

# A Strong Łojasiewicz Inequality and Real Analytic Milnor Fibrations\*

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## Abstract

We give a strong version of a classic inequality of Łojasiewicz; one which collapses to the usual inequality in the complex analytic case. We show that this inequality for a pair of real analytic functions allows us to construct a real Milnor fibration.

## 1 Introduction

Throughout this paper,  $\mathcal{U}$  denotes a connected, open neighborhood of the origin in  $\mathbb{R}^N$ , and  $p$  denotes a point in  $\mathcal{U}$ .

There is a classic inequality of Łojasiewicz [8] (see also [1]).

**Theorem 1.1.** *Let  $f : \mathcal{U} \rightarrow \mathbb{R}$  be real analytic. Then, there exists an open neighborhood  $\mathcal{W}$  of  $p$  in  $\mathcal{U}$ , and  $c, \theta \in \mathbb{R}$  such that  $c > 0$ ,  $0 < \theta < 1$ , and, for all  $x \in \mathcal{W}$ ,*

$$|f(x) - f(p)|^\theta \leq c |\nabla f(x)|,$$

where  $\nabla f$  is the gradient vector.

**Remark 1.2.** The phrasing above is classical, and convenient in some arguments. However, one can also fix the value of the constant  $c$  above to be any  $c > 0$ , e.g.,  $c = 1$ . In other words, one may remove the reference to  $c$  in the statement Theorem 1.1 and simply use the inequality

$$|f(x) - f(p)|^\theta \leq |\nabla f(x)|.$$

To see this, suppose that one has a  $\mathcal{W}$ ,  $c$ , and  $\theta$  as in the theorem. Let  $\theta'$  be such that  $\theta < \theta' < 1$ . We need to show that there exists  $\mathcal{W}'$ , a (connected) open neighborhood of  $p$ , such that, for all  $x \in \mathcal{W}'$  such that  $\nabla f(x) \neq 0$ ,  $|f(x) - f(p)|^{\theta'} / |\nabla f(x)| \leq 1$ .

This is trivial, since

$$\frac{|f(x) - f(p)|^{\theta'}}{|\nabla f(x)|} = \frac{|f(x) - f(p)|^{\theta'}}{|f(x) - f(p)|^\theta} \cdot \frac{|f(x) - f(p)|^\theta}{|\nabla f(x)|}.$$

As  $x$  approaches  $p$ , through  $x$ 's such that  $f(x) \neq f(p)$ , the left factor above approaches zero, while the absolute value of the right factor above is bounded by  $c$ . The desired conclusion follows.

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Theorem 1.1 implies a well-known complex analytic version of itself. One can easily obtain this complex version by replacing  $f$  by the square of the norm of the complex analytic function. However, we shall prove a not so well-known more general corollary, which immediately yields the complex analytic statement.

Let  $g$  and  $h$  be real analytic functions from  $\mathcal{U}$  to  $\mathbb{R}$ , and let  $f := (g, h) : \mathcal{U} \rightarrow \mathbb{R}^2$ .

**Corollary 1.3.** *There exists an open neighborhood  $\mathcal{W}$  of  $p$  in  $\mathcal{U}$ , and  $c, \theta \in \mathbb{R}$  such that  $c > 0$ ,  $0 < \theta < 1$ , and, for all  $x \in \mathcal{W}$ ,*

$$|f(x) - f(p)|^\theta \leq c\sqrt{|\nabla g(x)|^2 + |\nabla h(x)|^2}.$$

**Proof.** For convenience, we assume that  $f(p) = 0$ . We will prove that there exists  $\mathcal{W}$ ,  $c$ , and  $\theta$  as in the statement such that the inequality holds at all  $x \in \mathcal{W}$  such that  $f(x) \neq 0$ . This clearly suffices to prove the corollary. We will also assume that  $|\nabla g(p)|^2 + |\nabla h(p)|^2 = 0$  for, otherwise, the result is trivial.

Apply Theorem 1.1 to the function from  $\mathcal{U} \times \mathcal{U}$  to  $\mathbb{R}$  given by  $g^2(x) + h^2(w)$ . We conclude that there exists an open neighborhood  $\mathcal{W}$  of  $p$  in  $\mathcal{U}$ , and  $c, \theta \in \mathbb{R}$  such that  $c > 0$ ,  $0 < \theta < 1$ , and, for all  $(x, w) \in \mathcal{W} \times \mathcal{W}$ ,

$$|g^2(x) + h^2(w)|^\theta \leq c|2g(x)(\nabla g(x), 0) + 2h(w)(0, \nabla h(w))|.$$

Restricting to the diagonal, we obtain that, for all  $x \in \mathcal{W}$ ,

$$|g^2(x) + h^2(x)|^\theta \leq 2c\sqrt{g^2(x)|\nabla g(x)|^2 + h^2(x)|\nabla h(x)|^2}.$$

Applying the Cauchy-Schwarz inequality, we have, for all  $x \in \mathcal{W}$ ,

$$|g^2(x) + h^2(x)|^\theta \leq 2c\sqrt{|(g^2(x), h^2(x))| \cdot (|\nabla g(x)|^2 + |\nabla h(x)|^2)} = 2c(g^4(x) + h^4(x))^{1/4} (|\nabla g(x)|^4 + |\nabla h(x)|^4)^{1/4}.$$

For  $a, b, x > 0$ ,  $(a^x + b^x)^{1/x}$  is a decreasing function of  $x$ . Therefore, we conclude that, for all  $x \in \mathcal{W}$ ,

$$|(g(x), h(x))|^{2\theta} = |g^2(x) + h^2(x)|^\theta \leq 2c(g^2(x) + h^2(x))^{1/2} (|\nabla g(x)|^2 + |\nabla h(x)|^2)^{1/2},$$

and so, for all  $x \in \mathcal{W}$  such that  $f(x) \neq 0$ ,

$$|f(x)|^{2\theta-1} \leq 2c\sqrt{|\nabla g(x)|^2 + |\nabla h(x)|^2}.$$

This proves the result, except for the bounds on the exponent. As  $-1 < 2\theta - 1 < 1$ , we have only to eliminate the possibility that  $-1 < 2\theta - 1 \leq 0$ . However, this is immediate, as we are assuming that  $f(p) = 0$  and  $|\nabla g(p)|^2 + |\nabla h(p)|^2 = 0$ .  $\square$

The following corollary is immediate.

**Corollary 1.4.** *Suppose that, for all  $x \in \mathcal{U}$ ,  $\nabla g(x)$  and  $\nabla h(x)$  have the same magnitude.*

*Then, there exists an open neighborhood  $\mathcal{W}$  of  $p$  in  $\mathcal{U}$ , and  $c, \theta \in \mathbb{R}$  such that  $c > 0$ ,  $0 < \theta < 1$ , and, for all  $x \in \mathcal{W}$ ,*

$$|f(x) - f(p)|^\theta \leq c|\nabla g(x)| = c|\nabla h(x)|.$$

If  $N$  is even, say  $N = 2n$ , then we may consider the complexified version of  $f$  by defining  $f_c$  by

$$f_c(x_1 + iy_1, \dots, x_n + iy_n) := g(x_1, y_1, \dots, x_n, y_n) + ih(x_1, y_1, \dots, x_n, y_n).$$

Corollary 1.4 immediately yields:

**Corollary 1.5.** *Suppose that  $f_c$  is complex analytic. Then, there exists an open neighborhood  $\mathcal{W}$  of  $p$  in  $\mathcal{U} \subseteq \mathbb{C}^n$ , and  $c, \theta \in \mathbb{R}$  such that  $c > 0$ ,  $0 < \theta < 1$ , and, for all  $z \in \mathcal{W}$ ,*

$$|f_c(z) - f_c(p)|^\theta \leq c |\nabla f_c(z)|,$$

where  $\nabla f_c$  denotes the complex gradient.

The Lojasiewicz inequality is a useful tool in many arguments. Of particular importance in this paper is the use of Corollary 1.5 by Hamm and Lê in [4] to prove the existence of Thom stratifications in the complex analytic case. As Lê showed in [7], the existence of a Thom (or  $a_f$ ) stratification can be used to show that the Milnor fibration (inside a ball, not on a sphere) exists.

In proofs and some statements, it will frequently be convenient to assume that our point  $p$  is the origin, that  $f(\mathbf{0}) = \mathbf{0}$ , and that  $f \neq 0$ , i.e.,  $X := V(f) := f^{-1}(\mathbf{0})$  is not all of  $\mathcal{U}$  (which, by the principle of analytic continuation, is the same as assuming that  $X$  does not contain an open neighborhood of  $\mathbf{0}$ ).

We let  $\mathbf{r} := (r_1, \dots, r_N)$  denote the coordinate functions on  $\mathcal{U}$ . We write  $\Sigma f$  for the critical locus of  $f$ , by which we mean the set of points where  $f$  is not a submersion.

The now-classic work of Milnor in [10] deals primarily with the case where  $f_c$  is a complex analytic function, in which case, one has what is now called the Milnor fibration of  $f_c$  at  $p$ . The Milnor fibration is **the** fundamental device in the study of the topology of the hypersurface  $X$  defined by the vanishing of  $f_c$ .

In fact, there are two Milnor fibrations associated with complex analytic  $f_c$ : one defined on small spheres, and one defined inside small open balls. Both of these are referred to as **the** Milnor fibration because the two fibrations are diffeomorphic. We wish to be precise.

For  $\epsilon > 0$ , let  $S_\epsilon$  (resp.,  $B_\epsilon$ ,  $\overset{\circ}{B}_\epsilon$ ) denote the sphere (resp., closed ball, open ball) of radius  $\epsilon$ , centered at the origin. In the special case of balls in  $\mathbb{R}^2 \cong \mathbb{C}$ , we write  $\mathbb{D}_\epsilon$  in place of  $B_\epsilon$  and  $\overset{\circ}{\mathbb{D}}_\epsilon$  in place of  $\overset{\circ}{B}_\epsilon$ . Finally, we write  $\mathbb{D}_\epsilon^*$  in place of  $\mathbb{D}_\epsilon - \{\mathbf{0}\}$ , and  $\overset{\circ}{\mathbb{D}}_\epsilon^*$  in place of  $\overset{\circ}{\mathbb{D}}_\epsilon - \{\mathbf{0}\}$ .

One version of the Milnor fibration (at the origin) is given by: there exists  $\epsilon_0 > 0$  such that, for all  $\epsilon$  such that  $0 < \epsilon \leq \epsilon_0$ , the map  $f_c/|f_c| : S_\epsilon - S_\epsilon \cap X \rightarrow S^1 \subseteq \mathbb{C}$  is a smooth, locally-trivially fibration, whose diffeomorphism-type is independent of the choice of  $\epsilon$  (see [10]).

The second, diffeomorphic version of the Milnor fibration is given by: there exists  $\epsilon_0 > 0$  such that, for all  $\epsilon$  such that  $0 < \epsilon \leq \epsilon_0$ , there exists a  $\delta_0 > 0$  such that, for all  $\delta$  such that  $0 < \delta \leq \delta_0$ , the map  $f_c : \overset{\circ}{B}_\epsilon \cap f_c^{-1}(\partial\mathbb{D}_\delta) \rightarrow \partial\mathbb{D}_\delta \cong S^1 \subseteq \mathbb{C}$  is a smooth, locally-trivially fibration, whose diffeomorphism-type is independent of the choice of  $\epsilon$  and (sufficiently small)  $\delta$  (see Theorem 5.11 of [10] and [7]). The primary advantage to this second characterization of the Milnor fibration is that it compactifies nicely to yield a locally-trivial fibration  $f_c : B_\epsilon \cap f_c^{-1}(\partial\mathbb{D}_\delta) \rightarrow \partial\mathbb{D}_\delta$ , which, up to homotopy, is equivalent to either of the two previously-defined Milnor fibrations.

We shall not summarize the important properties of the Milnor fibration here, but refer the reader to [10], [2], [12], and the Introduction to [9]. We wish to emphasize again that our discussion of the Milnor fibration above assumes that  $f_c$  is a complex analytic function.

The question is: what can one say about “the” Milnor fibration of  $f$ , even if  $N$  is odd or  $f_c$  is not a complex analytic function?

In Chapter 11 of [10], Milnor discusses, fairly briefly, some results in the case of the real analytic function  $f = (g, h)$ . He considers the very special case where  $f$  has an isolated critical point at the origin, and shows that, while the restriction of  $f$  still yields a fibration over a small circle inside the ball,  $f/|f|$  does not necessarily yield the projection map of a fibration from  $S_\epsilon - S_\epsilon \cap X$  to  $S^1$ ; see p. 99 of [10].

There are basic questions which are of interest in the real case. Can one relax the condition that  $f$  has an isolated critical point and still obtain a locally-trivial fibration  $f : \circ B_\epsilon \cap f^{-1}(\partial\mathbb{D}_\delta) \rightarrow \partial\mathbb{D}_\delta$ ? Are there reasonable conditions that guarantee that  $f/|f| : S_\epsilon - S_\epsilon \cap X \rightarrow S^1$  is a locally-trivial fibration which is diffeomorphic to the fibration inside the ball? Such questions have been investigated by a number of researchers; see [10], [6], [11], [14], [13], and [3].

In this short paper, our goal is very modest. We will give a version of the Lojasiewicz inequality for  $f$ , one which reduces to the usual Lojasiewicz inequality in the case of complex analytic functions, and which implies the existence of the Milnor fibration inside the ball.

**Definition 1.6.** *We say that  $f = (g, h)$  satisfies the **strong Lojasiewicz inequality at  $p$**  or that  $f$  is **Lanalytic at  $p$**  if and only if there exists an open neighborhood  $\mathcal{W}$  of  $p$  in  $\mathcal{U}$ , and  $c, \theta \in \mathbb{R}$  such that  $c > 0$ ,  $0 < \theta < 1$ , and, for all  $x \in \mathcal{W}$ ,*

$$|f(x) - f(p)|^\theta \leq c \cdot \min_{|(a,b)=1} |a\nabla g(x) + b\nabla h(x)|.$$

Our main result, Corollary 4.4, is:

**Main Result.** *Suppose that  $f(\mathbf{0}) = \mathbf{0}$ , and that  $f \neq 0$ . Assume that  $f$  is Lanalytic at  $\mathbf{0}$ .*

*Then, for all  $0 < \delta \ll \epsilon \ll 1$ ,  $f : B_\epsilon \cap f^{-1}(\partial\mathbb{D}_\delta) \rightarrow \partial\mathbb{D}_\delta$  is a proper, surjective, stratified submersion, and so  $f : B_\epsilon \cap f^{-1}(\partial\mathbb{D}_\delta) \rightarrow \partial\mathbb{D}_\delta$  and  $f : \circ B_\epsilon \cap f^{-1}(\partial\mathbb{D}_\delta) \rightarrow \partial\mathbb{D}_\delta$  are locally-trivial fibrations.*

*Moreover, the diffeomorphism-type of  $f : \circ B_\epsilon \cap f^{-1}(\partial\mathbb{D}_\delta) \rightarrow \partial\mathbb{D}_\delta$  is independent of the appropriately small choices of  $\epsilon$  and  $\delta$ .*

## 2 Milnor's conditions (a) and (b)

We continue with our notation from the introduction.

In this section, we can expand our discussion to include arbitrary tuples of real analytic functions and, for convenience, restrict our attention to the origin. Thus, throughout this section, we let  $G := (g_1, \dots, g_k) : (\mathcal{U}, \mathbf{0}) \rightarrow (\mathbb{R}^k, \mathbf{0})$  be real analytic, and assume that  $G$  is not identically zero. Let  $X := G^{-1}(\mathbf{0}) = V(G)$ .

Let  $\mathfrak{A} = \Sigma G$  denote the closed, analytic set of points in  $\mathcal{U}$  at which the gradients  $\nabla g_1, \nabla g_2, \dots, \nabla g_k$  are linearly dependent. Let  $\rho$  denote the function given by the square of the distance from the origin, and let  $\mathfrak{B}$  denote the closed, analytic set of points in  $\mathcal{U}$  at which the gradients  $\nabla \rho, \nabla g_1, \nabla g_2, \dots, \nabla g_k$  are linearly dependent. Of course,  $\mathfrak{A} \subseteq \mathfrak{B}$ .

**Definition 2.1.** *We say that the map  $G$  satisfies **Milnor's condition (a) at  $\mathbf{0}$**  (or that  $\mathbf{0}$  is an **isolated critical value of  $G$  near  $\mathbf{0}$** ) if and only if  $\mathbf{0} \notin \overline{\mathfrak{A} - X}$ , i.e., if  $\Sigma G \subseteq V(G)$  near  $\mathbf{0}$ .*

*We say that the map  $G$  satisfies **Milnor's condition (b) at  $\mathbf{0}$**  (or the **pseudo- $a_G$  condition at  $\mathbf{0}$** ) if and only if  $\mathbf{0}$  is an isolated point of (or, is not in)  $X \cap \overline{\mathfrak{B} - X}$ .*

If  $G$  satisfies Milnor's conditions (a) and (b), then we say that  $\epsilon > 0$  is a **Milnor radius for  $G$  at  $\mathbf{0}$**  provided that  $B_\epsilon \cap (\overline{\mathfrak{A} - X}) = \emptyset$  and  $B_\epsilon \cap X \cap (\overline{\mathfrak{B} - X}) \subseteq \{\mathbf{0}\}$ .

**Remark 2.2.** Using our notation from the introduction, if  $f_c$  is complex analytic, then  $f = (g, h)$  satisfies Milnor's conditions (a) and (b). Condition (a) is well-known and follows easily from a curve selection argument. Condition (b) follows from the existence of good (or  $a_f$ ) stratifications of  $V(f)$ ; see [5] and [7].

Below,  $\circ B_{\delta_0}^*$  denotes  $\circ B_{\delta_0} - \{\mathbf{0}\}$  (in  $\mathbb{R}^k$ ).

**Theorem 2.3.** *Suppose that  $G$  satisfies Milnor's conditions (a) and (b), and let  $\epsilon$  be a Milnor radius for  $G$  at  $\mathbf{0}$ .*

*Then, there exists  $\delta_0 > 0$  such that  $G : B_\epsilon \cap G^{-1}(\circ B_{\delta_0}^*) \rightarrow \circ B_{\delta_0}^*$  is a surjective, smooth, proper, stratified submersion and, hence, a locally-trivial fibration. Hence,  $G : \circ B_\epsilon \cap G^{-1}(\circ B_{\delta_0}^*) \rightarrow \circ B_{\delta_0}^*$  is a locally-trivial fibration.*

*It follows that, for all  $\delta$  such that  $0 < \delta < \delta_0$ ,  $G : B_\epsilon \cap G^{-1}(\partial B_\delta) \rightarrow \partial B_\delta$  and  $G : \circ B_\epsilon \cap G^{-1}(\circ B_{\delta_0}^*) \rightarrow \circ B_{\delta_0}^*$  are locally-trivial fibrations.*

**Proof.** That there exists  $\delta_0 > 0$  such that  $G : B_\epsilon \cap G^{-1}(\circ B_{\delta_0}^*) \rightarrow \circ B_{\delta_0}^*$  is a smooth, proper, stratified submersion is immediate from Milnor's conditions (a) and (b); condition (a) guarantees that there are no interior critical points and condition (b) guarantees that there are no boundary critical points for sufficiently small  $\delta_0 > 0$ . By Thom's 1st isotopy lemma,  $G : B_\epsilon \cap G^{-1}(\circ B_{\delta_0}^*) \rightarrow \circ B_{\delta_0}^*$  is a locally-trivial fibration, in which the local trivializations can be chosen to respect the strata. Therefore,  $G : B_\epsilon \cap G^{-1}(\circ B_{\delta_0}^*) \rightarrow \circ B_{\delta_0}^*$  and  $G : \circ B_\epsilon \cap G^{-1}(\circ B_{\delta_0}^*) \rightarrow \circ B_{\delta_0}^*$  are locally trivial fibrations.

A locally trivial fibration, with a non-empty total space, over a connected base space, is surjective; our total space is not empty, since  $G$  cannot be identically zero on a neighborhood of the origin (by our assumption that  $G \not\equiv 0$  and analytic continuation).

That  $G : B_\epsilon \cap G^{-1}(\partial B_{\delta_0}) \rightarrow \partial B_{\delta_0}$  is a smooth, proper, stratified submersion is immediate. For fixed a fixed Milnor radius  $\epsilon$ , the diffeomorphism-type of this fibration is independent of the choice of  $\delta < \delta_0$  because  $\circ B_{\delta_0}^*$  is homotopy-equivalent to  $\partial B_{\delta_0}$ .  $\square$

**Corollary 2.4.** *Suppose that  $G$  satisfies Milnor's conditions (a) and (b). Then,  $G$  maps an open neighborhood of the origin onto an open neighborhood of the origin.*

**Proof.** Theorem 2.3 implies that  $G : \circ B_\epsilon \cap G^{-1}(\circ B_{\delta_0}^*) \rightarrow \circ B_{\delta_0}^*$  is surjective.  $\square$

**Definition 2.5.** *Suppose that  $G$  satisfies Milnor's conditions (a) and (b), and that  $\epsilon$  is a Milnor radius for  $G$  at  $\mathbf{0}$ . If  $\delta > 0$ , then  $(\epsilon, \delta)$  is a **Milnor pair for  $G$  at  $\mathbf{0}$**  if and only if there exists a  $\hat{\delta} > \delta$  such that  $G : B_\epsilon \cap G^{-1}(\circ B_{\hat{\delta}}^*) \rightarrow \circ B_{\hat{\delta}}^*$  is a surjective, stratified submersion (which is, of course, smooth and proper).*

**Remark 2.6.** Whenever we write that  $(\epsilon, \delta)$  is a Milnor pair for  $G$  at  $\mathbf{0}$ , we are assuming that  $G$  satisfies Milnor's conditions (a) and (b) at  $\mathbf{0}$ .

**Theorem 2.7.** *For all Milnor pairs  $(\epsilon, \delta)$  for  $G$  at  $\mathbf{0}$ , the diffeomorphism-types of the locally trivial fibrations  $G : B_\epsilon \cap G^{-1}(\partial B_\delta) \rightarrow \partial B_\delta$  and  $G : \circ B_\epsilon \cap G^{-1}(\partial B_\delta) \rightarrow \partial B_\delta$  are independent of the Milnor pair.*

**Proof.** Suppose that  $(\epsilon_1, \delta_1)$  and  $(\epsilon_2, \delta_2)$  are Milnor pairs for  $G$ . If  $\epsilon_1 = \epsilon_2$ , then the desired conclusion follows trivially by considering a  $\delta$  smaller than both  $\delta_1$  and  $\delta_2$ . Assume then that  $\epsilon_2 < \epsilon_1$ . Since  $B_{\epsilon_1} \cap X \cap (\overline{\mathfrak{B}} - \overline{X}) \subseteq \{0\}$ , we may select  $\delta$  smaller than both  $\delta_1$  and  $\delta_2$  and such that  $\rho : (B_{\epsilon_1} - \circ B_{\epsilon_2}) \cap G^{-1}(\partial B_\delta) \rightarrow [\epsilon_2^2, \epsilon_1^2]$  has no critical points. The desired conclusion follows (in the open ball case, one uses collarings of the boundaries).  $\square$

**Definition 2.8.** If  $G$  satisfies Milnor's conditions (a) and (b) at  $0$ , we refer to the two fibrations (or their diffeomorphism classes) in Theorem 2.7 as the **Milnor fibration of  $G$  at  $0$  inside a ball** and the **compact Milnor fibration of  $G$  at  $0$  (inside a ball)**.

### 3 Łanalytic Functions

We continue with our notation from the previous sections. In particular,  $\mathcal{U}$  is a connected, open neighborhood of the origin in  $\mathbb{R}^N$ ,  $g$  and  $h$  are real analytic functions from  $\mathcal{U}$  to  $\mathbb{R}$ , and  $f := (g, h)$ .

Throughout this section, we will use the results and definitions from the previous section, but we will use the more general statements which refer to the arbitrary point  $p$ , instead of merely the origin, and where the  $G$  of the previous section is replaced by  $G - G(p)$ .

**Definition 3.1.** We say that  $f$  satisfies the **strong Łojasiewicz inequality at  $p$**  or that  $f$  is **Łanalytic at  $p$**  if and only if there exists an open neighborhood  $\mathcal{W}$  of  $p$  in  $\mathcal{U}$ , and  $c, \theta \in \mathbb{R}$  such that  $c > 0$ ,  $0 < \theta < 1$ , and, for all  $x \in \mathcal{W}$ ,

$$|f(x) - f(p)|^\theta \leq c \cdot \min_{|(a,b)=1} |a\nabla g(x) + b\nabla h(x)|.$$

We say that  $f$  is **Łanalytic** if and only if  $f$  is Łanalytic at each point  $p \in \mathcal{U}$ .

**Remark 3.2.** One can do the Calculus problem and find that, explicitly, the minimum in Definition 3.1 is

$$\sqrt{\frac{|\nabla g(x)|^2 + |\nabla h(x)|^2 - \sqrt{(|\nabla g(x)|^2 - |\nabla h(x)|^2)^2 + 4(\nabla g(x) \cdot \nabla h(x))^2}}{2}},$$

which can also be written

$$\sqrt{\frac{|\nabla g(x)|^2 + |\nabla h(x)|^2 - \sqrt{(|\nabla g(x)|^2 + |\nabla h(x)|^2)^2 - 4[|\nabla g(x)|^2|\nabla h(x)|^2 - (\nabla g(x) \cdot \nabla h(x))^2]}}{2}},$$

Note that if  $g$  and  $h$  are such that  $\nabla g(x)$  and  $\nabla h(x)$  are always orthogonal to each other and have the same length, then the inequality collapses to

$$|f(x) - f(p)|^\theta \leq c|\nabla g(x)|,$$

which, as we saw in Corollary 1.3, is automatically satisfied.

**Definition 3.3.** When  $g$  and  $h$  are such that  $\nabla g(x)$  and  $\nabla h(x)$  are always orthogonal to each other and have the same length, we say that  $f = (g, h)$  is a **simple Łanalytic function**.

**Definition 3.4.** At each point  $x \in \mathcal{U}$  where  $|\nabla g(x)|^2 + |\nabla h(x)|^2 \neq 0$ , define

$$\rho_f(x) := \frac{2|\nabla g(x)||\nabla h(x)| \sin \eta(x)}{|\nabla g(x)|^2 + |\nabla h(x)|^2},$$

where  $\eta(x)$  equals the angle between  $\nabla g(x)$  and  $\nabla h(x)$ .

If  $|\nabla g(x)|^2 + |\nabla h(x)|^2$  is not identically zero, we define the **sL-weight** of  $f$ ,  $\rho_f^{\text{inf}}$ , to be the infimum of  $\rho_f(x)$  over all  $x \in \mathcal{U}$  such that  $|\nabla g(x)|^2 + |\nabla h(x)|^2 \neq 0$ .

If  $|\nabla g(x)|^2 + |\nabla h(x)|^2$  is identically zero, so that  $f$  is constant on  $\mathcal{U}$ , we set  $\rho_f^{\text{inf}}$  equal to 1.

**Remark 3.5.** One easily shows that  $0 \leq \rho_f(x) \leq 1$ , and that  $\rho_f(x) = 1$  if and only if  $\nabla g(x)$  and  $\nabla h(x)$  are orthogonal to each other and have the same length. In addition,  $\rho_f(x) = 0$  if and only if  $\nabla g(x)$  and  $\nabla h(x)$  are linearly dependent. Hence,  $0 \leq \rho_f^{\text{inf}} \leq 1$ , and  $\rho_f^{\text{inf}} = 1$  if and only if  $f$  is simple Lanalytic.

The following proposition makes it substantially easier to calculate whether a given  $f$  is Lanalytic or not.

**Proposition 3.6.** The map  $f$  is Lanalytic at  $p$  if and only if there exists an open neighborhood  $\mathcal{W}$  of  $p$  in  $\mathcal{U}$ , and  $c, \theta \in \mathbb{R}$  such that  $c > 0$ ,  $0 < \theta < 1$ , and, for all  $x \in \mathcal{W}$  such that  $|\nabla g(x)|^2 + |\nabla h(x)|^2 \neq 0$ ,

$$|f(x) - f(p)|^\theta \leq c \cdot \sqrt{\frac{|\nabla g(x)|^2 |\nabla h(x)|^2 - (\nabla g(x) \cdot \nabla h(x))^2}{|\nabla g(x)|^2 + |\nabla h(x)|^2}} = c \cdot \frac{|\nabla g(x)||\nabla h(x)| \sin \eta(x)}{\sqrt{|\nabla g(x)|^2 + |\nabla h(x)|^2}}.$$

**Proof.** Suppose that  $|\nabla g(x)|^2 + |\nabla h(x)|^2 \neq 0$ . Let

$$\gamma(x) := \frac{4[|\nabla g(x)|^2 |\nabla h(x)|^2 - (\nabla g(x) \cdot \nabla h(x))^2]}{(|\nabla g(x)|^2 + |\nabla h(x)|^2)^2}.$$

Then, one easily shows that  $0 \leq \gamma(x) \leq 1$  and, from Remark 3.2, one finds that

$$m := \min_{|(a,b)|=1} |a\nabla g(x) + b\nabla h(x)| = \sqrt{\frac{(|\nabla g(x)|^2 + |\nabla h(x)|^2)(1 - \sqrt{1 - \gamma(x)})}{2}}.$$

Hence,  $\frac{1}{2} \leq \frac{1}{1 + \sqrt{1 - \gamma(x)}} \leq 1$ , and

$$m = \sqrt{\frac{(|\nabla g(x)|^2 + |\nabla h(x)|^2)\gamma(x)}{2(1 + \sqrt{1 - \gamma(x)})}}.$$

Therefore,

$$\sqrt{\frac{(|\nabla g(x)|^2 + |\nabla h(x)|^2)\gamma(x)}{4}} \leq m \leq \sqrt{2} \cdot \sqrt{\frac{(|\nabla g(x)|^2 + |\nabla h(x)|^2)\gamma(x)}{4}}.$$

As

$$\sqrt{\frac{(|\nabla g(x)|^2 + |\nabla h(x)|^2)\gamma(x)}{4}} = \sqrt{\frac{|\nabla g(x)|^2 |\nabla h(x)|^2 - (\nabla g(x) \cdot \nabla h(x))^2}{|\nabla g(x)|^2 + |\nabla h(x)|^2}},$$

the proposition follows immediately.  $\square$

**Theorem 3.7.** *Suppose that the  $sL$ -weight of  $f$  is positive. Then,  $f$  is  $L$ analytic.*

**Proof.** Suppose that  $\rho_f^{\text{inf}} = c > 0$ . Then, for all  $x$  such that  $|\nabla g(x)|^2 + |\nabla h(x)|^2 \neq 0$ ,

$$c\sqrt{|\nabla g(x)|^2 + |\nabla h(x)|^2} \leq \frac{|\nabla g(x)||\nabla h(x)| \sin \eta(x)}{\sqrt{|\nabla g(x)|^2 + |\nabla h(x)|^2}}.$$

Now, combining Corollary 1.3 with Proposition 3.6 yields the theorem.  $\square$

Now that we have Theorem 3.7, we can give a nice class of  $L$ analytic functions, other than simple  $L$ analytic functions; thus, we give a class of  $L$ analytic functions that do not come from holomorphic or antiholomorphic functions.

**Example 3.8.** Suppose that  $f = (g, h)$  is a simple  $L$ analytic function, e.g., a pair coming from a holomorphic or antiholomorphic function.

Let  $(a, b) \in \mathbb{R}^2$ , and let  $f_{(a,b)} := (g + ah, h + bg)$ . Then, one trivially calculates that  $\rho_{f_{(a,b)}}(x)$  is identically equal to

$$\frac{2|1 - ab|}{2 + a^2 + b^2},$$

and, hence, this number is the  $sL$ -weight of  $f_{(a,b)}$ . This number is positive if  $ab \neq 1$  and, unless  $a = -b$ , this number is strictly less than 1, and so  $f_{(a,b)}$  itself is not simple  $L$ analytic.

## 4 The Main Theorem

Now that we have finished our general discussion of  $L$ analytic functions, we wish to investigate how they are related to Milnor's conditions (a) and (b).

The following proposition is trivial to conclude.

**Proposition 4.1.** *If  $f$  is  $L$ analytic at  $p$ , then, near  $p$ ,  $\Sigma f \subseteq f^{-1}(f(p))$ , i.e.,  $f - f(p)$  satisfies Milnor's condition (a) at  $p$ .*

Our goal is to prove that if  $f$  is  $L$ analytic at  $p$ , then  $f - f(p)$  also satisfies Milnor's condition (b) at  $p$ , for then Theorem 2.3 will tell us that the Milnor fibrations inside a ball exist.

The main lemma is:

**Lemma 4.2.** *Suppose that  $f$  is  $L$ analytic at  $\mathbf{0}$ , and that  $f(\mathbf{0}) = \mathbf{0}$ . Then, exists an open neighborhood  $\mathcal{W}$  of  $\mathbf{0}$  in  $\mathcal{U}$  and a real analytic stratification  $\mathfrak{S}$  of  $\mathcal{W} \cap X$  such that, for all  $S \in \mathfrak{S}$ , the pair  $(\mathcal{W} - X, S)$  satisfies Thom's  $a_f$  condition, i.e., if  $p_j \rightarrow p \in S \in \mathfrak{S}$ , where  $p_j \in \mathcal{W} - X$ , and  $T_{p_j} f^{-1} f(p_j)$  converges to some linear subspace  $\mathcal{T}$  in the Grassmanian of  $(N - 2)$ -dimensional linear subspaces of  $\mathbb{R}^N$ , then  $T_p S \subseteq \mathcal{T}$ .*

**Proof.** This proof is essentially identical to that of Theorem 1.2.1 of [5]. Let  $\mathcal{W}$ ,  $c$ , and  $\theta$  be as in Definition 3.1.

Let  $\mathfrak{l}$  be an integer greater than  $1/(1-\theta)$ . Consider the function  $f_{\mathfrak{l}} : \mathcal{U} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by

$$f_{\mathfrak{l}}(x, u, v) := (g(x) + \operatorname{Re}\{(u + iv)^{\mathfrak{l}}\}, h(x) + \operatorname{Im}\{(u + iv)^{\mathfrak{l}}\}).$$

Let  $\mathfrak{S}_{\mathfrak{l}}$  be a real analytic stratification of  $V(f_{\mathfrak{l}})$  which satisfies Whitney's condition (a) and such that  $V(f) \times \{(0, 0)\}$  is a union of strata. Let  $\mathfrak{S} := \{S \mid S \times \{(0, 0)\} \in \mathfrak{S}_{\mathfrak{l}}\}$ . We claim that  $\mathcal{W}$  and  $\mathfrak{S}$  satisfy the conclusion of the lemma.

It will be convenient to deal with the conormal formulation of the  $a_f$  condition. Suppose that we have a sequence of points  $p_j \rightarrow p \in S \in \mathfrak{S}$ , where  $p_j \in \mathcal{W} - X$ , and a sequence  $(a_j, b_j) \in \mathbb{R}^2$  such that  $a_j d_{p_j} g + b_j d_{p_j} h$  converge to a cotangent vector  $\eta$  (in the fiber of  $T^*\mathcal{U}$  over  $p$ ). We wish to show that  $\eta(T_p S) \equiv 0$ .

If  $\eta \equiv 0$ , there is nothing to show; so we assume that  $\eta \neq 0$ . We will work with vectors, instead of covectors. In terms of vectors, we are assuming that  $a_j \nabla g(p_j) + b_j \nabla h(p_j)$  converges to a non-zero vector  $\mathbf{w}$ . Note that we may assume that, for all  $j$ ,  $(a_j, b_j) \neq (0, 0)$ .

Let  $(u_j, v_j)$  be such that  $u_j + iv_j$  is an  $\mathfrak{l}$ -th root of  $-(g(p_j) + ih(p_j))$ , i.e., such that  $(p_j, u_j, v_j) \in V(f_{\mathfrak{l}})$ . Note that  $(u_j, v_j) \rightarrow (0, 0)$ . By taking a subsequence, if necessary, we may assume that the projective class

$$[a_j d_{(p_j, u_j, v_j)}(g + \operatorname{Re}\{(u + iv)^{\mathfrak{l}}\}) + b_j d_{(p_j, u_j, v_j)}(h + \operatorname{Im}\{(u + iv)^{\mathfrak{l}}\})]$$

converges to some  $[\omega]$ , where  $\omega = q_1 dr_1 + \dots + q_N dr_N + s du + t dv \neq 0$ . By Whitney's condition (a), we have that  $\omega(T_p S \times (0, 0)) \equiv 0$ . If we could show that the projective classes  $[\mathbf{w} \times (0, 0)]$  and  $[(q_1, \dots, q_N, s, t)]$  are equal, then we would be finished.

In terms of vectors, we are assuming that

$$[a_j (\nabla g(p_j), \mathfrak{l} \operatorname{Re}\{(u_j + iv_j)^{\mathfrak{l}-1}\}, -\mathfrak{l} \operatorname{Im}\{(u_j + iv_j)^{\mathfrak{l}-1}\}) + b_j (\nabla h(p_j), \mathfrak{l} \operatorname{Im}\{(u_j + iv_j)^{\mathfrak{l}-1}\}, \mathfrak{l} \operatorname{Re}\{(u_j + iv_j)^{\mathfrak{l}-1}\})] \rightarrow [(q_1, \dots, q_N, s, t)].$$

To show then that  $[\mathbf{w} \times (0, 0)]$  and  $[(q_1, \dots, q_N, s, t)]$  are equal, it suffices to show that

$$\frac{|a_j (\mathfrak{l} \operatorname{Re}\{(u_j + iv_j)^{\mathfrak{l}-1}\}, -\mathfrak{l} \operatorname{Im}\{(u_j + iv_j)^{\mathfrak{l}-1}\}) + b_j (\mathfrak{l} \operatorname{Im}\{(u_j + iv_j)^{\mathfrak{l}-1}\}, \mathfrak{l} \operatorname{Re}\{(u_j + iv_j)^{\mathfrak{l}-1}\})|}{|a_j \nabla g(p_j) + b_j \nabla h(p_j)|} \rightarrow 0.$$

Eliminating the factor of  $\mathfrak{l}$ , and simplifying the numerator, we want to show that

$$\frac{|(a_j, b_j)| \cdot |u_j + iv_j|^{\mathfrak{l}-1}}{|a_j \nabla g(p_j) + b_j \nabla h(p_j)|} \rightarrow 0.$$

Dividing the numerator and denominator by  $|(a_j, b_j)|$ , we see that we may assume that  $|(a_j, b_j)| = 1$  and we want to show that

$$\frac{|u_j + iv_j|^{\mathfrak{l}-1}}{|a_j \nabla g(p_j) + b_j \nabla h(p_j)|} \rightarrow 0.$$

Recall now that our choice of  $(u_j, v_j)$  implies that  $|u_j + iv_j|^{\mathfrak{l}-1} = |(g(p_j), h(p_j))|^{\frac{\mathfrak{l}-1}{\mathfrak{l}}}$ . Thus, we need to show that

$$\frac{|(g(p_j), h(p_j))|^{\frac{\mathfrak{l}-1}{\mathfrak{l}}}}{|a_j \nabla g(p_j) + b_j \nabla h(p_j)|} \rightarrow 0.$$

If  $p_j \in \mathcal{W} - X$ , then the strong Lojasiewicz inequality implies that

$$\frac{|(g, h)|^{\frac{\mathfrak{l}-1}{\mathfrak{l}}}}{|a_j \nabla g(p_j) + b_j \nabla h(p_j)|} \leq \frac{|(g(p_j), h(p_j))|^{\frac{\mathfrak{l}-1}{\mathfrak{l}}}}{|(g(p_j), h(p_j))|^{\theta}}.$$

As  $p_j \rightarrow p \in V(f)$ , the righthand term above will approach zero if  $\frac{l-1}{l} > \theta$ , which follows at once from our choice of  $l$ .  $\square$

The Main Theorem follows easily.

**Theorem 4.3.** *Suppose that  $f$  is Lanalytic at  $p$ . Then, Milnor's condition's (a) and (b) hold at  $p$ .*

**Proof.** We assume, without loss of generality, that  $p = \mathbf{0}$  and  $f(\mathbf{0}) = \mathbf{0}$ .

Milnor's condition (a) is immediate, as we stated in Proposition 4.1. Milnor's condition (b) follows from Lemma 4.2 by using precisely the argument of Lê in [7].

Let  $\epsilon_0 > 0$  be such that, for all  $\epsilon$  such that  $0 < \epsilon \leq \epsilon_0$ , the sphere  $S_\epsilon$  transversely intersects all of the strata of the  $a_f$  stratification whose existence is guaranteed by Lemma 4.2. Also assume that  $\mathbf{0}$  is the only critical value of  $f$  at points in  $B_{\epsilon_0}$ . Suppose there were a sequence of points  $p_i \in B_{\epsilon_0}$  which approaches a point  $p \in S_\epsilon \cap X$ . Let  $\epsilon_i$  denote the distance of  $p_i$  from the origin, and suppose that  $T_{p_i}(g^{-1}g(p_i) \cap h^{-1}h(p_i)) \subseteq T_{p_i}S_{\epsilon_i}$  (we used Milnor's condition (a) here). By taking a subsequence, we may assume that  $T_{p_i}(g^{-1}g(p_i) \cap h^{-1}h(p_i))$  approaches a limit  $\mathcal{T}$ . If we let  $M$  denote the stratum of our  $a_f$  stratification which contains  $p$ , then we find that  $T_p M \subseteq \mathcal{T}$  and  $\mathcal{T} \subseteq T_p S_\epsilon$ . As  $S_\epsilon$  transversely intersects  $M$ , we have a contradiction, which proves the desired result.  $\square$

**Corollary 4.4.** *Suppose that  $f$  is Lanalytic at  $p$ . Then, the Milnor fibrations for  $f - f(p)$  inside a ball, centered at  $p$ , exist.*

**Proof.** As we saw in Section 2, this follows immediately from Milnor's condition's (a) and (b).  $\square$

We include the following just to emphasize that Lanalytic have many properties that one associates with complex analytic functions.

**Corollary 4.5.** *Suppose that  $f$  is a non-constant Lanalytic function. Then,  $f$  is an open map.*

**Proof.** This follows at once from Theorem 4.3 and Corollary 2.4.  $\square$

## 5 Examples, Comments, and Questions

**Example 5.1.** If  $g$  and  $h$  are the real and imaginary parts of a holomorphic or anti-holomorphic function, then  $f = (g, h)$  is a simple Lanalytic function.

One can also mix holomorphic and anti-holomorphic functions. Let  $z = x + iy$  and  $w = u + iv$ , and consider the real and imaginary parts of  $\bar{z}w^2$ , i.e., let  $g = x(u^2 - v^2) + 2yuv$  and  $h = 2xuv - y(u^2 - v^2)$ . It is trivial to verify that  $(g, h)$  is simple Lanalytic.

More generally, if  $(g_1, h_1)$  and  $(g_2, h_2)$  are simple Lanalytic, then the real and imaginary parts of

$$(g_1(\mathbf{z}) + ih_1(\mathbf{z}))(g_2(\mathbf{w}) + ih_2(\mathbf{w}))$$

will yield a new simple Lanalytic function.

**Question 5.2.** Are there other “nice” classes of Lanalytic functions?

Of course, one can also ask about the Milnor fibration **on the sphere**, i.e., about a real analytic analog of the fibration  $f_c/|f_c| : S_\epsilon - S_\epsilon \cap X \rightarrow S^1 \subseteq \mathbb{C}$ , which exists when  $f_c$  is complex analytic. As we discussed in the introduction,  $f/|f|$  does not necessarily yield the projection map of a fibration from  $S_\epsilon - S_\epsilon \cap X$  to  $S^1$ ; when this map is, in fact, a locally trivial fibration (for all sufficiently small  $\epsilon$ ), one says that  $f$  satisfies the **strong Milnor condition** at the origin (see VII.2 of [14]).

It is not terribly difficult to try to mimic Milnor’s proof that the Milnor fibration inside the ball and the one on the sphere are diffeomorphic; one wants to integrate an appropriate vector field. In the case where  $f = (g, h)$  is real analytic and satisfies Milnor’s conditions (a) and (b), one finds that one needs a further condition:

**Definition 5.3.** Let  $\omega := -h\nabla g + g\nabla h$ , and assume that  $f(\mathbf{0}) = \mathbf{0}$ . We say that  $f$  **satisfies Milnor’s condition (c) at  $\mathbf{0}$**  if and only if there exists an open neighborhood  $\mathcal{W}$  of  $\mathbf{0}$  in  $\mathcal{U}$  such that, at all points  $\mathbf{x}$  of  $\mathcal{W} - X$ , if  $\mathbf{x}$  is a linear combination of  $\nabla g$  and  $\nabla h$ , then

$$|\omega(\mathbf{x})|^2(\mathbf{x} \cdot \nabla|f|^2(\mathbf{x})) > (\nabla|f|^2(\mathbf{x}) \cdot \omega(\mathbf{x}))(\mathbf{x} \cdot \omega(\mathbf{x})).$$

It is easy to show that there exists an open neighborhood  $\mathcal{W}$  of  $\mathbf{0}$  in  $\mathcal{U}$  such that, at all points  $\mathbf{x}$  of  $\mathcal{W} - X$ ,  $\mathbf{x} \cdot \nabla|f|^2(\mathbf{x}) > 0$ . It is also trivial that a simple Lanalytic function has  $\nabla|f|^2(\mathbf{x}) \cdot \omega(\mathbf{x}) = 0$ . Milnor’s condition (a) implies that  $\mathcal{W}$  can be chosen so that  $\omega(\mathbf{x}) \neq 0$  for  $\mathbf{x} \in \mathcal{W} - X$ . Thus, we conclude that simple Lanalytic functions satisfy the strong Milnor condition.

However, the real question is:

**Question 5.4.** Do general Lanalytic functions satisfy Milnor’s condition (c), and so satisfy the strong Milnor condition? What about functions with positive sL-weight?

It is trivial to extend the definition of Lanalytic to  $k$ -tuples of functions, instead of merely pairs, by using the inequality

$$|(g_1, \dots, g_k)(x) - (g_1, \dots, g_k)(p)|^\theta \leq c \cdot \min_{|(a_1, \dots, a_k)|=1} |a_1 \nabla g_1(x) + \dots + a_k \nabla g_k(x)|.$$

Such a definition clearly implies Milnor’s condition (a), but the proof of a lemma such Lemma 4.2 is unclear, and so one is led to ask:

**Question 5.5.** Do Lanalytic  $k$ -tuples satisfy Milnor’s condition (b)?

Finally, having seen in this paper that Lanalytic pairs of functions share some important properties with complex analytic functions, one is led to ask:

**Question 5.6.** Are Lanalytic functions an interesting class of functions to study for reasons having nothing to do with Milnor fibrations?

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