

PERVERSE COHOMOLOGY AND THE VANISHING INDEX THEOREM

DAVID B. MASSEY

ABSTRACT. The characteristic cycle of a complex of sheaves on a complex analytic space provides weak information about the complex; essentially, it yields the Euler characteristics of the hypercohomology of normal data to strata. We show how perverse cohomology actually allows one to extract the individual Betti numbers of the hypercohomology of normal data to strata, not merely the Euler characteristics. We apply this to the “calculation” of the vanishing cycles of a complex, and relate this to the work of Parusiński and Briançon, Maisonobe, and Merle on Thom’s a_f condition.

§0. Introduction

Let X be a d -dimensional complex analytic space contained in some open subset \mathcal{U} of some \mathbb{C}^{n+1} . Let $\tilde{f} : \mathcal{U} \rightarrow \mathbb{C}$ be a complex analytic function and $f := \tilde{f}|_X$. Let $\mathcal{S} = \{S_\alpha\}$ be a Whitney stratification of X with connected strata, and let d_α denote the dimension of S_α .

Let R be a base ring which is a p.i.d., and let \mathbf{F}^\bullet be a bounded complex of sheaves of R -modules which is constructible with respect to \mathcal{S} ; we write $\mathbf{F}^\bullet \in D_{\mathcal{S}}^b(X)$. Note that our assumptions about R guarantee that such an \mathbf{F}^\bullet is perfect (see [G-M1], 1.4 and [K-S], 8.4.3). We denote the full subcategory of $D^b(X)$ of (complex) constructible complexes by $D_c^b(X)$.

The main question which is addressed in this paper is: given the complex \mathbf{F}^\bullet , how does one calculate the vanishing cycle complex $\phi_f \mathbf{F}^\bullet$?

Of course, one should immediately ask: what does it mean to be “given” \mathbf{F}^\bullet and what does it mean to “calculate” $\phi_f \mathbf{F}^\bullet$? In Theorem 2.10 of [M1], we give an algebraic method for calculating the characteristic cycle $\text{Ch}(\phi_f \mathbf{F}^\bullet)$ when one starts with $\text{Ch}(\mathbf{F}^\bullet)$ – we refer to this result as *the vanishing index theorem*. However, giving the characteristic cycle data is a far cry from describing the structure of a complex of sheaves; the characteristic cycle supplies only Euler characteristic information about the strata.

In Theorem 3.1 of this paper, we show how to calculate the hypercohomology of normal slices mod complex links of strata of $\phi_f \mathbf{F}^\bullet$, starting with analogous information about the complex \mathbf{F}^\bullet . While this is still fairly coarse data to associate to \mathbf{F}^\bullet and $\phi_f \mathbf{F}^\bullet$, it is certainly a substantial improvement over the characteristic cycle information.

Our technique for deriving our formulas is simple, but technical: we apply the tool of *perverse cohomology* (see Section 2, and [BBD] and [K-S]) to the vanishing index theorem, and the Euler characteristic data almost magically turns into Betti number data. In [M3], we demonstrated this technique in the simple case in which the vanishing cycles are supported at an isolated point.

Almost by definition of what we are trying to calculate, we do not need to begin with Whitney stratifications, but rather analytic partitions for which the normal data to “strata” is well-defined,

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and then we care about only those strata which have non-trivial hypercohomological normal data. Hence, we define \mathbf{F}^\bullet -visible strata and \mathbf{F}^\bullet -normal partitionings of X in 2.3 and 2.7.

In the final section of this paper, we relate the main theorem (3.1) to Thom's a_f condition. In Theorem 4.4, we show that the vanishing cycles along f control Thom's a_f condition. We then show how our results relate to the results of Parusiński [P] and Briançon, Maisonobe, and Merle [BMM] that Whitney stratifications adapted to $V(f)$ are a_f stratifications.

§1. The Vanishing Index Theorem

We continue with the notation from the introduction.

In this section, we are going to give a general result which describes the characteristic cycle of $\phi_f \mathbf{F}^\bullet$ in terms of blowing-up the image of $d\tilde{f}$ inside the conormal spaces to strata. This is Theorem 2.10 of [M1].

Because d is the global dimension of X , and we are not assuming that X is pure-dimensional, or that f is not constant on a d -dimensional component of X , if $v \in \mathbb{C}$, then the dimension of $V(f - v)$ could be anything between 0 and d . Hence, we let $\hat{d}_v := 1 + \dim V(f - v)$, and will usually denote \hat{d}_0 by simply \hat{d} . Of course, if we work locally, or assume that X is pure-dimensional, and require f not to vanish on a component of X , then \hat{d} will have attain its “expected” value of d .

For each stratum $S_\alpha \in \mathcal{S}$, there is a pair $(\mathbb{N}_\alpha, \mathbb{L}_\alpha)$ consisting of a normal slice and the complex link of S_α ; the isomorphism class of the hypercohomology $\mathbb{H}^*(\mathbb{N}_\alpha, \mathbb{L}_\alpha; \mathbf{F}^\bullet)$ is well-defined, and we refer to it as the *normal data of S_α with respect to \mathbf{F}^\bullet* (see [G-M2]).

Definition 1.1. Recall that the *characteristic cycle*, $\text{Ch}(\mathbf{F}^\bullet)$, of \mathbf{F}^\bullet in $T^*\mathcal{U}$ is the linear combination $\sum_\alpha m_\alpha(\mathbf{F}^\bullet) \left[\overline{T_{S_\alpha}^* \mathcal{U}} \right]$, where the $m_\alpha(\mathbf{F}^\bullet)$ are integers determined by the Euler characteristic:

$$\begin{aligned} m_\alpha(\mathbf{F}^\bullet) &:= (-1)^{d-d_\alpha} \chi(\phi_{L|_X}[-1]\mathbf{F}^\bullet)_{\mathbf{x}} = (-1)^{d-d_\alpha} \chi(\phi_{L|\mathbb{N}_\alpha}[-1]\mathbf{F}^\bullet|_{\mathbb{N}_\alpha}[-d_\alpha])_{\mathbf{x}} = \\ &(-1)^{d-d_\alpha} \chi(\mathbb{H}^*(\mathbb{N}_\alpha, \mathbb{L}_\alpha; \mathbf{F}^\bullet)) \end{aligned}$$

for any point \mathbf{x} in S_α , with normal slice \mathbb{N}_α at \mathbf{x} , and any $L : (\mathcal{U}, x) \rightarrow (\mathbb{C}, 0)$ such that $d_{\mathbf{x}}L$ is a non-degenerate covector at \mathbf{x} (with respect to our fixed stratification; see [G-M2]) and $L|_{S_\alpha}$ has a Morse singularity at \mathbf{x} . This cycle is independent of all the choices made (see, for instance, [K-S, Chapter IX]).

Using the isomorphism, $T^*\mathcal{U} \cong \mathcal{U} \times \mathbb{C}^{n+1}$, we consider $\text{Ch}(\mathbf{F}^\bullet)$ as a cycle in $X \times \mathbb{C}^{n+1}$; we use $\mathbf{z} := (z_0, \dots, z_n)$ as coordinates on \mathcal{U} and $\mathbf{w} := (w_0, \dots, w_n)$ as the cotangent coordinates.

Let I denote the sheaf of ideals on \mathcal{U} given by the image of $d\tilde{f}$, i.e., $I = \langle w_0 - \frac{\partial \tilde{f}}{\partial z_0}, \dots, w_n - \frac{\partial \tilde{f}}{\partial z_n} \rangle$. For all α , let $B_\alpha = \text{Bl}_{\text{im } d\tilde{f}} \overline{T_{S_\alpha}^* \mathcal{U}}$ denote the blow-up of $\overline{T_{S_\alpha}^* \mathcal{U}}$ along the image of I in $\overline{T_{S_\alpha}^* \mathcal{U}}$, and let E_α denote the corresponding exceptional divisor. For all α , we have $E_\alpha \subseteq B_\alpha \subseteq X \times \mathbb{C}^{n+1} \times \mathbb{P}^n$. Let $\pi : X \times \mathbb{C}^{n+1} \times \mathbb{P}^n \rightarrow X \times \mathbb{P}^n$ denote the projection. Note that, if $(\mathbf{x}, \mathbf{w}, [\eta]) \in E_\alpha$, then $\mathbf{w} = d_{\mathbf{x}}\tilde{f}$ and so, for all α , π induces an isomorphism from E_α to $\pi(E_\alpha)$. We refer to $E := \sum_\alpha m_\alpha E_\alpha$ as the *total exceptional divisor* inside the total blow-up $\text{Bl}_{\text{im } d\tilde{f}} \text{Ch}(\mathbf{F}^\bullet) := \sum_\alpha m_\alpha \text{Bl}_{\text{im } d\tilde{f}} \left[\overline{T_{S_\alpha}^* \mathcal{U}} \right]$.

Theorem 2.10 of [M1] is:

Theorem 1.2 (Vanishing Index Theorem). *The projection π induces an isomorphism between the total exceptional divisor $E \subseteq \mathrm{Bl}_{\mathrm{im} d\tilde{f}} \mathrm{Ch}(\mathbf{F}^\bullet)$ and the sum over all $v \in \mathbb{C}$ of the (shifted) projectivized characteristic cycles of the sheaves of vanishing cycles of \mathbf{F}^\bullet along $f - v$, i.e.,*

$$E \cong \pi_*(E) = \sum_{v \in \mathbb{C}} (-1)^{d-\tilde{d}_v} \mathbb{P}(\mathrm{Ch}(\phi_{f-v} \mathbf{F}^\bullet)).$$

Remark 1.3. Actually, there are two mild differences between the above statement and that of 2.10 of [M1].

First of all, in [M1], we assumed that our base ring R was the complex field. However, the proof is entirely Morse-theoretic, and hence goes through without change when R is a p.i.d. (The reason for needing a p.i.d. is so that the rank of a finitely-generated R -module is well-defined and is additive over long exact sequences.)

Secondly, we have introduced the factor $(-1)^{d-\tilde{d}_v}$; the lack of this factor was an error in [M1]. Essentially, we had assumed that the dimension of $V(f)$ was always $d - 1$. The proof is unchanged.

We shall need one other result which follows from our work in [M1].

Proposition 1.4. *If R is a p.i.d., then*

$$\{\mathfrak{p} \in X \mid f(\mathfrak{p}) = 0, (\mathfrak{p}, d_{\mathfrak{p}} \tilde{f}) \in |\mathrm{Ch}(\mathbf{F}^\bullet)|\} \subseteq \mathrm{supp} \phi_f \mathbf{F}^\bullet.$$

If R is a field and \mathbf{P}^\bullet is a perverse sheaf on X , then

$$\{\mathfrak{p} \in X \mid f(\mathfrak{p}) = 0, (\mathfrak{p}, d_{\mathfrak{p}} \tilde{f}) \in |\mathrm{Ch}(\mathbf{P}^\bullet)|\} = \mathrm{supp} \phi_f \mathbf{P}^\bullet.$$

Proof. The first statement is immediate from Theorem 2.10 of [M1]. The second statement follows trivially from Theorem 3.2 of [M1]. \square

§2. Perverse Cohomology and Visible Strata

We wish to show how one can use perverse cohomology to extract Betti number information from Theorem 2.1. We list some properties of the perverse cohomology and of vanishing cycles that we will need later. The reader is referred to [BBD] and [K-S3].

The perverse cohomology functor (using middle perversity, μ) on X , ${}^\mu H^0$, is a functor from $D_{\mathbb{C}}^b(X)$ to the Abelian category of perverse sheaves on X . One lets ${}^\mu H^i(\mathbf{F}^\bullet)$ denote ${}^\mu H^0(\mathbf{F}^\bullet[i])$.

If \mathbf{F}^\bullet is constructible with respect to \mathcal{S} , then ${}^\mu H^0(\mathbf{F}^\bullet)$ is also constructible with respect to \mathcal{S} , and $({}^\mu H^0(\mathbf{F}^\bullet))|_{\mathbb{N}_\alpha}[-d_\alpha]$ is naturally isomorphic to ${}^\mu H^0(\mathbf{F}^\bullet|_{\mathbb{N}_\alpha}[-d_\alpha])$.

The functor ${}^{\mu}H^0$, applied to a perverse sheaf \mathbf{P}^{\bullet} is canonically isomorphic to \mathbf{P}^{\bullet} . In addition, a bounded, constructible complex of sheaves \mathbf{F}^{\bullet} is perverse if and only if ${}^{\mu}H^k(\mathbf{F}^{\bullet}) = 0$ for all $k \neq 0$. In particular, if X is a local complete intersection, then ${}^{\mu}H^{\dim X}(\mathbb{Z}_X^{\bullet}) \cong \mathbb{Z}_X^{\bullet}[\dim X]$ and ${}^{\mu}H^k(\mathbb{Z}_X^{\bullet}) = 0$ if $k \neq \dim X$.

The functor ${}^{\mu}H^0$ commutes with vanishing cycles with a shift of -1 , nearby cycles with a shift of -1 , and Verdier dualizing. That is, there are natural isomorphisms

$${}^{\mu}H^0 \circ \phi_f[-1] \cong \phi_f[-1] \circ {}^{\mu}H^0, \quad {}^{\mu}H^0 \circ \psi_f[-1] \cong \psi_f[-1] \circ {}^{\mu}H^0, \quad \text{and } \mathcal{D} \circ {}^{\mu}H^0 \cong {}^{\mu}H^0 \circ \mathcal{D}.$$

Let \mathbf{F}^{\bullet} be a bounded complex of sheaves on X which is constructible with respect to \mathcal{S} . Let S_{\max} be a maximal stratum (i.e., one not contained in the closure of another) which is contained in the support of \mathbf{F}^{\bullet} , and let $m = \dim S_{\max}$. Then, $({}^{\mu}H^0(\mathbf{F}^{\bullet}))|_{S_{\max}}$ is isomorphic (in the derived category) to the complex which has $(\mathbf{H}^{-m}(\mathbf{F}^{\bullet}))|_{S_{\max}}$ in degree $-m$ and zero in all other degrees.

In particular, $\text{supp } \mathbf{F}^{\bullet} = \bigcup_i \text{supp } {}^{\mu}H^i(\mathbf{F}^{\bullet})$, and if \mathbf{F}^{\bullet} is supported on an isolated point, \mathbf{q} , then $H^0({}^{\mu}H^0(\mathbf{F}^{\bullet}))_{\mathbf{q}} \cong H^0(\mathbf{F}^{\bullet})_{\mathbf{q}}$. Also, $\text{Ch}(\mathbf{F}^{\bullet}) = \sum_i (-1)^i \text{Ch}({}^{\mu}H^i(\mathbf{F}^{\bullet}))$.

A distinguished triangle

$$\mathbf{A}^{\bullet} \rightarrow \mathbf{B}^{\bullet} \rightarrow \mathbf{C}^{\bullet} \rightarrow \mathbf{A}^{\bullet}[1]$$

determines a long exact sequence in the Abelian category of perverse sheaves

$$\dots \rightarrow {}^{\mu}H^{-1}(\mathbf{B}^{\bullet}) \rightarrow {}^{\mu}H^{-1}(\mathbf{C}^{\bullet}) \rightarrow {}^{\mu}H^0(\mathbf{A}^{\bullet}) \rightarrow {}^{\mu}H^0(\mathbf{B}^{\bullet}) \rightarrow {}^{\mu}H^0(\mathbf{C}^{\bullet}) \rightarrow {}^{\mu}H^1(\mathbf{A}^{\bullet}) \rightarrow \dots$$

Switching Base Rings

In order to detect torsion by using Euler characteristics, we have to be able to switch base rings for our complexes. For each prime ideal \mathfrak{p} of R , let $k_{\mathfrak{p}}$ denote the field of fractions of R/\mathfrak{p} , i.e., k_0 is the field of fractions of R , and for $\mathfrak{p} \neq 0$, $k_{\mathfrak{p}} = R/\mathfrak{p}$. There are the obvious functors $\delta_{\mathfrak{p}} : \mathbf{D}_{\mathbb{C}}^b(R_X) \rightarrow \mathbf{D}_{\mathbb{C}}^b((k_{\mathfrak{p}})_X)$, which sends \mathbf{F}^{\bullet} to $\mathbf{F}^{\bullet} \otimes^L (k_{\mathfrak{p}})_X$, and $\epsilon_{\mathfrak{p}} : \mathbf{D}_{\mathbb{C}}^b((k_{\mathfrak{p}})_X) \rightarrow \mathbf{D}_{\mathbb{C}}^b(R_X)$, which considers $k_{\mathfrak{p}}$ -vector spaces as R -modules.

If \mathbf{A}^{\bullet} is a complex of $k_{\mathfrak{p}}$ -vector spaces, we may consider the perverse cohomology of \mathbf{A}^{\bullet} , ${}^{\mu}H_{k_{\mathfrak{p}}}^i(\mathbf{A}^{\bullet})$, or the perverse cohomology of $\epsilon(\mathbf{A}^{\bullet})$, which we denote by ${}^{\mu}H_R^i(\mathbf{A}^{\bullet})$. If $\mathbf{A}^{\bullet} \in \mathbf{D}_{\mathbb{C}}^b((k_{\mathfrak{p}})_X)$ and S_{\max} is a maximal stratum contained in the support of \mathbf{A}^{\bullet} , then there is a canonical isomorphism

$$\epsilon(({}^{\mu}H_{k_{\mathfrak{p}}}^i(\mathbf{A}^{\bullet}))|_{S_{\alpha}}) \cong ({}^{\mu}H_R^i(\mathbf{A}^{\bullet}))|_{S_{\alpha}};$$

in particular, $\text{supp } {}^{\mu}H_{k_{\mathfrak{p}}}^i(\mathbf{A}^{\bullet}) = \text{supp } {}^{\mu}H_R^i(\mathbf{A}^{\bullet})$.

If $\mathbf{F}^{\bullet} \in \mathbf{D}_{\mathbb{C}}^b(R_X)$, S_{\max} is a maximal stratum contained in the support of \mathbf{F}^{\bullet} , and $\mathbf{x} \in S_{\max}$, then for some prime ideal $\mathfrak{p} \subset R$ and for some integer i , $H^i(\mathbf{F}^{\bullet})_{\mathbf{x}} \otimes k_{\mathfrak{p}} \neq 0$; it follows that S_{\max} is also a maximal stratum in the support of $\mathbf{F}^{\bullet} \otimes^L (k_{\mathfrak{p}})_X$. Thus,

$$\text{supp } \mathbf{F}^{\bullet} = \bigcup_{\mathfrak{p}} \text{supp}(\mathbf{F}^{\bullet} \otimes^L (k_{\mathfrak{p}})_X)$$

and so

$$\text{supp } \mathbf{F}^{\bullet} = \bigcup_{i, \mathfrak{p}} \text{supp } {}^{\mu}H_{k_{\mathfrak{p}}}^i(\mathbf{F}^{\bullet} \otimes^L (k_{\mathfrak{p}})_X),$$

where the boundedness and constructibility of \mathbf{F}^{\bullet} imply that this union is locally finite.

For each prime ideal \mathfrak{p} , there is a natural isomorphism in $\mathbf{D}_c^b(R_{V(f)})$ given by

$$\phi_f(\mathbf{F}^\bullet \overset{L}{\otimes} (k_{\mathfrak{p}})_X^\bullet) \cong (\phi_f \mathbf{F}^\bullet) \overset{L}{\otimes} (k_{\mathfrak{p}})_{V(f)}^\bullet$$

(this is a particularly trivial case of the Sebastiani-Thom Isomorphism of [M4]), and hence, the stalk cohomology is given by

$$H^i(\phi_f(\mathbf{F}^\bullet \overset{L}{\otimes} (k_{\mathfrak{p}})_X^\bullet))_{\mathbf{x}} \cong (H^i(\phi_f \mathbf{F}^\bullet)_{\mathbf{x}} \otimes k_{\mathfrak{p}}) \oplus \mathrm{Tor}(H^{i+1}(\phi_f \mathbf{F}^\bullet)_{\mathbf{x}}, k_{\mathfrak{p}}).$$

Below, we use b_j to denote the j -th Betti number, e.g., $b_j(\mathbb{N}_\alpha, \mathbb{L}_\alpha; \mathbf{F}^\bullet) = \mathrm{rk}_R \mathbb{H}^j(\mathbb{N}_\alpha, \mathbb{L}_\alpha; \mathbf{F}^\bullet)$.

Proposition 2.1. *For all integers i and for all prime ideals \mathfrak{p} in R , the characteristic cycle of the perverse cohomology of the sheaf of $\mathbf{F}^\bullet[i]$ is given by*

$$\mathrm{Ch}(\mu H^i(\mathbf{F}^\bullet)) = (-1)^{\dim X} \sum_{\alpha} b_{i-d_\alpha}(\mathbb{N}_\alpha, \mathbb{L}_\alpha; \mathbf{F}^\bullet) \left[\overline{T_{S_\alpha}^* \mathcal{U}} \right],$$

and the characteristic cycle of the perverse cohomology of the sheaf of $k_{\mathfrak{p}}$ -vector spaces $\mathbf{F}^\bullet[i] \overset{L}{\otimes} (k_{\mathfrak{p}})_X^\bullet$ is given by

$$\mathrm{Ch}(\mu H_{k_{\mathfrak{p}}}^i(\mathbf{F}^\bullet \overset{L}{\otimes} (k_{\mathfrak{p}})_X^\bullet)) = (-1)^{\dim X} \sum_{\alpha} c_{i-d_\alpha}^{\mathfrak{p}}(\mathbb{N}_\alpha, \mathbb{L}_\alpha; \mathbf{F}^\bullet) \left[\overline{T_{S_\alpha}^* \mathcal{U}} \right],$$

where $c_j^{\mathfrak{p}}(\mathbb{N}_\alpha, \mathbb{L}_\alpha; \mathbf{F}^\bullet) := \dim_{k_{\mathfrak{p}}} (H^j(\mathbb{N}_\alpha, \mathbb{L}_\alpha; \mathbf{F}^\bullet) \otimes k_{\mathfrak{p}}) + \dim_{k_{\mathfrak{p}}} \mathrm{Tor}(H^{j+1}(\mathbb{N}_\alpha, \mathbb{L}_\alpha; \mathbf{F}^\bullet), k_{\mathfrak{p}})$.

Proof. The first formula is derived in the same way as the second, except that one has no “ $\overset{L}{\otimes} (k_{\mathfrak{p}})_X^\bullet$ ”’s anywhere; alternatively, one can deduce the first formula quickly from the second by letting $\mathfrak{p} = \langle 0 \rangle$. Thus, we shall derive the second formula only.

Let $\mathbf{x} \in S_\alpha$. Then,

$$\begin{aligned} m_\alpha(\mu H_{k_{\mathfrak{p}}}^i(\mathbf{F}^\bullet \overset{L}{\otimes} (k_{\mathfrak{p}})_X^\bullet)) &= (-1)^{\dim X - d_\alpha - 1} \chi(\phi_{L|_{\mathbb{N}_\alpha}}(\mu H^0(\mathbf{F}^\bullet[i] \overset{L}{\otimes} (k_{\mathfrak{p}})_X^\bullet))_{|\mathbb{N}_\alpha})_{\mathbf{x}} = \\ &= (-1)^{\dim X} \chi(\phi_{L|_{\mathbb{N}_\alpha}}[-1](\mu H^0(\mathbf{F}^\bullet[i] \overset{L}{\otimes} (k_{\mathfrak{p}})_X^\bullet))_{|\mathbb{N}_\alpha}[-d_\alpha])_{\mathbf{x}} = \\ &= (-1)^{\dim X} \chi(\phi_{L|_{\mathbb{N}_\alpha}}[-1] \mu H^0((\mathbf{F}^\bullet_{|\mathbb{N}_\alpha} \overset{L}{\otimes} (k_{\mathfrak{p}})_{\mathbb{N}_\alpha}^\bullet)[i - d_\alpha]))_{\mathbf{x}} = \\ &= (-1)^{\dim X} \dim_{k_{\mathfrak{p}}} H^0(\phi_{L|_{\mathbb{N}_\alpha}}[-1]((\mathbf{F}^\bullet_{|\mathbb{N}_\alpha} \overset{L}{\otimes} (k_{\mathfrak{p}})_{\mathbb{N}_\alpha}^\bullet)[i - d_\alpha]))_{\mathbf{x}} = \\ &= (-1)^{\dim X} \dim_{k_{\mathfrak{p}}} \left[(H^{i-d_\alpha}(\phi_{L|_{\mathbb{N}_\alpha}}[-1] \mathbf{F}^\bullet_{|\mathbb{N}_\alpha})_{\mathbf{x}} \otimes k_{\mathfrak{p}}) \oplus \mathrm{Tor}(H^{i-d_\alpha+1}(\phi_{L|_{\mathbb{N}_\alpha}}[-1] \mathbf{F}^\bullet_{|\mathbb{N}_\alpha})_{\mathbf{x}}, k_{\mathfrak{p}}) \right]. \end{aligned}$$

The proof of the first statement is identical, with $k_{\mathfrak{p}}$ replaced by R . \square

By combining 2.1 with 1.4, we obtain

Proposition 2.2. *The following three conditions are equivalent:*

- a) $\mathbf{x} \in \text{supp } \phi_{f-f(\mathbf{x})} \mathbf{F}^\bullet$;
- b) *there exists an integer i and the prime ideal \mathfrak{p} in R such that*

$$\mathbf{x} \in \text{supp } \phi_{f-f(\mathbf{x})} {}^\mu H_{k_{\mathfrak{p}}}^i (\mathbf{F}^\bullet \otimes^L (k_{\mathfrak{p}})_{\mathbf{x}}^\bullet);$$

and

- c) *there exists an integer i and the prime ideal \mathfrak{p} in R such that*

$$(\mathbf{x}, d_{\mathbf{x}} \tilde{f}) \in |\text{Ch}({}^\mu H_{k_{\mathfrak{p}}}^i (\mathbf{F}^\bullet \otimes^L (k_{\mathfrak{p}})_{\mathbf{x}}^\bullet))|.$$

Proof. By 1.4, b) and c) are equivalent. Now, a) and b) are equivalent because

$$\begin{aligned} \text{supp } \phi_{f-f(\mathbf{x})} \mathbf{F}^\bullet &= \bigcup_{i, \mathfrak{p}} \text{supp } {}^\mu H_{k_{\mathfrak{p}}}^i (\phi_{f-f(\mathbf{x})} \mathbf{F}^\bullet \otimes^L (k_{\mathfrak{p}})_{V(f-f(\mathbf{x}))}^\bullet) = \\ &= \bigcup_{i, \mathfrak{p}} \text{supp } {}^\mu H_{k_{\mathfrak{p}}}^i (\phi_{f-f(\mathbf{x})} (\mathbf{F}^\bullet \otimes^L (k_{\mathfrak{p}})_{\mathbf{x}}^\bullet)) = \bigcup_{i, \mathfrak{p}} \text{supp } \phi_{f-f(\mathbf{x})} {}^\mu H_{k_{\mathfrak{p}}}^i (\mathbf{F}^\bullet \otimes^L (k_{\mathfrak{p}})_{\mathbf{x}}^\bullet). \quad \square \end{aligned}$$

Looking at 2.1 and 2.2, we see that any stratum S_α for which $H^*(\mathbb{N}_\alpha, \mathbb{L}_\alpha; \mathbf{F}^\bullet) = 0$ is essentially irrelevant as far as vanishing cycles are concerned. Hence, we make the following definition.

Definition 2.3. A stratum S_α is \mathbf{F}^\bullet -invisible if $H^*(\mathbb{N}_\alpha, \mathbb{L}_\alpha; \mathbf{F}^\bullet) = 0$. Otherwise, we say that S_α is \mathbf{F}^\bullet -visible.

The basic principle is that \mathbf{F}^\bullet -invisible strata do not contribute any cohomology in Morse Theory arguments. As an example, we have

Theorem 2.4.

$$\bigcup_{v \in \mathbb{C}} \text{supp } \phi_{f-v} \mathbf{F}^\bullet = \left\{ \mathbf{x} \in X \mid (x, d_{\mathbf{x}} \tilde{f}) \in \bigcup_{\substack{\mathbf{F}^\bullet\text{-visible} \\ S_\alpha}} \overline{T_{S_\alpha}^* \mathcal{U}} \right\}.$$

Proof. This is immediate from the equivalence of a) and c) in 2.2, and the description of the characteristic cycle given in 2.1. \square

Remark 2.5. The union on the left side above is not just locally finite, but, in fact, locally consists of a single support, i.e., near a point $\mathbf{p} \in X$, $\text{supp } \phi_{f-v} \mathbf{F}^\bullet = \emptyset$ unless $v = f(\mathbf{p})$.

Note that Theorem 2.4 immediately implies a stronger version of itself. One does not need to begin with a Whitney stratification, but merely any stratification for which the normal data of strata with respect to \mathbf{F}^\bullet is “well-defined”, i.e., stratifications in which the normal data in normal slices to strata locally trivializes along the strata. Such stratifications only require refinement by including \mathbf{F}^\bullet -invisible strata in order to obtain a Whitney stratification.

This is essentially what is required by Briançon, Maisonobe, and Merle in [BMM], where they use stratifications which satisfy Whitney’s condition a) and the *property of local stratified triviality*.

Actually, for our purposes, we do not even need to have a “stratification” – that is, we do not need the condition of the frontier.

Definition 2.6. A (complex analytic) *partitioning* of X is a locally finite decomposition of X into disjoint analytic submanifolds of \mathcal{U} , which we still call *strata*, such that, for each stratum W_β , $\overline{W_\beta}$ and $\overline{W_\beta} - W_\beta$ are closed complex analytic subsets of X .

Proposition/Definition 2.7. Suppose that $\mathbf{F}^\bullet \in D_c^b(X)$. Let $\mathcal{W} := \{W_\beta\}$ be a complex analytic partitioning of X with connected strata. Then, the following are equivalent:

a) there exists a refinement $\mathcal{S} := \{S_\alpha\}$ of \mathcal{W} to a Whitney stratification with connected strata such that $\mathbf{F}^\bullet \in D_s^b(X)$, and such that, for all S_α such that $\overline{S_\alpha} \not\subseteq \{\overline{W_\beta} \mid W_\beta \in \mathcal{W}\}$, S_α is \mathbf{F}^\bullet -invisible;

b) if $\mathcal{S} := \{S_\alpha\}$ is a refinement of \mathcal{W} to a Whitney stratification with connected strata such that $\mathbf{F}^\bullet \in D_s^b(X)$, then, for all S_α such that $\overline{S_\alpha} \not\subseteq \{\overline{W_\beta} \mid W_\beta \in \mathcal{W}\}$, S_α is \mathbf{F}^\bullet -invisible;

c) for all integers i and for all prime ideals \mathfrak{p} in R ,

$$|\text{Ch}({}^\mu H_{k_{\mathfrak{p}}}^i(\mathbf{F}^\bullet \otimes_{(k_{\mathfrak{p}})_X}^L))| \subseteq \bigcup_{\beta} \overline{T_{W_\beta}^* \mathcal{U}}.$$

We refer to a partitioning \mathcal{W} which satisfies these equivalent conditions (and has connected strata) as an \mathbf{F}^\bullet -normal partitioning of X . Naturally, if \mathcal{W} is, in fact, a stratification of X which is \mathbf{F}^\bullet -normal, then we refer to \mathcal{W} as an \mathbf{F}^\bullet -normal stratification.

Let \mathcal{W} be an \mathbf{F}^\bullet -normal partitioning of X , and $W_\beta \in \mathcal{W}$. Let \mathcal{S} be a Whitney refinement of \mathcal{W} such that $\mathbf{F}^\bullet \in D_s^b(X)$, and let S_α be the unique stratum of \mathcal{S} such that $\overline{W_\beta} = \overline{S_\alpha}$; we refer to such an S_α as an \mathbf{F}^\bullet -Whitney stratum associated to W_β . We define the *normal data of W_β with respect to \mathbf{F}^\bullet* to be the isomorphism class of the normal data of such an S_α with respect to \mathbf{F}^\bullet , we write $\mathbb{H}^*(\mathbb{N}_\beta, \mathbb{L}_\beta; \mathbf{F}^\bullet)$ for this normal data, and we say that W_β is \mathbf{F}^\bullet -visible if and only if it has non-zero normal data.

The definition of normal data of W_β is independent of the refinement \mathcal{S} , and W_β is \mathbf{F}^\bullet -visible if and only if there exists an integer i and a prime ideal \mathfrak{p} in R such that

$$\overline{T_{W_\beta}^* \mathcal{U}} \subseteq |\text{Ch}({}^\mu H_{k_{\mathfrak{p}}}^i(\mathbf{F}^\bullet \otimes_{(k_{\mathfrak{p}})_X}^L))|.$$

Proof. Let $\mathcal{S} := \{S_\alpha\}$ be a refinement of \mathcal{W} to a Whitney stratification with connected strata such that $\mathbf{F}^\bullet \in D_S^b(X)$. By 2.1,

$$(†) \quad \bigcup_{\substack{\mathbf{F}^\bullet\text{-visible} \\ S_\alpha}} \overline{T_{S_\alpha}^* \mathcal{U}} = \bigcup_{i, \mathfrak{p}} |\mathrm{Ch}(\mu_{H_{k_{\mathfrak{p}}}}^i(\mathbf{F}^\bullet \otimes^L (k_{\mathfrak{p}})_X^\bullet))|.$$

Now,

$$\bigcup_{\substack{\mathbf{F}^\bullet\text{-visible} \\ S_\alpha}} \overline{T_{S_\alpha}^* \mathcal{U}} \subseteq \bigcup_{\beta} \overline{T_{W_\beta}^* \mathcal{U}}$$

if and only if for all S_α such that $\overline{S_\alpha} \not\subseteq \{\overline{W_\beta} \mid W_\beta \in \mathcal{W}\}$, S_α is \mathbf{F}^\bullet -invisible.

All of the conclusions follow immediately. \square

The point, of course, is that in results dealing with characteristic cycles, one needs to consider only visible strata of \mathbf{F}^\bullet -normal partitions. Note that, as a result of the characterization in part c), if \mathcal{W} is an \mathbf{F}^\bullet -normal partitioning of X , then $|\mathrm{Ch}(\mathbf{F}^\bullet)| \subseteq \bigcup_{\beta} \overline{T_{W_\beta}^* \mathcal{U}}$.

§3. The Main Theorem

In this section, we are going to combine the Vanishing Index Theorem (1.2) and the characteristic cycle calculation of 2.1 to extract the normal data of the vanishing cycles. We continue with the notations from the previous sections; especially recall the notations used in Theorem 1.2 and in Proposition 2.1.

In what follows, if C is an analytic cycle in \mathcal{U} , and V is a reduced and irreducible analytic subvariety of \mathcal{U} , then we write C_V for the coefficient of V in C . Also, if Y and Z are closed analytic subsets of \mathcal{U} , $\mathcal{W} = \{W_\gamma\}$ is an analytic partitioning of Y , and $\Omega = \sum_{\gamma} m_\gamma [\overline{T_{W_\gamma}^* \mathcal{U}}]$, then we let

$$\Omega_{\subseteq Z} := \sum_{W_\gamma \subseteq Z} m_\gamma [\overline{T_{W_\gamma}^* \mathcal{U}}],$$

and, on the set level, we define $|\Omega|_{\subseteq Z} := |\Omega_{\subseteq Z}|$.

Theorem 3.1. *Let $\mathcal{S} = \{S_\alpha\}$ be an \mathbf{F}^\bullet -normal partitioning of X , and let $\mathcal{W} = \{W_\beta\}$ be a complex analytic partitioning of $V(f)$. Let $d_\alpha := \dim S_\alpha$ and $d_\beta := \dim W_\beta$.*

Then, \mathcal{W} is a $\phi_f \mathbf{F}^\bullet$ -normal partitioning of $V(f)$ if and only if for all \mathbf{F}^\bullet -visible S_α ,

$$|\pi(E_\alpha)|_{\subseteq V(f)} \subseteq \bigcup_{\beta} \mathbb{P}(\overline{T_{W_\beta}^* \mathcal{U}}),$$

and whenever \mathcal{W} is a $\phi_f \mathbf{F}^\bullet$ -normal partitioning of $V(f)$, for all β , for all i , for all prime ideals \mathfrak{p} in R ,

$$b_{i-d_\beta}(\mathbb{N}_\beta, \mathbb{L}_\beta; \phi_f[-1] \mathbf{F}^\bullet) = \sum_{\alpha} b_{i-d_\alpha}(\mathbb{N}_\alpha, \mathbb{L}_\alpha; \mathbf{F}^\bullet) [\pi(E_\alpha)]_{\mathbb{P}(\overline{T_{W_\beta}^* \mathcal{U}})},$$

and

$$c_{i-d_\beta}^{\mathbf{p}}(\mathbb{N}_\beta, \mathbb{L}_\beta; \phi_f[-1]\mathbf{F}^\bullet) = \sum_{\alpha} c_{i-d_\alpha}^{\mathbf{p}}(\mathbb{N}_\alpha, \mathbb{L}_\alpha; \mathbf{F}^\bullet) [\pi(E_\alpha)]_{\mathbb{P}(\overline{T_{W_\beta}^* \mathcal{U}})}.$$

In particular,

$$\bigcup_{i, \mathbf{p}} \left| \mathbb{P} \left(\text{Ch} \left({}^\mu H_{k_{\mathbf{p}}}^i (\phi_f[-1]\mathbf{F}^\bullet \otimes^L (k_{\mathbf{p}})_{V(f)}^\bullet) \right) \right) \right| = \bigcup_{\substack{\mathbf{F}^\bullet\text{-visible} \\ S_\alpha}} |\pi(E_\alpha)|_{\subseteq V(f)}.$$

Proof. The two formulas in the theorem are derived in the same way, using Proposition 2.1. Hence, we shall prove the notationally simpler statement about b_* , and conclude the statement about $c_*^{\mathbf{p}}$ at the same time.

Let $\mathcal{Z} = \{Z_\gamma\}$ be a common Whitney refinement, with connected strata, of \mathcal{S} and \mathcal{W} such that \mathbf{F}^\bullet is constructible with respect to \mathcal{Z} and $\phi_f[-1]\mathbf{F}^\bullet$ is constructible with respect to the strata of \mathcal{Z} which are contained in $V(f)$. Let $d_\gamma := \dim Z_\gamma$.

Then, the characteristic cycle of $\mathbf{P}^\bullet := {}^\mu H^i(\mathbf{F}^\bullet)$ is given by

$$\text{Ch}(\mathbf{P}^\bullet) = (-1)^d \sum_{\gamma} b_{i-d_\gamma}(\mathbb{N}_\gamma, \mathbb{L}_\gamma; \mathbf{F}^\bullet) \left[\overline{T_{Z_\gamma}^* \mathcal{U}} \right].$$

By the Vanishing Index Theorem (1.2),

$$(-1)^{d-\hat{d}} \mathbb{P}(\text{Ch}(\phi_f \mathbf{P}^\bullet)) = (-1)^d \sum_{\gamma} b_{i-d_\gamma}(\mathbb{N}_\gamma, \mathbb{L}_\gamma; \mathbf{F}^\bullet) (\pi(E_\gamma))_{\subseteq V(f)}.$$

On the other hand,

$$\begin{aligned} (-1)^{d-\hat{d}} \mathbb{P}(\text{Ch}(\phi_f \mathbf{P}^\bullet)) &= (-1)^{d-\hat{d}} \mathbb{P}(\text{Ch}(\phi_f {}^\mu H^i(\mathbf{F}^\bullet))) = (-1)^{d-\hat{d}-1} \mathbb{P}(\text{Ch}({}^\mu H^i(\phi_f[-1]\mathbf{F}^\bullet))) = \\ &= (-1)^d \sum_{Z_\gamma \subseteq V(f)} b_{i-d_\gamma}(\mathbb{N}_\gamma, \mathbb{L}_\gamma; \phi_f[-1]\mathbf{F}^\bullet) \left[\overline{T_{Z_\gamma}^* \mathcal{U}} \right]. \end{aligned}$$

We conclude that if $Z_{\gamma_0} \subseteq V(f)$, then

$$b_{i-d_{\gamma_0}}(\mathbb{N}_{\gamma_0}, \mathbb{L}_{\gamma_0}; \phi_f[-1]\mathbf{F}^\bullet) = \sum_{\gamma} b_{i-d_\gamma}(\mathbb{N}_\gamma, \mathbb{L}_\gamma; \mathbf{F}^\bullet) [\pi(E_\gamma)]_{\mathbb{P}(\overline{T_{Z_{\gamma_0}}^* \mathcal{U}})}.$$

In addition, since \mathcal{S} is an \mathbf{F}^\bullet -normal stratification, we may replace the right-hand side by

$$\sum_{\alpha} b_{i-d_\alpha}(\mathbb{N}_\alpha, \mathbb{L}_\alpha; \mathbf{F}^\bullet) [\pi(E_\alpha)]_{\mathbb{P}(\overline{T_{Z_{\gamma_0}}^* \mathcal{U}})},$$

and we also conclude that the analogous formulas hold with b_{i-d_γ} replaced by $c_{i-d_\gamma}^{\mathbf{p}}$.

Using 2.7.c, it follows immediately that \mathcal{W} is a $\phi_f \mathbf{F}^\bullet$ -normal partitioning of $V(f)$ if and only if for all \mathbf{F}^\bullet -visible S_α ,

$$|\pi(E_\alpha)|_{\subseteq V(f)} \subseteq \bigcup_{\beta} \mathbb{P}(\overline{T_{W_\beta}^* \mathcal{U}}),$$

and we are finished. \square

Remark 3.2. The coefficient $[\pi(E_\alpha)]_{\mathbb{P}(\overline{T_{W_\beta}^* \mathcal{U}})}$ can be calculated by moving to a generic point, \mathbf{x} , of W_β and taking a normal slice \mathbb{N}_β . One is then reduced to the case where W_β consists of a single point; say $\mathbf{0}$. By Lemma 2.7 of [M1], the coefficient of $\mathbb{P}(\overline{T_0^* \mathcal{U}})$ in $[\pi(E_\alpha)]$ can be calculated by considering the relative polar curve of $f|_{S_\alpha}$ with respect to a generic linear form L ; the formula obtained is

$$[\pi(E_\alpha)]_{\mathbb{P}(\overline{T_0^* \mathcal{U}})} = (\Gamma_{f|_{S_\alpha}, L}^1 \cdot V(f))_{\mathbf{0}} - (\Gamma_{f|_{S_\alpha}, L}^1 \cdot V(L))_{\mathbf{0}}.$$

§4. Applications to Thom's a_f condition

In this section, we will use Theorem 3.1 to show that the vanishing cycles along f control Thom's a_f condition. We continue with the notations from the previous sections.

Throughout this section, we make the simplifying assumption that R is a field – this is no real restriction, since we can always reduce ourselves from the p.i.d. case to the field case by tensoring with $k_{\mathfrak{p}}$.

Moreover, in this section, we will usually assume that are starting with a perverse sheaf \mathbf{P}^\bullet on X ; thus; the only possibly non-zero ${}^i H^i(\mathbf{P}^\bullet)$ occurs when $i = 0$. These two assumptions – that R is a field and that \mathbf{P}^\bullet is perverse – greatly simplify many statements, for they imply:

- a) an analytic partitioning $\mathcal{W} := \{W_\beta\}$ of X is \mathbf{P}^\bullet -normal if and only if $|\mathrm{Ch}(\mathbf{P}^\bullet)| \subseteq \bigcup_\beta \overline{T_{W_\beta}^* \mathcal{U}}$, and
- b) a stratum W_β in a \mathbf{P}^\bullet -normal partitioning of X is \mathbf{P}^\bullet -visible if and only if $\overline{T_{W_\beta}^* \mathcal{U}} \subseteq |\mathrm{Ch}(\mathbf{P}^\bullet)|$.

Moreover, if \mathbf{P}^\bullet is perverse, then so is $\phi_f[-1]\mathbf{P}^\bullet$, and so statements about the vanishing cycles also simplify greatly; for example, the final statement of Theorem 3.1 becomes

$$|\mathbb{P}(\mathrm{Ch}(\phi_f[-1]\mathbf{P}^\bullet))| = \bigcup_{\substack{\mathbf{P}^\bullet\text{-visible} \\ S_\alpha}} |\pi(E_\alpha)|_{\subseteq V(f)}.$$

Note that, if $\mathbf{x} \in V(f)$, then there is an equality of fibres $(|\pi(E_\alpha)|_{\subseteq V(f)})_{\mathbf{x}} = |\pi(E_\alpha)|_{\mathbf{x}}$, since, locally, the stratified critical values of f are isolated.

Recall the definitions of the relative conormal space, and of Thom's a_f condition in its relative conormal formulation.

Definition 4.1. If M is an analytic submanifold of \mathcal{U} and $M \subseteq X$, then the *relative conormal space* (of M with respect to f in \mathcal{U}), $T_{f|M}^* \mathcal{U}$, is given by

$$\begin{aligned} T_{f|M}^* \mathcal{U} &:= \{(\mathbf{x}, \eta) \in T^* \mathcal{U} \mid \mathbf{x} \in M, \eta(\ker d_{\mathbf{x}}(f|_M)) = 0\} = \\ &\{(\mathbf{x}, \eta) \in T^* \mathcal{U} \mid \mathbf{x} \in M, \eta(T_{\mathbf{x}} M \cap \ker d_{\mathbf{x}} \tilde{f}) = 0\}. \end{aligned}$$

Let M and N be analytic submanifolds of X such that f has constant rank on N . Then, the pair (M, N) satisfies Thom's a_f condition at a point $\mathbf{x} \in N$ if and only if we have the containment $(\overline{T_{f|M}^* \mathcal{U}})_{\mathbf{x}} \subseteq (T_{f|N}^* \mathcal{U})_{\mathbf{x}}$ of fibres over \mathbf{x} .

In particular, if f is, in fact, constant on N , then the pair (M, N) satisfies Thom's a_f condition at a point $\mathbf{x} \in N$ if and only if we have the containment $\left(\overline{T_{f|_M}^* \mathcal{U}}\right)_{\mathbf{x}} \subseteq \left(T_N^* \mathcal{U}\right)_{\mathbf{x}}$ of fibres over \mathbf{x} .

Remark 4.2. We have been slightly more general in the above definition than is sometimes the case; we have not required that the rank of f be constant on M . Thus, if X is an analytic space, we may write that (X_{reg}, N) satisfies the a_f condition, instead of writing the much more cumbersome $(X_{\text{reg}} - \Sigma(f|_{X_{\text{reg}}}), N)$ satisfies the a_f condition. If f is not constant on any irreducible component of X , it is trivial to see that these statements are equivalent: let $\overset{\circ}{X} := X_{\text{reg}} - \Sigma(f|_{X_{\text{reg}}})$, which is dense in X_{reg} (as f is not constant on any irreducible components of X), and then one shows easily that $\overline{T_{f|_{\overset{\circ}{X}}}^* \mathcal{U}} = \overline{T_{f|_{X_{\text{reg}}}}^* \mathcal{U}}$.

Proposition 4.3. *Suppose that f is not constant on any irreducible component of X . Let E denote the exceptional divisor in $\text{Bl}_{\text{im } d\tilde{f}} \overline{T_{X_{\text{reg}}}^* \mathcal{U}} \subseteq \mathcal{U} \times \mathbb{C}^{n+1} \times \mathbb{P}^n$. Suppose that $N \subseteq X$ is an analytic submanifold of \mathcal{U} and that $\mathbf{x} \in N$ is such that (X_{reg}, N) satisfies Whitney's condition a) at \mathbf{x} and such that $d_{\mathbf{x}}(f|_N) \equiv 0$.*

Then, (X_{reg}, N) satisfies Whitney's a_f condition at \mathbf{x} if and only if there is the containment of fibres above \mathbf{x} given by

$$\left(\pi(E)\right)_{\mathbf{x}} \subseteq \left(\mathbb{P}(T_N^* \mathcal{U})\right)_{\mathbf{x}}.$$

Proof. This is a modification of our proof of Theorem 4.2 of [M1] – we put back the Whitney a) assumption that we went to some effort to remove in [M1].

In Theorem 4.2 of [M1], we showed that there is always a containment

$$\left(\pi(E)\right)_{\mathbf{x}} \subseteq \left(\mathbb{P}\left(\overline{T_{f|_{X_{\text{reg}}}}^* \mathcal{U}}\right)\right)_{\mathbf{x}}.$$

It follows immediately that if (X_{reg}, N) satisfies Whitney's a_f condition at \mathbf{x} , then $\left(\pi(E)\right)_{\mathbf{x}} \subseteq \left(\mathbb{P}(T_N^* \mathcal{U})\right)_{\mathbf{x}}$. We must now show the converse.

Assume that $\left(\pi(E)\right)_{\mathbf{x}} \subseteq \left(\mathbb{P}(T_N^* \mathcal{U})\right)_{\mathbf{x}}$. Let $\overset{\circ}{X} := X_{\text{reg}} - \Sigma(f|_{X_{\text{reg}}})$. Suppose that $[\eta] \in \left(\mathbb{P}\left(\overline{T_{f|_{\overset{\circ}{X}}}^* \mathcal{U}}\right)\right)_{\mathbf{x}}$.

We must show that $[\eta] \in \left(\mathbb{P}(T_N^* \mathcal{U})\right)_{\mathbf{x}}$.

There exists a complex analytic path $\alpha(t) = (\mathbf{x}(t), \eta_t) \in \overline{T_{f|_{\overset{\circ}{X}}}^* \mathcal{U}}$ such that $\alpha(0) = (\mathbf{x}, \eta)$ and $\alpha(t) \in T_{f|_{\overset{\circ}{X}}}^* \mathcal{U}$ for $t \neq 0$. As f has no critical points on $\overset{\circ}{X}$, each η_t can be written uniquely as $\eta_t = \omega_t + \lambda(\mathbf{x}(t)) d_{\mathbf{x}(t)} \tilde{f}$, where $\omega_t \in (T_{\overset{\circ}{X}}^* \mathcal{U})_{\mathbf{x}(t)}$ and $\lambda(\mathbf{x}(t))$ is a scalar. By evaluating each side on $\mathbf{x}'(t)$, we find that $\lambda(\mathbf{x}(t)) = \frac{\eta_t(\mathbf{x}'(t))}{\frac{d}{dt} f(\mathbf{x}(t))}$.

Thus, as $\lambda(\mathbf{x}(t))$ is a quotient of two analytic functions, there are only two possibilities for what happens to $\lambda(\mathbf{x}(t))$ as $t \rightarrow 0$.

Case 1: $|\lambda(\mathbf{x}(t))| \rightarrow \infty$ as $t \rightarrow 0$.

In this case, since $\eta_t \rightarrow \eta$, it follows that $\frac{\eta_t}{\lambda(\mathbf{x}(t))} \rightarrow 0$ and, hence, $-\frac{\omega_t}{\lambda(\mathbf{x}(t))} \rightarrow d_{\mathbf{x}}\tilde{f}$. Therefore,

$$\left(\mathbf{x}(t), -\frac{\omega_t}{\lambda(\mathbf{x}(t))}, \left[-\frac{\omega_t}{\lambda(\mathbf{x}(t))} - d_{\mathbf{x}(t)}\tilde{f} \right] \right) = \left(\mathbf{x}(t), -\frac{\omega_t}{\lambda(\mathbf{x}(t))}, [\eta_t(\mathbf{x}(t))] \right) \rightarrow (\mathbf{x}, d_{\mathbf{x}}\tilde{f}, [\eta]),$$

and so $(\mathbf{x}, [\eta]) \in \pi(E)$. Thus, $[\eta] \in (\pi(E))_{\mathbf{x}} \subseteq (\mathbb{P}(T_N^*\mathcal{U}))_{\mathbf{x}}$.

Case 2: $\lambda(\mathbf{x}(t)) \rightarrow \lambda_0$ as $t \rightarrow 0$.

In this case, $\omega_t \rightarrow \omega \in (\overline{T_x^*\mathcal{U}})_{\mathbf{x}}$ and $\eta = \omega + \lambda_0 d_{\mathbf{x}}\tilde{f}$. As (X_{reg}, N) satisfies Whitney's condition a) at \mathbf{x} , $\omega \in (T_N^*\mathcal{U})_{\mathbf{x}}$. As $d_{\mathbf{x}}(f|_N) \equiv 0$, $d_{\mathbf{x}}\tilde{f} \in (T_N^*\mathcal{U})_{\mathbf{x}}$. Thus, $\eta \in (T_N^*\mathcal{U})_{\mathbf{x}}$. \square

Theorem 4.4. *Let \mathbf{P}^\bullet be a perverse sheaf on X . Let \mathcal{W} be a Whitney a) \mathbf{P}^\bullet -normal partitioning of X such that $V(f)$ is a union of strata and such that $\widehat{\mathcal{W}} := \{W_\beta \mid W_\beta \subseteq V(f)\}$ is an analytic partitioning of $V(f)$. Let $M \subseteq V(f)$ be an analytic submanifold of \mathcal{U} .*

Then, $((\overline{W_\beta})_{\text{reg}}, M)$ satisfies the a_f condition of all \mathbf{P}^\bullet -visible W_β if and only if for all $\mathbf{x} \in M$, $|\text{Ch}(\phi_f[-1]\mathbf{P}^\bullet)|_{\mathbf{x}} \subseteq (T_M^\mathcal{U})_{\mathbf{x}}$.*

Proof. The theorem would follow immediately from Proposition 4.3 and the fact that

$$|\mathbb{P}(\text{Ch}(\phi_f[-1]\mathbf{P}^\bullet))| = \bigcup_{\substack{\mathbf{P}^\bullet\text{-visible} \\ W_\beta}} |\pi(E_\beta)|_{\subseteq V(f)},$$

except that one must worry about strata for which $f|_{W_\beta} \equiv 0$; however, in this case, $\pi(E_\beta) = \mathbb{P}(\overline{T_{W_\beta}^*\mathcal{U}})$ (see Lemma 2.9 of [M1]). The desired conclusion follows immediately. \square

Thus, we see that, for a perverse sheaf, the vanishing cycles control Thom's a_f condition; moreover, we could start with an arbitrary $\mathbf{F}^\bullet \in D_c^b(X)$ and an arbitrary p.i.d. for the base ring, and then apply Theorem 4.4 to each $\mathbf{P}^\bullet := {}^\mu H_{k_p}^i(\mathbf{F}^\bullet \otimes^L (k_p)_X)$.

Given the results of Parusiński [P] and Briançon, Maisonobe, and Merle [BMM] that Whitney stratifications adapted to $V(f)$ are a_f stratifications (the *Whitney- a_f result*), one might wonder if we can recover this result via the work above. The answer is: yes and no – basically we have to use one of the main results of [BMM] to reach the desired conclusion. This is not really terribly surprising; our work above has the same flavor and uses the same tools – characteristic cycles, nearby and vanishing cycles, perverse sheaves – as the approach used in [BMM]. Nonetheless, as Briançon, Maisonobe, and Merle concentrate on the nearby cycles, and we wish to promote the vanishing cycles as the “correct” object of study, we will now indicate how to use Theorem 4.4 to conclude the Whitney- a_f result.

Let $i : X - V(f) \hookrightarrow X$ and $j : V(f) \hookrightarrow X$ denote the inclusions. The following proposition

follows immediately from Theorem 3.4.2 of [BMM] (alternatively, it follows easily from Corollary 4.6 of [M2]).

Proposition 4.5. *If $\mathbf{P}^\bullet \in D_c^b(X)$ is such that all of the coefficients of $\text{Ch}(\mathbf{P}^\bullet)$ have the same sign (e.g., if \mathbf{P}^\bullet is perverse), then*

$$|\text{Ch}(i_! i^! \mathbf{P}^\bullet)|_{\subseteq V(f)} = |\text{Ch}(\psi_f[-1] \mathbf{P}^\bullet)|.$$

Proof. Using the notation of [BMM], the characteristic cycle of the \mathcal{D} -module $M[1/f]$ is equal to $\text{Ch}(i_! i^! \mathbf{P}^\bullet)$ (see, for instance, the proof of 4.2.1 of [BMM]). If all of the coefficients of $\text{Ch}(\mathbf{P}^\bullet)$ have the same sign, there can be no cancellations in the sums appearing in 3.4.2 of [BMM]. \square

Proposition 4.6. *If \mathbf{P}^\bullet is a perverse sheaf on X , and $\widehat{\mathcal{W}}$ is a $j^* \mathbf{P}^\bullet$ -normal partitioning of $V(f)$, then*

$$|\text{Ch}(\phi_f[-1] \mathbf{P}^\bullet)| \subseteq |\text{Ch}(i_! i^! \mathbf{P}^\bullet)|_{\subseteq V(f)} \cup \bigcup_{W_\beta \in \widehat{\mathcal{W}}} \overline{T_{W_\beta}^* \mathcal{U}}.$$

Moreover, if $\mathcal{W} := \{W_\beta\}$ is a \mathbf{P}^\bullet -normal partitioning of X such that $\widehat{\mathcal{W}} := \{W_\beta \mid W_\beta \subseteq V(f)\}$ is a $j^ \mathbf{P}^\bullet$ -normal partitioning of $V(f)$, then*

$$|\text{Ch}(\phi_f[-1] \mathbf{P}^\bullet)| \subseteq \bigcup_{W_\beta \in \widehat{\mathcal{W}}} \overline{T_{W_\beta}^* \mathcal{U}}.$$

Proof. There is the fundamental distinguished triangle

$$j^* \mathbf{P}^\bullet[-1] \rightarrow \psi_f[-1] \mathbf{P}^\bullet \rightarrow \phi_f[-1] \mathbf{P}^\bullet \rightarrow j^* \mathbf{P}^\bullet$$

which yields an equality

$$\text{Ch}(\phi_f[-1] \mathbf{P}^\bullet) = \text{Ch}(\psi_f[-1] \mathbf{P}^\bullet) - \text{Ch}(j^* \mathbf{P}^\bullet[-1]).$$

Thus,

$$|\text{Ch}(\phi_f[-1] \mathbf{P}^\bullet)| \subseteq |\text{Ch}(\psi_f[-1] \mathbf{P}^\bullet)| \cup |\text{Ch}(j^* \mathbf{P}^\bullet[-1])|.$$

The first statement now follows immediately from 4.5 and the fact that $\widehat{\mathcal{W}}$ is a $j^* \mathbf{P}^\bullet$ -normal partitioning of $V(f)$.

There is another distinguished triangle

$$i_! i^! \mathbf{P}^\bullet \rightarrow \mathbf{P}^\bullet \rightarrow j_* j^* \mathbf{P}^\bullet \rightarrow i_! i^! \mathbf{P}^\bullet[1],$$

which yields the equality

$$\text{Ch}(\mathbf{P}^\bullet) = \text{Ch}(i_! i^! \mathbf{P}^\bullet) + \text{Ch}(j_* j^* \mathbf{P}^\bullet).$$

As $\text{Ch}(j_* j^* \mathbf{P}^\bullet) = \pm \text{Ch}(j^* \mathbf{P}^\bullet)$, we have $|\text{Ch}(j_* j^* \mathbf{P}^\bullet)| = |\text{Ch}(j^* \mathbf{P}^\bullet)|$. Thus,

$$|\text{Ch}(i_! i^! \mathbf{P}^\bullet)| \subseteq |\text{Ch}(\mathbf{P}^\bullet)| \cup |\text{Ch}(j^* \mathbf{P}^\bullet)|,$$

which is contained in $\bigcup_{\beta} \overline{T_{W_{\beta}}^* \mathcal{U}}$ due to our normal partitioning assumptions. Therefore,

$$|\mathrm{Ch}(i_1^! \mathbf{P}^{\bullet})|_{\subseteq V(f)} \subseteq \bigcup_{W_{\beta} \in \widehat{\mathcal{W}}} \overline{T_{W_{\beta}}^* \mathcal{U}},$$

and the second statement now follows from the first. \square

Corollary 4.7. *Let \mathbf{P}^{\bullet} be a perverse sheaf on X . Let \mathcal{W} be a Whitney a) \mathbf{P}^{\bullet} -normal partitioning of X such that $V(f)$ is a union of strata and such that $\widehat{\mathcal{W}} := \{W_{\beta} \mid W_{\beta} \subseteq V(f)\}$ is a $j^* \mathbf{P}^{\bullet}$ -normal partitioning of $V(f)$.*

Let $W_{\alpha} \in \mathcal{W}$ be \mathbf{P}^{\bullet} -visible and let $W_{\beta} \in \widehat{\mathcal{W}}$. Then, $((\overline{W_{\alpha}})_{\mathrm{reg}}, W_{\beta})$ satisfies the a_f condition.

Proof. Let $\mathbf{x} \in W_{\beta}$. By Theorem 4.4, what we need to show is that $|\mathrm{Ch}(\phi_f[-1] \mathbf{P}^{\bullet})|_{\mathbf{x}} \subseteq (T_{W_{\beta}}^* \mathcal{U})_{\mathbf{x}}$.

By 4.6,

$$|\mathrm{Ch}(\phi_f[-1] \mathbf{P}^{\bullet})| \subseteq \bigcup_{W_{\gamma} \in \widehat{\mathcal{W}}} \overline{T_{W_{\gamma}}^* \mathcal{U}}.$$

Now, Whitney's condition a) tells us that

$$\left(\bigcup_{W_{\gamma} \in \widehat{\mathcal{W}}} \overline{T_{W_{\gamma}}^* \mathcal{U}} \right)_{\mathbf{x}} = (T_{W_{\beta}}^* \mathcal{U})_{\mathbf{x}}. \quad \square$$

We can now give our own proof of part of the results of Parusiński in [P] and Briangon, Maisonobe, and Merle in [BMM].

Corollary 4.8. *Let \mathcal{W} be a Whitney stratification of X such that $V(f)$ is a union of strata. Let $W_{\alpha}, W_{\beta} \in \mathcal{W}$ be such that $W_{\beta} \subseteq V(f)$. Then, $((\overline{W_{\alpha}})_{\mathrm{reg}}, W_{\beta})$ satisfies the a_f condition.*

Proof. Let $k : \overline{W_{\alpha}} \hookrightarrow X$ denote the inclusion. Apply 4.7 to $\mathbf{P}^{\bullet} := k_! {}^{\mu}H^0(\mathbb{C}_{\overline{W_{\alpha}}}^{\bullet}[\dim W_{\alpha}]) \cong {}^{\mu}H^0(k_! \mathbb{C}_{\overline{W_{\alpha}}}^{\bullet}[\dim W_{\alpha}])$.

As $k_! \mathbb{C}_{\overline{W_{\alpha}}}^{\bullet}[\dim W_{\alpha}]$ is constructible with respect to \mathcal{W} , ${}^{\mu}H^0(k_! \mathbb{C}_{\overline{W_{\alpha}}}^{\bullet}[\dim W_{\alpha}])$ is also constructible with respect to \mathcal{W} . Therefore, \mathcal{W} is a \mathbf{P}^{\bullet} -normal partitioning of X such that $V(f)$ is a union of strata and such that $\widehat{\mathcal{W}} := \{W_{\beta} \mid W_{\beta} \subseteq V(f)\}$ is a $j^* \mathbf{P}^{\bullet}$ -normal partitioning of $V(f)$. Moreover, by construction, W_{α} is \mathbf{P}^{\bullet} -visible. The result follows. \square

REFERENCES

- [BBD] A. Beilinson, J. Bernstein, and P. Deligne, *Faisceaux Pervers*, Astérisque **100**, Soc. Math. de France, 1983.
- [BMM] J. Briançon, P. Maisonobe, and M. Merle, *Localisation de systèmes différentiels, stratifications de Whitney et condition de Thom*, Invent. Math. **117** (1994), 531–550.
- [G-M1] M. Goresky and R. MacPherson, *Intersection homology II*, Inv. Math **71** (1983), 77–129.
- [G-M2] ———, *Stratified Morse Theory*, Ergebnisse der Math. 14, Springer-Verlag, Berlin, 1988.
- [K-S] M. Kashiwara and P. Schapira, *Sheaves on Manifolds*, Grund. der math. Wiss. 292, Springer - Verlag, 1990.
- [M1] D. Massey, *Critical Points of Functions on Singular Spaces*, Top. and Appl. **103** (2000), 55–93.
- [M2] ———, *Hypercohomology of Milnor Fibres*, Topology **35** (1996), 969–1003.
- [M3] ———, *A Little Microlocal Morse Theory*, preprint (2000).
- [M4] ———, *The Sebastiani-Thom Isomorphism in the Derived Category*, Compos. Math. (to appear).
- [P] A. Parusiński, *Limits of Tangent Spaces to Fibres and the w_f Condition*, Duke Math. J. **72** (1993), 99–108.

DAVID B. MASSEY, DEPT. OF MATHEMATICS, NORTHEASTERN UNIVERSITY, BOSTON, MA, 02115, USA
E-mail address: DMASSEY@NEU.edu