Vanishing Cycles and Thom's a_f Condition*

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Abstract

We give a complete description of the relationship between the vanishing cycles of a complex of sheaves along a function f and Thom's a_f condition.

1 Introduction

Let $\mathcal{U}$ be an open subset of $\mathbb{C}^{n+1}$, and let $\tilde{f}: \mathcal{U} \to \mathbb{C}$ be a complex analytic function. We let $\Sigma\tilde{f}$ denote the critical locus of $\tilde{f}$. Suppose that M and N are complex submanifolds of $\mathcal{U}$.

Thom's $a_{\tilde{f}}$ condition (see, for instance, [20]) is a relative Whitney (a) condition. The $a_{\tilde{f}}$ is condition is important for several reasons. First, it is an hypothesis of Thom's second isotopy lemma, which allows one to conclude that maps trivialize; see [20]. Second, the $a_{\tilde{f}}$ condition, and the existence of stratifications in which all pairs of strata satisfy the $a_{\tilde{f}}$ condition, is essential in arguments such as that used by Lê in [12] to prove that Milnor fibrations exist even when the domain is an arbitrarily singular space. Third, the $a_{\tilde{f}}$ condition is closely related to constancy of the Milnor number in families of isolated hypersurface singularities; see [13] and below.

There are at least two important general results about the $a_{\tilde{f}}$ condition. There is the above-mentioned existence of $a_{\tilde{f}}$ stratifications, proved first in the affine setting above by Hamm and Lê, following an argument of F. Pham, in Theorem 1.2.1 of [5], and then in a different manner for an arbitrary analytic domain by Hironaka in [6], and there is the theorem that Whitney stratifications in which $V(\tilde{f}) := \tilde{f}^{-1}(0)$ is a union of strata are $a_{\tilde{f}}$ stratifications, proved independently by Parusiński in [21], and Briançon, P. Maisonobe, and M. Merle in [1].

We wish to formulate the $a_{\tilde{f}}$ condition in conormal terms. So, we need a preliminary definition.

Definition 1.1. *The relative conormal space $T^*_{\tilde{f}_{|_M}}\mathcal{U}$ is given by*

$$T^*_{\tilde{f}_{|_M}}\mathcal{U} := \{(x,\eta) \in T^*\mathcal{U} \mid \eta(T_xM \cap \ker d_x\tilde{f}) = 0\}.$$

Remark 1.2. Note that $T^*_{\tilde{f}_{|_M}}\mathcal{U}$ equals the conormal space $T^*_M\mathcal{U} := \{(x,\eta) \in T^*\mathcal{U} \mid \eta(T_xM) = 0\}$ if and only if $d(\tilde{f}_{|_M})$ has constant rank zero, i.e., if and only if $\tilde{f}$ is locally constant on M.

Now, we can give the conormal definition of Thom's $a_{\tilde{f}}$ condition.

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Definition 1.3. *The* **pair** (M, N) **satisfies Thom's** $a_{\tilde f}$ **condition at a point** $x \in N$ *if and only if there is an inclusion, of fibers over* x, $\left(\overline{T^*_{\tilde f_{|_M}}\mathcal U}\right)_x \subseteq \left(T^*_{\tilde f_{|_N}}\mathcal U\right)_x$.

The **pair** (M, N) **satisfies Thom's** $a_{\tilde f}$ **condition** *if and only if it satisfies the* $a_{\tilde f}$ *condition at each point* $x \in N$.

Remark 1.4. Note that if $\tilde f$ is a locally constant function, then the $a_{\tilde f}$ condition reduces to condition (a) of Whitney.

In this paper, we prove what is essentially a generalization of the result of Lê and Saito in [13]; let us recall this result, and then give the formulation which generalizes nicely.

Let $(z_0, \dots, z_n)$ be coordinates on $\mathcal U$, let $Y := \mathcal U \cap (\mathbb C \times \{\mathbf 0\})$, and assume that $Y \subseteq V(\tilde f)$. For small $a \in \mathbb C$, define the family $\tilde f_a : (\mathcal U \cap V(z_0 - a), \mathbf 0) \to (\mathbb C, 0)$ by $\tilde f_a(z_1, \dots, z_n) := \tilde f(a, z_1, \dots, z_n)$. Assume that $\dim_{\mathbf 0} \Sigma \tilde f_0 = 0$.

Theorem 1.5.(Lê-Saito, [13]) *For all small* a*, the Milnor number* $\mu_{\tilde f_a}(\mathbf 0)$ *is independent of* a *if and only if the only component of the critical locus of* $\tilde f$, $\Sigma \tilde f$*, containing the origin is* Y *and* $(\mathcal U - Y, Y)$ *satisfies Thom's* $a_{\tilde f}$ *condition at* $\mathbf 0$.

Remark 1.6. We remark that, in the above setting, if $(\mathcal U - Y, Y)$ satisfies Thom's $a_{\tilde f}$ condition at $\mathbf 0$, then the only component of $\Sigma \tilde f$ containing the origin is Y. However, the proof of this requires the non-splitting result proved independently by Gabrielov [3], Lazzeri [9], and Lê [11].

Using the main result of Lê in [10], together with the non-splitting result of Remark 1.6, we can reformulate the result of Lê and Saito as:

Theorem 1.7.(2nd version of Lê-Saito Theorem, [13]) *For all small* a*, there is an inclusion of the Milnor fiber of* $\tilde f_a$ *at* $(a, \mathbf 0)$ *into the Milnor fiber of* $\tilde f$ *at* $\mathbf 0$ *which induces an isomorphism on integral cohomology if and only if* $(\mathcal U - Y, Y)$ *satisfies Thom's* $a_{\tilde f}$ *condition at* $\mathbf 0$.

We wish to reformulate the result of Lê-Saito in terms of vanishing cycles. For the remainder of this paper, we let X be a complex analytic subspace of $\mathcal U$, and let $f := \tilde f_{|_X}$.

Fix a base ring R, which is regular, Noetherian, and has finite Krull dimension, e.g., $\mathbb Z$, $\mathbb Q$, or $\mathbb C$. Let $\mathbf A^\bullet$ be a bounded, constructible complex of sheaves of R-modules on X.

We shall use the nearby cycles, $\psi_f \mathbf A^\bullet$, and the vanishing cycles, $\phi_f \mathbf A^\bullet$, of $\mathbf A^\bullet$ along f; we refer the reader to [18], Appendix B and [2]. More technical references are [7] and [22]. We shall almost always include a shift by -1 when we apply the nearby and vanishing cycles, and we remind the reader that $\psi_f[-1]\mathbf A^\bullet$ and $\phi_f[-1]\mathbf A^\bullet$ are complexes of sheaves of R-modules on $V(f)$, with stalk cohomologies at a point $x \in V(f)$ given by hypercohomology and relative hypercohomology of the Milnor fiber as follows:

$$H^k(\psi_f[-1]\mathbf A^\bullet)_x \cong \mathbb H^{k-1}(B_\epsilon \cap X \cap f^{-1}(a);\ \mathbf A^\bullet),$$

and

$$H^k(\phi_f[-1]\mathbf A^\bullet)_x \cong \mathbb H^k(B_\epsilon(x) \cap X, B_\epsilon \cap X \cap f^{-1}(a);\ \mathbf A^\bullet),$$

where $B_\epsilon(x)$ is a small ball (open or closed) of radius ϵ centered at x in $\mathcal{U}$, and $0 < |a| \ll \epsilon$. In the familiar case where $\mathbf{A}^\bullet = \mathbb{Z}^\bullet_X$, this means that the stalk cohomology in degree k of $\psi_f \mathbf{A}^\bullet$ (respectively, $\phi_f \mathbb{Z}^\bullet_X$) (without the shift) at a point $x \in V(f)$ is isomorphic to the (respectively, reduced) cohomology in degree k of the Milnor fiber of f at x. We also remind the reader that, in the case where $X = \mathcal{U}$ and $\mathbf{A}^\bullet = \mathbb{Z}^\bullet_{\mathcal{U}}$, the support of $\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}$ is contained in $V(f) \cap \Sigma f$.

Below, and later, we will consider iterated vanishing cycles of the form $\phi_g[-1](\phi_f[-1]\mathbf{A}^\bullet)$; when the domain of g is $\mathcal{U}$ or $\mathbb{C}^{n+1}$, we shall continue to write simply g in place of $g_{|_{V(f)}}$. What is the point of considering such iterated vanishing cycles?

Consider the case where $\mathbf{A}^\bullet = \mathbb{Z}^\bullet_X$ and g is the restriction to X of a non-zero linear form $\mathfrak{l} : \mathbb{C}^{n+1} \to \mathbb{C}$. Then, the stalk cohomology $H^k(\phi_{\mathfrak{l}}[-1](\phi_f[-1]\mathbb{Z}^\bullet_X))_{\mathbf{0}}$ is isomorphic to the relative hypercohomology module

$$\mathbb{H}^k\big(B_\epsilon \cap X, B_\epsilon \cap X \cap V(\mathfrak{l} - a);\ \phi_f[-1]\mathbb{Z}^\bullet_X\big),$$

where $0 < |a| \ll \epsilon \ll 1$. This module describes, on the level of cohomology, how the Milnor fibers of f in a nearby hyperplane section include into the Milnor fiber at f at $\mathbf{0}$.

In terms of such iterated vanishing cycles, Theorem 1.7 becomes:

Theorem 1.8. (3rd version of Lê-Saito Theorem, [13]) *$(\mathcal{U} - Y, Y)$ satisfies Thom's $a_{\tilde{f}}$ condition at $\mathbf{0}$ if and only if there exists a non-zero linear form $\mathfrak{l} : \mathbb{C}^{n+1} \to \mathbb{C}$ such that $H^*(\phi_{\mathfrak{l}}[-1]\phi_{\tilde{f}}[-1]\mathbb{Z}^\bullet_{\mathcal{U}})_{\mathbf{0}} = 0$.*

It is this result that we generalize to the setting of arbitrary $f : X \to \mathbb{C}$ and with coefficients in $\mathbf{A}^\bullet$. First, in Definition 2.3, we define what it means for $\mathbf{A}^\bullet$ to be ϕ-*constructible* along a submanifold $M \subseteq \mathcal{U}$; intuitively, this notion means that the cohomology of X, with coefficients in $\mathbf{A}^\bullet$, is "trivial" along M. Next, we define complex analytic stratifications, or merely complex analytic partitions (see Definition 2.1 and Definition 2.7), which may be weaker than Whitney stratifications, for which the Morse data associated to strata, with coefficients in $\mathbf{A}^\bullet$, is still defined; a stratum in such a partition, which has a non-trivial Morse module in some degree, is called $\mathbf{A}^\bullet$-*visible* (see Definition 2.7).

Our main theorem, Theorem 3.13, is:

Main Theorem. *Let M be a complex submanifold of $\mathcal{U}$ such that $M \subseteq V(f)$. Let $\mathcal{W}$ be a complex analytic stratification (or partition) of X such that $\mathbf{A}^\bullet$ is ϕ-constructible along each stratum of $\mathcal{W}$.*

Then, for all $\mathbf{A}^\bullet$-visible $W \in \mathcal{W}$, (W, M) satisfies the a_f condition if and only if, for all $\mathbf{A}^\bullet$-visible $W \in \mathcal{W}$, (W, M) satisfies the Whitney (a) condition, and $\phi_f[-1]\mathbf{A}^\bullet$ is ϕ-constructible along M.

The theorem above may look hopelessly abstract. We wish to put the reader on familiar ground by explaining what our main theorem says in the case where $\mathbf{A}^\bullet$ is the constant sheaf $\mathbb{Z}^\bullet_X$, and the set of points where the Milnor fiber is non-trivial is 1-dimensional (i.e., the support of the vanishing cycles is 1-dimensional).

Let M be a complex submanifold of $\mathcal{U}$ such that $M \subseteq X$. Then, $\mathbb{Z}^\bullet_X$ is ϕ-constructible along M if and only if, for all $x \in M$, for all representatives g of complex analytic germs from (X, x) to $(\mathbb{C}, 0)$ such that x is a regular point of $g_{|_M}$ (i.e., such that $d_x(g_{|_M})$ is a surjection), the Milnor fiber of g at x has the integral cohomology of a point. In this definition, one can use simply restrictions of affine linear forms in place of more general germs g; we show this in Corollary 2.11. If M is 0-dimensional, then $\mathbb{Z}^\bullet_X$ is ϕ-constructible along M, since the condition is vacuously satisfied. If $\dim M > 0$, and M is a stratum in some Whitney stratification of X, then $\mathbb{Z}^\bullet_X$ is ϕ-constructible along M; however, requiring $\mathbb{Z}^\bullet_X$ to be ϕ-constructible along M is, in general, weaker than requiring Whitney conditions.

Now, suppose that $\mathcal{W}$ is a complex analytic partition of X into analytic submanifolds of $\mathcal{U}$ (see Definition 2.1) such that $\mathbb{Z}^\bullet_X$ is ϕ-constructible along each $W \in \mathcal{W}$. As we may refine $\mathcal{W}$ to obtain a Whitney stratification, it follows that, on a generic subset of each $W \in \mathcal{W}$, there is a well-defined normal slice and complex link to the stratum (in the sense of Goresky and MacPherson [4]). We say that a $W \in \mathcal{W}$ is $\mathbb{Z}^\bullet_X$*-visible* if and only if the complex link (at a generic point) of W does **not** have the cohomology of a point.

Let $\Sigma_{\mathbb{Z}} f$ denote the *cohomological critical locus of* f, i.e., the set of points $x \in X$ such that the Milnor fiber of $f - f(x)$ at x does not have the integral cohomology of a point. Suppose that $\dim V(f) \cap \overline{\Sigma_{\mathbb{Z}} f} = 1$, and that M is a smooth complex analytic curve contained in one of the irreducible components of $V(f) \cap \overline{\Sigma_{\mathbb{Z}} f}$. Then, $\phi_f[-1]\mathbb{Z}^\bullet_X$ is ϕ-constructible along M if and only if, for all $x \in M$, for all representatives g of complex analytic germs from (X, x) to $(\mathbb{C}, 0)$ such that x is a regular point of $g_{|_M}$, the inclusion of the Milnor fiber of f at the unique point of $M \cap V(g - a)$ near x, for small $a \neq 0$, into the Milnor fiber of f at x induces an isomorphism on cohomology. In fact, in Theorem 3.5, we show that, since $\dim V(f) \cap \overline{\Sigma_{\mathbb{Z}} f} = 1$, $\phi_f[-1]\mathbb{Z}^\bullet_X$ is ϕ-constructible along M if and only if, for all $x \in M$, there exists a single non-zero affine linear form $\mathfrak{l} : (\mathbb{C}^{n+1}, x) \to (\mathbb{C}, 0)$ such that x is a regular point of $\mathfrak{l}_{|_M}$ and such that the inclusion of the Milnor fiber of f at the unique point of $M \cap V(\mathfrak{l} - a)$ near x, for small $a \neq 0$, into the Milnor fiber of f at x induces an isomorphism on cohomology.

Therefore, our main theorem, Theorem 3.13, which we stated above, combined with Theorem 3.5, tells us, in our current situation, that the following are equivalent:

1. for all $\mathbb{Z}^\bullet_X$-visible $W \in \mathcal{W}$, (W, M) satisfies the a_f condition;

2. for all $\mathbb{Z}^\bullet_X$-visible $W \in \mathcal{W}$, (W, M) satisfies the Whitney (a) condition, and for all $x \in M$, there exists a non-zero affine linear form $\mathfrak{l} : (\mathbb{C}^{n+1}, x) \to (\mathbb{C}, 0)$ such that x is a regular point of $\mathfrak{l}_{|_M}$ and such that the inclusion of the Milnor fiber of f at the unique point of $M \cap V(\mathfrak{l} - a)$ near x, for small $a \neq 0$, into the Milnor fiber of f at x induces an isomorphism on cohomology;

3. for all $\mathbb{Z}^\bullet_X$-visible $W \in \mathcal{W}$, (W, M) satisfies the Whitney (a) condition, and for all $x \in M$, for all representatives g of complex analytic germs from (X, x) to $(\mathbb{C}, 0)$ such that x is a regular point of $g_{|_M}$, the inclusion of the Milnor fiber of f at the unique point of $M \cap V(g - a)$ near x, for small $a \neq 0$, into the Milnor fiber of f at x induces an isomorphism on cohomology.

We recover the theorem of Lê and Saito by letting $X = \mathcal{U}$ and $\mathcal{W} = \{\mathcal{U}\}$.

Before proving Theorem 3.13, we will first discuss, in Section 2, basic definitions and results. In Section 3, we will prove our main theorem, and related results. Also in Section 3, we recall results from other papers which are essential to our proofs. In Section 4, we shall discuss the relations between the results and techniques of this paper and those of Briançon, Maisonobe, and Merle in [1].

2 Basic Definitions and Results

As in the introduction, we let X be an analytic subspace of $\mathcal{U}$, $f := \tilde{f}_{|_X}$, and let $\mathbf{A}^\bullet$ be a bounded, constructible complex of sheaves of R-modules on X. If M and N are complex submanifolds of $\mathcal{U}$, which are contained in X, then the $a_{\tilde{f}}$ condition for (M, N) depends only on f, and not on the extension $\tilde{f}$; hence, we refer simply to the a_f condition.

Definition 2.1. *A collection $\mathcal{W}$ of subsets of X is a* **(complex analytic) partition** *of X if and only if $\mathcal{W}$ is a locally finite disjoint collection of analytic submanifolds of $\mathcal{U}$, which we call strata, whose union is all of X, and such that, for each stratum $W \in \mathcal{W}$, $\overline{W}$ and $\overline{W} - W$ are closed complex analytic subsets of X.*

Throughout this paper, we assume that all partitions have connected strata.

A partition $\mathcal{W}$ is a **stratification** *if and only if it satisfies the condition of the frontier, i.e., for all $W \in \mathcal{W}$, $\overline{W}$ is a union of elements of $\mathcal{W}$.*

Note that, even when $\mathcal{W}$ is not a stratification, we nonetheless refer to elements of a partition $\mathcal{W}$ as strata.

Lemma 2.2. *Suppose that $\mathcal{W}$ and $\mathcal{W}'$ are partitions of X. Let $W \in \mathcal{W}$. Then, there exists a unique $W' \in \mathcal{W}'$ such that $\overline{W \cap W'} = \overline{W}$.*

Proof. This is easy. Let $p \in W$. Then, by local finiteness of $\mathcal{W}'$ and as $X = \bigcup_{W' \in \mathcal{W}} W'$, there exists an open neighborhood Θ of p in X, and $W'_1, \dots, W'_d \in \mathcal{W}'$, such that $\Theta \cap W \subseteq \bigcup_{i=1}^d (\Theta \cap W'_i)$, i.e.,

$$\Theta \cap W = \bigcup_{i=1}^{d} (\Theta \cap W \cap W'_i).$$

As W and the W'_i are analytically constructible, this implies that at least one of the $\Theta \cap W \cap W'_i$ is an analytically Zariski open dense subset of $\Theta \cap W$. As the elements of $\mathcal{W}'$ are disjoint, there must be a unique such W'_i; call it W'_p. Now, one uses the connectedness of W to conclude that W'_p is, in fact, the same element of $\mathcal{W}'$ for all p. The desired conclusion follows. □

For most cohomological results, we do **not** need a Whitney stratification of X with respect to which $\mathbf{A}^\bullet$ is constructible. We need merely a partition of X such that the cohomology of X, with coefficients in $\mathbf{A}^\bullet$, is "trivial" along the strata. Thus, we make the following definition.

Definition 2.3. *Let M be a complex submanifold of $\mathcal{U}$ such that $M \subseteq X$. We say that* $\mathbf{A}^\bullet$ **is ϕ-constructible along** *M if and only if, for all $x \in M$, for all representatives g of complex analytic germs from (X, x) to $(\mathbb{C}, 0)$ such that x is a regular point of $g_{|_M}$ (i.e., such that $d_x(g_{|_M})$ is a surjection), x is not contained in the support,* $\operatorname{supp}(\phi_g[-1]\mathbf{A}^\bullet)$*, of $\phi_g[-1]\mathbf{A}^\bullet$, i.e., there exists an open neighborhood Θ of x in $V(g)$ such that, for all $p \in \Theta$, $H^*(\phi_g[-1]\mathbf{A}^\bullet)_p = 0$.*

Let $\mathcal{W}$ be a partition of X. Then, $\mathbf{A}^\bullet$ **is ϕ-constructible with respect to** *$\mathcal{W}$ if and only if, for all $W \in \mathcal{W}$, $\mathbf{A}^\bullet$ is ϕ-constructible along W.*

Remark 2.4. The point of ϕ-constructibility is that it is a purely cohomological "replacement" for ordinary constructibility; one which does not need to refer to a Whitney stratification.

Of course, if $\mathfrak{S}$ is a Whitney stratification of X, with connected strata, then it is trivial to see that $\mathbf{A}^\bullet$ is ϕ-constructible with respect to $\mathfrak{S}$ if and only if $\mathbf{A}^\bullet$ is constructible with respect to $\mathfrak{S}$.

We wish to compare ϕ-constructibility with more standard notions. So, let $\mathfrak{S}$ denote a complex analytic Whitney stratification of X, with connected strata, with respect to which $\mathbf{A}^\bullet$ is constructible. For $S \in \mathfrak{S}$, we let $\mathbb{N}_S$ and $\mathbb{L}_S$ denote, respectively, the normal slice and link of the stratum S; see [4].

Definition 2.5. *A stratum $S \in \mathfrak{S}$ is* **$\mathbf{A}^\bullet$-visible** *if and only if the hypercohomology* $\mathbb{H}^*(\mathbb{N}_S, \mathbb{L}_S;\ \mathbf{A}^\bullet) \neq 0$. *We let* $\mathfrak{S}(\mathbf{A}^\bullet) := \{S \in \mathfrak{S} \mid S \text{ is } \mathbf{A}^\bullet\textit{-visible}\}$.

The point of defining $\mathbf{A}^\bullet$-visible strata is that, in most cohomological results, only the visible strata matter. In particular, if one refines $\mathfrak{S}$, i.e., simply throws in some extra strata, then the extra strata will be invisible; that is, the only possibly $\mathbf{A}^\bullet$-visible strata in the refinement are those whose closures are equal to closures of strata in $\mathfrak{S}$.

Throughout the remainder of this paper, the *micro-support*, $SS(\mathbf{A}^\bullet)$, of $\mathbf{A}^\bullet$ will be used extensively; see [8]. One may also use the proposition below as the definition of $SS(\mathbf{A}^\bullet)$ throughout this paper.

Proposition 2.6. ([16], Theorem 4.13) *The micro-support $SS(\mathbf{A}^\bullet)$ is equal to* $\displaystyle\bigcup_{S\in\mathfrak{S}(\mathbf{A}^\bullet)} \overline{T^*_S\mathcal{U}}$.

Let $\tau : T^*\mathcal{U} \to \mathcal{U}$ be the projection. For $Y \subseteq X$, we let $SS_Y(\mathbf{A}^\bullet) := \tau^{-1}(Y) \cap SS(\mathbf{A}^\bullet)$.

Now, we extend our definition of a "visible stratum" to certain kinds of partitions.

Definition 2.7. *A partition $\mathcal{W}$ of X is an* **$\mathbf{A}^\bullet$-partition** *provided that*

$$SS(\mathbf{A}^\bullet) \subseteq \bigcup_{W\in\mathcal{W}} \overline{T^*_W\mathcal{U}}.$$

If $\mathcal{W}$ is an **$\mathbf{A}^\bullet$-partition**, *then a stratum $W \in \mathcal{W}$ is* **$\mathbf{A}^\bullet$-visible** *if and only if* $\overline{T^*_W\mathcal{U}} \subseteq SS(\mathbf{A}^\bullet)$. *We let* $\mathcal{W}(\mathbf{A}^\bullet) := \{W \in \mathcal{W} \mid W \text{ is } \mathbf{A}^\bullet\textit{-visible}\}$.

Suppose that $\mathcal{W}$ is an **$\mathbf{A}^\bullet$-partition** *of X, and M is a complex submanifold of $\mathcal{U}$. Then, $(\mathcal{W}, M)$* **satisfies the $\mathbf{A}^\bullet$-visible Whitney (a) condition** *(respectively, the* **$\mathbf{A}^\bullet$-visible Thom a_f condition***) if and only if, for all $\mathbf{A}^\bullet$-visible $W \in \mathcal{W}$, (W, M) satisfies Whitney's condition (a) (respectively, Thom's a_f condition).*

Suppose that $\mathcal{W}$ is an **$\mathbf{A}^\bullet$-partition** *of X, and $\mathcal{W}'$ is a partition of a closed analytic subset of X. Then, $(\mathcal{W}, \mathcal{W}')$* **satisfies the $\mathbf{A}^\bullet$-visible Whitney (a) condition** *(respectively, the* **$\mathbf{A}^\bullet$-visible Thom a_f condition***) if and only if, for all $W' \in \mathcal{W}'$, $(\mathcal{W}, W')$ satisfies the $\mathbf{A}^\bullet$-visible Whitney (a) condition (respectively, the $\mathbf{A}^\bullet$-visible Thom a_f condition)*

Remark 2.8. The reader should understand that the point of an $\mathbf{A}^\bullet$-partition $\mathcal{W}$ is that, for each $\mathbf{A}^\bullet$-visible stratum S in $\mathfrak{S}$, there exists a unique $W \in \mathcal{W}$ such that $\overline{S} = \overline{W}$ and, hence, $\overline{T^*_S\mathcal{U}} = \overline{T^*_W\mathcal{U}}$. It follows at once from this, and the definition of $\mathbf{A}^\bullet$-visible strata of $\mathcal{W}$, that, if $\mathcal{W}$ is an $\mathbf{A}^\bullet$-partition, then

$$SS(\mathbf{A}^\bullet) \;=\; \bigcup_{W\in\mathcal{W}(\mathbf{A}^\bullet)} \overline{T^*_W\mathcal{U}}.$$

We could, of course, define an $\mathbf{A}^\bullet$-partition without using the conormal formulation in Definition 2.7. If we define the set $E(\mathbf{A}^\bullet)$ of $\mathbf{A}^\bullet$-*essential varieties* by $E(\mathbf{A}^\bullet) := \{\overline{S} \mid S \in \mathfrak{S}(\mathbf{A}^\bullet)\}$, then a partition $\mathcal{W}$ of X is an $\mathbf{A}^\bullet$-partition if and only if $E(\mathbf{A}^\bullet) \subseteq \{\overline{W} \mid W \in \mathcal{W}\}$. However, the conormal characterization in Definition 2.7 will be very useful later.

We should also remark that in [17], we referred to $\mathbf{A}^\bullet$-partitions as $\mathbf{A}^\bullet$-*normal partitionings*.

In [15], we made the following definition:

Definition 2.9. *The $\mathbf{A}^\bullet$-critical locus of f, $\Sigma_{\mathbf{A}^\bullet} f$, is $\{x \in X \mid H^*(\phi_{f-f(x)}[-1]\mathbf{A}^\bullet)_x \neq 0\}$.*

The support of $\phi_f[-1]\mathbf{A}^\bullet$ can be "calculated" as follows:

Theorem 2.10. ([19], Theorem 3.4)

$$\operatorname{supp}\phi_f[-1]\mathbf{A}^\bullet \;=\; V(f)\cap\overline{\Sigma_{\mathbf{A}^\bullet} f} \;=\; \{x\in V(f) \mid (x, d_x\tilde{f}) \in SS(\mathbf{A}^\bullet)\}.$$

Corollary 2.11. *In Definition 2.3, one may replace each reference to a complex analytic germ g by the restriction to X of an affine linear form and obtain a characterization of ϕ-constructibility.*

To be precise, let M be a complex submanifold of $\mathcal{U}$ such that $M \subseteq X$. Suppose that, for all $x \in M$, for all linear forms $\mathfrak{l} : \mathbb{C}^{n+1} \to \mathbb{C}$ such that x is a regular point of $\mathfrak{l}_{|_M}$, x is not contained in $\operatorname{supp}(\phi_{\mathfrak{l}-\mathfrak{l}(x)}[-1]\mathbf{A}^\bullet)$. *Then, $\mathbf{A}^\bullet$ is ϕ-constructible along M.*

Proof. By Theorem 2.10, $x \in \operatorname{supp}\phi_f[-1]\mathbf{A}^\bullet$ if and only if $x \in \operatorname{supp}\phi_{\mathfrak{l}-\mathfrak{l}(x)}[-1]\mathbf{A}^\bullet$ where $\mathfrak{l} := d_x\tilde{f}$ (where we have identified $\mathbb{C}^{n+1}$ with its tangent space at x). The corollary is immediate. □

3 Main Theorems

The following result is closely related to Proposition 8.6.4 of [8].

Proposition 3.1. *Let M be a complex submanifold of $\mathcal{U}$ which is contained in X. Then, $\mathbf{A}^\bullet$ is ϕ-constructible along M if and only if $SS_M(\mathbf{A}^\bullet) \subseteq T^*_M\mathcal{U}$.*

Proof. By Theorem 2.10, $p \notin \operatorname{supp}\phi_g[-1]\mathbf{A}^\bullet$ if and only if $d_p\tilde{g} \notin SS_p(\mathbf{A}^\bullet)$, where $\tilde{g}$ is a local extension of g to $\mathcal{U}$. The conclusion is immediate. □

Below, we once again identify $T_p\mathcal{U}$ with the ambient $\mathbb{C}^{n+1}$, and so identify elements of $(T^*\mathcal{U})_p$ with linear forms on the ambient space.

Lemma 3.2. *Let $\mathcal{W}$ be a partition of X such that, for all $W \in \mathcal{W}$ such that* $\dim W > 0$*, for all $p \in W$, there exists a projective algebraic set $V_p \subseteq \mathbb{P}\big((T^*\mathcal{U})_p\big)$ such that* $\dim V_p = \dim W - 1$*, and such that, for all projective classes $[\mathfrak{l}] \in V_p$, p is not contained in* $\operatorname{supp}(\phi_{\mathfrak{l}-\mathfrak{l}(p)}[-1]\mathbf{A}^\bullet)$.

Then, $\mathcal{W}$ is an $\mathbf{A}^\bullet$-partition;

Proof. Let $S \in \mathfrak{S}(\mathbf{A}^\bullet)$. Let W be the unique element of $\mathcal{W}$ such that $\overline{S \cap W} = \overline{S}$. Then, $\dim W \geq \dim S$. We claim that $\dim W = \dim S$, which implies that $\overline{W} = \overline{S}$ and $\overline{T^*_W\mathcal{U}} = \overline{T^*_S\mathcal{U}}$; this would prove the lemma.

If $\dim S = n+1$, there is nothing to show. So assume that $\dim S \leq n$.

Suppose that $\dim W > \dim S$. Let $p \in S \cap W$. Let V_p be as in the statement of the lemma. Then,

$$n - \dim \mathbb{P}\big((T^*_S\mathcal{U})_p\big) = \dim S < \dim W = \dim V_p + 1,$$

i.e., $n - 1 < \dim \mathbb{P}\big((T^*_S\mathcal{U})_p\big) + \dim V_p$. Thus, the projective algebraic subsets $\mathbb{P}\big((T^*_S\mathcal{U})_p\big)$ and V_p in $\mathbb{P}\big((T^*\mathcal{U})_p\big) \cong \mathbb{P}^n$ have a non-empty intersection, i.e., there exists $[\mathfrak{l}] \in \mathbb{P}\big((T^*_S\mathcal{U})_p\big) \cap V_p$. By Theorem 2.10, $p \in \operatorname{supp}(\phi_{\mathfrak{l}-\mathfrak{l}(p)}[-1]\mathbf{A}^\bullet)$, which contradicts that $[\mathfrak{l}] \in V_p$. □

Definition 3.3. *If $\mathcal{W}$ is a partition which satisfies the hypothesis of Lemma 3.2, we say that* $\mathbf{A}^\bullet$ **is weakly ϕ-constructible with respect to** $\mathcal{W}$.

Remark 3.4. Note that Lemma 3.2 enables us to talk about $\mathbf{A}^\bullet$-visible strata when $\mathbf{A}^\bullet$ is weakly ϕ-constructible with respect to $\mathcal{W}$.

Theorem 3.5. *Let $\mathcal{W}$ be a partition of X. Then, the following are equivalent:*

1. $\mathbf{A}^\bullet$ *is ϕ-constructible with respect to $\mathcal{W}$;*
2. $SS(\mathbf{A}^\bullet) \subseteq \bigcup_{W \in \mathcal{W}} T^*_W\mathcal{U}$*;*
3. *$\mathcal{W}$ is an $\mathbf{A}^\bullet$-partition such that $(\mathcal{W}, \mathcal{W})$ satisfies the $\mathbf{A}^\bullet$-visible Whitney (a) condition;*
4. $\mathbf{A}^\bullet$ *is weakly ϕ-constructible with respect to $\mathcal{W}$, and $(\mathcal{W}, \mathcal{W})$ satisfies the $\mathbf{A}^\bullet$-visible Whitney (a) condition.*

Proof. That (1) and (2) are equivalent follows immediately from Proposition 3.1.

If $\mathcal{W}$ is an $\mathbf{A}^\bullet$-partition, then $SS(\mathbf{A}^\bullet) = \bigcup_{W \in \mathcal{W}(\mathbf{A}^\bullet)} \overline{T^*_W\mathcal{U}}$, and $(\mathcal{W}, \mathcal{W})$ satisfies the visible Whitney (a) condition if and only if

$$\bigcup_{W \in \mathcal{W}(\mathbf{A}^\bullet)} \overline{T^*_W\mathcal{U}} \subseteq \bigcup_{W \in \mathcal{W}} T^*_W\mathcal{U}.$$

Thus, (2) and (3) are equivalent.

Now, (1) and (3) are equivalent, and clearly, together, they imply (4). Finally, Lemma 3.2 tells us that (4) implies (3). □

Example 3.6. In order to see why the $\mathbf{A}^\bullet$-visible Whitney (a) condition is important in the above theorem, consider the following example. Let $X=\mathcal{U}=\mathbb{C}^2$, and use y and z for coordinates. Let $\mathbf{B}^\bullet$ be the constant sheaf (over $\mathbb{Z}$) on the y-axis, extended by zero to all of $\mathbb{C}^2$. Let $\mathbf{C}^\bullet$ be the constant sheaf (over $\mathbb{Z}$) on the z-axis, extended by zero to all of $\mathbb{C}^2$. Let $\mathbf{A}^\bullet=\mathbf{B}^\bullet\oplus\mathbf{C}^\bullet$.

Then, $SS(\mathbf{A}^\bullet)=T^*_{V(z)}\mathcal{U}\cup T^*_{V(y)}\mathcal{U}$. The conormal to the origin $T^*_{\mathbf{0}}\mathcal{U}$ does **not** appear in $SS(\mathbf{A}^\bullet)$, because the stalk of $\mathbf{A}^\bullet$ at the origin is $\mathbb{Z}\oplus\mathbb{Z}$ and so is the stalk cohomology of the Milnor fiber of a generic linear form, and the comparison map is an isomorphism; it follows that the vanishing cycles of $\mathbf{A}^\bullet$ along a generic linear form are zero at the origin.

Thus, the partition $\mathcal{W}=\{\mathbb{C}^2-V(yz), V(z), V(y)-\{\mathbf{0}\}\}$ is an $\mathbf{A}^\bullet$-partition of $\mathbb{C}^2$. Note that we have **not** included $\{\mathbf{0}\}$ as a stratum. The $\mathbf{A}^\bullet$-visible strata of $\mathcal{W}$ are $V(z)$ and $V(y)-\{\mathbf{0}\}$. The paragraph above tells us that $\mathbf{A}^\bullet$ is weakly ϕ-constructible with respect to $\mathcal{W}$. However, $(\mathcal{W},\mathcal{W})$ does not satisfy the $\mathbf{A}^\bullet$-visible Whitney (a) condition. According to Theorem 3.5, $\mathbf{A}^\bullet$ is not ϕ-constructible with respect to $\mathcal{W}$.

This is easy to see in our current example. The origin is a regular point of the function given by the restriction of y to $V(z)$, and yet $H^*(\phi_y\mathbf{A}^\bullet)_{\mathbf{0}}\neq 0$, since the nearby fiber of the function y is a single point which has a single $\mathbb{Z}$ for its cohomology (in degree 0).

Below, for the sake of self-containment, we state the results from other papers that we need.

We identify $T^*\mathcal{U}$ with $\mathcal{U}\times\mathbb{C}^{n+1}$. Recall that $\tau: T^*\mathcal{U}\to\mathcal{U}$ is the projection.

If $\mathbf{B}^\bullet$ is a bounded, constructible complex of sheaves on X, and $Y\subseteq X$, we let $SS(\mathbf{B}^\bullet)_{\subseteq Y}$ denote the components of $SS(\mathbf{B}^\bullet)$ which lie above Y (i.e., project by τ to a set contained in Y). Note that, in general, $SS(\mathbf{B}^\bullet)_{\subseteq Y}\subseteq SS_Y(\mathbf{B}^\bullet)$, and the inclusion can easily be proper. Let $i: X-V(f)\hookrightarrow X$ denote the inclusion.

The equality involving $SS(i_!i^!\mathbf{A}^\bullet)_{\subseteq V(f)}$ in the next theorem can be proved using Theorem 4.2 of [14]. In Proposition 4.5 of [17], we used 4.2.1 and 3.4.2 of [1] to prove the result when $\mathbf{A}^\bullet$ is a perverse sheaf; in the perverse sheaf case, the characteristic cycle is equal to the micro-support. Our primary reason for including results about $SS(i_!i^!\mathbf{A}^\bullet)_{\subseteq V(f)}$ will be discussed in the next section.

Theorem 3.7.([19], Theorem 3.2 and [17], Proposition 4.5) *There is the following equality of subspaces of the cotangent space, $T^*\mathcal{U}$:*

$$SS(\psi_f[-1]\mathbf{A}^\bullet) \;=\; SS(i_!i^!\mathbf{A}^\bullet)_{\subseteq V(f)} \;=\; \tau^{-1}(V(f)) \,\cap\, \Big(\bigcup_{\substack{S\in\mathfrak{S}(\mathbf{A}^\bullet)\\ f_{|_S}\neq \text{ const.}}} \overline{T^*_{f_{|_S}}\mathcal{U}}\Big).$$

Corollary 3.8. *Let $\mathcal{W}$ be an $\mathbf{A}^\bullet$-partition of X. Let M be a complex submanifold of $\mathcal{U}$ such that $M\subseteq V(f)$. Then, the following are equivalent:*

1. *for all $W\in\mathcal{W}(\mathbf{A}^\bullet)$ such that $W\not\subseteq V(f)$, (W,M) satisfies the a_f condition;*
2. *$\psi_f[-1]\mathbf{A}^\bullet$ is ϕ-constructible along M;*
3. *$\tau^{-1}(M)\cap SS(i_!i^!\mathbf{A}^\bullet)_{\subseteq V(f)}\subseteq T^*_M\mathcal{U}$.*

In addition, these equivalent conditions imply that, for all $W\in\mathcal{W}(\mathbf{A}^\bullet)$ such that $W\not\subseteq V(f)$, (W,M) satisfies the Whitney (a) condition

Proof. Combine the theorem with Proposition 3.1 and the conormal characterization of the a_f condition. $\square$

Corollary 3.9. *Let $\mathcal{W}$ be an $\mathbf{A}^\bullet$-partition of X. Let $\mathcal{W}'$ be a Whitney (a) partition of $V(f)$. Then, the following are equivalent:*

1. *for all $W \in \mathcal{W}(\mathbf{A}^\bullet)$ such that $W \not\subseteq V(f)$, for all $W' \in \mathcal{W}'$, (W, W') satisfies the a_f condition;*
2. *$\psi_f[-1]\mathbf{A}^\bullet$ is ϕ-constructible with respect to $\mathcal{W}'$;*
3. *$\psi_f[-1]\mathbf{A}^\bullet$ is weakly ϕ-constructible with respect to $\mathcal{W}'$.*

Proof. This follows at once from Corollary 3.8 and Theorem 3.5. $\square$

Remark 3.10. We wish to discuss the problem in using Corollary 3.9 in practice; a problem that is removed by replacing the nearby cycles with the vanishing cycles.

Our primary goal in this paper, as we discussed in the introduction, is to provide a generalization of the result of Lê and Saito, in the form given in Theorem 1.8. We could obtain an analogous statement, using the nearby cycles in place of the vanishing cycles, by using the equivalence of Items 1 and 3 above. The problem is that we are required to begin with a Whitney (a) partition of all of $V(f)$, instead of merely a partition of Σf. Requiring that the smooth part of $V(f)$ satisfy the Whitney (a) condition with respect to strata of Σf is an unacceptable assumption, as such an assumption does not appear in the theorem of Lê and Saito.

The way that we will fix this problem is to use the vanishing cycles, whose support is contained in the critical locus.

We let $\pi : \mathcal{U} \times \mathbb{C}^{n+1} \times \mathbb{P}^n \to \mathcal{U} \times \mathbb{P}^n$ and $\nu : \mathcal{U} \times \mathbb{P}^n \to \mathcal{U}$ denote the respective projections. Recall that $\tilde{f}$ is our global extension of f to all of $\mathcal{U}$ (though we could use local extensions at each point). We let $\operatorname{im} d\tilde{f}$ denote the image of $d\tilde{f}$ in $T^*\mathcal{U}$.

Theorem 3.11.([17], Proposition 4.3) *Suppose that Y is an analytic subset of X. Suppose that f is not constant on any irreducible component of Y. Let E denote the exceptional divisor in $\mathrm{Bl}_{\operatorname{im} df}\overline{T^*_{Y_{\mathrm{reg}}}\mathcal{U}} \subseteq \mathcal{U} \times \mathbb{C}^{n+1} \times \mathbb{P}^n$. Suppose that $M \subseteq X$ is a complex analytic submanifold of $\mathcal{U}$ and that $x \in M$ is such that (X_{reg}, M) satisfies Whitney's condition a) at x and such that $d_x(f_{|_M}) \equiv 0$.*

Then, (Y_{reg}, M) satisfies Whitney's a_f condition at x if and only if there is the containment of fibres above x given by

$$\big(\pi(E)\big)_x \subseteq \big(\mathbb{P}(T^*_M\mathcal{U})\big)_x.$$

Now, for each $S \in \mathfrak{S}$ or $S \in \mathcal{W}$, let E_S denote the exceptional divisor of $\mathrm{Bl}_{\operatorname{im} d\tilde{f}}\overline{T^*_S\mathcal{U}}$. Then, in our current notation, the second equality of Theorem 3.4 of [19] tells us:

Theorem 3.12.([19], Theorem 3.4) *There is the following equality of subspaces of the projectivized cotangent space, $\mathbb{P}(T^*\mathcal{U})$:*

$$\mathbb{P}\big(SS(\phi_f[-1]\mathbf{A}^\bullet)\big) = \mathbb{P}\big(SS(\phi'_f[-1]\mathbf{A}^\bullet)\big) = \nu^{-1}(V(f)) \ \cap \ \pi\Big(\bigcup_{S\in\mathfrak{S}(\mathbf{A}^\bullet)} E_S\Big).$$

Note that, if $\mathcal{W}$ is an $\mathbf{A}^\bullet$-partition, then, in Theorem 3.12, we could have replaced $\bigcup_{S\in\mathfrak{S}(\mathbf{A}^\bullet)} E_S$ by $\bigcup_{W\in\mathcal{W}(\mathbf{A}^\bullet)} E_W$.

In previous papers, we have proved two results along the lines of our main theorem below. In Theorem 4.4 of [17], we proved a form of this result in the case where $\mathbf{A}^\bullet$ is a *perverse sheaf*. In the case of general $\mathbf{A}^\bullet$, we proved one direction of this result in Theorem 6.5 of [19]. In addition to containing less general results than our current paper, both [17] and [19] are so abstract that the reader would have difficulty extracting the relevant results. Also, Theorem 6.5 of [19] is proved using Theorem 4.8 of that paper; Theorem 4.8 is misstated (though is fine in the case where it is used). For all of these reasons, we prove both directions of the theorem below.

Theorem 3.13. *Let $\mathcal{W}$ be an $\mathbf{A}^\bullet$-partition of X. Let M be a complex submanifold of $\mathcal{U}$ such that $M \subseteq V(f)$.*

Then, $(\mathcal{W}, M)$ satisfies the $\mathbf{A}^\bullet$-visible a_f condition if and only if $(\mathcal{W}, M)$ satisfies the $\mathbf{A}^\bullet$-visible Whitney (a) condition and $\phi_f[-1]\mathbf{A}^\bullet$ is ϕ-constructible along M.

Proof. Let $W \in \mathcal{W}$. From Theorem 3.11, it follows easily that: (†) the pair (W, M) satisfies the a_f condition if and only if (W, M) satisfies Whitney's condition (a) and $\nu^{-1}(M) \cap \pi(E_W) \subseteq \mathbb{P}(T^*_M\mathcal{U})$.

Proof of ⇒:

Now, suppose that $(\mathcal{W}, M)$ satisfies the $\mathbf{A}^\bullet$-visible a_f condition. Then, (†) immediately implies that $(\mathcal{W}, M)$ satisfies the $\mathbf{A}^\bullet$-visible Whitney (a) condition. Combining Theorem 3.12 with (†), we find that

$$\mathbb{P}\big(SS_M(\phi_f[-1]\mathbf{A}^\bullet)\big) = \nu^{-1}(M) \ \cap \ \pi\Big(\bigcup_{W\in\mathcal{W}(\mathbf{A}^\bullet)} E_W\Big) \subseteq \mathbb{P}\big(T^*_M\mathcal{U}\big).$$

By Proposition 3.1, this is equivalent to $\phi_f[-1]\mathbf{A}^\bullet$ being ϕ-constructible along M.

Proof of ⇐:

Suppose that $(\mathcal{W}, M)$ satisfies the $\mathbf{A}^\bullet$-visible Whitney (a) condition, and that $\phi_f[-1]\mathbf{A}^\bullet$ is ϕ-constructible along M. Then, as above,

$$\mathbb{P}\big(SS_M(\phi_f[-1]\mathbf{A}^\bullet)\big) = \nu^{-1}(M) \ \cap \ \pi\Big(\bigcup_{W\in\mathcal{W}(\mathbf{A}^\bullet)} E_W\Big) \subseteq \mathbb{P}\big(T^*_M\mathcal{U}\big).$$

Let $W \in \mathcal{W}(\mathbf{A}^\bullet)$. By the $\mathbf{A}^\bullet$-visible Whitney (a) condition and (†), what we need to show is that $\nu^{-1}(M) \cap \pi(E_W) \subseteq \mathbb{P}\big(T^*_M\mathcal{U}\big)$, which follows from the above. □

Let $\phi'_f[-1]\mathbf{A}^\bullet$ denote the complex of sheaves on $V(f) \cap \overline{\Sigma_{\mathbf{A}^\bullet} f}$ obtained by restricting $\phi_f[-1]\mathbf{A}^\bullet$ to its support. Note that $SS(\phi'_f[-1]\mathbf{A}^\bullet) = SS(\phi_f[-1]\mathbf{A}^\bullet)$.

Corollary 3.14. *Let $\mathcal{W}$ be an $\mathbf{A}^\bullet$-partition of X. Let $\mathcal{W}'$ be a Whitney (a) partition of $V(f) \cap \overline{\Sigma_{\mathbf{A}^\bullet} f}$. Suppose that $(\mathcal{W}, \mathcal{W}')$ satisfies the $\mathbf{A}^\bullet$-visible Whitney (a) condition.*

Then, the following are equivalent:

1. *$(\mathcal{W}, \mathcal{W}')$ satisfies the $\mathbf{A}^\bullet$-visible a_f condition;*

2. $\phi'_f[-1]\mathbf{A}^\bullet$ *is* ϕ*-constructible with respect to* $\mathcal{W}'$*;*

3. $\phi'_f[-1]\mathbf{A}^\bullet$ *is weakly* ϕ*-constructible with respect to* $\mathcal{W}'$*;*

and, if $\mathcal{W}'$*, is, in fact, a Whitney stratification, these are equivalent to:*

4. $\phi'_f[-1]\mathbf{A}^\bullet$ *is constructible with respect to* $\mathcal{W}'$*.*

Proof. To obtain the equivalences of Items 1, 2, and 3, simply combine Theorem 3.13 with Theorem 3.5. If $\mathcal{W}'$, is a Whitney stratification, then the equivalence of Items 2 and 4 follows from Remark 2.4. □

Example 3.15. Let us return to the result of Lê and Saito, which we discussed at length in the introduction. We use the assumptions and notation that we used in Theorem 1.5.

Let $X = \mathcal{U}$, $\mathcal{W} = \{\mathcal{U}\}$, $f = \tilde{f}$, and $\mathbf{A}^\bullet = \mathbb{Z}^\bullet_{\mathcal{U}}$. Then, the critical locus of f near the origin is equal to $V(f) \cap \overline{\Sigma_{\mathbb{Z}^\bullet_{\mathcal{U}}} f}$, which we suppose is simply $Y = \mathcal{U} \cap (\mathbb{C} \times \{\mathbf{0}\})$. Let $\mathcal{W}' = \{Y\}$.

Then, the hypotheses of Corollary 3.14 are satisfied. In addition, the pair $(\mathcal{U}, Y)$ satisfies the a_f **generically** along Y; the only question, near the origin is: "what happens at the origin?". As Y is 1-dimensional, to know that $\phi'_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}}$ is weakly ϕ-constructible, we need to have a single non-zero linear form $\mathfrak{l} : \mathbb{C}^{n+1} \to \mathbb{C}$ such that $H^*(\phi_{\mathfrak{l}}[-1](\phi'_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}})) = H^*(\phi_{\mathfrak{l}}[-1](\phi_f[-1]\mathbb{Z}^\bullet_{\mathcal{U}})) = 0$ (recall Remark 3.4).

Therefore, the conclusion of Corollary 3.14 is precisely our third version of the Lê-Saito Theorem, which we stated in Theorem 1.8.

4 Relations with the Work of Briançon, Maisonobe, and Merle

In [1], Briançon, Maisonobe, and Merle introduce the *condition of local, stratified triviality* – a condition on a Whitney (a) stratification. The condition is that, for any point x in a stratum S, every analytic transverse slice to S at x (of any dimension) yields a stratified homeomorphism between an open neighborhood of x and the product of the slice with a open ball. See Definition 4.1.1 of [1].

Thus, if one has a stratification, $\mathcal{W}$, of X of which satisfies the condition of local, stratified triviality and $\mathbf{A}^\bullet$ is a bounded, constructible complex of sheaves on X whose local structure depends only on the local stratified topological-type of X, then $\mathbf{A}^\bullet$ will be ϕ-constructible with respect to $\mathcal{W}$.

Consider now a Whitney stratification $\mathfrak{S}$ of X such that $V(f)$ is a union of strata. Recall that $i : X - V(f) \hookrightarrow X$ denotes the inclusion. For each stratum $S \in \mathfrak{S}$ such that $S \not\subseteq V(f)$, let $\mathbf{A}^\bullet_S$ denote the extension by zero, to all of X, of the constant sheaf $\mathbb{Z}^\bullet_S$. Then, as $\mathfrak{S}$ is a Whitney stratification, $i_!i^!\mathbf{A}^\bullet_S$ is ϕ-constructible with respect $\mathfrak{S}$, and certainly S is $(i_!i^!\mathbf{A}^\bullet_S)$-visible.

Therefore, if M is a stratum of $\mathfrak{S}$ and $M \subseteq V(f)$, then Proposition 3.1 tells us that

$$\tau^{-1}(M) \cap SS(i_!i^!\mathbf{A}^\bullet)_{\subseteq V(f)} \ \subseteq \ SS_M(i_!i^!\mathbf{A}^\bullet) \ \subseteq \ T^*_M\mathcal{U},$$

and Corollary 3.8 tells us that the pair (S, M) satisfies the a_f condition.

The above is precisely the argument used in [1] to prove that Whitney stratifications, in which $V(\tilde{f}) := \tilde{f}^{-1}(0)$ is a union of strata, are $a_{\tilde{f}}$ stratifications. We remark again that this result was proved independently by Parusiński in [21]. We should also remark that, because Briançon, Maisonobe, and Merle used characteristic cycles, instead of micro-supports, in some parts of their paper, they needed to use a perverse sheaf for our $\mathbf{A}^\bullet_S$ above. Hence, rather than use the extension by zero of the constant sheaf, they used the extension by zero of the intersection cohomology complex (with constant coefficients) on $\overline{S}$.

The reader should understand that we included results on $\psi_f[-1]\mathbf{A}^\bullet$ and $SS(i_!i^!\mathbf{A}^\bullet)_{\subseteq V(f)}$ in this paper in order to show how ϕ-constructible partitions arise in the proof of the main theorem of [1]; most of these results appeared in some form in [1]. However, the results of [1] do not give us the desired generalization of the result of Lê and Saito; for that, we need our results on the vanishing cycles in Theorem 3.11, Theorem 3.12, Theorem 3.13, and Corollary 3.14.

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