

Vanishing Vanishing Cycles

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Abstract

If \mathbf{A}^\bullet is a bounded, constructible complex of sheaves on a complex analytic space X , and $f : X \rightarrow \mathbb{C}$ and $g : X \rightarrow \mathbb{C}$ are complex analytic functions, then the iterated vanishing cycles $\phi_g[-1](\phi_f[-1]\mathbf{A}^\bullet)$ are important for a number of reasons. We give a formula for the stalk cohomology $H^*(\phi_g[-1]\phi_f[-1]\mathbf{A}^\bullet)_x$ in terms of relative polar curves, algebra, and the normal Morse data and micro-support of \mathbf{A}^\bullet .

1 Introduction

Let \mathcal{U} be an open neighborhood of the origin in \mathbb{C}^{n+1} , and let $\tilde{f} : (\mathcal{U}, \mathbf{0}) \rightarrow (\mathbb{C}, 0)$ be a complex analytic function. Suppose that X is a complex analytic subspace of \mathcal{U} , and, for convenience, assume that $\mathbf{0} \in X$. Let f denote the restrictions of \tilde{f} to X . We write $V(f)$ for $f^{-1}(0)$.

Let \mathfrak{S} be a complex analytic Whitney stratification of X with connected strata. For $S \in \mathfrak{S}$, we let \mathbb{N}_S and \mathbb{L}_S denote, respectively, the normal slice and complex link of the stratum S ; see [8]. We let $d_S := \dim S$. Let R be a regular, Noetherian ring with finite Krull dimension (e.g., \mathbb{Z} , \mathbb{Q} , or \mathbb{C}). Let \mathbf{A}^\bullet be a bounded complex of sheaves of R -modules on X , whose cohomology is constructible with respect to \mathfrak{S} .

For each $S \in \mathfrak{S}$ and $k \in \mathbb{Z}$, define the *degree k Morse module of S with respect to \mathbf{A}^\bullet* to be the hypercohomology $m_S^k(\mathbf{A}^\bullet) := \mathbb{H}^{k-d_S}(\mathbb{N}_S, \mathbb{L}_S; \mathbf{A}^\bullet)$. We say that a stratum $S \in \mathfrak{S}$ is *\mathbf{A}^\bullet -visible* if $m_S^*(\mathbf{A}^\bullet) \neq 0$, and we let $\mathfrak{S}(\mathbf{A}^\bullet)$ denote the set of \mathbf{A}^\bullet -visible strata and let $\mathfrak{S}_0(\mathbf{A}^\bullet)$ denote the strata in $\mathfrak{S}(\mathbf{A}^\bullet)$ which contain $\mathbf{0}$ in their closures. It is a theorem, Theorem 4.13 of [18], that the *microsupport*, $\text{SS}(\mathbf{A}^\bullet)$, of \mathbf{A}^\bullet (see [10]) is equal to $\bigcup_{S \in \mathfrak{S}(\mathbf{A}^\bullet)} \overline{T_S^* \mathcal{U}}$.

If R is an integral domain, define

$$c_S(\mathbf{A}^\bullet) := (-1)^{\dim X} \left(\sum_{k \in \mathbb{Z}} (-1)^k \text{rank}(m_S^k(\mathbf{A}^\bullet)) \right),$$

and define the *characteristic cycle of \mathbf{A}^\bullet* , $\text{CC}(\mathbf{A}^\bullet)$, to be the analytic cycle in the cotangent space $T^* \mathcal{U}$ given by

$$\text{CC}(\mathbf{A}^\bullet) := \sum_{S \in \mathfrak{S}} c_S(\mathbf{A}^\bullet) [\overline{T_S^* \mathcal{U}}],$$

where $T_S^* \mathcal{U}$ is the conormal space to S in \mathcal{U} . (The reader should be aware that there are two or three different definitions of the characteristic cycle, differing by sign; see, for instance, [10] and [23].) Throughout this paper, whenever we make a statement involving the characteristic cycle, we are assuming that R is an integral domain, even though it will **not** be explicitly mentioned.

The cycle $\text{CC}(\mathbf{A}^\bullet)$, and the underlying set, $|\text{CC}(\mathbf{A}^\bullet)|$, are important geometric/topological objects associated to \mathbf{A}^\bullet . What $\text{CC}(\mathbf{A}^\bullet)$ tells one is the hypercohomology Euler characteristic data associated to the

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attaching of higher-dimensional strata to lower-dimensional strata. The characteristic cycle is topologically important for it is related to the local Euler obstruction (see [4]), the absolute polar varieties and local Morse inequalities (see Theorem 7.5 and Corollary 5.5 of [16]), and index formulas for the (shifted) vanishing cycles $\phi_f[-1]\mathbf{A}^\bullet$ (see [7], [22], [14], and below). In addition, the characteristic cycle of the intersection cohomology complex is of great importance in representation theory (see [11] and [2]).

By taking a normal slice to a stratum, the calculation of the coefficient of $\overline{T_S^*U}$ in $\text{CC}(\mathbf{A}^\bullet)$ is reduced to the calculation of the coefficient of the closure of the conormal space to a point-stratum. We use the origin as a convenient point and point out that, by definition, $c_0(\mathbf{A}^\bullet)$, the coefficient of T_0^*U in $\text{CC}(\mathbf{A}^\bullet)$ is given in terms of the Euler characteristic of the stalk cohomology of the vanishing cycles by $c_0(\mathbf{A}^\bullet) = (-1)^{\dim X} \chi(\phi_l[-1]\mathbf{A}^\bullet)_0$, where l is a generic linear form.

Let $\text{im } d\tilde{f}$ denote the image of $d\tilde{f}$, and let us recall the aforementioned index formulas for $\phi_f[-1]\mathbf{A}^\bullet$, conjectured by Deligne, and proved independently by Ginsburg, Sabbah, and L  :

Theorem 1.1. ([7], [22], [14]) *Suppose that $\mathbf{0}$ is an isolated point in the support of $\phi_f[-1]\mathbf{A}^\bullet$.*

Then, $(\mathbf{0}, d_0\tilde{f})$ is an isolated point in the intersection $|\text{CC}(\mathbf{A}^\bullet)| \cap \text{im } d\tilde{f}$, and the Euler characteristic of the stalk cohomology of the vanishing cycles of f is related to the intersection multiplicity of $\text{CC}(\mathbf{A}^\bullet)$ and image of $d\tilde{f}$ by

$$\chi(\phi_f[-1]\mathbf{A}^\bullet)_0 = (-1)^{\dim X} \left(\text{CC}(\mathbf{A}^\bullet) \cdot \text{im } d\tilde{f} \right)_{(\mathbf{0}, d_0\tilde{f})} = (-1)^{\dim X} \sum_{S \in \mathfrak{S}_0(\mathbf{A}^\bullet)} c_S(\mathbf{A}^\bullet) \left(\overline{T_S^*X} \cdot \text{im } d\tilde{f} \right)_{(\mathbf{0}, d_0\tilde{f})}.$$

Now, suppose that one wants the stalk cohomology modules of $\phi_f[-1]\mathbf{A}^\bullet$, not merely the Euler characteristic. Then, one must begin with analogous data for \mathbf{A}^\bullet ; that is, one needs not simply $\text{CC}(\mathbf{A}^\bullet)$, but rather the closures of conormals spaces to strata together with the Morse modules of the strata with coefficients in \mathbf{A}^\bullet .

The refinement of Theorem 1.1 that we proved in Theorem 5.3 of [18] was:

Theorem 1.2. ([18]) *The origin is an isolated point in the support of $\phi_f[-1]\mathbf{A}^\bullet$ if and only if $(\mathbf{0}, d_0\tilde{f})$ is an isolated point in the intersection $\text{SS}(\mathbf{A}^\bullet) \cap \text{im } d\tilde{f}$, and when these equivalent conditions hold, for all k ,*

$$H^k(\phi_f[-1]\mathbf{A}^\bullet)_0 \cong \bigoplus_{S \in \mathfrak{S}_0(\mathbf{A}^\bullet)} (m_S^k(\mathbf{A}^\bullet) \otimes_R R^{\delta_S}),$$

where $\delta_S := \left(\overline{T_S^*X} \cdot \text{im } d\tilde{f} \right)_{(\mathbf{0}, d_0\tilde{f})}$.

As we showed in Theorem 3.2 of [17], the intersection number δ_S of Theorem 1.2 can be calculated in terms of the relative polar curve (see [9], [25], [12], [13], and Section 2), $\Gamma_{f|_S}^1(S) := \Gamma_{f|_S, l}^1$, where l is a generic linear form, and $f|_S$ is not constant. The formula that one obtains for a stratum S of dimension at least 1 is

$$\delta_S = (\Gamma_{f|_S}^1(S) \cdot V(f))_0 - (\Gamma_{f|_S}^1(S) \cdot V(l))_0.$$

In the case above, where $(\mathbf{0}, d_0\tilde{f})$ is an isolated point in the intersection $\text{SS}(\mathbf{A}^\bullet) \cap \text{im } d\tilde{f}$, it is immediate that, if $S \in \mathfrak{S}_0(\mathbf{A}^\bullet)$ and $d_S \geq 1$, then $f|_S$ is not constant. However, in the case below, it will be useful to let

$\mathfrak{S}^f(\mathbf{A}^\bullet)$ denote the set of strata in $\mathfrak{S}(\mathbf{A}^\bullet)$ on which f is not constant and let $\mathfrak{S}_0^f(\mathbf{A}^\bullet) := \mathfrak{S}^f(\mathbf{A}^\bullet) \cap \mathfrak{S}_0(\mathbf{A}^\bullet)$, i.e., the set of \mathbf{A}^\bullet -visible strata whose closures contain $\mathbf{0}$ but which are not contained in $V(f)$.

How can one generalize Theorem 1.1 and Theorem 1.2 to the case where the support of $\phi_f[-1]\mathbf{A}^\bullet$ is of arbitrary dimension? Suppose that \mathfrak{l} is a generic linear form (we shall not distinguish notationally between \mathfrak{l} and $\mathfrak{l}|_{V(f)}$). If $\mathbf{0}$ is an isolated point in the support of $\phi_f[-1]\mathbf{A}^\bullet$, then it is trivial that there is an isomorphism of stalk cohomologies $H^k(\phi_{\mathfrak{l}}[-1]\phi_f[-1]\mathbf{A}^\bullet)_{\mathbf{0}} \cong H^k(\phi_f[-1]\mathbf{A}^\bullet)_{\mathbf{0}}$.

Thus, for generic linear \mathfrak{l} , Theorem 1.1 and Theorem 1.2 can be viewed as statements about $\chi(\phi_{\mathfrak{l}}[-1]\phi_f[-1]\mathbf{A}^\bullet)_{\mathbf{0}} = (-1)^{\dim V(f)} c_{\mathbf{0}}(\phi_f[-1]\mathbf{A}^\bullet)$ and $H^k(\phi_{\mathfrak{l}}[-1]\phi_f[-1]\mathbf{A}^\bullet)_{\mathbf{0}}$, where δ_S is now **defined** to be the value of $(\Gamma_{f,\mathfrak{l}}^1(S) \cdot V(f))_{\mathbf{0}} - (\Gamma_{f,\mathfrak{l}}^1(S) \cdot V(\mathfrak{l}))_{\mathbf{0}}$ provided that $S \in \mathfrak{S}_0^f(\mathbf{A}^\bullet)$. With these interpretations, do we obtain a generalization of Theorem 1.1 and Theorem 1.2? The answer is: yes.

We prove in Theorem 3.1 of this paper:

Main Theorem A. *Let \mathfrak{l} be a generic linear form, and let δ_S be the value of $(\Gamma_{f,\mathfrak{l}}^1(S) \cdot V(f))_{\mathbf{0}} - (\Gamma_{f,\mathfrak{l}}^1(S) \cdot V(\mathfrak{l}))_{\mathbf{0}}$ provided that $S \in \mathfrak{S}_0^f(\mathbf{A}^\bullet)$.*

Then,

$$c_{\mathbf{0}}(\phi_f[-1]\mathbf{A}^\bullet) = (-1)^{\dim X - \dim V(f)} \left(c_{\mathbf{0}}(\mathbf{A}^\bullet) + \sum_{S \in \mathfrak{S}_0^f(\mathbf{A}^\bullet)} \delta_S \cdot c_S(\mathbf{A}^\bullet) \right).$$

In fact, for all $k \in \mathbb{Z}$,

$$H^k(\phi_{\mathfrak{l}}[-1]\phi_f[-1]\mathbf{A}^\bullet)_{\mathbf{0}} \cong H^k(\phi_{\mathfrak{l}}[-1]\mathbf{A}^\bullet)_{\mathbf{0}} \oplus \bigoplus_{S \in \mathfrak{S}_0^f(\mathbf{A}^\bullet)} (m_S^k(\mathbf{A}^\bullet) \otimes R^{\delta_S}).$$

As we discuss in Remark 3.2, Theorem A allows us to quickly obtain Theorem 1 of [2].

Seeing that Theorem A provides a formula for the stalk cohomology of the iterated vanishing cycles $\phi_{\mathfrak{l}}[-1]\phi_f[-1]\mathbf{A}^\bullet$ in terms of Morse modules and the quantities $(\Gamma_{f,\mathfrak{l}}^1(S) \cdot V(f))_{\mathbf{0}} - (\Gamma_{f,\mathfrak{l}}^1(S) \cdot V(\mathfrak{l}))_{\mathbf{0}}$, one is naturally led to ask if there is a more general formula where all occurrences of \mathfrak{l} are replaced by a more general function g . The answer is: yes.

Suppose that we give ourselves a second complex analytic function $\tilde{g} : (\mathcal{U}, \mathbf{0}) \rightarrow (\mathbb{C}, 0)$, and let g denote the restriction of \tilde{g} to X . For each stratum $S \in \mathfrak{S}^f(\mathbf{A}^\bullet)$, we define in Section 2, following our work in [21], a *relative polar set* $|\Gamma_{f,\tilde{g}}(S)|$ and, if this set is 1-dimensional, we define a corresponding *relative polar curve* (as a cycle) $\Gamma_{f,\tilde{g}}^1(S)$; this relative polar curve agrees with the traditional one in the case where \tilde{g} is a generic linear form.

We also define $\widehat{\Gamma}_{f,\tilde{g}}^1(S)$ to be the sum of the components of the cycle $\Gamma_{f,\tilde{g}}^1(S)$ which contain the origin, but which are not contained in $V(g)$. We define $|\Gamma_{f,\tilde{g}}(\mathbf{A}^\bullet)| := \bigcup_{S \in \mathfrak{S}^f(\mathbf{A}^\bullet)} |\Gamma_{f,\tilde{g}}(S)|$. For all $S \in \mathfrak{S}(\mathbf{A}^\bullet)$, we define

$$\hat{\delta}_{f,\tilde{g}}(S) := \sum_{C \text{ comp. of } \widehat{\Gamma}_{f,\tilde{g}}^1(S)} \left((C \cdot V(f))_{\mathbf{0}} - \min \{ (C \cdot V(f))_{\mathbf{0}}, (C \cdot V(g))_{\mathbf{0}} \} \right).$$

In the case where \tilde{g} is a generic linear form, $\hat{\delta}_{f,\tilde{g}}(S)$ will agree with δ_S from Theorem A.

We prove in Theorem 4.1:

Main Theorem B. *Suppose that $\dim_{\mathbf{0}} V(f) \cap |\Gamma_{f,\tilde{g}}(\mathbf{A}^\bullet)| \leq 0$.*

Then, for all $\lambda \in \mathbb{C}$ such that λ is non-zero and $|\lambda|$ is sufficiently small, for all $k \in \mathbb{Z}$,

$$H^k(\phi_g[-1]\phi_f[-1]\mathbf{A}^\bullet)_{\mathbf{0}} \cong H^k(\phi_{f+\lambda g}[-1]\mathbf{A}^\bullet)_{\mathbf{0}} \oplus \bigoplus_{S \in \mathfrak{S}^f(\mathbf{A}^\bullet)} (m_S^k(\mathbf{A}^\bullet) \otimes R^{\hat{\delta}_{f,\tilde{g}}(S)}).$$

If \tilde{g} is a generic linear form l , then $H^k(\phi_{f+\lambda g}[-1]\mathbf{A}^\bullet)_{\mathbf{0}} \cong H^k(\phi_l[-1]\mathbf{A}^\bullet)_{\mathbf{0}}$ and $\hat{\delta}_{f,\tilde{g}}(S)$ is equal to δ_S from Theorem A; thus, Theorem B reduces to Theorem A.

Pencils of Milnor fibrations of the form $f + \lambda g$ have been studied in detail by Caubel in [5] and [6]. Caubel’s main technique, the “tilting in the Cerf diagram”, first used by Lê and Perron in [15], is an major part of what we use in our proof of Theorem B.

2 The General Relative Polar Curve

We continue with all of the notation from Section 1. Our treatment of the relative polar here follows [21], except that we avoid discussing enriched cycles.

Suppose that M is a complex submanifold of \mathcal{U} . Recall:

Definition 2.1. *The relative conormal space $T_{f|_M}^* \mathcal{U}$ is given by*

$$T_{f|_M}^* \mathcal{U} := \{(x, \eta) \in T^*\mathcal{U} \mid \eta(T_x M \cap \ker d_x \tilde{f}) = 0\}.$$

If $M \subseteq X$, then $T_{f|_M}^ \mathcal{U}$ depends on f , but not on the particular extension \tilde{f} . In this case, we write $T_{f|_M}^* \mathcal{U}$ in place of $T_{\tilde{f}|_M}^* \mathcal{U}$.*

Let $\pi : T^*\mathcal{U} \rightarrow \mathcal{U}$ denote the projection.

We are going to define a relative polar set $|\Gamma_{f,\tilde{g}}(S)|$ and a relative polar cycle $\Gamma_{f,\tilde{g}}^1(S)$, as we did in [21]. If \tilde{g} is a generic linear form, and $f|_S$ is not constant, it is easy to show that our $|\Gamma_{f,\tilde{g}}(S)|$ is purely 1-dimensional and that $\Gamma_{f,\tilde{g}}^1(S)$ is reduced, and agrees with all of the definitions/characterizations of the relative polar curve used in [9], [25], [12], [13] by Hamm, Lê, and Teissier. The point of Definition 2.2 is that it seems to be the “correct” definition of the relative polar curve even when \tilde{g} is not so generic. Note that, in the traditional case where \tilde{g} is a non-zero linear form, $d_x \tilde{g}$ is a “constant” non-zero covector.

Definition 2.2. *If $S \in \mathfrak{S}$ and $f|_S$ is not constant, we define the **relative polar set**, $|\Gamma_{f,\tilde{g}}(S)|$, to be $\pi\left(\overline{T_{f|_S}^* \mathcal{U}} \cap \text{im } d\tilde{g}\right)$. The **relative polar set**, $|\Gamma_{f,\tilde{g}}(\mathbf{A}^\bullet)|$, is defined by*

$$|\Gamma_{f,\tilde{g}}(\mathbf{A}^\bullet)| := \bigcup_{S \in \mathfrak{S}^f(\mathbf{A}^\bullet)} |\Gamma_{f,\tilde{g}}(S)|.$$

If C is a (reduced) 1-dimensional component of $|\Gamma_{f,\tilde{g}}(S)|$, then $\overline{T_{f|_S}^ \mathcal{U}}$ and $\text{im } d\tilde{g}$ intersect properly along a unique 1-dimensional component $C' := d\tilde{g}(C) = \{(x, d_x \tilde{g}) \mid x \in C\}$ such that $\pi(C') = C$. Thus, the intersection number $p_C(S) := \left(\overline{T_{f|_S}^* \mathcal{U}} \cdot [\text{im } d\tilde{g}]\right)_{C'}$ is well-defined.*

In particular, if $|\Gamma_{f,\tilde{g}}(S)|$ is purely 1-dimensional, then we may, and do, define the **relative polar curve**, $\Gamma_{f,\tilde{g}}^1(S)$, to be the properly pushed-forward cycle

$$\pi_* \left(\left[\overline{T_{f_1 S}^* \mathcal{U}} \right] \cdot [\text{im } d\tilde{g}] \right) = \sum_C p_C(S)[C],$$

where the sum is over all of the components of $|\Gamma_{f,\tilde{g}}(S)|$.

If $|\Gamma_{f,\tilde{g}}(\mathbf{A}^\bullet)|$ is purely 1-dimensional (respectively, is 1-dimensional at the origin), then we define the **relative polar curve**, as a cycle (respectively, as a cycle germ at the origin) to be

$$\Gamma_{f,\tilde{g}}^1(\mathbf{A}^\bullet) := \sum_{S \in \mathfrak{S}^f(\mathbf{A}^\bullet)} \Gamma_{f,\tilde{g}}^1(S).$$

Remark 2.3. It is trivial to show that $|\Gamma_{f,\tilde{g}}(S)|$ has no zero-dimensional components. Thus, using the convention that the empty set has dimension $-\infty$, the condition that $\dim_{\mathbf{0}} |\Gamma_{f,\tilde{g}}(\mathbf{A}^\bullet)| \leq 1$ is equivalent to $|\Gamma_{f,\tilde{g}}(\mathbf{A}^\bullet)|$ being purely one-dimensional at $\mathbf{0}$.

Definition 2.4. If $\dim_{\mathbf{0}} V(f) \cap |\Gamma_{f,\tilde{g}}(\mathbf{A}^\bullet)| \leq 0$ and $\dim_{\mathbf{0}} V(g) \cap |\Gamma_{f,\tilde{g}}(\mathbf{A}^\bullet)| \leq 0$, then, for all $S \in \mathfrak{S}^f(\mathbf{A}^\bullet)$, we define $\delta_{f,\tilde{g}}(S)$ to be the difference of intersection numbers $(\Gamma_{f,\tilde{g}}^1(S) \cdot V(f))_{\mathbf{0}} - (\Gamma_{f,\tilde{g}}^1(S) \cdot V(g))_{\mathbf{0}}$.

Remark 2.5. Note that the two dimension hypotheses of Definition 2.4 are satisfied when \tilde{g} is a generic linear form (see [21], Proposition 3.13). Also, by the work of Hamm, Lê, and Teissier, if \mathfrak{l} is a generic linear form, then $\Gamma_{f,\mathfrak{l}}^1(S)$ is reduced and $(\Gamma_{f,\mathfrak{l}}^1(S) \cdot V(\mathfrak{l}))_{\mathbf{0}}$ is the multiplicity of $\Gamma_{f,\mathfrak{l}}^1(S)$ at the origin; thus, $\delta_{f,\tilde{g}}(S) \geq 0$, when \tilde{g} is a generic linear form.

3 The Generic Linear Form Case

We now prove our first main theorem:

Theorem 3.1. Let \mathfrak{l} be a linear form, which is generic with respect to the fixed f and \mathfrak{S} . For all $S \in \mathfrak{S}_{\mathbf{0}}^f(\mathbf{A}^\bullet)$, let $\delta_S := (\Gamma_{f,\mathfrak{l}}^1(S) \cdot V(f))_{\mathbf{0}} - (\Gamma_{f,\mathfrak{l}}^1(S) \cdot V(\mathfrak{l}))_{\mathbf{0}}$.

Then,

$$c_{\mathbf{0}}(\phi_f[-1]\mathbf{A}^\bullet) = (-1)^{\dim X - \dim V(f)} \left(c_{\mathbf{0}}(\mathbf{A}^\bullet) + \sum_{S \in \mathfrak{S}_{\mathbf{0}}^f(\mathbf{A}^\bullet)} \delta_S \cdot c_S(\mathbf{A}^\bullet) \right).$$

In fact, for all $k \in \mathbb{Z}$,

$$H^k(\phi_{\mathfrak{l}}[-1]\phi_f[-1]\mathbf{A}^\bullet)_{\mathbf{0}} \cong H^k(\phi_{\mathfrak{l}}[-1]\mathbf{A}^\bullet)_{\mathbf{0}} \oplus \bigoplus_{S \in \mathfrak{S}_{\mathbf{0}}^f(\mathbf{A}^\bullet)} (m_S^k(\mathbf{A}^\bullet) \otimes R^{\delta_S}).$$

Proof. The statement about the coefficients in the characteristic cycle is proved in Corollary 4.6 of [17] (the last formula in the statement); however, it also follows quickly from Theorems 3.3 and 5.5 of [7] and Theorem 3.4.2 of [3].

If the base ring R is a field, the stalk cohomology statement follows at once by combining the characteristic cycle coefficient formula with Lemma 2.3 of [18], where we used perverse cohomology to extract individual

Betti numbers from characteristic cycle formulas. That is, in the first formula of the above theorem, we replace \mathbf{A}^\bullet by the perverse cohomology ${}^\mu H^k(\mathbf{A}^\bullet)$ (see [1], [10], or, for a quick summary, [18]), and use that

$$CC({}^\mu H^k(\mathbf{A}^\bullet)) = (-1)^{\dim X} \sum_{S \in \mathfrak{S}} b_{k-d_S}(\mathbb{N}_S, \mathbb{L}_S; \mathbf{A}^\bullet) [\overline{T_S^* \mathcal{U}}],$$

where $b_j(\mathbb{N}_S, \mathbb{L}_S; \mathbf{A}^\bullet)$ denotes the j -th relative Betti number, i.e., $b_j(\mathbb{N}_S, \mathbb{L}_S; \mathbf{A}^\bullet) := \dim_R \mathbb{H}^j(\mathbb{N}_S, \mathbb{L}_S; \mathbf{A}^\bullet)$.

For general R , the isomorphism is obtained by “enriching” the proof of Lemma 3.7 in [19]. That is, one uses precisely the proof of Lemma 3.7, except that one replaces cycles by enriched cycles and ordinary intersection theory by enriched intersection theory; such enriched proofs are discussed in [20].

We will also recover the given isomorphism as a special case of Theorem 4.1 of this paper, which has an independent proof. \square

Remark 3.2. Suppose that \tilde{f} is itself a linear form and that, at the origin, for generic linear \mathfrak{l} , $|\Gamma_{f,\mathfrak{l}}(\mathbf{A}^\bullet)|$ is a collection of lines. Then, for generic linear \mathfrak{l} , for all $S \in \mathfrak{S}_0^f(\mathbf{A}^\bullet)$, $\delta_S = 0$, and so $H^k(\phi_{\mathfrak{l}}[-1]\phi_f[-1]\mathbf{A}^\bullet)_{\mathbf{0}} \cong H^k(\phi_{\mathfrak{l}}[-1]\mathbf{A}^\bullet)_{\mathbf{0}}$ and $c_0(\phi_f[-1]\mathbf{A}^\bullet) = (-1)^{\dim X - \dim V(f)} c_0(\mathbf{A}^\bullet)$. Thus, we recover Theorem 1 of [2] (with a different convention on the signs of the characteristic cycle).

4 The General Main Theorem

We continue using all of the notation from the previous two sections.

Suppose that $\dim_{\mathbf{0}} |\Gamma_{f,\tilde{g}}(\mathbf{A}^\bullet)| \leq 1$. If $S \in \mathfrak{S}_0^f(\mathbf{A}^\bullet)$, we let $|\widehat{\Gamma}_{f,\tilde{g}}(S)|$ denote the union of the components of $|\Gamma_{f,\tilde{g}}(S)|$ which contain the origin, but are not contained in $V(g)$; let $\widehat{\Gamma}_{f,\tilde{g}}^1(S)$ denote the corresponding cycle. Let $|\widehat{\Gamma}_{f,\tilde{g}}(\mathbf{A}^\bullet)| := \bigcup_{S \in \mathfrak{S}_0^f(\mathbf{A}^\bullet)} |\widehat{\Gamma}_{f,\tilde{g}}(S)|$. Note that Lemma 3.10 of [21] implies that no component of $|\widehat{\Gamma}_{f,\tilde{g}}(\mathbf{A}^\bullet)|$ is contained in $V(f)$.

Throughout this section, C will denote a possibly non-reduced component of the cycle $\widehat{\Gamma}_{f,\tilde{g}}^1(\mathbf{A}^\bullet)$; we will write $|C|$ for the underlying analytic set of C (i.e., C with its reduced structure). Thus,

$$C = \left(\sum_{S \in \mathfrak{S}^f(\mathbf{A}^\bullet)} p_{|C|}(S) \right) |C|,$$

where $p_{|C|}(S)$ is as in Definition 2.2.

For all $S \in \mathfrak{S}(\mathbf{A}^\bullet)$, define

$$\hat{\delta}_{f,\tilde{g}}(S) := \sum_{C \text{ comp. of } \widehat{\Gamma}_{f,\tilde{g}}^1(S)} \left((C \cdot V(f))_{\mathbf{0}} - \min \{ (C \cdot V(f))_{\mathbf{0}}, (C \cdot V(g))_{\mathbf{0}} \} \right).$$

Note that, in the formula above, if $\mathbf{0}$ is not in the closure of S , then $\hat{\delta}_{f,\tilde{g}}(S)$ is automatically equal to 0.

We now prove a significant generalization of Theorem 3.1. Our method is to analyze the situation using the discriminant and Cerf diagram, the same method used first by Lê in [12], and subsequently in many works by various authors, such as Lê and Perron in [15], Tibăr in [26], [27], and [28], and Caubel in his thesis [5] and [6]. In particular, the recent related work of Tibăr in [28], and the “tilting” in the Cerf diagram that we use and the study of pencils of the form $f + \lambda g$ is closely related to work of Caubel in his thesis [5] and in [6].

Theorem 4.1. *Suppose that $\dim_{\mathbf{0}} V(f) \cap |\Gamma_{f,\bar{g}}(\mathbf{A}^\bullet)| \leq 0$.*

Then, for all $\lambda \in \mathbb{C}$ such that λ is non-zero and $|\lambda|$ is sufficiently small, for all $k \in \mathbb{Z}$,

$$H^k(\phi_g[-1]\phi_f[-1]\mathbf{A}^\bullet)_{\mathbf{0}} \cong H^k(\phi_{f+\lambda g}[-1]\mathbf{A}^\bullet)_{\mathbf{0}} \oplus \bigoplus_{S \in \mathfrak{S}^f(\mathbf{A}^\bullet)} (m_S^k(\mathbf{A}^\bullet) \otimes R^{\hat{\delta}_{f,\bar{g}}(S)}).$$

Proof. Suppose that $0 < a \ll b \ll \epsilon \ll 1$. We first show that

$$(*) \quad H^k(\phi_g[-1]\phi_f[-1]\mathbf{A}^\bullet)_{\mathbf{0}} \cong \mathbb{H}^k(B, B \cap (g^{-1}([b, \infty)) \cup f^{-1}([a, \infty))); \mathbf{A}^\bullet) \cong \mathbb{H}^k(B, B \cap (g^{-1}(b) \cup f^{-1}(a))); \mathbf{A}^\bullet).$$

Actually, we will demonstrate one of the isomorphisms; the proof of the other is obtained by homotoping $[a, \infty)$ to a and $[b, \infty)$ to b (or by replacing a with $[a, \infty)$ and b with $[b, \infty)$ in the proof below).

Recall some equivalent characterizations of the vanishing cycles. Suppose that Y is a complex analytic subspace of \mathcal{U} , that $h : Y \rightarrow \mathbb{C}$ is complex analytic, and that \mathbf{F}^\bullet is a bounded, constructible complex of R -modules on Y . Consider the inclusions $m : h^{-1}(0) \hookrightarrow h^{-1}([0, \infty))$, and $l : h^{-1}([0, \infty)) \hookrightarrow Y$. Then, up to isomorphism, the shifted vanishing cycles $\phi_h[-1]\mathbf{F}^\bullet$ are given by $m^!l^*\mathbf{F}^\bullet$. Equivalently, if we let $Z := \{y \in Y \mid \operatorname{Re}(h(y)) \leq 0\}$, then there is an isomorphism $\phi_h[-1]\mathbf{F}^\bullet \cong (R\Gamma_Z(\mathbf{F}^\bullet))|_{h^{-1}(0)}$. See [10], Exer. VIII.13 and [19], Appendix B, §3. These yield the well-known “formula”, which we referred to in the Introduction, for the stalk cohomology of $\phi_h[-1]\mathbf{F}^\bullet$: if $y \in V(h)$, then, for all k ,

$$H^k(\phi_h[-1]\mathbf{F}^\bullet)_y \cong \mathbb{H}^k(B_\epsilon(y) \cap Y, B_\epsilon(y) \cap Y \cap h^{-1}(b); \mathbf{F}^\bullet),$$

where $0 < b \ll \epsilon \ll 1$.

Thus, if $\gamma : B_\epsilon(y) \cap Y - B_\epsilon(y) \cap Y \cap h^{-1}(b) \hookrightarrow B_\epsilon(y) \cap Y$ is the inclusion map, then

$$H^k(\phi_h[-1]\mathbf{F}^\bullet)_y \cong \mathbb{H}^k(B_\epsilon(y) \cap Y; \gamma_! \gamma^! \mathbf{F}^\bullet).$$

Now, let us apply this to the case of iterated vanishing cycles. It follows from the discussion above that $H^k(\phi_g[-1]\phi_f[-1]\mathbf{A}^\bullet)_{\mathbf{0}}$ is isomorphic to

$$(\dagger) \quad \mathbb{H}^k(B_\epsilon(\mathbf{0}) \cap V(f), B_\epsilon(\mathbf{0}) \cap V(f) \cap g^{-1}(b); \phi_f[-1]\mathbf{A}^\bullet),$$

where $0 < b \ll \epsilon \ll 1$.

Suppose that $0 < a \ll b \ll \epsilon \ll 1$. Let $\beta : B - B \cap g^{-1}(b) \hookrightarrow B$ and $\alpha : B - B \cap f^{-1}(a) \hookrightarrow B$ be the inclusion maps. Let $B := B_\epsilon(\mathbf{0}) \cap X$. Then, (\dagger) is isomorphic to the “hypercohomology of the pair of pairs”

$$(\ddagger) \quad \mathbb{H}^k((B, B \cap f^{-1}(a)), (B \cap g^{-1}(b), B \cap g^{-1}(b) \cap f^{-1}(a)); \mathbf{A}^\bullet) := \mathbb{H}^k(B; \beta_! \beta^! \alpha_! \alpha^! \mathbf{A}^\bullet).$$

Let $\hat{\alpha} : B - B \cap (g^{-1}(b) \cup f^{-1}(a)) \hookrightarrow B - B \cap g^{-1}(b)$ and let $\hat{\beta} : B - B \cap (g^{-1}(b) \cup f^{-1}(a)) \hookrightarrow B - B \cap f^{-1}(a)$. Note that $\alpha, \beta, \hat{\alpha}$, and $\hat{\beta}$ are all open inclusions, so that there are natural isomorphisms $\alpha^* \cong \alpha^!$, $\beta^* \cong \beta^!$, $\hat{\alpha}^* \cong \hat{\alpha}^!$, and $\hat{\beta}^* \cong \hat{\beta}^!$. In addition, because $\alpha, \beta, \hat{\alpha}$, and $\hat{\beta}$ form a pull-back diagram, $\beta^* \alpha_! \cong \hat{\alpha}_! \hat{\beta}^*$.

Therefore,

$$\begin{aligned} H^k(\phi_g[-1]\phi_f[-1]\mathbf{A}^\bullet)_{\mathbf{0}} &\cong \mathbb{H}^k(B; \beta_! \beta^! \alpha_! \alpha^! \mathbf{A}^\bullet) \cong \mathbb{H}^k(B; \beta_! \beta^* \alpha_! \alpha^! \mathbf{A}^\bullet) \cong \mathbb{H}^k(B; \beta_! \hat{\alpha}_! \hat{\beta}^* \alpha^! \mathbf{A}^\bullet) \cong \\ &\mathbb{H}^k(B; (\beta \circ \hat{\alpha})_! \hat{\beta}^! \alpha^! \mathbf{A}^\bullet) \cong \mathbb{H}^k(B; (\beta \circ \hat{\alpha})_! (\alpha \circ \hat{\beta})^! \mathbf{A}^\bullet) = \mathbb{H}^k(B; (\beta \circ \hat{\alpha})_! (\beta \circ \hat{\alpha})^! \mathbf{A}^\bullet) \cong \end{aligned}$$

$$\mathbb{H}^k(B, B \cap (g^{-1}(b) \cup f^{-1}(a)); \mathbf{A}^\bullet),$$

as we claimed in (*).

Now, we proved in Theorem 4.13 of [21], the hypothesis that $\dim_{\mathbf{0}} V(f) \cap |\Gamma_{f,\tilde{g}}(\mathbf{A}^\bullet)| \leq 0$ implies that $(\mathbf{A}^\bullet)|_B$ pushes forward “nicely” by the restriction, J , of (g, f) to B . Let $\Delta := J(|\widehat{\Gamma}_{f,\tilde{g}}(\mathbf{A}^\bullet)|)$.

Let u and v be coordinates on \mathbb{C}^2 . Then, the result of Theorem 4.13 of [21] is that, in the interior P° of a polydisk $P := \mathbb{D}_\sigma \times \mathbb{D}_\tau$ around the origin in \mathbb{C}^2 , where $0 < \tau \ll \sigma \ll 1$, $\mathbf{F}^\bullet := RJ_*((\mathbf{A}^\bullet)|_B)$ is complex analytically constructible with respect to the stratification given by $P^\circ - \Delta - V(uv)$, $P^\circ \cap V(u) - \{\mathbf{0}\}$, $P^\circ \cap V(v) - \{\mathbf{0}\}$, $\{\mathbf{0}\}$, and the connected components of $P^\circ \cap \Delta - \{\mathbf{0}\}$. In addition, the inclusion of P° into P induces an isomorphism $\mathbb{H}^k(P; \mathbf{F}^\bullet) \cong \mathbb{H}^k(P^\circ; \mathbf{F}^\bullet)$. More precisely, the boundary strata introduced by replacing P° with P are irrelevant in our discussion below.

This is the derived category version of the Cerf diagram set-up. The main difference between the classical situation and our current situation is that now the components of the Cerf diagram are not necessarily tangent to either of the axes. The point of the Cerf diagram is that many quantities in which we are interested “upstairs” are easier to view geometrically “downstairs”.

Combining our choices in the paragraphs above, let a, b, τ , and σ be such that $0 < |a| < \tau \ll |b| < \sigma \ll \epsilon \ll 1$. Consider the isomorphisms

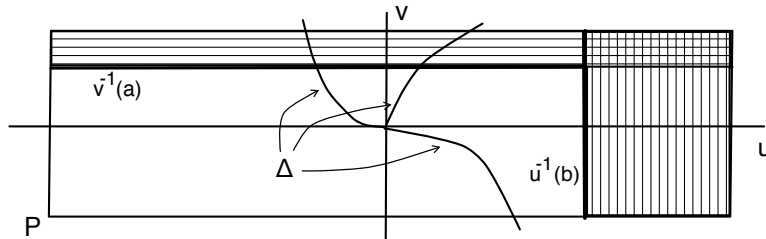
$$H^k(\phi_g[-1]\phi_f[-1]\mathbf{A}^\bullet)_0 \cong \mathbb{H}^k(B, B \cap (g^{-1}(b) \cup f^{-1}(a)); \mathbf{A}^\bullet) \cong \mathbb{H}^k(P, P \cap (u^{-1}(b) \cup v^{-1}(a)); \mathbf{F}^\bullet),$$

where the first isomorphism is what we showed above, and the second isomorphism is the canonical push-forward isomorphism.

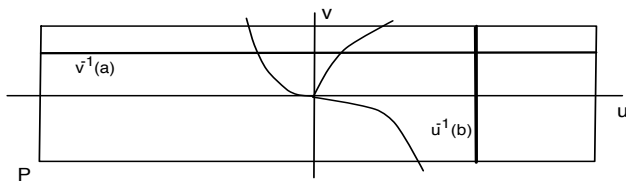
We will use a hypercohomology splitting lemma to show that $\mathbb{H}^k(P, P \cap (u^{-1}(b) \cup v^{-1}(a)); \mathbf{F}^\bullet)$ decomposes as a direct sum. This splitting lemma is Lemma 2.2 of [17], and is a hypercohomology version of a splitting argument used by Siersma in [24]. We will show that there is a map θ from P to \mathbb{D}_σ which is cohomologically trivial except over a finite number of stratified critical values and that the corresponding stratified critical points are the stratified critical points of $v + \lambda u$, where $\lambda \in \mathbb{C}$ is such that $0 < |\lambda| \ll |a|$.

We will not, however, describe the map θ via formulas. Rather, we will draw pictures of typical fibers of θ in the discriminant, and the u -value of the point of intersection of each fiber with the u -axis will be the value of θ on that fiber. Of course, our pictures are actually in \mathbb{R}^2 , but should be thought of as representing the situation in \mathbb{C}^2 .

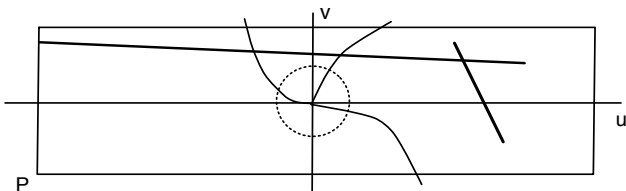
We begin with the following diagram, in which we indicate the local triviality of θ in a neighborhood of $P \cap (u^{-1}(b) \cup v^{-1}(a))$.



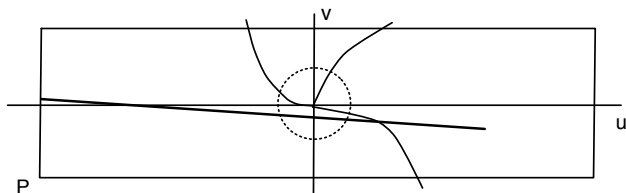
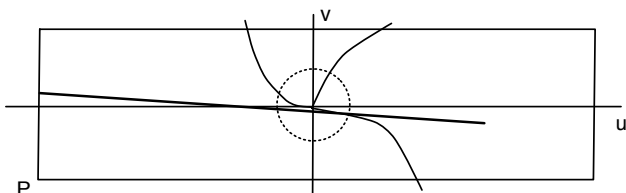
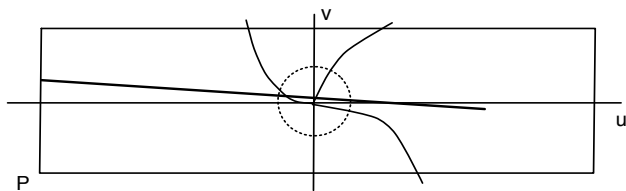
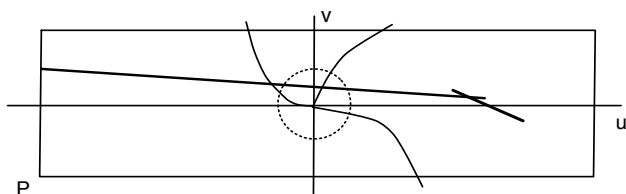
The fiber $\theta^{-1}(b)$ is shown in the following diagram.

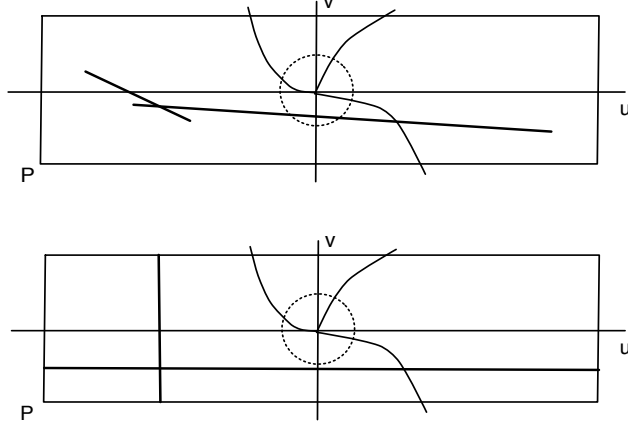


We “tilt” the line segments $P \cap v^{-1}(a)$ and $P \cap u^{-1}(b)$. The nearly horizontal line segment below is a portion of $(v + \lambda u)^{-1}(a)$ for a non-zero value of λ of small magnitude. We fix such a λ .



We consider typical fibers of θ in the following diagrams.





What these diagrams show is that $H^k(\phi_g[-1]\phi_f[-1]\mathbf{A}^\bullet)_\mathbf{0} \cong \mathbb{H}^k(P, P \cap (u^{-1}(b) \cup v^{-1}(a)); \mathbf{F}^\bullet)$ is isomorphic to the direct sum of the vanishing cycles of \mathbf{F}^\bullet along $v + \lambda u - r_i$, where $\lambda \in \mathbb{C}$ is sufficiently small in magnitude, the sum is over the isolated (stratified) critical points p_i of $v + \lambda u$ restricted to $P \cap \Delta$, and $r_i := (v + \lambda u)(p_i)$.

The contribution at the origin to this direct sum is precisely $H^k(\phi_{v+\lambda u}[-1]\mathbf{F}^\bullet)_\mathbf{0} \cong H^k(\phi_{f+\lambda g}[-1]\mathbf{A}^\bullet)_\mathbf{0}$. What remains for us to do is to “count” the contribution from the critical points of $v + \lambda u$ on $\Delta - \{\mathbf{0}\}$. We need to know two things for each irreducible component E of Δ : the Morse module $m_E^k(\mathbf{F}^\bullet) := m_{E-\{\mathbf{0}\}}^k(\mathbf{F}^\bullet)$ of $E - \{\mathbf{0}\}$ with respect to \mathbf{F}^\bullet , and the number b_E (counted with multiplicity) of critical points of $v + \lambda u$ restricted to $E - \{\mathbf{0}\}$. In terms of $m_E^k(\mathbf{F}^\bullet)$ and b_E , what we showed above is that

$$(\star) \quad H^k(\phi_g[-1]\phi_f[-1]\mathbf{A}^\bullet)_\mathbf{0} \cong H^k(\phi_{f+\lambda g}[-1]\mathbf{A}^\bullet)_\mathbf{0} \oplus \bigoplus_E (m_E^k(\mathbf{F}^\bullet) \otimes R^{b_E}),$$

where the sum is over the irreducible components E of Δ .

We first calculate b_E . Let E be an irreducible component of Δ , complex analytically parameterized by $\mathbf{p}(t)$, where $\mathbf{p}(0) = \mathbf{0}$. Inside P , the number of critical points of v restricted to E is equal to the number of zeroes, counted with multiplicity of $(v(\mathbf{p}(t)))'$; the only zero is located at the origin and its multiplicity is $\text{mult}_\mathbf{0} (v(\mathbf{p}(t)))' = -1 + \text{mult}_\mathbf{0} v(\mathbf{p}(t))$. Thus, for small λ , the number of critical points of $v + \lambda u$ restricted to E , inside P , is also $-1 + \text{mult}_\mathbf{0} v(\mathbf{p}(t))$; however, the multiplicity of the origin is now

$$-1 + \text{mult}_\mathbf{0}(v + \lambda u)(\mathbf{p}(t)) = -1 + \min \{ \text{mult}_\mathbf{0} v(\mathbf{p}(t)), \text{mult}_\mathbf{0} u(\mathbf{p}(t)) \}.$$

Thus,

$$b_E = (-1 + \text{mult}_\mathbf{0} v(\mathbf{p}(t))) - (-1 + \min \{ \text{mult}_\mathbf{0} v(\mathbf{p}(t)), \text{mult}_\mathbf{0} u(\mathbf{p}(t)) \}) = (E \cdot V(v))_\mathbf{0} - \min \{ (E \cdot V(v))_\mathbf{0}, (E \cdot V(u))_\mathbf{0} \}.$$

Now, we need to describe the Morse modules of \mathbf{F}^\bullet for the connected components of $\Delta - \{\mathbf{0}\}$. Let E be an irreducible component of Δ . Then, there exists an $S \in \mathfrak{S}^f(\mathbf{A}^\bullet)$ (which need not be unique) and an irreducible component $|C|$ of $|\widehat{\Gamma}_{f,\bar{g}}(S)|$ such that $J(|C|) = E$; let $d_{|C|}$ denote the degree of J restricted to $|C|$. By the definition of \mathbf{F}^\bullet and Δ , it follows at once that, for all k , the degree k Morse module of the stratum $E - \{\mathbf{0}\}$, which we denote simply by $m_E^k(\mathbf{F}^\bullet)$, is given by

$$m_E^k(\mathbf{F}^\bullet) \cong \bigoplus_{S \in \mathfrak{S}^f(\mathbf{A}^\bullet)} (m_S^k(\mathbf{A}^\bullet) \otimes R^{q_E(S)}),$$

where

$$q_E(S) := \sum_{\substack{|C| \text{ comp. of } |\widehat{\Gamma}_{f,\bar{g}}(S)| \\ J(|C|)=E}} d_{|C|} \cdot p_{|C|}(S).$$

Therefore, the term $\bigoplus_E (m_E^k(\mathbf{F}^\bullet) \otimes R^{b_E})$ is (\star) is isomorphic to

$$\bigoplus_E \left(\left(\bigoplus_{S \in \mathfrak{S}^f(\mathbf{A}^\bullet)} (m_S^k(\mathbf{A}^\bullet) \otimes R^{q_E(S)}) \right) \otimes R^{b_E} \right) \cong \bigoplus_{S \in \mathfrak{S}^f(\mathbf{A}^\bullet)} \bigoplus_E (m_S^k(\mathbf{A}^\bullet) \otimes R^{q_E(S)b_E}).$$

We claim that, for fixed $S \in \mathfrak{S}^f(\mathbf{A}^\bullet)$, $\hat{\delta}_{f,\bar{g}}(S) = \sum_E q_E(S)b_E$, which would finish the proof.

This follows from the proper push-forward formula. For each component C of $\widehat{\Gamma}_{f,\bar{g}}^1(S)$, the proper push-forward $J_*(C) = d_{|C|} \cdot p_{|C|}(S)J(|C|)$, and, hence,

$$\begin{aligned} \sum_E q_E(S)b_E &= \sum_E \sum_{\substack{|C| \text{ comp. of } |\widehat{\Gamma}_{f,\bar{g}}(S)| \\ J(|C|)=E}} d_{|C|} \cdot p_{|C|}(S) \left((E \cdot V(v))_{\mathbf{0}} - \min \{ (E \cdot V(v))_{\mathbf{0}}, (E \cdot V(u))_{\mathbf{0}} \} \right) = \\ &= \sum_{C \text{ comp. of } \widehat{\Gamma}_{f,\bar{g}}^1(S)} \left((J_*(C) \cdot V(v))_{\mathbf{0}} - \min \{ (J_*(C) \cdot V(v))_{\mathbf{0}}, (J_*(C) \cdot V(u))_{\mathbf{0}} \} \right) = \\ &= \sum_{C \text{ comp. of } \widehat{\Gamma}_{f,\bar{g}}^1(S)} \left((C \cdot V(f))_{\mathbf{0}} - \min \{ (C \cdot V(f))_{\mathbf{0}}, (C \cdot V(g))_{\mathbf{0}} \} \right) = \hat{\delta}_{f,\bar{g}}(S). \end{aligned}$$

□

Remark 4.2. It is an important part of Theorem 4.1 that the amount of genericity that we need for g is precisely that $\dim_{\mathbf{0}} V(f) \cap |\Gamma_{f,\bar{g}}(\mathbf{A}^\bullet)| \leq 0$. This dimension condition can, in fact, be checked in practice. In particular, we do **not** need for the origin to be an isolated point in the support of $\phi_{f+\lambda g}[-1]\mathbf{A}^\bullet$. However, if the origin is, in fact, an isolated point in the support of $\phi_{f+\lambda g}[-1]\mathbf{A}^\bullet$, then its stalk cohomology at the origin can be calculated via Theorem 1.2.

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