SEMI-SIMPLE CARROUSELS AND THE MONODROMY

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Abstract. Let $\mathcal{U}$ be an open neighborhood of the origin in $\mathbb{C}^{n+1}$ and let $f : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, 0)$ be complex analytic. Let z_0 be a generic linear form on $\mathbb{C}^{n+1}$. If the relative polar curve Γ^1_{f,z_0} at the origin is irreducible and the intersection number $\left(\Gamma^1_{f,z_0} \cdot V(f)\right)_\mathbf{0}$ is prime, then there are severe restrictions on the possible degree n cohomology of the Milnor fiber at the origin. We also obtain some interesting, weaker, results when $\left(\Gamma^1_{f,z_0} \cdot V(f)\right)_\mathbf{0}$ is not prime.

§0. Introduction

In [**Lê2**] and [**Lê3**], Lê introduces his carrousel as a tool for analyzing the relative monodromy of the Milnor fiber of a function, f, modulo a hyperplane slice. In [**T1**] and [**T2**], Tibăr gives a careful presentation of Lê's carrousel and uses it to obtain interesting results. Outside of the work of Lê and Tibăr, the carrousel seems to be a largely unused device. This is due in part to the complicated nature of the carrousel description.

In this short paper, we look at some interesting special cases that occur and, in particular, look at the case where the relative polar curve, Γ^1_{f,z_0}, has a single component such that the intersection number $\left(\Gamma^1_{f,z_0} \cdot V(f)\right)_\mathbf{0}$ is prime. In this case, we show, in Theorem 2.3, how Lê's carrousel tells one a great deal about the middle-dimensional homology/cohomology groups of the Milnor fiber of f, regardless of the dimension of the critical locus.

§1. Lê's Playground

Let $\mathcal{U}$ be an open neighborhood of the origin in $\mathbb{C}^{n+1}$ and let $f : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, 0)$ be a complex analytic function which has a critical point at the origin.

Recall that a *good stratification for* f is a stratification $\mathcal{S}$ of $V(f)$ which contains $V(f) - \Sigma f$, and such that, for all $S \in \mathcal{S}$, the pair $(\mathcal{U} - V(f), S)$ satisfies the a_f condition. After a linear change of coordinates, we may assume that the first coordinate, z_0, is a prepolar form (or coordinate) for f at $\mathbf{0}$ (see [**M1**]); this means that there exists a neighborhood, $\mathcal{W} \subseteq \mathcal{U}$, of $\mathbf{0}$ such that, inside $\mathcal{W} - \{\mathbf{0}\}$, $V(z_0)$ transversely intersects all of the strata of a good stratification of $V(f)$ (we do **not** need the condition of the frontier here – we could simply use a good partition). Then, at the origin, the relative polar curve Γ^1_{f,z_0} (see [**M1**]) is purely one-dimensional (or empty), and Γ^1_{f,z_0} properly intersects both $V(f)$ and $V(z_0)$ (again, see [**M1**]). We always consider Γ^1_{f,z_0} with its cycle structure (see [**M1**]). We assume that $\mathcal{U}$ is small enough so that every component of Γ^1_{f,z_0} passes through the origin.

Let D be a component of the cycle Γ^1_{f,z_0} (with either its reduced structure or its cycle structure). We have the following well-known formula, originally due to Teissier,

$$\left(D \cdot V(f)\right)_\mathbf{0} = \left(D \cdot V(z_0)\right)_\mathbf{0} + \left(D \cdot V\left(\frac{\partial f}{\partial z_0}\right)\right)_\mathbf{0}.$$

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As $\mathbf{0}$ is a critical point of f, it follows that $\left(D \cdot V\left(\frac{\partial f}{\partial z_0}\right)\right)_{\mathbf{0}} > 0$, and so

$$(\dagger) \qquad n_D := \big(D \cdot V(f)\big)_{\mathbf{0}} > \big(D \cdot V(z_0)\big)_{\mathbf{0}} =: m_D.$$

Lê's Attaching Theorem

Let B_ϵ (resp., $\mathbb{D}_\delta$) denote a closed ball of radius ϵ (resp., δ) centered at the origin in $\mathbb{C}^n$ (resp., $\mathbb{C}$). Assume that $0 < \eta \ll \delta \ll \epsilon \ll 1$. Let $\xi \in \mathbb{C}$ be such that $0 < |\xi| \leqslant \eta$. Then,

$$F_f := \big(\mathbb{D}_\delta \times B_\epsilon\big) \cap f^{-1}(\xi)$$

is (up to homotopy) the Milnor fiber of f at $\mathbf{0}$, and $F_{f_0} := V(z_0) \cap F_{f,\mathbf{0}}$ is the Milnor fiber of $f_0 := f_{|_{V(z_0)}}$ at the origin. The main theorem of [**Lê1**] (see, also, [**M1**]) is:

Theorem 1.1. (Lê) *The Milnor fiber F_f is obtained from F_{f_0} by attaching $\tau_{f,z_0} := \big(\Gamma^1_{f,z_0} \cdot V(f)\big)_{\mathbf{0}}$ n-handles (n-cells, up to homotopy).*

Thus, $H^k(F_f, F_{f_0}) = 0$ if $k \neq n$, there is an isomorphism

$$\omega_{f,z_0} : H^n(F_f, F_{f_0}) \xrightarrow{\cong} \bigoplus_D \mathbb{Z}^{n_D} \ \cong \ \mathbb{Z}^{\tau_{f,z_0}},$$

where D ranges over the (possibly non-reduced) components of Γ^1_{f,z_0}, and there is a map on reduced, integral cohomology $\widetilde{H}^{n-1}(F_{f_0}) \xrightarrow{\partial_{f,z_0}} H^n(F_f, F_{f_0})$ such that $\ker \partial_{f,z_0} \cong \widetilde{H}^{n-1}(F_f)$ and $\operatorname{coker} \partial_{f,z_0} \cong \widetilde{H}^n(F_f)$.

We refer to the above result as *Lê's Attaching Theorem*.

Remark 1.2. By the naturality of the Milnor monodromy, the map ∂_{f,z_0} commutes with the respect Milnor monodromies on $\widetilde{H}^{n-1}(F_{f_0})$ and $H^n(F_f, F_{f_0})$. In particular, the image of ∂_{f,z_0}, $\operatorname{im} \partial_{f,z_0}$, is a free Abelian submodule which is invariant under the monodromy.

In addition, the main theorem of A'Campo in [**A'C**] tells us that the trace of the monodromy action on $H^n(F_f, F_{f_0})$ is 0. Therefore, the trace of the monodromy action on $\operatorname{im} \partial_{f,z_0}$ is negative the trace of the monodromy action on the free part of $\widetilde{H}^n(F_f)$.

For all k, we denote the rank of $\widetilde{H}^k(F_f)$ (i.e., the k-th reduced Betti number) by $\tilde{b}_k(f)$. Thus, the rank of $\operatorname{im} \partial_{f,z_0}$ is $e_{f,z_0} := \tau_{f,z_0} - \tilde{b}_n(f)$. We denote the characteristic polynomials of the monodromy action on $\operatorname{im} \partial_{f,z_0}$, on $H^n(F_f, F_{f_0})$, and on the free part of $\widetilde{H}^n(F_f)$ (or on $\widetilde{H}_n(F_f)$) by $\operatorname{char}_{\operatorname{im} \partial_{f,z_0}}(\lambda)$, $\operatorname{char}_{\operatorname{rel}_{f,z_0}}(\lambda)$, and $\operatorname{char}^n_f(\lambda)$, respectively. Of course, we have the equality

$$\operatorname{char}_{\operatorname{rel}_{f,z_0}}(\lambda) = \operatorname{char}_{\operatorname{im} \partial_{f,z_0}}(\lambda) \cdot \operatorname{char}^n_f(\lambda).$$

The Swing

In [**L-P**], Lê and Perron use the "swing" to more carefully analyze the image of the attaching map ∂_{f,z_0} above. They do this in the case where $\dim_{\mathbf{0}} \Sigma f = 1$. However, in [**M1**], we showed that their argument works regardless of the dimension of the critical locus.

What the swing shows is:

Theorem 1.3. (Lê and Perron) *The rank of the free Abelian module* $\operatorname{im}\partial_{f,z_0}$ *is at least* $\gamma^1_{f,z_0} := \big(\Gamma^1_{f,z_0}\cdot V(z_0)\big)_{\mathbf{0}}$. *Thus, the rank of* $\widetilde{H}^n(F_f)$ *is at most*

$$\lambda^0_{f,z_0} := \tau_{f,z_0} - \gamma^1_{f,z_0} = \left(\Gamma^1_{f,z_0}\cdot V\left(\frac{\partial f}{\partial z_0}\right)\right)_{\mathbf{0}}.$$

In fact, for each component D *of* Γ^1_{f,z_0}, *there is a submodule* $E_D \subseteq \mathbb{Z}^{n_D}$ *which is generated by* m_D *of the basis elements of* $\mathbb{Z}^{n_D}$ *such that, if* π *denotes the projection from* $\bigoplus_D \mathbb{Z}^{n_D}$ *onto* $\bigoplus_D E_D \cong \mathbb{Z}^{\gamma^1_{f,z_0}}$, *then* $\pi\circ\omega_{f,z_0}\circ\partial_{f,z_0}$ *is a surjection (where* ∂_{f,z_0} *and* ω_{f,z_0} *are defined in Theorem 1.1).*

The number λ^0_{f,z_0} defined above is the 0*-th Lê number of* f *(at the origin with respect to* z_0*)* (see [**M1**]).

Lê's Monodromy Carrousel

Lê's Carrousel (see [**Lê2**] and [**T2**]) gives a geometric description of the monodromy action on $H^n(F_f, F_{f_0})$. Let us briefly recall the set-up and some features of Lê's Carrousel.

Let Θ denote the map (z_0, f) from $\big(\mathbb{D}_\delta\times B_\epsilon\big)\cap f^{-1}(\mathbb{D}_\eta)$ onto $\mathbb{D}_\delta\times\mathbb{D}_\eta$, and use (u,v) for coordinates on the codomain. Then, $C_{f,z_0} := \Theta\big(\Gamma^1_{f,z_0}\big)$ is the *Cerf diagram* of f with respect to z_0. Each component of C_{f,z_0} is tangent to the u-axis at the origin (this follows from (†) above). The map $\Theta_{|_{\Gamma^1_{f,z_0}}}$ is finite, and we endow C_{f,z_0} with a cycle structure via the proper push-forward. It follows that

$$\tau_{f,z_0} = \big(\Gamma^1_{f,z_0}\cdot V(f)\big)_{\mathbf{0}} = \big(C_{f,z_0}\cdot V(v)\big)_{\mathbf{0}}$$

and

$$\gamma^1_{f,z_0} = \big(\Gamma^1_{f,z_0}\cdot V(z_0)\big)_{\mathbf{0}} = \big(C_{f,z_0}\cdot V(u)\big)_{\mathbf{0}}.$$

However, Lê's Carrousel description requires that Γ^1_{f,z_0} be reduced and that $\Theta_{|_{\Gamma^1_{f,z_0}}}$ be one-to-one, i.e., that the cycle C_{f,z_0} is reduced, and we have **not** assumed that z_0 is generic enough to make this happen.

Definition 1.4. The linear form z_0 is a *carrousel form (for* f *at* **0***)* if and only if z_0 is a prepolar form for f at **0** and, at the origin, the cycle C_{f,z_0} is reduced.

We assume throughout the remainder of this section that z_0 is a carrousel form.

Suppose, again, that D is an irreducible component of Γ^1_{f,z_0}. Then, $C := \Theta(D)$ is an irreducible component of C_{f,z_0}, and every component of C_{f,z_0} is obtained in this manner. Moreover,

$$m_C := \big(C\cdot V(u)\big)_{\mathbf{0}} = \big(D\cdot V(z_0)\big)_{\mathbf{0}} = m_D,$$

and

$$n_C := \big(C\cdot V(v)\big)_{\mathbf{0}} = \big(D\cdot V(f)\big)_{\mathbf{0}} = n_D.$$

Let $g_C := \gcd(m_C, n_C)$, let $p_C := m_C/g_C$, and let $q_C := n_C/g_C$. The curve C has a local parameterization of the form

$$v = t^{n_C}, \quad u = \alpha_{C} t^{m_C} + \text{ higher order terms }.$$

The *carrousel approximation of* C is the curve given by

$$v = t^{q_C}, \quad u = \alpha_{C} t^{p_C},$$

i.e., the curve $\widehat{C} := V(u^{q_C} - \alpha_{C}^{q_C} v^{p_C})$. We refer to $\beta_{C} := \alpha_{C}^{q_C}$ as the *carrousel coefficient of* C.

Definition 1.5. We say that *carrousel of* f *with respect to (the carrousel form)* z_0 *is semi-simple* provided that:

i) for all components C of C_{f,z_0}, m_C and n_C are relatively prime (and so, $(m_C, n_C) = (p_C, q_C)$);

ii) distinct components of C_{f,z_0} have distinct carrousel approximations, i.e., if $C_1 \neq C_2$, then $(p_{C_1}, q_{C_1}, \beta_{C_1}) \neq (p_{C_2}, q_{C_2}, \beta_{C_2})$.

Before we state the next theorem, we need to give one more piece of terminology. We refer to the automorphism of $\mathbb{Z}^k$ given by $(a_1, a_2, \dots, a_k) \mapsto (-a_k, a_1, a_2, \dots, a_{k-1})$ as *cyclic anti-permutation.* The characteristic polynomial of cyclic anti-permutation is $\lambda^k + 1$. Cyclic anti-permutation has -1 as an eigenvalue if and only if k is odd and, in this case, the *anti-diagonal* $\widehat{\Delta} := \mathbb{Z}(1, -1, 1, -1, \dots, -1, 1)$ is the eigenspace of -1. One shows easily that $\mathbb{Z}^k/\widehat{\Delta} \cong \mathbb{Z}^{k-1}$.

We use the terminology *semi-simple* because Lê's carrousel study in [**Lê2**] immediately implies:

Theorem 1.6. (Lê) *If the carrousel of* f *with respect to* z_0 *is semi-simple, then the isomorphism of Theorem 1.1,*

$$\omega_{f,z_0} : H^n(F_f, F_{f_0}) \xrightarrow{\cong} \bigoplus_D \mathbb{Z}^{n_D},$$

can be chosen so that each direct summand is invariant under the Milnor monodromy (i.e., the monodromy breaks up into blocks), and the action on each block is either cyclic permutation or cyclic anti-permutation. In particular, the characteristic polynomial of the monodromy action on $H^n(F_f, F_{f_0})$ *is*

$$\mathrm{char}_{\mathrm{rel}}(\lambda) = \prod_D (\lambda^{n_D} \pm 1).$$

Proof. One refers to the proofs in [**Lê2**].

Condition i) of being semi-simple implies, for a given component C of C_{f,z_0}, that each carrousel disk contains at most one point of C; this implies that in one "turn of the carrousel" there is no interaction between different points on C.

Condition ii) of being semi-simple says that distinct components of C_{f,z_0} have distinct carrousel approximations and, hence, the carrousel points of distinct components do not interact as the carrousel turns.

Now, the carrousel disks are permuted cyclically by the monodromy, and each carrousel disk centered at a point of a Cerf component C contributes one copy of $\mathbb{Z}$ as a direct summand in $\mathbb{Z}^{n_C} = \mathbb{Z}^{n_d}$.

However, after a carrousel disk returns to itself after $n_C = n_D$ iterations, the corresponding copy of $\mathbb{Z}$ may be mapped to itself by either plus or minus the identity. Hence, the induced map on cohomology is either cyclic permutation or anti-permutation, and the conclusion follows immediately. $\square$

§2. Prime Polar Curves

In this section, we continue with our notation from Section 1, and **we continue to assume that z_0 is prepolar for f at $\mathbf{0}$, but we no longer assume that z_0 is a carrousel form**.

Definition 2.1. Let D be a (possibly non-reduced) component of the cycle Γ^1_{f,z_0}.

D is called *relatively prime* provided that $(D\cdot V(z_0))_{\mathbf{0}}$ and $(D\cdot V(f))_{\mathbf{0}}$ are relatively prime.

D is called *unitary* provided that $(D\cdot V(z_0))_{\mathbf{0}} = 1$, i.e., D is reduced, D is smooth, and D is transversely intersected at $\mathbf{0}$ by $V(z_0)$.

D is called *prime of order* $\mathfrak{p}$ provided that $\mathfrak{p} := (D\cdot V(f))_{\mathbf{0}}$ is a prime number.

Note that, as $(D\cdot V(z_0))_{\mathbf{0}} < (D\cdot V(f))_{\mathbf{0}}$, unitary and prime components are also relatively prime.

The cycle Γ^1_{f,z_0} is itself said to be *relatively prime* (resp., *unitary*, resp., *prime of order* $\mathfrak{p}$) provided that Γ^1_{f,z_0} has one irreducible component and that component is relatively prime (resp., unitary, resp., prime of order $\mathfrak{p}$). Note that if Γ^1_{f,z_0} is unitary or prime, then it is relatively prime.

Proposition 2.2. *Suppose that every component of Γ^1_{f,z_0} is relatively prime. Then, z_0 is a carrousel form. In particular, Γ^1_{f,z_0} is reduced.*

If Γ^1_{f,z_0} is itself relatively prime, then the carrousel of f with respect to z_0 is semi-simple.

Proof. Suppose that D is an irreducible component of Γ^1_{f,z_0}, which has a possibly non-reduced cycle structure. Let C be the proper push-forward of D by Θ, i.e., $C := \Theta_*(D)$. Then, $\big(C\cdot V(u)\big)_{\mathbf{0}} = \big(D\cdot V(z_0)\big)_{\mathbf{0}}$, and $\big(C\cdot V(v)\big)_{\mathbf{0}} = \big(D\cdot V(f)\big)_{\mathbf{0}}$. We must show that C is reduced.

Suppose that, as cycles, $C = kC'$, where C' is reduced. Then k must divide both $\big(C\cdot V(u)\big)_{\mathbf{0}}$ and $\big(C\cdot V(v)\big)_{\mathbf{0}}$, which are relatively prime. Thus, $k=1$, and z_0 is a carrousel form.

The remaining claim follows immediately from the definition of a semi-simple carrousel. $\square$

We wish to recall now the notion of the suspension of f (see, for instance, [**M1**]and [**M3**]). Suppose that $f = f(\mathbf{z})$. Suppose that w is a variable disjoint from $\mathbf{z}$. Then, the function $f + w^2$ on $\mathcal{U}\times\mathbb{C}$ is called the *suspension of* f.

It is trivial to show that $\Sigma(f+w^2) = \Sigma f \times \{0\}$, that $\Gamma^1_{f+w^2,z_0} = \Gamma^1_{f,z_0}\times\{0\}$, and that if z_0 is prepolar for f at $\mathbf{0}$, then z_0 is prepolar for $f+w^2$ at $\mathbf{0}$. See [**M1**]. It follows easily that $\gamma^1_{f,z_0} = \gamma^1_{f+w^2,z_0}$ and $\tau_{f,z_0} = \tau_{f+w^2,z_0}$. Therefore, Γ^1_{f,z_0} is prime of order $\mathfrak{p}$ if and only if $\Gamma^1_{f+w^2,z_0}$ is prime of order $\mathfrak{p}$

By the Sebastiani-Thom result (for references to this result, in many various cases, see [**M1**] and [**M3**]), for all k, $\widetilde{H}^{k+1}(F_{f+w^2}) \cong \widetilde{H}^k(f_f)$ and, under this isomorphism, the Milnor monodromy action on $\widetilde{H}^{k+1}(F_{f+w^2})$ is negative the monodromy action on $\widetilde{H}^k(F_f)$. one then recovers (as we saw above) the isomorphism

$$\widetilde{H}^{n+1}(F_{f+w^2}, F_{f_0+w^2}) \cong \widetilde{H}^{n+1}(F_f, F_{f_0})$$

and finds that, under this isomorphism, the Milnor monodromy action on $\widetilde{H}^{n+1}(F_{f+w^2}, F_{f_0+w^2})$ is negative the monodromy action on $\widetilde{H}^{n+1}(F_f, F_{f_0})$. Thus, we have the following relationships between characteristic polynomials of the Milnor monodromy actions:

$$\mathrm{char}_{\mathrm{im}\,\partial_{f+w^2,z_0}}(\lambda) = (-1)^{e_{f,z_0}} \cdot \mathrm{char}_{\mathrm{im}\,\partial_{f,z_0}}(-\lambda)$$

$$\mathrm{char}_{\mathrm{rel}_{f+w^2,z_0}}(\lambda) = (-1)^{\tau_{f,z_0}} \cdot \mathrm{char}_{\mathrm{rel}_{f,z_0}}(-\lambda),$$

and

$$\mathrm{char}^{n+1}_{f+w^2}(\lambda) = (-1)^{\tilde{b}_n(f)} \cdot \mathrm{char}^n_f(-\lambda).$$

Below, we state a result in terms of the homology of F_f, instead of cohomology. While, in general, we prefer to think in cohomological terms, discussions of the monodromy action on $\widetilde{H}^n(F_f)$ are more complicated by the possible presence of torsion. However, $\widetilde{H}_n(F_f)$ is free Abelian and is thus isomorphic to the free part of $\widetilde{H}^n(F_f)$.

Theorem 2.3. *Suppose that Γ^1_{f,z_0} is prime of order $\mathfrak{p}$. Then, we are in one of the following non-overlapping cases:*

Case 0: $\widetilde{H}_n(F_f) = 0$, $\operatorname{rank} \widetilde{H}_{n-1}(F_{f_0}) \geqslant \mathfrak{p}$, and $\dim_{\mathbf{0}} \Sigma f \geqslant 1$;

Case 1: $\widetilde{H}_n(F_f) \cong \mathbb{Z}$, and the monodromy action on $\widetilde{H}_n(F_f)$ is either a) *the identity or* b) *negative the identity;*

Case 2: $\mathfrak{p} \neq 2$, $\widetilde{H}_n(F_f) \cong \mathbb{Z}^{\mathfrak{p}-1}$, $\widetilde{H}_{n-1}(F_f)$ is free Abelian, Γ^1_{f,z_0} is unitary, and the characteristic polynomial of the monodromy action on $\widetilde{H}_n(F_f)$ is either a) $(\lambda^{\mathfrak{p}} - 1)/(\lambda - 1)$ *or* b) $(\lambda^{\mathfrak{p}} + 1)/(\lambda + 1)$.

Moreover, if $\mathfrak{p} = 2$ and $\widetilde{H}_n(F_f) \cong \mathbb{Z}$, then $\widetilde{H}_{n-1}(F_f)$ is free Abelian, and Γ^1_{f,z_0} is unitary.

In addition, suspending f (and using the "same" coordinate z_0) leaves one in the same case, but interchanges the subcases a) *and* b) *in Cases 1 and 2.*

Proof. By Proposition 2.2, z_0 is a carrousel form and the carrousel of f with respect to z_0 is semi-simple. Therefore, by Theorem 1.6, either $\mathrm{char}_{\mathrm{rel}_{f,z_0}}(\lambda) = \lambda^{\mathfrak{p}} - 1$ or $\mathrm{char}_{\mathrm{rel}_{f,z_0}}(\lambda) = \lambda^{\mathfrak{p}} + 1$.

By 1.3, $\mathrm{im}\,\partial_{f,z_0}$ is non-zero. Therefore, Remark 1.2 implies that $\widetilde{H}_n(F_f) = 0$, or that $\mathrm{char}^n_f(\lambda)$ is $\lambda - 1$, $\lambda + 1$, $(\lambda^{\mathfrak{p}} - 1)/(\lambda - 1)$, or $(\lambda^{\mathfrak{p}} + 1)/(\lambda + 1)$, where this last characteristic polynomial cannot occur if $\mathfrak{p} = 2$.

Case 0: Suppose that $\widetilde{H}_n(F_f) = 0$; this is equivalent to $\operatorname{rank}(\mathrm{im}\,\partial_{f,z_0}) = \mathfrak{p}$. This certainly implies that $\operatorname{rank} \widetilde{H}_{n-1}(F_{f_0}) \geqslant \mathfrak{p}$. We claim that it follows that f cannot have an isolated critical at the origin. Suppose to the contrary that $\dim_{\mathbf{0}} \Sigma f = 0$. Then, by the formula of Lê and Greuel, $\mu_{\mathbf{0}}(f) + \mu_{\mathbf{0}}(f_0) = \mathfrak{p}$, where μ denotes the Milnor number. As $\mu_{\mathbf{0}}(f) > 0$, $\mu_{\mathbf{0}}(f_0) = \operatorname{rank} \widetilde{H}_{n-1}(F_{f_0}) < \mathfrak{p}$, and we are finished.

Case 1: $\operatorname{char}^n_f(\lambda) = \lambda \pm 1$. The claims follow immediately.

Case 2: $\operatorname{char}^n_f(\lambda)$ is $(\lambda^{\mathfrak{p}} - 1)/(\lambda - 1)$ or $(\lambda^{\mathfrak{p}} + 1)/(\lambda + 1)$. Then, $\operatorname{char}_{\operatorname{im} \partial_{f,z_0}}(\lambda)$ is $\lambda \pm 1$; thus, under the isomorphism ω_{f,z_0}, $\operatorname{im} \partial_{f,z_0}$ is contained in the diagonal or anti-diagonal of $\mathbb{Z}^{\mathfrak{p}}$. By the first statement of Theorem 1.3, it follows that $\left(\Gamma^1_{f,z_0} \cdot V(z_0)\right)_{\mathbf{0}} = 1$ and so Γ^1_{f,z_0} is unitary. Now, the last statement of Theorem 1.3 implies that $\operatorname{im} \partial_{f,z_0}$ must be the entire diagonal or anti-diagonal. It follows that $\operatorname{coker} \partial_{f,z_0} \cong \widetilde{H}^n(F_f)$ is free Abelian, which is equivalent to $\widetilde{H}_{n-1}(F_f)$ being free Abelian.

The suspension claim is immediate from the properties discussed prior to the theorem. □

Example 2.4. We will show here that all of the cases of Corollary 2.3 can occur.

Note that, if $\dim_{\mathbf{0}} \Sigma f \leqslant 1$, then z_0 is prepolar if and only if $\dim_{\mathbf{0}} \Sigma f_0 \leqslant 0$.

First, consider $f = z_0^2 + z_1^2 + \cdots + z_n^2$. Then, we know that $\widetilde{H}_{n-1}(F_{f_0}) \cong \mathbb{Z}$ and $\widetilde{H}_n(F_f) \cong \mathbb{Z}$. By A'Campo's main theorem in [**A'C**], the trace of the monodromy action on $\widetilde{H}_n(F_f)$ is $(-1)^{n+1}$.

Now, as a cycle,

$$\Gamma^1_{f,z_0} = V\left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right) = V(z_1, \ldots, z_n).$$

Therefore, Γ^1_{f,z_0} has a single component and $\tau_{f,z_0} = 2$ is prime, and so we can apply Theorem 2.3. By looking at the trace, we conclude that we are in Case 1a if n is odd, and in Case 1b if n is even.

Now, we will give examples of Case 2. Suppose that $\dim_{\mathbf{0}} \Sigma f = 0$, $\dim_{\mathbf{0}} \Sigma f_0 = 0$, and that Γ^1_{f,z_0} is prime of order $\mathfrak{p} \geqslant 3$. Then, $\widetilde{H}_n(F_f) \cong \mathbb{Z}^{\mu_{\mathbf{0}}(f)}$. Therefore, if $\mu_{\mathbf{0}}(f) \geqslant 2$, then we must be in Case 2, and the trace distinguishes subcases a) and b).

To give a specific example, let $f = y^2 - x^3$, where we use x in place of z_0. Then, $\mu_{\mathbf{0}}(f) = 2$, $\Gamma^1_{f,x} = V(y)$, and $\tau_{f,x} = 3 = \mathfrak{p}$. By A'Campo's Lefschetz number result, we must be in Case 2b. By suspending, we find that $g := w^2 + y^2 - x^3$ (again, using $z_0 = x$) would be an example of Case 2a.

The example that we use for Case 0 was first shown to us by Dirk Siersma. Consider $f = (x^2 + y^2 - z^2)(y - z)$. We use the coordinate z for z_0. The critical locus of f is the line $V(x, y - z)$. As cycles, we find

$$V\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = V(2x(y-z),\ 2y(y-z) + x^2 + y^2 - z^2) = 3V(x, y-z) + V(x, 3y+z).$$

Therefore, $\Gamma^1_{f,z} = V(x, 3y + z)$ and $\tau_{f,z_0} = 3$. One also finds that $\mu_{\mathbf{0}}(f_0) = 4$. Thus, it is at least possible that we are in Case 1, but we must show this.

After an analytic coordinate change, $f = (x^2 + st)t$. As f is homogeneous, F_f is diffeomorphic to $f^{-1}(1)$. Now we observe that $f^{-1}(1)$ is the set of points where $t \neq 0$ and $s = (1 - tx^2)/t^2$. Thus, F_f is diffeomorphic to $\mathbb{C} \times \mathbb{C}^*$, and so is homotopy-equivalent to S^1. It follows that $\widetilde{H}_2(F_f) = 0$.

We would like to show that Case 2 of Theorem 2.3 rarely occurs. For this, we will need the result below.

Proposition 2.5. *Suppose that the rank of $\widetilde{H}_n(F_f)$ equals λ^0_{f,z_0}.*

Then, the trace of the monodromy action on $\widetilde{H}_n(F_f)$ *is*

$$(-1)^{n+1}\big(1-\chi(\mathbb{L}_{\Sigma f, z_0})\big),$$

where χ *denotes the Euler characteristic and* $\mathbb{L}_{\Sigma f, z_0}$ *is the "complex link of* Σf *at the origin with respect to* z_0*", i.e.,*

$$\mathbb{L}_{\Sigma f, z_0} := \overset{\circ}{B}_\epsilon \cap \Sigma f \cap V(z_0-\delta),$$

where $0 \ll |\delta| \ll \epsilon \ll 1$ *and* $\overset{\circ}{B}_\epsilon$ *is an open ball of radius* ϵ *centered at* $\mathbf{0}$*. Thus,*

In particular, if Σf *itself is smooth and transversely intersected by* $V(z_0)$ *at the origin, then the trace of the monodromy action on* $\widetilde{H}_n(F_f)$ *is* 0*.*

Proof. Recall from Remark 1.2 that the trace of the monodromy action on $\operatorname{im}\partial_{f,z_0}$ is negative the trace of the monodromy action on $\widetilde{H}_n(F_f)$.

Suppose that $\operatorname{rank}\widetilde{H}_n(F_f) = \lambda^0_{f,z_0}$. In the case where $\dim_{\mathbf{0}}\Sigma f \leqslant 1$, the analysis of the "nexus diagram" in Application 2 of [**M2**] tells us that the trace of the monodromy action on $\operatorname{im}\partial_{f,z_0}$ is

$$(-1)^{n+1}\big((|\Sigma f|\cdot V(z_0))_{\mathbf{0}}-1\big) = (-1)^n\big(1-\chi(\mathbb{L}_{\Sigma f,z_0})\big).$$

As we commented at the end of [**M2**], when the dimension of Σf is arbitrary, the nexus diagram still exists in the Abelian category of perverse sheaves, and the exact proof that we used when $\dim_{\mathbf{0}}\Sigma f \leqslant 1$ tells us that there is an equality of Lefschetz numbers of the respective monodromy actions at the origin given by

$$\mathcal{L}_{\mathbf{0}}\{\operatorname{im}\partial_{f,z_0}\} = \mathcal{L}_{\mathbf{0}}\{\phi_{f_0}[-1]\mathbb{Z}^\bullet_{V(z_0)}[n]\} - \mathcal{L}_{\mathbf{0}}\{\psi_{\hat{z}_0}[-1]\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]\},$$

where $\mathcal{L}_{\mathbf{0}}\{\mathbf{A}^\bullet\}$ denotes the Lefschetz number at the origin of the Milnor monodromy action on the complex $\mathbf{A}^\bullet$, and $\hat{z}_0$ is the restriction of z_0 to $V(f)$. Now, $\operatorname{im}\partial_{f,z_0}$ is a sub-perverse sheaf of a perverse sheaf which is supported on a point; hence, $\mathcal{L}_{\mathbf{0}}\{\operatorname{im}\partial_{f,z_0}\}$ is simply the trace of the monodromy action on $\operatorname{im}\partial_{f,z_0}$. In addition, as we are assuming that f has a critical point at the origin, A'Campo's result in [**A'C**] implies that $\mathcal{L}_{\mathbf{0}}\{\phi_{f_0}[-1]\mathbb{Z}^\bullet_{V(z_0)}[n]\} = (-1)^n$. It remains for us to show that

$$(\dagger)\qquad \mathcal{L}_{\mathbf{0}}\{\psi_{\hat{z}_0}[-1]\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]\} = (-1)^n\chi(\mathbb{L}_{\Sigma f,z_0}).$$

Consider the fundamental short exact sequence of perverse sheaves:

$$0 \to \mathbb{Z}_{V(f)}[n] \to \psi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \to \phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1] \to 0.$$

Let $\check{z}_0$ be the restriction of z_0 to Σf. If we restrict this sequence to Σf, then apply $\psi_{\check{z}_0}[-1]$, and use that locally $\Sigma f \subseteq V(f)$, we obtain a distinguished triangle

$$\psi_{\check{z}_0}[-1]\big(\mathbb{Z}_{\Sigma f}[n]\big) \to \psi_{\check{z}_0}[-1]\big(\,(\psi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1])_{|_{\Sigma f}}\,\big) \to \psi_{\check{z}_0}[-1]\big(\,(\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1])_{|_{\Sigma f}}\,\big) \xrightarrow{[1]},$$

on which the monodromy acts compatibly. Using A'Campo's result again, we obtain that

$$\mathcal{L}_{\mathbf{0}}\big(\psi_{\check{z}_0}[-1]\big(\,(\psi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1])_{|_{\Sigma f}}\,\big)\big) = 0.$$

Thus, by additivity, we obtain that

$$\mathcal{L}_{\mathbf{0}}\big(\psi_{\check{z}_0}[-1]\big(\,(\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1])_{|_{\Sigma f}}\,\big)\big) = -\mathcal{L}_{\mathbf{0}}\big(\psi_{\check{z}_0}[-1]\big(\mathbb{Z}_{\Sigma f}[n]\big)\big).$$

As the support of $\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}[n+1]$ already lies in Σf, we obtain (†). □

Corollary 2.6. *Suppose that Γ^1_{f,z_0} is prime of order $\mathfrak{p}$, and $\chi(\mathbb{L}_{\Sigma f,z_0})$ does not equal 0 or 2. Then, Case 2 of Theorem 2.3 does not occur, nor does Case 1 if $\mathfrak{p}=2$.*

In particular, if Γ^1_{f,z_0} is prime and Σf is itself smooth and transversely intersected by $V(z_0)$ at $\mathbf{0}$, then the rank of $\widetilde{H}_n(F_f)$ is 0 or 1.

Proof. In Case 2 of Theorem 2.3, or in Case 1 if $\mathfrak{p}=2$, the rank of $\widetilde{H}_n(F_f)$ equals $\tau_{f,z_0}-1=\lambda^0_{f,z_0}$, while the trace of the monodromy is ± 1. The Corollary follows at once from Proposition 2.5. □

In Example 2.4, we gave an example of a hypersurface with a line singularity which is a Case 0 example of Theorem 2.3. We also gave Case 1 examples which had isolated singularities. Corollary 2.6 tells us that we cannot produce a Case 2 example with a line singularity. Below, we give an example of a hypersurface with a line singularity which is Case 1.

Example 2.7. Consider the classic presentation of the Whitney umbrella as a family of nodes degenerating to a cusp: $f=y^2-x^3-tx^2$, where we use t for our prepolar coordinate. Then, $\mu_{\mathbf{0}}(f)=2$, $\Gamma^1_{f,t}=V(y,3x+2t)$, $\tau_{f,t}=3=\mathfrak{p}$, and $\lambda^0_{f,t}=2$. Thus, up to isomorphism, $\partial_{f,t}$ is a map from $\mathbb{Z}^2$ to $\mathbb{Z}^3$. Therefore, rank $\widetilde{H}_2(F_f)\geqslant 1$. However, as our critical locus is a line, we must be in Case b) of Theorem 2.5, and so rank $\widetilde{H}_2(F_f)<2$. We conclude the well-known: $\widetilde{H}_2(F_f)\cong\mathbb{Z}$, i.e., this is an example of Case 1 of Theorem 2.3.

§3. More Complicated Examples

Example 3.1. Consider the family of examples $g(t,x,y):=y^2-x^a-t^cx^b$, where $a,b,c\geqslant 2$, and a and b are relatively prime. If $a\leqslant b$, then $g=y^2-x^a(1-t^cx^{b-a})$, which after an analytic change of coordinates at the origin becomes y^2-x^a; this is simply a cross-product of an isolated hypersurface singularity. So, assume that $a>b\geqslant 2$. We also assume that $a-b$ and c are relatively prime. Note that this example subsumes Example 2.7.

One easily shows that $\Sigma g=V(x,y)$, and $g_0:=g_{|_{V(t)}}$ has an isolated critical point at the origin. Hence, t is a prepolar coordinate for g.

Now, the Milnor number of g_0 at the origin is $a-1$ and, hence, the reduced cohomology of F_{g_0} is 0 in degree 0 and is isomorphic to $\mathbb{Z}^{a-1}$ in degree 1. We would like to know, $\mathrm{char}^1_{g_0}(\lambda)$, the characteristic polynomial of the monodromy action on $\widetilde{H}^1(F_{g_0})$.

The function g_0 is the suspension of the function $-x^a$ on $\mathbb{C}$. The Milnor fiber of $-x^a$ is a points, which are permuted cyclically by the Milnor monodromy. Thus, the characteristic polynomial of the monodromy on the **reduced** cohomology $\widetilde{H}^0(F_{-x^a})$ is $(\lambda^a-1)/(\lambda-1)$ and so,

$$\mathrm{char}^1_{g_0}(\lambda)=(\lambda^a-(-1)^a)/(\lambda+1).$$

Now select a small $t_0\neq 0$. In the main theorem of [**M2**], we proved that, if $\widetilde{H}^1(F_g)\neq 0$, then $\mathrm{char}^1_g(\lambda)$ not only divides $\mathrm{char}^1_{g_0}(\lambda)$, but also divides $\mathrm{char}^1_{g_{t_0}}(\lambda)$, where g_{t_0} denotes $g_{|_{V(t-t_0)}}$.

Now, after an analytic change of coordinates at the origin (in $V(t-t_0)$), $g_{t_0} = y^2 - x^b(x^{a-b} - t_0^c)$ becomes $y^2 - x^b$. As this is the suspension of $-x^b$, we may use an analysis like that above to conclude that $\mathrm{char}^1_{g_{t_0}}(\lambda)$ equals $(\lambda^b - (-1)^b)/(\lambda+1)$.

As a and b are relatively prime, we conclude that $\mathrm{char}^1_{g_0}(\lambda)$ and $\mathrm{char}^1_{g_{t_0}}(\lambda)$ have no common divisors. Therefore, we conclude that $\widetilde{H}^1(F_g) = 0$.

As $V\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right) = V(ax^{a-1} + bt^c x^{b-1}, y)$, we find that $\Gamma^1_{g,t} = V(ax^{a-b} + bt^c, y)$. Hence, $\gamma^1_{g,t} = \left(\Gamma^1_{g,t} \cdot V(t)\right)_{\mathbf{0}} = a - b$, and

$$\lambda^0_{g,t} = \left(\Gamma^1_{g,t} \cdot V\left(\frac{\partial g}{\partial t}\right)\right)_{\mathbf{0}} = \left(\Gamma^1_{g,t} \cdot V(t^{c-1}x^b)\right)_{\mathbf{0}} = (c-1)(a-b) + bc = ac - (a-b).$$

Thus, $\tau_{g,t} = \gamma^1_{g,t} + \lambda^0_{g,t} = ac$. As a and b are relatively prime, so are $a-b$ and a. Therefore, as $a-b$ and c are also relatively prime, we find that $\gamma^1_{g,t}$ and $\tau_{g,t}$ are relatively prime. In addition, since $a-b$ and c are relatively prime, the polar curve $\Gamma^1_{g,t} = V(ax^{a-b} + bt^c, y)$ has a single irreducible component.

We conclude that $\Gamma^1_{g,t}$ is relatively prime and, hence, the carrousel of g with respect to t is semi-simple. Thus, $\mathrm{char}_{\mathrm{rel}_{g,t}}(\lambda) = \lambda^{ac} \pm 1$.

So, we have the map $\partial_{g,t} : \widetilde{H}^1(F_{g_0}) \to H^2(F_g, F_{g_0})$, where $\widetilde{H}^1(F_{g_0}) \cong \mathbb{Z}^{a-1}$, $H^2(F_g, F_{g_0}) \cong \mathbb{Z}^{ac}$, $\mathrm{char}^1_{g_0}(\lambda) = (\lambda^a - (-1)^a)/(\lambda+1)$, $\mathrm{char}_{\mathrm{rel}_{g,t}}(\lambda) = \lambda^{ac} \pm 1$, and $\widetilde{H}^1(F_g) = 0$. It follows that $\widetilde{H}_2(F_g) \cong \mathbb{Z}^{ac-a+1}$ and that

$$\mathrm{char}^2_g(\lambda) = (\lambda^{ac} \pm 1)(\lambda+1)/(\lambda^a - (-1)^a).$$

Note that, if $b = 2$, this example is an *isolated line singularity*, as studied by Siersma in [**S**]. In this case, Siersma's work tells us a bit more: it says that F_g has the homotopy-type of a bouquet of $(ac - a + 1)$ 2-spheres.

Example 3.2. In this example, we will look at $f(s,t,x,y) = y^2 - x^4 + (s^3 - t^2)x^3$. One easily checks that $\Sigma f = V(x,y)$, and so f has a 2-dimensional critical locus. Note also that $f_0 := f_{|_{V(s)}}$ is a function of the form of g from Example 3.1, with $a = 4$, $b = 3$, and $c = 2$.

We wish to see what our results can tell us about the cohomology and the monodromy of the Milnor fiber.

Our first problem is to verify that s is a prepolar coordinate for f at the origin. This means that we must first produce a good stratification. For this, we use the Lê cycles and numbers, and apply Corollary 6.6. and Remark 6.7 of [**M1**]. We fix the coordinate system (s,t,x,y) and will suppress any further reference to the coordinates.

We proceed with the calculation of the polar and Lê cycles (see [**M1**]):

$$\Gamma^3_f = V\left(\frac{\partial f}{\partial y}\right) = V(y);$$

$$\Gamma^3_f \cdot V\left(\frac{\partial f}{\partial x}\right) = V(y) \cdot V\big(-4x^3 + 3(s^3 - t^2)x^2\big) =$$

$$V(y) \cdot \Big(V\big(-4x + 3(s^3 - t^2)\big) \ + \ 2V(x)\Big) = V(-4x + 3(s^3 - t^2), y) + 2V(x,y) =$$

$$\Gamma^2_f + \Lambda^2_f;$$

$$\Gamma^2_f \cdot V\left(\frac{\partial f}{\partial t}\right) = V(-4x + 3(s^3 - t^2), y) \cdot V(-2tx^3) =$$

$$V(-4x + 3(s^3 - t^2), y) \cdot \big(V(t) + 3V(x)\big) =$$

$$V(-4x + 3s^3, y, t) + 3V(s^3 - t^2, x, y) = \Gamma^1_f + \Lambda^1_f;$$

and, finally,

$$\Gamma^1_f \cdot V\left(\frac{\partial f}{\partial s}\right) = V(-4x + 3s^3, y, t) \cdot V(3s^2x^3) = 2[\mathbf{0}] + 3 \cdot 3[\mathbf{0}] = 11[\mathbf{0}] = \Lambda^0_f.$$

Thus, we have $\Lambda^2_f = 2V(x, y)$, $\Lambda^1_f = 3V(s^3 - t^2, x, y)$, and $\Lambda^0_f = 11[\mathbf{0}]$.

One easily calculates the Lê numbers at a point $\mathbf{p} := (s_0, t_0, x_0, y_0)$ near $\mathbf{0}$:

$\lambda^0_f(\mathbf{p})$ equals 11 at the origin, and equals 0 elsewhere;

$\lambda^1_f(\mathbf{p}) = \big(3V(s^3 - t^2, x, y) \cdot V(s - s_0)\big)_{\mathbf{p}}$ equals $3 \cdot 2 = 6$ at the origin, 3 at other points of $V(s^3 - t^2, x, y)$, and equals 0 elsewhere;

$\lambda^2_f(\mathbf{p})$ equals 2 at all points of $V(x, y)$, and equals 0 elsewhere.

Therefore, Corollary 6.6. of [**M1**] tells us that $V(f)$ has a good stratification at the origin:

$$\{V(f) - V(x, y),\ V(x, y) - V(s^3 - t^2, x, y),\ V(s^3 - t^2, x, y) - \{\mathbf{0}\}, \{\mathbf{0}\}\}.$$

Now, $f_0 = y^2 - x^4 - t^2x^3$ has a critical locus consisting of just the t-axis, i.e., $V(x, y)$ inside $V(s)$. We see then that $V(s)$ transversely intersects all of the good strata, except $\{\mathbf{0}\}$, in a neighborhood of the origin, i.e., s is prepolar for f at $\mathbf{0}$.

We continue to calculate (and continue to suppress the coordinates in the notation):

$$\gamma^1_f = \big(\Gamma^1_f \cdot V(s)\big)_{\mathbf{0}} = \big(V(-4x + 3s^3, y, t) \cdot V(s)\big)_{\mathbf{0}} = 1;$$

and

$$\tau_f = \gamma^1_f + \lambda^0_f = 1 + 11 = 12.$$

As $\gamma^1_f = 1$, Γ^1_f is unitary. Proposition 2.2 tells us that s is a carrousel coordinate and that the carrousel of f with respect to s is semi-simple.

Putting all of the above work together, including our result in Example 3.1, we find that the map $\partial_{f,s} : \widetilde{H}^2(F_{f_0}) \to H^3(F_f, F_{f_0})$ is a map from a $\mathbb{Z}$-module of rank 5 to a copy of $\mathbb{Z}^12$, and the respective characteristic polynomials of the monodromy, acting on the free parts, are $\mathrm{char}^2_f(\lambda) = (\lambda^8 \pm 1)(\lambda + 1)/(\lambda^4 - 1)$ and $\mathrm{char}_{\mathrm{rel}_{f,s}}(\lambda) = \lambda^{12} \pm 1$.

Thus, in $\mathrm{char}^2_f(\lambda)$, we must choose the minus sign, and so $\mathrm{char}^2_{f_0}(\lambda) = (\lambda+1)(\lambda^4+1)$. By Proposition 2.5, the rank of $\mathrm{im}(\partial_{f,s})$ cannot be 1. Therefore, we must have one of two cases:

i) $\widetilde{H}^0(F_f) = 0$, $\widetilde{H}^1(F_f) = 0$, $\mathrm{rank}\,\widetilde{H}^2(F_f) = 1$, $\mathrm{rank}\,\widetilde{H}^3(F_f) = 12 - 4 = 8$, $\mathrm{char}^2_f(\lambda) = \lambda + 1$, and $\mathrm{char}^3_f(\lambda) = (\lambda^{12} + 1)/(\lambda^4 + 1)$;

or

ii) $\widetilde{H}^0(F_f)=0$, $\widetilde{H}^1(F_f)=0$, $\operatorname{rank}\widetilde{H}^2(F_f)=0$, $\operatorname{rank}\widetilde{H}^3(F_f)=12-5=7$, and

$$\operatorname{char}^3_f(\lambda)=(\lambda^{12}+1)/[(\lambda+1)(\lambda^4+1)].$$

We do not, in fact, know which of these cases is the correct one.

§4. **Concluding Remarks**

The main point of this paper is that the single number $\tau_{f,z_0}=\left(\Gamma^1_{f,z_0}\cdot V(f)\right)_{\mathbf{0}}$ can tell one a great deal about $\widetilde{H}_n(F_f)$, at least when τ_{f,z_0} is prime.

However, the calculation of τ_{f,z_0} is not a simple algebra exercise for complicated f. If one wants to use Theorem 2.3, one must first "calculate" the polar curve, see that it has only one component, and then see that τ_{f,z_0} is prime. Moreover, as we saw in Example 3.2, if $\dim_{\mathbf{0}}\Sigma f\geqslant 2$, then it is nontrivial to verify that z_0 is a prepolar coordinate.

Nonetheless, the case where Γ^1_{f,z_0} is prime occurs in enough examples that we find Theorem 2.3 to be interesting.

Together with Lê Dũng Tráng, we believe that we can prove a generalization of Theorem 2.5. That generalization says that if f has a smooth 1-dimensional critical locus, and z_0 is a carrousel form, then either $\Gamma^1_{f,z_0}=\emptyset$ or $\operatorname{rank}\widetilde{H}_n(F_f)<\lambda^0_{f,z_0}$. This result requires a more detailed study of the carrousel and the swing.

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