimes B^j$ with $i + j = n$ (and some differentials that are irrelevant) and its derived bifunctor $A^\bullet \otimes^L B^\bullet$, with $H^i(A^\bullet \otimes^L B^\bullet) = \mathrm{Tor}^i(A^\bullet, B^\bullet)$, which is a triangulated functor in each variable. We also have the bifunctor (contravariant in the first argument) $\mathcal{H}\mathrm{om}(A^\bullet, B^\bullet)$, whose terms are $\mathcal{H}\mathrm{om}(A^\bullet, B^\bullet)^i(U) = \mathrm{Hom}(A^\bullet|_U, B^\bullet[i]|_U)$, and its derived bifunctor $R\,\mathcal{H}\mathrm{om}(A^\bullet, B^\bullet)$, with $H^i R\,\mathcal{H}\mathrm{om}(A^\bullet, B^\bullet) = \mathrm{Ext}^i(A^\bullet, B^\bullet)$, which is triangulated in each variable. Of course, these two have an adjunction:

$$R\,\mathcal{H}\mathrm{om}(A^\bullet \overset{L}{\otimes} B^\bullet, C^\bullet) \cong R\,\mathcal{H}\mathrm{om}(A^\bullet, R\,\mathcal{H}\mathrm{om}(B^\bullet, C^\bullet)).$$

For any Zariski-open subset $U \subset X$ with inclusion map j, there are triangulated functors $j_!, j_*\colon \mathbf{D}(U) \to \mathbf{D}(X)$ and $j^* = j^!\colon \mathbf{D}(X) \to \mathbf{D}(U)$; if i is the inclusion of its complement Z, then there are likewise maps $i^!, i^*\colon \mathbf{D}(X) \to \mathbf{D}(Z)$ and $i_* = i_!\colon \mathbf{D}(Z) \to \mathbf{D}(X)$. (Technically the operation j_* is only left exact on sheaves and we should write Rj_* for its derived functor, but we will never have occasion to use the plain version so we elide this extra notation.) They satisfy a number of important relations, of which we will only use one here: there is a functorial triangle in the complex $A^\bullet_X \in \mathbf{D}(X)$:

$$j_!j^*(A^\bullet_X) \to A^\bullet_X \to i_*i^*(A^\bullet_X) \to \tag{1}$$

We will generally forget about writing i_* and consider $\mathbf{D}(Z) \subset \mathbf{D}(X)$.

There is also a triangulated duality functor $\mathbb{D}\colon \mathbf{D}(X) \to \mathbf{D}(X)^{\mathrm{op}}$ which interchanges ! and $*$, in that $\mathbb{D}j_*(A_U^\bullet) = j_!(\mathbb{D}A_U^\bullet)$, etc., and is an involution. In fact, if we set $\mathcal{D}_X^\bullet = \mathbb{D}\underline{\mathbb{C}}$, then $\mathbb{D}(A^\bullet) = R\,\mathcal{H}\mathrm{om}(A^\bullet, \mathcal{D}_X^\bullet)$.

For any map $f\colon X \to Y$ of varieties, we have $f^!, f^*$ as well (also $f_!, f_*$, and none of them are equal), with the same relationships to $\mathbb{D}$, and the useful identity

$$f^! R\,\mathcal{H}\mathrm{om}(A_Y^\bullet, B_Y^\bullet) = R\,\mathcal{H}\mathrm{om}(f^* A_Y^\bullet, f^! B_Y^\bullet). \tag{2}$$

Note that by these properties, we have $f^!\mathcal{D}_Y^\bullet = \mathcal{D}_X^\bullet$.

Perverse sheaves. Here we give a detail-free overview of the formalism of perverse sheaves created in [4]. Within $\mathbf{D}(X)$ there is an abelian category $\mathbf{M}(X)$ of "perverse sheaves" which has nicer properties than the category of actual sheaves. It is specified by means of a "t-structure", namely, a pair of full subcategories ${}^p\mathbf{D}(X)^{\leqslant 0}$ and ${}^p\mathbf{D}(X)^{\geqslant 0}$, also satisfying some conditions we won't use, and such that

$$\mathbf{M}(X) = {}^p\mathbf{D}(X)^{\leqslant 0} \cap {}^p\mathbf{D}(X)^{\geqslant 0}.$$

There are truncation functors $\tau^{\leqslant 0}\colon \mathbf{D}(X) \to {}^p\mathbf{D}(X)^{\leqslant 0}$ and likewise for $\tau^{\geqslant 0}$, fitting into a distinguished triangle for any complex $A_X^\bullet \in \mathbf{D}(X)$:

$$\tau^{\leqslant 0} A_X^\bullet \to A_X^\bullet \to \tau^{>0} A_X^\bullet \to$$

(where $\tau^{>0} = \tau^{\geqslant 1} = [-1] \circ \tau^{\geqslant 0} \circ [1]$). This triangle is *unique* with respect to the property that the first term is in ${}^p\mathbf{D}(X)^{\leqslant 0}$ and the third is in ${}^p\mathbf{D}(X)^{>0}$. They have the obvious properties implied by the notation: $\tau^{\leqslant a}\tau^{\leqslant b} = \tau^{\leqslant a}$ if $a \le b$, and likewise for $\tau^{\geqslant ?}$. Furthermore, there are "perverse cohomology" functors ${}^pH^i\colon \mathbf{D}(X) \to \mathbf{M}(X)$, where of course ${}^pH^i(A^\bullet) = {}^pH^0(A^\bullet[i])$ and ${}^pH^0 = \tau^{\geqslant 0}\tau^{\leqslant 0} = \tau^{\leqslant 0}\tau^{\geqslant 0}$; these are cohomological just like the ordinary cohomology functors. The abelian category structure of $\mathbf{M}(X)$ is more or less determined by the fact that if we have a map $f\colon \mathcal{F} \to \mathcal{G}$ of perverse sheaves (this is the notation we will be using; we will not think of perverse sheaves as complexes), then

$$\ker f = {}^pH^{-1}\operatorname{Cone}(f) \qquad\qquad \operatorname{coker} f = {}^pH^0\operatorname{Cone}(f).$$

For notational convenience, we will write $\mathcal{M}$ for a perverse sheaf on U, $\mathcal{F}$ for one on X, and as usual, abandon i_* and just consider $\mathbf{M}(Z) \subset \mathbf{M}(X)$ (for the reason expressed immediately below, this is reasonable).

The category $\mathbf{M}(X)$ is closed under the duality functor $\mathbb{D}$, but not necessarily under the six functors defined for an open/closed pair of subvarieties. However, it is true that $j_*(\mathcal{M}), i^!(\mathcal{F}) \in {}^p\mathbf{D}^{\geqslant 0}$ and $j_!(\mathcal{M}), i^*(\mathcal{F}) \in {}^p\mathbf{D}^{\leqslant 0}$, while $j^*(\mathcal{F}), i_*(\mathcal{F}_Z) \in \mathbf{M}$ ($\mathcal{F}_Z$ a perverse sheaf on Z); we say these functors are right, left, or just "t-exact". Furthermore, when j is an affine morphism (the primary example being when Z is a Cartier divisor), both $j_!$ and j_* are t-exact, and thus their restriction to $\mathbf{M}(U)$ is exact with values in $\mathbf{M}(X)$. There is also a "minimal extension" functor $j_{!*}$, defined so that $j_{!*}(\mathcal{M})$ is the image of ${}^pH^0(j_!\mathcal{M})$ in ${}^pH^0(j_*\mathcal{M})$ along the natural map $j_! \to j_*$; it is the unique perverse sheaf such that $i^*j_{!*}\mathcal{M} \in {}^p\mathbf{D}^{<0}(Z)$ and $i^!j_{!*}\mathcal{M} \in {}^p\mathbf{D}^{>0}(Z)$, but for us the most useful property is that when j is an affine, open immersion, then we have a sequence of *perverse sheaves*

$$i^*j_{!*}\mathcal{M}[-1] \hookrightarrow j_!\mathcal{M} \twoheadrightarrow j_{!*}\mathcal{M} \hookrightarrow j_*\mathcal{M} \twoheadrightarrow i^!j_{!*}\mathcal{M}[1]; \tag{3}$$

i.e. $i^*j_{!*}\mathcal{M}[-1] = \ker(j_!\mathcal{M} \to j_*\mathcal{M})$ and $i^!j_{!*}\mathcal{M}[1] = \operatorname{coker}(j_!\mathcal{M} \to j_*\mathcal{M})$ are both perverse sheaves.

Perverse sheaves have good category-theoretic properties: $\mathbf{M}(X)$ is both artinian and noetherian, so every perverse sheaf has finite length. Finally, we will use the sheaf-theoretic fact that if $\mathcal{L}$ is a locally constant sheaf on X, then $\mathcal{F} \otimes \mathcal{L}$ is perverse whenever $\mathcal{F}$ is. Note that since $\mathcal{L}$ is locally free, it is flat, and therefore $\mathcal{F} \otimes \mathcal{L} = \mathcal{F} \otimes^L \mathcal{L}$.

Nearby cycles. If we have a map $f\colon X \to \mathbb{A}^1$ such that $Z = f^{-1}(0)$ (so $U = f^{-1}(\mathbb{A}^1 \setminus \{0\}) = f^{-1}(\mathbf{G_m})$), the "nearby cycles" functor $R\psi_f\colon \mathbf{D}(U) \to \mathbf{D}(Z)$ is defined. Namely, let $u\colon \widetilde{\mathbf{G_m}} \to \mathbf{G_m}$ be the universal cover of $\mathbf{G_m} = \mathbb{A}^1 \setminus \{0\}$, let $v\colon \widetilde{U} = U \times_{\mathbf{G_m}} \widetilde{\mathbf{G_m}} \to U$ be its pullback, forming a diagram

$$\begin{tikzcd}
Z \arrow[r, "i"] \arrow[d] & X \arrow[d, "f"] & U \arrow[l, "j"'] \arrow[d] & \widetilde{U} \arrow[l, "v"'] \arrow[d] \\
\{0\} \arrow[r] & \mathbb{A}^1 & \mathbf{G_m} \arrow[l] & \widetilde{\mathbf{G_m}} \arrow[l, "u"]
\end{tikzcd}$$

and set (in this one instance, explicitly writing j_* and v_* as non-derived functors)

$$R\psi_f = R(i^* j_* v_* v^*)\colon \mathbf{D}(U) \to \mathbf{D}(Z).$$

Since i^* and v^* are exact, indeed ψ_f is a left-exact functor from sheaves on U to sheaves on Z. Many sources (e.g. [13, §1.1.1]) give the definition $R\psi_f = i^* Rj_* Rv_* v^*$; in fact, they are the same: since v is a covering map, if $\mathcal{F}$ is a flasque sheaf on U, then $v^*\mathcal{F}$ is flasque on $\widetilde{U}$ and so acyclic for v_* (and $v_* v^* \mathcal{F}$ acyclic for j_*). Therefore we may form the derived functor before or after composition. Note that v is not an algebraic map, and therefore it is not *a priori* clear whether $R\psi_f$ preserves constructibility; that it does is a theorem of Deligne ([7], Exposé XIII, Théorème 2.3 for étale sheaves and Exposé XIV, Théorème 2.8 for the comparison with classical nearby cycles).

The fundamental group $\pi_1(\mathbf{G_m})$ acts on any $v^* A_U^\bullet$ via deck transformations of $\widetilde{\mathbf{G_m}}$ and therefore acts on ψ_f and $R\psi_f$. There is a natural map $i^* A_X^\bullet \to \psi_f(j^* A_X^\bullet)$, obtained from (v^*, v_*)-adjunction, on whose image $\pi_1(\mathbf{G_m})$ acts trivially. We set, by definition,

$$i^* A_X^\bullet \to \psi_f(j^* A_X^\bullet) \to \phi_f(A_X^\bullet) \to 0$$

where $\phi_f(A_X^\bullet)$ is the "vanishing cycles" sheaf. Using some homological algebra tricks the above sequence induces a natural distinguished triangle

$$i^* A_X^\bullet \to R\psi_f(j^* A_X^\bullet) \to R\phi_f(A_X^\bullet) \to$$

where $R\phi_f$ is (morally) the right derived functor of ϕ_f. Like $R\psi_f$, $R\phi_f$ has a monodromy action of $\pi_1(\mathbf{G_m})$; this action is one of the maps on the cone of the above triangle induced by the monodromy action on $R\psi_f$, but as this is not functorial, one should consult the real definition in [7] (given for the algebraic nearby cycles, but see also the second exposé).

Lemma 1.1. There exists a unique decomposition of $R\psi_f$ as $R\psi_f^{\mathrm{un}} \oplus R\psi_f^{\neq 1}$, where for any choice of generator t of $\pi_1(\mathbf{G_m})$, $1 - t$ acts nilpotently on $R\psi_f^{\mathrm{un}}(A_U^\bullet)$ for any complex $A_U^\bullet$ and is an automorphism of $R\psi_f^{\neq 1}$.

The part $R\psi_f^{\mathrm{un}}$ is called the functor of *unipotent nearby cycles*.

Proof. To start, we observe that for any sheaf $\mathcal{F}$ on U, we have $\psi_f(\mathcal{F}) = H^0 R\psi_f(\mathcal{F})$, and so if $\mathcal{F}$ is constructible, by the constructibility of nearby cycles so is $\psi_f(\mathcal{F})$; thus, for any open $V \subset X$, $\psi_f(\mathcal{F})(V)$ is finite-dimensional. Let $\psi_f^{\mathrm{un}} \subset \psi_f$ be the subfunctor such that for any sheaf $\mathcal{F}$ on U, $\psi_f^{\mathrm{un}}(\mathcal{F})$ is the subsheaf of $\psi_f(\mathcal{F})$ in which $1 - t$ is nilpotent, so for each V, $\psi_f^{\mathrm{un}}(\mathcal{F})(V)$ is the generalized eigenspace of t with eigenvalue 1.

Therefore it is actually a direct summand; we recall the general argument which works over any field k. If T is an endomorphism of a finite-dimensional vector space M, we view M as a $k[x]$-module with x acting as T. By the classification of modules over a principal ideal domain, we have $M \cong \bigoplus k[x]/p(x)$ for certain polynomials $p(x)$. The generalized eigenspace with eigenvalue 1 is then the sum of those pieces for which $p(x)$ is a power of $1 - x$, and the remaining summands are a T-invariant complement in which $1 - T$ acts invertibly. As the image of $(1 - T)^n$ $(n \gg 0)$, this complement is functorial in the category of finite-dimensional $k[x]$-modules and so we have the same decomposition in sheaves of finite-dimensional $k[x]$-modules.

Specializing again to the present situation, we get a decomposition $\psi_f(\mathcal{F}) \cong \psi_f^{\mathrm{un}}(\mathcal{F}) \oplus \psi_f^{\neq 1}$ (this is the definition of $\psi_f^{\neq 1}$). Both summands are *a fortiori* left-exact functors and taking derived functors, we obtain a decomposition:

$$R\psi_f \cong R\psi_f^{\mathrm{un}} \oplus R\psi_f^{\neq 1}.$$

Since $1 - t$ is nilpotent on any $\psi_f^{\mathrm{un}}(\mathcal{F})$, we may apply ψ_f^{un} to any constructible complex of injectives, thus computing $R\psi_f^{\mathrm{un}}(A_U^\bullet)$ for any complex $A_U^\bullet$, and conclude that $1 - t$ is nilpotent on each such; by general principles it is invertible on $R\psi_f^{\neq 1}$. This is what we want.

Uniqueness of the decomposition is clear; indeed, if in any category with a zero object we have objects x and y together with endomorphisms N and I respectively such that N is nilpotent and I invertible, then any map $g\colon x \to y$ intertwining N and I is zero: we have $gN = Ig$, so $g = I^{-1}gN = I^{-2}gN^2 = \cdots = I^{-n}gN^n = 0$ if $N^n = 0$. For morphisms $y \to x$ we work in the opposite category. In particular, if we have

$$R\psi_f \cong F \oplus G$$

as a sum of two functors as in the statement of the lemma, then the identity map on $R\psi_f$ has no G-component on $R\psi_f^{\mathrm{un}}$ and no F-component on $R\psi_f^{\neq 1}$, and so induces isomorphisms $R\psi_f^{\mathrm{un}} \cong F$ and $R\psi_f^{\neq 1} \cong G$. $\square$

We note that this lemma is a special case of Lemma 4.2 when the field of coefficients is algebraically closed. However, this decomposition is defined over any field.

There is a triangle, functorial in $A_X^\bullet$,

$$i^* j_* j^* A_X^\bullet \to R\psi_f(j^* A_X^\bullet) \xrightarrow{1-t} R\psi_f(j^* A_X^\bullet)$$

(see [5, Prop. 1.1; 13, eq. (5.88)]) which, taking $A_U^\bullet = j^* A_X^\bullet$ and inserting $R\psi_f^{\mathrm{un}}$ because the monodromy acts trivially on the first term, gives the extremely important (for us) triangle

$$i^* j_* A_U^\bullet \to R\psi_f^{\mathrm{un}}(A_U^\bullet) \xrightarrow{1-t} R\psi_f^{\mathrm{un}}(A_U^\bullet) \to \tag{4}$$

We also have a unipotent part of the vanishing cycles functor $R\phi_f$, and, again since the monodromy acts trivially on $i^* A_X^\bullet$, a corresponding triangle

$$i^* A_X^\bullet \to R\psi_f^{\mathrm{un}}(A_X^\bullet) \to R\phi_f^{\mathrm{un}}(A_X^\bullet) \to \tag{5}$$

If $\mathcal{L}$ is any locally constant sheaf on $\mathbf{G_m}$ with underlying vector space L and unipotent monodromy, then $R\psi_f^{\mathrm{un}}(A_U^\bullet \otimes f^*\mathcal{L}) \cong R\psi_f^{\mathrm{un}}(A_U^\bullet) \otimes L$, where $\pi_1(\mathbf{G_m})$ acts on the tensor product by acting on each factor (since $\mathcal{L}$ is trivialized on $\widetilde{\mathbf{G_m}}$).

We note the following fact, crucial to all computations in this paper:

j is an affine morphism.

Indeed, Z is cut out by a single algebraic equation. Although when Z is any Cartier divisor it is still locally defined by equations f and the inclusion j of its complement is again an affine morphism, it is not necessarily possible to glue nearby cycles which are locally defined as above; c.f. [6, Remark 5.5.4]; an explicit example will appear in [8]. However, it follows from Corollary 2.7 that when $\mathcal{M}$ is a perverse sheaf and $R\psi_f^{\mathrm{un}}(\mathcal{M})$ has trivial monodromy, it is in fact independent of f and gluing is indeed possible.

Triangle (4) already implies that nearby cycles preserve perverse sheaves.

Lemma 1.2. The functor $R\psi_f^{\mathrm{un}}[-1]$ sends ${}^p\mathbf{D}(U)^{\leqslant 0}$ to ${}^p\mathbf{D}(Z)^{\leqslant 0}$ and takes $\mathbf{M}(U)$ to $\mathbf{M}(Z)$.

Proof. Since j is affine and an open immersion, j_* and $j_!$ are t-exact, so for any $A_U^\bullet \in {}^p\mathbf{D}(U)^{\leqslant 0}$, $i^* j_* A_U^\bullet = \mathrm{Cone}(j_! A_U^\bullet \to j_* A_U^\bullet)$ is in ${}^p\mathbf{D}(Z)^{\leqslant 0}$. If we apply the long exact sequence of perverse cohomology to triangle (4), we therefore get in nonnegative degrees:

$$\begin{aligned}{}^pH^0(R\psi_f^{\mathrm{un}} A_U^\bullet) \xrightarrow{1-t} {}^pH^0(R\psi_f^{\mathrm{un}} A_U^\bullet) \to (0 = {}^pH^1(i^* j_* A_U^\bullet)) \to \\ {}^pH^1(R\psi_f^{\mathrm{un}} A_U^\bullet) \xrightarrow{1-t} {}^pH^1(R\psi_f^{\mathrm{un}} A_U^\bullet) \to (0 = {}^pH^2(i^* j_* A_U^\bullet)) \to \cdots\end{aligned}$$

For $i \geq 0$, the map ${}^pH^i(R\psi_f^{\mathrm{un}} A_U^\bullet) \to {}^pH^i(R\psi_f^{\mathrm{un}} A_U^\bullet)$ is both given by a *nilpotent* operator and is surjective, so zero. It follows that $R\psi_f^{\mathrm{un}}(A_U^\bullet) \in {}^p\mathbf{D}(Z)^{\leqslant -1}$, as promised.

Now let $\mathcal{M} \in \mathbf{M}(U)$ be a perverse sheaf. Then $i^* j_* \mathcal{M} \in {}^p\mathbf{D}(Z)^{[-1,0]}$ since its perverse cohomology sheaves are the kernel and cokernel of the map $j_! \mathcal{M} \to j_* \mathcal{M}$. In degrees ≤ -2, then, we have

$$\begin{aligned}\cdots \to (0 = {}^pH^{-3}(i^* j_* \mathcal{M})) \to {}^pH^{-3}(R\psi_f^{\mathrm{un}} \mathcal{M}) \xrightarrow{1-t} {}^pH^{-3}(R\psi_f^{\mathrm{un}} \mathcal{M}) \to \\ (0 = {}^pH^{-2}(i^* j_* \mathcal{M})) \to {}^pH^{-2}(R\psi_f^{\mathrm{un}} \mathcal{M}) \xrightarrow{1-t} {}^pH^{-2}(R\psi_f^{\mathrm{un}} \mathcal{M})\end{aligned}$$

This means that for $i \leq -2$, all the maps $1-t$ are injective and nilpotent, hence zero. Thus $R\psi_f^{\mathrm{un}}(\mathcal{M}) \in {}^p\mathbf{D}(Z)^{-1}$, as desired. □

Since $R\psi_f^{\mathrm{un}}[-1]$ acts on perverse sheaves, we will give it the abbreviated notation Ψ_f^{un}.

2. Construction of the unipotent nearby cycles functor

Let L^a be the vector space of dimension $a \geq 0$ together with the action of a matrix $J^a = [\delta_{ij} - \delta_{i,j-1}]$, a unipotent (variant of a) Jordan block of dimension a. Let $\mathcal{L}^a$ be the locally constant sheaf on $\mathbf{G_m}$ whose underlying space is L^a and in whose monodromy action a (hereafter fixed choice of) generator t of $\pi_1(\mathbf{G_m})$ acts by J^a. Since it is locally free, it is flat, so we will write $\otimes$ rather than $\otimes^L$ in tensor products with it (actually, with $f^*\mathcal{L}^a$). It has the following self-duality properties, where $\check{\mathcal{L}}^a = \mathcal{H}\mathrm{om}(\mathcal{L}^a, \underline{\mathbb{C}})$ is the dual local system and $(\mathcal{L}^a)^{-1}$ is the local system in whose monodromy t acts by $(J^a)^{-1}$:

Lemma 2.1. We have $\mathcal{L}^a \cong \check{\mathcal{L}}^a \cong (\mathcal{L}^a)^{-1}$, and $\mathbb{D}(A^\bullet_U \otimes f^*\mathcal{L}^a) \cong \mathbb{D}(A^\bullet_U) \otimes f^*\mathcal{L}^a$ for $A^\bullet_U \in \mathbf{D}(U)$.

Proof. Since $\check{\mathcal{L}}^a$ is the local system with vector space the dual $\check{L}^a$ and monodromy $((J^a)^t)^{-1}$, its monodromy is again unipotent with a single Jordan block of length a. We fix in L^a the given basis $\vec{e}_1, \dots, \vec{e}_a$ associated to J, and in $\check{L}^a$ we choose a generalized eigenbasis $\check{f}_1, \dots, \check{f}_n$ in which $((J^a)^t)^{-1}$ has the matrix J^a, so that sending $\vec{e}_i \mapsto \check{f}_i$ identifies J^a with $((J^a)^t)^{-1}$ and thus induces the desired map of local systems. The same proof shows that $\mathcal{L}^a \cong (\mathcal{L}^a)^{-1}$.

In general, then, we construct an isomorphism:

$$\mathbb{D}(A^\bullet_U) \otimes f^*\mathcal{L}^a \xrightarrow{\sim} \mathbb{D}(A^\bullet_U \otimes f^*\mathcal{L}^a), \tag{6}$$

where

$$\mathbb{D}(A^\bullet_U) \otimes f^*\mathcal{L}^a = R\,\mathcal{H}\mathrm{om}(A^\bullet_U, \mathcal{D}^\bullet) \overset{L}{\otimes} f^*\mathcal{L}^a,$$

$$\mathbb{D}(A^\bullet_U \otimes f^*\mathcal{L}^a) = R\,\mathcal{H}\mathrm{om}(A^\bullet_U \overset{L}{\otimes} f^*\mathcal{L}^a, \mathcal{D}^\bullet_U) = R\,\mathcal{H}\mathrm{om}(A^\bullet_U, \mathbb{D}f^*\mathcal{L}^a),$$

by constructing a map

$$R\,\mathcal{H}\mathrm{om}(A^\bullet_U, \mathcal{D}^\bullet_U) \overset{L}{\otimes} f^*\mathcal{L}^a \to R\,\mathcal{H}\mathrm{om}(A^\bullet_U, \mathbb{D}f^*\mathcal{L}^a).$$

Such a map can be obtained by applying $(\otimes^L, R\,\mathcal{H}\mathrm{om})$-adjunction to a map:

$$\begin{aligned} R\,\mathcal{H}\mathrm{om}(A^\bullet_U, \mathcal{D}^\bullet_U) &\to R\,\mathcal{H}\mathrm{om}(f^*\mathcal{L}^a, R\,\mathcal{H}\mathrm{om}(A^\bullet_U, \mathbb{D}f^*\mathcal{L}^a)) \\ &= R\,\mathcal{H}\mathrm{om}(A^\bullet_U, R\,\mathcal{H}\mathrm{om}(f^*\mathcal{L}^a, \mathbb{D}f^*\mathcal{L}^a)). \end{aligned} \tag{7}$$

Since $\mathbb{D}$ exchanges $!$ and $*$, we have $\mathbb{D}f^*\mathcal{L}^a = f^!\mathbb{D}\mathcal{L}^a$. By the property (2) of $f^!$, we have

$$R\,\mathcal{H}\mathrm{om}(f^*\mathcal{L}^a, f^!\mathbb{D}\mathcal{L}^a) = f^! R\,\mathcal{H}\mathrm{om}(\mathcal{L}^a, \mathbb{D}\mathcal{L}^a)$$

Note also that $\mathcal{D}^\bullet_U = f^!\mathcal{D}^\bullet_{\mathbf{G_m}}$ by definition and therefore the map (7) can be constructed by applying $R\,\mathcal{H}\mathrm{om}(A^\bullet_U, f^!\bullet)$ to a certain map on $\mathbf{G_m}$:

$$\mathcal{D}^\bullet_{\mathbf{G_m}} \to R\,\mathcal{H}\mathrm{om}(\mathcal{L}^a, \mathbb{D}\mathcal{L}^a) = \mathbb{D}(\mathcal{L}^a \overset{L}{\otimes} \mathcal{L}^a).$$

This map, in turn, is obtained by first replacing the $\otimes^L$ with $\otimes$ (since $\mathcal{L}^a$ is locally free) and applying $\mathbb{D}$ to the pairing

$$\mathcal{L}^a \otimes \mathcal{L}^a \to \underline{\mathbb{C}}$$

given by the isomorphism $\mathcal{L}^a \cong \check{\mathcal{L}}^a$ described in the first paragraph. Thus, locally (6) is the tautological isomorphism $(\mathbb{D}A^\bullet_U)^{\oplus a} \cong \mathbb{D}(A^\bullet_U)^{\oplus a}$. Since it is a local isomorphism, it is an isomorphism. $\square$

In the rest of this section, $\mathcal{M}$ is any object of $\mathbf{M}(U)$. The following construction is Beilinson's definition of the unipotent nearby cycles:

Proposition 2.2. Let $\alpha^a \colon j_!(\mathcal{M} \otimes f^*\mathcal{L}^a) \to j_*(\mathcal{M} \otimes f^*\mathcal{L}^a)$ be the natural map. Then there is an inclusion $\ker(\alpha^a) \hookrightarrow \Psi^{\mathrm{un}}_f(\mathcal{M})$, identifying the actions of $\pi_1(\mathbf{G_m})$, which is an isomorphism for all sufficiently large a. (In fact, it suffices to take a large enough that $(1-t)^a$ annihilates $\Psi^{\mathrm{un}}_f(\mathcal{M})$.)

Proof. We know by Lemma 1.2 that $\Psi_f^{\mathrm{un}}(\bullet)$ is a perverse sheaf, so taking together the triangle (4) with $A_U^\bullet = \mathcal{M} \otimes f^*\mathcal{L}^a$ and exact sequence (1) with $A_X^\bullet = j_* A_U^\bullet$, we see that $\ker \alpha = \ker(1-t)$, where $1-t$ is the map appearing in the former triangle shifted by -1. We also have

$$\Psi_f^{\mathrm{un}}(\mathcal{M} \otimes f^*\mathcal{L}^a) \cong \Psi_f^{\mathrm{un}}(\mathcal{M}) \otimes L^a \cong \bigoplus_{i=1}^{a} \Psi_f^{\mathrm{un}}(\mathcal{M})_{(i)},$$

where the i'th coordinate of the action of t is $t_{(i)} - t_{(i+1)}$, with $t_{(i)}$ the copy of $t \in \pi_1(\mathbf{G_m})$ acting on $\Psi_f^{\mathrm{un}}(\mathcal{M})$ considered as the i'th summand. That is, using elements, $(x_1, \dots, x_n) \in \Psi_f^{\mathrm{un}}(\mathcal{M} \otimes f^*\mathcal{L}^a)$ is sent by t to $(tx_1 - tx_2, tx_2 - tx_3, \dots, tx_n)$. Thus, for an element of $\ker(1-t)$, we have $x_{i+1} = (1-t^{-1})x_i$, or:

$$x_i = (1-t^{-1})^{i-1}x_1 \qquad\qquad -t(1-t^{-1})^a x_1 = (1-t)x_n = 0.$$

If we define a map $u \colon \Psi_f^{\mathrm{un}}(\mathcal{M}) \to \Psi_f^{\mathrm{un}}(\mathcal{M} \otimes f^*\mathcal{L}^a)$ by sending the element $x = x_1$ to the coordinates x_i defined by the first formula above, then u is injective and its image contains $\ker(1-t)$ (namely, that subspace satisfying the second equation). Since $1-t$ (hence $1-t^{-1}$) is nilpotent on $\Psi_f^{\mathrm{un}}(\mathcal{M})$, for a sufficiently large, $\mathrm{im}(u) = \ker(1-t)$. We claim that u intertwines the actions of t^{-1} and J^a:

$$\begin{aligned} J^a u(x) &= (x - (1-t^{-1})x, (1-t^{-1})x + (1-t^{-1})^2 x, \dots) \\ &= (t^{-1}x, (1-t^{-1})t^{-1}x, \dots) = u(t^{-1}x). \end{aligned}$$

Finally, we employ the isomorphism $\mathcal{L}^a \cong (\mathcal{L}^a)^{-1}$ of Lemma 2.1 to give an automorphism of $\ker(\alpha^a) \subset j_!(\mathcal{M} \otimes f^*\mathcal{L}^a)$ intertwining J^a and $(J^a)^{-1}$. □

Corollary 2.3. *There exists an integer N such that $(1-t)^N$ annihilates both $\ker \alpha^a$ and $\operatorname{coker} \alpha^a$ for all a.*

Proof. By Proposition 2.2, the kernel is contained in $\Psi_f^{\mathrm{un}}(\mathcal{M})$ and thus annihilated by that power of $1-t$ which annihilates the nearby cycles. Temporarily let $\alpha^a = \alpha^a_{\mathcal{M}}$; then $\mathbb{D}(\alpha^a_{\mathcal{M}}) = \alpha^a_{\mathbb{D}\mathcal{M}}$, so $\operatorname{coker}(\alpha^a_{\mathcal{M}}) = \mathbb{D}\ker(\alpha^a_{\mathbb{D}\mathcal{M}})$ is again annihilated by some $(1-t)^N$. □

In preparation for the next section, we give a generalization of this construction. For each $a, b \geq 0$ there is a natural short exact sequence

$$0 \to \mathcal{L}^a \xrightarrow{g^{a,b}} \mathcal{L}^{a+b} \xrightarrow{g^{a+b,-a}} \mathcal{L}^b \to 0;$$

that is, for any $r \in \mathbb{Z}$, $g^{a,r}$ sends $\mathcal{L}^a$ to the first a coordinates of $\mathcal{L}^{a+r}$ if $r \geq 0$, and to the quotient $\mathcal{L}^{a-(-r)}$ given by collapsing the first $-r$ coordinates if $-r \geq 0$ (that is, $r \leq 0$) and $a + r \geq 0$. This sequence respects the action of $\pi_1(\mathbf{G_m})$ on the terms and, via Lemma 2.1, the (a,b) sequence is dual to the (b,a) sequence.

Let $\mathcal{M} \in \mathbf{M}(U)$; then we have induced maps on the tensor products:

$$g^{a,r}_{\mathcal{M}} = \mathrm{id} \otimes g^{a,r} \colon \mathcal{M} \otimes f^*\mathcal{L}^a \to \mathcal{M} \otimes f^*\mathcal{L}^{a+r}$$

(we will often omit the subscript $\mathcal{M}$ when no confusion is possible). By Lemma 2.1, these satisfy

$$\mathbb{D}g^{a,r}_{\mathcal{M}} = g^{a+r,-r}_{\mathbb{D}\mathcal{M}}. \tag{8}$$

Note that since the $\mathcal{L}^a$ are locally free, the $g^{a,r}_{\mathcal{M}}$ are all injective when $r \geq 0$ and surjective when $r \leq 0$. Let $r \in \mathbb{Z}$ and set

$$\alpha^{a,r} = j_*(g^{a,r}) \circ \alpha^a = \alpha^{a+r} \circ j_!(g^{a,r}) \colon j_!(\mathcal{M} \otimes f^*\mathcal{L}^a) \to j_*(\mathcal{M} \otimes f^*\mathcal{L}^{a+r}).$$

We will use the following self-evident properties of the $g^{a,r}$:

Lemma 2.4. The $g^{a,r}$ satisfy:

(1) When $a+r \geq 0$, we have $g^{a,r} \circ g^{a+r,-r} = (1-t)^{|r|}$.
(2) When r and s have the same sign and $a+r+s \geq 0$, we have $g^{a,r+s} = g^{a+r,s} \circ g^{a,r}$.
(3) Let $r \geq 0$, $a \geq r$; then we have:

$$\ker(1-t)^r = \ker(g^{a,-r}_{\mathcal{M}}) \cong \mathcal{M} \otimes f^*\mathcal{L}^r, \quad \operatorname{im}(1-t)^r = \operatorname{im}(g^{a-r,r}_{\mathcal{M}}) \cong \mathcal{M} \otimes f^*\mathcal{L}^{a-r}.$$

Finally, by Corollary 2.3 and (3), for $r \geq 0$, $(1-t)^{N+r}$ annihilates $\ker(\alpha^{a,-r})$ and $\operatorname{coker}(\alpha^{a,r})$. □

From now on, we will assume $r \geq 0$.

Proposition 2.5. For $a \gg 0$, the natural maps $j_!(g^{a,1})$ and $j_*(g^{a+r,-1})$ respectively induce isomorphisms

$$\ker(\alpha^{a,-r}) \xrightarrow{\sim} \ker(\alpha^{a+1,-r}) \qquad \operatorname{coker}(\alpha^{a,r}) \xrightarrow{\sim} \operatorname{coker}(\alpha^{a-1,r})$$

and $j_!(g^{a,r})$ and $j_*(g^{a+r,-r})$ induce isomorphisms

$$\ker(\alpha^{a,r}) \xrightarrow{\sim} \ker(\alpha^{a+r}) \qquad \operatorname{coker}(\alpha^{a+r}) \xrightarrow{\sim} \operatorname{coker}(\alpha^{a+r,-r})$$

Proof. Using the maps $j_!(g^{a,1})$ and $j_*(g^{a-r,1})$ we get a square which, using Lemma 2.4(1,2), we verify is commutative:

$$\begin{tikzcd} j_!(\mathcal{M} \otimes f^*\mathcal{L}^a) \arrow[r, "\alpha^{a,-r}"] \arrow[d, "j_!(g^{a,1})"'] & j_*(\mathcal{M} \otimes f^*\mathcal{L}^{a-r}) \arrow[d, "j_*(g^{a-r,1})"] \\ j_!(\mathcal{M} \otimes f^*\mathcal{L}^{a+1}) \arrow[r, "\alpha^{a+1,-r}"] & j_*(\mathcal{M} \otimes f^*\mathcal{L}^{a-r+1}) \end{tikzcd}$$

showing that $j_!(g^{a,1})$ induces a map on kernels. Since it is injective, we get a long sequence of inclusions of kernels:

$$\cdots \subset \ker \alpha^{a-1,-r} \subset \ker \alpha^{a,-r} \subset \ker \alpha^{a+1,-r} \subset \cdots.$$

By Lemma 2.4, each kernel is annihilated by $(1-t)^{N+r}$, whose kernel is (for $a \geq N+r$) the perverse sheaf $j_!(\mathcal{M} \otimes f^*\mathcal{L}^{N+r})$; thus, this sequence is contained in this sheaf. Since perverse sheaves are noetherian, this chain must have a maximum, so the kernels stabilize. For the cokernels, we apply (8) to the argument of Corollary 2.3. (One can also argue directly using the artinian property of perverse sheaves.)

For the second statement concerning kernels, since $(1-t)^N$ annihilates $\ker(\alpha^{a+r})$, for $a \geq N$ it is contained in $\operatorname{im}(g^{a,r})$, and therefore by definition in $\ker(\alpha^{a,r})$. The statement on cokernels is again obtained by dualization and (8). (A direct argument employing a diagram chase is also possible, using the fact that $(1-t)^N \operatorname{coker}(\alpha^{a+r}) = 0$.) □

Departing slightly from Beilinson's notation, we denote these stable kernels and cokernels $\ker \alpha^{\infty,-r}$ and $\operatorname{coker} \alpha^{\infty,r}$ for $r \geq 0$; when $r = 0$ we drop it.

Proposition 2.6. There is a natural isomorphism $\ker \alpha^{\infty,-r} \xrightarrow{\sim} \operatorname{coker} \alpha^{\infty,r}$.

Proof. Consider the map of short exact sequences for any a and any $b \geq r$ (to eliminate clutter we have not written the superscripts on the maps g):

$$\begin{tikzcd}
0 \arrow[r] & j_!(\mathcal{M} \otimes f^*\mathcal{L}^a) \arrow[r, "j_!(g)"] \arrow[d, "\alpha^{a,r}"] & j_!(\mathcal{M} \otimes f^*\mathcal{L}^{a+b}) \arrow[r, "j_!(g)"] \arrow[d, "\alpha^{a+b}"] & j_!(\mathcal{M} \otimes f^*\mathcal{L}^b) \arrow[r] \arrow[d, "\alpha^{b,-r}"] & 0 \\
0 \arrow[r] & j_*(\mathcal{M} \otimes f^*\mathcal{L}^{a+r}) \arrow[r, "j_*(g)"] & j_*(\mathcal{M} \otimes f^*\mathcal{L}^{a+b}) \arrow[r, "j_*(g)"] & j_*(\mathcal{M} \otimes f^*\mathcal{L}^{b-r}) \arrow[r] & 0
\end{tikzcd}$$

By the snake lemma, we have an exact sequence of kernels and cokernels:

$$\begin{aligned}0 \to \ker(\alpha^{a,r}) \to \ker(\alpha^{a+b}) \to \ker(\alpha^{b,-r}) \xrightarrow{\gamma^{a,b;r}} \operatorname{coker}(\alpha^{a,r}) \\ \to \operatorname{coker}(\alpha^{a+b}) \to \operatorname{coker}(\alpha^{b,-r}) \to 0. \quad (9)\end{aligned}$$

If $a, b \gg 0$, then the first and last maps are, by the second part of Proposition 2.5, isomorphisms. Therefore $\gamma^{a,b;r}$ is an isomorphism. Since the long exact sequence of cohomology (9) is natural, we see that $\gamma^{a,b;r}$ is independent of a and b in the sense of the proposition:

$$\gamma^{a,b+1;r} \circ j_!(g^{b,1}) = \gamma^{a,b;r} \qquad j_*(g^{a+1,-1}) \circ \gamma^{a+1,b;r} = \gamma^{a,b;r}$$

where the requisite commutative diagrams are produced using Lemma 2.4(1,2). For the same reason, $\gamma^{a,b;r}$ is a natural transformation between the two functors

$$\mathcal{M} \mapsto \ker(\alpha^{b,-r}_{\mathcal{M}}), \qquad \mathcal{M} \mapsto \operatorname{coker}(\alpha^{a,r}_{\mathcal{M}}), \qquad \square$$

Because they are equal, we will give a single name $\Pi^r_f(\mathcal{M}) = \ker(\alpha^{\infty,-r}) \cong \operatorname{coker}(\alpha^{\infty,r})$ to the stable kernel and cokernel. These are thus exact functors, and by definition of $\alpha^{a,r}$ and (8), they commute with duality: $\mathbb{D}\Pi^r_f(\mathcal{M}) \cong \Pi^r_f(\mathbb{D}\mathcal{M})$. From Proposition 2.2 we conclude:

Corollary 2.7. For $a \gg 0$ we have $\ker(\alpha^a) \cong \Psi^{\mathrm{un}}_f(\mathcal{M}) \cong \operatorname{coker}(\alpha^a)$, and thus an isomorphism

$$\mathbb{D}\Psi^{\mathrm{un}}_f(\mathcal{M}) \cong \Psi^{\mathrm{un}}_f(\mathbb{D}\mathcal{M})$$

which is natural in the perverse sheaf $\mathcal{M}$. A more effective, equivalent construction is obtained as follows: suppose $(1-t)^N$ annihilates $\Psi^{\mathrm{un}}_f(\mathcal{M})$. Then we have by (3):

$$\Psi^{\mathrm{un}}_f(\mathcal{M}) = i^* j_{!*}(\mathcal{M} \otimes f^*\mathcal{L}^N)[-1] = i^! j_{!*}(\mathcal{M} \otimes f^*\mathcal{L}^N)[1].$$

Conversely, if these equations hold, then of course $(1-t)^N$ annihilates $\Psi^{\mathrm{un}}_f(\mathcal{M})$. $\square$

3. Vanishing cycles and gluing

We will refer to Π^1_f as Ξ^{un}_f, which Beilinson calls the "maximal extension functor" and denotes without the superscript. Although there is no independent, non-unipotent analogue, we have chosen to use this notation to match that for the nearby and (upcoming) vanishing cycles functors, which do have such analogues.

Proposition 3.1. There are two natural exact sequences exchanged by duality and $\mathcal{M} \leftrightarrow \mathbb{D}\mathcal{M}$:

$$0 \to j_!(\mathcal{M}) \xrightarrow{\alpha_-} \Xi^{\mathrm{un}}_f(\mathcal{M}) \xrightarrow{\beta_-} \Psi^{\mathrm{un}}_f(\mathcal{M}) \to 0$$

$$0 \to \Psi^{\mathrm{un}}_f(\mathcal{M}) \xrightarrow{\beta_+} \Xi^{\mathrm{un}}_f(\mathcal{M}) \xrightarrow{\alpha_+} j_*(\mathcal{M}) \to 0,$$

where $\alpha_+ \circ \alpha_- = \alpha$ and $\beta_- \circ \beta_+ = 1 - t$.

Proof. These sequences are, respectively, the last and first halves of (9). For the first one, take $b = r$, so $g^{b,-r} = 0$ and therefore $\alpha^{b,-r} = 0$; we get an exact sequence

$$0 \to \ker(\alpha^{a,r}) \to \ker(\alpha^{a+r}) \to j_!(\mathcal{M} \otimes f^*\mathcal{L}^r) \xrightarrow{\gamma^{a,r;r}} \operatorname{coker}(\alpha^{a,r}) \to \operatorname{coker}(\alpha^{a+r}) \to 0.$$

For $a \gg 0$, by the second part of Proposition 2.5, the first map is an isomorphism, and for $r = 1$ we obtain the first short exact sequence from the remaining three terms above. For the second short exact sequence, we apply the same reasoning to (9) with $a = 0$ and then $r = 1$, with $b \gg 0$:

$$0 \to \ker(\alpha^{b}) \to \ker(\alpha^{b,-r}) \xrightarrow{\gamma^{0,b;r}} j_*(\mathcal{M} \otimes f^*\mathcal{L}^r) \to \operatorname{coker}(\alpha^{b}) \to \operatorname{coker}(\alpha^{b,-r}) \to 0.$$

It is obvious from these constructions and (8) that the two short exact sequences are exchanged by duality. To show that $\alpha_+ \circ \alpha_- = \mathrm{id}$ and $\beta_- \circ \beta_+ = 1 - t$, we identify these maps in the above sequences and rewrite the claims as:

$$\left(\gamma^{0,b;1} \circ (\gamma^{a,b;1})^{-1} \circ \gamma^{a,1;1}\right)\big|_U = \mathrm{id}, \quad \gamma^{a,b;1}|_{\ker(\alpha^b)} \bmod \operatorname{im}(\alpha^{a+1}) = (1-t)\gamma^{a+1,b;0}.$$

For both, we use the fact that since $\alpha^a|_U = \mathrm{id}$, we have $\gamma^{a,b;r}|_U = (g^{a+b,-a} \circ g^{a+r,b-r})^{-1}$, as constructed in the familiar proof of the snake lemma, with the inverse interpreted as a multi-valued pullback. Then the claims are equivalent to

$$g^{a+b,-a} \circ g^{a+1,b-1} = g^{1,b-1} \circ g^{a+1,-a}$$
$$g^{a+b+1,-a-1} \circ g^{a+1,b} = (1-t)g^{a+b,-a}g^{a+1,b-1}$$

which follow from Lemma 2.4(1,2). □

The remainder of the paper is simply what Beilinson calls "linear algebra" (one might argue that this has already been the case for most of the preceding). Take $\mathcal{M} = j^*\mathcal{F}$ for a perverse sheaf $\mathcal{F} \in \mathbf{M}(X)$ in the above exact sequences. From the maps in these two sequences we can form a complex:

$$j_!j^*\mathcal{F} \xrightarrow{(\alpha_-,\gamma_-)} \Xi_f^{\mathrm{un}}(j^*\mathcal{F}) \oplus \mathcal{F} \xrightarrow{(\alpha_+,-\gamma_+)} j_*j^*\mathcal{F}, \tag{10}$$

where $\gamma_- \colon j_!j^*(\mathcal{F}) \to \mathcal{F}$ and $\gamma_+ \colon \mathcal{F} \to j_*j^*(\mathcal{F})$ are defined by the left- and right-adjunctions $(j_!, j^*)$ and (j^*, j_*) and the property that $j^*(\gamma_-) = j^*(\gamma_+) = \mathrm{id}$.

Proposition 3.2. The complex (10) is in fact a complex; let $\Phi_f^{\mathrm{un}}(\mathcal{F})$ be its cohomology sheaf. Then Φ_f^{un} is an exact functor $\mathbf{M}(X) \to \mathbf{M}(Z)$, and there are maps u, v such that $v \circ u = 1 - t$ as in the following diagram:

$$\Psi_f^{\mathrm{un}}(j^*\mathcal{F}) \xrightarrow{u} \Phi_f^{\mathrm{un}}(\mathcal{F}) \xrightarrow{v} \Psi_f^{\mathrm{un}}(j^*\mathcal{F}).$$

Proof. That (10) is a complex amounts to showing that $\gamma_+ \circ \gamma_- = \alpha = \alpha_+ \circ \alpha_-$, which is true by definition of the $\gamma_\pm$ and adjunction. To show that Φ_f^{un} is exact, suppose we have $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$, so that we get a short exact sequence of complexes

$$0 \to C^\bullet(\mathcal{F}_1) \to C^\bullet(\mathcal{F}_2) \to C^\bullet(\mathcal{F}_3) \to 0,$$

where by $C^\bullet(\mathcal{F})$ we have denoted the complex (10) padded with zeroes on both sides. Note that since α_- is injective and α_+ surjective, $C^\bullet(\mathcal{F})$ fails to be exact only at the middle term. Therefore we have a long exact sequence of cohomology sheaves:

$$\cdots (0 = H^{-1}C^\bullet(\mathcal{F}_3)) \to \Phi_f^{\mathrm{un}}(\mathcal{F}_1) \to \Phi_f^{\mathrm{un}}(\mathcal{F}_2) \to \Phi_f^{\mathrm{un}}(\mathcal{F}_3) \to (0 = H^1(C^\bullet(\mathcal{F}_1))) \cdots$$

which shows that Φ_f^{un} is functorial and an exact functor.

If we apply j^* to (10), it becomes simply (with $j^*\mathcal{F} = \mathcal{M}$)

$$\mathcal{M} \xrightarrow{(\mathrm{id},\mathrm{id})} \mathcal{M} \oplus \mathcal{M} \xrightarrow{(\mathrm{id},-\mathrm{id})} \mathcal{M}$$

which is actually exact, so $j^*\Phi_f^{\mathrm{un}}(\mathcal{F}) = 0$; i.e. $\Phi_f^{\mathrm{un}}(\mathcal{F})$ is supported on Z. Finally, to define u and v, let $\mathrm{pr}\colon \Xi_f^{\mathrm{un}}(j^*\mathcal{F}) \oplus \mathcal{F} \to \Xi_f^{\mathrm{un}}(j^*\mathcal{F})$, and set $u = (\beta_+, 0)$ in coordinates, and $v = \beta_- \circ \mathrm{pr}$. Since $\beta_- \circ \alpha_- = 0$, v factors through $\Phi_f^{\mathrm{un}}(\mathcal{F})$, and we have $v \circ u = \beta_- \circ \beta_+ = 1 - t$ by Proposition 3.1. □

Define a *vanishing cycles gluing data* for f to be a quadruple $(\mathcal{F}_U, \mathcal{F}_Z, u, v)$ as in Proposition 3.2; for any $\mathcal{F} \in \mathbf{M}(X)$, the quadruple $F_f(\mathcal{F}) = (j^*\mathcal{F}, \Phi_f^{\mathrm{un}}(\mathcal{F}), u, v)$ is such data. Let $\mathbf{M}_f(U, Z)$ be the category of gluing data; then $F_f\colon \mathbf{M}(X) \to \mathbf{M}_f(U, Z)$ is a functor. Conversely, given a vanishing cycles data

$$\Psi_f^{\mathrm{un}}(\mathcal{F}_U) \xrightarrow{u} \mathcal{F}_Z \xrightarrow{v} \Psi_f^{\mathrm{un}}(\mathcal{F}_U),$$

we can form the complex

$$\Psi_f^{\mathrm{un}}(\mathcal{F}_U) \xrightarrow{(\beta_+,u)} \Xi_f^{\mathrm{un}}(\mathcal{F}_U) \oplus \mathcal{F}_Z \xrightarrow{(\beta_-,-v)} \Psi_f^{\mathrm{un}}(\mathcal{F}_U) \tag{11}$$

since $v \circ u = 1 - t = \beta_- \circ \beta_+$, and let $G_f(\mathcal{F}_U, \mathcal{F}_Z, u, v)$ be its cohomology sheaf.

Beilinson gives an elegant framework for proving the equivalence of (10) and (11) in [1, Appendix]. Rather than proving Theorem 3.6 directly, we present his technique (with slightly modified terminology).

Definition 3.3. Let a *diad* be a complex of the form

$$D^\bullet = \left(\mathcal{F}_L \xrightarrow{L = (a_L, b_L)} \mathcal{A} \oplus \mathcal{B} \xrightarrow{R = (a_R, b_R)} \mathcal{F}_R\right)$$

in which a_L is injective and a_R is surjective (so it is exact on the ends). Let the category of diads be denoted $\mathbf{M}_2$. Let a *triad* be a short exact sequence of the form

$$S = \left(0 \to \mathcal{F}_- \xrightarrow{(c_-, d_-^1, d_-^2)} \mathcal{A} \oplus \mathcal{B}^1 \oplus \mathcal{B}^2 \xrightarrow{(c_+, d_+^1, d_+^2)} \mathcal{F}_+ \to 0\right)$$

in which both $(c_-, d_-^i)\colon \mathcal{F}_- \to \mathcal{A} \oplus \mathcal{B}^i$ are injections and both $(c_+, d_+^i)\colon \mathcal{A} \oplus \mathcal{B}^i \to \mathcal{F}_+$ are surjections. Let the category of triads be denoted $\mathbf{M}_3$; it has a *reflection functor* $r\colon \mathbf{M}_3 \to \mathbf{M}_3$ which invokes the natural symmetry $1 \leftrightarrow 2$, and is an involution.

We can define a map $T\colon \mathbf{M}_2 \to \mathbf{M}_3$ by setting

$$T(D) = \left(0 \to \ker(R) \xrightarrow{(\iota_A, \iota_B, h)} \mathcal{A} \oplus \mathcal{B} \oplus H(D^\bullet) \xrightarrow{(\pi_A, \pi_B, -k)} \mathrm{coker}(L) \to 0\right),$$

where the natural inclusion/projection (resp. projection/inclusion) are called:

$$\ker(R) \xrightarrow{\iota = (\iota_A, \iota_B)} \mathcal{A} \oplus \mathcal{B} \xrightarrow{\pi = (\pi_A, \pi_B)} \mathrm{coker}(L), \quad \ker(R) \xrightarrow{h} H(D^\bullet) \xrightarrow{k} \mathrm{coker}(L)$$

(note $\pi \circ \iota = k \circ h$). We define the inverse T^{-1} by the formula

$$T^{-1}(S) = \left(\ker(d_-^2) \xrightarrow{(c_-, d_-^1)} \mathcal{A} \oplus \mathcal{B}^1 \to \mathrm{coker}(c_-, d_-^1)\right).$$

Lemma 3.4. The functors T, T^{-1} are mutually inverse equivalences of $\mathbf{M}_2$ with $\mathbf{M}_3$.

Proof. Before beginning the verification of the many necessary facts, we observe that the property of a sequence S as above being in $\mathbf{M}_3$ is equivalent to the following diagram being cartesian

$$\begin{tikzcd} \mathcal{F}_- \arrow[r, "d^i_-"] \arrow[d, "{(c_-, d^{3-i}_-)}"'] & \mathcal{B}^i \arrow[d, "-d^i_+"] \\ \mathcal{A} \oplus \mathcal{B}^{3-i} \arrow[r, "{(c_+, d^{3-i}_+)}"'] & \mathcal{F}_+ \end{tikzcd} \tag{12}$$

and the following smaller sequence being exact

$$0 \to \mathcal{F}_- \xrightarrow{(c_-, d^i_-)} \mathcal{A} \oplus \mathcal{B}^i \xrightarrow{(c_+, d^i_+)} \mathcal{F}_+ \to 0. \tag{13}$$

for $i = 1, 2$. Indeed, for (12), the diagram is cartesian if and only if S is exact in the middle, and for (13), the arrows are respectively injective and surjective by hypothesis if $S \in \mathbf{M}_3$, while exactness in the middle follows from that of S. For readability, we continue the proof as several sub-lemmas.

$T(D^\bullet) \in \mathbf{M}_3$: It is easily verified that $T(D^\bullet)$ is an exact sequence. To see that $(\pi_A, -k)$ is surjective and (ι_A, h) injective, we consider the cartesian diagrams

$$\begin{tikzcd} \mathcal{A} \oplus H(D^\bullet) \arrow[r, "{(\pi_A, -k)}"] \arrow[d] & \operatorname{coker}(L) \arrow[d] \\ \mathcal{A} \arrow[r, "a_R"] & \mathcal{F}_R \end{tikzcd} \qquad \begin{tikzcd} \ker(R) \arrow[r, "{(\iota_A, h)}"] \arrow[d] & \mathcal{A} \oplus H(D^\bullet) \arrow[d] \\ \mathcal{F}_L \arrow[r, "a_L"] & \mathcal{A} \end{tikzcd}$$

and use that a_R is surjective and a_L is injective. Likewise, $(\pi_B, -k)$ is surjective and (ι_B, h) is injective.

$T^{-1}(S) \in \mathbf{M}_2$: Clearly, $T^{-1}(S)$ is a complex, since the sequence $\mathcal{F}_1 \to \mathcal{A} \oplus \mathcal{B}^1 \to \operatorname{coker}(c_-, d^1_-)$ is exact (hence a complex). Since (c_-, d^2_-) is injective by hypothesis, $c_-|_{\ker(d^2_-)}$ is injective. We must show that $\mathcal{A} \to \operatorname{coker}(c_-, d^1_-)$ is surjective, where by (13) with $i = 1$ we have $\operatorname{coker}(c_-, d^1_-) = \mathcal{F}_+$; consider the diagram

$$\begin{tikzcd} \mathcal{A} \oplus \mathcal{F}_- \arrow[r, "{(\mathrm{id}, -c_-)}"] \arrow[d, "\mathrm{id} \oplus d^1_-"'] & \mathcal{A} \arrow[d, "c_+"] \\ \mathcal{A} \oplus \mathcal{B}^1 \arrow[r, "{(c_+, d^1_+)}"] & \mathcal{F}_+ \end{tikzcd}$$

which is cartesian by exactness of (13). Since the bottom arrow is a surjection, for c_+ to be a surjection it suffices to show that the left arrow is. By (12) with $i = 1$, d^1_- is a surjection since the bottom arrow there is a surjection by hypothesis.

$T^{-1} \circ T \cong \mathrm{id}$: Its $\mathcal{F}_L$ is $\ker(h) = \operatorname{im}(L) = \mathcal{F}_L$; its $\mathcal{A}$ and $\mathcal{B}$ are indeed $\mathcal{A}$ and $\mathcal{B}$, and its $\mathcal{F}_R$ is $\operatorname{coker}(\iota) = \mathcal{F}_R$; one checks quickly that the maps are right as well.

$T \circ T^{-1} \cong \mathrm{id}$: Its $\mathcal{F}_-$ is $\ker(\mathcal{A} \oplus \mathcal{B}^1 \to \operatorname{coker}(c_-, d^1_-)) = \mathcal{F}_-$ since (c_-, d^1_-) is an injection; its $\mathcal{A}$ and $\mathcal{B}^1$ are obviously the original $\mathcal{A}$ and $\mathcal{B}^1$. The small sequence (13) with $i = 1$ shows that $\mathcal{F}_+$ is correct as well. Finally, for $\mathcal{B}^2$, we must show that $\mathcal{F}_- / \ker(d^2_-) = \mathcal{B}^2$, or in other words, that d^2_- is surjective, which follows from (12) with $i = 2$. $\square$

Clearly, both of the complexes (10) and (11) are diads. Comparing them, we find that the construction of the latter is given by:

Corollary 3.5. The reflection functor on a diad is the complex

$$r(D^\bullet) = \Big(\ker(a_R) \xrightarrow{(a'_L, b'_L)} \mathcal{A} \oplus H(D^\bullet) \xrightarrow{(a'_R, b'_R)} \operatorname{coker}(a_L)\Big),$$

where a'_L is the natural inclusion and a'_R the natural projection, $b'_L = h \circ (a'_L, 0)$, and b'_R factors $-k$ through $\operatorname{coker}(a_L) \subset \operatorname{coker}(L)$.

Proof. That is, $T^{-1}rT(D^\bullet) = r(D^\bullet)$ as defined above. We need to show that $\ker(\iota_B) = \ker(a_R)$ and $\operatorname{coker}(\iota_A, h) = \operatorname{coker}(a_L)$, and prove the identities of the morphisms. The first is easily verified directly, considering both as subobjects of $\mathcal{A} \oplus \mathcal{B}$, while for the second, we assert that the map

$$(\mathrm{id}, 0)\colon \mathcal{A} \to \mathcal{A} \oplus H(D^\bullet)$$

induces the desired isomorphism from the latter to the former. To show that it identifies $\operatorname{im}(a_L)$ with $\operatorname{im}(\iota_A, h)$, it suffices to check that the following diagram is cartesian:

$$\begin{tikzcd}
\mathcal{F}_L \arrow[r, "a_L"] \arrow[d, "L"'] & \mathcal{A} \arrow[d, "(\mathrm{id},0)"] \\
\ker(R) \arrow[r, "(\iota_A, h)"'] & \mathcal{A} \oplus H(D^\bullet)
\end{tikzcd}$$

which follows from the definition of $H(D^\bullet) = \ker(R)/\operatorname{im}(R)$. The identities of a'_L, b'_L, and a'_R are clear from these constructions, while for b'_R it is fastest to chase the above diagram. $\square$

Theorem 3.6. The gluing category $\mathbf{M}_f(U, Z)$ is abelian; $F_f\colon \mathbf{M}(X) \to \mathbf{M}_f(U, Z)$ and $G_f\colon \mathbf{M}_f(U, Z) \to \mathbf{M}(X)$ are mutually inverse exact functors, and so $\mathbf{M}_f(U, Z)$ is equivalent to $\mathbf{M}(X)$.

Proof. That $\mathbf{M}_f(U, Z)$ is abelian amounts to proving that taking coordinatewise kernels and cokernels works. That is, if we have $(\mathcal{M}, \mathcal{F}_Z, u, v)$ and $(\mathcal{M}', \mathcal{F}'_Z, u', v')$ with maps $a_U\colon \mathcal{M} \to \mathcal{M}'$, $a_Z\colon \mathcal{F}_Z \to \mathcal{F}'_Z$ and such that the following diagram commutes:

$$\begin{tikzcd}
\Psi_f^{\mathrm{un}}(\mathcal{M}) \arrow[r, "u"] \arrow[d, "\Psi_f^{\mathrm{un}}(a_U)"'] & \mathcal{F}_Z \arrow[r, "v"] \arrow[d, "a_Z"] & \Psi_f^{\mathrm{un}}(\mathcal{M}) \arrow[d, "\Psi_f^{\mathrm{un}}(a_U)"] \\
\Psi_f^{\mathrm{un}}(\mathcal{M}') \arrow[r, "u'"] & \mathcal{F}'_Z \arrow[r, "v'"] & \Psi_f^{\mathrm{un}}(\mathcal{M}')
\end{tikzcd}$$

then $(\ker a_U, \ker a_Z, \widetilde{u}, \widetilde{v})$ is a kernel for (a_U, a_V), where $\widetilde{u}$ and $\widetilde{v}$ are induced maps; likewise for the cokernel; and we must show that (a_U, a_V) is an isomorphism if and only if the kernel and cokernel vanish. The maps $\widetilde{u}$ and $\widetilde{v}$ are constructed from the natural sequence of kernels (or cokernels) in the above diagram, and the exactness of Ψ_f^{un}, and once they exist it is obvious from the definition of morphisms in $\mathbf{M}_f(U, Z)$ that the desired gluing data is a kernel (resp. cokernel). Since $\mathbf{M}(U)$ and $\mathbf{M}(Z)$ are abelian and kernels and cokernels are taken coordinatewise, the last claim follows.

To show that F_f and G_f are mutually inverse, we interpret $\mathbf{M}(X)$ and $\mathbf{M}_f(U, Z)$ as diad categories in the form given, respectively, by diagrams (10) and (11). The reflection functor is given by Corollary 3.5; by Proposition 3.1 and the definition of Φ^{un}, its value on (10) is that of the functor F_f. For the same reason, its value on (11) is that of G_f interpreted as a complex of type (10) (the $\mathcal{F}$ term is what we have previously called the value of G_f). Since the reflection functor is an involution, G_f and F_f are mutually inverse. $\square$

4. Comments

We conclude with some musings on the theory exposited here. In the previous version arXiv:1002.1686v2 of these notes, we gave a substantially different proof of Proposition 2.6, adhering closely to that given in [1, Key Lemma]. As that proof may better illuminate the two-sided limit formalism which we also omit, the curious reader is encouraged to consult it.

The vanishing cycles functor and Φ_f^{un}. The functor Φ_f^{un}, like Ψ_f^{un}, has a familiar identity.

Theorem 4.1. There is an isomorphism of functors $\Phi_f^{\mathrm{un}} \cong R\phi_f^{\mathrm{un}}[-1]$ and a natural distinguished triangle

$$\Psi_f^{\mathrm{un}}(j^*\mathcal{F}) \xrightarrow{u} \Phi_f^{\mathrm{un}}(\mathcal{F}) \to i^*\mathcal{F} \to$$

isomorphic to that in (5).

Proof. According to the definition of Φ_f^{un} in Proposition 3.2, we have a short exact sequence and, thus, a corresponding distinguished triangle of the same form:

$$0 \to j_!j^*\mathcal{F} \to \ker(\alpha_+, -\gamma_+) \to \Phi_f^{\mathrm{un}}(\mathcal{F}) \to 0.$$

Since $K = \ker(\alpha_+, -\gamma_+) \subset \Xi_f^{\mathrm{un}}(j^*\mathcal{F}) \oplus \mathcal{F}$, there is a projection map $\mathrm{pr}\colon K \to \mathcal{F}$ commuting with the inclusion of $j_!j^*\mathcal{F}$.

Now we apply the octahedral axiom of triangulated categories as given in [4, (1.1.7.1)]:

$$\begin{tikzcd}
 & & \Phi_f^{\mathrm{un}}(\mathcal{F}) \arrow[dr] & & \\
 & K \arrow[ur] \arrow[dr, "\mathrm{pr}"] & & i^*\mathcal{F} \arrow[dr] & \\
 & & \mathcal{F} \arrow[ur] \arrow[drr] & & \\
j_!j^*\mathcal{F} \arrow[uur] \arrow[urr] & & & & C
\end{tikzcd}$$

where all the straight lines are distinguished triangles, both the (geometric) triangles are commutative, and the square commutes. It is easy to see that pr must be surjective because α_+ is surjective; thus, since both K and $\mathcal{F}$ are perverse, $C[-1]$ is also perverse, and so we have an exact sequence

$$0 \to C[-1] \to K \xrightarrow{\mathrm{pr}} \mathcal{F} \to 0.$$

But by definition, $\ker(\mathrm{pr}) = \ker(\alpha_+) \oplus 0$, and therefore $C[-1] \cong \Psi_f^{\mathrm{un}}(j^*\mathcal{F})$. Note that the inclusion then becomes the map u, as defined in the proof of Proposition 3.2. Rotating the other triangle in the above octahedral diagram, we have

$$\Psi_f^{\mathrm{un}}(j^*\mathcal{F}) \xrightarrow{u} \Phi_f^{\mathrm{un}}(\mathcal{F}) \to i^*\mathcal{F} \to .$$

Comparing with (5), we find that $R\phi_f^{\mathrm{un}}(\mathcal{F})[-1] \cong \Phi_f^{\mathrm{un}}(\mathcal{F})$ is perverse. Conversely, starting from (5) in place of the above triangle, we conclude by the octahedral axiom that $R\phi_f^{\mathrm{un}}(\mathcal{F})[-1]$ is the cohomology of (10), which admits a unique extension to a functor of $\mathcal{F}$ compatible with the octahedral diagram. Therefore, we conclude an isomorphism of functors $\Phi_f^{\mathrm{un}} \cong R\phi_f^{\mathrm{un}}[-1]$. □

The full nearby cycles functor $R\psi_f$. As Beilinson observes, the full nearby cycles functor $R\psi_f(\mathcal{M})$, for $\mathcal{M} \in \mathbf{M}(U)$, can be recovered from $R\psi_f^{\mathrm{un}}$ as applied to variations of $\mathcal{M}$. Here we must assume that the field of coefficients is algebraically closed.

Lemma 4.2. There exists a unique isomorphism of functors $\mathbf{D}(U) \to \mathbf{D}(Z)$

$$R\psi_f = \bigoplus_{\lambda \in \mathbb{C}^*} R\psi_f^{\lambda} \tag{14}$$

where for any constructible complex $A_U^\bullet$ on U, $\lambda - t$ is nilpotent on $R\psi_f^{\lambda}(A_U^\bullet)$.

Proof. Simply pursue the line of reasoning in Lemma 1.1 but, since the field of coefficients is algebraically closed, produce the full Jordan decomposition rather than just the unipotent and non-unipotent parts. The lemma can also be deduced from [10, Lemme 3.2.5], which applies to the Jordan decomposition of an endomorphism of any complex in the derived category. □

Let $\mathcal{L}_\lambda$ be the local system of rank 1 on $\mathbf{G_m}$ with monodromy λ; then clearly, we have $R\psi_f^{\lambda}(\mathcal{M}) = R\psi_f^{\mathrm{un}}(\mathcal{M} \otimes f^*\mathcal{L}_\lambda^{-1}) \otimes L_\lambda$, where t acts as λ on the one-dimensional vector space L_λ. Substituting into (14), we obtain:

$$R\psi_f(\mathcal{M}) = \bigoplus_{\lambda} \Psi_f^{\mathrm{un}}(\mathcal{M} \otimes f^*\mathcal{L}_\lambda^{-1}) \otimes L_\lambda.$$

Thus, Corollary 2.7 gives a procedure for computing the full nearby cycles functor of perverse sheaves, and $R\psi_f[-1]$ sends perverse sheaves on U to perverse sheaves on X.

Using some general reasoning, we can extend the properties of $\Psi_f^{\mathrm{un}} = R\psi_f^{\mathrm{un}}[-1]$ from the subcategory of perverse sheaves to the entire derived category. To this end, let $T\colon \mathbf{C} \to \mathbf{D}$ be a triangulated functor between triangulated categories with t-structures, and let the respective cores be the abelian categories $\mathbf{A}$, $\mathbf{B}$. We will assume that the objects of $\mathbf{C}$ are *bounded above*, meaning that $\mathbf{C} = \bigcup_{b \in \mathbb{Z}} \mathbf{C}^{\leqslant b}$.

Lemma 4.3. Suppose T is right t-exact and that $T\mathbf{A} \subset \mathbf{B}$; then T is t-exact.

Proof. We will show that T commutes with all truncations. Suppose we have an object $x \in \mathbf{C}^{\leqslant b}$, so that there is a distinguished triangle

$$\tau^{<b}x \to x \to \tau^{\geqslant b}x \to$$

where by definition, $\tau^{\geqslant b}x = H^b(x)[-b] \in \mathbf{A}[-b]$. By hypothesis on T, we have $T(x) \in \mathbf{D}^{\leqslant b}$, $T(\tau^{<b}x) \in \mathbf{D}^{<b}$, and $T(H^b x[-b]) \in \mathbf{B}[-b] \subset \mathbf{D}^{\geqslant b}$. Since T is triangulated, there is a triangle

$$T(\tau^{<b}x) \to T(x) \to T(H^b x[-b]) \to$$

and therefore, by uniqueness of the truncation triangle, it must be that $T(\tau^{<b}x) = \tau^{<b}T(x)$. This is under the hypothesis that $x \in \mathbf{C}^{\leqslant b}$; since then $\tau^{<b}x \in \mathbf{C}^{\leqslant b-1}$ and since $\tau^{<b-1}\tau^{<b} = \tau^{<b-1}$, we can apply truncations-by-one repeatedly and conclude that for all n, we have $\tau^{\leqslant n}T(x) = T(\tau^{\leqslant n}x)$.

Now suppose we have any x, and for any n form the distinguished triangle

$$\tau^{<n}x \to x \to \tau^{\geqslant n} \to$$

to which we apply T. Since $T(\tau^{<n}x) = \tau^{<n}T(x)$, the cone of the resulting triangle

$$\tau^{<n}T(x) \to T(x) \to T(\tau^{\geqslant n}x) \to$$

must be isomorphic to $\tau^{\geqslant n}T(x)$, by uniqueness of cones and the truncation triangle for $T(x)$. Thus, $\tau^{\geqslant n}T(x) = T(\tau^{\geqslant n}x)$. Since then T commutes with all trunctions, it is *a fortiori* t-exact. $\square$

Take $T = R\psi_f^{\mathrm{un}}[-1]$; by Lemma 1.2, it satisfies the hypothesis of Lemma 4.3, and therefore we conclude:

Theorem 4.4. The functor $R\psi_f[-1]$ on the bounded derived category $\mathbf{D}^b(X)$ is t-exact for the perverse t-structure. Likewise, $R\phi_f[-1]$ is t-exact.

Proof. For the second statement, we must show that $R\phi_f[-1]$ is right t-exact and preserves perverse sheaves; the latter claim already follows from Theorem 4.1. For the former, we apply the long exact sequence to the triangle

$$i^*\mathcal{F} \to R\psi_f(j^*\mathcal{F}) \to R\phi_f(\mathcal{F}) \to$$

We have $i^*\mathcal{F} \in {}^p\mathbf{D}(X)^{[-1,0]}$ because of triangle (1), and we already know that $R\psi_f[-1]$ is right t-exact, so the long exact sequence of perverse cohomology shows that ${}^pH^i(R\phi_f\mathcal{F}) = 0$ when $i \geq 0$, as desired. $\square$

We will not prove here that $R\psi_f[-1]$ commutes with Verdier duality. This is significantly more difficult since it necessitates enlarging the domain of a certain natural transformation (the map $\gamma^{a,b;r}$ constructed in Proposition 2.6) from the core of the perverse t-structure to the entire derived category. This involves the interaction with both objects and morphisms:

- The natural maps must be defined for all objects, not just those in $\mathbf{M}(U)$;
- The maps thus obtained must commute with all morphisms, not just those between objects of $\mathbf{M}(U)$.

To see why this is difficult, consider showing merely that the $\gamma^{a,b;r}$ (and their translates) are natural with respect to maps of the form $g\colon \mathcal{M} \to \mathcal{N}[i]$, with $i \in \mathbb{N}$ and $\mathcal{M}, \mathcal{N} \in \mathbf{M}(U)$. Note that the argument given for the naturality of $\gamma^{a,b;r}$ is not valid in this context, since kernel and cokernel constructions in the abelian category of perverse sheaves are not functorial in the entire derived category.

If $i = 1$, this is easy; we necessarily have $\mathrm{Cone}(g) \in \mathbf{M}(U)[1]$, so rotating the distinguished triangle gives a short exact sequence

$$0 \to \mathcal{N} \to \mathrm{Cone}(g)[-1] \to \mathcal{M} \to 0.$$

Conversely, this sequence constructs the distinguished triangle $\mathcal{M} \to \mathcal{N}[1] \to \mathrm{Cone}(g)$ by the reverse procedure. Then, applying $\mathbb{D}\Psi_f^{\mathrm{un}}$ and $\Psi_f^{\mathrm{un}}\mathbb{D}$ to the sequence, we find by naturality of $\gamma^{a,b;r}$ that there is a commutative diagram of short exact sequences, which implies that $\gamma^{a,b;r}$ is natural with respect to g.

The analogue of this argument for $i > 1$ would involve finding a sequence of the form

$$0 \xrightarrow{h_{i+1}=0} (\mathcal{N} = \mathcal{A}^{-(i+1)}) \xrightarrow{h_i} \mathcal{A}^{-i} \to \cdots \to \mathcal{A}^{-1} \xrightarrow{h_0} (\mathcal{A}^0 = \mathcal{M}) \to 0$$

representing g. The manner in which such a sequence does represent such a map is clear; we get a collection of short exact sequences representing maps:

$$0 \to \operatorname{coker}(h_{j+1}) \to \mathcal{A}^{-j} \to \operatorname{coker}(h_j) \to 0, \quad g_j\colon \operatorname{coker}(h_j) \to \operatorname{coker}(h_{j+1})[1]$$

(where $\operatorname{coker}(h_{i+1}) = \mathcal{N}$ and $\operatorname{coker}(h_1) = \mathcal{M}$), and thus, by composition, a map $g\colon \mathcal{M} \to \mathcal{N}[i]$, as desired. This is Yoneda's realization of $\operatorname{Ext}^i(\mathcal{M}, \mathcal{N})$; it holds in the derived category of $\mathbf{M}(U)$. It is, however, a nontrivial theorem, proved in [2], that this is the same as $\mathbf{D}(U)$, and in fact it is describing the morphisms that occupies the entirety of the work in that paper. Of course, once we choose to cite this result, it is a trivial consequence of Corollary 2.7 that $R\psi_f[-1]$ commutes with $\mathbb{D}$, since it is then the derived functor of a self-dual *exact* functor on $\mathbf{M}(U)$. Thus, we do not expect that there will be as elementary an argument as for the perversity of nearby cycles.

In the recent preprint [12], autoduality of the nearby cycles functor is proven in complete generality in the complex analytic setting, and references are given there for prior results and those in the algebraic setting.

The maximal extension functor Ξ_f^{un}. We have used the term "maximal extension functor" without explanation (as did Beilinson), but Proposition 3.1 provides sufficient rationale: applying i^* to the first one and $i^!$ to the second one, the long exact sequence of perverse cohomology shows that $i^*\Xi_f^{\mathrm{un}}(\mathcal{M}) \cong \Psi_f^{\mathrm{un}}(\mathcal{M}) \cong i^!\Xi_f^{\mathrm{un}}(\mathcal{M})$ are both perverse sheaves, which is as far out (cohomologically) as they can be given that i^* is right t-exact and $i^!$ is left t-exact. This should be compared with the defining property of the "minimal extension" $j_{!*}(\mathcal{M})$, that $i^*j_{!*}(\mathcal{M})[-1]$ and $i^!j_{!*}(\mathcal{M})[1]$ are perverse, so that it has a minimal presence on X given that it extends $\mathcal{M}$. The condition that $i^*\Xi_f^{\mathrm{un}}(\mathcal{M})$ and $i^!\Xi_f^{\mathrm{un}}(\mathcal{M})$ are perverse does not uniquely characterize $\Xi_f^{\mathrm{un}}(\mathcal{M})$, as one could add any perverse sheaf supported on Z without changing it, but imposing Proposition 3.1 forbids such a modification. As we will see below, these sequences uniquely determine $\Xi_f^{\mathrm{un}}(\mathcal{M})$.

To do so, consider the pair of upper and lower "caps" of an octahedron:

$$\begin{tikzcd}
j_!\mathcal{M} \arrow[rr, "\alpha"] \arrow[dr, "\alpha_-"'] & & j_*\mathcal{M} \arrow[dd] \\
& \Xi_f^{\mathrm{un}}(\mathcal{M}) \arrow[ur, "\alpha_+"'] \arrow[dl, "\beta_-"] \arrow[u, phantom, "c"] \arrow[d, phantom, "c"] \arrow[l, phantom, "d"] \arrow[r, phantom, "d"] & \\
\Psi_f^{\mathrm{un}}(\mathcal{M}) \arrow[uu, "{[1]}"] & & \Psi_f^{\mathrm{un}}(\mathcal{M})[1] \arrow[ll, "(1-t)[1]"] \arrow[ul, "\beta_+{[1]}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
j_!\mathcal{M} \arrow[rr, "\alpha"] & & j_*\mathcal{M} \arrow[dd] \arrow[dl] \\
& i^*j_*\mathcal{M} \arrow[ul, "{[1]}"'] \arrow[dr] \arrow[u, phantom, "d"] \arrow[d, phantom, "d"] \arrow[l, phantom, "c"] \arrow[r, phantom, "c"] & \\
\Psi_f^{\mathrm{un}}(\mathcal{M}) \arrow[uu, "{[1]}"] \arrow[ur] & & \Psi_f^{\mathrm{un}}(\mathcal{M})[1] \arrow[ll, "(1-t)[1]"]
\end{tikzcd}$$

(upper cap) (lower cap)

The triangles marked "c" are commutative and those marked "d" are distinguished; the arrows marked [1] have their targets (but not their sources) shifted by 1. The octahedral axiom states that given any diagram of commutative and distinguished triangles as in (lower cap) we can construct a diagram as in (upper cap) and vice versa ([4, §1.1.6]). Using these diagrams, we can derive (4) and Proposition 3.1 from each other. This idea is also present in [6, §5.7.2].

Proposition 4.5. Suppose we have functors Ξ_f^{un} and Ψ_f^{un} from $\mathbf{M}(U)$ to $\mathbf{M}(X)$, where Ψ_f^{un} has a unipotent action of $\pi_1(\mathbf{G_m})$, and satisfying Proposition 3.1. Then (4) holds with $R\psi_f^{\mathrm{un}} = \Psi_f^{\mathrm{un}}[1]$.

Proof. Given Proposition 3.1, each exact sequence there corresponds to a unique distinguished triangle in $\mathbf{D}(X)$ with the same entries; these triangles appear in (upper cap), where the top and bottom maps are α and $(1-t)[1]$ since the triangles containing them are commutative. The octahedral axiom gives us (lower cap), and since the upper triangle is distinguished its cone (the middle term) must necessarily be $i^*j_*\mathcal{M}$ by (1). Therefore the bottom triangle is (4), as desired. Note that all the interior maps in (lower cap) are uniquely determined, since they correspond to the kernels and cokernels of the maps α and $1-t$ of perverse sheaves in the long exact sequence of cohomology. □

Proposition 4.6. Given only the triangle (4), both the functor Ξ_f^{un} and its extension classes in $\mathrm{Ext}^1(\Psi_f^{\mathrm{un}}(\mathcal{M}), j_!\mathcal{M})$ and $\mathrm{Ext}^1(j_*\mathcal{M}, \Psi_f^{\mathrm{un}}(\mathcal{M}))$ can be constructed with Proposition 3.1 satisfied (except for the duality statement). In particular, by Proposition 4.5, Ξ_f^{un} is uniquely determined by Proposition 3.1.

Proof. Given (4), since we have (1) canonically we can form all the vertices of (lower cap) and both distinguished triangles; the left and right maps are determined by the requirement that the triangles containing them be commutative. The octahedral axiom gives us (upper cap) and $\Xi_f^{\mathrm{un}}(\mathcal{M})$, identified at first only as an element of $\mathbf{D}(X)$. From Lemma 1.2 we know that $\Psi_f^{\mathrm{un}}(\mathcal{M})$ is perverse; then the long exact sequence of perverse cohomology associated to either distinguished triangle in (upper cap) shows that, in fact, $\Xi_f^{\mathrm{un}}(\mathcal{M})$ is perverse, and thus those triangles correspond to exact sequences as in Proposition 3.1. The equations $\alpha_+\alpha_- = \alpha$ and $\beta_-\beta_+ = 1-t$ can then be read off from the commutativity of the upper and lower triangles. Since the vertical arrows come from (lower cap), these distinguished triangles are uniquely determined up to isomorphism fixing $j_{*,!}\mathcal{M}$ and $\Psi_f^{\mathrm{un}}(\mathcal{M})$, as desired. □

The identity of Ξ_f^{un} is somewhat mysterious, but can be made precise using the gluing category. These computations are also given in [6, Example 5.7.8].

Proposition 4.7. For any perverse sheaf $\mathcal{M} \in \mathbf{M}(U)$, we have the following correspondences via the gluing construction:

$$j_!(\mathcal{M}) = (\mathcal{M}, \Psi_f^{\mathrm{un}}(\mathcal{M}), \mathrm{id}, 1-t) \qquad j_{!*}(\mathcal{M}) = (\mathcal{M}, \mathrm{im}(1-t), 1-t, \mathrm{incl})$$
$$j_*(\mathcal{M}) = (\mathcal{M}, \Psi_f^{\mathrm{un}}(\mathcal{M}), 1-t, \mathrm{id}) \qquad \Xi_f^{\mathrm{un}}(\mathcal{M}) = (\mathcal{M}, \Psi_f^{\mathrm{un}}(\mathcal{M} \otimes f^*\mathcal{L}^2), u, v);$$

where $\alpha\colon j_!(\mathcal{M}) \to j_*(\mathcal{M})$ is the map $(\mathrm{id}, 1-t)$ in the gluing category; in $j_{!*}(\mathcal{M})$, we mean $\mathrm{im}(1-t) \subset \Psi_f^{\mathrm{un}}(\mathcal{M})$; in $\Xi_f^{\mathrm{un}}(\mathcal{M})$, taking $\Psi_f^{\mathrm{un}}(\mathcal{M} \otimes f^*\mathcal{L}^a) = \Psi_f^{\mathrm{un}}(\mathcal{M}) \oplus \Psi_f^{\mathrm{un}}(\mathcal{M})$, we have $u = (\mathrm{id}, 1-t)$ and $v = \mathrm{pr}_2$.

Proof. Using the triangle of Theorem 4.1, we have

$$\Psi_f^{\mathrm{un}}(j^*j_!\mathcal{M}) \to \Phi_f^{\mathrm{un}}(j_!\mathcal{M}) \to i^*j_!(\mathcal{M}) \to$$

and since $i^*j_! = 0$, we get an isomorphism $\Phi_f^{\mathrm{un}}(j_!\mathcal{M}) \cong \Psi_f^{\mathrm{un}}(\mathcal{M})$; dualizing, we have $\Phi_f^{\mathrm{un}}(j_*\mathcal{M}) \cong \Psi_f^{\mathrm{un}}(\mathcal{M})$ also. Since u is the first map in this triangle, under this identification we have $u = \mathrm{id}$, and therefore $v = 1-t$ since $v \circ u = 1-t$. This gives the quadruple for $j_!(\mathcal{M})$; for $j_*(\mathcal{M})$, we dualize, since u and v are dual by their definition in Propositions 3.2 and 3.1. That the natural map is given by $(\mathrm{id}, 1-t)$ follows from the fact that this does define a map $j_!(\mathcal{M}) \to j_*(\mathcal{M})$ in the gluing category, and that its restriction to U is the identity.

For $j_{!*}(\mathcal{M})$, we use the fact that it is the image of the natural map $\alpha\colon j_!(\mathcal{M}) \to j_*(\mathcal{M})$; having already identified all the parties, this is clear from the quadruples just obtained.

For the identification of $\Xi_f^{\mathrm{un}}(\mathcal{M})$, obviously, $v \circ u = 1-t$; more importantly, u is injective and v surjective. Then the pair of exact sequences in Proposition 3.1 can be described on quadruples as being trivial over U, and over Z the maps α_- and α_+ are described by the following maps of quadruples:

$$\begin{tikzcd}
j_!(\mathcal{M}) : \arrow[d, "\alpha_-"'] & \Psi_f^{\mathrm{un}}(\mathcal{M}) \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & \Psi_f^{\mathrm{un}}(\mathcal{M}) \arrow[r, "1-t"] \arrow[d, "u"] & \Psi_f^{\mathrm{un}}(\mathcal{M}) \arrow[d, "\mathrm{id}"] \\
\Xi_f^{\mathrm{un}}(\mathcal{M}) : \arrow[d, "\alpha_+"'] & \Psi_f^{\mathrm{un}}(\mathcal{M}) \arrow[r, "u"] \arrow[d, "\mathrm{id}"] & \Psi_f^{\mathrm{un}}(\mathcal{M} \otimes f^*\mathcal{L}^2) \arrow[r, "v"] \arrow[d, "v"] & \Psi_f^{\mathrm{un}}(\mathcal{M}) \arrow[d, "\mathrm{id}"] \\
j_*(\mathcal{M}) : & \Psi_f^{\mathrm{un}}(\mathcal{M}) \arrow[r, "1-t"] & \Psi_f^{\mathrm{un}}(\mathcal{M}) \arrow[r, "\mathrm{id}"] & \Psi_f^{\mathrm{un}}(\mathcal{M})
\end{tikzcd}$$

We take β_- and β_+ to be the maps whose Z-parts (the U-parts are zero) are:

$$\beta_-(y, z) = (1-t)y - z \qquad\qquad \beta_+(x) = (x, 0).$$

Then it is clear from the definitions of u and v that we obtain the sequences of Proposition 3.1; by the uniqueness part of Proposition 4.6, this uniquely determines $\Xi_f^{\mathrm{un}}(\mathcal{M})$, completing the proof. $\square$

Since the entirety of Section 3 follows only from Proposition 3.1, Propositions 4.5 and 4.6 show that the constructions of Section 2 are irrelevant for constructing the gluing functor. Their purpose, as is evident from the order we have chosen for the theorems, is to exhibit the autoduality of Ψ_f^{un} and Ξ_f^{un} (and, thus, Φ_f^{un}). However, Beilinson's development has an aesthetic virtue (over just using the above short proof of Proposition 3.1): once Lemma 1.2 is proven, the entire theory takes place within the abelian category of perverse sheaves. In addition, Corollary 2.7 is an ingeniously elementary, insightful, and more useful definition of a functor whose actual definition is quite obscure.

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Families of Gauss indicatrices on smooth surfaces in pseudo-spheres in the Minkowski 4-space

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Abstract

We study families of Gauss indicatrices on surfaces in pseudo-spheres in the Minkowski 4-space and obtain the generic local models of the configurations of the foliations determined by the fibres of their principal curvatures functions.

1 Introduction

In [9], Izumiya-Pei-Sano defined the hyperbolic Gauss indicatrix of a hypersurface in the Minkowski space model of the hyperbolic space. The work in [9] set the foundations of applications of singularity theory to the extrinsic geometry of submanifolds in the hyperbolic space. Given a point p on a hypersurface M in the hyperbolic space $H^n_+(-1)$, there is a well defined (at least locally) unit normal vector $\boldsymbol{e}(p)$ to M at p; see §2. The vector $\boldsymbol{e}(p)$ is in the de Sitter space S^n_1 and defines the de Sitter Gauss indicatrix

$$\begin{array}{rccc} \mathbb{E}: & M & \to & S^n_1 \\ & p & \to & \boldsymbol{e}(p) \end{array}$$

The de Sitter Gauss-Kronecker curvature at p is $K_e(p) := \det(-(d\mathbb{E})_p)$ and the totally umbilic hypersurfaces with $K_e \equiv 0$ are the hyperplanes in $H^n_+(-1)$. The de Sitter Gauss indicatrix on M is related to the contact of M with hyperplanes ([9]).

Another Gauss indicatrix on M is introduced in [9] and is called the hyperbolic or lightcone Gauss indicatrix; see §2. The vector $p \pm \boldsymbol{e}(p)$ is lightlike (i.e., belongs to the lightcone LC^*) and defines the hyperbolic Gauss indicatrices

$$\begin{array}{rccc} \mathbb{L}^{\pm}: & M & \to & LC^* \\ & p & \to & p \pm \boldsymbol{e}(p) \end{array}$$

The hyperbolic Gauss-Kronecker curvature at p is $K_h(p) := \det(-(d\mathbb{L}^{\pm})_p)$ and the totally umbilic hypersurfaces with $K_h \equiv 0$ are the hyperhorospheres in $H^n_+(-1)$. The hyperbolic Gauss indicatrix on M is related to the contact of M with hyperhorospheres ([9]).

In [1] is constructed a 1-parameter family of Gauss indicatrices which links $\mathbb{E}$ and $\mathbb{L}^{\pm}$. The family is given by $N_\theta(p) = \cos\theta p \pm \boldsymbol{e}(p) \in S^n(\sin^2(\theta))$, $\theta \in [0, \pi/2]$, and is called the Slant Gauss indicatrix. Observe that $N_\theta(p)$ is always spacelike for $\theta \neq 0$. The above family links the geometry of M related to hyperplanes to that related to hyperhorospheres. See also [11] for slant geometry in the de Sitter space and in the lightcone.

The work in this paper is inspired by that in [1, 11]. A hypersurface M in $H^n_+(-1)$ can be viewed as a codimension 2 spacelike submanifold in $\mathbb{R}^{n+1}_1$. It has then a timelike normal plane in $\mathbb{R}^{n+1}_1$ at

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any of its points. For this reason, we consider normal vector fields (Gauss indicatrix) on M which are not necessarily spacelike. We define two families of Gauss indicatrices on M. One is spacelike and is given by $N_\theta^s = \tanh(\theta)p + \boldsymbol{e}(p)$ and the other is timelike and is given by $N_\theta^t = \tanh(\theta)^{-1}p + \boldsymbol{e}(p)$ (we use hyperbolic angles here, see [17] for definition and properties). The families N_θ^w, $w = s, t$ tend to $\mathbb{L}^\pm$ as θ tends to $\pm\infty$. We define the θ^w-Gauss-Kronecker curvature by $K_\theta^w(p) := \det(-(dN_\theta^w)_p)$.

We give in §3 general results about the families N_θ^w on hypersurfaces in $H_+^n(-1)$ and deal in more details with surfaces in $H_+^3(-1)$ in §3.1. We denote by κ_1 and κ_2 the eigenvalues of the de Sitter shape operator and call them the de Sitter principal curvatures. It turns out that the θ^w-parabolic sets (points where K_θ^w vanishes) are given by $\kappa_i = constant$. The θ^s-parabolic sets foliate the region in M where $|\kappa_i| < 1$ and the θ^t-parabolic sets foliate the region in M where $|\kappa_i| > 1$; see Theorem 3.2. (One motivation behind considering the timelike Gauss indicatrices is that the θ^s-parabolic sets do not cover the whole surface. The other is that N_θ^t gives information about the contact of M with hyperspheres.) Note that the parabolic sets of the limiting families $N_{\pm\infty}^w = \mathbb{L}^\pm$ are the horospherical parabolic sets given by $\kappa_i = \pm 1$. We obtain the generic local configurations of the foliations $\kappa_i = constant$, $i = 1, 2$ (Theorem 3.7), and characterise geometrically their singularities (Theorems 3.4, 3.5, 3.8).

One can view $M \subset H_+^3(-1)$ as a surface in $\mathbb{R}_1^4$. Asymptotic directions are defined via the contact of M with lines. They are metric independent and we have thus well defined asymptotic curves on M given by a quadratic binary differential equation (BDE for short). We show that these asymptotic curves are in fact the lines of the de Sitter principal curvature. This is true for any spacelike or timelike surface in a pseudo-sphere in the Minkowski 4-space (Theorem 3.9).

We consider in §4 families of Gauss indicatrices on timelike hypersurfaces in the de Sitter space S_1^n, with emphasis on timelike surfaces in S_1^3. The foliations $\kappa_i = constant$, $i = 1, 2$, behave differently from those on spacelike surfaces (Theorem 4.1). We recall in the Appendix §5 the classification of codimension ≤ 1 singularities of BDEs.

2 Preliminaries

We start by recalling some basic concepts in hyperbolic geometry (see for example [16, 19]). The *Minkowski* $(n+1)$*-space* $(\mathbb{R}_1^{n+1}, \langle , \rangle)$ is the $(n+1)$-dimensional vector space $\mathbb{R}^{n+1}$ endowed with the *pseudo scalar product* $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = -u_0 v_0 + \sum_{i=1}^n u_i v_i$, for any $\boldsymbol{u} = (u_0, \ldots, u_n)$ and $\boldsymbol{v} = (v_0, \ldots, v_n)$ in $\mathbb{R}_1^{n+1}$. We say that a vector $\boldsymbol{u}$ in $\mathbb{R}_1^{n+1} \setminus \{\boldsymbol{0}\}$ is *spacelike*, *lightlike* or *timelike* if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle > 0, = 0$ or < 0 respectively. The norm of a vector $\boldsymbol{u} \in \mathbb{R}_1^{n+1}$ is defined by $\|\boldsymbol{u}\| = \sqrt{|\langle \boldsymbol{u}, \boldsymbol{u} \rangle|}$. Given a vector $\boldsymbol{v} \in \mathbb{R}_1^{n+1}$ and a real number c, a hyperplane with pseudo normal $\boldsymbol{v}$ is defined by

$$HP(\boldsymbol{v}, c) = \{\boldsymbol{u} \in \mathbb{R}_1^{n+1} \mid \langle \boldsymbol{u}, \boldsymbol{v} \rangle = c\}.$$

We say that $HP(\boldsymbol{v}, c)$ is a *spacelike*, *timelike* or *lightlike hyperplane* if $\boldsymbol{v}$ is timelike, spacelike or lightlike respectively. We have the following pseudo-spheres in $\mathbb{R}_1^{n+1}$ with centre $p \in \mathbb{R}_1^{n+1}$ and radius $r > 0$,

$$\begin{array}{rcl} H^n(p, -r) & = & \{\boldsymbol{u} \in \mathbb{R}_1^{n+1} \mid \langle \boldsymbol{u} - p, \boldsymbol{u} - p \rangle = -r^2\}, \\ S^n(p, r) & = & \{\boldsymbol{u} \in \mathbb{R}_1^{n+1} \mid \langle \boldsymbol{u} - p, \boldsymbol{u} - p \rangle = r^2\}, \\ LC^*(p) & = & \{\boldsymbol{u} \in \mathbb{R}_1^{n+1} \mid \langle \boldsymbol{u} - p, \boldsymbol{u} - p \rangle = 0\}. \end{array}$$

We denote by $H^n(-r)$ and $S^n(r)$ the pseudo-spheres centred at the origin in $\mathbb{R}_1^{n+1}$. The pseudo sphere $H^n(-r)$ has two connected components. The hyperbolic space $H_+^n(-1)$ is the connected component of $H^n(-1)$ whose points $\boldsymbol{u}$ have positive coordinate u_0. The de Sitter space is $S_1^n = S^n(1)$ and the lightcone is $LC^* = LC^*(\boldsymbol{0})$.

A hypersurface given by the intersection of $H_+^n(-1)$ with a spacelike, timelike or lightlike hyperplane is called respectively *hypersphere*, *equidistant hypersurface* or *hyperhorosphere*. The intersection of a hypersurface with a timelike hyperplane through the origin is called simply a *hyperplane*.

The study of the extrinsic geometry of hypersurfaces in the hyperbolic space from the viewpoint of Legendrian singularities was initiated in [9]. Let $\boldsymbol{x} : U \to H^n_+(-1)$ be a local parametrisation of a hypersurface M embedded in $H^n_+(-1)$, where U is an open subset of $\mathbb{R}^{n-1}$. We write $M = \boldsymbol{x}(U)$. Since $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \equiv -1$, we have $\langle \boldsymbol{x}_{u_i}, \boldsymbol{x} \rangle \equiv 0$, for $i = 1, \ldots, n-1$, where $\underline{u} = (u_1, \ldots, u_{n-1}) \in U$. The spacelike unit normal vector $\boldsymbol{e}(\underline{u})$ to M at $\boldsymbol{x}(\underline{u})$ is defined by

$$\boldsymbol{e}(\underline{u}) = \frac{\boldsymbol{x}(\underline{u}) \wedge \boldsymbol{x}_{u_1}(\underline{u}) \wedge \ldots \wedge \boldsymbol{x}_{u_{n-1}}(\underline{u})}{\|\boldsymbol{x}(\underline{u}) \wedge \boldsymbol{x}_{u_1}(\underline{u}) \wedge \ldots \wedge \boldsymbol{x}_{u_{n-1}}(\underline{u})\|}.$$

It follows that $\boldsymbol{x}(\underline{u}) \pm \boldsymbol{e}(\underline{u})$ is a lightlike vector for all $\underline{u} \in U$. The de Sitter and hyperbolic Gauss indicatrices $\mathbb{E}$ and $\mathbb{L}^\pm$ respectively are defined in the introduction. The linear transformation $-(d\mathbb{E})_p$ at $p = \boldsymbol{x}(\underline{u})$ is called the *de Sitter shape operator*. Its eigenvalues κ_i, $i = 1, \ldots, n-1$, are called the *de Sitter principal curvatures* and the corresponding eigenvectors $\boldsymbol{p}_i$, $i = 1, \ldots, n-1$, are called the *de Sitter principal directions*. The linear transformation $-(d\mathbb{L}^\pm)_p$ is labelled the *hyperbolic shape operator* of M at p. It has the same eigenvectors as $-(d\mathbb{E})_p$ but has distinct eigenvalues. In fact the eigenvalues $\bar{\kappa}_i^\pm$ of $-(d\mathbb{L}^\pm)_p$ satisfy $\bar{\kappa}_i^\pm = -1 \pm \kappa_i$, $i = 1, \ldots, n-1$.

A smooth submanifold M of the Minkowski space is said to be *spacelike* (resp. *timelike*) if the induced metric on M is Riemannian (resp. Lorentzian, i.e., of signature 1). For a spacelike (resp. timelike) hypersurface in the de Sitter space S^n_1, the vector $\boldsymbol{e}(\underline{u})$ is timelike (resp. spacelike) and defines a Gauss indicatrix with values in the hyperbolic (resp. de Sitter) space.

3 Hypersurfaces in $H^n_+(-1)$

We start with some general results on hypersurfaces M in $H^n_+(-1)$. Let $\boldsymbol{x} : U \to M$ be a local parametrisation of M. At each point $\boldsymbol{x}(\underline{u})$, the normal plane $N_{\boldsymbol{x}(\underline{u})}M$ to M in $\mathbb{R}^{n+1}$ is timelike and is generated by $\boldsymbol{e}(\underline{u})$ and $\boldsymbol{x}(\underline{u})$. Any choice of a normal vector in $N_{\boldsymbol{x}(\underline{u})}M$ generates a Gauss indicatrix. For instance, the hyperbolic Gauss indicatrix $\mathbb{L}^\pm$ is given by $\boldsymbol{x}(\underline{u}) \pm \boldsymbol{e}(\underline{u})$. We can parametrise a circle of vectors in $N_{\boldsymbol{x}(\underline{u})}M$ by $\cos(\theta)\boldsymbol{x}(\underline{u}) + \sin(\theta)\boldsymbol{e}(\underline{u})$ and get a family of Gauss indicatrices. However, we would like the parameter to have some geometric meaning and also to distinguish between the timelike and spacelike normal vectors as these lead to the contact of M with different models of hypersurfaces. The differential of the Gauss indicatrix given by the vector $\boldsymbol{x}(\underline{u})$ is the identity map so does not give any geometric information. For these reasons, we define the family of spacelike Gauss indicatrices by

$$\begin{array}{rccl} N^s_\theta: & U & \to & S^n(\cosh(\theta)^{-2}) \\ & \underline{u} & \mapsto & \tanh(\theta)\boldsymbol{x}(\underline{u}) + \boldsymbol{e}(\underline{u}) \end{array}$$

where $\theta \in \mathbb{R}$ is the hyperbolic angle between $N^s_\theta(\underline{u})$ and $\boldsymbol{x}(\underline{u})$. If we take $\sinh(\theta)\boldsymbol{x}(\underline{u}) + \cosh(\theta)\boldsymbol{e}(\underline{u}) \in S^n_1$ as a unit normal spacelike vector we will not get the desired limit $N^s_\theta \to \mathbb{L}^\pm$ when $\theta \to \pm\infty$. We define the family of timelike Gauss indicatrices by

$$\begin{array}{rccl} N^t_\theta: & U & \to & H^n(-\sinh(\theta)^{-2}) \\ & \underline{u} & \mapsto & \tanh(\theta)^{-1}\boldsymbol{x}(\underline{u}) + \boldsymbol{e}(\underline{u}) \end{array}$$

where $\theta \in \mathbb{R} \setminus \{0\}$ is the hyperbolic angle between $N^t_\theta(\underline{u})$ and $\boldsymbol{x}(\underline{u})$. Again, if we take $\cosh(\theta)\boldsymbol{x}(\underline{u}) + \sinh(\theta)\boldsymbol{e}(\underline{u}) \in H^n(-1)$ as a unit normal timelike vector we will not get the desired limit $N^t_\theta \to \mathbb{L}^\pm$ when $\theta \to \pm\infty$. (Observe that $\boldsymbol{x}$ is not a member of the family N^t_θ.)

We have the following result which follows from the definitions of N^w_θ, $w = s, t$.

Theorem 3.1 *The differential map $-(dN^s_\theta)_p = -\tanh(\theta)I_p - (d\mathbb{E})_p$ is a self-adjoint operator on T_pM. Its eigenvalues are $\kappa^s_{\theta i} = -\tanh(\theta) + \kappa_i$, where κ_i are the de Sitter principal curvatures. The eigenvectors of $-(dN^s_\theta)_p$, for any $\theta \in \mathbb{R}$, coincide with those of the de Sitter shape operator $-(d\mathbb{E})_p$.*

Similarly, the differential map $-(dN^t_\theta)_p = -\tanh(\theta)^{-1}I_p - (d\mathbb{E})_p$ *is a self-adjoint operator on* T_pM*. Its eigenvalues are* $\kappa^t_{\theta i} = -\tanh(\theta)^{-1} + \kappa_i$*. The eigenvectors of* $-(dN^t_\theta)_p$*, for any* $\theta \in \mathbb{R}\setminus\{0\}$*, also coincide with those of the de Sitter shape operator.*

We call $\kappa^w_{\theta i}$, $w = s,t$, the θ^w-principal curvatures and call $K^w_\theta(p) = det(-(dN^w_\theta)_p) = \prod_{i=1}^{n-1}\kappa^w_{\theta i}$ the θ^w-Gauss-Kronecker curvature of M at $p = \boldsymbol{x}(\underline{u})$. A point p on M is called (spacelike) θ^w-umbilic ($w = s$ or t) if $\kappa^w_{\theta i} = \kappa^w_{\theta j}$ for all i,j at p. It is called θ^w-parabolic if $K^w_\theta(p) = 0$.

We are interested in hypersurfaces whose points are all θ^w-umbilics, which we label totally θ^w-umbilic hypersurfaces. These will form the “model” hypersurfaces in the hyperbolic space. One can characterised θ^w-umbilic hypersurfaces in the same way as in Proposition 2.3 in [9]. For instance, if a hypersurface $M \subset H^n_+(-1)$ is totally θ^w-umbilic, then $\kappa^w_{\theta i}$ are all equal to the same constant, say κ^w_θ, on M. Then M is a subset of the intersection of $H^n_+(-1)$ with a hyperplane and the type of the hyperplane is determined by the value of the constant κ^w_θ.

We consider the contact of M with model hypersurfaces in $H^n_+(-1)$. We define the family of spacelike height functions by

$$\begin{array}{cccc} H^s_\theta & U\times S^n(\cosh(\theta)^{-2}) & \to & \mathbb{R} \\ & (\underline{u},\boldsymbol{v}) & \mapsto & \langle \boldsymbol{x}(\underline{u}),\boldsymbol{v}\rangle + \tanh(\theta) \end{array}$$

This measures the contact of M with the equidistant hypersurfaces $HP(\boldsymbol{v}, -\tanh(\theta)) \cap H^n_+(-1)$. We have $H^s_\theta = \partial H^s_\theta/\partial u_i = 0$ if and only if $\boldsymbol{v} = N^s_\theta(\underline{u})$. A point $p = \boldsymbol{x}(\underline{u})$ is a θ^s-parabolic point if and only if the Hessian of $H^s_\theta(-,\boldsymbol{v})$, with $\boldsymbol{v} = N^s_\theta(\underline{u})$, is singular. This means that the θ^s-parabolic set is the set of points on M which correspond to the singular points of the discriminant of the family H^s_θ. (One can show, using the same arguments in the proof of Proposition 4.2 in [9] that H^s_θ is a Morse family. This yields a Legendrian immersion whose generating family is H^s_θ. The wavefront of the Legendrian immersion is the Gauss indicatrix N^s_θ.)

We also define the family of timelike height functions

$$\begin{array}{cccc} H^t_\theta & U\times H^n(-\sinh(\theta)^{-2}) & \to & \mathbb{R} \\ & (\underline{u},\boldsymbol{v}) & \mapsto & \langle \boldsymbol{x}(\underline{u}),\boldsymbol{v}\rangle + \tanh(\theta)^{-1} \end{array}$$

which measures the contact of M with the hyperspheres $HP(\boldsymbol{v}, -\tanh^{-1}(\theta)) \cap H^n_+(-1)$. We have similar results to those for the family H^s_θ.

3.1 Surfaces in $H^3_+(-1)$

We obtain in this section geometric information about the foliations determined by $\kappa^w_{\theta i} = constant$, $i = 1,2$, $w = s,t$. As the θ^w-principal curvatures define the same foliations, we work with the de Sitter curvatures κ_1 and κ_2. Let $\boldsymbol{x} : U \subset \mathbb{R}^2 \to M \subset H^3_+(-1)$ be a local parametrisation of M and denote by (u,v) the coordinates in U. In this paper, subscripts involving the parameters u,v refer to partial differentiation with respect to these parameters. The coefficients of the first fundamental form with respect to $\boldsymbol{x}$ are denoted by

$$E = \langle \boldsymbol{x}_u,\boldsymbol{x}_u\rangle,\quad F = \langle \boldsymbol{x}_u,\boldsymbol{x}_v\rangle,\quad G = \langle \boldsymbol{x}_v,\boldsymbol{x}_v\rangle.$$

The θ^w-second fundamental form ($w = s,t$) at $p = \boldsymbol{x}(u,v)$, with associated shape operator $-(dN^w_\theta)_p$, is given by $\mathrm{II}^w_\theta(\boldsymbol{u},\boldsymbol{v}) = \langle -(dN^w_\theta)_p(\boldsymbol{u}),\boldsymbol{v}\rangle$, for $\boldsymbol{u},\boldsymbol{v} \in T_pM$. We denote by

$$\begin{array}{ccccc} l^w_\theta & = & \langle -(dN^w_\theta)_p(\boldsymbol{x}_u),\boldsymbol{x}_u\rangle & = & \langle N^w_\theta,\boldsymbol{x}_{uu}\rangle, \\ m^w_\theta & = & \langle -(dN^w_\theta)_p(\boldsymbol{x}_u),\boldsymbol{x}_v\rangle & = & \langle N^w_\theta,\boldsymbol{x}_{uv}\rangle, \\ n^w_\theta & = & \langle -(dN^w_\theta)_p(\boldsymbol{x}_v),\boldsymbol{x}_v\rangle & = & \langle N^w_\theta,\boldsymbol{x}_{vv}\rangle \end{array}$$

its coefficients with respect to the basis $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$. We have

$$\begin{array}{rcl} l_\theta^w & = & -\tanh(\theta)^\epsilon E + l, \\ m_\theta^w & = & -\tanh(\theta)^\epsilon F + m, \\ n_\theta^w & = & -\tanh(\theta)^\epsilon G + n, \end{array}$$

with $\epsilon = 1$ if $w = s$ and $\epsilon = -1$ if $w = t$ and where l, m, n denote the coefficients of the second fundamental form associated to the de Sitter shape operator $-d\mathbb{E}$. Because the induced metric on M is Riemannian, $-(dN_\theta^w)_p$ has always two real eigenvalues. The θ^w-lines of principal curvature are the same for all θ and coincide with the de Sitter lines of principal curvature. These are given by a BDE that can be represented in the following determinant form

$$\begin{vmatrix} dv^2 & -dudv & du^2 \\ E & F & G \\ l & m & n \end{vmatrix} = 0. \tag{1}$$

For a generic surface, the discriminant of equation (1) (which is the set of points on the surface where the equation determines a unique direction, see §5 for details) consists of the isolated umbilic points. We write

$$K_e = \kappa_1\kappa_2 = \frac{ln - m^2}{EG - F^2},$$
$$H_e = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{lG - 2mF + nE}{2(EG - F^2)},$$

for the de Sitter Gauss-Kronecker curvature and the de Sitter mean curvature, respectively. We have the following result.

Theorem 3.2 (1) *The θ^s-parabolic set is given by*

$$\tanh^2(\theta) - 2H_e\tanh(\theta) + K_e = 0.$$

It consists of the curves (which could be empty) $\kappa_i = \tanh(\theta)$, $i = 1, 2$. Each family of these curves foliate the region of M where $|\kappa_i| < 1$ as θ varies in $\mathbb{R}$. The leaves of the foliations tend to the horospherical parabolic set $|\kappa_i| = 1$ as θ tends to $\pm\infty$.

(2) *The θ^t-parabolic set is given by*

$$\tanh^2(\theta)K_e - 2H_e\tanh(\theta) + 1 = 0.$$

It consists of the curves (which could be empty) $\kappa_i = \tanh(\theta)^{-1}$, $i = 1, 2$. Each family of these curves foliate the region of M where $|\kappa_i| > 1$ as θ varies in $\mathbb{R} \setminus \{0\}$. The leaves of the foliations tend to the horospherical parabolic set as θ tends to $\pm\infty$.

Proof The θ^w-Gauss-Kronecker curvature is given by

$$K_\theta^w = \det(-(dN_\theta^w)_p) = \frac{l_\theta^w n_\theta^w - (m_\theta^w)^2}{EG - F^2} = \kappa_{\theta 1}^w \kappa_{\theta 2}^w.$$

The equations for the θ^w-parabolic sets follow from the fact that $\kappa_{\theta i}^w = -\tanh(\theta)^\epsilon + \kappa_i$ with $\epsilon = 1$ if $w = s$ and $\epsilon = -1$ if $w = t$ and observing that $K_e = \kappa_1\kappa_2$ and $2H_e = \kappa_1 + \kappa_2$. If we take, for example $w = s$, it follows that the θ^s-parabolic sets consists of the curves $\kappa_i = \tanh(\theta)$, $i = 1, 2$. As $|\tanh(\theta)| < 1$, these curves foliate the regions where $|\kappa_i| < 1$ as θ varies in $\mathbb{R}$. The case $w = t$ follows similarly.

□

Remark 3.3 It follows from Theorem 3.2 that the θ^s-parabolic sets do not cover the whole surface M. This is one of the reasons why we need to consider the family N_θ^t of timelike Gauss indicatrices.

A direction $\boldsymbol{u} \in T_pM$ is said to be θ^w*-asymptotic*, $w = s, t$, if $\langle (dN_\theta^w)_p(\boldsymbol{u}), \boldsymbol{u}\rangle = 0$. The integral curves of the θ^w-asymptotic directions are called the θ^w*-asymptotic curves.* It is not hard to show that the θ^w-asymptotic curves are the solutions of the binary differential equation (BDE)

$$(A_\theta^w): \; n_\theta^w dv^2 + 2m_\theta^w dudv + l_\theta^w du^2 = 0. \tag{2}$$

Equation (2) determines two θ^w-asymptotic directions in the region where $\delta_\theta^w = l_\theta^w n_\theta^w - (m_\theta^w)^2 > 0$, none where $\delta_\theta^w < 0$, and a unique (double) θ^w-asymptotic direction on the θ^w-parabolic set $\delta_\theta^w = 0$. See Appendix (§5) for topological models of the solutions of a BDE.

We show below that the singularities of the foliations $\kappa_i = constant$, $i = 1, 2$, are picked up by the families of height functions and by the BDE (2). This will allow us to determine their configurations at their singular points. We start with the families of height functions. The contact group is denoted by $\mathcal{K}$, the $\mathcal{K}$-singularities A_k are modelled by $u^2 \pm v^{k+1}$ and the $\mathcal{K}$-singularities D_k by $u^2 v \pm v^{k-1}$.

Theorem 3.4 *Away from a discrete set of values of $\theta \in \mathbb{R}$, the height function $H_\theta^w(-, \boldsymbol{v})$, $w = s, t$, along $\boldsymbol{v} = N_\theta^w(p)$, has generically $\mathcal{K}$-singularities of type A_1, A_2 and A_3 at p. These are characterised geometrically as follows:*

A_1 : *p is not a θ^w-parabolic point.*

A_2 : *p is a θ^w-parabolic point and the unique θ^w-asymptotic direction at p is transverse to the θ^w-parabolic set.*

A_3 : *p is a θ^w-parabolic point and the unique θ^w-asymptotic direction at p is tangent to the θ^w-parabolic set.*

Proof The height function $H_\theta^w(-, \boldsymbol{v})$ is singular at (u_0, v_0) if $\boldsymbol{v} = N_\theta^w(u_0, v_0)$. (In fact it is singular at (u_0, v_0) if and only if $\boldsymbol{v}$ is a normal vector to M at $\boldsymbol{x}(u_0, v_0)$.) We suppose that (u_0, v_0) is a singularity of $H_\theta^w(-, \boldsymbol{v})$ with $\boldsymbol{v} = N_\theta^w(u_0, v_0)$, and write H_θ^w for $H_\theta^w(-, \boldsymbol{v})$.

At (u_0, v_0), $(H_\theta^w)_{uu} = l_\theta^w = -\tanh(\theta)^\epsilon E + l$, $(H_\theta^w)_{uv} = m_\theta^w = -\tanh(\theta)^\epsilon F + m$, and $(H_\theta^w)_{vv} = n_\theta^w = -\tanh(\theta)^\epsilon G + n$. Thus the Hessian of H_θ^w at (u_0, v_0) is singular if and only if $\boldsymbol{x}(u_0, v_0)$ is a θ^w-parabolic point. The singularity is of type A_2 if the cubic part of the Taylor expansion of H_θ^w at (u_0, v_0) does not divide Q, where Q^2 is its quadratic part. To make the conditions more apparent, we choose a special local parametrisation of M where the coordinate curves are the de Sitter lines of principal curvature. (We can do this away from the de Sitter umbilic points and we can assume this to be the case at $\boldsymbol{x}(u_0, v_0)$.) Then $F \equiv 0$, $m \equiv 0$ and (u_0, v_0) is a singularity of H_θ^w if and only if $(H_\theta^w)_{uu}(u_0, v_0) = 0$ or $(H_\theta^w)_{vv}(u_0, v_0) = 0$. If both are zero we get a D_4-singularity and this is dealt with in Theorem 3.5. Suppose that $(H_\theta^w)_{uu}(u_0, v_0) = 0$ and $(H_\theta^w)_{vv}(u_0, v_0) \neq 0$. Then the singularity is of type A_2 if and only if $(H_\theta^w)_{uuu}(u_0, v_0) \neq 0$. We have $H_{uu} = \langle \boldsymbol{x}_{uu}, \boldsymbol{v}\rangle$, so at (u_0, v_0)

$$\begin{aligned} H_{uuu} &= \langle \boldsymbol{x}_{uuu}, \boldsymbol{v}\rangle \\ &= \langle \boldsymbol{x}_{uuu}, \tanh(\theta)^\epsilon \boldsymbol{x} + \boldsymbol{e}\rangle \\ &= \tanh(\theta)^\epsilon \langle \boldsymbol{x}_{uuu}, \boldsymbol{x}\rangle + \langle \boldsymbol{x}_{uuu}, \boldsymbol{e}\rangle. \end{aligned}$$

By differentiating twice the identity $\langle \boldsymbol{x}, \boldsymbol{x}_u\rangle = 0$ we get

$$\langle \boldsymbol{x}_{uuu}, \boldsymbol{x}\rangle = -3\langle \boldsymbol{x}_u, \boldsymbol{x}_{uu}\rangle = -\frac{3}{2}E_u.$$

We have $\langle \boldsymbol{x}_{uu}, \boldsymbol{e}\rangle = l$, so $\langle \boldsymbol{x}_{uuu}, \boldsymbol{e}\rangle + \langle \boldsymbol{x}_{uu}, \boldsymbol{e}_u\rangle = l_u$. However,

$$\langle \boldsymbol{x}_{uu}, \boldsymbol{e}_u\rangle = \langle \boldsymbol{x}_{uu}, -\kappa_1 \boldsymbol{x}_u\rangle = -\frac{1}{2}\kappa_1 E_u.$$

Thus

$$\langle \boldsymbol{x}_{uuu}, \boldsymbol{e}\rangle = l_u + \frac{1}{2}\kappa_1 E_u.$$

We have $\kappa_1 = \tanh(\theta)^\epsilon$ at (u_0, v_0), so at this point

$$\begin{array}{rcl} H_{uuu} & = & \tanh(\theta)^\epsilon \langle \boldsymbol{x}_{uuu}, \boldsymbol{x}\rangle + \langle \boldsymbol{x}_{uuu}, \boldsymbol{e}\rangle \\ & = & -\frac{3}{2}\tanh(\theta)^\epsilon E_u + l_u + \frac{1}{2}\tanh(\theta)^\epsilon E_u \\ & = & -\tanh(\theta)^\epsilon E_u + l_u. \end{array}$$

Now the discriminant of the asymptotic curves (the θ^w-parabolic set) is given by $l^w_\theta = -\tanh^\epsilon(\theta)E + l = 0$ and the unique asymptotic direction at (u_0, v_0) is along $(1,0)$. The direction $(1,0)$ is transverse the the θ^w-parabolic set at (u_0, v_0) if and only if $(-\tanh^\epsilon(\theta)E_u + l_u)(u_0, v_0) \neq 0$, that is, if and only if $(H^w_\theta)_{uuu}(u_0, v_0) \neq 0$. When $(H^w_\theta)_{uu} = (H^w_\theta)_{uuu} = 0$ at (u_0, v_0), we get and A_3-singularity for generic θ.

For θ fixed, the family H^w_θ is a 3-parameter family. Therefore, for a generic embedding of M in $H^3_+(-1)$, only singularities of $\mathcal{K}$-codimension ≤ 3 can occur. (See for example [14]. We are interested in the discriminant of the family H^w_θ, this is why we consider the $\mathcal{K}$-codimension and not the $\mathcal{K}_e$-codimension.) These are the A_1, A_2 and A_3-singularities. If we let θ vary, we get generically singularities of $\mathcal{K}$-codimension 4 at isolated points, which can occur for a discrete set of values of θ. □

Denote by $\mathcal{S} = \{(\theta, \boldsymbol{v}) \in \mathbb{R} \times S^3(\cosh(\theta)^{-2})\}$ and $\mathcal{T} = \{(\theta, \boldsymbol{v}) \in \mathbb{R} \setminus \{0\} \times H^3(-\sinh(\theta)^{-2})\}$. We consider the "big" families of height functions given by

$$\begin{array}{cccc} H^s & U \times \mathcal{S} & \to & \mathbb{R} \\ & ((u,v),(\theta,\boldsymbol{v})) & \mapsto & \langle \boldsymbol{x}(u,v), \boldsymbol{v}\rangle + \tanh(\theta) \end{array}$$

and

$$\begin{array}{cccc} H^t & U \times \mathcal{T} & \to & \mathbb{R} \\ & ((u,v),(\theta,\boldsymbol{v})) & \mapsto & \langle \boldsymbol{x}(u,v), \boldsymbol{v}\rangle + \tanh(\theta)^{-1} \end{array}$$

For a generic embedding of the surface the big family H^w, $w = s, t$, along $N^w_\theta(p)$ can have the following local catastrophic events at p:

(i) an A_3-singularity which is not $\mathcal{K}$-versally unfolded by the family H^w_θ.
(ii) an A_4-singularity of H^w_θ.
(iii) a D_4-singularity of H^w_θ; this occurs at an umbilic point with $\kappa_1 = \kappa_2 = \tanh(\theta)^\epsilon$.

Theorem 3.5 (1) *The family H^w_θ, $w = s, t$, for θ fixed, is always a $\mathcal{K}$-versal unfolding of the A_1 and A_2 singularities of the height function at p along $\boldsymbol{v} = N^w_\theta(p)$. It fails to be a $\mathcal{K}$-versal unfolding of an A_3-singularity if and only if the θ^w-parabolic set is singular.*

(2) *The big family H^w is always a $\mathcal{K}$-versal unfolding of a non-versal A_3-singularity of H^w_θ along $\boldsymbol{v} = N^w_\theta(p)$.*

(3) *For a generic surface, the big family H^w is a $\mathcal{K}$-versal unfolding of an A_4-singularity of H^w_θ at p along $\boldsymbol{v} = N^w_\theta(p)$.*

(4) *For a generic surface, the big family H^w is a $\mathcal{K}$-versal unfolding of a D_4-singularity of H^w_θ at p along $\boldsymbol{v} = N^w_\theta(p)$.*

Proof The proof is similar to those given in [3] for families of height functions on surfaces in $\mathbb{R}^3$. We deal here with the D_4-singularity case and with $w = s$. This occurs when $\kappa_1 = \kappa_1 = \tanh(\theta_0)$, say at $(u_0, v_0) = (0,0)$. Every direction in $T_{\boldsymbol{x}(0,0)}M$ is a de Sitter principal direction, so we cannot take a parametrisation with $F \equiv 0, m \equiv 0$. We take without loss of generality, $j^1\boldsymbol{x}(u,v) = (1, u, v, 0)$, $\boldsymbol{e}(0,0) = (0,0,0,1)$ and $\boldsymbol{v}_0 = (\tanh(\theta_0), 0, 0, 1)$. We write $\boldsymbol{x} = (x_0, x_1, x_2, x_3)$. For

$\boldsymbol{v} = (v_0, v_1, v_2, v_3) \in S^3(\cosh(\theta)^{-2})$ near $\boldsymbol{v}_0$, we can write $v_3 = \sqrt{\cosh^{-2}(\theta) + v_0^2 - v_1^2 - v_2^2}$. Then the family H^s is a $\mathcal{K}$-versal deformation of the D_4-singularity of $H^s_{\theta_0}$ at $(0,0)$ if and only if

$$\mathcal{E}_2 \left\langle \frac{\partial H^s_{\theta_0}}{\partial u}, \frac{\partial H^s_{\theta_0}}{\partial v}, H^s_{\theta_0} \right\rangle + \mathbb{R}.\left\{ \frac{\partial H^s}{\partial v_0}, \frac{\partial H^s}{\partial v_1}, \frac{\partial H^s}{\partial v_2}, \frac{\partial H^s}{\partial \theta} \right\} = \mathcal{E}_2 \tag{3}$$

where $H^s_{\theta_0}$, $\partial H^s/\partial v_i$, $i = 1, 2, 3$, $\partial H^s/\partial\theta$ are evaluated at $(u, v, \theta_0, \boldsymbol{v}_0)$, and $\mathcal{E}_2$ denotes the ring of germs of smooth functions at $(0,0)$.

The 2-jet of $H^s_{\theta_0}$ is identically zero and its 3-jet is a non-degenerate cubic (the singularity is of type D_4). Therefore, it is 3-$\mathcal{K}$-determined. We can then work in the 3-jet space and show that all degree 3 monomials in u and v are in the left hand side of (3). For degree ≤ 2 we proceed as follows. We have

$$\boldsymbol{x}(u, v) = (1, u, v, 0) + \frac{1}{2}(\boldsymbol{x}_{uu}(0,0)u^2 + 2\boldsymbol{x}_{uv}(0,0)uv + 2\boldsymbol{x}_{vv}(0,0)v^2).$$

One can show that

$$\begin{array}{rcl}
\boldsymbol{x}_{uu}(0,0) & = & (-E, \frac{1}{2}E_u, F_v - \frac{1}{2}E_v, \tanh(\theta_0)E),\\
\boldsymbol{x}_{uv}(0,0) & = & (-F, \frac{1}{2}E_v, \frac{1}{2}G_u, \tanh(\theta_0)F),\\
\boldsymbol{x}_{vv}(0,0) & = & (-G, F_v - \frac{1}{2}G_u, \frac{1}{2}G_v, \tanh(\theta_0)G).
\end{array}$$

We have $\partial H^s/\partial\theta((u, v), (\theta_0, \boldsymbol{v}_0)) = \cosh(\theta_0)^{-2}$, so the constant terms are in the left hand side of (3) and we can work modulo these terms. We have

$$\begin{array}{rcl}
j^2 \frac{\partial H^s}{\partial v_0}((u, v), (\theta_0, \boldsymbol{v}_0)) - 1 & = & j^2(-x_0(u, v) + \tanh(\theta)x_3(u, v)) - 1\\
& = & \frac{1}{2}(1 + \tanh^2(\theta_0))(Eu^2 + 2Fuv + Gv^2).
\end{array}$$

Also, by similar calculations to those in the proof of Theorem 3.4,

$$\begin{array}{rl}
j^2(H^s_{\theta_0})_u(u, v) & = \frac{1}{2}((H^s_{\theta_0})_{uuu}u^2 + 2(H^s_{\theta_0})_{uuv}uv + (H^s_{\theta_0})_{uvv}v^2)\\
& = \frac{1}{2}((-\tanh(\theta_0)E_u + l_u)u^2 + 2(-\tanh(\theta_0)F_u + m_u)uv + (-\tanh(\theta_0)G_u + n_u)v^2),\\
j^2(H^s_{\theta_0})_v(u, v) & = \frac{1}{2}((H^s_{\theta_0})_{uuv}u^2 + 2(H^s_{\theta_0})_{uvv}uv + (H^s_{\theta_0})_{vvv}v^2)\\
& = \frac{1}{2}((-\tanh(\theta_0)E_v + l_v)u^2 + 2(-\tanh(\theta_0)F_v + m_v)uv + (-\tanh(\theta_0)G_v + n_v)v^2).
\end{array}$$

We put a multiple of the above three vectors in the following matrix form

$$\begin{array}{cccc}
 & u^2 & uv & v^2\\
\frac{2}{1+\tanh^2(\theta_0)} j^2 \frac{\partial H^s}{\partial v_0} & E & 2F & G\\
2j^2(H^s_{\theta_0})_u & -\tanh(\theta_0)E_u + l_u & 2(-\tanh(\theta_0)F_u + m_u) & -\tanh(\theta_0)G_u + n_u\\
2j^2(H^s_{\theta_0})_v & -\tanh(\theta_0)E_v + l_v & 2(-\tanh(\theta_0)F_v + m_v) & -\tanh(\theta_0)G_v + n_v
\end{array} \tag{4}$$

The determinant of the above matrix is not zero at a generic umbilic point. Therefore, $u^2, uv, , v^2$ are in the left hand side of (3). We can work now on the 1-jet level and obtain u, v using

$$j^1 \frac{\partial H^s}{\partial v_1}((u, v), (\theta_0, \boldsymbol{v}_0)) = j^1(x_1(u, v)) = u \text{ and } j^1 \frac{\partial H^s}{\partial v_2}((u, v), (\theta_0, \boldsymbol{v}_0)) = j^1 x_2(u, v) = v.$$

□

Remark 3.6 It follows from Theorem 3.5 that the de Sitter parabolic set can have singularities if it is considered as a member of the θ-parabolic sets. This means that there is nothing special about the de Sitter Gauss map $\mathbb{E}$ when considered as a member of the family N^s_θ.

Theorem 3.7 *The curves κ_i = constant, $i = 1, 2$, undergo Morse transitions at a non-versal A_3-singularity of the height function (Figure 1, first two figures) and remain smooth at an A_4-singularity. At a D_4^+ (resp. D_4^-)-singularity (i.e., at an umbilic point) the generic configuration is as in the third (resp. fourth) figure in Figure 1.*

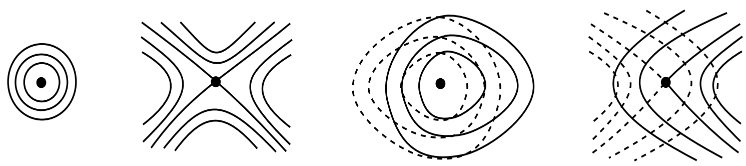

Figure 1: The foliation $\kappa_i = constant$ ($i = 1$ or 2) at a non-versal A_3-singularity (first two figures). The third (resp. fourth) figure is the generic configuration of the foliations $\kappa_i = constant, i = 1, 2$ at a D_4^+ (resp. D_4^-)-singularity, continuous lines for κ_i and dashed for $\kappa_j, j \neq i$.

Proof The first two statements are a consequence of Theorem 3.5. At an umbilic point (u_0, v_0) with $\tanh(\theta_0)^\epsilon = \kappa_1 = \kappa_2$, the foliations $\kappa_i = constant$ are given by $\tanh(\alpha)^{2\epsilon} - 2H_e \tanh(\alpha)^\epsilon + K_e = 0$ and $\tanh(\alpha)^\epsilon = constant$. The first equation determines a surface S in the (θ, u, v)-space which has a cone singularity at $q_0 = (\theta_0, u_0, v_0)$. The projection $\pi : S \to U$ maps diffeomorphically each connected component of $S \setminus \{q_0\}$ to $U \setminus \{(u_0, v_0)\}$. The foliations $\kappa_i = constant$ are the images by π of the traces of the planes $\theta = constant$ on S. The traces of these planes on one component on $S \setminus \{q_0\}$ project to $\kappa_1 = constant$ and those on the other component project to $\kappa_2 = constant$. The plane $\theta = \theta_0$ is generically not tangent to the cone, so we have two possible configurations for its trace on the cone: it is either an isolated point (this is the case when the height function $H^w_{\theta_0}$ along $N^w_{\theta_0}(p)$ has a D_4^+-singularity) or it is a pair of crossing curves (this is the case when the height function $H^w_{\theta_0}$ along $N^w_{\theta_0}(p)$ has a D_4^--singularity). As θ varies near θ_0 we obtain generic cone sections. If the cone sections are closed curves (resp. hyperbole), the configuration of their projections to the (u, v)-plane is as in Figure 2, third (resp. fourth) figure. □

We turn now the θ^w-asymptotic curves and their singularities (see Appendix for notation).

Theorem 3.8 *For a generic surface M in $H^3_+(-1)$, the BDE (A^w_θ) of the θ^w-asymptotic curves can have singularities of codimension ≤ 1.*

(1) *The BDE (A^w_θ) has a folded singularity (or worse) at p if and only if H^w_θ along $N^w_\theta(p)$ has an A_3-singularity (or worse) at p. The three types of the folded singularities of BDEs can occur in (A^w_θ) (Figure 3).*

(2) *The BDE (A^w_θ) has a folded saddle-node singularity at p for some $\theta = \theta_0$ if and only if $H^w_{\theta_0}$ has an A_4-singularity at p. The family (A^w_θ), as θ varies near θ_0, is generic if and only if the big family H^w is a versal unfolding of the A_4-singularity of $H^w_{\theta_0}$ (Figure 4, left).*

(3) *The BDE (A^w_θ) can have a node-focus change at p for some $\theta = \theta_0$. This is not detected by the family H^w_θ. The family (A^w_θ), as θ varies near θ_0, is generic for generic surfaces in $H^3_+(-1)$ (Figure 4, right).*

(4) *The BDE (A^w_θ) has a Morse Type 1 singularity at p for some $\theta = \theta_0$ if and only if $H^w_{\theta_0}$ has a non-versal A_3-singularity at p. The family (A^w_θ), as θ varies near θ_0, is always a generic family (Figure 5).*

(5) *At an umbilic point the BDE $A^w_{\theta_0}$ has a Morse Type 2 singularity with discriminant of type A_1^+ (Figure 6) or A_1^- (Figure 7). The family (A^w_θ) as θ varies near θ_0 is a generic family if and only if the family H^w is a versal unfolding of the D_4-singularity of $H^w_{\theta_0}$.*

Proof The proofs here are also similar to those for surfaces in $\mathbb{R}^3$ ([2, 6]). For the case (5), the condition for the family (A^s_θ) to be a generic family at an umbilic point is

$$\begin{vmatrix} a_\theta & b_\theta & c_\theta \\ a_u & b_u & c_u \\ a_v & b_v & c_v \end{vmatrix} \neq 0,$$

where $a, 2b, c$ are the coefficients of (A^s_θ) (see [6]). The above determinant is, up to a scalar multiple, the determinant of the matrix (4) in the proof of Theorem 3.5. □

3.2 Surfaces in $H^3_+(-1)$ viewed as surfaces in $\mathbb{R}^4_1$

In §3.1 we defined a θ^w-asymptotic direction $\boldsymbol{u} \in T_pM$ by $\langle (dN^w_\theta)_p(\boldsymbol{u}), \boldsymbol{u}\rangle = 0$. This notion depends on the shape operator $-dN^w_\theta$. For surfaces in $\mathbb{R}^4$, there is another notion of asymptotic directions which is defined in terms of the contact of the surface with lines and hyperplanes ([4, 13]; see also [12] for their definition in terms of the curvature ellipse). For this reason, these asymptotic directions and their integral curves (the asymptotic curves) are affine properties of the surface, i.e., they do not depend on the metric on $\mathbb{R}^4$ and can be defined in the same way on a surface in $\mathbb{R}^4_1$.

Let $\mathbf{r} : U \subset \mathbb{R}^2 \to M \subset \mathbb{R}^4_1$ be a local parametrisation of a spacelike or timelike surface M. We have a well defined second fundamental form on M using the Levi-Civita connection on $\mathbb{R}^4_1$ (see for example [16]). Let $\{e_3, e_4\}$ be a frame in the normal plane N_pM. Then the coefficient of this second fundamental form are given by

$$a_i = \langle e_i, \mathbf{r}_{uu}\rangle, \quad b_i = \langle e_i, \mathbf{r}_{uv}\rangle, \quad c_i = \langle e_i, \mathbf{r}_{vv}\rangle, \quad i = 3, 4.$$

Given any normal vector field μ, with coordinates (α, β) in the normal space N_pM, the shape operator $S_\mu : T_pM \to T_pM$ along μ is represented, with respect to the basis $\{\mathbf{r}_u, \mathbf{r}_v\}$, by the matrix

$$S_\mu = \frac{1}{EG - F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} \alpha a_3 + \beta a_4 & \alpha b_3 + \beta b_4 \\ \alpha b_3 + \beta b_4 & \alpha c_3 + \beta c_4 \end{pmatrix}.$$

We denote by

$$[S_\mu] = \begin{pmatrix} \alpha a_3 + \beta a_4 & \alpha b_3 + \beta b_4 \\ \alpha b_3 + \beta b_4 & \alpha c_3 + \beta c_4 \end{pmatrix}$$

the symmetric matrix associated to S_μ (it completely determines S_μ). We call the eigenvectors of S_μ (when they exist) the μ-principal directions and call their integral curves the μ-principal curves. These are given by the binary differential equation

$$\begin{vmatrix} dv^2 & -dudv & du^2 \\ E & F & G \\ \alpha a_3 + \beta a_4 & \alpha b_3 + \beta b_4 & \alpha c_3 + \beta c_4 \end{vmatrix} = 0. \tag{5}$$

Following [4], we say that a direction $\boldsymbol{u} \in T_pM$ is asymptotic if the projection of M along $\boldsymbol{u}$ to a transverse hyperplane has an $\mathcal{A}$-singularity more degenerate than a cross-cap at p. It is not difficult to show that the asymptotic curves on $M \subset \mathbb{R}^4_1$ are given by a BDE which has the same form as that of a surface in $\mathbb{R}^4$, namely

$$(A):\ (b_3c_4 - b_4c_3)dv^2 + (a_3c_4 - a_4c_3)dudv + (a_3b_4 - a_4b_3)du^2 = 0 \tag{6}$$

where a_i, b_i, c_i, $i = 3, 4$, are the coefficients of the second fundamental form at (u, v). This equation can also be written in a determinant form

$$\begin{vmatrix} dv^2 & -dudv & du^2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix} = 0. \tag{7}$$

We follow the notation for surfaces in $\mathbb{R}^4$ and label the discriminant of equation (6) by Δ. Points where Δ is singular (generically a Morse singularity $A_1^{\pm}$) are labelled *inflection points.* The generic configurations of the asymptotic curves at inflection points are the same as those for surfaces in $\mathbb{R}^4$, top figures in Figure 6 and Figure 7 (see [5, 7]).

Theorem 3.9 *Let M be a spacelike or timelike surface contained in a pseudo-sphere in $\mathbb{R}^4_1$. Then the μ-principal curves coincide for all normal vector fields μ on M and are precisely the asymptotic curves of M when viewed as a surface in $\mathbb{R}^4_1$.*

Proof Let $\boldsymbol{x} : U \to M$ be a local parametrisation of M. Because the metric on M is not degenerate, $\{\boldsymbol{x}(u,v), \boldsymbol{e}(u,v)\}$ is a basis of the normal plane N_pM at all points $p = \boldsymbol{x}(u,v)$. The coefficients of the second fundamental form (with respect to $\{\boldsymbol{x}, \boldsymbol{e}\}$) are given by

$$\begin{array}{ll} a_3 = \langle \boldsymbol{x}, \boldsymbol{x}_{uu} \rangle = -\langle \boldsymbol{x}_u, \boldsymbol{x}_u \rangle = -E, & a_4 = \langle \boldsymbol{e}, \boldsymbol{x}_{uu} \rangle = l, \\ b_3 = \langle \boldsymbol{x}, \boldsymbol{x}_{uv} \rangle = -\langle \boldsymbol{x}_u, \boldsymbol{x}_v \rangle = -F, & b_4 = \langle \boldsymbol{e}, \boldsymbol{x}_{uv} \rangle = m, \\ c_3 = \langle \boldsymbol{x}, \boldsymbol{x}_{vv} \rangle = -\langle \boldsymbol{x}_v, \boldsymbol{x}_v \rangle = -G, & c_4 = \langle \boldsymbol{e}, \boldsymbol{x}_{vv} \rangle = n. \end{array}$$

Let $\mu = \alpha \boldsymbol{x} + \beta \boldsymbol{e}$ be a normal vector field to M (we assume that $\beta \neq 0$). Then the equation of the μ-lines of principal curvature is given by

$$\begin{vmatrix} dv^2 & -dudv & du^2 \\ E & F & G \\ -\alpha E + \beta l & -\alpha F + \beta m & -\alpha G + \beta n \end{vmatrix} = 0 = \begin{vmatrix} dv^2 & -dudv & du^2 \\ E & F & G \\ l & m & n \end{vmatrix}.$$

The last determinant above is equation (7) of the asymptotic curves of M when viewed as a surface in $\mathbb{R}^4_1$. □

Remark 3.10 The proof of Theorem 3.9 is an alternative to that in [18] for surfaces in the Euclidean 4-space and for spacelike surfaces in the Minkowski 4-space [8].

We shall not distinguish between a general BDE (9) (see Appendix) and its non-zero multiples, so at each point $(u,v) \in U$ we can view the BDE as a quadratic form $a\beta^2 + 2b\beta\gamma + c\gamma^2 = 0$ ($\beta = dv$ and $\gamma = du$) and represent it by the point $Q = (a : 2b : c)$ in the projective plane $\mathbb{R}P^2$. In $\mathbb{R}P^2$ there is a conic $\Gamma = \{Q : b^2 - ac = 0\}$ of singular quadratic forms. These can be put in the form $(a_1\beta + b_1\gamma)^2$.

The *polar line* $\widehat{Q}$ of a point Q (with respect to the conic Γ) is the line that contains all points O such that Q and O are harmonic conjugate points with respect to the intersection points R_1 and R_2 of the conic Γ and a variable line through Q. Geometrically, if the polar line $\widehat{Q}$ meets Γ, then the tangents to Γ at the points of intersection meet at Q.

The symmetric matrix $[S_\mu]$ associated to the shape operator S_μ can be represented by a point $S_\mu = (\alpha a_3 + \beta a_4 : \alpha b_3 + \beta b_4 : \alpha c_3 + \beta c_4) \in \mathbb{R}P^2$. Then these points trace at each point $p \in M$ a pencil in $\mathbb{R}P^2$ (by varying α, β). This pencil is precisely the polar line $\hat{A}$ of the asymptotic BDE (6), [15, 21]. We also represent the metric $Gdv^2 + 2Fdudv + Edu^2$ by the point $L = (G : F : E)$.

Corollary 3.11 *Let M be a surface in $H^3_+(-1)$. The families of shape operators $-dN^w_\theta$, $\theta \in \mathbb{R}$, trace the polar line of the de Sitter lines of principal curvature with the points $\mathbb{L}^{\pm}$ and L removed. The family $-dN^s_\theta$ (resp. $-dN^t_\theta$) trace the part of the polar line corresponding to spacelike (resp. timelike) shape operators. The hyperbolic shape operators $\mathbb{L}^+$ and $\mathbb{L}^-$ form an obstruction for joining spacelike and timelike shape operators.*

4 Timelike hypersurfaces in S_1^n

Let M be a hypersurface in the de Sitter space S_1^n. If M is spacelike, then its normal plane in $\mathbb{R}_1^{n+1}$ is timelike and we have similar results to those in §3 for a hypersurface in the hyperbolic space. We deal here with the case when M is timelike. Then the normal plane N_pM in $\mathbb{R}_1^{n+1}$ is spacelike for all $p \in M$. The vectors $\boldsymbol{x}(\underline{u})$ and $\boldsymbol{e}(\underline{u})$ form an orthonormal basis of N_pM. Therefore, we can parametrise the unit normal vectors in N_pM by $\sin(\alpha)\boldsymbol{x}(\underline{u}) + \cos(\alpha)\boldsymbol{e}(\underline{u})$. However, the derivative of the Gauss indicatrix $\boldsymbol{x}(\underline{u})$ is the identity map on $T_{\boldsymbol{x}(\underline{u})}M$, so all points on M are umbilic points with respect to this Gauss indicatrix. This is why we define the family of (spacelike) Gauss indicatrices by

$$\begin{array}{rccc} N_\alpha : & U & \to & S_1^n(\cos(\alpha)^{-2}) \\ & \underline{u} & \mapsto & \tan(\alpha)\boldsymbol{x}(\underline{u}) + \boldsymbol{e}(\underline{u}) \end{array}$$

where $\alpha \in (-\pi/2, \pi/2)$ is the angle between $N_\alpha(\underline{u})$ and $\boldsymbol{e}(\underline{u})$. The family N_α does not contain the normal vector $\boldsymbol{x}$. We associate the same notions to $-(dN_\alpha)_p$ as those associated to $-(dN_\theta^w)_p$ in §3. We have, for instance, the α-principal curvatures given by $\kappa_{\alpha i} = -\tan(\alpha) + \kappa_i$. The α-principal directions do not depend on α.

We define the family of height functions

$$\begin{array}{rccc} H_\alpha & U \times S_1^n(\cos(\alpha)^{-2}) & \to & \mathbb{R} \\ & (\underline{u}, \boldsymbol{v}) & \mapsto & \langle \boldsymbol{x}(\underline{u}), \boldsymbol{v} \rangle - \tan(\alpha) \end{array}$$

We have similar results to those in §3 concerning the families N_α and H_α. In this section we deal mainly with timelike surfaces in S_1^3 and give only the results that are distinct from those in §3.1.

4.1 Surfaces in S_1^3

Let $\boldsymbol{x} : U \subset \mathbb{R}^2 \to M \subset S_1^3$ be a local parametrisation of M and let

$$\begin{array}{rcccl} l_\alpha & = & \langle -(dN_\alpha)_p(\boldsymbol{x}_u), \boldsymbol{x}_u \rangle & = & \langle N_\alpha, \boldsymbol{x}_{uu} \rangle, \\ m_\alpha & = & \langle -(dN_\alpha)_p(\boldsymbol{x}_u), \boldsymbol{x}_v \rangle & = & \langle N_\alpha, \boldsymbol{x}_{uv} \rangle, \\ n_\alpha & = & \langle -(dN_\alpha)_p(\boldsymbol{x}_v), \boldsymbol{x}_v \rangle & = & \langle N_\alpha, \boldsymbol{x}_{uv} \rangle, \end{array}$$

denote the coefficients of the α-second fundamental form at $p = \boldsymbol{x}(u, v)$ associated to the shape operator $-(dN_\alpha)_p$. We have

$$l_\alpha = -\tan(\alpha)E + l, \; m_\alpha = -\tan(\alpha)F + m, \; n_\alpha = -\tan(\alpha)G + n,$$

where l, m, n denote the coefficients of second fundamental form associated to the de Sitter shape operator $-d\mathbb{E}$. We denote, as in §3.1, by K_e and H_e the de Sitter Gauss-Kronecker curvature and the de Sitter mean curvature, respectively.

The (de Sitter) lines of principal curvature are given by the same equation as for the case of a surface in $H_+^3(-1)$ (i.e., equation (1)). The difference here is that the induced metric on the surface M is Lorentzian, so $-dN_\alpha$ does not always have two real eigenvalues. For a generic surface, the discriminant of the lines of principal curvature is a smooth curve except possibly at isolated points where it has Morse singularities of type A_1^- (node) ([10]). This discriminant is denoted by the LPL in [10] (Lightlike Principal Locus) and consists of points where the two principal directions coincide and become lightlike. The singular points of the LPL are labelled *timelike umbilic points*. In view of Theorem 3.9, the LPL is precisely the Δ-set of M as a surface in $\mathbb{R}_1^4$.

Theorem 4.1 *The α-parabolic set, $\alpha \in (-\pi/2, \pi/2)$, is given by*

$$\tan^2(\alpha) - 2H_e \tan(\alpha) + K_e = 0.$$

It consists of the curves $\kappa_i = \tan(\alpha)$, $i = 1, 2$. Each of these curves foliate, as α varies in $(-\pi/2, \pi/2)$, the region of M where there are two principal directions.

Proof The proof is similar to that of Theorem 3.2. Here the de Sitter principal curvatures κ_1 and κ_2 may be complex conjugate but $K_e = \kappa_1\kappa_2$ and $H_e = (\kappa_1 + \kappa_2)/2$ are always real numbers. $\square$

The α-asymptotic curves (which we define following §3.1) are given by

$$(A_\alpha):\ n_\alpha dv^2 + 2m_\alpha dudv + l_\alpha du^2 = 0. \tag{8}$$

The α-parabolic set is the discriminant of equation (8). Away from the LPL, the α-parabolic sets behave as the θ-parabolic sets in §3.1 (we have similar results to those in Theorems 3.4, 3.5, 3.8). We shall consider their behaviour at points on the LPL. We observe that the generic configurations of the lines of principal curvature at points on the LPL are obtained in [10].

Theorem 4.2 *Let M be a timelike surface in S^3_1 and p a point on the LPL of M.*

At most points on the LPL the height function H_α along the normal direction N_α has an A_2-singularity.

The singularity is of type A_3 if and only if p is a folded singularity of the de Sitter lines of curvature (and hence of all α-lines of curvature) and of an α-asymptotic curves.

The singularity is of type D_4 if and only if p is a timelike umbilic point (i.e., a singularity of the LPL) and $\tan(\alpha) = \kappa_1 = \kappa_2$. *At such point, the de Sitter lines of curvature has a Morse Type 2 singularity with a discriminant having a singularity of type A_1^- (Figure 7, top figures). The α-asymptotic curves have a Morse Type 2 singularity with the discriminant of type A_1^+ (Figure 6, top figures) or A_1^- (Figure 7, top figures).*

Proof We take a special parametrisation of the surface where the coordinate curves coincide with the lightlike curves, so $E \equiv 0$, $F \equiv 0$. The equation of the de Sitter lines of curvature becomes

$$ndv^2 - ldu^2 = 0,$$

and its discriminant (the LPL) is given by $ln = 0$. Suppose that p is a smooth point on the LPL, and assume that $l = 0$ and $n \neq 0$. Then the de Sitter lines of curvature have (generically) a folded singularity if and only if $l_u = 0$.

At a singular point of the LPL ($l = n = 0$) both coefficients of the de Sitter lines of curvature vanish. Thus, the de Sitter lines of curvature have generically a Morse Type 2 singularity with a discriminant ($ln = 0$) having a singularity of type A_1^-. The five generic configurations in Figure 7 (top figures) can occur.

The α-asymptotic curves are given by

$$ndv^2 + 2(-\tan(\alpha)F + m)dudv + ldu^2 = 0,$$

and the α-parabolic set (its discriminant) is given by $(-\tan(\alpha)F + m)^2 - ln = 0$. With the same setting as above, a smooth point $\boldsymbol{x}(u_0, v_0)$ on the α-parabolic set is also on the LPL if $\tan(\alpha) = (m/F)(u_0, v_0)$. Then the α-asymptotic curves, with $\tan(\alpha) = (m/F)(u_0, v_0)$, have (generically) a folded singularity if and only if $l_u = 0$.

At a singular point of the LPL, all the coefficients of the α-asymptotic curves BDE, with $\tan(\alpha) = (m/F)(u_0, v_0)$, vanish. The discriminant can have either an A_1^+ or an A_1^- singularity, so the α-asymptotic curves have generically a Morse Type 2 singularity with both discriminant types. All the generic configurations of Morse Type 2 singularities can occur (Figures 6, 7, top figures).

The height function $H_\alpha(-, \boldsymbol{v})$ is singular at (u_0, v_0) if $\boldsymbol{v} = N_\alpha(u_0, v_0)$. We write H_α for $H_\alpha(-, \boldsymbol{v})$. We have at (u_0, v_0), $(H_\alpha)_{uu} = l$, $(H_\alpha)_{uv} = -\tan(\alpha)F + m$, and $(H_\alpha)_{vv} = n$, so on the LPL (and with the setting above), $(H_\alpha)_{uu} = 0$ and the Hessian of H_α is degenerate if and only if $(H_\alpha)_{uv} = 0$, that is, $\tan(\alpha) = (m/F)(u_0, v_0)$. Calculations similar to those in the proof of Theorem 3.4 show that the singularity is of type A_2 if and only if $(H^w_\theta)_{uuu}(u_0, v_0) = l_u(u_0, v_0) \neq 0$. When $l_u = 0$, we get generically an A_3-singularity. At a timelike umbilic point $l = n = 0$, and with $\tan(\alpha) = (m/F)(u_0, v_0)$, the 2-jet of H vanishes, so the singularity is generically of type D_4. $\square$

Theorem 4.3 *Let M be a timelike surface in S^3_1 and p a point on the LPL of M.*

(1) *At most points on the LPL the foliations $\kappa_i = constant$, $i = 1, 2$ are as in Figure 2, top left. The leaves of $\kappa_1 = constant$ join those of $\kappa_2 = constant$ on the LPL and form smooth curves which have ordinary tangency with the LPL. At isolated points on the smooth part of the LPL the foliation $\kappa_i = constant$, $i = 1, 2$ are as in Figure 2, top right. These points are generically distinct from the folded singularities of the de Sitter lines of principal curvature.*

(3) *There are generically three configurations of the foliations $\kappa_i = constant$, $i = 1, 2$ at a timelike umbilic point. These are as in Figure 2, bottom figures.*

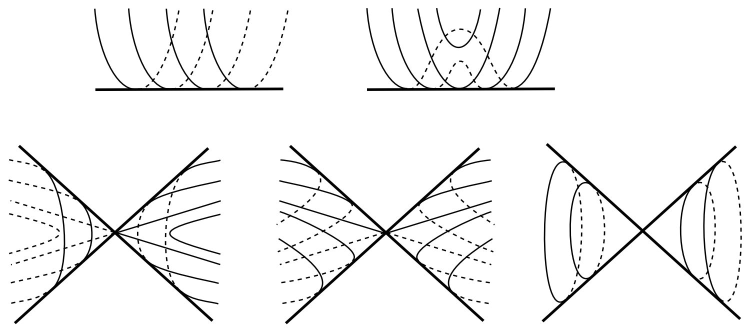

Figure 2: Generic configurations of the foliations $\kappa_i = constant, i = 1, 2$ at points on the LPL (continuous lines for κ_i and dashed for $\kappa_j, j \neq i$).

Proof The α-parabolic sets, which give the foliations $\kappa_i = constant$, are given by $\tan(\alpha)^2 - 2H_e \tan(\alpha) + K_e = 0$. In a local chart with $E \equiv 0$ and $F \equiv 0$, this becomes $(-\tan(\alpha)F + m)^2 - ln = 0$. To simplify notation, we denote by

$$\phi(u, v, \lambda) = (-\lambda F + m)^2 - ln,$$

where $\lambda = \tan(\alpha)$. The surface $\phi^{-1}(0)$ is smooth at (u, v, λ) if and only if $p = \boldsymbol{x}(u, v)$ is not a timelike umbilic point. At a timelike umbilic point with $\lambda = m/F$, $\phi^{-1}(0)$ is generically diffeomorphic to a cone. The projection $\pi : \phi^{-1}(0) \to U$ is a fold map at (u, v, λ) when $p = \boldsymbol{x}(u, v) \in LPL$ and is not a timelike umbilic point. The discriminant of π is the LPL. We call *criminant* the critical set of π.

Suppose that $p \in LPL$ is not a timelike umbilic point. The α-parabolic sets are the images by π of the intersection of $\phi^{-1}(0)$ with the planes $\lambda = constant$. These planes are transverse to $\phi^{-1}(0)$. Therefore their traces on $\phi^{-1}(0)$ is a family of smooth curves. We have two possible generic configurations of their projections to the (u, v)-plane (i.e., of the α-parabolic sets) depending on whether the criminant is transverse to the plane $\lambda = constant$ (Figure 2, top left) or tangent to it (Figure 2, top right). A condition for tangency is $\phi_{u\lambda}\phi_v - \phi_{v\lambda}\phi_u = 0$ (the tangency is ordinary in general) and is distinct from that for having a folded singularity of the de Sitter lines of curvature. The criminant splits $\phi^{-1}(0)$ locally into two components. The projections of the traces of $\lambda = constant$ in one component give the foliation $\kappa_1 = constant$ and those in the other component give the foliation $\kappa_2 = constant$.

We consider now the case when $p = \boldsymbol{x}(u_0, v_0)$ is a timelike umbilic point with $\lambda_0 = \tan(\alpha_0) = (m/F)(u_0, v_0)$. Then $\phi^{-1}(0)$ is a cone at (u_0, v_0, λ_0). The plane $\lambda = \lambda_0$ is not tangent to the cone, so we have two possible configurations for its trace on the cone: it is either an isolated point (this is the case when the α-parabolic set has a singularity of type A_1^+) or it is a pair of crossing curves (this is the case when the α-parabolic set has a singularity of type A_1^-). As λ varies near λ_0 we obtain generic cone sections. The LPL lifts to two smooth curves on $\phi^{-1}(0)$. If the cone sections are closed curves, we have one possible configuration for their projections to the (u, v)-plane (Figure 2, last bottom figure). If the cone sections are hyperbole, then we have two possible configurations depending on the position of the lift of the LPL with respect to the plane $\lambda = \lambda_0$. If both components of the LPL in a connected component of the cone with the singularity removed are on one side of the plane $\lambda = \lambda_0$, then the projections to the (u, v)-plane of the $\lambda = \textit{constant}$ sections are as in Figure 2, first figure of the bottom row. Otherwise they are as in Figure 2, middle figure of the bottom row. If we take the special parametrisation $E \equiv 0, G \equiv 0$, the last two types of configurations are distinguished by the sign of

$$((m_u l_v - m_v l_u)F - (F_u l_v - F_v l_u)m)\,((m_u n_v - m_v n_u)F - (F_u n_v - F_v n_u)m)$$

at (u_0, v_0), positive for the first case and negative for the second. $\square$

5 Appendix: singularities of BDEs

We give a brief summary of results concerning the singularities of quadratic Binary Differential Equations (BDEs) and their bifurcations (see [20] for a survey article and references). A BDE is given in the form

$$a(u, v)dv^2 + 2b(u, v)dudv + c(u, v)du^2 = 0, \tag{9}$$

with $(u, v) \in U \subset \mathbb{R}^2$. It determines a pair of transverse foliations away from the discriminant curve, which is the set of points where the function $\delta = b^2 - ac$ vanishes. The pair of foliations together with the discriminant curve are called the *configuration* of the solutions of the BDE. In all the figures, we draw one foliation in continuous line and the other in dashed line. The discriminant curve is drawn in thick black.

We consider here topological equivalence among BDEs and say that two BDEs are topologically equivalent if there is a local homeomorphism in the plane taking the configuration of one equation to the configuration of the other. We suppose the point of interest to be the origin. There are two cases to consider depending on whether all the coefficients of the BDE vanish or not at the origin.

When the coefficients do not all vanish at the origin, the stable configurations are as shown in Figure 3. The last three figures are called folded saddle, folded node and folded focus in that order. Folded singularities occur when the unique direction determined by the BDE on the discriminant is tangent to the discriminant.

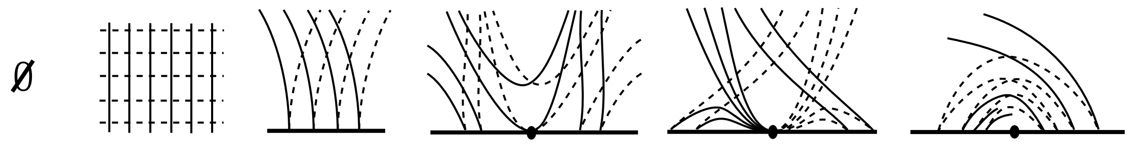

Figure 3: Stable configurations of BDEs: last three figures are the folded saddle, folded node and folded focus respectively.

Codimension 1 singularities can occur in three ways: (i) a folded saddle and a folded node coming together and disappearing (folded saddle-node singularity) on a smooth discriminant (Figure 4, left); (ii) a change from a folded saddle to a folded node on a smooth discriminant (Figure 4, right); (iii) the discriminant undergoes a Morse transition of type A_1^+ or A_1^-. For each type we have two

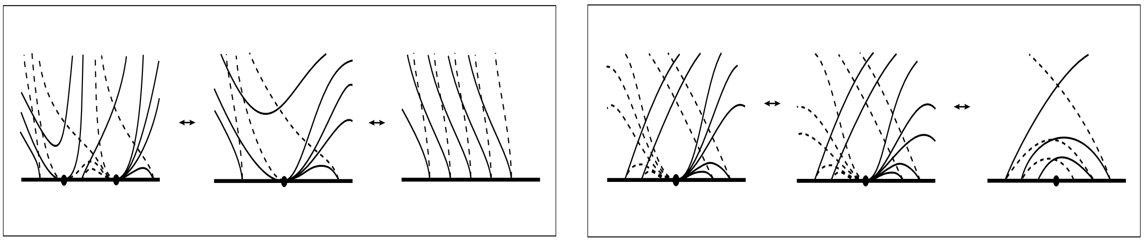

Figure 4: Folded saddle-node bifurcations (left) and a folded node-focus change (right).

cases depending on whether two folded saddles or two folded foci appear in the bifurcations. These singularities are label Morse Type 1 (Figure 5).

When the coefficients of the BDE all vanish at the origin, the singularities are automatically of codimension ≥ 1. If the discriminant has a Morse singularity, then we label the singularities of the BDE Morse Type 2 singularities. We have three generic configurations when the singularity of the discriminant is of type A_1^+ (Figure 6) and five (one case splits into two sub-cases when deformed) when it is of type A_1^- (Figure 7). In Figures 6 and 7 only one side of the transition is drawn the other side is symmetrical.

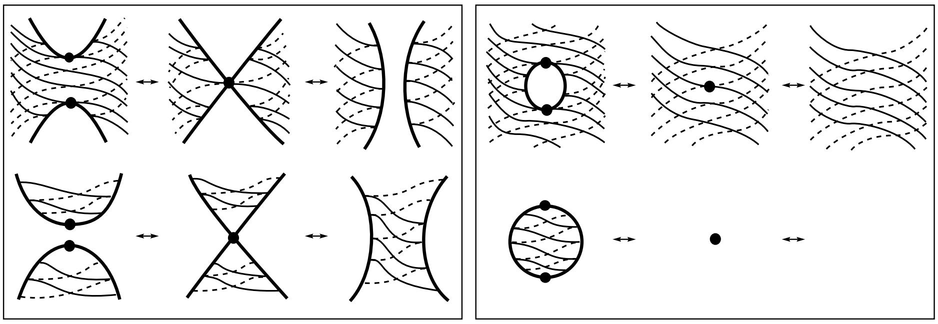

Figure 5: Bifurcations at a Morse Type 1 singularity: A_1^- left and A_1^+ right.

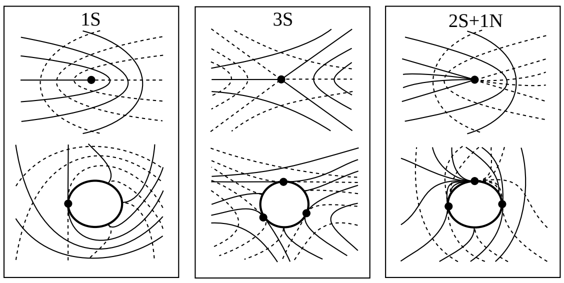

Figure 6: Bifurcations at a Morse Type 2 singularity (A_1^+).

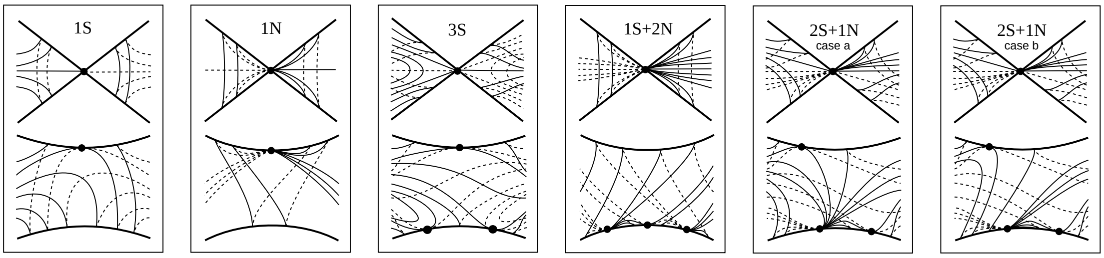

Figure 7: Bifurcations at a Morse Type 2 singularity (A_1^-).

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ON BORDISM AND COBORDISM GROUPS OF MORIN MAPS

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Abstract. We prove that the unoriented cobordism groups of Morin maps are 2-primary in nearly all cases. In the second part we define and compute a ring structure on the rational cobordism group of oriented fold maps.

1. Introduction

We will consider Morin maps of n-manifolds into $(n+k)$-manifolds, $k>0$. That is, all maps will be assumed to be locally generic and have differentials of rank n or $n-1$ everywhere. It is known [5] that at each point a Morin map is either regular or has a singularity of type Σ^{1_r} for some $r \geq 1$, $r \leq \frac{n}{k+1}$, in which case it has the local form

$$(t_1,\dots,t_{n-1},x) \mapsto (t_1,\dots,t_{n-1},\ t_1x+\dots+t_rx^r,\dots,t_{(k-1)r+1}x+\dots+t_{kr}x^r,\\ t_{kr+1}x+\dots+t_{kr+r-1}x^{r-1}+x^{r+1}).$$

Definition 1. *A Morin map will be called Σ^{1_r}-map if it has no singularities Σ^{1_s} for $s>r$. The cobordism group of Σ^{1_r}-maps of n-manifolds into $\mathbb{R}^{n+k}$ can be defined in a natural way (see [11]). Let us denote this group by $\Sigma^{1_r}(n,k)$ (no orientability is required). This group for $r=1$ will be denoted by $Fold(n,k)$, and its oriented version will be $Fold^{SO}(n,k)$. For $r=0$ (i.e., for the cobordism groups of immersions), the notation will be $Imm(n,k)$ and $Imm^{SO}(n,k)$, respectively.*

Our goal in the first part is to evaluate the cobordism groups $\Sigma^{1_r}(n,k)$ modulo finite 2-primary groups. In the second part we will define and compute the ring structure on the bigraded group

$$\bigoplus_{n,k} Fold^{SO}(n,k)\otimes\mathbb{Q}.$$

2. The unoriented cobordism groups of Morin maps

Theorem 1.

(1) The cobordism group $\Sigma^{1_r}(n,k)$ of Morin maps without singularities Σ^{1_s} for $s>r$ is a finite 2-primary group if
  - *$r=\infty$ (i.e., we allow all Morin maps),*
  - *r is arbitrary and k is odd,*
  - *$r \not\equiv 0 \bmod 4$ and k is even.*

(2) For $r\equiv 0 \bmod 4$ and k even the rank of the free part of the group $\Sigma^{1_r}(n,k)$ is equal to that of $H^{n-r(k+1)-k}(BO(k);\mathbb{Z})$. (Recall that the latter is equal to the number of partitions of $\frac{n-r(k+1)-k}{4}$ with entries not greater than $k/2$, in particular it is zero when $\frac{n-r(k+1)-k}{4}$ is not an integer).

This theorem improves [10, Theorem 1]. We do not know whether the groups $\Sigma^{1_r}(n,k)$ have odd torsion in the last case above or not.

The proof will be based on the so called Kazarian conjecture proved by the second author in [11]. In order to formulate this conjecture we recall the so called Kazarian space (considered already by R. Thom). For a given list τ of allowed singularities the Kazarian space $\mathcal{K}_\tau$ is the subset of the universal jet bundle corresponding to the allowed singularities τ. (This space $\mathcal{K}_\tau$ is very useful in computing the so called Thom polynomials giving the homology classes of different singularity strata.) On the other hand the second author constructed a space X_τ, whose homotopy groups give the cobordism groups of the so called τ-maps, i.e., maps with singularities only from the list τ (see [11] and the references there). For τ-maps a universal (virtual) normal bundle can be defined, it is a virtual bundle over the Kazarian space $\mathcal{K}_\tau$ and it will be denoted by ν. The (strengthened version of) Kazarian conjecture says that

$$X_\tau \cong \Omega^\infty S^\infty T\nu$$

where $T\nu$ is the "Thom space" of the virtual bundle ν. Note that although the Thom space of a virtual bundle is not defined, the space $\Omega^\infty S^\infty T\nu$ is well-defined (see [11]).

Recall that the Kazarian space is glued together from "blocks", one for each allowed monosingularity η. The block for η is the total space of the vector bundle over BG_η associated to the source representation λ_η, where G_η is the maximal compact subgroup of the symmetry group of η. We shall denote this vector bundle by ξ_η. Recall from [7],[11] that this bundle is the universal normal bundle of the η-stratum in the source manifold. Analogously, the vector bundle over BG_η associated to the target representation $\tilde{\lambda}_\eta$ is the universal normal bundle of the η-stratum in the target manifold, and we shall denote this vector bundle by $\tilde{\xi}_\eta$. For $\eta = \Sigma^{1_r}$ we abbreviate $\xi_{\Sigma^{1_r}}$ to ξ_r and $\tilde{\xi}_{\Sigma^{1_r}}$ to $\tilde{\xi}_r$. For Morin maps, these data have been calculated in [9], [10], [7] (more necessary information on the Kazarian spaces $\mathcal{K}_\tau$ will be given in the proof of Theorem 6).

We will first investigate the cohomology of the (stable) Thom space $T\nu$ of the virtual normal bundle ν over the Kazarian space. To obtain that, we use twisted Thom isomorphism to reduce the question to the determination of the cohomology groups of the Kazarian space with coefficients twisted by ν.

Along with the classes of maps mentioned above, we will need to use prim (i.e., *pr*ojected *im*mersion) maps. Recall that a Morin map f is called a *prim map,* if the one dimensional vector bundle $\ker df$ over the set of singular points is trivial, and moreover it is trivialised. The name prim comes from the property that being a prim map is equivalent to being the *pr*ojection of an *im*mersion into the product of the original target manifold with the real line. The analogue of the bundle ξ_r for prim maps will be denoted by $\bar{\xi}_r$.

3. Calculation

We will consider coefficients from a ring R where there is division by 2, and we will denote the local system twisted by the determinant bundle of some vector bundle ζ by R_ζ. We shall say also that R_ζ is twisted by the class $w_1(\zeta)$.

We will need to compute $H^*(T\nu;R)$, where ν is the virtual normal bundle over the Kazarian space $\mathcal{K}_\tau$. Using the twisted Thom isomorphism we have: $H^*(T\nu;R) \cong H^{*-k}(\mathcal{K}_\tau;R_\nu)$.

We shall consider the Kazarian spectral sequence [11] for prim maps and then for arbitrary Morin maps. Recall that the Kazarian spectral sequence starts from the cohomologies of BG_η and converges to those of $\mathcal{K}_\tau$. This time we need the spectral sequence that converges to the cohomologies of the Kazarian space with coefficients in R_ν.

3.1. **Case of even k (k = codimension of the map).**

First, consider the prim case. We know that in this case the Kazarian spectral sequence converges to the cohomology of the Kazarian space for $(k+1)$-codimensional immersions, which is $BO(k+1)$. This is so because a codimension k prim map can be identified with its codimension $k+1$ lift to an immersion. The virtual normal bundle is hence stably the same as the canonical bundle over $BO(k+1)$. We claim that for k even these cohomology groups vanish.

Lemma 1. *The twisted cohomology $H^*(BO(k+1); R_\nu)$ coincides with the group of classes which are anti-invariant under the deck transformation of the covering map $\pi : BSO(k+1) \to BO(k+1)$ (i.e., those which the deck transformation sends to their negatives) and hence $H^*(BO(k+1); R_\nu)$ is zero when k is even.*

Proof. Note that the local system R_ν is the same as R_π, because $w_1(\nu) = w_1(\pi)$. From the Leray spectral sequence applied to the covering π it follows that

$$H^*(BSO(k+1); R) = H^*(BO(k+1); \pi_*(R)).$$

Here $\pi_*(R)$ is the push-forward of the untwisted local system R on $BSO(k+1)$. Hence this is locally $R \oplus R$ at each point, and the non-trivial loop-class acts on it by interchanging the summands. Therefore it can be decomposed as the sum of the invariant and anti-invariant part: $\pi_*(R) = R \oplus R_\pi$. Thus

$$\begin{aligned} H^*(BSO(k+1); R) &= H^*(BO(k+1); \pi_*(R)) = \\ &= H^*(BO(k+1); R) \oplus H^*(BO(k+1); R_\nu). \end{aligned}$$

Since the groups $H^*(BSO(k+1); R)$ and $H^*(BO(k+1); R)$ are isomorphic (both groups are generated by the Pontrjagin classes) we obtain that $H^*(BO(k+1); R_\nu) = 0$. □

Completely analogously one obtains the twisted cohomologies of $BO(k)$ for k even.

Lemma 2. *If $k = 2t$, then $H^*(BO(k); R_\nu) = \chi \cup R[p_1, \dots, p_t]$, where χ is the twisted Euler class.* □

Now knowing where the Kazarian spectral sequence for prim maps and for k even converges to let us have a look at its starting term.

The E_1 term of the Kazarian spectral sequence can be described in this (prim maps) case as follows. The r-th column contains the twisted cohomology groups of the pair (D_r, S_r), where D_r and S_r are the total spaces of the disc bundle and the sphere bundle associated to the vector bundle $\bar{\xi}_r$, which is the universal source bundle over the Σ^{1_r}-points for prim maps. This universal bundle is $\bar{\xi}_r = r(\gamma^k \oplus \varepsilon^1)$ (over $BO(k)$ as its base space), see [9], [7]. The virtual normal bundle ν over the base space of $\bar{\xi}_r$ is stably the same as the canonical bundle γ^k over $BO(k)$. Hence $w_1(\bar{\xi}_r) = rw_1(\gamma^k) = rw_1(\nu)$.

Using the twisted Thom isomorphism again, we have

$$H^*(D_r, S_r; R_\nu) \cong H^{*-r(k+1)}(BO(k); R_\nu^{r+1}),$$

where $R_\nu^{r+1} = R_\nu \otimes \cdots \otimes R_\nu$, $r+1$ copies. Recall that we consider the case when k is even, say $k = 2t$. Let us denote by A the ring $R[p_1, \dots, p_t]$. It is well known that the untwisted cohomology ring of $BO(k)$ is isomorphic to this ring A, with p_j identified with the jth Pontrjagin class. Summarizing the computation we see that each column of the E_1 term of the Kazarian spectral sequence is the ring A but shifted differently: in the columns number $2h$ and $2h+1$ it is shifted by $(2h+1)k$. It follows that this spectral sequence must converge to zero in a very controlled way. Namely the first differential d_1 must be an isomorphism between the p-th and $(p+1)$-th column for $p = 0, 2, 4, \dots$. Indeed for $p = 0$ this follows from the facts that

- the elements of lowest degree in the 0-th column must be mapped onto those in the next column by d_1 isomorphically because this is the only chance for these elements to disappear, and they do disappear since the E_∞ term vanishes. Hence $d_1(\chi) = U$, where U is the twisted Thom class of the bundle $\bar{\xi}_1$, while χ is the twisted Euler class in $H^*(BO(k); R_\nu)$ (note that in this case R_ν and R_γ are isomorphic).
- the differential d_1 is multiplicative in the sense that $d_1(\chi \cup p_I) = d_1(\chi) \cup p_I$, see [11, Section 13.1].

The argument can then be repeated for each even p.

Second case, general (not necessarily prim) Morin maps, $k = 2t$. The previous spectral sequence (for the prim case) has a $\mathbb{Z}_2$ action corresponding to changing the orientation of the kernel bundle, and we need to know what are the eigenspaces corresponding to the two possible eigenvalues of this action. For this, we need to understand what happens with the coefficient system $R_\nu \otimes R_{\xi_r} = R_{\tilde{\xi}_r}$ for various values of r. From [7] one knows that the target representation is

$$\tilde{\lambda}(\varepsilon, Q) = \varepsilon^{r+1} \oplus Q \oplus \left\lceil \frac{r-1}{2} \right\rceil 1 \oplus \left\lfloor \frac{r-1}{2} \right\rfloor \varepsilon \oplus \left\lfloor \frac{r}{2} \right\rfloor Q \oplus \left\lceil \frac{r}{2} \right\rceil \varepsilon Q$$

for $Q \in O(k)$, $\varepsilon \in O(1)$. When ε changes its sign then $\tilde{\lambda}$ changes orientation exactly when $r + 1 + \lfloor \frac{r-1}{2} \rfloor$ is odd, i.e., when $r \equiv 2, 3 \bmod 4$.

When $r \equiv 0, 1 \bmod 4$, then the action discussed above (induced by $-id : \varepsilon \to \varepsilon$) is identical and the columns as well as the differentials between them remain the same as for prim maps. When $r \equiv 2, 3 \bmod 4$, then the action is changing the signs of all cohomology classes, so these columns in the spectral sequence for general Morin maps vanish.

Since the spectral sequences for prim maps and arbitrary Morin maps can be mapped into each other, and for the columns $r \equiv 0, 1 \bmod 4$ will be mapped onto each other isomorphically we obtain that the differential d_1 will be isomorphism again, and so the final E_∞ term vanishes again. We obtain that the cohomology groups of the Kazarian space for arbitrary Morin maps (with coefficients twisted by $w_1(\nu)$) vanish if k is even (under the assumption that in the coefficient ring 2 is invertible).

If we truncate the previous spectral sequence at the column corresponding to Σ^{1_r}, i.e., we consider the spectral sequence for Σ^{1_r}-maps, then the differential $d_1^{p,*}$

still remains an isomorphism between the p-th and $(p+1)$-th column for p even except when $p = r$ and $r \equiv 0 \bmod 4$.

3.2. Case of odd k.

Here, the coefficient system on $BO(k)$ is twisted by $w_1(\gamma^k)$, and in the column corresponding to Σ^{1_r} the coefficients are twisted corresponding to

$$w_1(\tilde{\xi}) = \left(r + 1 + \left\lfloor \frac{r-1}{2} \right\rfloor + \left\lceil \frac{r}{2} \right\rceil\right) w_1(\gamma^1) + (r+1)w_1(\gamma^k) =$$

$$= rw_1(\gamma^1) + (r+1)w_1(\gamma^k).$$

In particular, for no column is the coefficient system twisted trivially. Notice that $H^*(BSO(k);R) \cong R[p_1, \ldots, p_{\lfloor k/2 \rfloor}]$ is isomorphic to $H^*(BO(k) \times \mathbb{R}P^\infty; R)$, with the natural projection $BSO(k) \times \mathbb{S}^\infty \to BO(k) \times \mathbb{R}P^\infty$ inducing an isomorphism. This implies that the groups $H^*(BO(k) \times \mathbb{R}P^\infty; R_\zeta)$ are all 0 except when ζ is a trivial line bundle by the same argument as in Lemma 1. So the Kazarian spectral sequence with coefficients in R_ν starts from the empty state $E_1^{**} = 0$ in this case and hence $H^*(\mathcal{K}_{\Sigma^{1_r}}; R_\nu) = 0$ for all $0 \leq r \leq \infty$.

3.3. A geometric argument.

In addition to the previous computations we prove the following proposition:

Theorem 2. *The image of the forgetting map $\rho : Imm(n,k) \to Fold(n,k)$ from the unoriented cobordism group of immersions into that of fold maps contains only elements of order 2 and the zero element. That is, $2\rho(Imm(n,k)) = 0$.*

Proof. Let $i : M^n \to P^{n+k}$ be a k codimensional immersion. Choose a section v of the normal bundle ν_i transverse to the zero section; we consider v as a vector field along i. Our claim is that the map $j : M \times [-1,1] \to P \times [-1,1]$, $j(x,t) = (i(x) + tv(x), t^2)$ is a fold map if v is small enough. At the points where v is nonzero, the map j is clearly an immersion, and hence the only singular points can be of the form $(p, 0)$ with $p \in M$ and $v(p) = 0$. Since v is transverse to the zero section, the set of points p where $v(p) = 0$ is a k codimensional submanifold Z of M and v establishes an isomorphism between the normal bundle of Z in M and the normal bundle of $i(M)$ in P (restricted to Z). Thus for any point p that satisfies $v(p) = 0$ we can choose coordinate neighbourhoods $U \cong \mathbb{R}^k \times \mathbb{R}^{n-k}$ of p and $V \cong \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^{n-k}$ of $i(p)$ such that on U the immersion i has the form $(x,y) \mapsto (0,x,y)$ with $x \in \mathbb{R}^k, y \in \mathbb{R}^{n-k}$, and the vector field v has the form $(0,x,y) \mapsto (x,0,0)$. In these coordinates

$$j(x,y,t) = (i(x,y) + tv(x,y), t^2) = (tx, x, y, t^2)$$

has the normal form of a $\Sigma^{1,0}$ singularity multiplied by $id_{\mathbb{R}^{n-k}}$. Consequently j is a fold map as claimed. The boundary of j consists of two immersions regularly homotopic to i, proving our initial claim. □

Remark: This means that the embedding $\mathcal{K}_{imm} \to \mathcal{K}_{fold}$ of the Kazarian space of immersions into that of fold maps is 0 modulo 2-torsion in (co)homology with coefficients twisted by ν, and there is only a single way of getting this result in the Kazarian spectral sequence, by having the first differential surjective in cohomology (injective in homology). But the ranks of the corresponding groups are the same, so these surjections are actually isomorphisms. Hence we obtained a geometric proof of the fact that we have seen above, that the differential d_1 in the Kazarian spectral sequence maps the zero column isomorphically onto the next column.

4. Proof of Theorem 1

As a consequence of the computations in Section 3, we see that for k even and $r \not\equiv 0 \bmod 4$, as well as for k odd, the stable space $T\nu$ (where ν is the virtual normal bundle over the Kazarian space $\mathcal{K}_\tau$) has the same (co)homology groups modulo 2-primary groups as a contractible space. By the mod $\mathcal{C}$ Hurewicz theorem (due to Serre [8]), this implies that the stable homotopy groups of $T\nu$ are also the same as those of a point modulo 2-primary groups, that is, they are all 2-primary groups. But the stable homotopy groups of $T\nu$ are the homotopy groups of the classifying space $X_\tau \cong \Omega^\infty S^\infty T\nu$. Applying the mod $\mathcal{C}$ Hurewicz theorem again, we obtain the statement of the Theorem.

When k is even and $r \equiv 0 \bmod 4$, we have

$$H^*(T\nu;\mathbb{Q}) \cong H^{*-k}(\mathcal{K}_{\Sigma^{1_r}};\mathbb{Q}_\nu) \cong H^{*-r(k+1)-k}(BO(k);\mathbb{Q}_\nu) \cong$$
$$\cong H^{*-r(k+1)-2k}(BO(k);\mathbb{Q})$$

as the Kazarian spectral sequence degenerates with only column number r being nonzero. Since stable homotopy groups of any space have the same rank as the rational homology of the same space, we have

$$\dim \pi_{n+k}(X_\tau) \otimes \mathbb{Q} = \dim \pi^S_{n+k}(T\nu) \otimes \mathbb{Q} = \dim H_{n+k}(T\nu;\mathbb{Q}) =$$
$$= \dim H^{n-r(k+1)-k}(BO(k);\mathbb{Q})$$

as claimed.

5. Left-right bordism groups of τ-maps.

Definition 2. *The so called left-right bordism groups of τ-maps were defined in* [11]*. In this case we allow to change the target manifold also by a cobordism, and two τ-maps are equivalent (in this case bordant) if their sources and targets are cobordant and there is a τ map from the cobordism between the sources into that of the targets joining the original maps. The corresponding group is denoted by $Bord_\tau(n,k)$. (Here the singularities in the list τ are those of codimension k maps, the sources are n-dimensional and the target manifolds are $(n+k)$-dimensional.)*

Remark: A version of the Pontrjagin - Thom construction for singular maps implies that

$$Bord_\tau(n,k) \approx \mathfrak{N}_{n+k}(X_\tau).$$

These groups are vector spaces over $\mathbb{Z}_2$. In [12] these groups were computed for the simplest set of multisingularities. Here we shall consider the following versions of these groups:

- The targets and their cobordisms are oriented, but the sources might be non-orientable. These groups are denoted by $Bord_\tau^{target-or}(n,k)$. They are isomorphic to the oriented bordism groups of the unoriented classifying spaces X_τ:

$$Bord_\tau^{target-or}(n,k) \cong \Omega_{n+k}(X_\tau).$$

- Both the target and the source, as well as their bordisms, are oriented. These groups are denoted by $Bord_\tau^{SO}(n,k)$. They are isomorphic to the oriented bordism groups of the oriented version of the classifying space X_τ^{SO}:[1]

$$Bord_\tau^{SO}(n,k) \cong \Omega_{n+k}(X_\tau^{SO}).$$

Theorem 3. *Let τ be the collection of all multisingularities of k codimensional maps, $k \geq 2$, composed from*

- *all Morin monosingularities, or*
- *Σ^{1_s}, $s \leq r$ for some $r \geq 0$, $r \not\equiv 0 \bmod 4$ and k is even, or*
- *Σ^{1_s}, $s \leq r$ for some $r \geq 0$, and k is odd.*

Then the τ-bordism groups with oriented target $Bord_\tau^{target-or}(n,k)$ are isomorphic modulo 2-primary groups to Ω_{n+k}, the oriented cobordism group of $(n+k)$-manifolds. The mapping $Bord_\tau^{target-or}(n,k) \to \Omega_{n+k}$ that associates to a map the cobordism class of its target is a $\bmod 2$ *isomorphism.*

Proof. The bordism groups $Bord_\tau^{target-or}(n,k) \cong \Omega_{n+k}(X_\tau)$ can be computed using the Atiyah-Hirzebruch spectral sequence (see [1]). The first page of the spectral sequence is $H_p(X_\tau;\Omega_q)$. Since the homotopy groups of X_τ are 2-primary groups and the space X_τ is $(k-1)$-connected, its reduced integral homology groups are also finite 2-primary groups for $k \geq 2$; hence the first page modulo 2-primary groups vanishes apart from the first column (that is, $p = 0$), which corresponds to the cobordism class of the target. The spectral sequence degenerates and we get the statement of the theorem. □

Remark: For $k = 1$ the space X_τ might be (and will be) non-simply connected and therefore we cannot use the mod $\mathcal{C}$ Hurewicz theorem of [8] to deduce that its homology groups are 2-primary groups.

The remaining case when k is even and r is divisible by 4 can be handled more conveniently with the following notation. Let A^* be a graded Abelian group of finite type, $A^* = \underset{j}{\oplus} A^j$. Denote by $\mathrm{SP}(A^*)$ the graded skew-commutative ring multiplicatively freely generated by the additive generators of A^*. That is,

$$\mathrm{SP}(A^*) = \left(\wedge(\underset{j \text{ odd}}{\oplus} A^j)\right) \otimes \left(Sym(\underset{j \text{ even}}{\oplus} A^j)\right),$$

where for a vector space V the symmetric algebra $Sym(V)$ is defined as

$$Sym(V) = \bigoplus_{j=0}^{\infty} V^{\otimes j}/(v_1 \otimes \cdots \otimes v_j - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(j)})_{v_1,\ldots,v_j \in V, \sigma \in S_j},$$

and $\wedge V$ is the free skew algebra generated by V.

Recall that for a topological space X the infinite symmetric power of X is also denoted by $\mathrm{SP}\, X$.

Lemma 3.

$$H^*(\mathrm{SP}\, X;\mathbb{Q}) \cong \mathrm{SP}\, H^*(X;\mathbb{Q}).$$

[1] The space X_τ was not considered in [11], and X_τ^{SO} was denoted there by X_τ.

Proof. By [3, page 472] the space $\mathrm{SP}\,X$ is weakly homotopically equivalent to $\prod_{j\geq 0} K(H_j(X;\mathbb{Z}),j)$. Hence the rational homology groups of $\mathrm{SP}\,X$ are those of $\prod_{j\geq 0} K(\mathbb{Z}^{b_j},j)$, where b_j is the rank of $H_j(X;\mathbb{Z})$. It is a well-known result of Serre (see e.g. [2]) that $H^*(K(\mathbb{Z},j);\mathbb{Q})$ is generated by a j-dimensional free, respectively skew generator depending on whether j is even or odd. Hence the left hand side of the statement of the Lemma is the free skew symmetric algebra on the generators of $H^*(X;\mathbb{Q})$, and this coincides with the right hand side by definition. $\square$

Theorem 4. *Let τ be the collection of all multisingularities of k codimensional maps composed from Σ^{1_s}, $s \leq r$ where k is even and r is divisible by 4. Then the free part of the τ-bordism group with oriented target has dimension*

$$\dim Bord_\tau^{target-or}(n,k)\otimes\mathbb{Q} \;=\; \dim\left(\mathrm{SP}\left(H^*(T\nu;\mathbb{Q})\right)\otimes\Omega_*\right)_{n+k} \;=$$

$$\sum_{\frac{n+k}{4}=j+\sum_l la_l} q_j \prod_{\substack{l\\ a_l>0}} \binom{q_{l-\frac{(k+1)r+2k}{4}}+a_l-1}{a_l},$$

where q_m is the number of partitions of m into positive integers.

Proof. Just as before, we have

$$Bord_\tau^{target-or}(n,k)\otimes\mathbb{Q} \cong \Omega_{n+k}(X_\tau)\otimes\mathbb{Q} \cong \sum_{i=0}^{n+k} H_{n+k-i}(X_\tau;\mathbb{Q})\otimes\Omega_i \cong$$

$$\cong \sum_{j=0}^{\lfloor\frac{n+k}{4}\rfloor} H_{n+k-4j}(\Gamma T\nu;\mathbb{Q})\otimes\mathbb{Q}^{q_j}.$$

For any virtual cell complex L the spaces ΓL and $\mathrm{SP}\,L$ are rationally homotopy equivalent, see e.g. [11, Lemma 81]. In particular, their rational homology groups are isomorphic and we can replace $\Gamma T\nu$ by $\mathrm{SP}\,T\nu$ in the last expression above, obtaining the first claimed equality. Extend the notation q_α by setting $q_\alpha = 0$ if α is not an integer. Then

$$\dim H_m(T\nu;\mathbb{Q}) = \dim H^{m-(k+1)r-2k}(BO(k);\mathbb{Q}) = q_{\frac{m-(k+1)r-2k}{4}};$$

note that in particular $4 \mid r$ and $2 \mid k$ imply that $\dim H_m(T\nu;\mathbb{Q}) = 0$ unless $4 \mid m$. Therefore, for any n, we have

$$\dim Bord^{target-or}(n,k)\otimes\mathbb{Q} =$$

$$\sum_{j=0}^{\lfloor\frac{n+k}{4}\rfloor} \dim\left(\left(\wedge\left(\bigoplus_{u\text{ odd}} H^u(T\nu;\mathbb{Q})\right)\otimes Sym\left(\bigoplus_{u\text{ even}} H^u(T\nu;\mathbb{Q})\right)\right)_{n+k-4j}\otimes\mathbb{Q}^{q_j}\right) =$$

$$\sum_{j=0}^{\lfloor\frac{n+k}{4}\rfloor} \sum_{\substack{n+k-4j=\\ \sum ub_u}} q_j \prod_{\substack{u\text{even}\\ b_u>0}} \binom{\dim H_u(T\nu;\mathbb{Q})+b_u-1}{b_u} =$$

$$\sum_{\substack{n+k= \\ 4j+\sum 4la_l}} q_j \prod_{\substack{l \\ a_l>0}} \binom{\dim H_{4l}(T\nu;\mathbb{Q})+a_l-1}{a_l} =$$

$$\sum_{\frac{n+k}{4}=j+\sum la_l} q_j \prod_{\substack{l \\ a_l>0}} \binom{q_{l-\frac{(k+1)r+2k}{4}}+a_l-1}{a_l},$$

which is our second claimed equality. □

6. Ring structure on the direct sum of oriented cobordism groups of fold maps $\bigoplus\limits_{n,k} Fold^{SO}(n,k)\otimes\mathbb{Q}$

Let us recall first Wells' theorem from [13] on the ring of immersions. By the product of two immersion-cobordism classes $[f : M^m \to \mathbb{R}^{m+k}]$ and $[g : N^n \to \mathbb{R}^{n+l}]$ we mean the cobordism class of the product of the representatives[2]: $[f : M^m \to \mathbb{R}^{m+k}] \times [g : N^n \to \mathbb{R}^{n+l}] = [f\times g : M^m \times N^n \to \mathbb{R}^{m+k}\times\mathbb{R}^{n+l}]$. Let s_i be the characteristic class corresponding to the symmetric polynomial $x_1^{2i}+x_2^{2i}+\dots$, where the total Pontrjagin class is $1+p_1+p_2+\dots=\prod\limits_j(1+x_j^2)$, see [4].

Theorem 5 (Wells, [13])**.**

$$\bigoplus_{n,k} \mathrm{Imm}(n,k)\otimes\mathbb{Q} = \mathbb{Q}\big[[f_0],[f_1],\dots\big],$$

where $[f_i]$ is the cobordism class of an immersion $f_i : M^{4i+2} \looparrowright \mathbb{R}^{4i+4}$ such that $\langle e\cup s_i(p_1,\dots,p_i),[M]\rangle \neq 0$. Here e denotes the twisted normal Euler class of f_i, and $[M]$ is the twisted integer valued fundamental class of the unoriented manifold M, finally s_i is the characteristic class described above.

Definition 3. *Given two cobordism classes of oriented fold maps $[f : M^m \to \mathbb{R}^{m+k}]$ and $[g : N^n \to \mathbb{R}^{n+l}]$ we define their product as follows. The representatives f and g can be chosen to be immersions. Their product is an immersion, and we define $[f]\times[g]$ to be the oriented fold-cobordism class of this product.*

Theorem 6.

a) The above definition does not depend on the involved choices, that is, it gives a well-defined product on the direct sum $\bigoplus\limits_{n,k} Fold^{SO}(n,k)\otimes\mathbb{Q}$.

b)

$$\bigoplus_{n,k} Fold^{SO}(n,k)\otimes\mathbb{Q} = \mathbb{Q}[g_0,h_1,h_2,\dots],$$

where $h_i : \mathbb{C}P^{2i} \to \mathbb{R}^{6i}$ is any generic map, and g_0 is the inclusion of a point into the line.

Proof. Part a) follows from the facts that:

1) the natural map $H_n(BSO(k);\mathbb{Q}) \to H_n(BO(k);\mathbb{Q})$ is onto;
2) $Fold^{SO}(n,k)\otimes\mathbb{Q} \cong H_n(BO(k);\mathbb{Q})$ (see [11]); and

[2] In [6] a different multiplication was considered: $f*g : M^m\times N^n \to R^{m+k}\times R^{n+l}\times R^1$. For arbitrary Morin maps only that definition seemed to be possible, besides this operation made the singularities multpilicative. Here we consider the most natural product operation, it turns out to be possible to define it for *fold* maps, more precisely for their rational cobordism classes.

3) $Imm^{SO}(n,k)\otimes\mathbb{Q} \cong H_n(BSO(k);\mathbb{Q})$ (see e.g. the proof of Corollary 7 below).

Part b). We have seen in [11] that the Kazarian space $\mathcal{K}_\tau$ for $\tau = \{[\Sigma^0],[\Sigma^{1,0}]\}$ (i.e., $\mathcal{K}_\tau = \mathcal{K}_{fold}$) has the same rational homology groups as $BO(k)$. Recall that

$$H^*(BO(k);\mathbb{Q}) = \mathbb{Q}\big[p_1,\dots,p_{[\frac{k}{2}]}\big] = \mathbb{Q}\big[s_1,\dots,s_{[\frac{k}{2}]}\big].$$

It is well known [4] that the cobordism class of a manifold M^{4i} is irreducible in $\Omega_*\otimes\mathbb{Q}$ if and only if $s_i[M^{4i}] \neq 0$. In particular the even complex dimensional projective spaces $\mathbb{C}P^{2i}$ satisfy this property, hence $\Omega_*\otimes\mathbb{Q} = \mathbb{Q}[[\mathbb{C}P^2],[\mathbb{C}P^4],\dots]$.

For the convenience of the reader we give here a short summary of the properties of the Kazarian spaces that we use. We deal with the case of cooriented maps, for the unoriented version of the space replace all occurrences of the group SO with O. The space $\mathcal{K}_\tau$ depends on the set τ, which is the list of allowed singularities. Recall that a τ-map is a map such that all its singularities have a type from the list τ. The space $\mathcal{K}_\tau$ is universal in the following sense. It has a stratification according to the singularity types, each stratum corresponds to an element of τ. Further for each cooriented τ-map $f : M^n \to P^{n+k}$ there arises a map $\kappa_f : M^n \to \mathcal{K}_\tau$ of the source manifold M into the Kazarian space. This map κ_f is transverse to each stratum of $\mathcal{K}_\tau$, hence the preimages of the strata induce a stratification on M which coincides with the one induced by f (that is, the pulled-back strata coincide with the singularity strata of f). Additionally, $\mathcal{K}_\tau$ is the total space of a fibre bundle over BSO,

$$\pi : \mathcal{K}_\tau \to BSO.$$

Pulling back the canonical bundle γ (the limit $\lim\limits_{m\to\infty}\gamma^m$) to $\mathcal{K}_\tau$ one obtains the bundle $\nu = \pi^*\gamma$, which is the universal virtual normal bundle of a τ-map in the sense outlined below. If $f : M^n \to P^{n+k}$ is a τ-map, and ν_f is its virtual normal bundle, then

$$\kappa_f^*\nu \cong \nu_f.$$

Hence there is a homotopically commutative diagram

$$\begin{tikzcd} & \mathcal{K}_\tau \arrow[d, "\pi"] \\ M \arrow[ur, "\kappa_f"] \arrow[r, "\nu_f"] & BSO \end{tikzcd}$$

so κ_f is a lift of the map $\nu_f : M \to BSO$ to $M \to \mathcal{K}_\tau$.

Now recall that in [11] for any τ-map $f : M^n \to \mathbb{R}^{n+k}$ and for any $x \in H^n(\mathcal{K}_\tau;\mathbb{Q})$ the x-characteristic number of f, that is, $x[f] \stackrel{\text{def}}{=\!=} \langle\kappa_f^*(x),[M^n]\rangle$ has been defined and that these characteristic numbers give an isomorphism:

$$\begin{aligned} H_n(\mathcal{K}_\tau;\mathbb{Q}) = \operatorname{Hom}(H^n(\mathcal{K}_\tau;\mathbb{Q}),\mathbb{Q}) &\longleftarrow Cob_\tau(n,k)\otimes\mathbb{Q} \\ (x \longmapsto x[f]) \quad &\longleftarrow\!\!\shortmid \quad [f]. \end{aligned}$$

If $h_i : \mathbb{C}P^{2i} \to \mathbb{R}^{6i}$ is any generic map, then it is a fold map. Suppose that $k \geq 2i$. Let us consider the map $h_i^{(k)} = g_0^{k-2i}\cdot h_i$, i.e., the composition $\mathbb{C}P^{2i} \xrightarrow{h_i} \mathbb{R}^{6i} \hookrightarrow \mathbb{R}^{4i+k}$. In [11] we have shown that for $\tau = \{\Sigma^0,\Sigma^{1,0}\}$ (when τ-maps are precisely the fold maps) the space $\mathcal{K}_\tau$ has the same rational homotopy groups as $BO(k)$ does, and $\pi : \mathcal{K}_\tau \to BSO$ is the standard mapping $BO(k) \to BSO$ (induced by $O(k) \overset{(id,\det)}{\to} SO(k+1) \to SO$) up to a rational homotopy. Hence π^* maps all the

Pontrjagin classes $p_i \in H^{4i}(BSO;\mathbb{Q})$ to those in $H^{4i}(BO(k);\mathbb{Q})$ for $i \le \frac{k}{2}$, while for $i > \frac{k}{2}$ the class p_i goes to zero.

Beyond the characteristic classes s_i mentioned in the previous proof we shall also consider the classes $\bar{s}_i$ that we define as $s_i(\bar{p}_1,\dots)$, where $\bar{p}_j$ are the normal Pontrjagin classes defined by $1+\bar{p}_1+\bar{p}_2+\dots = (1+p_1+p_2+\dots)^{-1}$.

Lemma 4. *For $x=\bar{s}_i$, that is, $x(p_1,\dots)=s_i(\bar{p}_1,\dots)$, the x-characteristic number of $h_i^{(k)}$ is*

$$\bar{s}_i[h_i^{(k)}] = s_i[\mathbb{C}P^{2i}] \neq 0.$$

Proof.

$$\begin{aligned}\langle s_i(p_1,\dots,p_m),[\mathbb{C}P^{2i}]\rangle &= \langle \nu_f^* s_i(\bar{p}_1,\dots,\bar{p}_m),[\mathbb{C}P^{2i}]\rangle = \\ &= \langle \kappa_f^*\pi^* s_i(\bar{p}_1,\dots,\bar{p}_m),[\mathbb{C}P^{2i}]\rangle = \langle \kappa_f^* s_i(\bar{p}_1,\dots,\bar{p}_m),[\mathbb{C}P^{2i}]\rangle = \\ &= \langle \bar{s}_i(\kappa_f^* p_1,\dots,\kappa_f^* p_m),[\mathbb{C}P^{2i}]\rangle = \bar{s}_i[h_i^{(k)}].\end{aligned}$$

□

By the multiplicative property of the classes $\bar{s}_I = \bar{s}_{i_1}\dots\bar{s}_{i_r}$ we have that the cobordism classes of fold maps $g_0^{k-2|I|}h_I$, where $I=(i_1,\dots,i_r)$ is a multiindex $h_I = h_{i_1}\times h_{i_2}\times\dots\times h_{i_r}$ and $|I| = i_1+\dots+i_r$, are linearly independent. Indeed, the matrix $a_{JI} = \left(\bar{s}_J\left[g_0^{k-2|I|}\cdot h_I\right]\right)$ is non-degenerated. Here both multiindices J and I run over all the partitions of all the numbers $0,1,2,\dots,\left[\frac{k}{2}\right]$. Hence

$$\bigoplus_{n,k} Cob_{\Sigma^{1,0}}(n,k)\otimes\mathbb{Q} \cong \mathbb{Q}[g_0,h_1,h_2,\dots].$$

□

Corollary 7. *The ring $\bigoplus_{n,k} Imm^{SO}(n,k)\otimes\mathbb{Q}$ is isomorphic to the direct sum of the rings $\mathbb{Q}[g_0,h_1,h_2,\dots]$ and $\mathbb{Q}[f_0,f_1,f_2,\dots]$. Here f_i is the map defined in Wells' theorem. If $\alpha\in\mathbb{Q}[g_0,h_1,h_2,\dots]$ and $\beta\in\mathbb{Q}[f_0,f_1,f_2,\dots]$, then $\alpha\cdot\beta=0$.*

Proof. By Wells' theorem $Imm^{SO}(n,k)\approx\pi^s_{n+k}(T\gamma_k^{SO})$, where γ_k^{SO} is the universal oriented bundle of rank k. By Serre's theorem

$$\pi^s_{n+k}(T\gamma_k^{SO})\otimes\mathbb{Q}\approx H_{n+k}(T\gamma_k^{SO};\mathbb{Q}).$$

By the Thom isomorphism this is isomorphic to $H_n(BSO(k);\mathbb{Q})$, which is isomorphic to $H^n(BSO(k);\mathbb{Q})$, since $\mathbb{Q}$ is a field. Now let us consider again the double cover $\pi: BSO(k)\to BO(k)$ and the decomposition arising from it (see Lemma 1):

$$H_*(BSO(k);\mathbb{Q}) = H_*(BO(k);\mathbb{Q})\oplus H_*(BO(k);\mathbb{Q}_\pi).$$

Wells has shown that the second summand is isomorphic to the cobordism group of unoriented immersion of n-manifolds into $\mathbb{R}^{n+k}$, while in [11] it has been shown that the first summand is isomorphic to the oriented cobordism group of fold maps of oriented n-manifolds into $\mathbb{R}^{n+k}$.

Hence the theorem holds at least additively. But then we have also an isomorphism of rings if we define the multiplication on the direct sum as described in the theorem. This follows from the following obvious lemma.

Lemma 5. *Let A, B, C be rings. Let $\varphi : A \to B$ and $\psi : A \to C$ be ring epimorphisms such that their direct sum $\varphi \oplus \psi : A \to B \oplus C$ is an additive isomorphism. Then $\varphi \oplus \psi$ is a ring isomorphism between A and $B \oplus C$, where the product on the latter is defined so that for $\beta \in B$ and $\gamma \in C$ the equality $\beta \cdot \gamma = 0$ holds.*

Proof. The obtained map is obviously a ring homomorphism and additive isomorphism, hence it is a ring isomorphism. □

□

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MONODROMY OF PLANE CURVES AND QUASI-ORDINARY SURFACES

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Abstract

We establish an explicit recursive formula for the vertical monodromies of an irreducible quasi-ordinary surface in $\mathbf{C}^3$. The calculation employs a local description of the singularity at the generic point of each singular component in terms of a "truncation" and a "derived" surface. These objects are also used to retrieve a formula for the (classical) horizontal monodromy in recursive terms.

Consider an irreducible germ of analytic surface S in $\mathbf{C}^3$, arranged so that the projection $\pi : (x, y, z) \mapsto (x, y)$ has its discriminant locus contained in the coordinate axes. This is the local picture of a *quasi-ordinary surface*. The theory of such surfaces (which we briefly recall in section 3) says that each sheet may be expressed in the following way:

$$\zeta = \sum c_{\lambda\mu} x^{\lambda} y^{\mu},$$

where the exponents range over certain non-negative rational numbers with a common denominator. Let d denote the number of sheets (equivalently the number of conjugates of ζ). One can write a function defining S by taking a product over all conjugates:

$$f(x, y, z) = \prod_{k=1}^{d} (z - \zeta_k).$$

In general the singular locus of such a surface is one-dimensional, with at most two components. In almost all instances, the x-axis is one such component. A transverse slice $x = C$ (where C is a small nonzero constant) cuts out a singular plane curve. The Milnor fiber of this curve undergoes a monodromy transformation when C loops around the origin; the action on its homology groups is called the *vertical monodromy*. In this article we show how to explicitly calculate this monodromy. Our formula is expressed recursively, by associating to our surface two related quasi-ordinary surfaces which we call its *truncation* S_1 and its *derived surface* S', and then expressing the vertical monodromy of S via the monodromies of S_1 and of S'.

As is well known, there is another fibration over a circle, called the *Milnor fibration*; here the action on homology is called the *horizontal monodromy*. In the course of working out our recursion for vertical monodromy, we have discovered what appears to be a new viewpoint about the horizontal monodromy, expressed in a similar recursion which again invokes the same two associated surfaces. In fact this recursion makes sense even outside the context of quasi-ordinary surfaces, and thus we have found a novel way to express the monodromy associated to the Milnor fibration of a singular plane curve. (There are known formulas for this monodromy, e.g. Theorem 2 of [3] and formula (6.1) of [4], as well as quasi-ordinary analogs presented in [7] and [14], but they are not framed in the same recursive manner.)

We begin by working out this situation, to motivate our later setup and to provide a model for the more elaborate calculation.

As a corollary to our formulas, we have found that from the vertical and horizontal monodromies (one pair for each component of the singular locus), together with the surface monodromy formula worked out in [14] and [7], one can recover the complete set of characteristic pairs of a quasi-ordinary surface. Since these data depend only on the embedded topology of the surface, we thus have a new proof of Gau's theorem [6] in the 2-dimensional case. As another application, we can employ a theorem of Steenbrink [17] (extended to the non-isolated case by M. Saito [15]) which relates the horizontal and vertical monodromies to the spectrum of the surface and to the spectrum of any member of the Yomdin series. Since the spectrum of an isolated singularity is computable in principle, we expect that the monodromies worked out here may be exploited to calculate the spectrum of a quasi-ordinary surface. We intend to explicate these two applications in subsequent papers. We have also begun, along with Mirel Caibăr and Manuel González Villa, to investigate whether our recursion has a motivic incarnation akin to that of [4]; we believe that it does.

We begin in section 1 with two "approximation lemmas" that allow us to replace one function by another when studying their associated fibrations. In section 2 we work out the monodromy of the Milnor fiber of a plane curve singularity. Everything in this section is well-known (although it is not usually presented in a recursive framework), and we present it merely as a prototype for our original contributions in subsequent sections. In section 3 we briefly recall the basic notions of quasi-ordinary surfaces and introduce the "transverse Milnor fiber." Section 4 formulates and proves our main results. In these results we assume that a certain characteristic exponent μ_1 does not vanish; our last (very brief) section discusses the case $\mu_1 = 0$.

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1. Approximation lemmas

In the proofs of our recursive formulas we use the following lemmas. For ease of reference, we give two separate formulations, but clearly the first lemma follows from the second.

Lemma 1.1. *Suppose that f and g are two holomorphic functions on a smooth compact analytic surface S with boundary. Suppose that they have the same divisor D, and that D_{red} is transverse to the boundary. Suppose that the unit $u = f/g$ always has positive real part. Then, for sufficiently small σ, the fibration over the circle $|\epsilon| = \sigma$ with fibers $f = \epsilon$ is smoothly isotopic to the fibration with fibers $g = \epsilon$.*

Lemma 1.2. *Over a circle $|x| = \rho$, let S be the total space of a continuous family of smooth compact analytic surfaces S_x with boundary. Suppose that f and g are two continuous functions such that, for each x, their restrictions f_x and g_x are holomorphic functions on S_x having the same divisor D_x. Suppose that each D_x is transverse to the boundary. Suppose that the unit $u = f/g$ always has positive real part. Then, for sufficiently small σ, the fibration over the torus $|x| = \rho, |\epsilon| = \sigma$ with fibers $f_x = \epsilon$ is isotopic to the fibration with fibers $g_x = \epsilon$.*

Proof. Let D be the union of the divisors D_x. We argue that in a punctured neighborhood of D, the interpolation $F_t = tf + (1-t)g$ (with $0 \leq t \leq 1$) has a non-vanishing gradient (as does its restriction to the boundary). Then by the Ehresmann fibration theorem, F_t provides a locally trivial fibration.

There is a neighborhood of D on which, away from D itself, the relative gradient ∇g does not vanish. Indeed, let V be the variety on which ∇g vanishes. Then g must be constant on each component of V, and each such component either misses D or is completely contained within it. Similarly, we claim that there is a (punctured) neighborhood of D on which ∇f is never a negative multiple of ∇g. To see this, consider the variety V on which the two gradients are linearly dependent; note that D is contained in V. Then the quotient $\lambda = \nabla f / \nabla g$ is a well-defined analytic function on V at least away from D. Suppose we have a map $\gamma : (C, p) \to V$ from a nonsingular curve germ, with $\gamma(p) \in D$. Then on C we have

$$\lambda = f'/g' = u + \frac{g}{g'} u'.$$

The quotient g/g' has a removable singularity at p and vanishes there. Thus we have $\lambda(p) = u(p)$. Since the curve C is arbitrary, this shows that λ is well-defined on D and agrees with u there. Thus there is a neighborhood of V in which the real part of λ cannot be negative; in the punctured neighborhood ∇F_t does not vanish.

Finally, since each D_x is transverse to the boundary, we can find a local trivialization of a neighborhood of $D_x \cap \partial S$ in ∂S, with fibers isomorphic to the complex disk. Then a similar argument as above applies to f and g restricted to the boundary. □

2. Plane curves

The material in this section is well-known. We present it to establish notations, to isolate certain technical details for later reference, and to elucidate our recursive point of view.

Consider a germ at the origin of an irreducible analytic plane curve defined by $f(y, z) = 0$; we will simply call it a "curve." (For basic notions and facts about singular plane curves see [5] or [18].) The *Milnor fiber* F is the set of points (y, z) obtained by the following process:

(1) requiring that $\|(y, z)\| \leq \delta$, a sufficiently small radius,
(2) then requiring that $f(y, z) = \epsilon$, a number sufficiently close to zero.

The boundary of the Milnor fiber is a link in the sphere. Letting ϵ vary over a circle centered at 0 we obtain the *Milnor fibration* (which we will also call the *horizontal fibration*). Let $h_q : H_q(F; \mathbf{Q}) \to H_q(F; \mathbf{Q})$ be the monodromy operator. The graded characteristic function

$$\mathbf{H}(t) = \frac{\det(tI - h_0)}{\det(tI - h_1)}$$

is called the *horizontal monodromy.* (In the literature it is often called the *monodromy zeta function.*) Taking its degree computes the Euler characteristic χ of F.

Assuming that the curve is not the axis $y = 0$, there is a parametrization

$$y = t^d, \quad z = \sum_j c_j t^j,$$

where the exponents (taken all together) are relatively prime positive integers, and all coefficients are nonzero. The integer d (which we call the *degree*) is the number of sheets for the projection $\pi : (y, z) \mapsto y$, and over a slitted neighborhood of 0 we may parametrize each sheet by

$$\zeta = \sum_j c_j y^{j/d},$$

having chosen one of the d possible roots. We prefer to write this as follows:

$$\zeta = \sum c_\mu y^\mu, \tag{2.1}$$

where the sum is now over certain positive rational numbers with common denominator d (arranged in increasing order); this is called the *Puiseux series* of the curve. One can recover f by forming a product over all conjugates:

$$f(y, z) = \prod^d (z - \zeta).$$

(Note our notation for recording the number of conjugates.)

An exponent of the Puiseux series is called *essential* (or *characteristic*) if its denominator does not divide the common denominator of the previous exponents. In particular (by the convention that the least common multiple of the empty set is 1) all integer exponents are inessential, but the first noninteger exponent is essential. Clearly there are only finitely many essential exponents $\mu_1 < \mu_2 < \cdots < \mu_e$. The sum

$$\sum_{i=1}^e y^{\mu_i} \tag{2.2}$$

parametrizes the d sheets of a singular curve which we call the *prototype*.

Theorem 2.1. *A curve and its prototype have the same horizontal monodromy.*

(As an example, if there are no essential exponents then the curve is nonsingular at the origin, its prototype is $z = 0$, and the horizontal monodromy is $t - 1$.) This theorem is well-known; see for example [16]. We will prove Theorem 2.1 by induction on e, at the same time that we prove a set of recursive formulas. To this end, we define the *truncation* of a singular curve with prototype

$$\sum_{i=1}^e y^{\mu_i}$$

to be the curve with Puiseux series

$$\zeta_1 = y^{\mu_1} = y^{n/m}$$

(where the second equation defines the relatively prime integers m and n). Its *derived curve* is the curve with Puiseux series

$$\zeta' = \sum_{i=1}^{e-1} y^{\mu'_i},$$

with the new exponents computed by

$$\mu'_i = m(\mu_{i+1} - \mu_1 + n).$$

Example 2.2. Suppose we begin with the curve whose Puiseux series is

$$\zeta = y^{3/2} + y^{7/4} + y^{11/6}.$$

Then its truncation is parametrized by $\zeta_1 = y^{3/2}$, and its derived curve is parametrized by

$$\zeta' = y^{13/2} + y^{20/3}.$$

Repeating this process, we obtain truncation $\zeta_1' = y^{13/2}$ and second derived curve

$$\zeta'' = y^{79/3}.$$

Let d_1 and d' denote the degrees of the truncation and the derived curve, respectively. Similarly, let χ_1 and χ' denote the Euler characteristics of their Milnor fibers; let $\mathbf{H}_1$ and $\mathbf{H}'$ denote their horizontal monodromies.

Theorem 2.3. *The degree, Euler characteristic, and horizontal monodromy are determined by these formulas:*

(1) $d_1 = m$
(2) $d = d_1 d'$
(3) $\chi_1 = m + n - mn$
(4) $\chi = d'(\chi_1 - 1) + \chi'$
(5)
$$\mathbf{H}_1(t) = \frac{(t^m - 1)(t^n - 1)}{t^{mn} - 1}$$
(6)
$$\mathbf{H}(t) = \frac{\mathbf{H}_1(t^{d'}) \cdot \mathbf{H}'(t)}{t^{d'} - 1}$$

These formulas may be compared with the well-known (non-recursive) versions in the literature; see e.g. [18].

For the curve of Example 2.2, the first two formulas tell us that $d = 2d' = 4d'' = 12$. By formulas (3) and (4), the Euler characteristic of the Milnor fiber is

$$\chi = d'(\chi_1 - 1) + d''(\chi_1' - 1) + \chi'' = 6(-2) + 3(-12) + (-155) = -203.$$

By formulas (5) and (6), the horizontal monodromy is

$$\mathbf{H}(t) = \frac{\mathbf{H}_1(t^{d'})}{t^{d'} - 1} \cdot \frac{\mathbf{H}_1(t^{d''})}{t^{d''} - 1} \cdot \mathbf{H}''(t) = \frac{(t^{12} - 1)(t^{18} - 1)(t^{39} - 1)(t^{79} - 1)}{(t^{36} - 1)(t^{78} - 1)(t^{237} - 1)}.$$

Before embarking on the proof of Theorems 2.1 and 2.3, we describe its key idea, and elaborate it by working out the details of Example 2.2. As is well known, one may obtain an embedded resolution of a curve singularity by a resolution process whose steps are dictated by the Puiseux exponents, and from such a resolution one can compute the monodromy by invoking a formula of A'Campo [2]. Our proof does not use this full process of resolution, but just the first step of it: the toric transformation prescribed by the leading exponent. In the example, the toric transformation is given by

$$y = u^2 v$$
$$z = u^3 v^2.$$

Pulling back

$$f(y, z) = \prod^{12} \left(z - \left[y^{3/2} + y^{7/4} + y^{11/6} \right] \right)$$

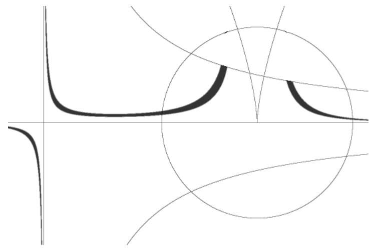

FIGURE 1. The Milnor fiber (the thickened curve) is divided into two pieces by the boundary of N (indicated by a circle). The rupture component is horizontal, and another exceptional divisor is shown vertically. The strict transform enters from above.

by this transformation and factoring, we see that

$$f = u^{36} v^{18} \prod^{12} \left(v^{1/2} - \left[1 + u^{1/2} v^{1/4} + u^{2/3} v^{1/3} \right] \right).$$

Thus there are two exceptional divisors of multiplicities 36 and 18; the former is called the *rupture component*. There is another exceptional divisor with multiplicity 12, not visible in the selected chart. Note that we have not achieved an embedded resolution, nor do we wish to do so; we are content to work with this "partial resolution." (Other authors have also used this idea of partial resolution, e.g. [8].)

The product of 12 conjugates defines the strict transform, and we note that it hits the rupture component at two different points, namely $(u, v) = (0, \pm 1)$. To focus attention at the point $(0, 1)$, we introduce two new variables y' and w. We let B denote a small ball $\|(y', w)\| \leq \delta'$ centered at the origin, and map it to a neighborhood N of $(u, v) = (0, 1)$ by letting $u = \frac{y'}{w+1}$ and $v = (w+1)^2$. When pulled back via this map, just one of the two values $v^{1/2}$ becomes $w + 1$. Thus six of the 12 conjugates become units, and our function f is thus a unit times the following function:

$$(y')^{36} \prod^{6} \left(w - \left[(y')^{1/2} + (y')^{2/3} \right] \right). \tag{2.3}$$

Our Milnor fiber is thus divided into two pieces: the piece inside N and the outside piece; see Figure 1. Our decomposition is coarser than the usual decomposition of the Milnor fiber, as explained in [2]. Those pieces in the usual decomposition coming from the first sequence of blowups, i.e., dictated by the first characteristic exponent, constitute our outside piece, while the remaining pieces constitute our inside piece. As we show in our proof of Theorem 2.3, the outside piece consists of six copies of the Milnor fiber of the curve $z^2 = y^3$, i.e., the truncation.

To understand the inside piece, we observe that the configuration of curves defined by the vanishing of 2.3, consisting of the strict transform together with the rupture component, can be interpreted as the total transform of a new singular curve. The blowing down map is $(y', w) \mapsto (y', (y')^6 w)$, and the resulting curve has Puiseux series

$$\zeta' = (y')^{13/2} + (y')^{20/3}; \tag{2.4}$$

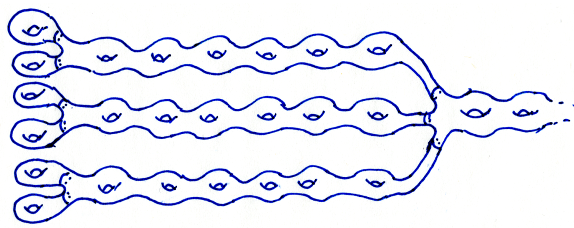

FIGURE 2. The Milnor fiber for Example 2.2 consists of six copies of the Milnor fiber for $z^2 = y^3$ attached to a single copy of the Milnor fiber of its derived curve. In turn, the Milnor fiber of the derived curve consists of three copies of the Milnor fiber for $z^2 = y^{13}$ attached to a single copy of the Milnor fiber of the second derived curve $z^3 = y^{79}$.

this is the derived curve. The blowing down map misses six small disks, and we observe that these disks are cyclically permuted by the monodromy. Figure 2 gives another picture of our decomposition, and indicates how the recursion will continue.

Proof. As indicated, we will simultaneously provide an inductive proof of Theorem 2.1 (inducting on the number of essential exponents) and a recursive proof of Theorem 2.3.

The Milnor fiber of the truncation, which is defined by $z^m - y^n = \epsilon$, is projected by π onto a neighborhood of 0 on the y-line, with total ramification above the nth roots of $-\epsilon$. This neighborhood can be retracted onto the union L of line segments from 0 to these points, in such a way that there is a compatible retraction of the Milnor fiber onto $\pi^{-1}L$, which is the complete bigraph on the n points $((-\epsilon)^{1/n}, 0)$ and the m points $(0, \epsilon^{1/m})$. As ϵ goes around a circle, each set of points is cyclically permuted. Since m and n are relatively prime, the mn edges of the graph are likewise cyclically permuted. Thus the odd-numbered formulas are confirmed.

To verify the recursive formulas and to handle the inductive step in the proof of Theorem 2.1, suppose we are given a curve with Puiseux series (2.1) and prototype (2.2). We first replace

$$\frac{z - \sum_{\mu \in \mathbf{Z}} c_\mu y^\mu}{c_{\mu_1}}.$$

by z. In the new coordinate system, the curve is defined by the vanishing of

$$f = \prod^{d} \left(z - \left[y^{n/m} + \sum_{\mu > n/m} c_\mu y^\mu \right] \right),$$

(where for simplicity the coefficients have been renamed). The truncation is defined by the vanishing of

$$f_1 = \prod^{m} (z - y^{n/m}) = z^m - y^n.$$

Note that m divides d, and that, as we vary the dth root of y, each value of $y^{1/m}$ occurs d/m times. Thus

$$\frac{f}{f_1^{d/m}} = \prod^{d} \left(1 - \frac{\sum_{\mu>n/m} c_\mu y^\mu}{z - y^{n/m}}\right). \tag{2.5}$$

One can obtain an embedded resolution of the truncation by a sequence of blowups dictated by its exponent $\mu_1 = n/m$ and the Euclidean algorithm. The total transform will consist of a chain of exceptional divisors occurring with certain multiplicities, together with a strict transform meeting just one such exceptional divisor, which we call the *rupture component*. Along this chain the function z^m/y^n has no indeterminacy, and in fact except along the rupture component its value is either 0 or ∞. In either case one immediately verifies that the value of (2.5) is 1.

To work in a chart containing the rupture component, we use substitutions dictated by the matrix

$$\begin{bmatrix} m & n \\ r & s \end{bmatrix},$$

where r and s are the smallest positive integers for which the determinant is 1, namely

$$\begin{aligned} y &= u^m v^r \\ z &= u^n v^s. \end{aligned}$$

We find that in this chart the total transform of the truncation is defined by the vanishing of

$$f_1 = u^{mn} v^{rn} (v - 1),$$

and its strict transform is defined by the vanishing of the last factor. Note that it meets the v-axis at the point $(u, v) = (0, 1)$. The total transform of the given curve is defined by the vanishing of

$$f = \prod^{d} \left(u^n v^s - \left[u^n v^{rn/m} + \sum_{\mu>n/m} c_\mu u^{m\mu} v^{r\mu}\right]\right)$$

which may be rewritten as

$$f = u^{nd} v^{rnd/m} \prod^{d} \left(v^{1/m} - \left[1 + \sum_{\mu>n/m} c_\mu u^{m\mu-n} v^{r(m\mu-n)/m}\right]\right). \tag{2.6}$$

The strict transform is defined by the vanishing of the last d factors, and again it meets the v-axis at $(0, 1)$ (as well as at $m - 1$ other points). Note that

$$\frac{f}{f_1^{d/m}} = \prod^{d} \left(1 - \frac{\sum_{\mu>n/m} c_\mu u^{m\mu-n} v^{r(m\mu-n)/m}}{v^{1/m} - 1}\right),$$

which is indeterminate at $(0, 1)$ but whose value elsewhere on the rupture component is 1.

Introducing two new variables y' and w, let B denote a small ball $\|(y', w)\| \le \delta'$ centered at the origin, and map it to a neighborhood N of $(u, v) = (0, 1)$ by letting $u = \frac{y'}{(w+1)^r}$ and $v = (w + 1)^m$. Note that this map is nonsingular at the origin.

When pulled back via this map, just one of the values $v^{1/m}$ becomes $w+1$. Thus d/m of the factors at the end of (2.6) become

$$w - \sum_{\mu > n/m} c_\mu (y')^{m\mu - n},$$

whereas the remaining $d - d/m$ factors become units.

We can regard the Milnor fiber of our original curve as a subset of the surface obtained by the sequence of blowups. Let us assume that the choices of δ and ϵ made in defining the Milnor fiber are subsequent to the choice of δ'. We claim that by choosing δ sufficiently small we can guarantee that the strict transform of the original curve germ lies entirely within N. Indeed, we note that on the strict transform the following equation holds:

$$v^{1/m} = 1 + \sum_{\mu > n/m} c_\mu y^{\mu - n/m}$$

(for some choice of conjugate). Thus we can force v to be arbitrarily close to 1 by choosing δ sufficiently small, and since $u^m = y/v^r$ we can likewise force u arbitrarily close to 0. Then by appropriate choice of ϵ we can arrange that the Milnor fiber of our curve is transverse to the boundary of N, and that its boundary lies completely within N. Our Milnor fiber is thus divided into two pieces. (See Figure 1.)

Consider first the piece of the Milnor fiber lying outside of N. Having excluded the points of indeterminacy of $f/f_1^{d/m}$, we may apply the approximation lemma 1.1 to conclude that the monodromy of f is the same as the monodromy of $f_1^{d/m}$. The Milnor fiber has d/m connected components corresponding to all possible values of $\epsilon^{m/d}$, and each one is a copy of the Milnor fiber for f_1. Fixing one such value η, we see as above that the corresponding component can be contracted onto the complete bigraph on the n points $((-\eta)^{1/n}, 0)$ and the m points $(0, \eta^{1/m})$. As ϵ goes around a circle the values of $\epsilon^{m/d}$ are cyclically permuted; thus the components are likewise permuted. As ϵ goes around this circle d/m times, however, each η goes once around a circle. Thus the monodromy of this piece is $\mathbf{H}_1(t^{d/m})$.

Now consider the piece of the Milnor fiber lying inside N. Note that it has two sorts of boundary components: the components of the original link L and those components created by its intersection with the boundary sphere of N. To analyze it, we look at its inverse image in the ball B. By the approximation lemma 1.1, we may ignore all unit factors in f. Thus we may assume that the function defining this piece of the Milnor fiber is

$$(y')^{nd} \prod^{d/m} \left(w - \sum_{\mu > n/m} c_\mu (y')^{m\mu - n} \right).$$

The map $(y', w) \mapsto (y', (y')^{nm} w)$ takes this piece to the Milnor fiber of the curve with Puiseux series

$$\sum_{\mu > n/m} c_\mu (y')^{m\mu - n + nm}, \tag{2.7}$$

but it misses disks centered at the d/m points $(0, \epsilon^{m/d})$. Note that these disks are cyclically permuted by the monodromy. In (2.7) there are $e-1$ essential terms, whereas our original Puiseux series had e essential terms. By the inductive hypothesis, the monodromy of this curve is the same as that of its prototype, which has

Puiseux series

$$\sum_{i=2}^{e}(y')^{m(\mu_i-\mu_1+n)};$$

by reindexing we obtain the Puiseux series of the derived curve. Thus $d' = d/m$, confirming formula (2) of the theorem, and the monodromy of this piece of the Milnor fiber is

$$\frac{\mathbf{H}'(t)}{t^{d'}-1}.$$

Combining this with our conclusion about the monodromy of the first piece, we obtain formula (6). Finally we obtain formula (4) by computing the degree of both sides of (6). □

3. Quasi-ordinary surfaces

We now turn to quasi-ordinary surfaces, beginning with a compressed account of the essential facts and definitions. A reader seeking more information should consult [1, 4, 10, 11, 12].

We suppose that S is a germ at the origin of an irreducible analytic surface defined by the vanishing of a function $f(x, y, z)$. The quasi-ordinary condition means that we can arrange a projection $\pi : (x, y, z) \mapsto (x, y)$ so that $\pi|_S$ has discriminant locus contained in the coordinate axes $xy = 0$. In particular $\pi|_S$ is a finite covering space map over the complement of the axes, whose fundamental group is $\mathbf{Z}\times\mathbf{Z}$. It is known that S has many curve-like properties. Foremost among them is the existence of a fractional-exponent power series

$$\zeta(x, y) = \sum c_{\lambda\mu}x^{\lambda}y^{\mu} \tag{3.1}$$

which parametrizes S via $(x, y) \mapsto (x, y, \zeta(x, y))$, where we vary the conjugate of ζ so as to obtain the various sheets of the cover. The exponents can all be taken to have a common denominator, and we write only those terms in which $c_{\lambda\mu} \neq 0$. One can recover f by forming a product over all conjugates:

$$f(x, y, z) = \prod^{d}(z - \zeta(x, y)).$$

(Here d denotes the number of conjugates and thus the number of sheets.)

Define an ordering on pairs of exponents as follows: we say that $(\lambda, \mu) < (\lambda^*, \mu^*)$ if $\lambda \leq \lambda^*$, $\mu \leq \mu^*$, and they are not the same pair. The restriction on the discriminant locus implies that among the exponent pairs of (3.1) we may find a finite sequence of *characteristic pairs*

$$(\lambda_1, \mu_1) < (\lambda_2, \mu_2) < \cdots < (\lambda_e, \mu_e) \tag{3.2}$$

with these properties:

(1) $(0, 0) < (\lambda_1, \mu_1)$.
(2) Each (λ_i, μ_i) is not contained in the subgroup of $\mathbf{Q}\times\mathbf{Q}$ generated by $\mathbf{Z}\times\mathbf{Z}$ and by the previous characteristic pairs.
(3) If (λ, μ) is a noncharacteristic pair, then it is contained in the subgroup generated by those characteristic pairs for which $(\lambda_i, \mu_i) < (\lambda, \mu)$.

In our analysis we will assume that $\mu_1 \neq 0$. (Note that this covers the case of a *reduced* quasi-ordinary surface as defined in [12], viz., a surface for which $\lambda_1\mu_1 \neq 0$.) In this case one immediately verifies that the intersection of the surface with the plane $y = 0$ is the x-axis; except in trivial cases the x-axis is actually a component of the singular locus. For such a surface we define the *Milnor fiber of a transverse slice* to be the set of points (x, y, z) obtained by the following process:

(1) requiring that $\|(x, y, z)\| \leq \delta$, a sufficiently small radius,
(2) then requiring that x be a fixed number sufficiently close to (but different from) zero,
(3) then requiring that $f(x, y, z) = \epsilon$, a number sufficiently close to (but different from) zero.

Denote this transverse Milnor fiber by F and its Euler characteristic by χ. We should point out a subtlety in the definition: the tranverse slice (obtained by the first two steps but then staying on the surface $f = 0$) may be a plane curve with several branches. For example, the transverse slice of $z^2 = x^3y^2$ is a pair of lines, and thus its transverse Milnor fiber has two boundary components.

By keeping x fixed but letting ϵ vary over a circle centered at 0, we obtain the *horizontal fibration*. Keeping ϵ fixed but letting x vary over a circle centered at 0, we obtain the *vertical fibration*. Thus we have a fibration over a torus. Let $h_q : H_q(F; \mathbf{Q}) \to H_q(F; \mathbf{Q})$ and $v_q : H_q(F; \mathbf{Q}) \to H_q(F; \mathbf{Q})$ be the respective monodromy operators. We call the graded characteristic functions

$$\mathbf{H}(t) = \frac{\det(tI - h_0)}{\det(tI - h_1)} \qquad \text{and} \qquad \mathbf{V}(t) = \frac{\det(tI - v_0)}{\det(tI - v_1)}$$

the *horizontal monodromy* and *vertical monodromy*; in the literature they are often called *zeta functions*.

For a quasi-ordinary surface with $\mu_1 = 0$, the definitions of horizontal and vertical monodromy need to be formulated in a slightly different way. We discuss this case in the last section of the paper. In all circumstances our definitions agree with those of Kulikov [9], p. 137 (except in those cases where the surface is not singular along or above the x-axis, in which case our formulas yield trivial monodromy).

4. Recursive formulas for horizontal and vertical monodromy

Suppose we begin with a series (3.1) defining the germ at the origin of an irreducible quasi-ordinary surface S. As in the case of plane curves, we create a new series using just the characteristic pairs,

$$\sum_{i=1}^{e} x^{\lambda_i} y^{\mu_i}, \tag{4.1}$$

and call the corresponding surface the *prototype*.

Theorem 4.1. *A quasi-ordinary surface (with $\mu_1 \neq 0$) and its prototype have the same horizontal monodromy and the same vertical monodromy.*

We will establish this as in the case of plane curves: by induction on e, while simultaneously proving a set of recursive formulas. The case $e = 0$ is trivial, and henceforth we assume that $e > 0$. We define the *truncation* to be the surface S_1 determined by

$$\zeta_1 = x^{\lambda_1} y^{\mu_1} = x^{\frac{a}{mb}} y^{\frac{n}{m}},$$

where n and m are relatively prime, as are a and b.

As before, let r and s be the smallest nonnegative integers so that

$$\begin{bmatrix} m & n \\ r & s \end{bmatrix}$$

has determinant 1. The *derived surface* is the surface S' determined by

$$\zeta' = \sum_{i=1}^{e-1} x^{\lambda_i'} y^{\mu_i'},$$

where the new exponents are computed by these formulas:

$$\begin{aligned} \mu_i' &= m(\mu_{i+1} - \mu_1 + mb\mu_1) \\ \lambda_i' &= b(\lambda_{i+1} - \lambda_1 + mb\lambda_1 + r\mu_i'\lambda_1). \end{aligned}$$

Example 4.2. For the quasi-ordinary surface with branch

$$\zeta = x^{1/2}y^{4/3} + x^{2/3}y^{4/3} + x^{11/12}y^{4/3},$$

the derived surface is determined by the branch

$$\zeta' = x^{163/3}y^{24} + x^{329/6}y^{24}.$$

For the truncation, let d_1, χ_1, $\mathbf{H}_1$, and $\mathbf{V}_1$ denote its degree, the Euler characteristic of its transverse Milnor fiber, and its horizontal and vertical monodromies. Let d', χ', $\mathbf{H}'$, and $\mathbf{V}'$ denote the same things for the derived surface. Let (n, a) denote the greatest common divisor.

Theorem 4.3. *For a quasi-ordinary surface germ (with* $\mu_1 \neq 0$*), its degree, the Euler characteristic of its transverse Milnor fiber, its horizontal monodromy, and its vertical monodromy are determined by the following formulas.*

(1) $d_1 = mb$
(2) $d = d_1 d'$
(3) $\chi_1 = mb + nb - mnb^2$
(4) $\chi = d'(\chi_1 - b) + b\chi' = d'\chi_1 + b(\chi' - d')$
(5)
$$\mathbf{H}_1(t) = \frac{(t^{mb}-1)(t^{nb}-1)}{(t^{mnb}-1)^b}$$
(6)
$$\mathbf{H}(t) = \frac{\mathbf{H}_1(t^{d'})(\mathbf{H}'(t))^b}{(t^{d'}-1)^b}$$
(7)
$$\mathbf{V}_1(t) = \frac{(t-1)^{mb}}{(t^{nb/(n,a)}-1)^{(n,a)(mb-1)}}$$
(8)
$$\mathbf{V}(t) = \frac{(\mathbf{V}_1(t))^{d'}\mathbf{V}'(t^b)}{(t^b-1)^{d'}}$$

Before embarking on the proof, we will illustrate its ideas by working out the details of Example 4.2, the surface with branch

$$\zeta = x^{1/2}y^{4/3} + x^{2/3}y^{4/3} + x^{11/12}y^{4/3}.$$

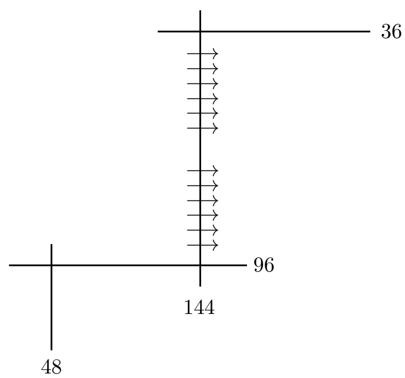

FIGURE 3. The resolution diagram for the transverse slice of the surface of Example 4.2, with multiplicities indicated. The rupture component meets the strict transform at 12 points.

Its intrinsic equation is a polynomial f of degree 36 in z (whose coefficients are functions of x and y):

$$f = \prod^{36} \left(z - \left[x^{1/2} y^{4/3} + x^{2/3} y^{4/3} + x^{11/12} y^{4/3} \right] \right).$$

As x moves on a circle of small radius ρ, each value of x determines a transverse slice of the surface. All of our constructions will be done equivariantly, i.e., by doing the same thing simultaneously to all transverse slices. First, in each transverse slice, we perform the series of blowups dictated by $\mu_1 = 4/3$ and the Euclidean algorithm: this in fact gives an embedded resolution of each transverse slice, with the resolution diagram shown in Figure 3. (This happens because $\mu_1 = \mu_2 = \mu_3$. In general this first set of blowups will only begin the resolution process, and the strict transform will continue to be singular.)

The exceptional divisor meeting the strict transform is called the *rupture component*, and to study it we examine the chart given by

$$y = u^3 v^2$$
$$z = u^4 v^3.$$

The pullback of f is a product of 36 conjugates:

$$f = \prod^{36} \left(u^4 v^3 - \left[x^{1/2} u^4 v^{8/3} + x^{2/3} u^4 v^{8/3} + x^{11/12} u^4 v^{8/3} \right] \right),$$

which we factor as follows

$$f = u^{144} v^{96} x^{18} \prod^{36} \left(\left(\frac{v}{x^{3/2}} \right)^{1/3} - \left[1 + x^{1/6} + x^{5/12} \right] \right). \tag{4.2}$$

Here the rupture component is the v-axis, and the strict transform meets it at the twelve points determined by the values

$$v = (1 + x^{1/6} + x^{5/12}) x^{3/2}.$$

As shown in Figure 4, these twelve points are clustered around the two points where the torus knot $v^2 = x^3$ meets our transverse slice.

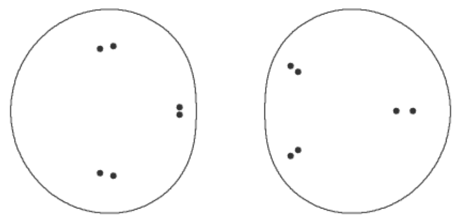

FIGURE 4. The strict transform of a transverse slice of the quasi-ordinary surface $\zeta = x^{1/2}y^{4/3} + x^{2/3}y^{4/3} + x^{11/12}y^{4/3}$ meets the rupture component in twelve points. The tubular neighborhood N meets the rupture component in two topological disks.

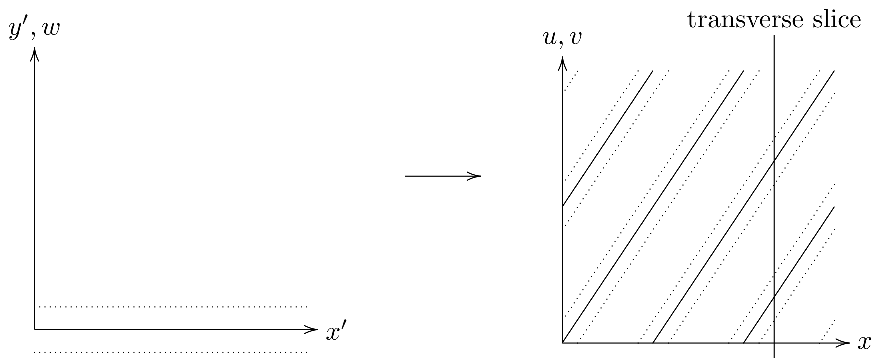

FIGURE 5. A tubular neighborhood B of the circle $\|x'\| = \rho^{1/b}$ is mapped onto a tubular neighborhood N of the torus knot $v^b = x^a$ (where $u = 0$, and x moves on the circle of radius ρ). Each transverse slice $x =$ constant meets N in b disjoint topological balls. In this example, $a = 3$ and $b = 2$.

Introducing three new variables x', y', and w, let B denote the product of the circle $\|x'\| = \rho^{1/2}$ and the 4-ball $\|(y', w)\| \leq \delta'$, where δ' is sufficiently small. We map B to a tubular neighborhood N of the torus knot as follows:

$$x = (x')^2$$
$$u = \frac{y'}{(w+1)^2\rho}$$
$$v = (w+1)^3(x')^3,$$

thus mapping the core circle of B to the knot. Figure 5 illustrates this map. In Figure 4, one sees that N meets the rupture component in two topological disks.

The Milnor fiber of the transverse slice is thus divided into two pieces: the piece lying within N, and the piece lying outside N. Our proof will show that the outside piece is unchanged if in (4.2) we replace f by

$$u^{144}v^{96}(v^{12} - x^{18}),$$

i.e., the pullback of $z^{36}-x^{18}y^{48}$. Thus this piece has six connected components, each of which is a copy of the transverse Milnor fiber of the truncation, the surface with branch

$$\zeta_1 = x^{1/2}y^{4/3}.$$

As for the inside piece, we will argue that it is the same as the transverse Milnor fiber of a new singular surface. When pulled back to B, thirty of the 36 factors at the end of (4.2) become units. To see this, first observe that we can force the value in square brackets to be arbitrarily close to 1 by choosing sufficiently small radii δ' and ρ. To obtain a non-unit, we must therefore pick the "principal value" of $x^{1/2}$ for which it equals x' and then similarly pick the appropriate cube root of $v/(x')^3$ so that

$$\left(\frac{v}{(x')^3}\right)^{1/3} = w+1;$$

these choices can be made uniformly throughout B. Thus the inside piece is defined by the vanishing of

$$(x')^{324}(y')^{144}\prod^{6}\left(w-\left[(x')^{1/3}+(x')^{5/6}\right]\right)$$

The map $(x',y',w)\mapsto(x',y',(x')^{54}(y')^{24}w)$ takes this piece to the transverse Milnor fiber of the quasi-ordinary surface with branch

$$\zeta' = (x')^{163/3}(y')^{24}+(x')^{329/6}(y')^{24},$$

in accordance with our general formula. The image of the map misses six small disks centered at the points $(x',0,\epsilon^{1/6})$.

Proof. As indicated, we will simultaneously provide an inductive proof of Theorem 4.1 (inducting on the number of characteristic pairs) and a recursive proof of Theorem 4.3.

Fixing a value of x, consider the transverse Milnor fiber of the truncation, defined by $z^{mb}-x^a y^{nb}=\epsilon$, and its image under the projection π. There is total ramification above the (nb)th roots of $(-\epsilon/x^a)$. We can retract a neighborhood of 0 onto the union L_x of line segments from 0 to these points, in such a way that there is a compatible retraction of the Milnor fiber onto $\pi^{-1}L_x$, which is the complete bigraph on the nb points

$$\left(\sqrt[nb]{-\epsilon/x^a},0\right) \tag{4.3}$$

and the mb points

$$\left(0,\ \sqrt[mb]{\epsilon}\right). \tag{4.4}$$

As ϵ goes around a circle, each set of points is cyclically permuted. Since m and n are relatively prime, the mnb^2 edges of the graph fall into b orbits of length mnb. This confirms formula (5). If ϵ is fixed but x varies, the retractions of the Milnor fibers fit together continuously. The points (4.4) are fixed but the points (4.3) fall into (n,a) orbits each of size $nb/(n,a)$. For the edges of the graph the orbits likewise have this size, and there are $(n,a)mb$ such orbits. This confirms formula (7). Formula (3) follows by taking the degree, and formula (1) is trivial.

To verify the recursive formulas and to handle the inductive step in the proof of Theorem 4.1, suppose we are given a curve with series (3.1) and prototype (4.1). We first replace

$$\frac{z - \sum_{(\lambda,\mu)\in\mathbf{Z}\times\mathbf{Z}} c_{\lambda\mu}x^{\lambda}y^{\mu}}{c_{\lambda_1\mu_1}}.$$

by z. In the new coordinate system, the surface is defined by the vanishing of

$$f = \prod^{d}\left(z - \left[x^{\frac{a}{mb}}y^{\frac{n}{m}} + \sum_{(\lambda,\mu)>\left(\frac{a}{mb},\frac{n}{m}\right)} c_{\lambda\mu}x^{\lambda}y^{\mu}\right]\right), \tag{4.5}$$

(where for simplicity the coefficients have been renamed). The truncation is defined by the vanishing of

$$f_1 = \prod^{mb}(z - x^{\frac{a}{mb}}y^{\frac{n}{m}}) = z^{mb} - x^a y^{nb}. \tag{4.6}$$

Dividing (4.5) by a power of (4.6), we claim that

$$\frac{f}{f_1^{d/(mb)}} = \prod^{d}\left(1 - \frac{\sum_{(\lambda,\mu)>\left(\frac{a}{mb},\frac{n}{m}\right)} c_{\lambda\mu}x^{\lambda}y^{\mu}}{z - x^{\frac{a}{mb}}y^{\frac{n}{m}}}\right). \tag{4.7}$$

To justify this we argue as follows. Let (x,y) be a point close to the origin but not lying on the x- or y-axis. Let d_x be the common denominator of all x-exponents appearing in (4.5); similarly let d_y be the common denominator of all y-exponents. Fix a value $\bar{x} = x^{1/d_x}$ and similarly a value $\bar{y} = y^{1/d_y}$. Then there is a map from the product of two groups of roots of unity:

$$\mu_{d_x} \times \mu_{d_y} \to \text{points on the surface projecting to } (x,y)$$

whose last coordinate is given by

$$(\alpha,\beta) \mapsto (\alpha\bar{x})^{ad_x/(mb)}(\beta\bar{y})^{nd_y/m} + \sum_{(\lambda,\mu)>\left(\frac{a}{mb},\frac{n}{m}\right)} c_{\lambda\mu}(\alpha\bar{x})^{\lambda d_x}(\beta\bar{y})^{\mu d_y}. \tag{4.8}$$

(Note that all exponents are integers.) This map factors through the quotient $(\mu_{d_x} \times \mu_{d_y})/K$, where K consists of all elements determining the same point as $(1,1)$. This quotient group has order d. Similarly there is a map

$$(\alpha,\beta) \mapsto (\alpha\bar{x})^{ad_x/(mb)}(\beta\bar{y})^{nd_y/m}$$

onto the points of the truncation surface, with kernel K_1 and with quotient group $(\mu_{d_x} \times \mu_{d_y})/K_1$ of order mb. A fiber of the homomorphism

$$(\mu_{d_x} \times \mu_{d_y})/K \to (\mu_{d_x} \times \mu_{d_y})/K_1$$

(i.e, a coset of the kernel K_1/K) corresponds to all distinct series in (4.8) compatible with a specified first term. Since these fibers all have the same cardinality $d/(mb)$, the calculation leading to (4.7) is justified.

Now we suppose that x moves on the circle of radius ρ. All of our constructions will be done equivariantly, i.e., by doing the same thing simultaneously to all transverse slices. First, in each transverse slice, we perform the series of blowups dictated by $\mu_1 = n/m$ and the Euclidean algorithm. Doing this for the truncation, we obtain (for each transverse slice) a total transform consisting of certain exceptional divisors occurring with certain multiplicities, together with a strict transform meeting just one exceptional divisor, which we call the *rupture component*. Along this chain the function z^m/y^n has no indeterminacy, and in fact except along the rupture component its value is either 0 or ∞.

If all of the exponents μ appearing in (4.7) were strictly greater than n/m, then we could argue, as in the earlier proof of Theorem 2.3, that the value of (4.7) along a non-rupture exceptional divisor is 1. But since there may be a repetition of exponents (even in the characteristic pairs) we need to be more careful. If $z^m/y^n = 0$, then

$$\frac{f}{f_1^{d/(mb)}} = \prod^d \left(1 + \sum_{(\lambda,\mu)>\left(\frac{a}{mb},\frac{n}{m}\right)} c_{\lambda\mu} x^{\lambda - a/(mb)} y^{\mu - n/m} \right),$$

and since y vanishes everywhere along the exceptional divisors we find that

$$\frac{f}{f_1^{d/(mb)}} = \prod^d \left(1 + \sum_{\lambda > \frac{a}{mb}} c_{\lambda\mu_1} x^{\lambda - a/(mb)} \right).$$

Note that by choosing x sufficiently close to 0 we can guarantee that this value has positive real part. If $z^m/y^n = \infty$, i.e. $y^n/z^m = 0$, then a similar calculation shows that the value of (4.7) is 1.

To work in a chart containing the rupture component, we use substitutions dictated by the matrix

$$\begin{bmatrix} m & n \\ r & s \end{bmatrix},$$

where r and s are the smallest positive integers for which the determinant is 1, namely

$$y = u^m v^r$$
$$z = u^n v^s.$$

We find that in this chart the total transform of the truncation is defined by the vanishing of

$$f_1 = u^{mnb} v^{rnb} (v^b - x^a),$$

and its strict transform is defined by the vanishing of the last factor. Note that it meets the v-axis in b points, and that as x travels around a small circle these points trace out the torus knot $v^b = x^a$. The total transform of the given surface is defined by the vanishing of

$$f = \prod^d \left(u^n v^s - \left[x^{\frac{a}{mb}} u^n v^{rn/m} + \sum_{(\lambda,\mu)>\left(\frac{a}{mb},\frac{n}{m}\right)} c_{\lambda\mu} x^\lambda u^{m\mu} v^{r\mu} \right] \right)$$

which may be rewritten as

(4.9)

$$f = u^{nd} v^{rnd/m} x^{ad/(mb)} \prod^d \left(\left(\frac{v}{x^{a/b}} \right)^{1/m} - \left[1 + \sum_{(\lambda,\mu)>\left(\frac{a}{mb},\frac{n}{m}\right)} c_{\lambda\mu} x^{\lambda - a/(mb)} u^{m\mu - n} v^{r(m\mu - n)/m} \right] \right).$$

Again if all the values of μ appearing in (4.9) are strictly greater than n/m, then we can assert that the strict transform meets the v-axis in the same set of b points, but if there is a repetition of exponents then we find that the strict transform meets this axis at all points at which (for some choice of conjugate)

$$v^b = \left(1 + \sum_{\lambda > \frac{a}{mb}} c_{\lambda\mu_1} x^{\lambda - a/(mb)}\right)^{mb} x^a. \tag{4.10}$$

We also note that

$$\frac{f}{f_1^{d/(mb)}} = \prod^{d} \left(1 - \frac{\sum_{(\lambda,\mu) > \left(\frac{a}{mb}, \frac{n}{m}\right)} c_{\lambda\mu} x^{\lambda - a/(mb)} u^{m\mu - n} v^{r(m\mu - n)/m}}{\left(\frac{v}{x^{a/b}}\right)^{1/m} - 1}\right),$$

and that its restriction to the rupture component is

$$\prod^{d} \left(1 - \frac{\sum_{\lambda > \frac{a}{mb}} c_{\lambda\mu_1} x^{\lambda - a/(mb)}}{\left(\frac{v}{x^{a/b}}\right)^{1/m} - 1}\right). \tag{4.11}$$

Introducing three new variables x', y', and w, let B denote the product of the circle $\|x'\| = \rho^{1/b}$ and the 4-ball $\|(y', w)\| \leq \delta'$. Map this product to a neighborhood N of the torus knot as follows:

$$x = (x')^b$$
$$u = \frac{y'}{(w+1)^r \rho^{ar/(mb)}}$$
$$v = (w+1)^m (x')^a$$

(See Figure 5.) Note that the circle $(y', w) = (0, 0)$ is mapped onto the knot. We claim that if δ' is sufficiently small then the map is injective (regardless of the value of ρ). Indeed, suppose that (x'_1, y'_1, w_1) and (x'_2, y'_2, w_2) are two points whose images agree. Then

$$\left(\frac{w_2 + 1}{w_1 + 1}\right)^m = \left(\frac{x'_1}{x'_2}\right)^a,$$

where the quantity on the right is a bth root of 1. If w_1 and w_2 are sufficiently close to 0 then this root must be 1 itself. Since a and b are relatively prime, this implies that $x'_1/x'_2 = 1$. Since the map $w \mapsto (w+1)^m$ is injective near 0, we see that $w_1 = w_2$ and then that $y'_1 = y'_2$.

Thus N is a tubular neighborhood of the torus knot: its intersection with each transverse plane consists of b disjoint topological disks, each of which encloses one of the points where the torus knot meets the plane.

We can regard each transverse Milnor fiber as a subset of the surface obtained from the transverse plane $x =$ constant by the sequence of blowups. Let us assume that the choices of δ, x, and ϵ which determine the transverse Milnor fiber are made subsequent to the choice of δ'. We claim that we can make these choices so as to guarantee that the strict transform of the surface lies entirely within N. Indeed, we note that on the strict transform

$$w = \sum_{(\lambda,\mu) > \left(\frac{a}{mb}, \frac{n}{m}\right)} c_{\lambda\mu} x^{\lambda - a/(mb)} y^{\mu - n/m},$$

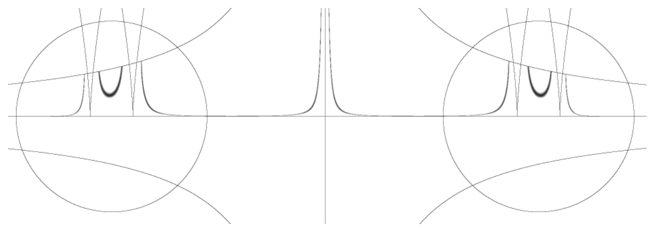

FIGURE 6. The transverse Milnor fiber is divided into two pieces by the boundary of N (indicated by two circles). The rupture component is horizontal, and another exceptional divisor is shown vertically. The strict transform enters from above.

where in each term at least one of the exponents is positive. Thus by choosing δ and $\|x\|$ sufficiently small we may force w arbitrarily close to 0. Now observe that

$$(y')^m = y\left(\frac{x'}{\rho^{1/b}}\right)^{-ar}$$

and that $\|x'/\rho^{1/b}\| = 1$. Thus we may also force $\|y'\|$ to be arbitrarily small. Note in particular that N will contain the points where the strict transform meets the v-axis (as determined by equation (4.10)); Figure 4 shows an example.

Looking at formula (4.11), we note that outside of N the value of $\left(\frac{v}{x^{a/b}}\right)^{1/m}$ along the rupture component is bounded away from 1, with the bound being independent of the choice of x; thus by choosing x sufficiently close to 0 we can guarantee that the value of (4.11) has positive real part. Finally by choosing ϵ sufficiently close to 0, we can guarantee that the Milnor fiber is transverse to the boundary of N and that its boundary lies entirely within N. Our transverse Milnor fiber is thus divided into two pieces. (See Figure 6.)

Consider first the piece of the Milnor fiber lying outside of N. By the approximation lemma 1.2, the monodromies of f and $f_1^{d/(mb)}$ are the same for this piece. The Milnor fiber has $d/(mb)$ connected components corresponding to all possible values of $\eta = \epsilon^{mb/d}$, and each one is a copy of the Milnor fiber for f_1. As ϵ goes around a circle, these copies are cyclically permuted. As ϵ goes around this circle $d/(mb)$ times, however, each η goes once around a circle. Thus the horizontal monodromy of this piece is $\mathbf{H}_1(t^{d/(mb)})$. But if ϵ is fixed and x varies, then each copy is individually acted upon by the vertical monodromy, so that the contribution from this piece is $(\mathbf{V}_1(t))^{d/(mb)}$.

Now consider the piece of the Milnor fiber lying inside N. Note that it has two sorts of boundary components: the components of the original link and those components created by its intersection with the boundary sphere of N. To analyze it, we look at its inverse image in B, which is contained in the b disjoint balls centered at the points $(x', y', w) = (x^{1/b}, 0, 0)$ (allowing all possible roots).

When pulled back to B, most of the d factors at the end of (4.9) become units. To see this, first observe that we can force the value in square brackets to be arbitrarily close to 1 by choosing sufficiently small radii δ' and ρ. To obtain a non-unit, we must therefore pick the "principal value" of $x^{1/b}$ for which it equals x' and then similarly pick the appropriate mth root of $v/(x')^a$ so that

$$\left(\frac{v}{(x')^a}\right)^{1/m} = w + 1;$$

note that these choices can be made uniformly throughout B. Thus $d/(mb)$ of the factors at the end of (4.9) become

$$w - \sum_{(\lambda,\mu)>\left(\frac{a}{mb},\frac{n}{m}\right)} c'_{\lambda\mu}(x')^{b\lambda - a/m + ar(m\mu-n)/m}(y')^{m\mu-n}$$

(where $c'_{\lambda\mu} = c_{\lambda\mu}\rho^{-ar(m\mu-n)/(mb)}$), whereas the remaining $d - d/(mb)$ factors become units. Each such unit takes its values in an arbitrarily small neighborhood of some $e-1$, where e is a nontrivial (mb)th root of unity. Thus by the approximation lemma 1.2, we may ignore all unit factors in f. Thus we may assume that the function defining this piece of the Milnor fiber is

$$(x')^{ads}(y')^{nd}\prod^{d/(mb)}\left(w - \sum_{(\lambda,\mu)>\left(\frac{a}{mb},\frac{n}{m}\right)} c'_{\lambda\mu}(x')^{b\lambda - a/m + ar(m\mu-n)/m}(y')^{m\mu-n}\right).$$

The map $(x', y', w) \mapsto (x', y', (x')^{asmb}(y')^{nmb}w)$ takes this piece to the transverse Milnor fiber of the quasi-ordinary surface with series

$$\sum_{(\lambda,\mu)>\left(\frac{a}{mb},\frac{n}{m}\right)} c'_{\lambda\mu}(x')^{b\lambda - a/m + ar(m\mu-n)/m + ambs}(y')^{m\mu-n+nmb}, \tag{4.12}$$

but it misses disks centered at the $d/(mb)$ points

$$(x', 0, \epsilon^{d/(mb)}). \tag{4.13}$$

The horizontal monodromy permutes these disks. In (4.12) there are $e-1$ characteristic pairs, whereas our original series had e characteristic pairs. By the inductive hypothesis, the horizontal monodromy of this curve is the same as that of its prototype, which has series

$$\sum_{i=2}^{e}(x')^{b[\lambda_i - \lambda_1 + mb\lambda_1 + rm(\mu_i - \mu_1 + mb\mu_1)\lambda_1]}(y')^{m(\mu_i-\mu_1+mb\mu_1)}.$$

(In calculating the first exponent we have used $ms = rn + 1$.) By reindexing we obtain the series of the derived surface. (Note that all of the exponents on y' are positive; thus we are still in the hypothesized case.) Thus $d' = d/(mb)$, confirming formula (2) of the theorem. Since there are b copies of this situation (one for each bth root of x), the monodromy of this piece of the transverse Milnor fiber is

$$\left(\frac{\mathbf{H}'(t)}{t^{d'}-1}\right)^b.$$

Combining this with our conclusion about the monodromy of the first piece, we obtain formula (6). Then we obtain formula (4) by computing the degree of both sides of (6).

Turning to the vertical monodromy, we remark that it cyclically permutes the individual pieces of the Milnor fiber cut out by the b disjoint balls. Its bth power acts on each such piece by the vertical monodromy of the derived surface, in such a way that the disks of (4.13) are cyclically permuted. Thus the contribution to the vertical monodromy of our original surface is

$$\frac{\mathbf{V}'(T)}{(T-1)^{d'}}$$

where $T = t^b$. Combining this with our conclusion about the vertical monodromy of the first piece, we obtain formula (8). □

Here is another example. If we begin with the surface parametrized by

$$\zeta = x^{1/2}y^{3/2} + x^{1/2}y^{7/4} + x^{2/3}y^{11/6},$$

then its truncation and derived surface are parametrized by

$$\zeta_1 = x^{1/2}y^{3/2} \qquad \text{and} \qquad \zeta' = x^{17/4}y^{13/2} + x^{9/2}y^{20/3}.$$

Repeating the process, the new truncation and the second derived surface are parametrized by

$$\zeta_1' = x^{17/4}y^{13/2} \qquad \text{and} \qquad \zeta'' = x^{1438/3}y^{157/3}.$$

By repeated use of the first two formulas in Theorem 4.3, we find that the degree of the quasi-ordinary surface is

$$d = d_1 d_1' d'' = 2 \cdot 4 \cdot 3 = 24.$$

By formulas (3) and (4), the Euler characteristic of the transverse Milnor fiber is

$$\chi = d'(\chi_1 - b) + d''(\chi_1' - b') + b'\chi'' = 12(-1-1) + 3(-74-2) + 2(-311) = -874.$$

By formulas (5) and (6), the horizontal monodromy is

$$\begin{aligned}\mathbf{H}(t) &= \frac{\mathbf{H}_1(t^{d'})}{(t^{d'}-1)^b}\left[\frac{\mathbf{H}_1'(t^{d''})}{(t^{d''}-1)^{b'}}\right]^b [\mathbf{H}''(t)]^{bb'} \\ &= \frac{(t^{24}-1)(t^{36}-1)}{(t^{72}-1)(t^{12}-1)}\left[\frac{(t^{12}-1)(t^{78}-1)}{(t^{156}-1)^2(t^3-1)^2}\right]^1 \left[\frac{(t^3-1)(t^{157}-1)}{t^{471}-1}\right]^2.\end{aligned} \tag{4.14}$$

By formulas (7) and (8), the vertical monodromy is

$$\begin{aligned}\mathbf{V}(t) &= \left[\frac{\mathbf{V}_1(t)}{t^b-1}\right]^{d'}\left[\frac{\mathbf{V}_1'(t^b)}{(t^{bb'}-1)}\right]^{d''} \cdot \mathbf{V}''(t^{bb'}) \\ &= \left[\frac{(t-1)^2}{(t^3-1)(t-1)}\right]^{12}\left[\frac{(t-1)^4}{(t^{26}-1)^3(t^2-1)}\right]^3 \cdot \frac{(t^2-1)^3}{(t^{314}-1)^2}.\end{aligned} \tag{4.15}$$

5. Quasi-ordinary surfaces for which $\mu_1 = 0$

Suppose that in (3.2) we have $\mu_i = 0$ for $1 \le i \le s < e$. Then the singular locus of S may contain a curve which does not lie in the x-y plane, namely the intersection of S with the plane $y = 0$. This curve projects to the x-axis, and if we restrict our attention to those points lying over a small circle we see an N-sheeted covering $C \to S^1$, where N is the least common denominator of $\{\lambda_i\}_{i=1}^s$. The transverse slice of S (as defined in section 3) will then be a curve with N singularities. For example, on the surface parametrized by $\zeta = x^{3/2} + x^2y^{3/2}$ the curve $z^2 = x^3$ is a component of the singular locus. A transverse slice is shown in Figure 7.

In this case, the correct definitions of the horizontal and vertical fibrations use Milnor fibers at the points of C. Such a Milnor fiber consists of those points within

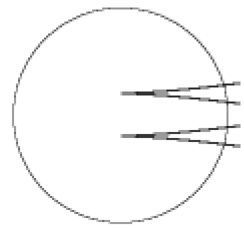

FIGURE 7. The real points of the transverse slice of the quasi-ordinary surface parametrized by $\zeta = x^{3/2} + x^2 y^{3/2}$. Here $N = 2$.

a transverse slice, within a sufficiently small neighborhood of the specified point of C, and satisfying $f = \epsilon$ (for sufficiently small ϵ). Each transverse slice will contain N such Milnor fibers, and they form the fibers of a fibration over $C \times S^1$ (the latter factor consisting of all ϵ on a small circle). One obtains the horizontal or vertical fibration by fixing (respectively) the point of C or the value of ϵ.

Lipman [12] (p. 65 ff.) shows that we can find a different quasi-ordinary surface S' with characteristic pairs $\{(\lambda'_i, \mu'_i) = (N\lambda_{i+s}, \mu_{i+s})\}$, $1 \leq i \leq e - s$, so that the horizontal and vertical fibrations of S (as just defined) are the same as those of S' (as defined in section 3). Thus the characteristic pairs $\{(\lambda_i, 0)\}_{i=1}^{s}$ are invisible in these monodromies, but they are precisely what is recovered by the topological zeta function of the two-dimensional singularity; see [14] and [13].

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